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Mathematics Area - PhD course in
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Variation of gluing in homological mirror symmetry

Candidate:
Hayato Morimura

Advisor:
Nicolò Sibilla

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ABSTRACT

This thesis is a collection of seven papers concerned with the relationship between variation of gluing spaces and categories in homological mirror symmetry(HMS). We divide it into three parts according to how we vary gluing what. The first part consists of three papers on algebraic deformations of Calabi–Yau 3-folds(CY3s), where we vary complex structures to glue locally trivial deformations. The second part consists of two papers on cut-and-reglue procedure for relative Jacobians of generic elliptic 3-folds, where we vary Brauer classes to glue smooth elliptic 3-folds with sections. The third part consists of two papers on local-to-global principle for wrapped Fukaya categories of very affine hypersurfaces(VAHs), where we vary Liouville structures to glue pairs of pants. Our main goal of the first two parts is to construct new Fourier–Mukai partners(FMPs), nonbirational derived-equivalent CY3s. While birational CY3s are derived-equivalent, FMPs give highly nontrivial multiple mirrors to the dual manifolds. Our main goal of the third part is to establish HMS for complete intersections of VAHs. Recently, Gammage–Shende established HMS for VAHs under some assumption essential to construct a global skeleton, which allows them to reduce gluing wrapped Fukaya categories to gluing local skeleta. For several reasons we need a different approach to remove their assumption.

In the first paper, we prove that the derived equivalence of CY3s extends to their versal deformations over an affine complex variety. This is fundamental for our deformation methods to construct new examples of FMPs. Due to the main theorem of the second paper, the derived category of the generic fiber of a flat proper family can be described as a certain Verdier quotient. As a consequence, the derived equivalence of the above versal deformations is inherited to their generic fibers. We analyze some good cases where also nonbirationality is inherited, establishing a deformation method to construct new FMPs from known examples. Conversely, the description enables us to prove specialization, i.e., the derived equivalence of the generic fibers extends to general fibers, completing all the relevant inductions of the derived equivalence of CY3s through deformations. The main theorem of the third paper gives a rigorous explanation of these phenomena. Namely, deformations of a CY3 are equivalent to Morita deformations of its dg category of perfect complexes. We also prove that, analogous to isomorphisms of schemes, the derived equivalence is inherited from effectivizations to their enough close approximations. This is an improvement of the main theorem of the first paper, expected from the equivalence of the two deformation theories.

In the fourth paper, we prove that any flat projective family must be what we call an *almost coprime twisted power*, whenever it is linear derived-equivalent over the base to a generic elliptic CY3. This should be the best possible reconstruction result for generic elliptic CY3s. Combining with the main theorem of the first paper, we obtain a family of pairs of coprime twisted powers whose closed fibers are nonbirational whenever they are nonisomorphic. Unwinding our arguments, one sees that generic elliptic CY3s are linear derived-equivalent over the base if and only if their generic fibers are derived-equivalent. This is the key observation for the fifth paper where we give affirmative answers to two of the four conjectures raised by Knapp–Scheidegger–Schimannek. Namely, we prove that each of 12 pairs of elliptic CY3s constructed by them share the relative Jacobian and linear derived-equivalent over the base. Except one self-dual pair, the closed fibers of the family obtained by the above combination are nonisomorphic. Hence we obtain families of new FMPs, establishing another deformation

method to construct FMPs. As far as we know, this is the first systematic construction of (families of) FMPs. Moreover, it works for elliptic CY3s with higher multisections, whose examples some string theorists have been looking for.

In the sixth paper, we establish HMS for complete intersections of VAHs. The main challenge is computing wrapped Fukaya categories of complete intersections. With the aid of equivariantization/de-equivariantization, we reduce it to unimodular case. Proving that locally complete intersections are products of lower dimensional pairs of pants, we reduce it further to hypersurface case without the assumption imposed on the previous result by Gammage–Shende. We extend it by inductive argument following Pascaleff–Sibilla which does not require any global skeleton. Besides the invariance of wrapped Fukaya categories under simple Liouville homotopies, one key is to find Weinstein structures on the initial exact symplectic manifold and the additional pair of pants which glue to yield that on the gluing, everytime we proceed the inductive argument. Another is to show that also their wrapped Fukaya categories glue to yield that of the gluing. Our method should work to compute wrapped Fukaya categories in other relevant settings. Finally, we glue HMS for pairs of pants along the global combinatorial duality over the tropical hypersurface. The geometry of VAHs is further studied in the seventh paper, where we complete the missing A-side of the SYZ picture over fanifolds. This can be regarded as a generalization of that over tropical hypersurfaces.

BIBLIOGRAPHICAL SKETCH

Hayato Morimura was born the son of Hiromi and Midori in Japan in May 1987. He got a master's degree in Agricultural Science and a veterinary medical license at the University of Tokyo in March 2013. After working as a veterinarian caring livestock in Gunma for three years, he got another master's degree in Mathematics at the University of Tokyo in March 2019. Within a half year he moved to Trieste, Italy to start a PhD program in Mathematics at SISSA. He has been married to Flavia since November 2017. They currently live in Trieste with Kristilend Chronos, a small Dachshund with gargantuan attitude.

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Jacopo Stoppa, a professor in SISSA. When graduating the master course, for bad financial issues I was wondering whether to continue mathematics. I met him at a conference held in Kyoto in January 2019. He was an invited speaker among great mathematicians. During the social dinner, out of nowhere and just because he is Italian same as my wife, I spoke to him in terrible English about my depressing situation. Nevertheless, he carefully listened to me and later sent a notification of the selection for this PhD program to Ueda, remembering my advisor's name. Also, he was supportive of me to start a new life in SISSA.

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ALGEBRAIC DEFORMATIONS AND FOURIER–MUKAI TRANSFORMS FOR CALABI–YAU MANIFOLDS

HAYATO MORIMURA

Abstract. Given a pair of derived-equivalent Calabi–Yau manifolds of dimension more than two, we prove that the derived equivalence can be extended to general fibers of versal deformations. As an application, we give a new proof of the Pfaffian–Grassmannian derived equivalence.

1. Introduction

Let X_0 be a Calabi–Yau manifold of dimension more than two in the strict sense, i.e., a smooth projective variety over a field $\mathbf{k}$ with trivial canonical bundle and $\mathrm{H}^i(X_0, \mathcal{O}_{X_0}) = 0$ for $0 < i < \dim X_0$. Then the deformation functor

$$F_{X_0} = \mathrm{Def}_{X_0} \colon \mathbf{Art_k} \to \mathbf{Set}$$

of X_0 has a universal formal family (R, ξ), which is effective by [GD61, Theorem III 5.4.5] and there exists an effectivization X_R flat and projective over R, whose formal completion along the closed fiber X_0 is isomorphic to ξ. Since deformations of Calabi–Yau manifolds are unobstructed, the complete local noetherian ring R is regular and we have

$$R \cong \mathbf{k}[\![t_1, \ldots, t_d]\!],$$

where $d = \dim_{\mathbf{k}} \mathrm{H}^1(X_0, \mathcal{T}_{X_0})$. By [Art69b, Theorem 1.6] there exists a versal deformation X_S flat and of finite type over S, where S is an algebraic $\mathbf{k}$-scheme with a distinguished closed point s such that the formal completion along the closed fiber X_0 over s is isomorphic to ξ. It is known that the triple (S, s, X_S) is unique only locally around s in the étale topology. Unwinding the construction, one finds a nonsingular affine variety S over which the versal deformation X_S is smooth projective. Our main result is the following:

Theorem 1.1 (Theorem 4.1). *Let X_0 and X_0' be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then there exists a nonsingular affine variety S over $\mathbf{k}$ such that general fibers of smooth projective versal deformations X_S and X_S' over S are derived-equivalent. In particular, after possible shrinking of the base scheme S, the schemes X_S and X_S' are derived-equivalent.*

The relationship between deformations and Fourier–Mukai transforms has been addressed in [Tod09] for first order deformations of smooth projective varieties, in [BBP07] for formal deformations of complex tori, and in [HMS09] for formal deformations of K3 surfaces by deforming Fourier–Mukai kernels. In the above cases, a relative Fourier–Mukai transform of n-th order deformations induces an isomorphism which associates to the direction of a $(n+1)$-th order deformation of one side that of the other side. So the fiber product deforms along the pair of the directions to yield the fiber product of the $(n+1)$-th order deformations. Then it is natural to ask whether one can deform the Fourier–Mukai kernel to a perfect complex on the fiber product of the $(n+1)$-th order deformations, and the relative integral functor defined by the deformed perfect complex is an equivalence.

SISSA, via Bonomea 265, 34136 Trieste, Italy
E-mail address: hmorimur@sissa.it.

For Calabi–Yau manifolds of dimension more than two, the isomorphism induced by a relative Fourier–Mukai transform connects a pair of $(n+1)$-th order deformations of complex structures. Moreover, since the effectivizations X_R and X'_R are smooth over R, the obstruction class to deforming a perfect complex [Low05, Lie06] is given by the product of the relative Atiyah class and the relative Kodaira–Spencer class [HT10, Corollary 3.4]. In this paper, based on the argument in [HMS09, Section 3], we deform a Fourier–Mukai kernel defining the derived equivalence of X_0 and X'_0 along the sequence of the natural quotient maps

$$\cdots \to R/\mathfrak{m}_R^{n+2} \to R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^{n} \to \cdots .$$

From the compatible system of deformed Fourier–Mukai kernels, we obtain an effectivization as a perfect complex by [Lie06, Proposition 3.6.1]. Passing through a filtered inductive system of finitely generated subalgebras of R whose colimit is R, we obtain a perfect complex on $X_S \times_S X'_S$ which restricts to the Fourier–Mukai kernel on $X_0 \times X'_0$ by [Lie06, Proposition 2.2.1] and the construction of the versal deformations. Then the standard argument shows that the relative integral functor defined by the perfect complex is an equivalence. One can also show the derived equivalence of effectivizations of universal formal families, in particular, that of formal deformations of X_0 and X'_0.

As an application, we give a new (but slightly weaker) proof of the Pfaffian–Grassmannian equivalence, which is conjectured in [Rød00], explained in [HT07] from a physical perspective, and proved in [BC09, Kuz, ADS15]. Via Theorem 1.1, the derived equivalence is induced by that of the complete intersections of G_2-Grassmannians [Kuz18, Ued19]. Similarly, due to [IIM19, Proposition 4.7] the derived equivalence of the intersections of two Grassmannians in $\mathbb{P}^9$ [BCP20] is induced by that of the complete intersections in $G(2,5)$ [KR19, Mor21]. We expect to find a new example of Fourier–Mukai partners through deformation methods using Theorem 1.1.

Notations and conventions. *We work over an algebraically closed field* $\mathbf{k}$ *of characteristic 0 throughout the paper. For an augmented* $\mathbf{k}$*-algebra* A*, by* $\mathfrak{m}_A$ *we denote its augmentation ideal. For a noetherian formal scheme* $\mathscr{X}$*, by* $D^b(\mathscr{X})$ *we denote the bounded derived category of the abelian category* $\mathrm{Coh}(\mathscr{X})$ *of coherent sheaves on* $\mathscr{X}$*.*

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2. Smooth projective versal deformations

When it comes to deformations, Calabi–Yau manifolds are equipped with nice geometric features. In this section, after reviewing some basics on deformation theory of schemes, we explain how to construct smooth projective versal deformations of Calabi–Yau manifolds of dimension more than two.

2.1. **Infinitesimal deformations of schemes.** Let X be a $\mathbf{k}$-scheme. A deformation of X over a local artinian $\mathbf{k}$-algebra A with residue field $\mathbf{k}$ is a pair (X_A, i_A), where X_A is a scheme flat over A and $i_A \colon X \hookrightarrow X_A$ is a closed immersion such that the induced map $X \to X_A \times_A \mathbf{k}$ is an isomorphism. Two deformations (X_A, i_A) and (Y_A, j_A) are said to be equivalent if there is an A-isomorphism $X_A \to Y_A$ compatible with i_A and j_A. The deformation functor $F_X = \mathrm{Def}_X \colon \mathbf{Art_k} \to \mathbf{Set}$ sends each $A \in \mathbf{Art_k}$ to the set of equivalence classes of deformations of X over A.

Assume that X is projective over $\mathbf{k}$. Then F_X satisfies Schlessinger's criterion [Sch68] and there exists a miniversal formal family (R, ξ) for F_X [Har10, Theorem 18.1], where R is a complete local noetherian $\mathbf{k}$-algebra with residue field $\mathbf{k}$, and ξ belongs to the limit

$$\hat{F}_X(R) = \varprojlim F_X(R/\mathfrak{m}_R^n)$$

of the inverse system

$$\cdots \to F_X(R/\mathfrak{m}_R^{n+2}) \to F_X(R/\mathfrak{m}_R^{n+1}) \to F_X(R/\mathfrak{m}_R^n) \to \cdots$$

induced by the natural quotient maps $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$. The formal family ξ corresponds to a natural transformation

$$h_R = \mathrm{Hom}_{\mathbf{k}\text{-alg}}(R, -) \to F_X, \tag{2.1}$$

which sends each homomorphism $f \in h_R(A)$ factorizing through $R \to R/\mathfrak{m}_R^{n+1} \xrightarrow{g} A$ to $F_X(g)(\xi_n)$ [Har10, Proposition 15.1]. The functor (2.1) is strongly surjective by versality of ξ. So for every surjection $B \to A$ in $\mathbf{Art}_\mathbf{k}$ the map

$$h_R(B) \to h_R(A) \times_{F_X(A)} F_X(B)$$

is surjective. In particular, the map $h_R(A) \to F_X(A)$ is surjective for each $A \in \mathbf{Art}_\mathbf{k}$.

Let X_n be the schemes which define ξ_n. Then by [Har10, Proposition 21.1] there is a noetherian formal scheme $\mathscr{X}$ over R such that $X_n \cong \mathscr{X} \times_R R/\mathfrak{m}_R^{n+1}$ for each n. By abuse of notation, we use the same symbol ξ to denote the formal scheme $\mathscr{X}$. Thus any scheme which defines an equivalence class $[X_A, i_A]$ can be obtained as the pullback of ξ along some morphism of noetherian formal schemes $\operatorname{Spec} A \to \operatorname{Spf} R$. If X is regular, then the Zariski tangent space of $\operatorname{Spec} R$ at the closed point is $\mathrm{H}^1(X, \mathcal{T}_X)$. Assume further that $\mathrm{H}^0(X, \mathcal{T}_X) = 0$, i.e., the scheme X has no infinitesimal automorphisms which restrict to the identity of X. Then every equivalence class $[X_A, i_A]$ is just a deformation (X_A, i_A) and we have $h_R \simeq F_X$ [Har10, Corollary 18.3]. In this case, the functor F_X is said to be pro-representable and (R, ξ) a universal formal family for F_X.

2.2. **Algebraization.** Towards an algebraic family of deformations of X, the first step is to find a scheme X_R flat and of finite type over R whose formal completion along the closed fiber X is isomorphic to ξ. If X is projective and $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, i.e., deformations of any invertible sheaf on X are unobstructed, then by [GD61, Theorem III5.4.5] there exists such a scheme X_R. In this case, the formal family (R, ξ) is said to be effective. One sees that the scheme X_R appeared in the proof of [GD61, Theorem III5.4.5] is projective over R. We will call such X_R an effectivization of ξ.

The next step is to find an algebraic $\mathbf{k}$-scheme S with a distinguished closed point $s \in S$, and a scheme X_S flat and of finite type over S whose formal completion along the closed fiber X over s is isomorphic to ξ. The deformation functor F_X can naturally be extended to a functor defined on the category $\mathbf{k\text{-}alg}^{\mathrm{aug}}$ of augmented noetherian $\mathbf{k}$-algebras. By abuse of notation, we use the same symbol F_X to denote the extended functor, which sends each $(B, \mathfrak{m}_B) \in \mathbf{k\text{-}alg}^{\mathrm{aug}}$ to the set of equivalence classes of deformations over $(B, \mathfrak{m}_B)$. The following is the well-known fact necessary to prove the existence of such a triple (S, s, X_S). We provide a proof as we could not find any reference in the literature.

Lemma 2.1. *Let X be an algebraic $\mathbf{k}$-scheme. Then the functor F_X is locally of finite presentation, i.e., for every filtered inductive system of augmented noetherian $\mathbf{k}$-algebras $\{(B_i, \mathfrak{m}_{B_i})\}_{i \in I}$ whose colimit is B, the canonical map*

$$\varinjlim F_X((B_i, \mathfrak{m}_{B_i})) \to F_X((B, \mathfrak{m}_B))$$

is bijective.

Proof. To show the surjectivity, let $[X_B, i_B]$ be an element in $F_X((B, \mathfrak{m}_B))$. By [GD66, Corollary IV11.2.7] for some index $\lambda \in I$ there exists a scheme X_{B_λ} flat and of finite type over B_λ with a B-isomorphism $X_B \to X_{B_\lambda} \times_{B_\lambda} B$. Then an element $\{[X_{B_k}, i_{B_k}]\}_{k\geq\lambda} \in \varinjlim F_X((B_i, \mathfrak{m}_{B_i}))$ is sent to $[X_B, i_B]$ by the canonical map.

