THE ZEROS OF THE PARTITION FUNCTION OF THE PINNING MODEL

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Abstract. We aim at understanding for which (complex) values of the potential the pinning partition function vanishes. The pinning model is a Gibbs measure based on discrete renewal processes with power law inter-arrival distributions. We obtain some results for rather general inter-arrival laws, but we achieve a substantially more complete understanding for a specific one parameter family of inter-arrivals. We show, for such a specific family, that the zeros asymptotically lie on (and densely fill) a closed curve that, unsurprisingly, touches the real axis only in one point (the critical point of the model). We also perform a sharper analysis of the zeros close to the critical point and we exploit this analysis to approach the challenging problem of Griffiths singularities for the disordered pinning model. The techniques we exploit are both probabilistic and analytical. Regarding the first, a central role is played by limit theorems for heavy tail random variables. As for the second, potential theory and singularity analysis of generating functions, along with their interplay, will be at the heart of several of our arguments.

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1. Introduction and results

1.1. **The pinning model: general framework.** We denote by $\tau = (\tau_j)_{j=0,1,2,\ldots}$ a discrete recurrent renewal process with $\tau_0 = 0$. So $(\tau_{j+1} - \tau_j)_{j=0,1,2,\ldots}$ is a sequence of IID random variables taking values in $\mathbb{N} := \{1, 2, \ldots\}$. Often one considers the rather general framework that for $n \in \mathbb{N}$

$$K(n) := \mathbf{P}(\tau_1 = n) \overset{n\to\infty}{\sim} \frac{c}{n^{1+\alpha}} \text{ and } K(1) > 0\,, \tag{1.1}$$

with $\alpha \in (0, 1)$ and $c > 0$ and, unless otherwise stated, we make the choice $c = 1/(-\Gamma(-\alpha))$. Of course $K(\cdot)$, called *inter-arrival distribution*, determines the law of τ. The requirement $K(1) > 0$ is not essential, but allowing $K(1)$ to be zero does complicate some arguments and notations (see comment after (1.5)). Note that any of the two requirements in (1.1) implies that τ is aperiodic and that recurrence means that $\sum_n K(n) = 1$.

Remark 1.1. *The precise value of c has only a minor impact on the model. The choice we make of c is customary when dealing with stable laws because it simplifies some expressions. In fact* [12, pp. 448-449] *we know that there exists a sequence of positive real numbers* (a_n) *such that* τ_n/a_n *converges to the (stable) limit law with support on the positive semi-axis*

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and with Laplace transform $s \mapsto \exp(-s^\alpha)$. *The normalizing sequence* (a_n) *turns out to be asymptotically proportional to* $n^{1/\alpha}$, *so we can choose* a_n *equal to* $n^{1/\alpha}$ *times a positive constant: this constant is equal to* 1 *if* $c = 1/(-\Gamma(-\alpha))$.

We consider

$$Z_{N,h} := \mathbf{E}\left[\exp\left(h\sum_{n=1}^{N}\delta_n\right)\delta_N\right], \tag{1.2}$$

where $\delta_n := \mathbf{1}_{n\in\tau}$: we are viewing $\tau = \{\tau_0, \tau_1, \ldots\}$ as a random subset of $\mathbb{N}\cup\{0\}$. We will work with $h \in \mathbb{C}$, but let us first consider the case $h \in \mathbb{R}$. It is straightforward to see that, in this case, $(\log Z_{N,h})$ is a super-additive sequence, so the limit

$$\textsc{f}(h) := \lim_N \frac{1}{N}\log Z_{N,h}, \tag{1.3}$$

exists for every $h \in \mathbb{R}$ and it is equal to the supremum of the sequence. Moreover, $\textsc{f}(h)$ can be identified via an elementary computation, e.g. [19, pp. 7 and 8] and the result is that

$$\textsc{f}(h) = \begin{cases} \text{unique solution } \textsc{f} \text{ of } \sum_n K(n)\exp(-n\textsc{f}) = \exp(-h) & \text{if } h \geq 0, \\ 0 & \text{if } h < 0. \end{cases} \tag{1.4}$$

It can be seen from (1.4) that $h \mapsto \textsc{f}(h)$ is (strictly) increasing and strictly convex on the positive semi-axis, while it is non decreasing and convex over all $\mathbb{R}$: all these properties can be extracted also directly from (1.3). It is also clear that $\textsc{f}(\cdot)$ is not (real) analytic at the origin. What (1.4) tells us beyond this is that the origin is the only singular point (*critical point*): $\textsc{f} \mapsto \sum_n K(n)\exp(-n\textsc{f})$ is a real analytic invertible map from $(0,\infty)$ to $(0,1)$ so $h \mapsto \textsc{f}(h)$ is real analytic simply because $\textsc{f}(h)$ is obtained by applying the inverse of the map to $\exp(-h)$.

On the other hand, $Z_{N,h}$ is just a polynomial of degree N in $\exp(h)$ and in fact it is sometimes more practical to use the polynomial notation

$$P_N(w) := Z_{N,\log w}. \tag{1.5}$$

Note that the degree of $P_N(w)$ is N because we are assuming $K(1) > 0$. If $K(1) = 0$ the degree would be smaller, for example if $K(1) = 0$ and $K(2) > 0$ then $P_N(w)$ is a polynomial of degree $\lfloor N/2 \rfloor$. Choosing $K(1) > 0$ hence simplifies the normalization of the empirical probability of the zeros. Other (non essential, albeit welcome) simplifications due to this choice are connected to $P_N(w) \sim K(1)w$ for w small.

Remark 1.2. *We mostly work with the variable* h *which is more natural in the statistical mechanics language. It must be however noted that* $Z_{N,h}$ *is* 2π*-periodic in* $\Im(h)$*: this periodicity is just the periodicity of the exponential function in the imaginary direction. So, strictly speaking* $Z_{N,h}$ *always has infinitely many zeros, but they are just periodic copies of the* $N-1$ *zeros in* $\mathbb{C}$ *with imaginary part (say) in* $(-\pi,\pi]$*: note that the origin is a simple zero of* $P_N(w)$ *(again,* $K(1) > 0$*), but this zero is at* $-\infty$ *for* $Z_{N,h}$*. It is therefore natural to introduce* $\mathbb{C}_{2\pi} := \mathbb{C}/(2\pi i\mathbb{Z})$*, which we will identify with* $\{z \in \mathbb{C} : \Im(z) \in (-\pi,\pi]\}$*, and restrict ourselves to this set when dealing with the* $N-1$ *zeros of* $Z_{N,h}$.

We refer to [14, 17, 19, 24] for a thorough discussion of the model in statistical mechanics terms: the *critical point* $h = 0$ captures a delocalization ($h < 0$) to localization ($h > 0$) transition. Taking the Lee-Yang viewpoint [26], we remark that $(1/N)\log Z_{N,h}$ is real analytic on the whole of $\mathbb{R}$. But $Z_{N,h}$ is an entire function and the singularities in the complex plane of $(1/N)\log Z_{N,h}$ are due to the zeros of $Z_{N,h}$: in the limit $N\to\infty$ these singularities may accumulate on the real axis. In our case, they are going to accumulate on the real axis only at zero. Our purpose is to determine the location of the zeros of $Z_{N,h}$ for $N\to\infty$. We stress that, unless otherwise stated, by $\log(\cdot)$ we mean the principal branch of the complex logarithm: this is discussed more in detail after (1.8) where we introduce the notation $\mathrm{Log}(\cdot)$ for the principal branch, but $\mathrm{Log}(\cdot)$ will be used only when strictly needed.

Even if results for pinning models with $h\in\mathbb{R}$ are typically obtained assuming only (1.1) (or in even wider frameworks: for example regularly varying inter-arrival distributions [17]) and the results *essentially* depend only on α, it appears to be really challenging to extend such a universal behavior to $h\in\mathbb{C}$. So, we will give some results assuming only (1.1), but we are able to obtain a good control on the location of the zeros when working with a much more restrictive choice of $K(\cdot)$. But let us start with a general result that holds in our most general framework:

Proposition 1.3. *We fix $K(\cdot)$ that satisfies* (1.1)*. Then there exists $C>0$ and, for every $\varepsilon>0$ and $\varepsilon'>0$, there exist $N_{\varepsilon,\varepsilon'}\in\mathbb{N}$, $C_{\varepsilon'}>0$ and a subset $V_{\varepsilon,\varepsilon'}$ of the complex plane that contains*

(1) the half plane $\Re(h)\le -\varepsilon$;
(2) the half plane $\Re(h)\ge C$;
(3) the set of h's with $\Re(h)>\varepsilon'$ and $|\Im(h)|<C_{\varepsilon'}$;

such that $Z_{N,h}\neq 0$ for every $h\in V_{\varepsilon,\varepsilon'}$ and every $N\ge N_{\varepsilon,\varepsilon'}$.

This statement is visualized in Figure 1.

We will give a sketch of the proof of Proposition 1.3 in § 1.6: while Proposition 1.3 is rather rough, § 1.6 will be of help in understanding some of the tools we repeatedly use (also in the proof of much sharper results) and why in the general framework we are limited to Proposition 1.3.

1.2. **Special inter-arrival distributions.** A special choice of $K(\cdot)$ for which we are able to go much farther is

$$K(n)\,=\,\frac{\Gamma(n-\alpha)}{-\Gamma(-\alpha)\,n!}\,=\,\frac{-(n-1-\alpha)(n-2-\alpha)\cdots(1-\alpha)(-\alpha)}{n!}\stackrel{n\to\infty}{\sim}\frac{n^{-(1+\alpha)}}{-\Gamma(-\alpha)}\,. \tag{1.6}$$

We remark also that $K(1)=\alpha$. One of the important features of this distribution is that its z-transform (or characteristic function) has an explicit expression:

$$\widehat{K}(z)\,:=\,\sum_{j=1}^{\infty}z^jK(j)\,=\,1-(1-z)^{\alpha}\,. \tag{1.7}$$

The power series defining $\widehat{K}(z)$ has radius of convergence 1, and this is of course true also for the general framework (1.1). The explicit expression in (1.7) is saying that, with the special choice (1.6), $\widehat{K}(z)$ can be extended to $\mathbb{C}\setminus\{z:\Re(z)\ge 1\}$.

Remark 1.4. *This is not a generic feature: in fact, under the hypothesis* (1.1)*, one can exhibit $K(\cdot)$ such that $\widehat{K}(z)$ has a natural boundary on the unit circle. For example, if $K_A(n)=C\mathbf{1}_A(n)/n^2$ with $A=\{n_j:\,j\in\mathbb{N}\}$ with $n_j/j\to\infty$, $C>0$ chosen so that $\sum_n K_A(n)=1$ and if we further assume that n_j does not diverge too fast so that the series defining $\widehat{K}_A(z)$ has 1 as radius of convergence (take for example $n_j=j^2$) then*

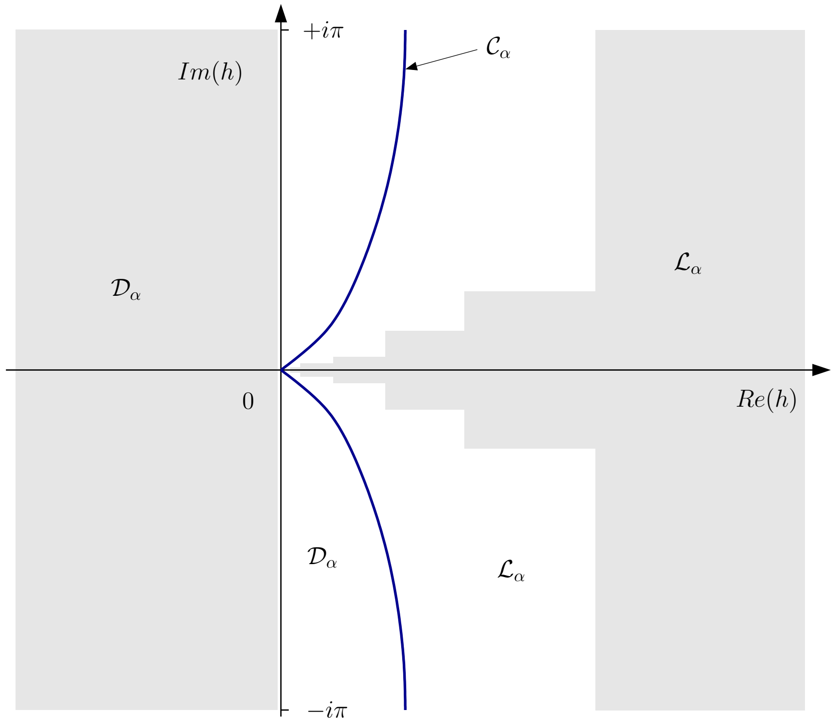


FIGURE 1. A pictorial vision of the content of Proposition 1.3, and beyond: there is no zero in the shadowed region for N sufficiently large. The shadowed region is obtained by applying Proposition 1.3 for more than one value of ε', in particular a very small value. In the restricted set-up with inter-arrival distribution (1.6) we can show that the zeros asymptotically lie on a *critical* curve $\mathcal{C}_\alpha$ which splits $\mathbb{C}_{2\pi}$ into a *delocalized* region $\mathcal{D}_\alpha$, in which $|Z_{N,h}|$ does not grow exponentially, and a *localized* one $\mathcal{L}_\alpha$ in which $|Z_{N,h}|$ grows exponentially. We stress that the lower bound we obtain on $C_{\varepsilon'}$, see Proposition 1.3, is $o(\varepsilon')$, so, in the general framework, we do not establish that there exists $m > 0$ such that there is no zero in the cone $\{z \in \mathbb{C} : \Re(z) > 0 \text{ and } |\Im(z)| < m\Re(z)\}$. This feature however does hold in the restricted set-up (1.6).

$\widehat{K}_A(\cdot)$ *is singular everywhere on* $\partial B_0(1)$ [9, Ch. XI, in particular p. 373]. *So the inter-arrival distribution* $K(n) = (K_1(n) + K_A(n))/2$, *with* $K_1(\cdot)$ *given in* (1.6), *satisfies* (1.1) *(except possibly for the value of c, but this can easily be fixed) and has* $\partial B_0(1)$ *as natural boundary.*

Let us also observe right away that by applying the formula for $\textsc{f}(h)$ given right after (1.3) we have that the choice (1.6) yields for $h > 0$

$$\textsc{f}(h) \,=\, -\log\left(1 - (1 - \exp(-h))^{1/\alpha}\right) . \tag{1.8}$$

How far into $\mathbb{C}$ can this function be analytically continued? One problem comes from $h \mapsto (1 - \exp(-h))^{1/\alpha}$ that has a cut discontinuity starting at origin (unless $1/\alpha = 2, 3, \ldots$ for which $h \mapsto (1 - \exp(-h))^{1/\alpha}$ is entire). Unless otherwise stated, by z^c, $c \in \mathbb{R}$, we mean $\exp(c \operatorname{Log} z)$ with $\operatorname{Log}(\cdot)$ the principal branch of the logarithm (for $z \in (-\infty, 0)$ we set $\operatorname{Log} z := \log|z| + i\pi$) . In particular, with this choice, the cut of $h \mapsto (1 - \exp(-h))^{1/\alpha}$ is on $(-\infty, 0)$. Therefore, if $\log_R(\cdot)$ is the logarithm defined from its natural Riemann surface (infinitely many copies of $\mathbb{C}$) to $\mathbb{C}$ [1, Ch. 8], we have that

$$\textsc{f}(h) := -\log_R\left(1-\exp\left((1/\alpha)\,\mathrm{Log}\,(1-\exp(-h))^{1/\alpha}\right)\right), \tag{1.9}$$

is analytic on $\mathbb{C}\setminus(-\infty,0]$ and coincides with $\textsc{f}(h)$ for $h>0$.

On the other hand $\textsc{f}(h)=0$ for $h<0$, which of course can be continued to the whole $\mathbb{C}$: so, at this stage there is no reason to believe that the continuation defined in (1.9) is relevant, at least not over the whole region where we have defined it.

In terms of *continuation* of the real free energy outside $\mathbb{R}$ the related *harmonic continuation* is a priori more straightforward: $\Re(\textsc{f}(h))$ (cf. (1.9)) is harmonic on $\mathbb{C}\setminus(-\infty,0]$ when viewed as a function of two real variables (just set $h=x+iy$). Such harmonic extension is at this stage equivalent with the analytic one, but it avoids the arbitrary choice of the branch of the logarithm. The problem of choosing the branch of the log is present also at fixed N, but it is avoided if we simply consider the harmonic function $\Re(\log Z_{N,h})=\log|Z_{N,h}|$. We will see (Section 4) that $\lim(1/N)\log|Z_{N,h}|$ is not only uniquely defined, but it also converges in the whole of $\mathbb{C}$. The limit coincides with the harmonic continuation $\Re(\textsc{f}(h))$ of the free energy on the positive real axis up to where it vanishes, or in other words where it matches the harmonic continuation from the negative real axis. More precisely, we are going to show that the connected component of the set $\{h:\Re(\textsc{f}(h))>0\}$ that contains the positive real axis is a subset of the half plane $\{z\in\mathbb{C}:\Re(z)>0\}$ and the *relevant continuation* of $\textsc{f}(\cdot)$, defined on $\mathbb{R}$, is $\Re(\textsc{f}(h))(>0)$ on this connected component, and it is zero on the rest of $\mathbb{C}$. Hence the *critical region* is identified by the values of h with $\Re(h)\geq 0$ and $\textsc{f}(h)=i\theta$ for some $\theta\in\mathbb{R}$. It is not too difficult to see that this set can be written more explicitly as

$$\mathcal{C}_\alpha := \{-\mathrm{Log}\,(1-(1-\exp(-i\theta))^\alpha):\theta\in[0,2\pi)\}. \tag{1.10}$$

This set appears in Figures 1, 2 and 4). Here are some properties:

Lemma 1.5. *We have that $\mathcal{C}_\alpha$ (see Fig. 1 and Fig. 2) is invariant under complex conjugation, that $\mathcal{C}_\alpha$ is a subset of the strip $0\leq\Re(h)\leq-\log(2^\alpha-1)$ and touches the boundary of this strip only at the origin and at $-\log(2^\alpha-1)+i\pi$. Moreover $\mathcal{C}_\alpha$ is a simple closed curve in the cylinder $\mathbb{C}_{2\pi}$. This curve is smooth, except at 0, of finite length and it is not homotopic to a point: hence $\mathbb{C}_{2\pi}\setminus\mathcal{C}_\alpha$ is the union of two disjoint connected sets that we call $\mathcal{L}_\alpha$ and $\mathcal{D}_\alpha$. $\mathcal{L}_\alpha$ contains the positive real axis and $\mathcal{D}_\alpha$ contains the $\Re(h)<0$ half plane.*

Remark 1.6. *The pinning model transition that we observe at $h=0$, see for example* (1.4)*, is a (de)localization transition: this is discussed at length for example in* [14, 17, 19]*. Of course $\mathcal{L}_\alpha$, $\mathcal{D}_\alpha$ and $\mathcal{C}_\alpha$ are, respectively, the continuation in the complex plane of the delocalized region $(-\infty,0)$, of the localized region $(0,\infty)$ and of the critical point 0. We stress that we do not know how to do this continuation (at least, not in such a complete sense) in the general framework* (1.1)*. Moreover we do not attach a pathwise sense to the notion of (de)localization for $h\in\mathbb{C}\setminus\mathbb{R}$. Nevertheless it may be natural to identify $\mathcal{L}_\alpha$ (in the general context) as the region in which $\liminf_N(1/N)\log|Z_{N,h}|>0$, see notably the caption of Figure 1, the content of Section 2.2 and Remark 1.12.*

It turns out that the zeros of $Z_{N,h}$ accumulate on $\mathcal{C}_\alpha$: this is what we explain next.

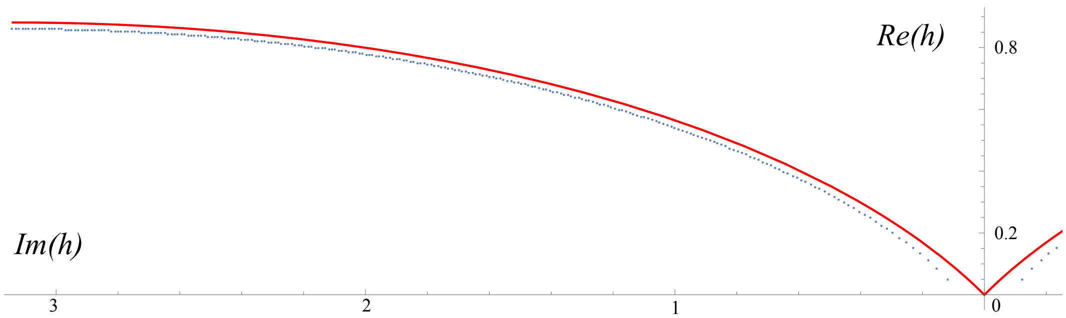


FIGURE 2. In red the plot of the critical curve $\mathcal{C}_{1/2}$, hence we are working with the inter-arrival law (1.6): the complex axes are rotated by 90 degrees and we cut the part with $\Im(h) < 0$ which is just the specular image of $\Im(h) > 0$. The blue dots mark the locations of the zeros of $Z_{500,h}$. Note that the black dots in Figure 5 show comparable zeroes, corresponding to a different inter-arrival law (1.1), but still with $\alpha = 1/2$. More precisely the black dots in Figure 5 mark the zeros of $P_{500}(w)$, so the black dots in Figure 5 should be compared to the exponential of the blue dots in here (and the qualitative behavior is the same).