To show the injectivity, let $\{[X_{B_k}, i_{B_k}]\}_{k\geq j}$ and $\{[Y_{B_k}, j_{B_k}]\}_{k\geq j}$ be two elements sent to the same equivalence class $[X_B, i_B] = [Y_B, j_B]$. By [GD66, Theorem IV8.8.2, Corollary IV8.8.2.4] for some index $l \geq j$ there is a B_l-isomorphism $X_{B_l} \to Y_{B_l}$ sent to the B-isomorphism $X_B \to Y_B$. Since we have $[X_B, i_B] = [Y_B, j_B]$, the isomorphism is compatible with i_{B_l} and j_{B_l}. Thus $\{[X_{B_k}, i_{B_k}]\}_{k\geq l}$ and $\{[Y_{B_k}, j_{B_k}]\}_{k\geq l}$ define the same element in $\varinjlim F_X((B_i, \mathfrak{m}_{B_i}))$. □

By Lemma 2.1 one can apply [Art69b, Theorem 1.6] to obtain such a triple (S, s, X_S). The scheme X_S is said to be a versal deformation over S and the miniversal formal family (R, ξ) is said to be algebraizable. Since some details are necessary in the sequel, we show the existence of a versal deformation when X is a higher dimensional Calabi–Yau manifold.

Theorem 2.2. *Let X_0 be a Calabi–Yau manifold of dimension more than two. Then every effective universal formal family (R, ξ) for F_{X_0} is algebraizable.*

Proof. Let $T = \operatorname{Spec} \mathbf{k}[t_1, \ldots, t_d]$ and $t \in T$ be the closed point corresponding to a maximal ideal $(t_1, \ldots, t_d)$. Since the formal completion of $\mathcal{O}_T(T)$ along $(t_1, \ldots, t_d)$ is isomorphic to R, there is a filtered inductive system $\{R_i\}_{i\in I}$ of finitely generated $\mathcal{O}_T(T)$-subalgebras of R whose colimit is R. Choose a finite type presentation

$$R_i = \mathcal{O}_T(T)[Y]/(f(Y)),$$

where $Y = (Y_1, \ldots, Y_N)$ and $f = (f_1, \ldots, f_m)$. Then we have the solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_N)$ of $f(Y) = 0$ in R corresponding to the canonical homomorphism $R_i \to R$ [Art69a, Corollary 1.6]. Since F_{X_0} is locally of finite presentation, $[X_R, i_R]$ is the image of some element $\zeta_i \in F_{X_0}((R_i, \mathfrak{m}_{R_i}))$ by the canonical map $F_{X_0}((R_i, \mathfrak{m}_{R_i})) \to F_{X_0}(R)$. By [Art69a, Corollary 2.1] there exist an étale neighborhood S of t in T, and a solution $y = (y_1, \ldots, y_N)$ in $\mathcal{O}_S(S)$ with

$$y_i \equiv \hat{y}_i \pmod{\mathfrak{m}_R^2}, \tag{2.2}$$

i.e., y and $\hat{y}$ induce the same element in $F_{X_0}(R/\mathfrak{m}_R^2)$. Let $\varphi\colon R_i \to \mathcal{O}_S(S)$ be the homomorphism corresponding to the solution y, and let $[X_S, i_S]$ be the image of ζ_i by the map $F_{X_0}(\varphi)$ and $\{\eta_n\}_{n\in\mathbb{N}}$ the formal family induced by $[X_S, i_S]$. From (2.2) it follows

$$F_{X_0}(\psi_1)([X_R, i_R]) = \xi_1 = \eta_1,$$

where $\psi_1\colon R \to R/\mathfrak{m}_R^2$ is the natural surjection. By versality of (R, ξ) there is a compatible sequence of homomorphisms $\psi_n\colon R \to R/\mathfrak{m}_R^{n+1}$ lifting ψ_{n-1} and such that $F_{X_0}(\psi_n)([X_R, i_R]) = \eta_n$ for every positive integer n. The sequence $\{\psi_n\}_{n\in\mathbb{N}}$ induces a homomorphism $\psi\colon R \to R$ such that

$$F_{X_0}(\psi)([X_R, i_R]) \equiv \eta_n \pmod{\mathfrak{m}_R^{n+1}}.$$

Since ψ is the identity modulo $\mathfrak{m}_R^2$, it is an automorphism. Thus the formal completion of X_S along the closed fiber X_0 is isomorphic to ξ. □

2.3. **Smoothness and projectivity.** If S' is an étale neighborhood of s in S, then the scheme $X_{S'}$ obtained in the same way gives another versal deformation. The following lemma is crucial for the rest of the paper.

Lemma 2.3. *Let X_0 be a Calabi–Yau manifold of dimension more than two. Then there exists a nonsingular affine variety S over $\mathbf{k}$ with a versal deformation X_S which is projective and smooth of relative dimension* $\dim X_0$ *over S.*

Proof. An étale neighborhood of t in T is smooth over $\mathbf{k}$. Since an open immersion is étale, we may assume that S is connected. Then S must be irreducible, otherwise the local ring $\mathcal{O}_{S,s}$ has more than one minimal prime ideal for every point s in the intersection of irreducible components. We already know that X_S is flat over S. Since X_R is projective over R, by [GD66, Theorem IV8.10.5] there exists an index j such that for all $k \geq j$ the schemes X_{R_k} are projective over R_k. A base change of projective morphism is projective [SP, Tag 02V6].

Since S is irreducible and $\pi_S : X_S \to S$ is flat and proper, the restriction of π_S to each irreducible component of X_S is surjective. In particular, each irreducible component contains the closed fiber X_0 and we have

$$\operatorname{rel.dim}(\pi_{\mathrm{S}}) = \dim X_0.$$

Note that the function

$$n_{X_S/S} : S \to \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

which sends every point $s \in S$ to the dimension of the fiber over s is locally constant, since π_S is flat and proper [SP, Tag 0D4J]. Again, we have used the irreducibility of S.

Due to Lemma 2.4 below, the morphism $\pi_R \colon X_R \to \operatorname{Spec} R$ is smooth. We claim that there is an index l such that for all $k \geq l$ the morphisms $X_{R_k} \to \operatorname{Spec} R_k$ are smooth. To show this, we may assume that X_{R_i} are affine. Let $R \to B$ be the ring homomorphism corresponding to π_R. Then there exists a finitely generated $\mathbb{Z}$-subalgebra R_0 of R and a smooth ring homomorphism $R_0 \to B_0$ such that $B \cong B_0 \otimes_{R_0} R$ [SP, Tag 00TP]. By [SP, Tag 07C3] the inclusion $R_0 \to R$ factors through R_l for some index l. Since smoothness is stable under base change, the claim follows. □

Lemma 2.4. *The scheme X_R is regular and the morphism $\pi_R \colon X_R \to \operatorname{Spec} R$ is smooth of relative dimension* $\dim X_0$.

Proof. We adapt the proof of [Har77, Proposition III10.4]. Since the base scheme $\operatorname{Spec} R$ has the only one closed point, every closed point $x \in X_R$ belongs to the closed fiber X_0. By [SP, Tag 031E] the local ring $\mathcal{O}_{X_R,x}$ is regular, so X_R is regular. Note that X_R is irreducible. From the proof of [SP, Tag 031E] one sees that π_R induces the injection $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_{\mathcal{O}_{X_R,x}}/\mathfrak{m}_{\mathcal{O}_{X_R,x}}^2$, which is dual to the surjection

$$T_{\pi_R} \colon T_x X_R \to T_{\pi_R(x)} R$$

of Zariski tangent spaces. It follows that

$$\dim_{\mathbf{k}(x)} \left(\Omega_{X_R/R} \otimes \mathbf{k}(x)\right) = \dim X_0$$

for every closed point $x \in X_R$. Since π_R is flat and of finite type, we also have

$$\dim_{\mathbf{k}(\zeta)} \left(\Omega_{X_R/R} \otimes \mathbf{k}(\zeta)\right) = \dim X_0$$

for the generic point ζ of X_R [Har77, Theorem I4.8A, Theorem II8.6A]. Then by [Har77, Lemma II8.9] the coherent sheaf $\Omega_{X_R/R}$ is locally free of rank $\dim X_0$. □

Remark 2.1. From the proof of Lemma 2.3, one sees that the scheme X_S must be irreducible, otherwise X_0 becomes disconnected.

3. Deformations of Fourier–Mukai kernels

In order to define a relative integral functor from $D^b(X_S)$ to $D^b(X'_S)$, we deform the Fourier–Mukai kernel $\mathcal{P}_0$ to a perfect complex $\mathcal{P}_S$ on the fiber product $X_S \times_S X'_S$ of smooth projective versal deformations. Although applying the main theorem in [Lie06] might suffice, we adopt a more concrete approach based on [HMS09, HT10], still using some results from [Lie06]. Here, for a deformation X_B of a $\mathbf{k}$-scheme X over an augmented noetherian $\mathbf{k}$-algebra $(B, \mathfrak{m}_B)$, by a deformation of a perfect complex E on X over $(B, \mathfrak{m}_B)$ we mean a pair (E_B, u_B), where $E_B \in D^b(X_B)$ and $u_B\colon E_B \otimes^L_B \mathbf{k} \to E$ is an isomorphism.

3.1. **Derived equivalence and relative Hochschild cohomology.** Let X_n and X'_n be derived-equivalent schemes smooth and projective over R_n. The schemes X_n, X'_n, and their fiber product $X_n \times_{R_n} X'_n$ over R_n form the following diagram

$$\begin{tikzcd} & X_n \times_{R_n} X'_n \arrow[dl, "q_n"'] \arrow[dr, "p_n"] & \\ X_{R_n} & & X'_{R_n} \end{tikzcd}$$

with the natural projections q_n and p_n. For any perfect complex $\mathcal{P}_n$ on $X_n \times_{R_n} X'_n$, the relative integral functor

$$\Phi_{\mathcal{P}_n}(-) = Rp_{n*}\left(\mathcal{P}_n \otimes^L q_n^*(-)\right)$$

sends each object in $D^b(X_n)$ to $D^b(X'_n)$. Due to the Grothendieck–Verdier duality the functor $\Phi_{\mathcal{P}_n}$ has the right adjoint, which we denote by $\Phi_{(\mathcal{P}_n)_R}$.

Assume that $\Phi_{\mathcal{P}_n}$ is an equivalence. Then two functors

$$\begin{gathered} \Psi_1\colon D^b(X_n \times_{R_n} X_n) \to D^b(X_n \times_{R_n} X'_n) \\ \mathcal{G} \mapsto \mathcal{P}_n * \mathcal{G}, \end{gathered}$$

$$\begin{gathered} \Psi_2\colon D^b(X_n \times_{R_n} X'_n) \to D^b(X'_n \times_{R_n} X'_n) \\ \mathcal{G}' \mapsto \mathcal{G}' * (\mathcal{P}_n)_R \end{gathered}$$

respectively induce isomorphisms

$$\psi_1\colon \operatorname{Ext}_{X_n \times_{R_n} X_n}(-,-) \to \operatorname{Ext}_{X_n \times_{R_n} X'_n}(\Psi_1(-), \Psi_1(-)),$$

$$\psi_2\colon \operatorname{Ext}_{X_n \times_{R_n} X'_n}(-,-) \to \operatorname{Ext}_{X'_n \times_{R_n} X'_n}(\Psi_2(-), \Psi_2(-)).$$

The composition defines the isomorphism

$$\Phi^{\mathrm{HH}^*}_{\mathcal{P}_n} = \psi_2 \circ \psi_1\colon \mathrm{HH}^*(X_n/R_n) \to \mathrm{HH}^*(X'_n/R_n)$$

of the relative Hochschild cohomology complex [Căl10], which gives rise to the isomorphism

$$\Phi^{\mathrm{HT}^*}_{\mathcal{P}_n} = (\mathrm{I}^{\mathrm{HKR}}_{X'_n})^{-1} \circ \Phi^{\mathrm{HH}^*}_{\mathcal{P}_n} \circ \mathrm{I}^{\mathrm{HKR}}_{X_n}\colon \mathrm{HT}^*(X_n/R_n) \to \mathrm{HT}^*(X'_n/R_n),$$

where

$$\mathrm{I}^{\mathrm{HKR}}_{X_n}\colon \mathrm{HT}^*(X_n/R_n) \to \mathrm{HH}^*(X_n/R_n),$$

$$\mathrm{I}^{\mathrm{HKR}}_{X'_n}\colon \mathrm{HT}^*(X'_n/R_n) \to \mathrm{HH}^*(X'_n/R_n)$$

are the relative Hochschild–Kostant–Rosenberg isomorphisms. Namely, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{HT}^*(X_n/R_n) \arrow[r, "\mathrm{I}^{\mathrm{HKR}}_{X_n}"] \arrow[d, "\Phi^{\mathrm{HT}^*}_{\mathcal{P}_n}"'] & \operatorname{Ext}^*_{X_n \times_{R_n} X_n}(\mathcal{O}_{\Delta_n}, \mathcal{O}_{\Delta_n}) \arrow[d, "\Phi^{\mathrm{HH}^*}_{\mathcal{P}_n} = \psi_2 \circ \psi_1"] \\ \mathrm{HT}^*(X'_n/R_n) & \operatorname{Ext}^*_{X'_n \times_{R_n} X'_n}(\mathcal{O}_{\Delta'_n}, \mathcal{O}_{\Delta'_n}). \arrow[l, "\left(\mathrm{I}^{\mathrm{HKR}}_{X'_n}\right)^{-1}"'] \end{tikzcd} \tag{3.1}$$

3.2. **Relative Atiyah class and HKR isomorphism.** For a perfect complex E_n on X_n the relative Atiyah class is the element

$$A(E_n) \in \mathrm{Ext}^1_{X_n}(E_n, E_n \otimes \Omega_{\pi_n})$$

induced by the boundary morphism of the short exact sequence

$$0 \to I_n/I_n^2 \to \mathcal{O}_{X_n \times_{R_n} X_n}/I_n \to \mathcal{O}_{\Delta_n} \to 0,$$

where I_n is the defining ideal sheaf of the relative diagonal. Composition in $D^b(X_n)$ and exterior product $\Omega_{\pi_n}^{\otimes i} \to \Omega^i_{\pi_n}$ yield the exponential

$$\exp(A(E_n)) \in \bigoplus \mathrm{Ext}^i_{X_n}(E_n, E_n \otimes \Omega^i_{\pi_n}).$$

Consider the automorphism $\tau\colon X_n \times_{R_n} X_n \to X_n \times_{R_n} X_n$ interchanging the two factors. Since the conormal bundle I_n/I_n^2 consists of elements of the form $x \otimes_{R_n} 1 - 1 \otimes_{R_n} x$, the pullback τ^* acts on $\mathrm{H}^p(X_n, \wedge^q \mathcal{T}_{\pi_n})$ by $(-1)^q$. Then as a straightforward generalization of [Tod09, Lemma 5.8], one obtains two commutative diagrams

$$(3.2)\qquad \begin{array}{ccc} \mathrm{HT}^*(X_n/R_n) & \xrightarrow{\tau^* \circ \mathrm{I}^{\mathrm{HKR}}_{X_n}} & \mathrm{Ext}^*_{X_n \times_{R_n} X_n}(\mathcal{O}_{\Delta_n}, \mathcal{O}_{\Delta_n}) \\ \downarrow{\scriptstyle q_n^*} & & \downarrow{\scriptstyle \psi_1} \\ \mathrm{HT}^*(X_n \times_{R_n} X'_n/R_n) & \xrightarrow{\exp(A(\mathcal{P}_n))\cdot} & \mathrm{Ext}^*_{X_n \times_{R_n} X'_n}(\mathcal{P}_n, \mathcal{P}_n), \end{array}$$

$$(3.3)\qquad \begin{array}{ccc} \mathrm{HT}^*(X'_n/R_n) & \xrightarrow{\mathrm{I}'^{\mathrm{HKR}}_{X'_n}} & \mathrm{Ext}^*_{X'_n \times_{R_n} X'_n}(\mathcal{O}_{\Delta'_n}, \mathcal{O}_{\Delta'_n}) \\ \downarrow{\scriptstyle p_n^*} & & \downarrow{\scriptstyle \psi_2^{-1}} \\ \mathrm{HT}^*(X_n \times_{R_n} X'_n/R_n) & \xrightarrow{\exp(A(\mathcal{P}_n))\cdot} & \mathrm{Ext}^*_{X_n \times_{R_n} X'_n}(\mathcal{P}_n, \mathcal{P}_n). \end{array}$$

3.3. **Obstruction class.** There exists an obstruction for a perfect complex E_n on X_n to deform to some perfect complex on X_{n+1} [Low05, Lie06]. By [HT10, Corollary 3.4] it has the explicit expression as the product of the truncated Atiyah class of E_n and the truncated Kodaira–Spencer class of the thickenings $X_n \hookrightarrow X_{n+1}$ defined by a square zero ideal. In our setting, the deformation X_{n+1} is smooth over R_{n+1}. So the truncated Atiyah and Kodaira–Spencer classes coincide with the relative Atiyah and Kodaira–Spencer classes respectively. Then the obstruction class is given by

$$\varpi(E_n) = (\mathrm{id}_{E_n} \otimes \kappa_n) \circ A(E_n) \in \mathrm{Ext}^2_{X_n}(E_n, E_n^{\oplus l_n}),$$

where $\kappa_n \in \mathrm{Ext}^1_{X_n}(\Omega_{\pi_n}, \mathcal{O}_{X_n}^{\oplus l_n})$ denotes the relative Kodaira–Spencer class, which is the extension class of the short exact sequence

$$0 \to \mathcal{O}_{X_n}^{\oplus l_n} \xrightarrow{\cdot^t(ds_1 \cdots ds_{l_n})} \Omega_{X_{n+1}}|_{X_n} \to \Omega_{\pi_n} \to 0.$$

Here $l_n = \dim_{\mathbf{k}} \mathfrak{m}_R^{n+1}/\mathfrak{m}_R^{n+2}$ and $\{s_1, \ldots, s_{l_n}\}$ is a fixed basis of the $\mathbf{k}$-vector space $\mathfrak{m}_R^{n+1}/\mathfrak{m}_R^{n+2}$.

Suppose that there exists a thickening $X'_n \hookrightarrow X'_{n+1}$ whose relative Kodaira–Spencer class is $\kappa'_n = \left(\Phi^{\mathrm{HT}^2}_{\mathcal{P}_n}\right)^{\oplus l_n}(\kappa_n) \in \mathrm{H}^1(\mathcal{T}_{\pi'_n})^{\oplus l_n}$. Let $\kappa_n \boxplus \kappa'_n = q_n^*\kappa_n + p_n^*\kappa'_n \in \mathrm{H}^1(\mathcal{T}_{\pi_n \times \pi'_n})^{\oplus l_n}$. Adapting [HMS09, Lemma 3.7] to our setting in a straightforward way, we obtain the following:

Lemma 3.1. *Under the above assumption there exists a perfect complex $\mathcal{P}_{n+1}$ on $X_{n+1} \times_{R_{n+1}} X'_{n+1}$ with an isomorphism $\mathcal{P}_{n+1} \otimes^L_{R_{n+1}} R_n \cong \mathcal{P}_n$ such that the integral functor $\Phi_{\mathcal{P}_{n+1}}\colon D^b(X_{n+1}) \to D^b(X'_{n+1})$ is an equivalence.*

Proof. We show the vanishing of the obstruction class

$$A(\mathcal{P}_n) \cdot (\kappa_n \boxplus \kappa'_n) \in \operatorname{Ext}^2_{X_n \times_{R_n} X'_n}(\mathcal{P}_n, \mathcal{P}_n)^{\oplus l_n}.$$

We write κ_n, κ'_n as $\kappa_n = (\kappa_n^1, \dots, \kappa_n^{l_n}), \kappa'_n = (\kappa_n'^1, \dots, \kappa_n'^{l_n})$ with respect to the fixed basis. By commutativity of the diagrams (3.1), (3.2), and (3.3), we have

$$\begin{aligned}
A(\mathcal{P}_n) &= A(\mathcal{P}_n) \cdot (\kappa_n^i \boxplus \kappa_n'^i) \\
&= \psi_1\left(\tau^*\left(I^{\mathrm{HKR}}_{X_n}\left(\kappa_n^i\right)\right)\right) + \psi_2^{-1}\left(I'^{\mathrm{HKR}}_{X'_n}\left(\kappa_n'^i\right)\right) \\
&= \psi_2^{-1} I'^{\mathrm{HKR}}_{X'_n}\left(-\left(I'^{\mathrm{HKR}}_{X'_n}\right)^{-1}\left(\psi_2\left(\psi_1\left(I^{\mathrm{HKR}}_{X_n}\left(\kappa_n^i\right)\right)\right)\right) + \kappa_n'^i\right) \\
&= \psi_2^{-1} I'^{\mathrm{HKR}}_{X'_n}\left(-\Phi^{\mathrm{HT}^2}_{\mathcal{P}_n}\left(\kappa_n^i\right) + \kappa_n'^i\right) = 0
\end{aligned}$$

for each i. Note that, as mentioned above, the pullback τ^* acts on $\mathrm{H}^1(X_n \times_{R_n} X'_n, \mathcal{T}_{\pi_n \times \pi'_n})$ by -1. So there exists a deformation $\mathcal{P}_{n+1}$ of $\mathcal{P}_n$. Then by [LST13, Proposition 1.3] the functor $\Phi_{\mathcal{P}_{n+1}}$ is an equivalence, since $\mathcal{P}_0$ defines an equivalence and π_n, π'_n are smooth projective. □

Combining Lemma 3.1 and Lemma 3.2 below, one sees that if the closed fibers X_0 and X'_0 are higher dimensional Calabi–Yau manifolds, then one can always deform a Fourier–Mukai kernel on $X_0 \times X'_0$ to some Fourier–Mukai kernel on $X_n \times_{R_n} X'_n$ for arbitrary order n.

Lemma 3.2. *If the closed fiber of X'_0 is a Calabi–Yau manifold of dimension more than two, then there exists a thickening $X'_n \hookrightarrow X'_{n+1}$ whose relative Kodaira–Spencer class is $\kappa'_n = \left(\Phi^{HT^2}_{\mathcal{P}_n}\right)^{\oplus l_n}(\kappa_n)$.*

Proof. First, we show the vanishing of cohomology $H^0(\wedge^2 \mathcal{T}_{\pi'_n})$ and $H^2(\mathcal{O}_{X'_n})$. Since π'_n is a projective morphism of noetherian schemes and the sheaves $\wedge^2 \mathcal{T}_{\pi'_n}$, $\mathcal{O}_{X'_n}$ are flat over R_n, by [Har77, Theorem III12.8] there is a Zariski open neighborhood $U \subset \operatorname{Spec} R_n$ of the closed point such that $\dim_{k(y)} H^0(\wedge^2 \mathcal{T}_{\pi'_n, y}) = 0$ and $\dim_{k(y)} H^0(\mathcal{O}_{X'_n, y}) = 0$ for all $y \in U$. Then we have $U = \operatorname{Spec} R_n$, as the complement does not contain the only closed point of $\operatorname{Spec} R_n$.

Next, we construct the thickening $X'_n \hookrightarrow X'_{n+1}$. Fix an affine open covering $\{U_i\}$ of X'_n. The element $\kappa'_n \in \mathrm{H}^1(\mathcal{T}_{\pi'_n})^{\oplus l_n}$ is represented by 2-cocycles $\{\theta_{ij}\}$ with respect to $\{U_i\}$. By [Har10, Propsotion 3.6, Exercise 5.2] these cocycles define automorphisms of the trivial deformations $U_{ij} \times_{R_n} \operatorname{Spec} R_{n+1}$ that can be glued to make a global deformation X'_{n+1} of X'_n. By definition, the relative Kodaira–Spencer class of this thickening is κ'_n. □

3.4. **Algebraization.** Now, we have a system of deformations $\mathcal{P}_n \in \operatorname{perf}(X_{R_n} \times_{R_n} X'_{R_n})$ of $\mathcal{P}_0$ with compatible isomorphisms $\mathcal{P}_{n+1} \otimes^L_{R_{n+1}} R_n \to \mathcal{P}_n$. By [Lie06, Proposition 3.6.1] there exists an effectivization, i.e., a perfect complex $\mathcal{P}_R$ on $X_R \times_R X'_R$ with compatible isomorphisms $\mathcal{P}_R \otimes^L_R R_n \to \mathcal{P}_n$. Recall that in Section 2 to algebrize X_R we have used a filtered inductive system $\{R_i\}_{i \in I}$ of finitely generated $\mathcal{O}_T(T)$-subalgebras of R whose colimit is R. For a sufficiently large index i, there are deformations X_{R_i}, X'_{R_i} of X_0, X'_0 over R_i whose pullback along the canonical homomorphism $R_i \to R$ are X_R, X'_R. So we have

$$X_R \times_R X'_R \cong \left(X_{R_i} \times_{R_i} X'_{R_i}\right) \times_{R_i} R.$$

By [Lie06, Proposition 2.2.1] there exists a perfect complex $\mathcal{P}_{R_i}$ on $X_{R_i} \times_{R_i} X'_{R_i}$ with an isomorphism $\mathcal{P}_{R_i} \otimes^L_{R_i} R \to \mathcal{P}_R$. Then the derived pullback $\mathcal{P}_S \in \operatorname{perf}(X_S \times_S X'_S)$ along $R_i \to \mathcal{O}_S(S)$ is a deformation of $\mathcal{P}_0$. Finally, we obtain the following:

Proposition 3.3. *Let $\mathcal{P}_0$ be a Fourier–Mukai kernel defining the derived equivalence of Calabi–Yau manifolds X_0 and X'_0 of dimension more than two. Then there exists a perfect complex $\mathcal{P}_S$*

on the fiber product $X_S \times_S X'_S$ of smooth projective versal deformations with an isomorphism $\mathcal{P}_S \otimes^L_{\mathcal{O}_S(S)} \mathbf{k} \to \mathcal{P}_0$.

Remark 3.1. For fixed universal formal families (R, ξ) and (R, ξ') of X_0 and X'_0, the deformations X'_n associated with the images $\kappa'_n = \left(\Phi^{\mathrm{HT}^2}_{\mathcal{P}_n}\right)^{\oplus l_n}(\kappa_n)$ determine another universal formal family $(R, \tilde{\xi}')$ of X'_0. Since two universal formal families (R, ξ') and $(R, \tilde{\xi}')$ are isomorphic up to unique isomorphism, by the construction of our versal deformations we may algebrize X_R and X'_R simultaneously.

4. Proof of the main theorem

The smooth projective versal deformations X_S, X'_S, and their fiber product $X_S \times_S X'_S$ over S form the following diagram

$$\begin{tikzcd} & X_S \times_S X'_S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X_S & & X'_S \end{tikzcd}$$

with the natural projections q and p. The relative integral functor

$$\Phi_{\mathcal{P}_S}(-) = Rp_*\left(\mathcal{P}_S \otimes^L q^*(-)\right)$$

sends each object in $D^b(X_S)$ to $D^b(X'_S)$, where $\mathcal{P}_S \in \mathrm{perf}(X_S \times_S X'_S)$ is a deformation of $\mathcal{P}_0$ over $(\mathcal{O}_S(S), \mathfrak{m}_{\mathcal{O}_S(S)})$. In this section, after possible shrinking of the base scheme S, we show that the functor $\Phi_{\mathcal{P}_S}$ is an equivalence. One can also show similar results for formal deformations and their effectivizations.

Theorem 4.1. *Let X_0 and X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then there exists a nonsingular affine variety S over $\mathbf{k}$ such that general fibers of smooth projective versal deformations X_S and X'_S over S are derived-equivalent. In particular, after possible shrinking of the base scheme S, the schemes X_S and X'_S are derived-equivalent.*

Proof. Due to the Grothendieck–Verdier duality the functor $\Phi_{\mathcal{P}_S}$ has the left adjoint, which we denote by $\Phi_{(\mathcal{P}_S)_L}$. For every object $E \in D^b(X_S)$ the counit morphism $\eta\colon \Phi_{(\mathcal{P}_S)_L} \circ \Phi_{\mathcal{P}_S} \to \mathrm{id}_{D^b(X_S)}$ gives a distinguished triangle

$$\Phi_{(\mathcal{P}_S)_L} \circ \Phi_{\mathcal{P}_S}(E) \to E \to F := \mathrm{Cone}\left(\eta(E)\right). \tag{4.1}$$

We may assume that E and F are perfect complexes on X_S. Let $i_s\colon X_s \hookrightarrow X_S$, $i'_s\colon X'_s \hookrightarrow X'_S$, and $j_s = i_s \times i'_s\colon X_s \times X'_s \hookrightarrow X_S \times_S X'_S$ be the closed immersions for every closed point $s \in S$. By the derived flat base change we have

$$Li'^*_s \Phi_{\mathcal{P}_S}(E) \cong \Phi_s(E_s),$$

where $E_s = Li^*_s(E)$ and $\Phi_s = \Phi_{j^*_s\mathcal{P}_S}$. We also denote by $(\Phi_s)_L$ the left adjoint of Φ_s with kernel $(j^*_s\mathcal{P}_S)_L$. Then (4.1) restricts to a distinguished triangle

$$(\Phi_s)_L \circ \Phi_s(E_s) \to E_s \to F_s.$$

Note that the restriction of the counit morphism is the counit morphism. Since $\Phi_{\mathcal{P}_0}$ is an equivalence, the restriction of F to X_0 is quasi-isomorphic to 0. So the support $\mathrm{supp}(F) = \bigcup \mathrm{supp}\,\mathcal{H}^l(F)$ of the perfect complex F is a proper Zariski closed subset of X_S. Let $U \subset S$ be the complement of the image $\pi_S(\mathrm{supp}(F))$. Since U contains the image of the closed fiber X_0, it is a nonempty open subset of S and $\pi_S^{-1}(U)$ does not intersect with $\mathrm{supp}(F)$. In particular, we have $F_s \cong 0$ for every closed point $s \in U$. If E is a strong generator of $D^b(X_S)$, this implies that $\Phi_{\mathcal{P}_U}$ is fully faithful. Here $\mathcal{P}_U$ denotes the restriction of $\mathcal{P}_S$ to $q^{-1}\pi_S^{-1}(U)$.

Recall that a triangulated category is strongly finitely generated if there exist an object E and nonnegative integer k such that every object can be obtained from E by taking isomorphisms, direct summands, shifts, and not more than k times cones. Since X_S is noetherian, separated, and regular, $D^b(X_S)$ is strongly finitely generated by [BB03, Theorem 3.1.4]. Since $\Phi_{\mathcal{P}_S}$ and $\Phi_{(\mathcal{P}_S)_L}$ commute with direct sums on $D^b(X_S)$ by [BB03, Corollary 3.3.4], we may assume that E has no nontrivial direct summands. Using the cohomology long exact sequence induced by a distinguished triangle, one inductively sees that on $\pi_S^{-1}(U)$ the cone of the counit morphism for any object is quasi-isomorphic to 0. Similarly, one finds a Zariski open subset $V \subset S$ such that $(\Phi_{\mathcal{P}_V})_L$ is fully faithful. Finally, we obtain an equivalence $\Phi_s \colon D^b(X_s) \to D^b(X'_s)$ for every closed point $s \in U \cap V \neq \emptyset$, as a fully faithful functor which admits either fully faithful left or right adjoint is an equivalence. In particular, $\Phi_{\mathcal{P}_S}$ is an equivalence after possible shrinking of the base scheme S. □

Corollary 4.2. *Let X_0 and X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then all effectivizations X_R and X'_R of universal formal families ξ and ξ' projective over R are derived-equivalent.*

Proof. Replace X_S, X'_S, and S by X_R, X'_R, and R in the above proof. Then $\pi_R(\operatorname{supp} F)$ is a Zariski closed subset of $\operatorname{Spec} R$ which does not contain the only one closed point. This implies that $\operatorname{supp} F$ is empty. It remains to show that the derived equivalence does not depend on the choice of X_R and X'_R. Given an effectivization $\pi_R \colon X_R \to \operatorname{Spec} R$ of ξ, we have the following pullback diagram

$$\begin{tikzcd}
\xi \cong \hat{X}_R \arrow[r, hook] \arrow[d, "\hat{\pi}"'] & X_R \arrow[d, "\pi_R"] \\
\operatorname{Spf} R \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

of noetherian formal schemes, where X_R is considered as the formal completion along itself. Since π_R is projective and R is a complete local noetherian ring, by [GD61, Corollary III5.1.6] the functor

$$\operatorname{coh}(X_R) \to \operatorname{Coh}(\hat{X}_R),$$

which sends each coherent sheaf $\mathcal{F}$ on X_R to its formal completion $\hat{\mathcal{F}}$ along the closed fiber is an equivalence of abelian categories. So we have

$$D^b(X_R) \simeq D^b(\hat{X}_R).$$

In particular, all effectivizations of ξ are derived-equivalent. □

Corollary 4.3. *Let X_0 and X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then for any formal deformation $\mathcal{X} = \hat{X}_{\mathbf{k}[[t]]}$ of X_0 there exists a formal deformation $\mathcal{X}' = \hat{X}'_{\mathbf{k}[[t]]}$ of X'_0 which is derived-equivalent to $\mathcal{X}$.*

Proof. From the argument in Section 3 and the above proof, it follows immediately. □

5. Pfaffian–Grassmannian equivalence

By Theorem 4.1 the derived equivalence of central fibers of versal deformations can be extended to that of general fibers. After studying deformations of the relevant Calabi–Yau 3-folds, we show that the Pfaffian–Grassmannian derived equivalence is induced by the derived equivalence of IMOU varieties.