1.3. **The zeros of the partition function: macroscopic limit.** We call $h_{N,1}, h_{N,2}, \ldots, h_{N,N-1}$ the $N-1 \geq 1$ zeros of $Z_{N,h}$ in $\mathbb{C}_{2\pi}$ and we introduce the empirical probability

$$\mu_N := \frac{1}{N-1} \sum_{j=1}^{N-1} \delta_{h_{N,j}} , \tag{1.11}$$

where δ_z is the probability on $\mathbb{C}_{2\pi}$ that is concentrated on z. We remark that if $Z_{N,h} = 0$ then $Z_{N,\bar{h}} = 0$, so μ_N is symmetric with respect to conjugation.

The notion of convergence of probability measures is the standard notion of convergence in law: $\lim_N \mu_N = \mu$ if $\lim \int_{\mathbb{C}_{2\pi}} f \, \mathrm{d}\mu_n = \int_{\mathbb{C}_{2\pi}} f \, \mathrm{d}\mu$ for every $f : \mathbb{C}_{2\pi} \to \mathbb{R}$ which is continuous and bounded. In particular, μ is a probability too.

Theorem 1.7. *With the inter-arrival distribution* (1.6) *we have that* $\lim_N \mu_N = \mu$. *The support of* μ *coincides with* $\mathcal{C}_\alpha$. *Moreover* μ *is absolutely continuous with respect to the arc-length measure on* $\mathcal{C}_\alpha$ *and its density vanishes only at* 0.

The argument of proof of Theorem 1.7 – i.e., sharp asymptotic control on $Z_{N,h}$ – directly yields the following result that makes more evident the role of the harmonic continuation of the free energy.

Theorem 1.8. *With the inter-arrival distribution* (1.6) *we have that*

(1) if $h \in \mathcal{D}_\alpha$ *then* $\lim_N (1/N) \log |Z_{N,h}| = 0$;

(2) if $h \in \mathcal{L}_\alpha$ *then*

$$\lim_{N\to\infty} \frac{1}{N} \log |Z_{N,h}| = \Re \left(\mathrm{F}(h) \right) , \tag{1.12}$$

where $\mathrm{F}(h)$ *is given in* (1.9).

Both results hold uniformly if h *is chosen bounded away from* $\mathcal{C}_\alpha$.

Moreover $\lim_N (1/N) \log |Z_{N,h}| = 0$ *also if* $h \in \mathcal{C}_\alpha$.

1.4. **The zeros of the partition function: local control.** Theorem 1.7 appears to be definitely sharper than Proposition 1.3, but we draw the attention on the fact that Theorem 1.7 is just about the empirical probability of the zeros and $o(N)$ of the zeros may behave in an arbitrary way without affecting the empirical probability. However, we do have also a stronger result

Theorem 1.9. *With the inter-arrival distribution* (1.6)*, for every* $\varepsilon > 0$ *there exists* $N_\varepsilon \in \mathbb{N}$ *such that the distance between the support of* μ_N *and* $\mathcal{C}_\alpha$ *is smaller than* ε *for* $N \geq N_\varepsilon$.

Theorem 1.9 is therefore saying that all the zeros are at distance ε from the support of the limit empirical probability, and this is an important improvement on Theorem 1.7. But it is still a very imprecise result near the most interesting point, that is $h = 0$. In particular, it is straightforward to check that $\mathcal{C}_\alpha$ near the origin is asymptotically close to the angle $\{z : |\operatorname{Arg}(z)| = \alpha\pi/2\}$. It thus seems natural to conjecture that the zeros are close to $\{z : |\operatorname{Arg}(z)| = \alpha\pi/2\}$ when we look at the zeros that are very close to the origin, i.e. the zeros that are at a distance from the origin that vanishes for $N \to \infty$: for $\alpha = 1/2$ we will show that the closest zeros are at a distance proportional to $1/\sqrt{N}$ from the origin and that in a ball of radius $r/\sqrt{N}$ we can find arbitrarily many zeros by choosing r large, in the limit $N \to \infty$. However the natural distance to consider on this scale is the one rescaled by the size of the neighborhood we are considering. Whether the zeros are close to $\{z : |\operatorname{Arg}(z)| = \alpha\pi/2\}$ or not on such intermediate scales depends on a finer analysis that one can in principle deal with for every $\alpha \in (0, 1)$, but there are some obstacles to this analysis. The most important one is the implicit characterization of the α-stable laws. There is however the notable exception of $\alpha = 1/2$ and, in fact, in this case we are able to go rather far. Here and in all the rest of the paper we order $(h_{N,j})$ so that $(|h_{N,j}|)$ is non decreasing. We actually choose $h_{N,1}$ one of the closest zeros to the origin with argument in $[0, \pi]$. Moreover $h_{N,2} := \overline{h_{N,1}}$, unless $h_{N,1}$ is real.

The next result is therefore restricted to $\alpha = 1/2$ (but we stress that it does not require the special set-up (1.6)). To state it we introduce the entire function

$$F_0(\zeta) := e^{\zeta^2} \zeta \left(1 + \operatorname{erf}(\zeta)\right) + \frac{1}{\sqrt{\pi}}, \tag{1.13}$$

where $\operatorname{erf}(\zeta) = (2/\sqrt{\pi}) \int_0^\zeta e^{-t^2} \, dt$ is the error function [29, Ch. 7]. We show in Lemma 5.5 that all the solutions $F_0(\zeta_j) = 0$ are given by an infinite sequence $(\zeta_j)_{j=1,2,\ldots}$ that can be ordered in such a way that the modulus is non decreasing and $\zeta_{2j-1} = \overline{\zeta_{2j}}$. With this we are implicitly saying that there is no real zero. Moreover, all the zeros have positive real part and they are all simple (cf. Remark 5.4).

Theorem 1.10. *With the inter-arrival distribution* (1.1), $\alpha = 1/2$ *and possibly by properly arranging the order of* $(h_{N,j})$ *we have that for every* j

$$h_{N,j} \overset{N \to \infty}{\sim} \frac{\zeta_j}{N^{1/2}}. \tag{1.14}$$

In particular we will see that $\zeta_1 = 1.225\ldots + i2.547\ldots$, that is $\operatorname{Arg}(\zeta_1) = 1.122\ldots = (\pi/4)1.429\ldots$. This is therefore saying that the closest zeros to the origin are not close to $\{z : |\operatorname{Arg}(z)| = \alpha\pi/2\}$.

It is highly plausible that $(|\zeta_{2j-1}|)$ is strictly increasing for every j and therefore so is $(|h_{N,2j-1}|)$ for N sufficiently large: if this is the case we avoid the nuisance of properly choosing the ordering of $(h_{N,j})$ when there are values of j such that $|h_{N,2j-1}| = |h_{N,2j+1}|$.

1.5. **The disordered pinning model and Griffiths singularities.** The partition function of the disordered version of the pinning model is

$$Z_{N,\omega,h} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\omega_n + h)\delta_n\right)\delta_N\right], \tag{1.15}$$

where $\omega = (\omega_n)_n$ is an IID sequence, and h a real parameter. Results on this disordered model have been obtained under mild conditions on the law of ω_1, but let us choose ω_1 that takes just two values. And, to be ready for the specific analysis we want to perform, we choose $\omega_n = sb_n$, with s a real number and (b_n) IID Bernoulli$(1-p)$, $p \in (0,1)$. Moreover, (b_n) and (τ_n) are independent and we are therefore working on a probability space that is the product of the space in which the disorder variables are defined, and the space on which the renewal process is defined: the probability is then the product probability of $\mathbb{P}$ and $\mathbf{P}$. Note that if $s = 0$ then the model is non disordered – sometimes called *pure* – and $Z_{N,\omega,h} = Z_{N,h}$. The reason for this atypical choice of disorder is that we are going to be interested in the limit $s \nearrow \infty$, but let us recall some general facts for now.

First of all the *free energy density* $\textsc{f}_s(h) = \lim_N (1/N)\mathbb{E}\log Z_{N,\omega,h}$ exists and it is a convex non decreasing function of h. In fact, $\textsc{f}_s(h) = 0$ for every $h \le h_c(s,p) \in \mathbb{R}$ and $h \mapsto \textsc{f}_s(h)$ is strictly increasing (and strictly convex) for $h > h_c(s,p)$. Many estimates are available on the value of $h_c(s,p)$, and in some cases it can even be computed exactly, but this will not be important for us here. While it is clear that $h_c(s,p)$ is a critical point, i.e. $h \mapsto \textsc{f}_s(h)$ is not real analytic at $h_c(s,p)$, the only result available on the regularity of $h \mapsto \textsc{f}_s(h)$ for $h > h_c(s,p)$ for $s \neq 0$ is that it is C^∞ [20]. On the other hand, for $s = 0$ (non disordered case), the free energy density $h \mapsto \textsc{f}_0(h) = \textsc{f}(h)$ is real analytic except at $h = h_c(0,p) = 0$. The transition at $h_c(s,p)$ is a delocalization to localization transition as explained in detail for example in [17, 19, 24]: we refer to [3] for an updated bibliography.

The obstruction to showing analyticity in the presence of disorder is not just a technical problem: R. B. Griffiths showed in 1969 that disorder may induce singularities. Griffiths' full argument was given for the ferromagnetic Ising model with dilution; that is, Ising model on a lattice, $\mathbb{Z}^d$, in which some bonds are deleted. In spite of a large amount of literature on the issue, the understanding of Griffiths singularities is still poor. In particular Griffiths singularities are expected to be rather generic, but their existence is proven only in very specific cases (for example, in presence of dilution, which corresponds to introducing infinite potentials in the system).

It is very natural to ask whether Griffiths singularities are present for the pinning model: is $h \mapsto \textsc{f}_s(h)$ analytic for $h > h_c(s,p)$ or are there other non analyticity points or regions? This question has been tackled in [27] by considering the $s \nearrow \infty$ limit of the model we just introduced. To deal with this limit it is practical to consider also the discrete renewal process $\sigma = (\sigma_n)$ that marks the sites where $b_n = 1$ and set $N_\sigma := \sup\{j : \sigma_j \le N\}$, with $\sigma_0 := 0$. By this we mean that, if $N_\sigma > 0$, $\{\sigma_1, \ldots, \sigma_{N_\sigma}\} = \{n = 1, \ldots, N : b_n = 1\}$. Otherwise $\{n = 1, \ldots, N : b_n = 1\}$ is empty. Separating out the contribution of the realizations where the renewal process τ visits all of the sites in σ, we have

$$Z_{N,\omega,h} = \exp(N_\sigma s)\left(\prod_{j=1}^{N_\sigma} Z_{\sigma_j - \sigma_{j-1},h}\right) Z_{N-\sigma_{N_\sigma},h} + O\left(\exp\left((N_\sigma - 1)s\right)\right) \tag{1.16}$$

for N fixed and $s \nearrow \infty$.

It is straightforward to check that $\lim_{s \nearrow \infty} h_c(s,p) = -\infty$, so the limit model is always localized. One can now consider as reduced model the first term in the right-hand side of (1.16) and the free energy density of this model is (of course a.s. $\lim_N N_\sigma/N = 1-p$)

$$s(1-p) + (1-p)^2 \sum_{n=1}^{\infty} p^{n-1} \log Z_{n,h} \,, \tag{1.17}$$

where we have used the Law of Large Numbers: $\mathbb{E}[|\log Z_{\sigma_1,h}|] < \infty$ because $e^h K(n) \leq Z_{n,h} \leq \exp(n \max(h,0))$. Note that the existence of a Griffiths singularity in this reduced model boils down to determining whether

$$h \mapsto \sum_{n=1}^{\infty} p^{n-1} \log Z_{n,h} =: \widetilde{\mathrm{F}}_p(h) \,, \tag{1.18}$$

is real analytic or not and the prediction is straightforward: $\widetilde{\mathrm{F}}_p(\cdot)$ does have a singularity in zero, because the zeros of $Z_{n,h}$ in the complex plane have a unique real accumulation point, as $n \to \infty$, in the origin.

The fact that the singularity is expected to happen at $h = h_c(0,p) = 0$ in this specific model is very much in the spirit of Griffiths' idea. The critical point of the pure model ($s = 0$) is $h = 0$. For $s > 0$ and large the system is essentially a collection of independent pure models pinned at the points on which $b_n = 1$ and its (localization) critical point is $h_c(s,p)$. All of the pure systems in the collection are finite, so their contribution is analytic, but in this collection there are systems that are arbitrarily large (the larger, the fewer). And the larger they are, the less their contribution can be continued outside the real line in the proximity of $h = 0$. Therefore the total contribution is not analytic, but the free energy turns out to be in any case C^∞ at $h = 0$ because the large pure systems in the collection are *exponentially rare*.

Here is the result that we have:

Theorem 1.11. *In the framework of* (1.6) *with* $\alpha = 1/2$, $h \mapsto \widetilde{\mathrm{F}}_p(h)$ *is real analytic except at* 0 *where for* $k \to \infty$

$$\left.\frac{\partial_h^k \widetilde{\mathrm{F}}_p(h)}{(k-1)!}\right|_{h=0} = C_1 C_2^k \exp(A\sqrt{k}) \Gamma\left(\frac{k}{2}+1\right) \left(\cos\left(\mathtt{a}k + \mathtt{b}\sqrt{k} + \mathtt{c}\right) + O\left(\frac{(\log k)^2}{\sqrt{k}}\right)\right), \tag{1.19}$$

where C_1, C_2, A, $\mathtt{a}$, $\mathtt{b}$ *and* $\mathtt{c}$ *are real constants that we give explicitly in the proof (see Remark 6.3). In particular, as a consequence of the fact that* $\mathtt{b} \neq 0$, *we have that there exists a* $\mathbf{N}_0 \subset \mathbb{N}$ *of density zero in* $\mathbb{N}$ *such that for* $k \to \infty$ *with* $k \notin \mathbf{N}_0$ *we have*

$$\left.\frac{\partial_h^k \widetilde{\mathrm{F}}_p(h)}{(k-1)!}\right|_{h=0} \sim C_1 C_2^k \exp(A\sqrt{k}) \Gamma\left(\frac{k}{2}+1\right) \cos\left(\mathtt{a}k + \mathtt{b}\sqrt{k} + \mathtt{c}\right). \tag{1.20}$$

Theorem 1.11 is strongly related to Theorem 1.10, notably to (1.14) for the case $j = 1$: the two zeros that are closest to the origin determine the leading behavior of the singularity. However, to obtain (1.19) we have employed a substantial refinement of (1.14) in the case $j = 1$: see Proposition 5.8.

A priori (1.19) may not be very informative because $\cos\left(\mathtt{a}k + \mathtt{b}\sqrt{k} + \mathtt{c}\right)$ may be arbitrarily close to zero and $O((\log k)^2/\sqrt{k})$ may become leading. But, as we will explain in the proof, $|\cos\left(\mathtt{a}k + \mathtt{b}\sqrt{k} + \mathtt{c}\right)| \gg (\log k)^2/\sqrt{k}$ except on a density zero subsequence of values of k. This is spelled out in (1.20), which implies non analyticity of $\textsc{f}_p(\cdot)$ at the origin because of the superexponential growth of the right-hand side in (1.20).

Theorem 1.11 also shows that the picture of the phenomenon (location of the zeros, Griffiths singularities) given in [27], while qualitatively to a certain extent correct, it is quantitatively imprecise. The limit of the analysis in [27] is that it plays on the fact that the zeros accumulate along the lines with slope $\pm\tan(\alpha\pi/2)$ near the origin. This is true in an appropriate *mesoscopic* sense, but, as we have seen, the leading Griffiths singularity of the reduced model (introduced in [27]) depends only on the two conjugate zeros closest to the origin, and they are not close (on the correct microscopic scale) to those two lines.

1.6. **About the tools we use, organization of proofs, perspectives.**

How we tackle the problem. Our approach mixes probabilistic tools and analytic ones. We discuss in some detail the proof of Proposition 1.3 because it contains some of the main tools we also use for the sharper results that follow. A direct consequence of a basic result in renewal theory [10, Th. A] is that for $h<0$ [17, Ch. 2]:

$$Z_{N,h} \overset{N\to\infty}{\sim} \frac{e^h}{(1-e^h)^2} K(N)\,. \tag{1.21}$$

Proposition 3.1 says that (1.21) holds also in the complex plane, provided that $\Re(h)<0$. Moreover this asymptotic behavior is uniform if $\Re(h)$ is bounded away from zero: this directly entails that, asymptotically, $Z_{N,h}\neq 0$ in the left complex half plane, and that no zero escapes to $-\infty$ as $N\to\infty$. The proof of Proposition 3.1 uses [10, Th. A] much in the same way as for (1.21).

On the other hand, we already know that $\log Z_{N,h} \sim N\textsc{f}(h)$, with $\textsc{f}(h)>0$ and increasing for $h>0$, so we definitely expect that also for $\Re(h)>0$ and $|\Im(h)|$ *somewhat small* with respect to $\Re(h)$ the partition function still grows exponentially. In fact, we will show that for $\Re(h)$ sufficiently large, exponential growth holds regardless of the value of $\Im(h)$. In order to make this concrete and quantitative we exploit the singularity analysis of the z-transform (characteristic function). Recall (1.7) for the notation: the z-transform of $(Z_{N,h})$ can be easily computed in terms of the z-transform of $(K(N))$. In fact with $Z_{0,h}:=1$ we have

$$\begin{aligned}\widehat{Z}_h(z) \,:=\, \sum_{n=0}^\infty z^n Z_{n,h} \,&=\, 1+\sum_{n=1}^\infty \sum_{k=1}^n \sum_{\substack{\ell\in\mathbb{N}^k:\\ \sum_j \ell_j=n}} \prod_{j=1}^k \left(e^h z^{\ell_j} K(\ell_j)\right)\\ &=\, 1+\sum_{k=1}^\infty \sum_{\ell\in\mathbb{N}^k} \prod_{j=1}^k \left(e^h z^{\ell_j} K(\ell_j)\right) \,=\, \frac{1}{1-e^h\sum_{j=1}^\infty z^j K(j)} \,=\, \frac{1}{1-e^h\widehat{K}(z)}\,. \end{aligned} \tag{1.22}$$

These steps are justified only for $|z|$ small, as it can be seen also from the rightmost term: the radius of convergence of $\widehat{K}(z)$ is one, so $\widehat{Z}_h(z)$ is meromorphic in the unit disk. The precise asymptotic behavior of $(Z_{N,h})$ can be obtained by analyzing the singularities of $\widehat{Z}_h(z)$: in particular it is well known [15] that, if $1-e^h\widehat{K}(z_0)=0$ for $|z_0|<1$ (let us assume that there is a unique zero with minimal modulus and that this zero, which we call z_0, is simple: of course general results are available) then the leading behavior of $Z_{N,h}$ for $N\to\infty$ is $|z_0|^{-N}$ times an explicit h dependent non zero constant. One can actually show that this result is uniform in a neighborhood of h and, as before, this excludes $Z_{N,h}=0$ in such a neighborhood and for N sufficiently large.

We are therefore at the level of the grey regions of Figure 1 and it is natural, in analogy with the real case, to dub as *localized* the region in which the free energy has exponential growth: we could therefore define $\mathcal{L}_\alpha$ as the values of h such that $1-e^h\widehat{K}(z)=0$ can be solved for $|z|<1$. We have chosen to introduce $\mathcal{L}_\alpha$ only for the special one parameter family of inter-arrival laws in (1.6) because our main focus is on the location of the zeros. As a matter of fact we have defined first the critical curve $\mathcal{C}_\alpha$ (Lemma 1.5), on which the zeros lie in the $N\to\infty$ limit, and this curve splits the whole space in two open regions $\mathcal{L}_\alpha$ and $\mathcal{D}_\alpha$ that are natural continuation of the localized and delocalized (non critical) real regions. A posteriori (see Section 2.2), we do verify that $h\in\mathcal{L}_\alpha$ if and only if $1-e^h\widehat{K}(z)=0$ can be solved for $|z|<1$.

But if we understand why $\mathcal{L}_\alpha$ is asymptotically zero free, Figure 1 is telling us that the fact that $\mathcal{L}_\alpha$ and $\{h:\Re(h)<0\}$ are zero free leaves open a substantial region on which the zeros may end up being. It turns out that complex analytic singularity analysis is useful in this region too, but only under the requirement of being able to analytically continue $1-e^h\widehat{K}(z)$ beyond the unit circle. Note that $\widehat{K}(z)$ has a singularity in 1 and that this singularity is not a pole, but this does not exclude a continuation to the centered ball of radius $R>1$ minus $[1,\infty)$ (or minus a proper cone containing $[1,\infty)$, see for example [15, Ch. VI]). In this case, singularity analysis does yield, again, sharp asymptotic control on $Z_{N,h}$ that excludes that $Z_{N,h}=0$ for N large.