5.1. **Grassmannian side.** Let $\mathcal{E}$ be a locally free sheaf on a smooth projective variety Z over $\mathbb{C}$ and let Y_0 be the zero scheme of a section $s \in H^0(Z, \mathcal{E})$ with $\operatorname{codim} Y_0 = \operatorname{rank} \mathcal{E}$ and the defining ideal sheaf $\mathcal{I}_{Y_0} \subset \mathcal{O}_Z$. By [Weh84] a sufficient condition for every algebraic deformation of Y_0 to be obtained by varying the section in $H^0(Z, \mathcal{E})$ is the vanishing of cohomology

$$H^1(Z, \mathcal{E} \otimes \mathcal{I}_{Y_0}), \quad H^1(Y_0, \mathcal{T}_Z|_{Y_0}).$$

Now, we will consider the case

$$Y_0 = \mathrm{Gr}(2, V_7)_{1^7} := \mathrm{Gr}(2, V_7) \cap \mathbb{P}(W),$$

where V_7 is a 7-dimensional complex vector space and W is a 14-dimensional general quotient vector space of $\wedge^2 V_7 \twoheadrightarrow W$.

Lemma 5.1. *Every deformation of $Y_0 = \mathrm{Gr}(2, V_7)_{1^7}$ can be obtained by varying the section s.*

Proof. It suffices to show the vanishing of cohomology

$$H^1(\mathrm{Gr}(2, V_7), \mathcal{O}_{\mathrm{Gr}(2,V_7)}(1)^{\oplus 7} \otimes \mathcal{I}_{Y_0}), \quad H^1(Y_0, \mathcal{T}_{\mathrm{Gr}(2,V_7)}|_{Y_0}).$$

By [Küc96, (1.4)] we have two spectral sequences

$$\begin{aligned} \mathrm{H}^p(\mathrm{Gr}(2, V_7), \mathcal{F} \otimes \wedge^{q+1}\mathcal{O}(-1)_{\mathrm{Gr}(2,V_7)}) &\Rightarrow \mathrm{H}^{p-q}\left(\mathrm{Gr}(2, V_7), \mathcal{F} \otimes \mathcal{I}_{Y_0}\right), q \geq 0, \\ \mathrm{H}^p\left(\mathrm{Gr}(2, V_7), \mathcal{F} \otimes \wedge^{q}\mathcal{O}(-1)_{\mathrm{Gr}(2,V_7)}\right) &\Rightarrow \mathrm{H}^{p-q}\left(\mathrm{Gr}(2, V_7), \mathcal{F}|_{Y_0}\right) \end{aligned}$$

for any locally free sheaf $\mathcal{F}$ on $\mathrm{Gr}(2, V_7)$. Then Borel–Bott–Weil theorem gives the desired result. □

5.2. **Pfaffian side.** Let $H_{Y_0'}\colon \mathbf{Art_k} \to \mathbf{Set}$ be the functor of embedded deformations of a projective scheme $Y_0' \subset \mathbb{P}^6$ over $\mathbb{C}$. Then we have the forgetful functor $H_{Y_0'} \to F_{Y_0'}$. Let $\mathrm{t}_{H_{Y_0'}} \to \mathrm{t}_{F_{Y_0'}}$ be the induced map of tangent spaces, which is given by

$$H_{Y_0'}(\mathbb{C}[t]/t^2) \to F_{Y_0'}(\mathbb{C}[t]/t^2).$$

Now, we will consider the case

$$Y_0' = \mathrm{Pf}(4, V_7) \cap \mathbb{P}(W^\perp) \cong \mathrm{Pf}(4, V_7) \cap \mathbb{P}^6,$$

where $W^\perp = \mathrm{Coker}(W^\vee \hookrightarrow \wedge^2 V_7^\vee)$.

Lemma 5.2. *The induced map of tangent spaces*

$$\mathrm{t}_{H_{Y_0'}} \to \mathrm{t}_{F_{Y_0'}}$$

is surjective.

Proof. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-7) \to 7\mathcal{O}_{\mathbb{P}^6}(-4) \to 7\mathcal{O}_{\mathbb{P}^6}(-3) \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{Y_0'} \to 0. \tag{5.1}$$

From the cohomology of (5.1) and the restriction of Euler sequence we obtain $\mathrm{H}^1(\mathcal{T}_{\mathbb{P}^6}|_{Y_0'}) \cong 0$. Since Y_0' is nonsingular, the short exact sequence

$$0 \to \mathcal{T}_{Y_0'} \to \mathcal{T}_{\mathbb{P}^6}|_{Y_0'} \to \mathcal{N}_{Y_0'/\mathbb{P}^6} \to 0$$

gives rise to a long exact sequence of cohomology

$$\begin{aligned} 0 \to \mathrm{H}^0(\mathcal{T}_{Y_0'}) \to \mathrm{H}^0(\mathcal{T}_{\mathbb{P}^6}|_{Y_0'}) \to \mathrm{H}^0(\mathcal{N}_{Y_0'/\mathbb{P}^6}) \\ \xrightarrow{\delta^0} \mathrm{H}^1(\mathcal{T}_{Y_0'}) \to \mathrm{H}^1(\mathcal{T}_{\mathbb{P}^6}|_{Y_0'}) \to \mathrm{H}^1(\mathcal{N}_{Y_0'/\mathbb{P}^6}) \to \cdots, \end{aligned}$$

where the boundary map δ^0 coincides with $\mathrm{t}_{H_{Y_0'}} \to \mathrm{t}_{F_{Y_0'}}$ by [Har10, Proposition 20.2]. □

Lemma 5.3. *Every deformation of $Y_0' = \mathrm{Pf}(4, V_7) \cap \mathbb{P}^6$ lifts to an embedded deformation in $\mathbb{P}^6$.*

Proof. It is well-known that $H_{Y_0'}$ is pro-representable and unobstructed. We also know that $F_{Y_0'}$ is pro-representable. Then Lemma 5.2 allows us to apply [Har10, Exercise 15.8] and the forgetful functor $H_{Y_0'} \to F_{Y_0'}$ is strongly surjective. In particular, it is surjective. $\square$

5.3. **Induced derived equivalence.** Let X_0 be the complete intersection in G_2-Grassmannian $\mathbf{G} = G_2/P$ associated with the crossed Dynkin diagram defined by an equivariant vector bundle $\mathcal{E}_{(1,1)} \cong G_2 \times_P V^P_{(1,1)}$, which is a flat degeneration of Y_0 [IIM19, Proposition 5.1]. Let X_0' be the complete intersection in G_2-Grassmannian $\mathbf{Q} = G_2/Q$ associated with the crossed Dynkin diagram defined by an equivariant vector bundle $\mathcal{F}_{(1,1)} \cong G_2 \times_Q V^Q_{(1,1)}$, which is a flat degeneration of Y_0' [KK16, Theorem 7.1]. It is known that the Calabi–Yau 3-folds X_0 and X_0' are derived-equivalent [Kuz18, Ued19].

Corollary 5.4. *The Calabi–Yau* 3*-folds* $Y_0 = \mathrm{Gr}(2, V_7)_{1^7}$ *and* $Y_0' = \mathrm{Pf}(4, V_7) \cap \mathbb{P}^6$ *are derived-equivalent.*

Proof. By definition of Y_0 and Lemma 5.1 general fibers of a versal deformation of X_0 are isomorphic to $\mathrm{Gr}(2, V_7)_{1^7}$. By [KK16, Corollary 6.3] and Lemma 5.3 general fibers of a versal deformation of X_0' are isomorphic to $\mathrm{Pf}(4, V_7) \cap \mathbb{P}^6$. Then $\mathrm{Gr}(2, V_7)_{1^7}$ and $\mathrm{Pf}(4, V_7) \cap \mathbb{P}^6$ are derived-equivalent by Theorem 4.1. $\square$

Remark 5.1. The derived-equivalent pair obtained here does not carry any information about W and $W^\perp$, while the original Pfaffian–Grassmannian equivalence connects $\mathrm{Gr}(2, V_7) \cap \mathbb{P}(W)$ with $\mathrm{Pf}(4, V_7) \cap \mathbb{P}(W^\perp)$ for every W. We have proved that for a generic choice of W the Y_0 is derived-equivalent to the Y_0' associated with some other W.

References

[ADS15] N. Addington, W. Donovan, and E. Segal, *The Pfaffian–Grassmannian equivalence revisited*, Algebr. Geom. 2(3), 332-364 (2015).

[Art69a] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math-Paris. 36, 23-58 (1969).

[Art69b] M. Artin, *Algebraization of formal moduli: I*, Global Analysis (Papers in Honor of K. Kodaira), University of Tokyo Press, 21-71 1969, ISBN:978-1-4008-7123-0.

[BB03] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3(1), 1-36 (2003).

[BBP07] O. Ben-Bassat, J. Block, and T. Pantev, *Non-commutative tori and Fourier-Mukai duality*, Compos. Math. 143(2), 423-475 (2007).

[BC09] L. Borisov and A. Căldăraru, *The Pfaffian–Grassmannian derived equivalence*, J. Algebraic. Geom. 18(2), 201-222 (2009).

[BCP20] L. Borisov, A. Căldăraru, and A. Perry, *Intersections of two Grassmannians in* $\mathbb{P}^9$, J. Reine. Angew. Math. 2020(760), 133-162 (2020).

[Căl10] A. Căldăraru, *The Mukai pairing, I: a categorical approach*, New York J. Math. 16, 61-98 (2010).

[GD61] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: III. Étude cohomologique des faisceaux cohérent, première partie.*, Publ. Math-Paris. 11, 5-167 (1961).

[GD66] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie.*, Publ. Math-Paris. 28, 5-255 (1966).

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, 1977, ISBN: 0-387-90244-9.

[Har10] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, 257, Springer-Verlag, 2010, ISBN: 978-1-4419-1596-2.

[HMS09] D. Huybrechts, E. Macrì, and P. Stellari, *Derived equivalences of K3 surfaces and orientation*, Duke Math. J. 149(3), 461-507 (2009).

[HT07] K. Hori and D. Tong, *Aspects of non-abelian gauge dynamics in two-dimensional* $\mathcal{N} = (2, 2)$, J. High Energy Phys. 79(5), 41pp (2007).

[HT10] D. Huybrechts, R. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*. Math. Ann. 346, 545 (2010).

[IIM19] D. Inoue, A. Ito, and M. Miura, *Complete intersection Calabi–Yau manifolds with respect to homogeneous vector bundles on Grassmannians*, Math. Z. 292(1-2), 677-703 (2019).

[KK16] G. Kapustka and M. Kapustka, *Calabi–Yau threefolds in* $\mathbb{P}^6$, Ann. Mat. Pur. Appl. 195(2), 529-556 (2016).

[KR19] M. Kapustka and M. Rampazzo, Marco, *Torelli problem for Calabi–Yau threefolds with GLSM description*, Commun. Num. Theory. Phys. 13(4), 725-761 (2019).

[Küc96] O. Küchle, *Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians*, Pac. J. Math. 175(1), 117-125 (1996).

[Kuz] A. Kuznetsov, *Homological projective duality for Grassmannians of lines*, arXiv:math/0610957

[Kuz18] A. Kuznetsov, *Derived equivalence of Ito–Miura–Okawa–Ueda Calabi–Yau* 3*-folds*, J. Math. Soc. Jap. 70(3), 1007-1013 (2018).

[Lie06] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15, 175-206 (2006).

[Low05] W. Lowen, *Obstruction theory for objects in abelian and derived categories*, Comm. Algebra. 33, 3195-3223 (2005).

[LST13] A.C. López Martín, D. Sánchez Gómez, and C. Tejero Prieto, *Relative Fourier–Mukai transforms for Weierstrass fibrations, abelian schemes and Fano fibrations*, Math. Proc. Cambridge. 155(1), 129-153 (2013).

[Mor21] H. Morimura, *Derived equivalences for the flops of type* C^2 *and* A_G^4 *via mutation of semiorthogonal decomposition*, Algebras Represent. Theory, https://doi.org/10.1007/s10468-021-10036-y

[Rød00] A. Rødland, *The Pfaffian Calabi–Yau, its mirror, and their link to the Grassmannian* $G(2,7)$, Compos. Math. 122(2), 135-149 (2000).

[Sch68] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. 130, 208-222 (1968).

[SP] The Stacks Project authors, *Stacks project*, https://stacks.math.columbia.edu.

[Ued19] K. Ueda, G_2*-Grassmannians and derived equivalences*, Manuscripta Math. 159(3-4), 549-559 (2019).

[Tod09] Y. Toda, *Deformations and Fourier–Mukai transforms*, J. Differential. Geom. 81(1), 197-224 (2009).

[Weh84] J. Weher, *Deformation of varieties defined by sections in homogeneous vector bundles*, Math. Ann. 268(4), 519-532 (1984).

CATEGORICAL GENERIC FIBER

HAYATO MORIMURA

Abstract. For flat proper families of algebraic varieties, we describe the abelian category of coherent sheaves on the generic fiber as a Serre quotient. As an application, we provide new examples of Fourier–Mukai partners via deformations. As another application, we prove that the derived equivalence of the generic fibers extends to that of general fibers.

1. Introduction

1.1. **The main result.** The *categorical general fiber* was introduced in [HMS11] to study the generic categorical behavior of formal deformations of $K3$ surfaces. When a formal deformation is effective, one can consult the generic fiber of an effectivization. In analytic setting, Raynaud constructed the generic fiber of formal deformations [Ray74]. Based on his idea, Huybrechts–Macrì–Stellari developed the categorical general fiber, providing a categorical analogue of the generic fiber for noneffective formal deformations of $K3$ surfaces. As shown in [HMS09], it captures the generic categorical behavior of formal deformations. Below we briefly review their work.

Let $\mathcal{X} \to \operatorname{Spf} \mathbf{k}[\![t]\!]$ be a formal deformation of a smooth projective $\mathbf{k}$-variety X_0. Recall that the *abelian category of coherent sheaves on the general fiber* is the Serre quotient

$$\operatorname{Coh}(\mathcal{X}_{\mathbf{k}((t))}) \coloneqq \operatorname{Coh}(\mathcal{X}) / \operatorname{Coh}(\mathcal{X})_0,$$

where $\operatorname{Coh}(\mathcal{X})_0 \subset \operatorname{Coh}(\mathcal{X})$ is the full abelian subcategory spanned by coherent $\mathbf{k}[\![t]\!]$-torsion $\mathscr{O}_{\mathcal{X}}$-modules. By [Miy91, Theorem 3.2] the derived category $D^b\left(\operatorname{Coh}\left(\mathcal{X}_{\mathbf{k}((t))}\right)\right)$ of the Serre quotient is equivalent to the Verdier quotient

$$D^b(\mathcal{X})/D^b_0(\mathcal{X}),$$

where $D^b_0(\mathcal{X})$ is the full triangulated subcategory spanned by complexes with coherent $\mathbf{k}[\![t]\!]$-torsion cohomology. Huybrechts–Macrì–Stellari showed the $\mathbf{k}((t))$-linear exact equivalence

$$D^b\left(\operatorname{Coh}\left(\mathcal{X}_{\mathbf{k}((t))}\right)\right) \simeq D^b_{\mathrm{c}}\left(\operatorname{Mod}\left(\mathscr{O}_{\mathcal{X}}\right)\right) / D^b_{\mathrm{c}_0}\left(\operatorname{Mod}\left(\mathscr{O}_{\mathcal{X}}\right)\right)$$

when X_0 is a $K3$ surface [HMS11, Theorem 1.1]. Here $D^b_{\mathrm{c}}\left(\operatorname{Mod}\left(\mathscr{O}_{\mathcal{X}}\right)\right)$ is the bounded derived category of $\mathscr{O}_{\mathcal{X}}$-modules with coherent cohomology and $D^b_{\mathrm{c}_0}\left(\operatorname{Mod}\left(\mathscr{O}_{\mathcal{X}}\right)\right)$ is its full triangulated subcategory spanned by complexes with coherent $\mathbf{k}[\![t]\!]$-torsion cohomology. The latter Verdier quotient is called the *derived category of the general fiber*.

If $\mathcal{X}$ is effective with a proper effectivization X, i.e., isomorphic to the formal completion of a proper $\mathbf{k}[\![t]\!]$-scheme X, then by [GD61, Corollory III5.1.6] we have

$$D^b(X) \simeq D^b(\mathcal{X})$$

and $D^b\left(\operatorname{Coh}\left(\mathcal{X}_{\mathbf{k}((t))}\right)\right)$ is equivalent to the Verdier quotient

$$D^b(X)/D^b_0(X),$$

SISSA, via Bonomea 265, 34136 Trieste, Italy
E-mail address: hmorimur@sissa.it.
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which is $\mathbf{k}((t))$-linear. This can be regarded as an effectivization of $D^b(\mathrm{Coh}(\mathcal{X}_{\mathbf{k}((t))}))$. On the other hand, the derived category of the generic fiber $X_{\mathbf{k}((t))}$ of the effectivization X gives another $\mathbf{k}((t))$-linear category. By [BFN10, Theorem 1.2] also $D^b(X_{\mathbf{k}((t))})$ is obtained from the $\mathbf{k}[\![t]\!]$-linear category $D^b(X)$. As both $D^b(X)/D^b_0(X)$ and $D^b(X_{\mathbf{k}((t))})$ carry sufficient information on the generic categorical behavior of formal deformations, it is natural to ask whether they are equivalent as a $\mathbf{k}((t))$-linear triangulated category. Motivated by this question, we prove the following.