Remark 1.12. *In the special framework* (1.6) $\mathcal{C}_\alpha=\partial\mathcal{L}_\alpha=\partial\mathcal{D}_\alpha$*, but we have no reason to believe that this holds in full generality. Our arguments heavily rely on a suitable analytic continuation and the general context does not grant this, see Remark 1.4. Moreover we do know that in different contexts the* critical region *on which the zeros accumulate is not a curve: see the end of Section 1.6, notably the considerations on* [7].

Let us go more deeply into the special framework of (1.6). In this case, by (1.7), we have

$$\widehat{Z}_h(z)\;=\;\frac{1}{1-e^h\left(1-(1-z)^\alpha\right)}\,, \tag{1.23}$$

and we readily see that $z\mapsto 1-e^h(1-(1-z)^\alpha)$ can be continued to an analytic function to the whole of $\mathbb{C}$ minus a *cut curve* that starts at 1. Singularity analysis, once again, yields the sharp asymptotic behavior from which we conclude that also $\mathcal{D}_\alpha$ (and even $\mathcal{C}_\alpha$!) is eventually (i.e., for N sufficiently large) zero free. But where are the zeros then? The point is that the results we obtain in $\mathcal{L}_\alpha$ and $\mathcal{D}_\alpha$ are uniform in h bounded away from the critical curve $\mathcal{C}_\alpha$. This leaves the door open to the possibility that the zeros asymptotically accumulate on their boundary $\mathcal{C}_\alpha$. And, by exploiting tools from potential theory, we do prove that this happens.

The limit of potential theory is that it yields only *macroscopic results*, much in the sense that controlling the free energy yields a control on macroscopic observables. But we may be interested in sharper aspects: the crucial relevance of sharper estimates is definitely clear for $h\in\mathbb{R}$ [17, 19], notably (but not only) for h close to the critical point, i.e. zero. And we are able to produce finer estimates precisely in a complex neighborhood of the origin: for this we exploit once again a probabilistic approach and identify the scaling behavior of $Z_{N,h}$ with h that tends to zero with N in a suitable way. We are thus able to understand the critical window in the complex plane (see [32] for the real case). Results here are mostly limited to $\alpha=1/2$ because of the non explicit character of the stable laws for $\alpha\neq 1/2$, even if we do not need to restrict to (1.6). The Argument Principle is exploited, in conjunction with the scaling limits, to identify the position of the zeros.

Corrections to the leading asymptotic locations of the zeros are obtained in the special framework of (1.6) (still assuming $\alpha = 1/2$) and this is central for proving the results in connection with Griffiths singularities.

Organization of the rest of the paper. In Section 2 we prove Lemma 1.5 and we provide alternative characterizations of $\mathcal{L}_\alpha$, $\mathcal{D}_\alpha$ and $\mathcal{C}_\alpha$ that we use in the sections that follow.

Section 3 exploits singularity analysis to obtain the sharp behavior of $(Z_{N,h})$. Proposition 3.1 is the only result in this section that does not rely on singularity analysis and Proposition 3.1 plus Proposition 3.2 provide a full proof of Proposition 1.3. The rest of the Section is devoted to the proof of Theorem 1.8: in fact, much more precise results are proven, see notably Proposition 3.3. Theorem 1.9 is also a direct consequence of Proposition 3.3.

Section 4 is devoted to the potential theory analysis. Theorem 1.7 is a direct corollary of Proposition 4.2, but several other estimates of independent interest, notably about the limit density of the zeros on $\mathcal{C}_\alpha$, are given.

Section 5 is devoted to the precise analysis of the zeros close to the origin. One finds here a proof of Theorem 1.10, which follows from the general result in Proposition 5.3 and the $\alpha = 1/2$ control on the scaling limit of Lemma 5.5. This section contains also the much sharper estimate of Proposition 5.8 which demands hypothesis (1.6) and is crucial for the Griffiths singularity analysis.

The Griffiths singularity analysis, with the proof of Theorem 1.11, is in Section 6.

Perspectives and open problems. The following are a few aspects of the related literature and plausible future developments.

- The pinning model may be considered the easiest exactly solvable statistical mechanics model. Yet, it does not enjoy the *surprisingly rigid* structure of the Lee-Yang Circle Theorem [26], see [16] and [31] for many developments and references. Nonetheless, in the special framework, the zeros do (asymptotically) lie on a closed curve that is smooth (except for the corner at the real critical point), but only in the limit. There is numerical evidence, see Figure 2, that the zeros approach the critical curve $\mathcal{C}_\alpha$ from the delocalized region $\mathcal{D}_\alpha$ and we believe that this is within reach of our tools (but we do not develop this aspect). Moreover, we do have (and present) a good control of the zeros which are at distance $O(1/\sqrt{N})$ from the origin when $\alpha = 1/2$, but results appear to be much more challenging if $\alpha \neq 1/2$, or even for $\alpha = 1/2$ but on an *intermediate scale*. By intermediate scale we mean studying the points close to the origin, but at a distance much larger than $1/\sqrt{N}$.
- What happens in the general framework of (1.1)? Theorem 1.10 does shed some light, but ultimately only for $\alpha = 1/2$ and, worse, only for the zeros at distance $O(1/N^\alpha)$ from the origin. This suffices to exclude the validity of the generalization to the pinning model, stated in [27], of the conjecture in [25] that the zeros should approach the real critical point close to the lines with slope $\pm\tan(\alpha\pi/2)$. However, this fact should hold on intermediate scales, i.e. for zeros that are far from the origin on the scale $1/N^\alpha$, but a distance $o(1)$ from the origin. But this is precisely the intermediate scale region on which the control is poor.
- In [7] (see also [8] for models on hierarchical lattices) the random energy model is analyzed and the zeros densely fill a subset of $\mathbb{C}$ with non empty interior. Can this type of phenomena happen also for pinning models? We do not know the answer, but

the fact that the critical region $\mathcal{C}_\alpha$ is a curve is by no means granted in the general framework (see Figure 5).
- Our analysis is restricted to the case of $\alpha \in (0,1)$. Larger values of α can be treated as well, at least to a certain extent, but it is lengthy and not straightforward. In particular, the case $\alpha \in (1,2)$ (inter-arrivals with finite first moment, but infinite variance) is different from the $\alpha \geq 2$ case, for which the inter-arrival law is in the domain of attraction of the Gaussian law.
- It is certainly possible to give a general statement for inter-arrival laws whose characteristic function satisfies a number of hypotheses, in particular suitable continuation properties, not only for the characteristic function but also for its inverse (defined a priori on the positive real axis). This is rather involved and, ultimately, we can verify the conditions only for (1.6), at least if we want to treat every $h \in \mathbb{C}$.
- It is very unfortunate that we control the Griffiths singularity only for the reduced model introduced in [27]. As it is claimed in [27], the result should be somewhat robust and should hold also for the original model (at least close to the limit in which the reduced model emerges). How to prove this remains a challenge. But this challenge is a special case of the (much more) general problem of showing the existence of Griffiths singularities for non diluted models.
- A number of works, e.g. [2, 5, 23], studied the dynamical counterpart of Griffiths singularities and rather sharp results have been obtained for some diluted lattice models. We cite also [6] for another type of dynamical phenomenon due to rare regions of Griffiths type. For pinning models the dynamical analysis is up to now limited to the nondisordered case: we cite [4] that deals with the localized phase, the one relevant for the Griffiths singularity, of the pinning model, but the results in [4] are without disorder.

Recurrent notations. We use $\overline{z}$ for the complex conjugate of z, $B_w(r) := \{z \in \mathbb{C} : |z-w| < r\}$ for the open ball of radius $r > 0$ centered in $w \in \mathbb{C}$ and $\mathrm{Sect}(\beta) := \{z : |\arg(z)| < \beta\}$ for the symmetric sector centered on the positive real axis, of angle opening 2β.

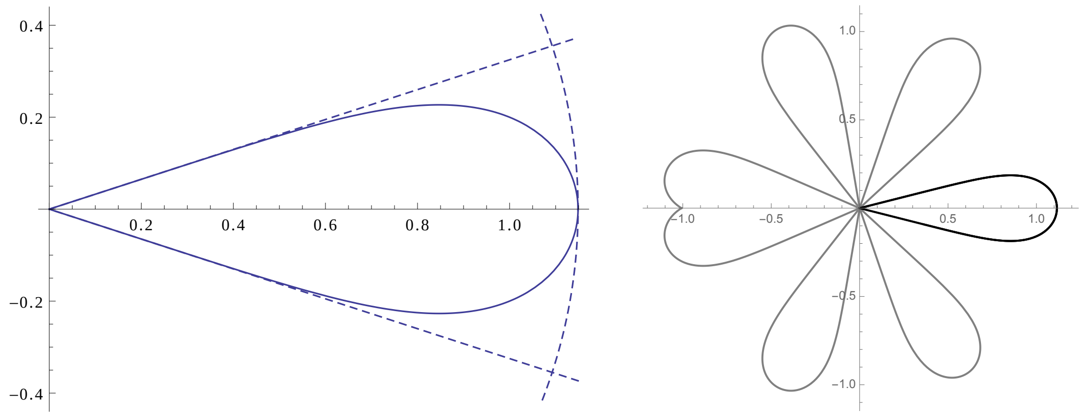
2. On the localized, delocalized and critical regions (assuming (1.6))

$\mathcal{L}_\alpha$, $\mathcal{D}_\alpha$ and $\mathcal{C}_\alpha$ are defined in Lemma 1.5, assuming (1.6): in this section we work only in this restricted framework. We start by giving a proof of Lemma 1.5, so $\mathcal{L}_\alpha$, $\mathcal{D}_\alpha$ and $\mathcal{C}_\alpha$ are well defined. Then we give alternative characterizations of these three sets.

2.1. **About the critical curve: proof of Lemma 1.5.** $\mathcal{C}_\alpha$ is just the image of $\theta \mapsto (1-\exp(-i\theta))^\alpha$ under the map $z \mapsto \mathrm{Log}(1-z)$. So we start with the following result:

Lemma 2.1. *The map $\theta \mapsto (1-\exp(-i\theta))^\alpha$ draws a simple closed curve in $\mathcal{C}$ when θ runs from 0 to 2π. This curve is invariant under complex conjugation, is contained in the closure of Sect$(\alpha\pi/2)$ and in the closure of $B_0(2^\alpha)$ (hence it is also contained in the strip $\{z : 0 \leq \Re(z) \leq 2^\alpha$, see Fig. 3(A)). For $\theta \searrow 0$ and $\theta \nearrow 2\pi$ the curve is tangent to the boundary of Sect$(\alpha\pi/2)$. Moreover, it is smooth, except at the origin.*

Proof. The proof follows by elementary arguments based on the fact that the curve is the map of the circle $\partial B_1(1) = \{1-\exp(-i\theta) : \theta \in [0,2\pi)\}$ under $z \mapsto z^\alpha$. □



(A) (B)

FIGURE 3. In (A) the curve drawn by the map map $\theta \mapsto (1-\exp(-i\theta))^\alpha$, with $\alpha = 0.2$. The dashed line are the bounds given in Lemma 2.1. In (B) we plot the set $\{\eta : |1-\eta^{1/\alpha}| = 1\}$ (recall that when we write $\eta^{1/\alpha}$ we mean that η is in the domain of $z \mapsto z^{1/\alpha}$, i.e. $\eta \notin (-\infty, 0])$ for $\alpha = 1/\sqrt{40} \approx 0.1581$.

Proof of Lemma 1.5. This follows directly from Lemma 2.1 and some elementary considerations: notably the fact that the curve of Lemma 2.1 is in $\mathrm{Sect}(\alpha\pi/2)$ and tangent to its boundary approaching the origin says that $\mathcal{C}_\alpha$ does not enter $\mathrm{Sect}(\alpha\pi/2)$ and it is also tangent to this set approaching the origin. Moreover the curve of Lemma 2.1 is in the closure of $B_0(2^\alpha)$ (in fact, the intersection with the boundary of $B_0(2^\alpha)$ is just the point 2^α) and this yields that $\mathcal{C}_\alpha$ is in the strip $0 \le \Re(h) \le -\log(2^\alpha - 1)$ (and that the point of contact with the boundary of the strip are only 0 and $-\log(2^\alpha-1)+i\pi$). The curve of Lemma 2.1 separates $\mathbb{C}$ into two connected components: the bounded one is mapped into $\mathcal{L}_\alpha$, and the unbounded one is mapped into $\mathcal{D}_\alpha$. □

Remark 2.2. *Figure 3(B) identifies a phenomenon we need to watch out for:* $\{(1-\exp(-i\theta))^\alpha : \theta \in \mathbb{R}\}$ *is a subset of* $\{\eta : |1-\eta^{1/\alpha}| = 1\}$ *and, unless* $\alpha \ge 2/3$*, it is a proper subset. This is due to the fact that* $(\eta \exp(2\pi i k\alpha))^{1/\alpha} = \eta^{1/\alpha}$ *if* $\eta \exp(2\pi i k \alpha) \in \mathbb{C}\setminus(-\infty, 0)$*. So* $\{\eta : |1-\eta^{1/\alpha}| = 1\}$ *in general contains several copies of* $\{(1-\exp(-i\theta))^\alpha : \theta \in \mathbb{R}\}$ *rotated by* $\exp(2\pi i k\alpha)$*, except that the phase* $2\pi\alpha + \arg(\eta)$ *of the points in the rotated copies must be in* $(-\pi, \pi]$*. In view of the (sharp) bounds in Lemma 1.5 we see that the two sets coincide if and only if the curve for* $k = 1$ *has empty intersection with the upper half plane (equivalently, the curve for* $k = -1$ *has empty intersection with the lower half plane). This amounts to* $2\pi\alpha - (\pi/2)\alpha \ge \pi$*, i.e.* $\alpha \ge 2/3$*.*

Another fact that follows directly from Lemma 2.1 is that $\mathcal{C}_\alpha$ can be seen as the graph of a function of the imaginary coordinate. It is actually an increasing (respectively, decreasing) function of the imaginary coordinate if the imaginary coordinate is positive (respectively, negative) as it is apparent from the curves on the right of Figure 4. This can be shown by making the parametric representation $\mathcal{C}_\alpha$ explicit: with a rather cumbersome

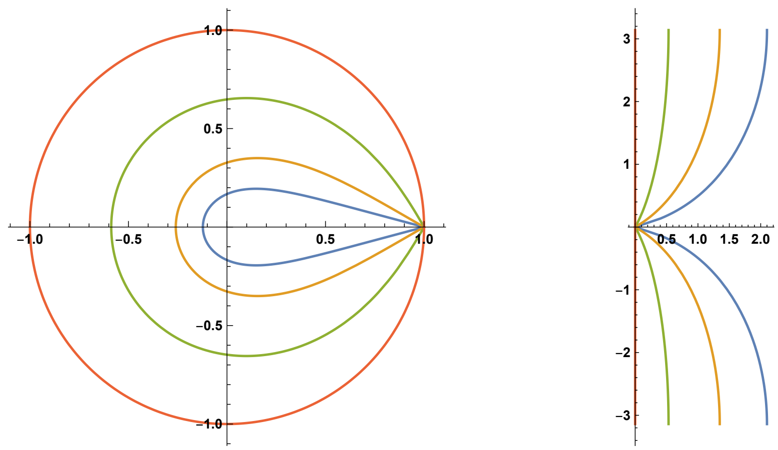


FIGURE 4. On the left the plot of the curve $\theta \to 1-(1-\exp(-i\theta))^\alpha$, for $\alpha = 1/6, 1/3, 2/3, 1$ (i.e., blue, yellow, green red). On the right the logarithm of the same curve: i.e., on the right we have $\mathcal{C}_\alpha$ for the same values of α. The monotonicity of the curves on the right and (say) in the first quadrant has the simple geometric interpretation that when one goes though one of the curves with $\alpha < 1$ on the left, the distance of the curve $\theta \mapsto 1-(1-\exp(-i\theta))^\alpha$ to the origin decreases for θ that goes from 0 to π. A proof of the monotonicity can be found in Lemma C.1.

computation we can write $\mathcal{C}_\alpha$ as $\{f_1(\theta)+if_2(\theta) : \theta \in [0,2\pi)\}$ (we set $a := 1-\alpha$) with

$$f_1(\theta) = -\frac{1}{2}\log\left(2^{2a}\sin^2\left(\frac{a(\pi-\theta)}{2}\right)\sin^{2a}\left(\frac{\theta}{2}\right) + \left(1-2^a\sin^a\left(\frac{\theta}{2}\right)\cos\left(\frac{a(\pi-\theta)}{2}\right)\right)^2\right), \tag{2.1}$$

and

$$f_2(\theta) = \arctan_0\left(\frac{2^a\sin\left(\frac{a(\pi-\theta)}{2}\right)\sin^a\left(\frac{\theta}{2}\right)}{1-2^a\sin^a\left(\frac{\theta}{2}\right)\cos\left(\frac{a(\pi-\theta)}{2}\right)}\right), \tag{2.2}$$

where $\arctan_0(\cdot) : \mathbb{R} \to [0,\pi]$ is a version of $\arctan(\cdot) : \mathbb{R} \to (-\pi/2, \pi/2)$ defined as $\arctan_0(t) = \arctan(t)$ if $t \ge 0$ and $\arctan_0(t) = \arctan(t) + \pi$ if $t < 0$. And now it is *just* a matter of showing that $f_1'(\theta) > 0$ for $\theta \in (0,\pi)$. In Lemma C.1 we show this along with an independent proof of $f_2'(\theta) > 0$.

2.2. **Alternative characterizations of the localized and delocalized regions.** We start by defining the open set

$$\mathcal{L}^\star_\alpha := \{h \in \mathbb{C} : \Re(\mathrm{F}(h)) > 0\}, \tag{2.3}$$

and remark that $(0,\infty) \subset \mathcal{L}^\star_\alpha$. We then introduce two more subsets of $\mathbb{C}$:

$$\mathcal{L}'_\alpha \text{ is the connected component of } \mathcal{L}^\star_\alpha \text{ that contains } (0,\infty), \tag{2.4}$$

and

$$\mathcal{L}''_\alpha := \left\{h \in \mathbb{C} : \text{ there exists } z \in B_0(1) \text{ such that } 1 - e^h(1-(1-z)^\alpha) = 0\right\}. \tag{2.5}$$

Let us point out from now that, for $h \in \mathcal{L}''_\alpha$, the solution to $1-e^h(1-(1-z)^\alpha) = 0$ is of course unique: in fact, for $z \in B_0(1)$, $1-e^{-h} = (1-z)^\alpha$ is equivalent to $z = 1-(1-e^{-h})^{1/\alpha}$. What is also straightforward is to check that z is a simple zero.

Lemma 2.3. $\mathcal{L}_\alpha = \mathcal{L}'_\alpha = \mathcal{L}''_\alpha$.

Proof. Since $h \mapsto 1 - e^{-h}$ and $z \mapsto 1 - z$ are both one-to-one, this is the same as asking whether $\mathcal{T}_\alpha = \mathcal{T}'_\alpha = \mathcal{T}''_\alpha$ with

(1) $\mathcal{T}_\alpha$ defined by considering the closed curve $\{(1 - \exp(i\theta))^\alpha : \theta \in [0, \pi)\}$ that splits $\mathbb{C}$ into two connected components: $\mathcal{T}_\alpha$ is the bounded one;
(2) $\mathcal{T}'_\alpha$ the connected component of $\left\{\eta : \left|1 - \eta^{1/\alpha}\right| < 1\right\}$ containing $(0, 1)$;
(3) $\mathcal{T}''_\alpha := \{\eta : \text{there exists } \zeta \text{ such that } |1 - \zeta| < 1 \text{ and } \zeta^\alpha = \eta\}$.

$\mathcal{T}_\alpha = \mathcal{T}'_\alpha$ is a direct consequence of Lemma 2.1.

Moreover we have $\mathcal{T}''_\alpha \supset \mathcal{T}'_\alpha$ because if $\eta \in \mathcal{T}'_\alpha$ we can set $\zeta = \eta^{1/\alpha}$, which is in $\mathrm{Sect}(\pi/2)$, hence $\zeta^\alpha = \eta$, besides of course $|1 - \zeta| < 1$. Therefore $\eta \in \mathcal{T}''_\alpha$.

For $\mathcal{T}''_\alpha \subset \mathcal{T}'_\alpha$ we start by claiming that $\mathcal{T}''_\alpha \subset \{\eta : |1 - \eta^{1/\alpha}| < 1\}$. In fact if $\eta \in \mathcal{T}''_\alpha$ there exists $\zeta \in B_1(1)$ such that $\zeta^\alpha = \eta$, so $\zeta = \eta^{1/\alpha}$. And taking $\eta^{1/\alpha} \in B_1(1)$ yields the claim. Now we remark that Lemma 2.1 implies that

$$\{\eta : |1 - \eta^{1/\alpha}| < 1\} \setminus \mathcal{T}'_\alpha \subset \left\{\eta : |\arg(\eta)| \geq 2\pi\alpha - \frac{\pi}{2}\alpha\right\} . \tag{2.6}$$

But $|1 - \zeta| < 1$ implies $|\arg(\zeta)| < \pi/2$, so $|\arg(\zeta^\alpha) < \alpha\pi/2$. So ζ^α is not contained in the set in the right-hand side of (2.6). Therefore $\{\eta : |1 - \eta^{1/\alpha}| < 1\} \setminus \mathcal{T}'_\alpha$ and $\mathcal{T}''_\alpha$ have empty intersection. Hence $\mathcal{T}''_\alpha \subset \mathcal{T}'_\alpha$ and the proof is complete. □

Lemma 2.3 implies that if $h \in \mathcal{D}_\alpha \cup \mathcal{C}_\alpha$ then $1 - e^h(1 - (1 - z)^\alpha) = 0$ has no solution $z \in B_0(1)$. We need to refine this statement:

Lemma 2.4. *(1) If $h \in \mathcal{C}_\alpha$ then there exists a unique solution z to $1 - e^h(1 - (1 - z)^\alpha) = 0$ and $|z| = 1$. Moreover z is a simple zero if $h \neq 0$.*
(2) For every $\varepsilon > 0$ there exists $r_\varepsilon > 1$ such that if $h \in \mathcal{D}_\alpha$ and $\mathrm{dist}(h, \mathcal{C}_\alpha) \geq \varepsilon$ then $1 - e^h(1 - (1 - z)^\alpha) \neq 0$ for every $z \in B_0(r_\varepsilon)$.

Proof. For (1) we see that the equation $1 - e^h(1 - (1 - z)^\alpha) = 0$, for $h = -\mathrm{Log}(1 - (1 - \exp(-i\theta))^\alpha)$, reduces to $z = \exp(-i\theta)$. Moreover $\partial_z(1 - e^h(1 - (1 - z)^\alpha)) = -e^h(1 - z)^{\alpha - 1}$ is clearly non zero for $z = \exp(-i\theta)$, $\theta \in (0, 2\pi)$, so the zero is simple. Let us remark that the problem with $h = 0$, i.e. $z = 1$, is that it is a singular point for $1 - e^h(1 - (1 - z)^\alpha)$.

For (2) we remark that if $r_\varepsilon > 1$ does not exist then we can find sequences (h_j) and (z_j) with $h_j \in \mathcal{D}_\alpha$, $\mathrm{dist}(h_j, \mathcal{C}_\alpha) > \varepsilon$, $|z_j| > 1$ and $1 - e^{h_j}(1 - (1 - z_j)^\alpha) = 0$ for every j, but $z_j \to z \in \partial B_0(1)$. Since $1 - e^{h_j}(1 - (1 - z_j)^\alpha) = 0$ and the fact that (z_j) stays in a compact set, $\Re(h_j)$ is bounded below. Therefore there is no loss of generality in assuming also $h_j \to h$ and of course h is in $\mathcal{D}_\alpha$ and at distance ε or more from the boundary $\mathcal{C}_\alpha$. But this implies that $1 - e^h(1 - (1 - z)^\alpha) = 0$, with $z \in \partial B_0(1)$, that is $h \in \mathcal{C}_\alpha$, which is impossible. So part (2) is proven. □

3. Sharp estimates on the partition function

In this section we mostly exploit complex analysis tools, except for the first result (Proposition 3.1) that is based on a more probabilistic estimate.

3.1. **Sharp estimates in the general framework.** In the general context (1.1), for $n \to \infty$ and uniformly in j such that $n/a_j \to \infty$, i.e. $j/n^\alpha \to 0$ (recall Remark 1.1 for the definition of (a_j)), we have [10, Th. A]

$$\mathbf{P}\left(\tau_j = n\right) \sim jK(n). \tag{3.1}$$

Proposition 3.1. *In the general context of* (1.1), *if* $\Re(h) < 0$ *we have*

$$Z_{N,h} \overset{N\to\infty}{\sim} K(N) \frac{\exp(h)}{(1-\exp(h))^2}, \tag{3.2}$$

and this result holds uniformly if $\Re(h)$ *is bounded away from* 0.

Proof. We write $Z_{N,h} = \sum_{j=1}^N e^{hj}\mathbf{P}(\tau_j = N)$ and for $\Re(h) < 0$ by (3.1) we have that if we choose a decreasing sequence (γ_N) of positive numbers, say $\gamma_N := 1/\log(N)$, then there exists (ε_N), $\varepsilon_N \searrow 0$, such that for N sufficiently large

$$\begin{aligned} \left| Z_{N,h} - K(N)\sum_{j=1}^N e^{hj} j \right| &\le \sum_{j=1}^N \exp(-j|\Re(h)|) \left|\mathbf{P}\left(\tau_j = N\right) - jK(N)\right| \\ &\le K(N) \sum_{j \le N^\alpha \gamma_N} j\exp(-j|\Re(h)|) \left| \frac{\mathbf{P}\left(\tau_j = N\right)}{jK(N)} - 1 \right| + 2 \sum_{j > N^\alpha\gamma_N} j \exp(-j|\Re(h)|) \\ &\le K(N)\varepsilon_N \frac{\exp(-|\Re(h)|)}{(1-\exp(-|\Re(h)|))^2} + 3N^\alpha \gamma_N |\Re(h)| \exp(-N^\alpha\gamma_N|\Re(h)|), \end{aligned} \tag{3.3}$$

where from the second to the third line we have used $jK(N) = O(1/N^\alpha) = o(1)$. Therefore if $\Re(h) \le -\varepsilon$ for an $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\left| Z_{N,h} - K(N)\frac{\exp(h)}{(1-\exp(h))^2} \right| \le c_\varepsilon K(N)\varepsilon_N \exp(-|\Re(h)|), \tag{3.4}$$

and this is the uniform estimates we claimed: since $\exp(h)/(1-\exp(h))^2 \sim \exp(h)$ for $\Re(h) \to -\infty$, for every $c > 0$ the ratio between the error term (i.e., the right-hand side of (3.4)) and the leading behavior of $Z_{N,h}$, i.e. $K(N)\exp(h)/(1-\exp(h))^2$, is $O(\varepsilon_N)$ uniformly in h such that $\Re(h) \le -c$. □

Proposition 3.2. *We fix* $K(\cdot)$ *which satisfies* (1.1) *and consider* $W \subset \mathbb{C}$ *which is the union of*

1. *the half plane with* $\Re(h) \ge C > 0$*;*
2. *the set of* h*'s with* $\Re(h) \ge a > 0$ *and* $|\Im(h)| < \varepsilon$.

If we choose C *suitably large and* ε *suitably small (*ε *depends on* a, C *does not) then for every* $h \in W$ *there exists a unique solution* $z = z_h \in B_0(1)$ *to* $\widehat{K}(z) = \exp(-h)$ *with minimal absolute value and such that, uniformly in* $h \in W$, *we have*

$$Z_{N,h} \overset{N\to\infty}{\sim} \frac{(1-\exp(-h))^{(1-\alpha)/\alpha}}{\alpha \exp(h) z_h} z_h^{-N}. \tag{3.5}$$

Proof. We proceed by obtaining the sharp asymptotic behavior of $Z_{N,h}$ for $N \to \infty$ and uniformly in h in appropriate subsets of $\mathbb{C}$. We will be in the case in which we can identify $r \in (0,1)$ such that $\widehat{Z}_{N,h}$ has only one pole, a single pole that we call $z_h \in B_0(r)$ and no pole on $\partial B_0(r)$, so we have (we recall that $\widehat{K}(z)$ is defined in (1.7) and $\widehat{Z}_h(z)$ in (1.22))

$$\widehat{Z}_h(z) \,=\, -\frac{\exp(-h)}{\widehat{K}'(z_h)(z-z_h)} + R_h(z)\,, \tag{3.6}$$

which defines $R_h(z)$. Therefore $z \mapsto R_h(z)$ is analytic in a neighborhood of the closure of $B_0(r)$ and

$$Z_{N,h} \,=\, \frac{\exp(-h)}{\widehat{K}'(z_h)} z_h^{-N-1} + \frac{1}{2\pi i}\oint \frac{R_h(z)}{z^{N+1}}\,\mathrm{d}z\,, \tag{3.7}$$

with z running, counterclockwise, on $\partial B_0(r)$.

We treat the two regions separately.

For case (1) we observe that for z small $\widehat{K}(z) \sim K(1)z$ and $\widehat{K}'(0) = K(1) > 0$. This entails that there exists $r > 0$ such that $\widehat{K}'(z) \neq 0$ for every $z \in B_0(r)$ and $\widehat{K} : B_0(r) \to \widehat{K}(B_0(r))$ is invertible. Of course $\widehat{K}(B_0(r))$ is a neighborhood of the origin. Therefore there exists $h_0 > 0$ such that $\exp(-h) \in \widehat{K}(B_0(r))$ for $\Re(h) > h_0$ and, for such values of h, $z_h = \widehat{K}^{-1}(\exp(-h))$ is the unique solution $B_0(r)$ of $1 - e^h\widehat{K}(z) = 0$. Possibly by replacing r by a smaller value, we can assume also that $\widehat{K}(z) \neq 0$ for every z with $|z| = r$, so $\inf_{z\in\partial B_0(r)} |\widehat{K}(z)| \geq c_r > 0$. Hence, always for $z \in \partial B_0(r)$, we have $|1 - \exp(h)\widehat{K}(z)| \geq \exp(\Re(h))c_r/2$ for $\Re(h) \geq h_0 > 0$, with suitable choice of h_0. Note that $z_h \sim \exp(-h)/K(1)$ for $\Re(h) \to \infty$ so, in particular, $\widehat{K}'(z_h) \sim K(0)$. Therefore, possibly by choosing h_0 smaller, we have $|z - z_h| \geq r/2$ and

$$\sup_z |R_h(z)| \,\leq\, \sup_z \left|\widehat{Z}_h(z)\right| + \sup_z \frac{e^{-\Re(h)}}{|\widehat{K}'(z_h)|\,|z-z_h|} \,\leq\, C_r e^{-\Re(h)}\,, \tag{3.8}$$

where the z runs in $\partial B_0(r)$ and C_r can be chosen equal to $2(1/c_r + 1/(rK(0)))$. So, by using (3.7) we obtain that for every h with $\Re(h) \geq h_0$ we have

$$\left| Z_{N,h} - \frac{\exp(-h)}{\widehat{K}'(z_h)} z_h^{-N-1}\right| \,\leq\, C_r e^{-\Re(h)} r^{-N}\,. \tag{3.9}$$

Such a uniform estimate guarantees that there exists h_0 and N_0 such that, if $\Re(h) \geq h_0$, $Z_{N,h} \neq 0$ for $N \geq N_0$.

For case (2) we start by recalling that $\widehat{K}'(z) > 0$ for $z \in (0,1)$, so $\widehat{K}(z) = \exp(-h)$ has a unique (positive) solution $z = z_h$ for $h > 0$. This may not be the unique solution in $\mathbb{C}$, but if there is another one, call it $w_h \in \mathbb{C}\setminus(0,\infty)$, then $|w_h| > z_h$. In fact if $|w_h| < z_h$ then $\exp(-h) = \widehat{K}(w_h) \leq \widehat{K}(|w_h|) < \widehat{K}(z_h) = \exp(-h)$, which is impossible. And $|w_h| = z_h$ is excluded by aperiodicity of $K(\cdot)$. Another immediate fact is that $\widehat{K}(z) = \exp(-h)$ can be solved for h in a neighborhood of the real axis and $z = z_h$ which is also in a neighborhood of the real axis: in fact, by the analytic inverse function theorem, z_h is an analytic function on $B_{a,b}(\varepsilon) := \{z \in \mathbb{C} : \inf_{x\in(a,b)} |z - x| < \varepsilon\}$, with $0 < a < b < \infty$ and $\varepsilon > 0$ sufficiently small. Let us fix a $\varepsilon > 0$ such we have that $|w_h| > |z_h|$ for every $h \in B_{a,b}(2\varepsilon)$ and such that z_h is analytic in $B_{a,b}(2\varepsilon)$. We aim at showing that there exists $\delta > 0$ such that, if there exists $w_h \neq z_h$ such that $\widehat{K}(w_h) = \exp(-h)$ for $h \in B_{a,b}(\varepsilon)$, then $|w_h| \geq |z_h| + \delta$. The proof is by contradiction: if this is false, then we can find (h_j) in $B_{a,b}(\varepsilon)$ for which w_{h_j}

exists for every j (so $|w_{h_j}| > |z_{h_j}|$) and $|w_{h_j}| - |z_{h_j}| \to 0$. Without loss of generality we can assume that these three sequences converge (for the limits we just omit the subscript). By passing to the limit we see that $|w_h| = |z_h|$ for h which is in the closure of $B_{a,b}(\varepsilon)$, hence in $B_{a,b}(2\varepsilon)$, which is impossible.

The proof now proceeds in the same way as case (1) and the final result is that for $0 < a < b < \infty$ there exists $\varepsilon > 0$ and a two positive constants c and C such that

$$\left| Z_{N,h} - \frac{\exp(-h)}{\widehat{K}'(z_h)} z_h^{-N-1} \right| \le C\left((1+c)|z_h|\right)^{-N-1}, \tag{3.10}$$

for every $h \in B_{a,b}(\varepsilon)$. This is of course sufficient to cover case (2) in view of case (1). $\square$

3.2. **Sharp estimates in the special framework.**

Proposition 3.3. *With the inter-arrival distribution* (1.6)

(1) if $h \in \mathcal{L}_\alpha$ then $z \mapsto 1 - \exp(h)(1-(1-z)^\alpha)$ has a unique zero $z_{\alpha,h}$ in the open unit disk around the origin and

$$Z_{N,h} \overset{N\to\infty}{\sim} \frac{(1-\exp(-h))^{(1-\alpha)/\alpha}}{\alpha \exp(h)} z_{\alpha,h}^{-(N+1)}, \tag{3.11}$$

and this result is uniform for h bounded away from $\mathcal{C}_\alpha$;

(2) if $h \in \mathcal{D}_\alpha$

$$Z_{N,h} \overset{N\to\infty}{\sim} K(N)\frac{e^h}{(1-e^h)^2}, \tag{3.12}$$

and also this result is uniform for h bounded away from $\mathcal{C}_\alpha$;

Proof. For case (1) let us first remark that, by Proposition 3.2, it suffices to show the result for $\Re(h) \le C$. So we focus on the compact set $V_\varepsilon := \{h \in \mathcal{L}_\alpha : \text{dist}(h, \mathcal{C}_\alpha) \ge \varepsilon$ and $\Re(h) \le C\} \subset \mathbb{C}_{2\pi}$. The denominator in (1.23), that we call $D(h,z)$ in this proof, for every $h \in \mathcal{L}_\alpha$ has a unique zero $z = z_h \in B_0(1)$. Consider now $r_\varepsilon := \sup_{h \in V_\varepsilon} |z_h| < 1$ and choose $\eta > 0$ such that $\exp(-\eta) > r_\varepsilon$. We can now use the same argument as in Proposition 3.2: we apply (3.6) and (3.7) with $r = \exp(-\eta)$. Since V_ε is compact we readily see that there exists $c_{\varepsilon,\eta} > 0$ such that $|R_h(z)| \le c_{\varepsilon,\eta}$ for every $h \in V_\varepsilon$ and $|z| = \exp(-\eta)$. This directly yields

$$\left| Z_{N,h} - \frac{\exp(-h)}{\widehat{K}'(z_h)} z_h^{-N-1} \right| \le c_{\varepsilon,\eta} \exp(\eta N). \tag{3.13}$$

This completes the proof in case (1).

For case (2) we follow closely [15, Section VI.3], also from the notational viewpoint. In particular, we use the fact that we can find $R > 1$ such that $\widehat{Z}_h(\cdot)$ is analytic in the open domain

$$\Delta(\phi, R) := \{z : |z| < R, z \ne 1, |\arg(z-1)| > \phi\}, \tag{3.14}$$

and this for any choice of $\phi \in (0, \pi/2)$. This follows from Lemma 2.4(1) and we can (and do) choose $R = R_\varepsilon = 1 + (r_\varepsilon - 1)/2$. This directly yields that $\sup_h \sup_{z:|z|=R} \widehat{Z}_h(z) < \infty$ where $h \in K_\varepsilon$ with K_ε a compact subset with two requirements: $K_\varepsilon \subset \mathcal{D}_\alpha$ and $\text{dist}(K_\varepsilon, \mathcal{C}_\alpha) \ge \varepsilon$. Moreover

$$\widehat{Z}_h(z) = \frac{1}{1 - e^h(1-(1-z)^\alpha)} = \frac{1}{1-e^h} - \frac{e^h(1-z)^\alpha}{(1-e^h)^2} + O\left(|1-z|^{2\alpha}\right), \tag{3.15}$$

uniformly for z in the intersection of $\Delta(\phi, R)$ with a neighborhood of 1 (since it suffices to show the result for ε small, A direct application of [15, Th. VI.3] (see also [15, Th. VI.1] for the details on how to extract the leading order term from the $(1-z)^\alpha$ term in (3.15)) yields

$$Z_{N,h} \overset{N\to\infty}{=} \frac{e^h}{(-\Gamma(-\alpha))\,(1-e^h)^2}\frac{1}{N^{1+\alpha}} + O\left(\frac{1}{N^{1+2\alpha}}\right) . \tag{3.16}$$

The proof of [15, Th. VI.3] (pp.131-132) is based on estimates on a contour integral involving $\widehat{Z}_h(z)$ and the uniform control we have just claimed on $\widehat{Z}_h(z)$ yields that (3.16) holds uniformly in $h \in K_\varepsilon$. □

Remark 3.4. *The argument we just presented can be upgraded to deal with K_ε non compact (still satisfying the two requirements of being a subset of $\mathcal{D}_\alpha$ which is bounded away from the boundary $\mathcal{C}_\alpha$): it is a matter of following carefully what happens for $-\Re(h)$ large. This provides an alternative argument for Proposition 3.1.*

Theorem 1.8 is a corollary of the sharp estimates we just established, except for the critical case.

Proof of Theorem 1.8. For $h \in \mathcal{L}_\alpha$ we use Proposition 3.3(1): $\lim_N (1/N)\log|Z_{N,h}| = -\log|z_{\alpha,h}|$ and, by making $z_{\alpha,h}$ explicit, we see that it is equal to $\Re(\mathrm{F}(h))$ in the whole of $\mathcal{L}_\alpha$. The case of $h \in \mathcal{D}_\alpha$ is even more straightforward and uses Proposition 3.3(2). And in both cases uniformity follows because Proposition 3.3 is proven uniformly, away from $\mathcal{C}_\alpha$.

In the critical case, by Lemma 2.4, there is a simple pole $z_{\alpha,h}$ on the unit ball (except for $h = 0$: in this case $Z_{N,0} \sim c_\alpha N^{1-\alpha}$ for an explicit $c_\alpha > 0$, see [10, Th. B] so $\lim_N (1/N)\log|Z_{N,0}| = 0$, even if this last result is easily established without sharp control on $Z_{N,0}$). Of course there is no pole in the unit circle, because that happens only for $h \in \mathcal{L}_\alpha$ (cf. Lemma 2.3). One can check that there is no other pole, but this is not very important because it is obvious that there is no other pole in the closure of $B_0(r)$ for some $r > 1$. This allows to choose a circuit of integration that coincides with $\partial B_0(r)$, except close to 1, where we have to use a circuit like the one in the proof of Proposition 3.3(2), see notably (3.14). Therefore the sharp asymptotic behavior of $Z_{N,h}$ in this case will be given by the dominant contributions among the pole (3.11) and the essential singularity in 1 (3.12). Actually, since $|z_{\alpha,h}| = 1$, the pole in this case just contributes an additive constant times an N dependent phase, while the essential singularity contributes a vanishing term (and eventual poles outside the unit circles would just contribute exponentially vanishing terms and the higher order contribution of the essential singularities would be more dominant anyways). If we sum up: for $h \in \mathcal{C}_\alpha \setminus \{0\}$ we have (recall (3.16))

$$Z_{N,h} = \frac{(1-\exp(-h))^{(1-\alpha)/\alpha}}{\alpha\exp(h)} e^{-(N+1)\arg(z_{\alpha,h})i} + \frac{e^h}{(-\Gamma(-\alpha))\,(1-e^h)^2}\frac{1}{N^{1+\alpha}} + O\left(\frac{1}{N^{1+2\alpha}}\right) . \tag{3.17}$$

Therefore $\lim_N (1/N)\log|Z_{N,h}| = 0$ for every $h \in \mathcal{C}_\alpha$. □

4. Potential theory and empirical measure analysis

While the set up of this section is general, all the results in the end depend on the control of $|Z_{N,h}|$ for every complex h (outside of the critical curve). They are therefore limited to the special framework (1.6).