Theorem 1.1 (Theorem 2.5)**.** *Let X_R be a smooth separated family over a noetherian connected regular affine $\mathbf{k}$-scheme* $\operatorname{Spec} R$ *whose closed points are $\mathbf{k}$-rational. Let K be the field of fractions of R and X_K the generic fiber. Then there exists a K-linear equivalence*

$$\mathrm{coh}(X_K) \simeq \mathrm{coh}(X_R)/\,\mathrm{coh}(X_R)_0$$

of abelian categories, where $\mathrm{coh}(X_R)_0$ *is the Serre subcategory spanned by R-torsion sheaves.*

Corollary 1.2 (Corollary 2.6)**.** *Under the same assumption as above, there exists an exact K-linear equivalence*

$$D^b(X_K) \simeq D^b(X_R)/D^b_0(X_R),$$

where $D^b_0(X_R)$ is the full triangulated subcategory spanned by complexes with R-torsion cohomology.

We impose the technical condition on $\operatorname{Spec} R$ to include smooth proper effectivizations of formal families over formal power series rings, besides smooth proper families over nonsingular affine $\mathbf{k}$-varieties.

1.2. **The first application.** One advantage to describe the derived category of the generic fiber as a Verdier quotient is that Fourier–Mukai machinery carries over easily. Suppose that

$$\Phi_{\mathcal{E}} \colon D^b(X_R) \to D^b(X'_R)$$

is a relative Fourier–Mukai transform of smooth proper families over R with kernel $\mathcal{E} \in D^b(X_R \times_R X'_R)$. Then $\Phi_{\mathcal{E}}$ induces a Fourier–Mukai transform

$$\Phi_{\mathcal{E}_K} \colon D^b(X_K) \to D^b(X'_K)$$

of the generic fibers, where the kernel $\mathcal{E}_K$ is the pullback along the canonical inclusion $R \hookrightarrow K$. By the standard argument the further base change to the closure defines a Fourier–Mukai transform

$$\Phi_{\mathcal{E}_{\bar{K}}} \colon D^b(X_{\bar{K}}) \to D^b(X'_{\bar{K}})$$

of the geometric generic fibers.

Typical examples for our results are given by deformations of higher dimensional Calabi–Yau manifolds. Recently, the author proved the following.

Theorem 1.3 ([Mor23, Theorem 1.1])**.** *Let X_0, X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then there exists a nonsingular affine $\mathbf{k}$-variety* $\operatorname{Spec} S$ *such that smooth projective versal deformations X_S, X'_S over S are derived-equivalent.*

Here, the derived equivalence is given by the relative Fourier–Mukai transform with kernel obtained by deformation of the original kernel for central fibers. Similarly, the Fourier–Mukai transform of central fibers extends to proper effectivizations of universal formal families. Combining with our current results, we obtain derived-equivalent geometric generic fibers of the versal deformations and the effectivizations respectively. One can check that they are Calabi–Yau manifolds of dimension $\dim X_0$. When the central fibers are nonbirational, in some special cases one can deduce the nonbirationality of the geometric generic fibers. See, also specialization of birational types over a smooth connected curve [KT19, Theorem 1.1]. To summarize, we obtain

Theorem 1.4 (Theorem 5.7)**.** *Let X_0, X_0' be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then the geometric generic fibers $X_{\bar{K}}, X'_{\bar{K}}$ of proper effectivizations and that $X_{\bar{Q}}, X'_{\bar{Q}}$ of smooth projective versal deformations are respectively derived-equivalent Calabi–Yau manifolds. If, in addition, we have either $\mathrm{NS}_{tor} X_0 \neq \mathrm{NS}_{tor} X_0'$, or $\rho(X_0) = \rho(X_0') = 1$ and $\deg(X_0) \neq \deg(X_0')$, then they are respectively nonbirational.*

Recall that *Fourier–Mukai partners* are pairs of nonbirational Calabi–Yau threefolds that are derived-equivalent. They are of considerable interest from the viewpoint of string theory and mirror symmetry. For instance, the Gross–Popescu pair [GP01a, Sch] and the Pfaffian–Grassmannian pair [BC09, Kuz] satisfy the first and the second conditions in Theorem 1.4 respectively. Thus we obtain new examples of Fourier–Mukai partners over the closure $\bar{K}, \bar{Q}$ of function fields. Note that if $\mathbf{k}$ is a universal domain, i.e., an algebraically closed field of infinite transcendence degree of the prime field, then there exists an isomorphism $\mathbf{k} \cong \bar{Q}$ [Via13, Lemma 2.1]. In particular, if $\mathbf{k} = \mathbb{C}$ then the new examples over $\bar{Q}$ can be regarded as complex manifolds.

When X_0, X_0' are the Pfaffian–Grassmannian pair, we demonstrate the subtle difference between $X_{\bar{Q}}, X'_{\bar{Q}}$ and known examples. The geometric generic fibers $X_{\bar{Q}}, X'_{\bar{Q}}$ are respectively isomorphic to X_0, X_0' as a scheme, but not as a variety. We will explain why any other known pair Y_0, Y_0' over $\mathbf{k}$ cannot be isomorphic to $X_{\bar{Q}}, X'_{\bar{Q}}$ even as a scheme. However, we emphasize that one can obtain $X_{\bar{Q}}, X'_{\bar{Q}}$ starting from IMOU varieties [IMOU, Kuz18] which are flat degenerations of X_0, X_0'.

1.3. **The second application.** Another, and probably the most important advantage to describe the derived category of the generic fiber as a Verdier quotient is that any object of $D^b(X_K)$ lifts to that of $D^b(X_R)$. The quotient description extends to nonaffine base case for flat proper families of $\mathbf{k}$-varieties.

Theorem 1.5 (Theorem 6.1)**.** *Let $\pi \colon X \to S$ be a flat proper morphism of $\mathbf{k}$-varieties. Then there exists a K-linear exact equivalence*

$$D^b(X)/\operatorname{Ker}(\bar{\iota}_K^*) \simeq D^b(X_K),$$

where $K = k(S)$ is the function field and $\bar{\iota}_K \colon X_K \to X$ is the canonical morphism.

In particular, there always exists a lift $\mathcal{E} \in D^b(X \times_S X')$ of a Fourier–Mukai kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$ whenever X_K, X'_K are derived-equivalent. It allows us to extend the derived equivalence of the generic fibers to that of general fibers.

Theorem 1.6 (Corollary 6.9)**.** *Let $\pi \colon X \to S, \pi' \colon X' \to S$ be flat proper morphisms of $\mathbf{k}$-varieties. Assume that their generic fibers X_K, X'_K are derived-equivalent. Then there exists an open subset $U \subset S$ over which the restrictions X_U, X'_U become derived-equivalent. In particular, over U any pair of closed fibers are derived-equivalent.*

This can be regarded as a categorical analogue of the fact that isomorphic generic fibers imply birational families in our setting. In classical algebraic geometry, the generic fiber often controls behaviors of general fibers. For instance, if the generic fiber satisfies a certain property which is constructible, then general fibers also satisfy the same property. Such characteristics of the generic fiber must be translated via Gabriel's theorem [Gab62] into the abelian category of coherent sheaves. Nevertheless, as there exist pairs of nonisomorphic derived-equivalent

K-varieties, it makes sense to wonder how the derived category of the generic fiber affects that of general fibers.

When $\mathbf{k}$ is a universal domain, very general fibers are obtained as base changes of the geometric generic fiber along isomorphisms $\mathbf{k} \cong \bar{K}$ from [Via13, Lemma 2.1]. Hence in this case the derived equivalence of the generic fibers induces that of very general fibers. The proof of [Mor23, Theorem 1.1] shows that Theorem 1.6 follows as soon as a lift of the Fourier–Mukai kernel induces the derived equivalence of a single pair of closed fibers. However, since the composition of the canonical morphisms $R \to K \to \bar{K}$ with a fixed $\bar{K} \cong \mathbf{k}$ does not coincide with the surjection $R \to \mathbf{k}$, the induced Fourier Mukai kernel on very general fibers should be different from the restriction of the lift.

Consider another description

$$D^b_{dg}(X_\xi) \simeq \mathrm{Perf}_{dg}(X_\xi) \simeq \mathrm{Perf}_{dg}(X) \otimes_{\mathrm{Perf}_{dg}(S)} \mathrm{Perf}_{dg}(K)$$

of a dg enhancement $D^b_{dg}(X_\xi)$ of $D^b(X_\xi)$ obtained from [BFN10, Theorem 1.2] and [Coh]. By [CP21] and [Miy91] the category $\mathrm{Perf}_{dg}(K)$ is a dg enhancement of the Verdier quotient

$$D^b(S)/D^b_{\leq \dim S-1}(S),$$

where $D^b_{\leq \dim S-1}(S)$ is the full triangulated subcategory spanned by objects with cohomology supported on dimension at most $\dim S - 1$. Thus removing the torsion support from $D^b(X)$ is equivalent to removing all closed fibers from the supports of objects of $D^b(X)$. In particular, from a collection of a finite number of objects and Hom-sets between them in $D^b(X)$, one can remove its torsion support by shrinking the base.

In order to show that the restriction of the lift to general fibers define equivalences, we apply the argument in the proof of [Mor23, Theorem 1.1] to a fixed strong generator E_U of $D^b(X_U)$, which always exists over sufficiently small open subset $U \subset S$. By shrinking U further if necessary, one can remove the torsion parts with respect to the base from E_U and its relevant Hom-sets. Then we invoke some basic categorical results to show that the value of the counit morphism on the trimmed strong generator is an isomorphism, which implies that the restriction of the counit morphism is a natural isomorphism.

Theorem 1.6 tells us that the derived category of the generic fiber determines an U-linear triangulated category $D^b(X_U) \simeq D^b(X)/D^b_Z(X)$ for some open subset $U \subset S$ and its complement Z, where $D^b_Z(X) \subset D^b(X)$ is the full S-linear triangulated subcategory with cohomology supported on X_Z. In general, the derived category of the generic fiber cannot recover that of the initial fiber as we have $D^b(X_K) \simeq D^b(X_R)/D^b(X_0)$ for $R = \mathbf{k}[\![t]\!]$. Rather, [HMS09] and our results suggest that it carries information on derived categories of general fibers. We expect that Theorem 1.6 provides a way to seek categorical constructible properties and their derived invariance.

Notations and conventions. *We work over an algebraically closed field* $\mathbf{k}$ *of characteristic* 0 *throughout the paper. Every time we apply* [Via13, Lemma 2.1] *we always assume* $\mathbf{k}$ *to be a universal domain with comments. A* $\mathbf{k}$*-variety is an integral separated* $\mathbf{k}$*-scheme of finite type. A Calabi–Yau manifold* X_0 *is a smooth projective* $\mathbf{k}$*-variety with trivial canonical bundle and* $H^i(X_0, \mathscr{O}_{X_0}) = 0$ *for* $0 < i < \dim X_0$. *For a noetherian formal scheme* $\mathcal{X}$ *by* $D^b(\mathcal{X})$ *we denote the bounded derived category of the abelian category* $\mathrm{Coh}(\mathcal{X})$ *of coherent sheaves on* $\mathcal{X}$.

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2. The derived category of the generic fiber

For a scheme X_R over an integral domain R, we have the following pullback diagram

$$\begin{tikzcd} X_K \arrow[r, "i"] \arrow[d, "\pi_K"'] & X_R \arrow[d, "\pi_R"] \\ \operatorname{Spec} K \arrow[r, "j"'] & \operatorname{Spec} R, \end{tikzcd}$$

where K is the field of fractions of R and X_K is the generic fiber, i.e., the fiber over the generic point $\xi \in \operatorname{Spec} R$. In the sequel, we assume the following conditions:

(i) X_R is connected and smooth separated over R.
(ii) $\operatorname{Spec} R$ is a noetherian connected regular affine $\mathbf{k}$-scheme whose closed points are $\mathbf{k}$-rational.

The assumption guarantees that X_R and X_K are noetherian separated regular. Indeed, $\mathscr{O}_{X_R,x}$ is regular for every closed point $x \in X_R$ by [SP, Tag 031E]. Under the assumption also X_K is connected. An example of X_R we have in mind is proper effectivizations of miniversal formal families of a smooth projective $\mathbf{k}$-variety whose deformations are unobstructed. Note that in this case $\operatorname{Spec} R$ is not of finite type over $\mathbf{k}$. We impose (ii) to include such examples, besides smooth proper families over nonsingular affine $\mathbf{k}$-varieties.

We denote by $\operatorname{coh}(X_R)_0 \subset \operatorname{coh}(X_R)$ the Serre subcategory spanned by R-torsion sheaves, i.e., for each $\mathscr{F} \in \operatorname{coh}(X_R)_0$ there is an element $r \in R$ such that $r\mathscr{F} = 0$. We write $C = \operatorname{coh}(X_R)/\operatorname{coh}(X_R)_0$ for the Serre quotient. The natural projection $p\colon \operatorname{coh}(X_R) \to C$ which sends $\mathscr{F}$ to $\mathscr{F}_K$ is known to be exact. By universality of Serre quotient, the exact functor

$$(-) \otimes_R K\colon \operatorname{coh}(X_R) \to \operatorname{coh}(X_K)$$

induces a unique exact functor

$$\Phi\colon C \to \operatorname{coh}(X_K)$$

such that $(-) \otimes_R K = \Phi \circ p$. Then Φ defines the derived functor

$$D^b(C) \to D^b(X_K),$$

which induces a functor

$$\Psi\colon D^b(X_R)/D^b_0(X_R) \to D^b(X_K)$$

via [Miy91, Theorem 3.2]. We show that Φ and Ψ are equivalences. In particular,

$$D^b(C) \simeq D^b(X_R)/D^b_0(X_R) \tag{2.1}$$

gives an alternative description of $D^b(X_K)$.

2.1. K-linear categorical quotients.

As expected from their constructions, both the Serre quotient $C = \operatorname{coh}(X_R)/\operatorname{coh}(X_R)_0$ and the Verdier quotient $D^b(X_R)/D^b_0(X_R)$ carry natural K-linear structures. To see this, one can adapt [HMS11, Proposition 2.3, 2.9] to our setting in a straightforward way. We include the proofs for reader's convenience.

Lemma 2.1 ([HMS11, Proposition 2.3]). *The abelian category C is K-linear and for all $\mathscr{E}, \mathscr{F} \in \operatorname{coh}(X_R)$ the natural projection $p\colon \operatorname{coh}(X_R) \to C$ induces a K-linear isomorphism*

$$\operatorname{Hom}_{X_R}(\mathscr{E}, \mathscr{F}) \otimes_R K \cong \operatorname{Hom}_C(\mathscr{E}_K, \mathscr{F}_K). \tag{2.2}$$

Proof. As a quotient of the R-linear category $\mathrm{coh}(X_R)$, the category C is also R-linear. The multiplication with r^{-1} for $r \in R$ is defined as follows. Let $f \in \mathrm{Hom}_C(\mathscr{E}_K, \mathscr{F}_K)$ be a morphism represented by $f\colon (\mathscr{E} \xleftarrow{s_0} \mathscr{E}_0 \xrightarrow{g} \mathscr{F})$ with $\mathrm{Ker}(s_0), \mathrm{Coker}(s_0) \in \mathrm{coh}(X_R)_0$. Then set $r^{-1}f\colon (\mathscr{E} \xleftarrow{rs_0} \mathscr{E}_0 \xrightarrow{g} \mathscr{F})$, which is well-defined in C, since $\mathrm{Ker}(rs_0), \mathrm{Coker}(rs_0)$ are in $\mathrm{coh}(X_R)_0$. Moreover, we have $r(r^{-1}f) = f$ due to the commutative diagram

$$\begin{tikzcd}
 & \mathscr{E}_0 \arrow[dl, "rs_0"'] \arrow[dr, "rg"] \arrow[dd, "r\cdot\mathrm{id}"] & \\
\mathscr{E} & & \mathscr{F} \\
 & \mathscr{E}_0 \arrow[ul, "s_0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

and the K-linearity of the composition is obvious. Recall that a morphism in the Serre quotient is an equivalence class of diagrams. In C, two morphisms $f\colon (\mathscr{E} \xleftarrow{s_0} \mathscr{E}_0 \xrightarrow{g} \mathscr{F})$ and $f'\colon (\mathscr{E} \xleftarrow{s_0'} \mathscr{E}_0' \xrightarrow{g'} \mathscr{F})$ are equivalent if there is a third diagram $f''\colon (\mathscr{E} \xleftarrow{s_0''} \mathscr{E}_0'' \xrightarrow{g''} \mathscr{F})$ with $\mathrm{Ker}(s_0''), \mathrm{Coker}(s_0'') \in \mathrm{coh}(X_R)_0$ and morphisms $u\colon \mathscr{E}_0'' \to \mathscr{E}_0, v\colon \mathscr{E}_0'' \to \mathscr{E}_0'$ in $\mathrm{coh}(X_R)$ which makes the diagram

$$\begin{tikzcd}
 & \mathscr{E}_0 \arrow[dl, "s_0"'] \arrow[dr, "g"] & \\
\mathscr{E} & \mathscr{E}'' \arrow[l, "s_0''"] \arrow[r, "g_0''"'] \arrow[u, "u"] \arrow[d, "v"'] & \mathscr{F} \\
 & \mathscr{E}_0' \arrow[ul, "s_0'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

commute.