In this section we begin by using the polynomial notation (1.5) and we point out that, since $\mathbf{P}(\tau_N = N) = K(1)^N$, we have

$$P_N(w) \,=\, K(1)^N w \prod_{j=1}^{N-1} (w - w_{N,j}) \,, \tag{4.1}$$

with $(w_{N,j})_{j=1,\ldots,N}$ the zeros of $P_N(\cdot)$ and $w_{N,N} = 0$. With $\mathtt{x}, \mathtt{y} \in \mathbb{R}$ we set

$$u_N(\mathtt{x}, \mathtt{y}) \,:=\, \frac{1}{N-1} \Re \log P_N(\mathtt{x} + i\mathtt{y}) \,=\, \frac{1}{N-1} \log |P_N(\mathtt{x} + i\mathtt{y})| \,. \tag{4.2}$$

With abuse of notation we write $u_N(w)$ also for $u_N(\mathtt{x}, \mathtt{y})$ when $w = \mathtt{x} + i\mathtt{y}$, in fact we identify w with $(\mathtt{x}, \mathtt{y})$. Note that $u_N(\cdot)$ is smooth out of the zeros and [30, Th. 3.7.8]

$$\Delta u_N \,=\, 2\pi \frac{1}{N-1} \sum_{j=1}^{N-1} \delta_{w_{N,j}} + \frac{2\pi}{N-1} \delta_0 \,=:\, 2\pi \nu_N + \frac{2\pi}{N-1} \delta_0 \,, \tag{4.3}$$

and this means that for every $g \in C_0^\infty$ (i.e., $g \in C^\infty$ and g is compactly supported) we have

$$\int_{\mathbb{R}^2} u_N \Delta g \, \mathrm{d}\lambda \,=\, 2\pi \int_{\mathbb{R}^2} g \, \mathrm{d}\nu_N + \frac{2\pi}{N-1} g(0) \,, \tag{4.4}$$

where λ is the Lebesgue measure on $\mathbb{R}^2$.

We now go back to our original coordinate systems. We have $u_N(\exp(h)) = \mathrm{F}_N(h)$ and, with $h = x + iy$ identified with (x, y)

$$\Delta \mathrm{F}_N \,=\, 2\pi \frac{1}{N-1} \sum_{j=1}^{N-1} \delta_{h_{N,j}} \,=:\, 2\pi \mu_N \,, \tag{4.5}$$

where $h_{N,j}$ is one of the $N-1$ zeros of $h \mapsto Z_{N,h}$ with $\Im(h) \in (-\pi, \pi]$: we can choose them so that $w_{N,j} = \exp(h_{N,j})$ and $w_{N,N} = 0$ is *pushed to* $-\infty$ in these variables. Equation (4.5) means

$$\int_{\mathbb{R}^2} \mathrm{F}_N(x, y) \Delta f(x, y) \, \mathrm{d}x \, \mathrm{d}y \,=\, 2\pi \int_{\mathbb{R}^2} f(x, y) \, \mathrm{d}\mu_N(x, y) \,, \tag{4.6}$$

for every $f : \mathbb{C}_{2\pi} \to \mathbb{R}$ which is smooth and compactly supported. Stepping from (4.3) to (4.5), i.e. from (4.4) to (4.6), is a computation, but let us note that to obtain (4.6) it suffices to consider (4.4) with $g \in C_0^\infty$ and whose support is bounded away from 0. In fact, with the change of variable $w = \exp(h)$, h in a compact subset of $\mathbb{C}_{2\pi}$ means w lives in a compact subset of $\mathbb{C} \setminus \{0\}$. The computation can be performed in $\mathbb{R}^2$, by this we mean that if $w = u + iv$ and $h = x + iy$, the change of coordinates is $u = e^x \cos(y)$ and $v = e^x \sin(y)$. The determinant of the Jacobian of this transformation is e^{2x}. On the other hand, $\Delta f(x, y) = (\partial_x^2 + \partial_y^2) g(e^x \cos(y), e^x \sin(y)) = e^{2x} (\Delta g)(e^x \cos(y), e^x \sin(y))$, so (4.6) follows.

Lemma 4.1. *Assume* (1.6)*. We have*

$$\lim_{N \to \infty} \int_{\mathbb{R}^2} \mathrm{F}_N(x, y) \Delta f(x, y) \, \mathrm{d}x \, \mathrm{d}y \,=\, \int_{\mathbb{R}^2} \mathrm{F}(x, y) \Delta f(x, y) \, \mathrm{d}x \, \mathrm{d}y \,. \tag{4.7}$$

Proof. By the uniform convergence of $\mathrm{F}_N(h)$ away from the simple (smooth, finite length) curve $\mathcal{C}_\alpha$, see Lemma 1.5 and Theorem 1.8, and because $\mathrm{F}(\cdot)$ is continuous, it suffices to show that, with $A_\varepsilon := \{(x,y) : x+iy \in \mathbb{C}_{2\pi} \text{ and } \mathrm{dist}(x+iy, \mathcal{C}_\alpha) < \varepsilon\}$, we have

$$\lim_{\varepsilon \searrow 0} \sup_N \left| \int_{A_\varepsilon} \mathrm{F}_N(x,y) \Delta f(x,y) \, \mathrm{d}x \, \mathrm{d}y \right| = 0 . \tag{4.8}$$

Now let us point out (Proposition 1.3) that there exists $C > 1$ such that, uniformly in N, all the zeros are in the compact set $K := \{h : |\Re(h)| \le C\}$ except of course for the zero at ∞ which however gives a contribution x/N to $\mathrm{F}_N(x,y)$ (without loss of generality we assume $A_\varepsilon \subset K$). We write

$$\mathrm{F}_N(x,y) = \log K(1) + \frac{x}{N} + \frac{1}{N} \sum_{j=1}^{N-1} \log |\exp(x+iy) - \exp(h_{N,j})| , \tag{4.9}$$

and focus on $\log|\exp(x+iy) - \exp(h_{N,j})|$, which is of course bounded above by $c_1 := \log 2 + \log C$. On the other hand $|\exp(h) - \exp(h_0)| \ge \varepsilon/2$, if $h, h_0 \in K$ and $|h - h_0| \ge \varepsilon$ (for ε sufficiently small). Therefore for every N, for every $j = 1, \ldots, N-1$ and every $(x,y) \in K$ with $|x+iy-h_{N,j}| \ge \varepsilon$ we have

$$|\log|\exp(x+iy) - \exp(h_{N,j})|| \le c + |\log \varepsilon| , \tag{4.10}$$

with $c = c_1 + \log 2$ On the other hand $\int_{B_0(\varepsilon)} \log|\exp(x+iy) - 1| \le 4\varepsilon^2 |\log \varepsilon|$ for ε small. By putting these estimates together we see that

$$\left| \int_{A_\varepsilon} \mathrm{F}_N(x,y) \Delta f(x,y) \, \mathrm{d}x \, \mathrm{d}y \right| \le (|\log K(1)| + (C/N) + c + |\log \varepsilon|) \, \lambda(A_\varepsilon) + 4\varepsilon^2 |\log \varepsilon| . \tag{4.11}$$

Since $\lambda(A_\varepsilon) = O(\varepsilon)$, (4.8) is proven. □

We can now conclude that

Proposition 4.2. *Assume* (1.6). *We have that* (μ_N) *converges to* μ *in distribution, i.e.* $\lim_N \int f \, \mathrm{d}\mu_N = \int f \, \mathrm{d}\mu$ *for every* f *continuous and bounded: in particular,* μ *is a probability. Moreover* μ *is identified by*

$$\int_{\mathbb{R}^2} \mathrm{F}(x,y) \Delta f(x,y) \, \mathrm{d}x \, \mathrm{d}y = 2\pi \int_{\mathbb{R}^2} f(x,y) \, \mathrm{d}\mu(x,y) \quad \text{for every } f \in C_0^\infty . \tag{4.12}$$

Proof. Proposition 1.3 confines all zeros to a compact set, so (μ_N) is tight. The limit points μ are therefore probabilities and Lemma 4.1 guarantees that they satisfy (4.12). But (4.12) uniquely identifies μ and we obtained the desired convergence. □

One can extract a number of facts from Proposition 4.2: we list some of them here, in an informal way.

- Proposition 4.2 directly yields that the support of μ is contained in $\mathcal{C}_\alpha$ because if F is C^2 in the open ball $B_{(x,y)}(r)$, then $\int f \, \mathrm{d}\mu = 0$ for every f supported in $B_{(x,y)}(r)$, and F is smooth outside of $\mathcal{C}_\alpha$ (that the support of μ is contained in $\mathcal{C}_\alpha$ can also be seen from Proposition 3.3). But in fact the support is exactly $\mathcal{C}_\alpha$ and one can show that μ has a density on its support (for example, by taking as reference measure the arc-length on $\mathcal{C}_\alpha$) and this density vanishes only at zero, which is the only singular point of the density. Establishing this is a bit cumbersome: it involves performing integration by parts on the left-hand side of (4.12) exploiting the monotonicity of the critical curve proven in Appendix C. The result is

$$\sum_{\sigma=\pm 1} \sigma \int_0^{x_0} \left(\partial_1 \mathrm{F}(x, \sigma Y(x)) Y'(x) - \partial_2 \mathrm{F}(x, \sigma Y(x))\right) f(x, \sigma Y(x))\, \mathrm{d}x\,, \tag{4.13}$$

where $x_0 = -\log(2^\alpha - 1)$ and the function $x \mapsto Y(x)$, with $x \in [0, X_0]$, is so that the union of $\{(x, Y(x)) : x \in [0, X_0]\}$ and $\{(x, -Y(x)) : x \in [0, X_0]\}$ yields C_α. From (4.13) one reads that the probability μ has a density on its support.

- One computation that can be performed in detail with moderate effort is the one that leads to the density of μ near 0. We call s the arc length computed (with sign) starting from the origin: let us restrict to the portion of $\mathcal{C}_\alpha$ with positive imaginary part and let $\gamma_\alpha(s)$, s from zero to the total length ℓ_α of the curve (with positive imaginary part), so $\gamma_\alpha(0) = 0$ and $\gamma_\alpha(\ell_\alpha) = -\log(2^\alpha - 1) + i\pi$. Then

$$\frac{\mathrm{d}\mu([0, \gamma(s)])}{\mathrm{d}s} \overset{s \searrow 0}{\sim} \frac{s^{(1-\alpha)/\alpha}}{\alpha \cos\left(\alpha \frac{\pi}{2}\right)}\,. \tag{4.14}$$

- One can work out more explicitly the case $\alpha = 1/2$: it is more practical to extract the density as a function of $x \in \left[0, \log\left(1 + \sqrt{2}\right)\right)$ and we obtain

$$\frac{8e^x \sinh(x)}{\sqrt{6e^{2x} - e^{4x} - 1}} \overset{x \searrow 0}{=} 4\,x + \frac{8}{3} x^3 + O\left(x^5\right)\,. \tag{4.15}$$

This density is convex and diverges approaching $\log\left(1 + \sqrt{2}\right)$, but this is an artefact of the parametrization: in the arc length parametrisation, that is if we divide by $(1 + (Y'(x))^2)^{1/2}$, we obtain the particularly simple formula

$$\sqrt{2}\left(1 - e^{-2s}\right) \overset{s \searrow 0}{=} 2\sqrt{2}\, s - 2\sqrt{2}\, s^2 + O\left(s^3\right)\,, \tag{4.16}$$

which is a concave bounded function and which is, of course, in agreement with, (4.14).

- A byproduct of the analysis developed in this section is the formula

$$\Re\left(\mathrm{F}(h)\right) = \log \alpha + \int \log |e^h - e^\zeta| \mu(\mathrm{d}\zeta) = \int \log \frac{|e^h - e^\zeta|}{|1 - e^\zeta|} \mu(\mathrm{d}\zeta)\,, \tag{4.17}$$

where the first expression follows from from (4.1) (or the middle term in (6.1), which is just (4.1) with $w = e^h$), Proposition 4.2 and the fact that the zeros are bound to a compact region (Proposition 1.3). For the second expression it suffices to use the rightmost term in (6.1) and $\log Z_{N,0} = o(N)$.

5. On the zeros close to the origin

5.1. **Results in the general setting.** We start off in the general setting of (1.1). The proof of the following result can be found in [21, Ch. 9, § 49 and § 50].

Theorem 5.1 (Local Limit Theorem)**.** *For $\alpha \in (0, 1)$ we set $a_j := j^{1/\alpha}$. In the general setting of* (1.1) *we have*

$$\lim_{j \to \infty} \sup_n \left| a_j \mathbf{P}\left(\tau_j = n\right) - g_\alpha\left(\frac{n}{a_j}\right)\right| = 0\,, \tag{5.1}$$

where $g_\alpha(\cdot)$ is the law of the positive stable law identified by $\int_0^\infty g_\alpha(y) e^{-ty}\, \mathrm{d}y = \exp(-t^\alpha)$ for $t > 0$.

We have [33, p. 99]

$$g_\alpha(x) \overset{x\searrow 0}{\sim} \frac{1}{\sqrt{2\pi\alpha(1-\alpha)}} \left(\frac{\alpha}{x}\right)^{\frac{2-\alpha}{2(1-\alpha)}} \exp\left(-(1-\alpha)\left(\frac{\alpha}{x}\right)^{\frac{\alpha}{1-\alpha}}\right), \tag{5.2}$$

and [33, p. 90]

$$g_\alpha(x) \overset{x\to\infty}{\sim} \frac{\Gamma(1+\alpha)\sin(\pi\alpha)}{\pi} x^{-(1+\alpha)}. \tag{5.3}$$

In the special case $\alpha = 1/2$ the asymptotic equivalence (5.2) becomes an equality (only truly explicit case, $\alpha = 1/3$ and $\alpha = 2/3$ can be expressed via McDonald functions): for every $x > 0$

$$g_{1/2}(x) = \frac{1}{\sqrt{4\pi x^3}} \exp\left(-\frac{1}{4x}\right). \tag{5.4}$$

Proposition 5.2. *Uniformly for ζ in compact subsets of $\mathbb{C}$*

$$Z_{N,\zeta/N^\alpha} \overset{N\to\infty}{\sim} \frac{1}{N^{1-\alpha}} \int_0^\infty \exp(\zeta x)\, g_\alpha\left(x^{-1/\alpha}\right) x^{-1/\alpha}\, \mathrm{d}x =: \frac{1}{N^{1-\alpha}} F_{0,\alpha}(\zeta), \tag{5.5}$$

and with $Z'_{N,h} = \partial_h Z_{N,h}$

$$Z'_{N,\zeta/N^\alpha} \overset{N\to\infty}{\sim} \frac{1}{N^{1-2\alpha}} \int_0^\infty \exp(\zeta x)\, g_\alpha\left(x^{-1/\alpha}\right) x^{1-1/\alpha}\, \mathrm{d}x = \frac{1}{N^{1-2\alpha}} F'_{0,\alpha}(\zeta). \tag{5.6}$$

Proposition 5.2 is a direct consequence of Theorem 5.1 and Riemann sum approximations, with a control of the tails of the sums. The arguments are standard, but we provide some details in App. A.

One direct consequence of Proposition 5.2 is that $F_{0,\alpha}(\cdot)$ is an entire function: in fact $\zeta \mapsto N^{1-\alpha} Z_{N,\zeta/N^\alpha}$ is entire and the uniform convergence implies that the limit is entire. It is of course easy to check that $F_{0,\alpha}(\cdot)$ is not constant, so it has only isolated zeros. The following result yields a non negligible control on the zeros of $Z_{N,\zeta/N^\alpha}$ that are at distance $O(1/N^\alpha)$ from the origin if we know where the zeros of $F_{0,\alpha}(\cdot)$ are.

Proposition 5.3. *Suppose that in a bounded simply connected open set D with smooth boundary there are exactly n zeros of $F_{0,\alpha}(\cdot)$ counted with their multiplicities. Then for N sufficiently large there are exactly n zeros of $h \mapsto Z_{N,h}$ in D/N^α.*

In particular, Proposition 5.3 says that if $F_{0,\alpha}(\zeta_0) = 0$ and if this zero is simple, then for every $\varepsilon > 0$ there exists N_ε such that $Z_{N,h} = 0$ for exactly one $h \in B_{\zeta_0/N^\alpha}(\varepsilon)$ if $N \geq N_\varepsilon$.

Proof. We have

$$\frac{1}{2\pi i}\oint_{\partial D/N^\alpha} \frac{Z'_{N,z}}{Z_{N,z}}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{\partial D} \frac{Z'_{N,\zeta/N^\alpha}}{N^\alpha Z_{N,\zeta/N^\alpha}}\,\mathrm{d}\zeta \overset{N\to\infty}{\longrightarrow} \frac{1}{2\pi i}\oint_{\partial D} \frac{F'_{0,\alpha}(\zeta)}{F_{0,\alpha}(\zeta)}\,\mathrm{d}\zeta = n, \tag{5.7}$$

where the last equality is the Argument Principle and the convergence step follows from Proposition 5.2. Since, again by the Argument Principle, the left-most term is the number of zeros of $h \mapsto Z_{N,h}$ in D/N^α, the proof is complete. □

5.2. **The $\alpha = 1/2$ case.** Unfortunately, solving $F_{0,\alpha}(\zeta) = 0$ appears to be too challenging. It should be possible to show that, for ζ large, the zeros will be in the first and fourth quadrants and close to the lines with directions $\exp(i\alpha\pi/2)$ (in analogy with (5.17) below). However this is not straightforward and, as we explained in Section 1.5, one is particularly interested in the zeros that are the closest (they come in pairs, unless they are real) to the origin.

Therefore, in order to go farther, we specialize to $\alpha = 1/2$. As we announced, in this case things get more explicit:

$$F_0(\zeta) \,:=\, F_{0,\frac{1}{2}}(\zeta) \,=\, e^{\zeta^2}\zeta\,(1+\mathrm{erf}(\zeta)) + \frac{1}{\sqrt{\pi}}\,, \tag{5.8}$$

and we record that

$$F_0'(\zeta) \,=\, \frac{2\zeta}{\sqrt{\pi}} + \exp(\zeta^2)\left(1+2\zeta^2\right)(1+\mathrm{erf}(\zeta))\,. \tag{5.9}$$

Remark 5.4. *Note that* $F_0'(\zeta) = (2\zeta + 1/\zeta)F_0(\zeta) - 1/(\sqrt{\pi}\zeta)$ *so*

$$F_0(\zeta) \,=\, 0 \quad\Longrightarrow\quad F_0'(\zeta) \,=\, -\frac{1}{\zeta\sqrt{\pi}}\,. \tag{5.10}$$

In particular, the zeros of F_0 *are simple. We record also, for later use, that* $F_0''(\zeta) = -2/\sqrt{\pi}$ *if* $F_0(\zeta) = 0$.

In spite of the rather explicit expression for $F_0(\cdot)$, it does not appear that $F_0(\zeta) = 0$ can be solved explicitly. What we are mostly interest in are the zeros that are closest to the origin: we can only identify them numerically. Nevertheless something can be said rigorously. Moreover the numerical approximations can be controlled rigorously, at least if we accept the assistance of the computer for symbolic computations.

For the statement, order the solution of ζ_j to $F_0(\zeta_j) = 0$ so that $|\zeta_j|$ is non decreasing in j. We can assume that $\Im(\zeta_1) \geq 0$, and set $\zeta_2 = \overline{\zeta}_1$ (unless ζ_1 is real). A priori there could still be more than one choice for ζ_1.

Lemma 5.5. $\Re(z_j) > 0$ *and* $\Im(z_j) \neq 0$ *for every* j *(hence we can stipulate that* $\Im(z_{2k-1}) = -\Im(z_{2k}) > 0$ *for* $k = 1, 2, \ldots$*). Moreover* ζ_1 *is well defined (i.e.,* $|z_1| < |z_3|$*). In fact,* $\zeta_1 = 1.225 + 2.547i + r_1$ *and* $z_3 = 2.026 + 3.162i + r_3$*, with* $|r_1|$ *and* $|r_3|$ *smaller than* 0.0005.

Of course Lemma 5.5 is also implicitly saying that $|\zeta_j| \geq |\zeta_3| = |\zeta_4|$ for $j = 5, 6, \ldots$.