Consider the induced K-linear map

$$\eta_K\colon\ \mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F}) \otimes_R K \to \mathrm{Hom}_C(\mathscr{E}_K, \mathscr{F}_K).$$

To prove the injectivity of η_K, let $f \in \mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F})$ be a morphism with $\eta_K(f) = 0$. There exists a commutative diagram

$$\begin{tikzcd}
 & \mathscr{E}' \arrow[dl, "s"'] \arrow[dr, "0"] & \\
\mathscr{E} \arrow[rr, "f"'] & & \mathscr{F},
\end{tikzcd}$$

with $\mathrm{Ker}(s), \mathrm{Coker}(s) \in \mathrm{coh}(X_R)_0$ and hence f factorizes through

$$f\colon \mathscr{E} \xrightarrow{q} \mathrm{Coker}(s) \xrightarrow{f'} \mathscr{F}.$$

If $r\,\mathrm{Coker}(s) = 0$, then this yields $rf = f' \circ (rq) = 0$. In particular, $f \otimes 1 \in \mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F}) \otimes_R K$ is trivial.

To prove the surjectivity of η_K, we have to show that for any $f \in \mathrm{Hom}_C(\mathscr{E}_K, \mathscr{F}_K)$ there exists an element $r \in R$ such that rf is induced by a morphism $\mathscr{E} \to \mathscr{F}$ in $\mathrm{coh}(X_R)$. Write $f\colon (\mathscr{E} \xleftarrow{s_0} \mathscr{E}_0 \xrightarrow{g} \mathscr{F})$ with $r_1\,\mathrm{Ker}(s_0) = r_2\,\mathrm{Coker}(s_0) = 0$ for some $r_1, r_2 \in R$. Consider the exact sequence

$$0 \to \mathrm{Hom}_{X_R}(\mathscr{E}', \mathscr{F}) \xrightarrow{\circ\tilde{p}} \mathrm{Hom}_{X_R}(\mathscr{E}_0, \mathscr{F}) \xrightarrow{\circ\tilde{i}} \mathrm{Hom}_{X_R}(\mathrm{Ker}(s_0), \mathscr{F})$$

induced by the natural projection $\tilde{p}\colon \mathscr{E}_0 \twoheadrightarrow \mathscr{E}' := \mathrm{Im}(s_0)$ and its kernel $\tilde{i}\colon\ \mathrm{Ker}(s_0) \hookrightarrow \mathscr{E}_0$. Since $(r_1 g) \circ \tilde{i} = g \circ (r_1 \tilde{i}) = 0$, there exists a unique homomorphism $g'\colon \mathscr{E}' \to \mathscr{F}$ such that $g' \circ \tilde{p} = r_1 g$.

This yields the commutative diagram

$$\begin{tikzcd}
& \mathscr{E}_0 \arrow[dl, "s_0"'] \arrow[dr, "r_1 g"] \arrow[dd, "\tilde{p}"] & \\
\mathscr{E} & & \mathscr{F}, \\
& \mathscr{E}' \arrow[ul, hook] \arrow[ur, "g'"'] &
\end{tikzcd}$$

which allows us to represent $r_1 f$ by $\mathscr{E} \hookleftarrow \mathscr{E}' \xrightarrow{g'} \mathscr{F}$.

As $\mathscr{E}/\mathscr{E}' \cong \mathrm{Coker}(s_0)$ is annihilated by r_2, the homomorphism $r_2 g' \colon \mathscr{E}' \to \mathscr{F}$ lifts to a homomorphism $g'' \colon \mathscr{E} \to \mathscr{F}$, i.e., $g''|_{\mathscr{E}'} = r_2 g'$. This yields the commutative diagram

$$\begin{tikzcd}
& \mathscr{E}' \arrow[dl, hook'] \arrow[dr, "r_2 g'"] \arrow[dd, hook] & \\
\mathscr{E} & & \mathscr{F}. \\
& \mathscr{E} \arrow[ul, "\mathrm{id}"] \arrow[ur, "g''"'] &
\end{tikzcd}$$

Hence $r_1 r_2 f$ is represented by $\left(\mathscr{E} \xleftarrow{\mathrm{id}} \mathscr{E} \xrightarrow{g''} \mathscr{F}\right)$, i.e., $r_1 r_2 f = \eta_K(g'')$. $\square$

Lemma 2.2 ([HMS11, Proposition 2.9]). *The triangulated category $D^b(C)$ is K-linear and for all $E, F \in D^b(X_R)$ the natural projection $Q \colon D^b(X_R) \to D^b(C)$ induces a K-linear isomorphism*

$$\mathrm{Hom}_{D^b(X_R)}(E, F) \otimes_R K \cong \mathrm{Hom}_{D^b(C)}(E_K, F_K).$$

Proof. Any morphism in $D^b(C)$ can be represented by a morphism of bounded complexes of objects in C, which is a collection of morphisms in C compatible with the differentials. Since by Lemma 2.1 both the morphisms and the differentials are K-linear, the representative is also K-linear. The K-linear isomorphism is a direct consequence of Corollary 2.6, whose proof does not rely on it. $\square$

Remark 2.1. In [HMS11] only the case where $R = \mathbf{k}[\![t]\!]$ was treated. This is because some results require R to be a DVR. Throughout the paper, we are free from the requirement and results which rely on it.

2.2. **Canonical functor from the Serre quotient.** Due to the natural K-linear structure, the Serre quotient C can be embedded in $\mathrm{coh}(X_K)$ via the exact functor Φ, which is induced by that $(-) \otimes_R K \colon \mathrm{coh}(X_R) \to \mathrm{coh}(X_K)$.

Proposition 2.3. *The functor $\Phi \colon C \to \mathrm{coh}(X_K)$ is fully faithful.*

Proof. The images of $\mathscr{E}_K, \mathscr{F}_K \in C$ by Φ are respectively isomorphic to the pullbacks $i^*\mathscr{E}, i^*\mathscr{F}$ of some coherent sheaves $\mathscr{E}, \mathscr{F}$ on X_R. We have

$$\begin{aligned}
\mathrm{Hom}_{X_K}(\Phi(\mathscr{E}_K), \Phi(\mathscr{F}_K)) &= \mathrm{Hom}_{X_K}(i^*\mathscr{E}, i^*\mathscr{F}) \\
&\cong \Gamma \circ i^* \underline{\mathrm{Hom}}_{X_R}(\mathscr{E}, \mathscr{F}) \\
&\cong j^* \circ (\pi_R)_* \underline{\mathrm{Hom}}_{X_R}(\mathscr{E}, \mathscr{F}) \\
&\cong \mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F}) \otimes_R K \\
&\cong \mathrm{Hom}_C(\mathscr{E}_K, \mathscr{F}_K),
\end{aligned}$$

where the second, the third, and the fourth isomorphisms follow from flat base change, Lemma 2.4 below, and Lemma 2.1 respectively. $\square$

Lemma 2.4. *For all $\mathscr{E}, \mathscr{F} \in \mathrm{coh}(X_R)$ we have an isomorphism*

$$(\pi_R)_* \underline{\mathrm{Hom}}_{X_R}(\mathscr{E}, \mathscr{F}) \otimes_R K \cong \mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F}) \otimes_R K.$$

Proof. We may consider $(\pi_R)_*\underline{\mathrm{Hom}}_{X_R}(\mathscr{E},\mathscr{F})\otimes_R K$ as the stalk of the sheaf $(\pi_R)_*\underline{\mathrm{Hom}}_{X_R}(\mathscr{E},\mathscr{F})$ at the generic point ξ of $\operatorname{Spec} R$. For an affine open cover $\operatorname{Spec} R = \{D(f)\},\ 0 \neq f \in R$, take any germ $\langle D(f), s\rangle$ of $(\pi_R)_*\underline{\mathrm{Hom}}_{X_R}(\mathscr{E},\mathscr{F})_\xi$. Since $(\pi_R)_*\underline{\mathrm{Hom}}_{X_R}(\mathscr{E},\mathscr{F})$ is a quasi-coherent sheaf on an affine scheme as X_R is noetherian, by [Har77, Lemma II5.3] there exists an integer $n \geq 0$ such that $f^n s$ becomes a global section. Let

$$\phi\colon (\pi_R)_*\underline{\mathrm{Hom}}_{X_R}(\mathscr{E},\mathscr{F})_\xi \to \mathrm{Hom}_{X_R}(\mathscr{E},\mathscr{F})\otimes_R K$$

be the homomorphism of R-algebras which sends $\langle D(f), s\rangle$ to $f^m s \otimes (1/f^m)$, where m is the minimum integer such that $f^m s$ becomes a global section. One can check that this is well-defined. The inverse ϕ^{-1} is given by the map which sends $v \otimes (g/f)$ to $\langle D(f), (gv/f)\rangle$ for $v \in \mathrm{Hom}_{X_R}(\mathscr{E},\mathscr{F})$ and $g \in R$. □

Theorem 2.5. *The functor $\Phi\colon C \to \mathrm{coh}(X_K)$ is a K-linear equivalence of abelian categories.*

Proof. It suffices to show that Φ is essentially surjective. By assumption X_K is connected. Let $\mathscr{F}_\xi$ be an object of $\mathrm{coh}(X_K)$. Since X_K is noetherian integral separated regular, by [Har77, Exercise III6.8] any coherent sheaf on X_K can be obtained as the cokernel of a morphism of locally free sheaves of finite rank. The essential image of Φ is a full abelian subcategory of $\mathrm{coh}(X_K)$. In particular, it is closed under taking cokernels. Hence we may assume $\mathscr{F}_\xi$ to be a locally free sheaf of finite rank.

Take an affine open cover $\{U_i\}_{i=1}^m, U_i = \operatorname{Spec} A_i$ of X_R such that the restriction of $\mathscr{F}_\xi$ to each affine open subset $V_i = U_i \times_R K$ of X_K is isomorphic to a finite rank free $\tilde{B}_i = \tilde{A}_i \otimes_R K$-module

$$F_i = \tilde{B}_i^{\oplus N}.$$

Let $\phi_{ij} = \phi_i \circ \phi_j^{-1}\colon F_j|_{V_{ij}} \to F_i|_{V_{ij}}$ be isomorphisms on $V_{ij} = V_i \cap V_j$ where $\phi_i\colon F_\xi|_{V_i} \to F_i$ are trivializations with their inverses $\phi_i^{-1}\colon F_i \to F_\xi|_{V_i}$. In other words, we have the commutative diagrams

$$\begin{tikzcd} F_\xi|_{V_{ij}} \arrow[r, equal] & F_\xi|_{V_{ij}} \arrow[d, "\phi_i"] \\ F_j|_{V_{ij}} \arrow[u, "\phi_j^{-1}"] \arrow[r, "\phi_{ij}"'] & F_i|_{V_{ij}}. \end{tikzcd}$$

From F_i we obtain a rank N free $\tilde{A}_i$-module

$$E_i = \tilde{A}_i^{\oplus N}$$

with the same generators. By construction tensoring K with E_i recovers F_i. Now, up to shrinking the base $\operatorname{Spec} R$, we glue E_i to construct a coherent sheaf $\mathscr{E}$ on X_R such that $\mathscr{E}\otimes_R K \cong \mathscr{F}_\xi$. By Lemma 2.1 there are lifts $\bar{\phi}_{ij}\colon E_j|_{U_{ij}} \to E_i|_{U_{ij}}$ on $U_{ij} = U_i \cap U_j$ of ϕ_{ij} along (2.2). Namely, we have

$$\bar{\phi}_{ij} \otimes_R 1/r_{ij} = \phi_{ij}$$

for some $r_{ij} \in R$.

Consider the affine open subset

$$\operatorname{Spec} T \subset \operatorname{Spec} R$$

defined by $r_{ij} \neq 0, 1 \leq i, j \leq m$. On the base changes $U_{ij,T} = U_{ij} \times_R T$ all r_{ij} become invertible. Hence ϕ_{ij} canonically lift to isomorphisms

$$r_{ij}^{-1}\bar{\phi}_{ij}\colon E_j|_{U_{ij,T}} \to E_i|_{U_{ij,T}}$$

which injectively map to $r_{ij}^{-1}\bar{\phi}_{ij} \otimes_T 1 = \phi_{ij}$ under

$$\mathrm{Hom}_{U_{ij,T}}(E_j|_{U_{ij,T}}, E_i|_{U_{ij,T}}) \xrightarrow{-\otimes_T K} \mathrm{Hom}_{U_{ij,T}}(E_j|_{U_{ij,T}}, E_i|_{U_{ij,T}}) \otimes_T K \cong \mathrm{Hom}_C((E_j)_K, (E_j)_K).$$

Clearly, the lifts satisfy cocycle condition. Thus $E_i|_{U_{i,T}}$ glue to yield a locally free sheaf $\tilde{\mathscr{E}}$ on $X_T = X_R \times_R T$ such that $\tilde{\mathscr{E}} \otimes_T K \cong \mathscr{F}_\xi$.

By [Har77, Exercise II5.15] the lift $\tilde{\mathscr{E}}$ extends to a coherent sheaf $\mathscr{E}$ on X_R. Since the exact functor $(-) \otimes_R K$ factorizes through

$$\mathrm{coh}(X_R) \to \mathrm{coh}(X_T) \to \mathrm{coh}(X_K)$$

and it sends $\mathscr{E}$ to $\mathscr{F}_\xi$, there is an object $\mathscr{E}_K \in C$ which maps to $\mathscr{F}_\xi$ under Φ. □

2.3. **Canonical functor from the Verdier quotient.** As the functor $\Phi\colon C \to \mathrm{coh}(X_K)$ is exact, termwise application of Φ defines the derived functor $D^b(C) \to D^b(X_K)$. By universality of Verdier quotient, the induced functor

$$\Psi\colon D^b(X_R)/D^b_0(X_R) \to D^b(X_K)$$

by (2.1) coincides with $D^b(C) \to D^b(X_K)$. From Theorem 2.5 we obtain

Corollary 2.6. *The functor $\Psi\colon D^b(X_R)/D^b_0(X_R) \to D^b(X_K)$ is a K-linear exact equivalence.*

In Section 6 we will extend Theorem 2.5 and Corollary 2.6 to nonaffine base case for flat proper families of $\mathbf{k}$-varieties.

3. Comparison with the categorical general fiber

3.1. **The abelian category of coherent sheaves on the general fiber.** Recall that in [HMS11] for a formal deformation $\mathcal{X}$ of a smooth projective $\mathbf{k}$-variety over a formal power series ring $\mathbf{k}[\![t]\!]$ the *abelian category of coherent sheaves on the general fiber* is defined as the Serre quotient

$$\mathrm{Coh}(\mathcal{X}_{\mathbf{k}((t))}) := \mathrm{Coh}(\mathcal{X})/\,\mathrm{Coh}(\mathcal{X})_0,$$

where $\mathrm{Coh}(\mathcal{X})$ is the abelian category of coherent $\mathscr{O}_\mathcal{X}$-modules and $\mathrm{Coh}(\mathcal{X})_0$ is the full abelian subcategory spanned by coherent $\mathbf{k}[\![t]\!]$-torsion $\mathscr{O}_\mathcal{X}$-modules. In the case where $\mathcal{X}$ is effective with a proper effectivization, one can obtain $\mathrm{Coh}(\mathcal{X}_{\mathbf{k}((t))})$ via formal completion along the closed fiber in the following sense.

Corollary 3.1. *Let $\mathcal{X} = \hat{X}_R \to \mathrm{Spf}\, R$ be an effective formal deformation of a smooth projective variety with a proper effectivization X_R. Then abelian category of coherent sheaves on the general fiber of $\mathcal{X}$ is equivalent to that on the generic fiber X_K of its effectivization, i.e., there exists a K-linear equivalence*

$$\mathrm{coh}(X_K) \simeq \mathrm{Coh}(\mathcal{X}_K)$$

of abelian categories.

Proof. We have the pullback diagram

$$\begin{tikzcd} \mathcal{X} = \hat{X}_R \arrow[r, "\iota"] \arrow[d, "\hat{\pi}"'] & X_R \arrow[d, "\pi_R"] \\ \mathrm{Spf}\, R \arrow[r] & \mathrm{Spec}\, R \end{tikzcd}$$

of noetherian formal schemes. Since R is a complete local noetherian ring, one can apply [GD61, Corollorary III5.1.6] to see that the functor

$$\mathrm{coh}(X_R) \to \mathrm{Coh}(\mathcal{X}), \tag{3.1}$$

which sends each coherent sheaf $\mathscr{F}$ on X_R to its formal completion $\hat{\mathscr{F}}$ along the closed fiber, is an R-linear equivalence of abelian categories. By universality of Serre quotient, we obtain the induced K-linear equivalence

$$\mathrm{coh}(X_R)/\,\mathrm{coh}(X_R)_0 \to \mathrm{Coh}(\mathcal{X})/\,\mathrm{Coh}(\mathcal{X})_0.$$

□

3.2. **Serre functor.** In the case where $\mathcal{X}$ is effective with a proper effectivization, the Serre functor from [HMS11, Theorem 1.1] constructed when $\mathcal{X}$ is a formal deformation of a $K3$ surface, extends to smooth projective varieties and formal power series rings of any finite dimension in a straightforward way.

Proposition 3.2. *Let $\mathcal{X} = \hat{X}_R \to \mathrm{Spf}\, R$ be an effective formal deformation of a d-dimensional smooth projective variety with a proper effectivization X_R. Then a Serre functor on $D^b(\mathcal{X}_K)$ is given by*

$$S(\hat{E}_K) = (\widehat{E \otimes \omega_{\pi_R}})_K[d],$$

where ω_{π_R} is the dualizing sheaf for π_R.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{D^b(\mathrm{Coh}(\mathcal{X}_K))}\left(\hat{E}_K, \hat{F}_K\right) &\cong \mathrm{Hom}_{D^b(\mathcal{X})}\left(\hat{E}, \hat{F}\right) \otimes_R K \\
&\cong \mathrm{Hom}_{D^b(X_R)}(E, F) \otimes_R K \\
&\cong \mathrm{Hom}_{D^b(X_R)}(F, E \otimes \omega_{\pi_R}[d])^\vee \otimes_R K \\
&\cong \mathrm{Hom}_{D^b(\mathcal{X})}\left(\hat{F}, \widehat{E \otimes \omega_{\pi_R}}[d]\right)^\vee \otimes_R K \\
&\cong \mathrm{Hom}_{D^b(\mathrm{Coh}(\mathcal{X}_K))}\left(\hat{F}_K, (\widehat{E \otimes \omega_{\pi_R}})_K[d]\right)^\vee,
\end{aligned}$$

where the first and the fifth, the second and the fourth, and the third isomorphisms follow from Lemma 2.2, the equivalence (3.1), and Serre duality for the smooth morphism π_R of relative dimension d respectively. □

3.3. **The derived category of the general fiber.** Recall that the *derived category of the general fiber* is defined as the Verdier quotient

$$D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_\mathcal{X}))\,/D^b_{\mathrm{c}_0}(\mathrm{Mod}(\mathscr{O}_\mathcal{X})),$$

where $D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_\mathcal{X}))$ is the bounded derived category of $\mathscr{O}_\mathcal{X}$-modules with coherent cohomology and $D^b_{\mathrm{c}_0}(\mathrm{Mod}(\mathscr{O}_\mathcal{X}))$ is the full triangulated subcategory spanned by complexes with coherent $\mathbf{k}[\![t]\!]$-torsion cohomology. By [HMS11, Theorem 1.1] we have

$$D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_\mathcal{X}))\,/D^b_{\mathrm{c}_0}(\mathrm{Mod}(\mathscr{O}_\mathcal{X})) \simeq D^b(\mathrm{Coh}(\mathcal{X}_{\mathbf{k}((t))}))$$

when $\mathcal{X}$ is a formal deformation of a K3 surface. This is deduced from the intermediate $\mathbf{k}((t))$-linear exact equivalence

$$D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_\mathcal{X}))\,/D^b_{\mathrm{c}_0}(\mathrm{Mod}(\mathscr{O}_\mathcal{X})) \simeq D^b(\mathcal{X})/D^b_0(\mathcal{X}) \tag{3.2}$$

established in the proof of [HMS11, Proposition 3.10].

While we have

$$D^b_{\mathrm{c}_0}(\mathrm{Mod}(\mathscr{O}_\mathcal{X})) \simeq D^b_0(\mathcal{X})$$

by [HMS11, Proposition 2.5], in general the natural inclusion

$$D^b(\mathcal{X}) \hookrightarrow D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_{\mathcal{X}}))$$

is not an equivalence. Hence one cannot expect (3.2) to hold for more general $\mathcal{X}$. However, in the case where $\mathcal{X}$ is effective with a proper effectivization, we have

$$D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_{\mathrm{X_R}}))/D^b_{\mathrm{c_0}}(\mathrm{Mod}(\mathscr{O}_{\mathrm{X_R}})) \simeq D^b(X_R)/D^b_0(X_R) \simeq D^b(X_K)$$

by Corollary 2.6. Note that the first equivalence follows from

$$D^b_{\mathrm{c}}(\mathrm{Mod}(\mathscr{O}_{\mathrm{X_R}})) \simeq D^b(X_R),$$

which holds for noetherian schemes. Unless the closed fiber of X_R is a $K3$ surface, in general one can only recover the part

$$D^b(\mathcal{X})/D^b_0(\mathcal{X}) \simeq D^b(\mathcal{X}_K)$$

via [Miy91, Theorem 3.2] and formal completion along the closed fiber in the sense of Corollary 3.1.

4. Induced Fourier–Mukai transforms

As mentioned in [HMS11], one advantage to describe the derived category of the generic fiber as a Verdier quotient is that the Fourier–Mukai machinery carries over easily. Given a relative integral functor $\Phi_{\mathcal{E}}\colon D^b(X_R) \to D^b(X'_R)$ on smooth proper families $\pi_R\colon X_R \to \operatorname{Spec} R$ and $\pi'_R\colon X'_R \to \operatorname{Spec} R$, we study the induced derived equivalence on thier generic fibers, geometric generic fibers, and formal completions. One will see that $\Phi_{\mathcal{E}}$ being equivalences when restricted to general fibers implies the derived equivalence of their generic and geometric generic fibers. We will discuss the opposite direction in Section 6 below.

4.1. Induced Functor from smooth proper families to generic fibers.

Proposition 4.1. *Let X_R, X'_R be smooth proper families over R. If $\Phi_{\mathcal{E}}\colon D^b(X_R) \to D^b(X'_R)$ is a relative Fourier–Mukai functor, then the induced integral functor $\Phi_{\mathcal{E}_K}\colon D^b(X_K) \to D^b(X'_K)$ is an equivalence. Here, $\mathcal{E}_K \in D^b(X_R \times_R X'_R)/D^b_0(X_R \times_R X'_R)$ is the image of $\mathcal{E}$ by the natural projection.*

Proof. Since objects of $D^b(X_R \times_R X'_R)/D^b_0(X_R \times_R X'_R)$ are the same as those of $D^b(X_R \times_R X'_R)$ [HMS11, Appendix], the R-linear functor $\Phi_{\mathcal{E}}$ induces an integral functor

$$\Phi_{\mathcal{E}_K}\colon D^b(X_R)/D^b_0(X_R) \to D^b(X'_R)/D^b_0(X'_R).$$

By Corollary 2.6 we have the commutative diagram

$$\begin{tikzcd} D^b(X_R) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "Q"'] & D^b(X'_R) \arrow[d, "Q"] \\ D^b(X_K) \arrow[r, "\Phi_{\mathcal{E}_K}"] & D^b(X'_K). \end{tikzcd}$$

The inverse functor $\Phi_{\mathcal{E}}^{-1}$ is a left adjoint to $\Phi_{\mathcal{E}}$ as $\Phi_{\mathcal{E}}$ is an equivalence. On the other hand, due to the Grothendieck–Verdier duality $\Phi_{\mathcal{E}}$ has a left adjoint $\Phi_{\mathcal{E}_L}$ with $\mathcal{E}_L$ a perfect complex on $X_R \times_R X'_R$. By uniqueness of left adjoint up to isomorphism, it follows $\Phi_{\mathcal{E}}^{-1} \cong \Phi_{\mathcal{E}_L}$. Then $\Phi_{\mathcal{E}}^{-1}$ induces an integral functor $\Phi_{(\mathcal{E}_L)_K}$ and we obtain natural isomorphisms $\Phi_{(\mathcal{E}_L)_K} \circ \Phi_{\mathcal{E}_K} \cong \mathrm{Id}_{\mathrm{D^b(X_K)}}$, $\Phi_{\mathcal{E}_K} \circ \Phi_{(\mathcal{E}_L)_K} \cong \mathrm{Id}_{\mathrm{D^b(X'_K)}}$. Thus the functor $\Phi_{\mathcal{E}_K}$ is an equivalence. □

Remark 4.1. By universality of Verdier quotient, Corollary 2.6 induces a mere K-linear equivalence $D^b(X_K) \simeq D^b(X'_K)$, while Proposition 4.1 preserves Fourier–Mukai kernels.

Corollary 4.2. *Let X_R, X'_R be smooth proper families over R. If $\Phi_{\mathcal{E}}\colon D^b(X_R) \to D^b(X'_R)$ is a relative integral functor whose restrictions to general fibers are equivalences, then the induced integral functor $\Phi_{\mathcal{E}_K}\colon D^b(X_K) \to D^b(X'_K)$ is an equivalence. Here, $\mathcal{E}_K \in D^b(X_R \times_R X'_R)/D^b_0(X_R \times_R X'_R)$ is the image of $\mathcal{E}$ by the natural projection.*

Proof. We are given a Zariski open subset $U \subset \operatorname{Spec} R$ such that any pair of closed fibers of X_R and X'_R over U are derived-equivalent. By [HLS09, Proposition 2.15] the relative integral functor $\Phi_{\mathcal{E}_U}$ is an equivalence, where $\mathcal{E}_U$ denotes the restriction of $\mathcal{E}$ to $\mathrm{pr}_1^{-1} \circ \pi_R^{-1}(U)$. Note that we do not need the assumption on $X_U = X_R \times_R U$ to be locally projective in the statement of [HLS09, Proposition 2.15], which guarantees the existence of a right adjoint to $\Phi_{\mathcal{E}_U}$, as π_R, π'_R are smooth proper. Now, the claim follows immediately from Proposition 4.1. □

Remark 4.2. The previous corollary provides from general to generic induction of Fourier–Mukai transforms. Conversely, by Corollary 2.6 any integral functor $\Phi_{\mathcal{E}_K}\colon D^b(X_\xi) \to D^b(X'_\xi)$ lifts to a relative integral functor $\Phi_{\mathcal{E}}\colon D^b(X_R) \to D^b(X'_R)$. In Section 6 we will prove that the induced functor is an equivalence when restricted to general fibers.

4.2. **Induced Functor from generic to geometric generic fibers.** Due to [Ola], one can slightly improve [Huy06, Exercise 5.18] and a well-known fact about the relation between field extensions and Fourier–Mukai transforms.

Lemma 4.3. *Let X_K, X'_K be smooth proper K-varieties and $\Phi_{\mathcal{E}_K}\colon D^b(X_K) \to D^b(X'_K)$ an integral functor. Then $\Phi_{\mathcal{E}_K}$ is an equivalence if and only if there are isomorphisms*

$$\mathcal{E}_K * (\mathcal{E}_K)_L \cong \mathcal{O}_\Delta,\ (\mathcal{E}_K)_L * \mathcal{E}_K \cong \mathcal{O}_{\Delta'}, \tag{4.1}$$

where $\Delta\colon X_K \hookrightarrow X_K \times X_K, \Delta'\colon X'_K \hookrightarrow X'_K \times X'_K$ are the diagonal embeddings.

Proof. Assume that we are given the isomorphisms (4.1). Regarding the isomorphic objects as kernels, we obtain natural isomorphisms

$$\Phi_{(\mathcal{E}_K)_L} \circ \Phi_{\mathcal{E}_K} \cong \mathrm{Id}_{\mathrm{D^b(X_K)}},\ \Phi_{\mathcal{E}_\mathrm{K}} \circ \Phi_{(\mathcal{E}_\mathrm{K})_\mathrm{L}} \cong \mathrm{Id}_{\mathrm{D^b(X'_K)}}. \tag{4.2}$$

Thus the functor $\Phi_{\mathcal{E}_K}$ is an equivalence.

Conversely, assume that $\Phi_{\mathcal{E}_K}$ is an equivalence. Let $(\Phi_{\mathcal{E}_K})^{-1}$ be its inverse. Then we have natural isomorphisms

$$(\Phi_{\mathcal{E}_K})^{-1} \circ \Phi_{\mathcal{E}_K} \cong \mathrm{Id}_{\mathrm{D^b(X_K)}},\ \Phi_{\mathcal{E}_\mathrm{K}} \circ (\Phi_{\mathcal{E}_\mathrm{K}})^{-1} \cong \mathrm{Id}_{\mathrm{D^b(X'_K)}}.$$

In particular, it follows that $(\Phi_{\mathcal{E}_K})^{-1}$ is a left adjoint of $\Phi_{\mathcal{E}_K}$. By uniqueness of left adjoint up to isomorphism, we obtain $(\Phi_{\mathcal{E}_K})^{-1} \cong \Phi_{(\mathcal{E}_K)_L}$ and (4.2). Thus two pairs of the kernel $\mathcal{E}_K * (\mathcal{E}_K)_L, \mathcal{O}_\Delta$ and $(\mathcal{E}_K)_L * \mathcal{E}_K, \mathcal{O}_{\Delta'}$ respectively define the same derived autoequivalence of $D^b(X_K)$ and $D^b(X'_K)$. Since any derived equivalence of smooth proper varieties is defined by a unique Fourier–Mukai kernel up to isomorphism [Ola], we obtain (4.1). □

Lemma 4.4. *Let X_K, X'_K be smooth proper K-varieties. If X_K, X'_K are derived-equivalent, then for any filed extension L_0/K the base changes X_{L_0}, X'_{L_0} are derived-equivalent.*

Proof. Let $\mathcal{E}_K \in D^b(X_K \times X'_K)$ be a Fourier–Mukai kernel, which is unique up to isomorphism. By Lemma 4.3 we have isomorphisms

$$\mathcal{E}_K * (\mathcal{E}_K)_L \cong \mathcal{O}_\Delta,\ (\mathcal{E}_K)_L * \mathcal{E}_K \cong \mathcal{O}_{\Delta'}.$$

As L_0 is a flat K-module, the pullback by $\bar{i}''\colon X_{L_0} \times X_{L_0} \to X_K \times X_K$ yields

$$\mathcal{E}_{L_0} * (\mathcal{E}_{L_0})_L \cong \bar{i}''^* (\mathcal{E}_K * (\mathcal{E}_K)_L) \cong \bar{i}''^* \mathcal{O}_\Delta \cong \mathcal{O}_{\bar{\Delta}},$$

where $\mathcal{E}_{L_0} = \bar{\iota}''^{*}\mathcal{E}_K$ and $\bar{\Delta}\colon X_{L_0} \hookrightarrow X_{L_0} \times X_{L_0}$ is the diagonal embedding. Here, we have used $\Omega_{X_{L_0}/L_0} \cong \Omega_{X_K/K} \otimes_K L_0$ [Har77, Proposition II8.10]. Similarly, we have $(\mathcal{E}_{L_0})_L * \mathcal{E}_{L_0} \cong \mathcal{O}_{\bar{\Delta}'}$. Again, by Lemma 4.3 we conclude that $\Phi_{\mathcal{E}_{L_0}}$ is an equivalence. □

Remark 4.3. When $\operatorname{Spec} R$ is an affine $\mathbf{k}$-variety and $\mathbf{k}$ is a universal domain, very general fibers of X_R, X'_R are dereived-equivalent if and only if so are their geometric generic fibers. Indeed, by [Via13, Lemma 2.1] there is an isomorphism $\mathbf{k} \to \bar{K}$ along which the pullback of $X_{\mathbf{k}} \times X'_{\mathbf{k}}$ is isomorphic to $X_{\bar{K}} \times X'_{\bar{K}}$. Here, $X_{\mathbf{k}}, X'_{\mathbf{k}}$ are very general fibers of X_R, X'_R. One can apply the same argument as in the proof of Lemma 4.4. Note that we assume $\mathbf{k}$ to be a universal domain to apply [Via13, Lemma 2.1].

4.3. **Induced Functors to effective formal families and their categorical general fibers.** Assume that the families $\pi_R\colon X_R \to \operatorname{Spec} R$, $\pi'_R\colon X'_R \to \operatorname{Spec} R$ are effectivizations of formal deformations $\mathcal{X}$, $\mathcal{X}'$ over R of smooth projective varieties X_0, X'_0 respectively. Assume further that π_R, π'_R are proper. Here, we will show that the induced Fourier–Mukai functor from smooth proper families to their generic fibers is compatible with formal completion along the closed fibers.