Proof. For $\Re(\zeta) < 0$ we use the representation

$$\frac{\pi}{2}F_0(\zeta) \,=\, \frac{1}{2}\int_0^\infty \frac{e^{-y}\sqrt{y}}{y+\zeta^2}\,\mathrm{d}y \,=\, \int_0^\infty \frac{e^{-x^2}x^2}{x^2+\zeta^2}\,\mathrm{d}x\,. \tag{5.11}$$

Remark 5.6. (5.11) *follows from* [29, (7.2.3) and (7.7.2)]. *To see this it is quicker to exploit also the complementary error function* $\mathrm{erfc}(\zeta) = 1 - \mathrm{erf}(\zeta)$. *So the symmetry* $\mathrm{erf}(-\zeta) = -\mathrm{erf}(\zeta)$ *that holds for* $erf(\cdot)$ *is equivalent to* $\mathrm{erfc}(-\zeta) = 2 - \mathrm{erf}(\zeta)$ *and*

$$F_0(\zeta) \,=\, \zeta e^{\zeta^2}\mathrm{erfc}(-\zeta) + \frac{1}{\sqrt{\pi}}\,. \tag{5.12}$$

The identities [29, (7.2.3) and (7.7.2)] *yield that for* $\Im(z) > 0$

$$e^{-z^2}\mathrm{erfc}(-iz) \;=\; \frac{2z}{\pi i}\int_0^\infty \frac{\exp(-t^2)}{t^2-z^2}\,\mathrm{d}t\,. \tag{5.13}$$

The identities (5.12) *and* (5.13) *imply* (5.11), *which holds for* $\Re(\zeta) < 0$, *and also that for* $\Re(\zeta) > 0$ *we have instead*

$$\frac{2}{\pi}\int_0^\infty \frac{e^{-x^2}x^2}{x^2+\zeta^2}\,\mathrm{d}x \;=\; F_0(\zeta) - 2\zeta e^{\zeta^2}\,. \tag{5.14}$$

With $\zeta = u + iv$ we see that the real and the imaginary part of the previous quantity are respectively

$$\int_0^\infty \frac{e^{-x^2}x^2\left(x^2+u^2-v^2\right)}{\left(x^2+u^2-v^2\right)^2+4u^2v^2}\,\mathrm{d}x \quad \text{and} \quad 2uv\int_0^\infty \frac{e^{-x^2}x^2}{\left(x^2+u^2-v^2\right)^2+4u^2v^2}\,\mathrm{d}x\,, \tag{5.15}$$

and, since we are assuming that $u < 0$, the second expression – the imaginary part – is zero if and only if $v = 0$. But in that case the first expression – the real part – is positive. Therefore $F_0(\zeta) \neq 0$ if $\Re(\zeta) < 0$.

For the case $u = \Re(\zeta) = 0$ we directly use (5.8) and we rewrite it as

$$F_0(iv) \;=\; \frac{1}{\sqrt{\pi}} - ve^{-v^2}\mathrm{erfi}(v) + ive^{-v^2}\,, \tag{5.16}$$

where $\mathrm{erfi}(v) := \mathrm{erf}(iv)/i = (1/\sqrt{\pi})\int_0^v e^{t^2}\,\mathrm{d}t$. So $\mathrm{erfi}(\cdot)$ is real and odd on the real axis. Moreover it is positive on the positive semiaxis. From this we readily infer that $F_0(iv) \neq 0$ for every v: in fact $|F_0(iv)|$ vanishes only for $|v| \to \infty$.

The fact that $F_0(\zeta) > 0$ for $\zeta > 0$, in fact $F_0(\zeta) > 1/\pi$, follows from $F_0'(\zeta) > 0$, see (5.9).

In order to determine ζ_1 and ζ_3 (in fact, every ζ_j in principle) we need to write a sufficiently precise polynomial approximation of $F_0(\cdot)$, with adequate control of the remainder, and use the Argument Principle (see for example the proof of Proposition 5.3). Implementing this approach in practice, however, is quite cumbersome and probably can only usefully be done on a computer.

In order to establish that there are infinitely many zeros one can adapt the approach in [13]. In fact, one can identify a sequence of simple zeros that satisfy

$$\zeta_n \;=\; \lambda_n - \frac{1}{4\lambda_n}\log\left(8\sqrt{2\pi}\lambda_n^3\right) + i\left(\lambda_n + \frac{1}{4\lambda_n}\log\left(8\sqrt{2\pi}\lambda_n^3\right)\right) + O\left(|\log n|^2/n^{3/2}\right)\,, \tag{5.17}$$

with $\lambda_n = (\pi(n+1/8))^{1/2}$. One can also show that, sufficiently far from the origin, there is no other zero (up to conjugation). We do not go into the lengthy details of this result that is not central for us, but one can use (5.14); a key point is that

$$\lim_{\substack{|\zeta|\to\infty:\\ \Re(\zeta)>0}} e^{\zeta^2}\zeta\,\mathrm{erfc}(\zeta) \;=\; \frac{1}{\sqrt{\pi}}\,. \tag{5.18}$$

In fact, by the continuous fraction expansion [29, (7.9.1)], we have that in the same limit

$$\frac{1}{\sqrt{\pi}} - e^{\zeta^2}\zeta\,\mathrm{erfc}(\zeta) \;=\; \frac{1}{2\sqrt{\pi}\zeta^2} + O\left(\frac{1}{\zeta^4}\right)\,. \tag{5.19}$$

By writing the analog of (5.15) for $\Re(\zeta > 0$ and using (5.18) and (5.19) one can see that the zeros (that are far from the origin) need to be close to the diagonal of the first and second quadrant. And a controlled perturbation analysis leading to (5.17).

The asymptotic formula (5.17) turns out to be surprisingly accurate even for n small: see Table 1. □

n	ζ_n	$\zeta_n^{\sim}$	$\lvert\zeta_n - \zeta_n^{\sim}\rvert$
1	$1.225 + 2.547\,i$	$1.229 + 2.531\,i$	0.017
2	$2.026 + 3.162\,i$	$2.018 + 3.149\,i$	0.015
3	$2.629 + 3.656\,i$	$2.621 + 3.646\,i$	0.013
4	$3.132 + 4.083\,i$	$3.125 + 4.075\,i$	0.011
5	$3.573 + 4.466\,i$	$3.566 + 4.459\,i$	0.010
6	$3.969 + 4.817\,i$	$3.963 + 4.810\,i$	0.009
7	$4.332 + 5.141\,i$	$4.326 + 5.136\,i$	0.008

TABLE 1. *Exact* (i.e., numerically evaluated) and approximate (i.e., $\zeta_n^{\sim}$ is the right-hand side of (5.17) without the remainder) location of the zeros of $F_{1/2}$. Here we consider only the zeros with positive imaginary parts, so ζ_n is an abuse of notation for ζ_{2n-1}.

By combining Proposition 5.3 and Lemma 5.5 we readily reach:

Corollary 5.7. *For N sufficiently large, $h_{N,1} = \overline{h}_{N,2} \sim \zeta_1/\sqrt{N}$ and for $j = 3, \ldots, N-1$*

$$\frac{|h_{N,j}|}{|h_{N,1}|} > 1 + \frac{1}{2}\left(\frac{|\zeta_3|}{|\zeta_1|} - 1\right) . \tag{5.20}$$

The factor $\frac{1}{2}$ is of course arbitrary and may be replaced by any number in $(0,1)$.

5.3. **Sharper control.** Can one go beyond Corollary 5.7? For example, sticking to $\alpha = 1/2$, one might wonder whether a development like $h_{N,j} = z_0/\sqrt{N} + z_1/N + z_2/N^{3/2} + \ldots$, of course with $z_0 = \zeta_j$, holds. It is not difficult to convince oneself that this cannot hold in the general framework of (1.1).

We develop this issue in the special case of (1.6) and our motivation is that such a precise estimate is needed in Section 6.

Proposition 5.8. *Assume that* (1.6) *holds with $\alpha = 1/2$ and Fix $j \in \mathbb{N}$. We have that*

$$h_{N,j} = \frac{z_0}{\sqrt{N}} + \frac{z_1}{N} + \frac{z_2}{N^{3/2}} + O\left(\frac{1}{N^2}\right) , \tag{5.21}$$

where $z_0 = \zeta_j$ and

$$z_1 = \frac{1}{2} z_0^2 \quad \text{and} \quad z_2 = \frac{1}{24}\sqrt{\pi} z_0 \left(12 z_0^4 + 2 z_0^2 - 3\right) . \tag{5.22}$$

One can push (5.21) to an arbitrary large order, at the price of more and more cumbersome computations: (5.21) suffices for our purposes.

It is not difficult to realize that in the restricted framework (1.6) one can get finer and finer approximations of $Z_{N,\zeta/\sqrt{N}}$ via Stirling expansion, but this turns out to be very involved. We have found it easier to exploit the representation of $Z_{N,h}$ recently given in

[11]: for the special case of (1.6) the partition function $Z_{N,h}$ is the N-th moment of a positive random variable:

$$Z_{N,h} = \int_{(0,\infty)} x^N \nu_h(\mathrm{d}x)\,, \tag{5.23}$$

where ν_h is a probability measure . For $\alpha = 1/2$ (see [11] for $\alpha \in (0,1)$)

$$\nu_h(\mathrm{d}x) := \frac{e^h}{\pi x} \frac{\sqrt{x(1-x)}}{(x(1-2e^h)+e^{2h})} \mathbf{1}_{(0,1)}(x)\,\mathrm{d}x + \frac{2(e^h-1)}{2e^h-1} \mathbf{1}_{(0,\infty)}(h) \delta_{e^{2h}/(2e^h-1)}(\mathrm{d}x)\,. \tag{5.24}$$

This result is at first sight surprising because $Z_{N,h}$ is a polynomial in $\exp(h)$, while the right-hand side in (5.23) has different expressions for $h > 0$ and $h < 0$ because of the delta contribution to ν_h that we can of course view as $\nu_h^{\mathrm{abs}} + \nu_h^{\mathrm{sing}}$ separating thus absolutely continuous and singular part of the measure. The subtlety here is that there is a singularity in the denominator of ν_h^{abs}: note that the density of the absolutely continuous part has a meaning also for $h \in \mathbb{C}$, even if of course it is no longer a probability density, while for the singular part the analytic continuation can be done only after integration. We can appreciate better this singularity by remarking that for $x \in [0,1]$ and $h \in \mathbb{C}$ small

$$x(1-2e^h) + e^{2h} = (1-x)(1+2h) + (2-x)h^2 + O(h^3)\,, \tag{5.25}$$

so for x near 1 the dominant contribution is $(1-x)+h^2$ (the remainder is $O((1-x)h) + O(h^3)$), which yields a non integrable singularity for imaginary h. As a matter of fact, one directly checks that the right-hand side in (5.23) is analytic for $\Re(h) < 0$ and for $\Re(h) > 0$. For $\Re(h) < 0$ and using the parametrization $h = \zeta/\sqrt{N}$ we have for $N \to \infty$

$$\begin{aligned}\int_{(0,1)} x^N \nu_{\zeta/\sqrt{N}}(\mathrm{d}x) &= \frac{e^{\zeta/\sqrt{N}}}{\pi} \int_{(0,1)} x^{N-1} \frac{\sqrt{x(1-x)}}{\left(x(1-2e^{\zeta/\sqrt{N}}) + e^{2\zeta/\sqrt{N}}\right)}\,\mathrm{d}x \\ &\sim \frac{1}{\pi} \int_{(0,1)} x^{N-1} \frac{\sqrt{(1-x)}}{(1-x)+\zeta^2/N}\,\mathrm{d}x \\ &\sim \frac{1}{\pi} \int_{(0,1)} \exp(-yN) \frac{\sqrt{y}}{y+\zeta^2/N}\,\mathrm{d}y \sim \frac{1}{\pi\sqrt{N}} \int_0^\infty \exp(-y) \frac{\sqrt{y}}{y+\zeta^2}\,\mathrm{d}y\,,\end{aligned} \tag{5.26}$$

where in the first asymptotic statement we have used (5.25) and the fact that the leading contribution to the integrals involved comes from x close to 1. The very same computation holds for $\Re(h) > 0$, hence $\Re(\zeta) > 0$, because we have restricted the integral to $(0,1)$, so we are effectively only integrating with respect to $\nu_{\zeta/\sqrt{N}}^{\mathrm{abs}}$. Without surprise we have that for $\Re(\zeta) < 0$ (see Remark 5.6)

$$\frac{1}{\pi} \int_0^\infty \exp(-y) \frac{\sqrt{y}}{y+\zeta^2}\,\mathrm{d}y = \frac{2}{\pi} \int_0^\infty \exp(-x^2) \frac{x^2}{x^2+\zeta^2}\,\mathrm{d}x = F_0(\zeta) \tag{5.27}$$

and for $\Re(\zeta) > 0$

$$\frac{1}{\pi} \int_0^\infty \exp(-y) \frac{\sqrt{y}}{y+\zeta^2}\,\mathrm{d}y = F_0(\zeta) - 2\zeta \exp\left(\zeta^2\right)\,. \tag{5.28}$$

One then easily verifies that

$$g_N(\zeta) := \int_{[1,\infty)} x^N \nu_{\zeta/\sqrt{N}}(\mathrm{d}x) = \frac{2\left(e^{z/\sqrt{N}}-1\right)e^{2z\sqrt{N}}}{\left(2e^{z/\sqrt{N}}-1\right)^{N+1}} \stackrel{N\to\infty}{\sim} \frac{2\zeta\exp\left(\zeta^2\right)}{\sqrt{N}}. \tag{5.29}$$

Therefore the steps (5.26)–(5.29) provide an alternative proof of (5.5) in the restricted set up of (1.6), only for $\alpha = 1/2$ and only for $\Re(z) \neq 0$. This of course is a very poor result with respect to Proposition 5.2. But (5.23) turns out to be very efficient when we want to obtain higher order corrections in $1/\sqrt{N}$ and that is why we use it now.

Proof of Proposition 5.8. We need to consider only the case $\Re(h) > 0$, but dealing at the same time with $\Re(h) < 0$ essentially affects only one formula, i.e. (5.31), and the estimates we do work just assuming that $\Re(h)$ is bounded away from 0. Therefore we treat both cases at the same time till (5.35). We set

$$f_N(\zeta, y) := \frac{e^{\zeta/\sqrt{N}}}{\pi N}\frac{(1-y/N)^N\sqrt{(y/N)/(1-y/N)}}{\left((1-y/N)(1-2e^{\zeta/\sqrt{N}})+e^{2\zeta/\sqrt{N}}\right)}, \tag{5.30}$$

and, by recalling (5.29), we see that

$$Z_{N,\zeta/\sqrt{N}} = \begin{cases} \int_0^N f_N(\zeta,y)\,\mathrm{d}y & \text{if } \Re(\zeta) < 0\,, \\ \int_0^N f_N(\zeta,y)\,\mathrm{d}y + g_N(\zeta) & \text{if } \Re(\zeta) > 0\,. \end{cases} \tag{5.31}$$

In what follows we consider ζ belonging to a compact subset $K \subset \mathbb{C}$. We have

$$\begin{aligned} g_N(\zeta) = {} & \frac{2}{\sqrt{N}}\zeta\exp\left(\zeta^2\right) - \frac{1}{N}\zeta^2(3+2\zeta^2)\exp\left(\zeta^2\right) \\ & + \frac{1}{6N^{3/2}}\zeta^3\left(26+31\zeta^2+6\zeta^4\right)\exp\left(\zeta^2\right) + O\left(\frac{1}{N^2}\right), \end{aligned} \tag{5.32}$$

and for every y, $\zeta \in K$ and for every sufficiently large N

$$\left| f_N(\zeta,y) - \frac{1}{\sqrt{N}}\frac{e^{-y}\sqrt{y}}{\pi\left(y-\zeta^2\right)} \right| \le \frac{C}{N}\frac{y}{|y+\zeta^2|^2}, \tag{5.33}$$

with $C = C_K$.

Lemma 5.9. *For every* $\zeta \in \mathbb{C}$ *we have* $\min\{|x+\zeta^2| : x \ge 0\} \ge (\Re(\zeta))^2$.

Proof. We have $|x+\zeta^2|^2 = x^2 + 2x\Re(\zeta^2) + |\zeta|^4$ so the minimum of this expression is reached at $x_0 = -\Re(\zeta^2) = (\Im(\zeta))^2 - (\Re\zeta)^2$ if $x_0 > 0$ and it is reached at $x = 0$ if $x_0 \le 0$. In the second case $\min(|x+\zeta^2|) = |\zeta|^2 \ge (\Re(\zeta))^2$. In the first case $\min(|x+\zeta^2|) = |(\Im(\zeta^2))| = 2|\Re(\zeta)||\Im(\zeta)|$ which is bounded below by $2(\Re(\zeta))^2$ because $x_0 > 0$. □

Lemma 5.9 tells us that the limit we are interested can be handled uniformly in $\zeta \in K$ and $|\Re(\zeta)|$ bounded away from zero. This leaves a strip out that a priori is non trivial to handle, but this is of course not a problem because we already know that the zeros are not there (cf., Lemma 5.5 and Corollary 5.7).

We use

$$\pi\sqrt{N}\frac{\exp(y)}{\sqrt{y}}f_N(\zeta,y) \;=\; \frac{1}{(y+\zeta^2)} - \frac{1}{\sqrt{N}}\frac{y\zeta}{(y+\zeta^2)^2} + \frac{1}{N}\frac{6y^3-6y^4+30y^2\zeta^2-12y^3\zeta^2+11y\zeta^4-6y^2\zeta^4-\zeta^6}{12\left(y+\zeta^2\right)^3} + \frac{1}{N^{3/2}}R_N(\zeta,y)\,, \quad (5.34)$$

where, for $\zeta \in K$ and $|\Re(\zeta)| \geq \delta > 0$ we have $|R_N(\zeta,y)| \leq C_K(1+y^5)/\delta^8$, with C_K a constant that depends on the choice of the compact set K. Therefore for $\zeta \in K$ and $\Re(\zeta) \geq \delta > 0$ (recall the contribution from (5.32))

$$\sqrt{N}Z_{N,\zeta/\sqrt{N}} \;=\; F_0(\zeta) + \frac{1}{\sqrt{N}}F_1(\zeta) + \frac{1}{N}F_2(\zeta) + O\left(\frac{1}{N^{3/2}}\right)\,, \quad (5.35)$$

where

$$F_0(\zeta) \;=\; e^{\zeta^2}\zeta\,\left(1+\mathrm{erf}(\zeta)\right) + \frac{1}{\sqrt{\pi}}\,, \quad (5.36)$$

is an entire function. Also F_1 and F_2 are entire functions whose rather awful expressions

$$F_1(\zeta) \;=\; -\frac{1}{2}\zeta\left(e^{\zeta^2}\zeta\left(2\zeta^2+3\right)\left(1+\mathrm{erf}(\zeta)\right) + \frac{2\left(\zeta^2+1\right)}{\sqrt{\pi}}\right)\,, \quad (5.37)$$

and

$$F_2(\zeta) \;=\; \frac{1}{24}\left(2e^{\zeta^2}\left(6\zeta^4+31\zeta^2+26\right)\zeta^3(1+\mathrm{erf}(\zeta)) + \frac{12\zeta^6+56\zeta^4+30\zeta^2-3}{\sqrt{\pi}}\right)\,, \quad (5.38)$$

considerably simplify for $\zeta = \zeta_j$, that is for ζ such that $F_0(\zeta) = 0$. It is more practical to introduce the notation z_0 for such values ζ_j (and this is the notation used in the statement of Proposition 5.8):

$$F_1(z_0) \;=\; \frac{1}{2\sqrt{\pi}}z_0 \quad \text{and} \quad F_2(z_0) \;=\; -\frac{6z_0^4+22z_0^2+3}{24\sqrt{\pi}}\,. \quad (5.39)$$

Recall that, by Corollary 5.7, $\sqrt{N}h_{N,j} \sim \zeta_j = z_0$ and that $F_0(z_0) = 0$ implies $F_0'(z_0) \neq 0$ (Remark 5.4). If we expand the left-hand side of $Z_{N,z_0/N^{1/2}+z_1/N+z_2/N^{3/2}+\ldots} = 0$ and solve the equation order by order, we are lead to guessing

$$\sqrt{N}h_{N,j} \;=\; z_0 + \frac{z_1}{\sqrt{N}} + \frac{z_2}{N} + O\left(\frac{1}{N^{3/2}}\right)\,, \quad (5.40)$$

with

$$z_1 = -\frac{F_1(z_0)}{F_0'(z_0)} = \frac{1}{2}z_0^2 \quad \text{and} \quad z_2 = \frac{(1/2)F_0''(z_0)z_1^2 - F_1'(z_0)z_1 - F_2(z_0)}{F_0'(z_0)}\,. \quad (5.41)$$

Since

$$F_0'(z_0) \;=\; -\frac{1}{z_0\sqrt{\pi}}, \quad F_0''(z) \;=\; -\frac{2}{\sqrt{\pi}} \quad \text{and} \quad F_1'(z_0) \;=\; \frac{z_0^2+2}{\sqrt{\pi}}\,, \quad (5.42)$$

we have

$$z_1 \;=\; \frac{1}{2}z_0^2 \quad \text{and} \quad z_2 \;=\; \frac{1}{24}\sqrt{\pi}z_0\left(12z_0^4+2z_0^2-3\right)\,. \quad (5.43)$$

In order to make (5.40) rigorous we need to show that there exists $R > 0$ and $N_0 > 0$ such that for $N > N_0$ there exists a (unique) $\zeta \in B_{z(N)}(R/N^{3/2})$, with

$$z(N) \;=\; z_0 + \frac{z_1}{\sqrt{N}} + \frac{z_2}{N}\,, \quad (5.44)$$

that solves $Z_{N,\zeta/\sqrt{N}} = 0$. For this we remark that, while (5.40) is formal, the procedure that leads to it (Taylor expansion) does yield

$$Z_{N,z(N)/\sqrt{N}} \,=\, O\left(\frac{1}{N^2}\right), \tag{5.45}$$

and, by applying the Argument principle with $C_N := \partial B_0(R/N^2)$, it suffices to show that

$$\frac{1}{2\pi i}\oint_{C_N} \frac{Z'_{N,z(N)/\sqrt{N}+\zeta}}{Z_{N,z(N)/\sqrt{N}+\zeta}}\,\mathrm{d}\zeta - 1 \,=\, 0\,, \tag{5.46}$$

for N sufficiently large. For this we write

$$Z_{N,z(N)/\sqrt{N}+e^{it}R/N^2} \,=\, F'(z_0)e^{it}\frac{R}{N^2} + r_N(R,t)\,, \tag{5.47}$$

and, by applying the first order Taylor expansion and using (5.45), there exists $c > 0$ such that for every R we have $|r_N(R,t)| \le c/N^2$ for every t and N sufficiently large (how large may depend on R). Moreover

$$Z'_{N,z(N)/\sqrt{N}+e^{it}R/N^2} \,\sim\, F'(z_0)\,, \tag{5.48}$$

uniformly in t. This means that for every $R > 2c$

$$\limsup_{N}\sup_{t}\left|\frac{Z'_{N,z(N)/\sqrt{N}+e^{it}R/N^2}}{N^2 Z_{N,z(N)/\sqrt{N}+e^{it}R/N^2}} - \frac{\exp(-it)}{R}\right| \,\le\, \frac{3c}{R^2}\,. \tag{5.49}$$

This means that, if we choose R properly large, we can make the absolute value of the left-hand side in (5.46) smaller than 1 for N sufficiently large. Hence, for such values of N, (5.46) holds and the proof of Proposition 5.8 is complete. □

6. A reduced model for Griffiths singularities: the proof

This section just deals with the special framework (1.6) and with $\alpha = 1/2$. However, several equations are more readable if we write α for 1/2, therefore we will do so.

Proof of Theorem 1.11. Recall that $(h_{N,j})$ are the $N-1$ zeros of $Z_{N,h}$, ordered with non decreasing modulus and $\Im(h_{N,j}) > 0$ (respectively, $\Im(h_{N,j}) < 0$) for j odd (respectively, even). In analogy with (4.1) we have

$$Z_{N,h} \,=\, K(1)^N \exp(h)\prod_{j=1}^{N-1}\left(e^h - e^{h_{N,j}}\right) \,=\, Z_{N,0}e^h\prod_{j=1}^{N-1}\left(\frac{e^h - e^{h_{N,j}}}{1-e^{h_{N,j}}}\right), \tag{6.1}$$

where $Z_{N,0} = \mathbf{P}(N\in\tau)$. Since

$$\left(1+\frac{e^h-1}{1-e^\eta}\right)\Big/\left(1-\frac{h}{\eta}\right) = \frac{e^h-e^\eta}{h-\eta}\frac{\eta}{1-e^\eta}\,, \tag{6.2}$$

has no zeros and only removable singularities for $\Im(h)\in(-\pi,\pi)$ and $\Im(\eta)\in(-\pi,\pi]$, for such η's we can extend

$$h \mapsto \log\left(1+\frac{e^h-1}{1-e^\eta}\right) - \log\left(1-\frac{h}{\eta}\right), \tag{6.3}$$

to an analytic function on the strip $\Im(h) \in (-\pi, \pi)$. Note that this function is also bounded for h and η in compact subsets of $\mathbb{C}$. As a result, we can study the regularity of

$$f : h \mapsto \sum_{n=n_0}^{\infty} p^n \sum_{j=1}^{n-1} \log\left(1 - \frac{h}{h_{n,j}}\right) , \tag{6.4}$$

with n_0 fixed, but arbitrary: we can neglect the contribution for $n < n_0$ because, it is straightforward to see that there exists no solution to $Z_{n,h} = 0$ for $h \in \mathbb{R}$: in fact, $Z_{n,h} > 0$ is even increasing in h. Therefore, for n fixed the (complex) solutions to $Z_{n,h} = 0$, say for h bounded away from $-\infty$ and ∞, are bounded away from the real axis and the neglected terms yield a real analytic contribution.

The fact that $f(\cdot)$ is real analytic away from the origin can be seen as consequence of Proposition 1.3 that guarantees that for every $\varepsilon > 0$ there exists n_0 such that

$$\inf_{n \ge n_0} \inf_{\substack{j=1,\ldots,n-1: \\ |\Re(h_{n,j})| > \varepsilon}} |\Im(h_{n,j})| \,>\, 0 \,. \tag{6.5}$$

The lack of real analyticity in the origin is more subtle, but let us first show that f is C^∞ also at the origin. The k-th derivative of f is

$$f^{(k)}(h) \,=\, -(k-1)! \sum_{n=n_0}^{\infty} p^n \sum_{j=1}^{n-1} (h_{n,j} - h)^{-k} , \tag{6.6}$$

a priori at least for $h \neq 0$. To see that f is C^k, for every k, also in 0 it suffices to find an appropriate bound on the internal sum, the one over j, for h is a neighborhood of the origin. For this we remark that, uniformly for h such that $|\Im(h)| \le 1/n^2$, we have

$$Z_{n,h} \,=\, \sum_{j=1}^{n} e^{j\Re(h)} \mathbf{P}(\tau_j = n) e^{j\Re(h)} e^{j\Im(h)} \,=\, Z_{n,\Re(h)} \left(1 + O(1/n)\right) , \tag{6.7}$$

which implies that $Z_{n,h}$ has no zero in the strip $|\Im(h)| \le 1/n^2$ for $n \ge n_0$ and n_0 appropriately chosen. Therefore, for $h \in \mathbb{R}$, we have that $|\sum_{j=1}^{n-1} (h_{n,j} - h)^{-k}| \le n^{2k+1}$ which suffices to show that $f \in C^k$ for every k.

We are now to the lack of real analyticity for which we identify the sharp $k \to \infty$ behavior of

$$\frac{1}{k!} f^{(k)}(0) \,=\, -\frac{1}{k} \sum_{n=n_0}^{\infty} p^n \sum_{j=1}^{n-1} (h_{n,j})^{-k} \,. \tag{6.8}$$

For this we start by setting

$$\eta_{n,j} := n^\alpha h_{n,j} \,, \tag{6.9}$$

so we can write

$$\frac{1}{k!} f^{(k)}(0) \,=\, -\frac{1}{k} \sum_{n=n_0}^{\infty} p^n n^{\alpha k} \sum_{j=1}^{n-1} (\eta_{n,j})^{-k} \,. \tag{6.10}$$

By Proposition 5.8 we know that

$$\eta_{n,1} \,=\, z_0 + \frac{z_1}{\sqrt{n}} + \frac{z_2}{n} + O\left(\frac{1}{n^{3/2}}\right) \,=\, z_0 \exp\left(\frac{z_1}{z_0\sqrt{n}} + \left(\frac{z_2}{z_0} - \frac{z_1^2}{2z_0^2}\right)\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right)\right) , \tag{6.11}$$

and that there exist $n_0 \in \mathbb{N}$ and $q \in (0,1)$ such that for $n \geq n_0$

$$\sup_{j=3,\dots,n-1} \frac{|\eta_{n,1}|}{|\eta_{n,j}|} \leq q\,. \tag{6.12}$$

From (6.11) we directly have also that $1-\varepsilon \leq |\eta_{n,1}|/|z_0| \leq 1+\varepsilon$ for every $n \geq n_0$: choosing ε close to zero amounts to choosing n_0 larger. It is therefore natural, in the limit $k \to \infty$, to single out the contribution due to $\eta_{N,1}$ and $\eta_{N,2} = \overline{\eta_{N,1}}$ so we write

$$\begin{aligned} \frac{1}{k!} f^{(k)}(0) \;&=\; -\frac{2}{k} \sum_{n=n_0}^{\infty} p^n n^{\alpha k} |\eta_{n,1}|^{-k} \cos\left(k \arg(\eta_{n,1})\right) - \frac{1}{k} \sum_{n=n_0}^{\infty} p^n n^{\alpha k} \sum_{j=3}^{n-1} (\eta_{n,j})^{-k} \\ &=\; T_k + E_k\,. \end{aligned} \tag{6.13}$$

We bound E_k by using (6.12), so the terms in the sum over j can be bounded by $q/|z_0(1-\varepsilon)|$ to the power k which does not depend on j. Therefore

$$|E_k| \;\leq\; \frac{1}{k} \left(\frac{q}{|z_0|(1-\varepsilon)} \right)^k \sum_{n=n_0}^{\infty} p^n n^{\alpha k+1}\,. \tag{6.14}$$

At this point it is useful to introduce the polylogarithm of parameter $s \in \mathbb{R}$:

$$\mathrm{Li}_s(z) \;=\; \sum_{n=1}^{\infty} \frac{z^n}{n^s}\,, \tag{6.15}$$

for z in the open unit ball. For $x \in (0,1)$ we have (see App. B for references and more details)

$$\mathrm{Li}_s(x) \overset{s \to -\infty}{\sim} \Gamma(1-s)(-\log x)^{s-1}\,. \tag{6.16}$$

This tells us that for $k \to \infty$

$$\begin{aligned} E_k \;&=\; O\left(\frac{1}{k} |\log p|^{-2-\alpha k} \Gamma(\alpha k + 2) \left(\frac{q}{|z_0|(1-\varepsilon)} \right)^k \right) \\ &=\; O\left(|\log p|^{-1-\alpha k} \Gamma(\alpha k + 1) \left(\frac{q}{|z_0|(1-\varepsilon)} \right)^k \right)\,. \end{aligned} \tag{6.17}$$

This is relevant because if we neglect the cosine modulation in T_k we have

$$T_k^+ \;:=\; \frac{1}{k} \sum_{n=n_0}^{\infty} p^n n^{\alpha k} |\eta_{n,1}|^{-k} \;\geq\; -c^k + \frac{1}{k} \left((1+\varepsilon)|z_0| \right)^{-k} \mathrm{Li}_{-\alpha k}(p)\,, \tag{6.18}$$

where the term c^k takes care of the first n_0 terms in the sum of the polylogarithm. Since $\mathrm{Li}_{-\alpha k}(p) \sim \Gamma(\alpha k + 1) |\log p|^{-1-\alpha k}$ for $k \to \infty$, by choosing ε adequately small we readily see that T_k^+ is much larger, in fact exponentially larger, than E_k:

$$E_k \;=\; O\left(k q^k (1+3\varepsilon)^k T_k^+ \right)\,. \tag{6.19}$$

Going on to estimating T_k turns out to be somewhat technical, so we move some of the estimates to App. B. We are going to see, as a byproduct of App. B that, if we introduce $\ell_k = \sqrt{k} \log k$ and $n_{p,k} := \lfloor \alpha k / |\log p| \rfloor$, T_k is asymptotically equivalent to the truncated sum

$$-\frac{2}{k} \sum_{n:\, |n - n_{p,k}| \leq \ell_k} p^n n^{\alpha k} |\eta_{n,1}|^{-k} \cos\left(k \arg(\eta_{n,1})\right)\,. \tag{6.20}$$

This motivates the following lemma.

Lemma 6.1. *We write $n = n_{p,k} + j$. There exists real constants a, b, c, d, A, B and C, whose explicit expressions are given in the proof, such that for with $|j| \le \ell_k = \sqrt{k}\log k$ and for $k \to \infty$*

$$\cos\left(k \arg\left(\eta_{n,1}\right)\right) \,=\, \cos\left(ak + \sqrt{k} + d\frac{j}{\sqrt{k}} + c\right) + O\left(\frac{(\log k)^2}{\sqrt{k}}\right), \tag{6.21}$$

and

$$|\eta_{n,1}|^{-k} \,=\, |z_0|^{-k}\exp\left(A\sqrt{k} + C\frac{j}{\sqrt{k}} + B\right)\left(1 + O\left(\frac{\log k)^2}{\sqrt{k}}\right)\right). \tag{6.22}$$

Proof. By Taylor expansion we obtain for $n \to \infty$

$$\begin{aligned}
\arg\left(\eta_{n,1}\right) \,=\, &\arg\left(z_0\right) + \frac{1}{\sqrt{n}}\frac{\Im(z_1)\Re(z_0) - \Im(z_0)\Re(z_1)}{|z_0|^2} \\
+ \frac{1}{n}\Bigg(&\frac{-\Re(z_2)\Im(z_0)^3 + \Im(z_0)^2\Im(z_1)\Re(z_1) + \Im(z_0)^2\Im(z_2)\Re(z_0) - \Im(z_0)\Im(z_1)^2\Re(z_0)}{|z_0|^4} \\
&+ \frac{-\Im(z_0)\Re(z_0)^2\Re(z_2) + \Im(z_0)\Re(z_0)\Re(z_1)^2 - \Im(z_1)\Re(z_0)^2\Re(z_1) + \Im(z_2)\Re(z_0)^3}{|z_0|^4}\Bigg) \\
&+ O\left(\frac{1}{n^{3/2}}\right) \,=:\, \arg\left(z_0\right) + \frac{1}{\sqrt{n}}b_1 + \frac{1}{n}b_2 + O\left(\frac{1}{n^{3/2}}\right).
\end{aligned} \tag{6.23}$$

Therefore with $n = n_{p,k} + j$, $|j| \le \ell_k$ and for $k \to \infty$ we have

$$\cos\left(k \arg\left(\eta_{n,1}\right)\right) \,=\, \cos\left(k\arg\left(z_0\right) + b_1\frac{k}{\sqrt{n}} + b_2\frac{k}{n} + O\left(\frac{1}{\sqrt{k}}\right)\right), \tag{6.24}$$

where this equation defines b_1 and b_2 by comparison with (6.23). If we set $c_p^2 := |\log p|/\alpha$ ($c_p > 0$) we have

$$\frac{k}{\sqrt{n}} \,=\, c_p\sqrt{k} - \frac{c_p^3}{2}\frac{j}{\sqrt{k}} + O\left(\frac{(\log k)^2}{\sqrt{k}}\right) \quad \text{and} \quad \frac{k}{n} \,=\, c_p^2 + O\left(\frac{\log k}{\sqrt{k}}\right), \tag{6.25}$$

so

$$\begin{aligned}
\cos\left(k \arg\left(\eta_{n,1}\right)\right) \,=\, &\cos\left(\arg(z_0)k + b_1c_p\sqrt{k} - \frac{1}{2}b_1c_p^3\frac{j}{\sqrt{k}} + b_2c_p^2\right) + O\left(\frac{(\log k)^2}{\sqrt{k}}\right) \\
=:\, &\cos\left(ak + b\sqrt{k} + d\frac{j}{\sqrt{k}} + c\right) + O\left(\frac{(\log k)^2}{\sqrt{k}}\right),
\end{aligned} \tag{6.26}$$

and the last line is the definition of the constants a, b, c and d. This completes the verification of (6.21).

A similar Taylor expansion computation yields (6.22). Here we give just the constants:

$$A := -\Re\left(\frac{z_1}{z_0}\right)\sqrt{\frac{|\log p|}{\alpha}}, \quad B := -\left(\frac{z_2}{z_0} - \frac{z_1^2}{2z_0^2}\right)\frac{|\log p|}{\alpha}, \; C := \frac{1}{2}\Re\left(\frac{z_1}{z_0}\right)\left(\frac{|\log p|}{\alpha}\right)^{3/2}. \tag{6.27}$$

□

It is now a matter of applying the results of Appendix B, notably (B.1) and (B.9), to see that (c_0 is a positive constant)

$$
\begin{aligned}
-\frac{2}{k}\sum_{n\geq n_0} p^n n^{\alpha k}|\eta_{n,1}|^{-k}\cos\left(k\arg(\eta_{n,1})\right) \,=\, \\
-\frac{2}{k}\sum_{n:\,|n-n_{p,k}|\leq \ell_k} p^n n^{\alpha k}|\eta_{n,1}|^{-k}\cos\left(k\arg(\eta_{n,1})\right)\left(1+O\left(e^{-c_0(\log k)^2}\right)\right) \\
=\, -\frac{2e^{A\sqrt{k}+B}}{k|z_0|^k}\times \\
\sum_{n:\,|n-n_{p,k}|\leq \ell_k} p^n n^{\alpha k} e^{C\frac{n-n_{p,k}}{\sqrt{k}}}\left(\cos\left(ak+b\sqrt{k}+d\frac{(n-n_{p,k})}{\sqrt{k}}+c\right)+O\left(\frac{(\log k)^2}{\sqrt{k}}\right)\right).
\end{aligned}
\tag{6.28}
$$

We now use

$$
\begin{aligned}
\cos\left(ak+b\sqrt{k}-\frac{dj}{\sqrt{k}}+c\right) \,=\, \\
\cos\left(ak+b\sqrt{k}+c\right)\cos\left(\frac{dj}{\sqrt{k}}\right)+\sin\left(ak+b\sqrt{k}+c\right)\sin\left(\frac{dj}{\sqrt{k}}\right),
\end{aligned}
\tag{6.29}
$$

and we are therefore left with estimating

$$
-\frac{2e^{A\sqrt{k}+B}}{k|z_0|^k}\cos\left(ak+b\sqrt{k}+c\right)\sum_{n:\,|n-n_{p,k}|\leq \ell_k} p^n n^{\alpha k} e^{C\frac{n-n_{p,k}}{\sqrt{k}}}\cos\left(d\frac{(n-n_{p,k})}{\sqrt{k}}\right), \tag{6.30}
$$

plus the analogous expression with $\cos(\cdot)$ replaced by $\sin(\cdot)$. For this we apply (B.1), (B.9) and (B.10), with $\beta=\alpha k$ (we remark also the very mild effect due to using ℓ_k instead go $\ell_{\alpha k}$). The net result is that the expression in (6.30) is equal, up to a multiplicative error of $1+O(\log k/\sqrt{k})$, to

$$
-\frac{2e^{A\sqrt{k}+B}}{k|z_0|^k}\cos\left(ak+b\sqrt{k}+c\right)\frac{\Gamma(1+k\alpha)}{|\log p|^{1+\alpha k}}\exp\left(\frac{(C^2-d^2)}{2(\log p)^2}\right)\cos\left(\frac{Cd}{(\log p)^2}\right), \tag{6.31}
$$

and for the sine case we obtain exactly the same expression, with the $\cos(\cdot)$ replaced by $\sin(\cdot)$ in the two occurrences. We therefore conclude that

$$
\begin{aligned}
-\frac{2}{k}\sum_{n\geq n_0} p^n n^{\alpha k}|\eta_{n,1}|^{-k}\cos\left(k\arg(\eta_{n,1})\right) \,=\, \\
-\frac{2e^{A\sqrt{k}+B}}{k|z_0|^k}\cos\left(ak+b\sqrt{k}+c+\frac{Cd}{(\log p)^2}\right)\frac{\Gamma(1+k\alpha)}{|\log p|^{1+\alpha k}}\exp\left(\frac{(C^2-d^2)}{2(\log p)^2}\right)(1+r_k)\,,
\end{aligned}
\tag{6.32}
$$

where $r_k=O(\log k/\sqrt{k})$.

Lemma 6.2. *For every $a,c\in\mathbb{R}$ and for $b\neq 0$ we have that*

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\delta_{(ak+b\sqrt{k}+c)\mathrm{mod}(2\pi)} \,=\, \lambda_{2\pi}\,, \tag{6.33}
$$

where $\lambda_{2\pi}$ is the uniform probability on the circle $\mathbb{R}/(2\pi\mathbb{Z})$ and the convergence is the usual convergence in distribution.

Proof. This is fully based on [28, Chapter 2]. By rotation invariance, we can and do assume that $c = 0$. The case $a = 0$ follows directly from [28, Theorem 2.5] while the case of $a/(2\pi) = p/q$ rational can be reduced to the case $a = 0$ by separating the sums into q terms. The case $a/(2\pi)$ irrational instead requires a different approach: this is treated by [28, Theorem 3.3]. We remark also that if $a/(2\pi)$ irrational the result holds also if $b = 0$. □

Lemma 6.2 can now be used in conjunction with (6.13), (6.20) and (6.32): it guarantees that the absolute value of the oscillating term $\cos\left(ak + b\sqrt{k} + c + \frac{Cd}{(\log p)^2}\right)$ is bounded away from 0 if k stays out of a set of a density that can be made arbitrarily small, and in this case (6.32) really gives the leading asymptotic behavior. One can then argue by contradiction to ensure that it suffices that k stays out of a suitably chosen zero density set, and the leading asymptotic behavior is still given by (6.32). This completes the proof of Theorem 1.11. □

Remark 6.3. *Here is a guide to reconstruct the constants in Theorem 1.11. First of all $z_0 = 1.22516\ldots + i\,2.54713 + \ldots$ so*

$$\mathtt{a} \,=\, \arg(z_0) \,=\, 1.12247\ldots \tag{6.34}$$

and (recall that $z_1 = z_0^2/2$)

$$\mathtt{b} \,=\, \sqrt{2|\log p|}\, b_1 \quad \textit{with} \quad b_1 = \frac{\Im(z_1)\Re(z_0) - \Im(z_0)\Re(z_1)}{|z_0|^2} = 1.27356\ldots, \tag{6.35}$$

with the important fact that $\mathtt{b} \neq 0$ (cf. Lemma 6.2). The precise value of the other constants is less crucial: we have

$$\mathtt{c} \,=\, 2|\log p| b_2 - \; C b_1 \sqrt{\frac{2}{|\log p|}}\,, \tag{6.36}$$

with C in (6.27) and b_2 in (6.23) (like b_1, which however is also in (6.35)). Moreover

$$C_1 \,=\, -\frac{2}{|\log p|} \exp\left(B + \frac{\left(C^2 - 2b_1^2 |\log p|^3\right)}{2(\log(p))^2}\right) \quad \textit{and} \quad C_2 \,=\, \frac{1}{|z_0|\sqrt{|\log p|}}\,, \tag{6.37}$$

and B is also in (6.27), as well as A.

Appendix A. Probability estimates

Proof of Proposition 5.2. We start with the leading asymptotic behavior of

$$Z_{n,\zeta/n^\alpha} \,=\, \sum_{j=1}^{n} \exp\left(j\zeta/n^\alpha\right) \mathbf{P}\left(\tau_j = n\right)\,. \tag{A.1}$$

and we are looking for a result that holds uniformly for ζ is chosen in a compact set: we will just say "uniformly in ζ". For a positive (large) constant L we split the sum according to whether $j \le n^\alpha/L$, $j \in (n^\alpha/L, Ln^\alpha)$ and $j \ge Ln^\alpha$. The intermediate segment, $j \in (n^\alpha/L, Ln^\alpha)$, can be treated by applying Theorem 5.1 obtaining the asymptotic behavior claimed in (5.5) with the integral spanning from $1/L$ to L, instead of from 0 to ∞. It is therefore sufficient to show that the remaining two terms are $\varepsilon_L O(n^{1-\alpha})$, with ε_L a positive constant that vanishes as $L \to \infty$.

For the case $j \le n^\alpha/L$ we are going to use that for $n \to \infty$ and uniformly in j such that $j/n^\alpha \longrightarrow 0$ we have that $\mathbf{P}(\tau_j = n) \sim j\mathbf{P}(\tau_1 = n)$ [10, Th. A] so that for n sufficiently large and uniformly in ζ

$$\left| \sum_{j \le n^\alpha/L} \exp\left(j\zeta/n^\alpha\right) \mathbf{P}\left(\tau_j = n\right) \right| \le 2\mathbf{P}(\tau_1 = n) \sum_{j \le n^\alpha/L} j \le \frac{2\mathsf{c}_K n^{1-\alpha}}{L^2}\,, \tag{A.2}$$

For $j \ge Ln^\alpha$ we use instead [10, Lemma 4] that directly yields that for an appropriate choice of $C > 0$, not depending on L, we have that for $j \in [Ln^\alpha, n/L)$ and n sufficiently large

$$\mathbf{P}(\tau_j = n) \le C\left(j/n^\alpha\right)^{1/(2(1-\alpha))} \exp\left(-\frac{1}{C}\left(j/n^\alpha\right)^{1/(1-\alpha)}\right)\,, \tag{A.3}$$

and that there exists $C_L > 0$ such that

$$\mathbf{P}(\tau_j = n) \le \exp\left(-j/C_L\right)\,, \tag{A.4}$$

for $j \ge n/L$. Therefore, with b an upper bound for $|\Re(z)|$, and using again Riemann sum approximation we have that for n sufficiently large

$$\left| \sum_{j \ge Ln^\alpha} \exp\left(j\zeta/n^\alpha\right) \mathbf{P}\left(\tau_j = n\right) \right| \le \\ \frac{2C}{n^{1-\alpha}} \int_L^\infty y^{1/(2(1-\alpha))} \exp\left(by - \frac{1}{C} y^{1/(1-\alpha)}\right) \mathrm{d}y + \sum_{j \ge n/L} \exp\left(b\frac{j}{n^\alpha} - \frac{j}{C_L}\right)\,, \tag{A.5}$$

and we see that the first term in the right-hand side is $O(1/n^{1-\alpha})$ times a term that can be made arbitrarily small by choosing L large. The second term instead is $O(\exp(-n/(2C_L))$ and it is therefore much smaller. This completes the proof of (5.5).

For the proof of proof of (5.6) we have to apply the very same arguments to

$$Z'_{n,\zeta/n^\alpha} = \sum_{j=1}^n \exp\left(j\zeta/n^\alpha\right) j\,\mathbf{P}\left(\tau_j = n\right)\,. \tag{A.6}$$

We skip the straightforward details. □

Appendix B. Asymptotic behavior of modified polylogarithms

For the standard polylogarithm we have

$$\sum_n p^n n^\beta \overset{\beta\to\infty}{\sim} T_\beta := \frac{\Gamma(1+\beta)}{|\log p|^{1+\beta}} = \frac{\exp(\beta\log\beta - \beta)}{|\log p|^{1+\beta}}\left(\sqrt{2\pi\beta} + O(1/\beta)\right)\,, \tag{B.1}$$

where $p \in (0,1)$. The first step in (B.1) follows from [29, (25.12.12)] and the last step is Stirling formula with first order reminder.

We now aim at recovering (B.1) by a direct saddle point analysis: the result will then be easily generalized to the case that interests us. For this we start by introducing $\ell_\beta := \sqrt{\beta}\log\beta$ and $n_\beta = \beta/|\log p|$. We start by observing that the ratio

$$\sum_{n \notin [n_\beta - \ell_\beta, n_\beta + \ell_\beta]} p^n n^\beta \Big/ \left(\int_0^{n_\beta - \ell_\beta} p^x x^\beta \,\mathrm{d}x + \int_{n_\beta + \ell_\beta}^\infty p^x x^\beta \,\mathrm{d}x\right)\,, \tag{B.2}$$

is bounded away from 0 and ∞ (in fact, it tends to one as $\beta \to \infty$, but (B.2) is only used for tail bounds, which do not need to be sharp). With $x = n_\beta + y$ we have

$$p^x x^\beta \,=\, \frac{\exp(\beta \log \beta - \beta)}{|\log p|^\beta} \exp\left(-\frac{y^2 |\log p|^2}{2\beta}\right)\left(1 + O\left(\frac{y^3}{\beta^2}\right)\right) . \tag{B.3}$$

The first application of this estimate is to show that the denominator, hence also the numerator, of (B.2) is much smaller than T_β: more precisely, for every $c > 0$ it is $O(T_\beta/\beta^{-c})$. For the first integral in the denominator of (B.2) we use that the integrand is increasing in the interval of integration and (B.3) with $y = -\ell_\beta$:

$$\begin{aligned}\int_0^{n_\beta-\ell_\beta} p^x x^\beta \,\mathrm{d}x \,\le\, p^{n_\beta-\ell_\beta}(n_\beta-\ell_\beta)^\beta \int_0^{n_\beta-\ell_\beta} \mathrm{d}x \,\le \\ \frac{2\beta\exp(\beta\log\beta-\beta)}{|\log p|^{1+\beta}} \exp\left(-\frac{(\log\beta)^2|\log p|^2}{2}\right) \,\le\, T_\beta \exp\left(-\frac{(\log\beta)^2|\log p|^2}{4}\right) .\end{aligned} \tag{B.4}$$

For the second integral we use that the integrand is this time decreasing: since the interval of integration is unbounded we consider separately the integral from $n_\beta + \ell_\beta$ to β^2 and from β^2 to ∞. We have

$$\int_{n_\beta+\ell_\beta}^{\beta^2} p^x x^\beta \,\mathrm{d}x \,\le\, \beta^2 p^{n_\beta+\ell_\beta}(n_\beta+\ell_\beta)^\beta \,\le\, \frac{\beta^2\exp(\beta\log\beta-\beta)}{|\log p|^\beta} \exp\left(-\frac{(\log\beta)^2|\log p|^2}{3}\right) , \tag{B.5}$$

and precisely the final bound in (B.4) is recovered. It is then straightforward to see that the integral from β^2 to ∞ vanishes as $\beta \to \infty$, yielding thus a negligible contribution.

We can then focus on

$$\sum_{n\in[n_\beta-\ell_\beta, n_\beta+\ell_\beta]} p^n n^\beta \,=\, \int_{n_\beta-\ell_\beta}^{n_\beta-\ell_\beta} p^x x^\beta \,\mathrm{d}x + E_\beta , \tag{B.6}$$

where E_β can be bounded (first order Euler-Maclaurin formula) in terms of the value of the integrand at the two boundary points, this gives a contribution $O(T_\beta \exp(-c(\log\beta)^2))$ for some $c > 0$ like in the previous estimates, plus the integral of the (absolute value) of the first derivative of the integrand. Since $|\partial_x(p^x x^\beta)| = p^x x^\beta O(\ell_\beta/\beta)$ we readily find that $|E_\beta| = O(T_\beta \log\beta/\sqrt{\beta})$. We can then work with the integral and, by (B.3), we see that

$$\begin{aligned}\int_{n_\beta-\ell_\beta}^{n_\beta+\ell_\beta} p^x x^\beta \,\mathrm{d}x \,&=\, \frac{\exp(\beta\log\beta-\beta)}{|\log p|^\beta} \int_{-\ell_\beta}^{\ell_\beta} \exp\left(-\frac{y^2|\log p|^2}{2\beta}\right) \mathrm{d}y \\ &=\, \frac{\exp(\beta\log\beta-\beta)}{|\log p|^\beta} \left(\int_{-\infty}^{\infty} \exp\left(-\frac{y^2|\log p|^2}{2\beta}\right) \mathrm{d}y + o(1)\right) \\ &=\, \frac{\exp(\beta\log\beta-\beta)}{|\log p|^{1+\beta}} \left(\sqrt{2\pi\beta} + o(1)\right) ,\end{aligned} \tag{B.7}$$

where $o(1)$ is actually $O\left(\exp\left(-(\log\beta)^2|\log p|^2/3\right)\right)$. Therefore we have recovered and strengthened (B.1): for $\beta \to \infty$

$$\sum_n p^n n^\beta \,=\, \sum_{n:\,|n-n_\beta|\le\ell_\beta} p^n n^\beta \left(1 + O\left(\exp\left(-c(\log\beta)^2\right)\right)\right) \,=\, T_\beta\left(1 + O\left(\frac{\log\beta}{\sqrt{\beta}}\right)\right) , \tag{B.8}$$

with $c = (\log p)^2/4$. We have developed in detail this procedure because of the control on the truncation error and because the steps generalize in a straightforward way to the following result: for $H(x) := \exp(Cx)h(x)$, $C \in \mathbb{R}$ and h a bounded function with bounded first derivative, we have

$$\begin{aligned}\sum_n p^n n^\beta H\left(\frac{n-n_\beta}{\sqrt{\beta}}\right) &= \sum_{n:\,|n-n_\beta|\le \ell_\beta} p^n n^\beta H\left(\frac{n-n_\beta}{\sqrt{\beta}}\right)\left(1+O\left(e^{-c(\log\beta)^2}\right)\right)\\ &= T_\beta\left(\mathbf{E}\left[H\left(\frac{Z}{|\log p|}\right)\right]+O\left(\frac{\log\beta}{\sqrt{\beta}}\right)\right),\end{aligned} \tag{B.9}$$

where Z is a standard Gaussian random variable. The steps in the proof of (B.9) are identical to those we performed for (B.8) because the modulating function we have introduced changes the bounds only by constants (depending on $|C|$, $\|h\|_\infty$ and $\|h'\|_\infty$).

We need this result to $h(\cdot) = \cos(d\,\cdot)$ and to $\sin(d\,\cdot)$, with $d \in \mathbb{R}$: with these two special choices of $h(\cdot)$ we have

$$\mathbf{E}\left[\exp\left(C\frac{Z}{|\log p|}\right)h\left(\frac{Z}{|\log p|}\right)\right] = \exp\left(\frac{C^2-d^2}{2(\log p)^2}\right)h\left(\frac{Cd}{(\log p)^2}\right). \tag{B.10}$$

Appendix C. Monotonicity of the critical curve

Recall $f_1(\cdot)$ from (2.1) and $f_2(\cdot)$ from (2.2). Recall moreover that $a = 1-\alpha \in (0,1)$. One directly checks that $f_1(\pi+\theta) = f_1(\pi-\theta)$ and $\tan(f_2(\pi+\theta)) = -\tan(f_2(\pi-\theta))$ for $\theta \in [0,\pi]$. This allows to focus on $\theta \in [0,\pi]$ and, by continuity, it suffices to show that both $f_1'(\theta)$ and $f_2'(\theta)$ are positive for $\theta \in (0,\pi)$.

Lemma C.1. *Both $f_1'(\theta) > 0$ and $f_2'(\theta) > 0$ hold for every $\theta \in (0,\pi)$ and every $\alpha \in (0,1)$.*

Proof. We start by analyzing $f_2(\cdot)$. If we differentiate the argument of the arctangent with respect to θ we find

$$\frac{2^{a-1}a\sin^{a-1}\left(\frac{\theta}{2}\right)\left(2^a\sin^{a+1}\left(\frac{\theta}{2}\right)+\cos\left(\frac12(a\theta-\pi a+\theta+\pi)\right)\right)}{\left(1-2^a\sin^a\left(\frac{\theta}{2}\right)\cos\left(\frac12 a(\pi-\theta)\right)\right)^2}, \tag{C.1}$$

so the sign of this term is positive for $\theta \in (0,\pi)$ if and only if

$$2^a\sin^{a+1}\left(\frac{\theta}{2}\right)+\cos\left(\frac12(a\theta-\pi a+\theta+\pi)\right) > 0, \tag{C.2}$$

which is equivalent to

$$(2\cos(\varphi))^{1+a} > 2\cos(\varphi(1+a)), \tag{C.3}$$

for $\varphi \in (0,\pi/2)$. Note now that it suffices to show this inequality for $\varphi \in (0,\pi/(2(1+a))]$, because otherwise the right-hand side is negative. So it suffices to show, with $b = 1+a \in (1,2)$ and $h(\cdot) := 2\cos(\cdot)$, that for $\varphi \in (0,\pi]$

$$(h(\varphi/b))^b > h(\varphi) \iff \frac{\log h(\varphi/b)}{\varphi/b} > \frac{\log h(\varphi)}{\varphi}, \tag{C.4}$$

and the inequality on the right holds because $\partial_\varphi \log h(\varphi) = -(\varphi\tan(\varphi)+\log(2\cos(\varphi))/t^2$ is negative ($\varphi\tan(\varphi)+\log(2\cos(\varphi)$ is equal to $\log 2 > 0$ for $\varphi = 0$ and its derivative is $\varphi/\cos(\varphi))^2 > 0$). This completes the proof that $f_2'(\theta) > 0$ for $\theta \in (0,\pi)$ and $a \in (0,1)$ because $\arctan(\cdot)$ is increasing.

Remark C.2. *The θ-derivative of the square root of the denominator is*

$$-2^{a-1}a\sin\left(\frac{1}{2}(a+1)(\pi-\theta)\right)\sin^{a-1}\left(\frac{\theta}{2}\right), \tag{C.5}$$

which is negative for $\theta\in[0,\pi)$ and it is zero at $\theta=\pi$. So $\theta=\pi$ is the minimum of the square denominator which takes value 1 in $\theta=0$ and value $1-2^a<0$ in $\theta=\pi$. Hence the denominator hits zero only in one point $\theta_a\in(0,\pi)$. At this point the expression for $f_2(\theta)$ would have a jump of $-\pi$ had we chosen $\arctan(\cdot)$ *instead of* $\arctan_0(\cdot)$.

We are then left with showing that

$$\frac{f_1'(\theta)}{f_2'(\theta)}=\frac{\sin\left(\frac{1}{2}(a\theta-\pi a+\theta+\pi)\right)-2^a\cos\left(\frac{\theta}{2}\right)\sin^a\left(\frac{\theta}{2}\right)}{2^a\sin^{a+1}\left(\frac{\theta}{2}\right)+\cos\left(\frac{1}{2}(a\theta-\pi a+\theta+\pi)\right)}, \tag{C.6}$$

for $\theta\in(0,\pi)$ and $a\in(0,1)$. By (C.2) it suffices to show positivity of the numerator and this is equivalent to showing

$$\sin((1+a)\varphi)\,>\,2^a\sin(\varphi)\cos(\varphi)\quad\Longleftrightarrow\quad 2^b\sin((2-b)\varphi)\,>\,\sin(2\varphi), \tag{C.7}$$

for every $\varphi\in(0,\pi/2)$ and $b=1-a\in(0,1)$. If $(2-b)\varphi\geq\pi/2$ the inequality holds because it holds even without the 2^b factor. So it suffices to focus on $\varphi\in(0,\pi/(2(2-b)))$. We then make the change of variable $\psi=(2-b)\varphi\in(0,\pi/2)$ and we boil down to the inequality

$$\sin(\psi)\,>\,2^{-b}\sin\left(\frac{\psi}{1-(b/2)}\right). \tag{C.8}$$

But $2^{-b}<1-(b/2)$ for $b\in(0,1)$ so we are done if we can show the previous inequality with 2^{-b} replaced by $1-(b/2)$. This amounts to showing that $\sin(\psi)>c\sin(\psi/c)$ for $c\in(1/2,1)$ and $\psi\in(0,\pi/2)$: this last inequality holds even for $c\in(0,1)$ as one directly verifies. The proof of Lemma C.1 is therefore complete. □

Appendix D. About numerics

As pointed out in [27], in the restricted framework of (1.6) with $\alpha=1/2$ there is the explicit formula for $\mathbf{P}(\tau_j=N)=(j/(2n-j))2^{-2n+j}C^n_{2n-j}$, with $C^k_n=n!/(k!(n-k)!)$. In general, one can obtain the coefficients

$$\mathcal{P}_N\,:=\,(\mathbf{P}(\tau_j=N))_{j=1,2,\ldots,N}, \tag{D.1}$$

by an iterative procedure that consists in building $\mathcal{P}_{n+1}$ from $(\mathcal{P}_k)_{k=1,\ldots,n}$ via

$$\mathbf{P}(\tau_{j+1}=n+1)\overset{j=1,\ldots,n}{=}\sum_{m=j}^{n}\mathbf{P}(\tau_j=m)K(n+1-m), \tag{D.2}$$

and $\mathbf{P}(\tau_1=n+1)=K(n+1)$, starting from $\mathcal{P}_1=(K(1))$. This way we can deal with arbitrary inter-arrival laws $K(\cdot)$ with N up to 500 with standard computers and moderate amount of time. This of course allows a large spectrum of numerical investigations even if the reachable N are still rather small to really guess what the $N=\infty$ behavior could be (see for example Figure 5).

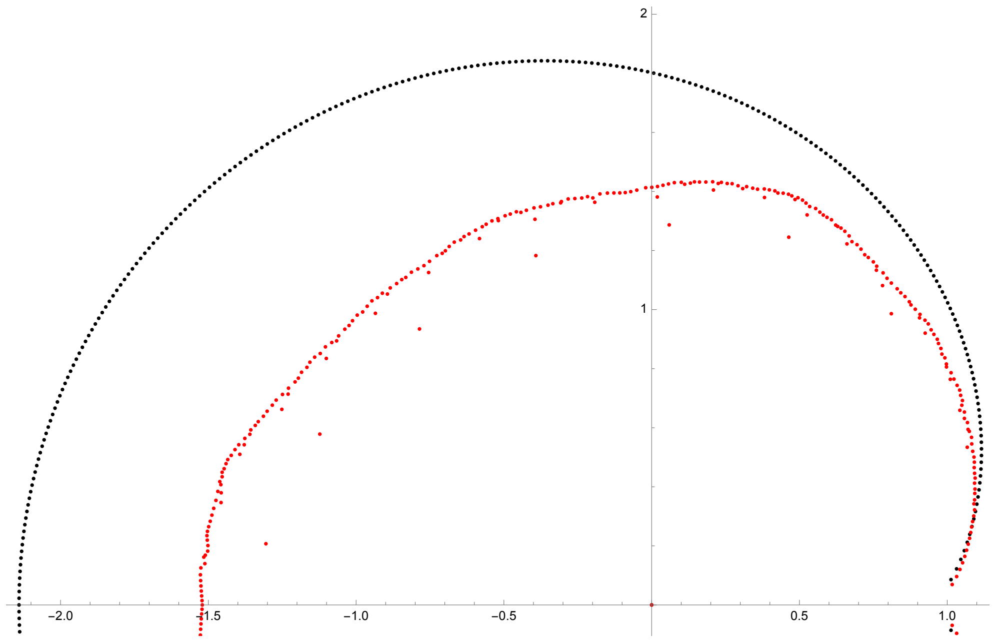


FIGURE 5. The black dots are the zeros of $P_{500}(w) = Z_{500,\mathrm{Log}\, w}$ with $K(n) = (K_1(n) + K_2(n))/2$ where $K_1(\cdot)$ is the inter-arrival law in (1.6) with $\alpha = 1/2$ and $K_2(n) = 1/(n^2\zeta(2))$, where the normalization $\zeta(\cdot)$ is the Riemann ζ function. The red dots instead are the zeros in the case in which $K_2(n) = 1/(n^2\zeta(4))$ if $\sqrt{n} \in \mathbb{N}$ and $K_2(n) = 0$ otherwise. We are therefore in the framework evacuated in Remark 1.4: the major effect of using a *lacunary* distribution (albeit subleading!) is apparent, even if much larger values of N would be needed to draw predictions from such a numerical observation (only 22 entries of $K_2(\cdot)$ are non zero).

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Competing Interests

The Authors declare they have no competing interests.

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