The schemes X_R, X'_R, their restrictions to the n-th order thickenings, and their formal completions along the closed fibers form the commutative diagram

$$\begin{tikzcd}
 & & X_n \times_{R_n} X'_n \arrow[dll, "q_n"'] \arrow[d, hook, "\tau''_n"] \arrow[drr, "p_n"] & & \\
X_n \arrow[d, hook, "\tau_n"] & & \mathcal{X} \times_R \mathcal{X}' \arrow[dll, "\hat{q}"'] \arrow[d, hook, "\iota''"] \arrow[drr, "\hat{p}"] & & X'_n \arrow[d, hook, "\tau'_n"] \\
\mathcal{X} \arrow[d, hook, "\iota"] & & X_R \times_R X'_R \arrow[dll, "q"'] \arrow[drr, "p"] & & \mathcal{X}' \arrow[d, hook, "\iota'"] \\
X_R & & & & X'_R
\end{tikzcd}$$

of noetherian formal schemes. Here, $\hat{q}, \hat{p}$ are canonically determined as the limit by the compatible collections of morphisms q_n, p_n of schemes, and compositions of two sequential vertical arrows give the canonical factorizations of the closed embeddings

$$\kappa_n = \iota \circ \tau_n, \quad \kappa''_n = \iota'' \circ \tau''_n, \quad \kappa'_n = \iota' \circ \tau'_n.$$

Proposition 4.5. *Given $\mathcal{E} \in D^b(X_R \times_R X'_R)$, the formal completion $\hat{\mathcal{E}}$ along the closed fiber $X_0 \times X'_0$ defines a relative integral functor*

$$\Phi_{\hat{\mathcal{E}}} = R\hat{p}_*(\hat{\mathcal{E}} \otimes^L \hat{q}^*(-))\colon D^b(\mathcal{X}) \to D^b(\mathcal{X}'),$$

which makes the diagram

$$\begin{tikzcd}
D^b(X_R) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "G"'] & D^b(X'_R) \arrow[d, "G"] \\
D^b(\mathcal{X}) \arrow[r, "\Phi_{\hat{\mathcal{E}}}"] & D^b(\mathcal{X}')
\end{tikzcd}$$

2*-commutative.*

Proof. Since objects of $D^b(\mathcal{X}'), D^b(\mathcal{X} \times_R \mathcal{X}')$ are quasi-isomorphic to perfect complexes, the functors $\hat{q}^*\colon D^b(\mathcal{X}) \to D^b(\mathcal{X} \times_R \mathcal{X}')$ and $\hat{\mathcal{E}} \otimes^L (-)\colon D^b(\mathcal{X} \times_R \mathcal{X}') \to D^b(\mathcal{X} \times_R \mathcal{X}')$ can be computed by termwise application after replacing objects with perfect complexes. It is known that $R\hat{p}_*$ is well-defined and sends bounded complexes to bounded complexes by the comparison theorem [GD61, Corollary 4.1.7] and Leray spectral sequence. Since by the equivalence (3.1) any object of $D^b(\mathcal{X})$ can be written as $\hat{E} = G(E) \cong L\iota^* E$ for some object $E \in D^b(X_R)$, we have

$$\begin{aligned}
\Phi_{\hat{\mathcal{E}}}(\hat{E}) \otimes^L_R R_n &= L\tau'^*_n R\hat{p}_*(\hat{\mathcal{E}} \otimes^L_R L\hat{q}^*\hat{E}) \\
&\cong Rp_{n*} L\tau''^*_n(\hat{\mathcal{E}} \otimes^L_R L\hat{q}^*\hat{E}) \\
&\cong Rp_{n*}(\mathcal{E}_n \otimes_{R_n} q_n^* E_n) \\
&\cong \Phi_{\mathcal{E}_n}(E_n),
\end{aligned}$$

where $R_n = R/\mathfrak{m}_R^{n+1}$, $E_n = \tau_n^*\hat{E}$, and the first isomorphism follows from the comparison theorem. We also have

$$\begin{aligned}
\Phi_{\mathcal{E}}(E) \otimes^L_R R_n &= L\kappa'^*_n Rp_*(\mathcal{E} \otimes^L_R Lq^* E) \\
&\cong Rp_{n*} L\kappa''^*_n(\mathcal{E} \otimes^L_R Lq^* E) \\
&\cong Rp_{n*}(\mathcal{E}_n \otimes_{R_n} q_n^* E_n) \\
&\cong \Phi_{\mathcal{E}_n}(E_n).
\end{aligned}$$

Thus we obtain isomorphisms

$$f_n \colon \Phi_{\hat{\mathcal{E}}}(\hat{E}) \otimes_R R_n \to \Phi_{\mathcal{E}}(E) \otimes_R R_n$$

for any positive integer n satisfying $f_{n+1} \otimes^L_{R_{n+1}} \mathrm{id}_{R_n} = f_n$. Note that $\Phi_{\hat{\mathcal{E}}}(\hat{E})$ is the formal completion of some perfect complex on X'_R, as it belongs to $D^b(\mathcal{X}') \simeq D^b(X'_R)$. Then, by the argument in the proof of [HMS11, Lemma 3.4], taking the limit yields an isomorphism

$$f \colon \Phi_{\hat{\mathcal{E}}}(\hat{E}) \to G\left(\Phi_{\mathcal{E}}(E)\right)$$

which completes the proof. □

Corollary 4.6. *Given a Fourier–Mukai kernel $\mathcal{E} \in D^b(X_R \times_R X'_R)$, the functors $\Phi_{\hat{\mathcal{E}}} \colon D^b(\mathcal{X}) \to D^b(\mathcal{X}')$ and $\Phi_{\hat{\mathcal{E}}_K} \colon D^b(\mathrm{Coh}(\mathcal{X}_K)) \to D^b(\mathrm{Coh}(\mathcal{X}'_K))$ are equivalences, where $\hat{\mathcal{E}}_K \in D^b(\mathcal{X} \times_R \mathcal{X}')/D^b_0(\mathcal{X} \times_R \mathcal{X}')$ is the image of $\hat{\mathcal{E}}$ by the natural projection.*

5. Fourier–Mukai partners over the closure of function fields

In this section, passing through the deformation theory, we provide new examples of Fourier–Mukai partners, pairs of nonbirational Calabi–Yau threefolds that are derived-equivalent. Our results play a role in deducing the derived equivalences. Let X_0, X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two. There exists a nonsingular affine **k**-variety $\operatorname{Spec} S$ such that smooth projective versal deformations X_S, X'_S over S are derived-equivalent [Mor23, Theorem 1.1]. Also effectivizations X_R, X'_R are shown to be derived-equivalent. These equivalences are given by deformed Fourier–Mukai kernels. From our results in Section 4 it immediately follows that the geometric generic fibers of X_S, X'_S and X_R, X'_R are respectively derived-equivalent. One can check that the geometric generic fibers are Calabi–Yau manifolds. If X_0, X'_0 satisfy either $\mathrm{NS}_{tor} X_0 \neq \mathrm{NS}_{tor} X'_0$, or $\rho(X_0) = \rho(X'_0) = 1$ and $\deg(X_0) \neq \deg(X'_0)$, then the geometric generic fibers are nonbirational as well as X_0, X'_0. Several pairs are known to satisfy one of the two conditions. Thus we obtain new examples of Fourier–Mukai partners over the closure $\bar{K}, \bar{Q}$ of the function fields. One may also pass to base changes over a smooth connected **k**-curve containing X_0, X'_0 to apply [KT19, Theorem 1.1]. If **k** is a universal domain, then $\bar{Q}$ is isomorphic to **k** but the Fourier–Mukai partner $X_{\bar{Q}}, X'_{\bar{Q}}$ are different as a variety from known examples. We demonstrate this subtle difference when X_0, X'_0 are the famous Pfaffian–Grassmannian pair.

5.1. **Deformations of Calabi–Yau manifolds.** Let X_0 be a Calabi–Yau manifold with dimension more than two. The deformation functor

$$\mathrm{Def}_{X_0}\colon\ \mathbf{Art_k} \to \mathbf{Set}$$

of X_0 has a universal formal family (R, ξ), where R is a formal power series ring with $\dim_{\mathbf{k}} \mathrm{H}^1(X_0, \mathscr{T}_{X_0})$ valuables, and ξ belongs to the limit

$$\widehat{\mathrm{Def}}_{X_0}(R) = \varprojlim \mathrm{Def}_{X_0}(R/\mathfrak{m}_R^n)$$

of the inverse system

$$\cdots \to \mathrm{Def}_{X_0}(R/\mathfrak{m}_R^{n+2}) \to \mathrm{Def}_{X_0}(R/\mathfrak{m}_R^{n+1}) \to \mathrm{Def}_{X_0}(R/\mathfrak{m}_R^{n}) \to \cdots$$

induced by the natural quotient maps $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$. Here, $\mathfrak{m}_R \subset R$ is the maximal ideal.

The formal family ξ corresponds to a natural transformation

$$h_R = \mathrm{Hom}_{\mathbf{k}\text{-alg}}(R, -) \to \mathrm{Def}_{X_0},$$

which sends each homomorphism $f \in h_R(A)$ factorizing through

$$R \to R/\mathfrak{m}_R^{n+1} \xrightarrow{g} A$$

to $\mathrm{Def}_{X_0}(g)(\xi_n)$. Let X_n be the schemes defining ξ_n. There is a noetherian formal scheme $\mathcal{X}$ over R such that $X_n \cong \mathcal{X} \times_R R/\mathfrak{m}_R^{n+1}$ for each n. Thus for a deformation (X_A, i_A) the scheme X_A can be obtained as the pullback of $\mathcal{X}$ along some morphism of noetherian formal schemes $\operatorname{Spec} A \to \operatorname{Spf} R$.

Now, we briefly recall how to algebrize $\mathcal{X}$. By [GD61, Theorem III5.4.5] there exists a scheme X_R flat projective over R whose formal completion along the closed fiber X_0 is isomorphic to $\mathcal{X}$. Moreover, X_R is smooth over R of relative dimension $\dim X_0$ [Mor23, Lemma 2.4]. We call X_R an effectivization of $\mathcal{X}$. Consider the extended functor

$$\mathrm{Def}_{X_0}\colon\ \mathbf{k\text{-}alg}^{\mathrm{aug}} \to \mathbf{Set}$$

from the category of augmented noetherian $\mathbf{k}$-algebras. Let $T = \mathbf{k}[t_1, \dots, t_d]$ and $t \in \operatorname{Spec} T$ be the closed point corresponding to maximal ideal $(t_1, \dots, t_d)$. There is a filtered inductive system $\{R_i\}_{i \in I}$ of finitely generated T-subalgebras of R whose colimit is R. Since Def_{X_0} is locally of finite presentation, $[X_R, i_R]$ is the image of some element $\zeta_i \in \mathrm{Def}_{X_0}((R_i, \mathfrak{m}_{R_i}))$ by the canonical map $\mathrm{Def}_{X_0}((R_i, \mathfrak{m}_{R_i})) \to \mathrm{Def}_{X_0}(R)$. By [Art69, Corollary 2.1] there exists an étale neighborhood $\operatorname{Spec} S$ of t in $\operatorname{Spec} T$ with first order approximation $\varphi\colon R_i \to S$ of $R_i \hookrightarrow R$. Let $[X_S, i_S]$ be the image of ζ_i by the map $\mathrm{Def}_{X_0}(\varphi)$. The formal completion of X_S along the closed fiber X_0 is isomorphic to $\mathcal{X}$.

The triple $(\operatorname{Spec} S, s_0, X_S)$, or sometimes X_S, is called a versal deformation of X_0. By construction, $\operatorname{Spec} S$ is an algebraic $\mathbf{k}$-variety with a distinguished closed point s_0 mapping to t, and X_S is flat of finite type over S whose closed fiber over s_0 is X_0. In our setting, one can find a versal deformation X_S smooth projective over a nonsingular affine $\mathbf{k}$-variety $\operatorname{Spec} S$ of relative dimension $\dim X_0$ [Mor23, Lemma 2.3]. Moreover, given another Calabi–Yau manifold X_0' derived-equivalent to X_0, one can find a smooth projective versal deformation X_S' over the same base. The construction passes through effectivizations. Namely, there are effectivizations X_R, X_R' of X_0, X_0' over the same regular affine $\mathbf{k}$-scheme $\operatorname{Spec} R$.

5.2. **Calabi–Yau geometric generic fibers.** First, we consider the effectivization X_R of $\mathcal{X}$ and its geometric generic fiber. We have the pullback diagram

$$\begin{tikzcd}
X_{\bar{K}} \arrow[r, "\bar{i}"] \arrow[d, "\pi_{\bar{K}}"'] & X_K \arrow[r, "i"] \arrow[d, "\pi_K"'] & X_R \arrow[d, "\pi_R"] \\
\operatorname{Spec} \bar{K} \arrow[r, "\bar{j}"'] & \operatorname{Spec} K \arrow[r, "j"'] & \operatorname{Spec} R,
\end{tikzcd}$$

where K is the field of fractions of R and $\bar{K}$ is the closure of K.

Lemma 5.1. *The geometric generic fiber $X_{\bar{K}}$ is a Calabi–Yau manifold over $\bar{K}$.*

Proof. Smoothness and projectivity follow from their being stable under base change. One can apply [GD66, Proposition IV15.5.7] to see that $X_{\bar{K}}$ is connected. Then $X_{\bar{K}}$ must be irreducible, as it is regular. By [Har77, Theorem III12.8] the function $h^0\colon \operatorname{Spec} R \to \mathbb{Z}$ defined as

$$h^0(r) = \dim_{\mathbf{k}} \mathrm{H}^0(X_r, \wedge^{\dim X_0 - 1} \mathscr{T}_{\pi_R} \otimes_R k(r))$$

for $r \in \operatorname{Spec} R$ is upper semicontinuous, where $\mathscr{T}_{\pi_R}$ is the relative tangent sheaf. It follows that there is an open neighborhood U of the closed point to which the restriction vanishes. Since R is a domain, U contains the generic point. By flat base change we obtain

$$\mathrm{H}^0(X_{\bar{K}}, \wedge^{\dim X_0 - 1} \mathscr{T}_{X_{\bar{K}}}) \cong \mathrm{H}^0(X_K, \wedge^{\dim X_0 - 1} \mathscr{T}_{X_K}) \otimes_K \bar{K} = 0.$$

Similarly, one can show the vanishing of all the other relevant cohomology.

It remains to show the triviality of the canonical bundle. Consider the formal completion $\hat{\omega}_{X_R/R}$ of the relative canonical sheaf on X_R along the closed fiber X_0. It is given by the limit of inverse system $\{\omega_{X_{R_n}/R_n}\}_{n\in\mathbb{N}}$ with $R_n = R/\mathfrak{m}_R^{n+1}$. Here, the inverse system consists of the sequence of deformations of ω_{X_0} along order by order square zero extensions. Since we have $\omega_{X_0} \cong \mathscr{O}_{X_0}$, the inverse system $\{\mathscr{O}_{X_{R_n}}\}_{n\in\mathbb{N}}$ also consists of the sequence of deformations of ω_{X_0}. On the other hand, by [Lie06, Theorem 3.1.1] freedom of deformations of ω_{X_0} as a perfect complex to X_1 is given by $\operatorname{Ext}^1_{X_0}(\omega_{X_0}, \omega_{X_0})^{\oplus l_1}$, where l_1 is the dimension of $\mathbf{k}$-vector space $\mathfrak{m}_R/\mathfrak{m}_R^2$. This is trivial by the assumption on X_0 and there is an isomorphism $\omega_{X_1/R_1} \cong \mathscr{O}_{X_{R_1}}$ respecting $\omega_{X_1/R_1} \otimes_{R_1} \mathbf{k} \cong \omega_{X_0}$ and $\mathscr{O}_{X_{R_1}} \otimes_{R_1} \mathbf{k} \cong \mathscr{O}_{X_0} \cong \omega_{X_0}$. Inductively, one finds isomorphisms $\omega_{X_n/R_n} \cong \mathscr{O}_{X_{R_n}}$ respecting $\omega_{X_n/R_n} \otimes_{R_n} R_{n-1} \cong \omega_{X_{n-1}}$ and $\mathscr{O}_{X_{R_n}} \otimes_{R_n} R_{n-1} \cong \mathscr{O}_{X_{n-1}} \cong \omega_{X_{n-1}}$. By universality of limit, we obtain $\hat{\omega}_{X_R/R} \cong \hat{\mathscr{O}}_{X_R}$, which in turn induces $\omega_{X_R/R} \cong \mathscr{O}_{X_R}$ via the equivalence (3.1). $\square$

Next, we consider the versal deformation X_S of X_0 and its geometric generic fiber. We have the pullback diagram

$$\begin{tikzcd}
X_{\bar{Q}} \arrow[r, "\bar{u}"] \arrow[d, "\pi_{\bar{Q}}"'] & X_{\bar{Q}} \arrow[r, "u"] \arrow[d, "\pi_Q"'] & X_S \arrow[d, "\pi_S"] \\
\operatorname{Spec} \bar{Q} \arrow[r, "\bar{v}"'] & \operatorname{Spec} Q \arrow[r, "v"'] & \operatorname{Spec} S,
\end{tikzcd}$$

where Q is the field of fractions of S and $\bar{Q}$ is the closure of Q.

Lemma 5.2. *The geometric generic fiber $X_{\bar{Q}}$ is a Calabi–Yau manifold over $\bar{Q}$.*

Proof. Nontrivial part is the triviality of the canonical bundle. Consider the collection $\{\omega_{X_{R_i}/R_i}\}_{i\in I}$ of relative canonical sheaves on X_{R_i}. It consists of the sequence of deformations of ω_{X_0}. Since we have $\omega_{X_0} \cong \mathscr{O}_{X_0}$, the collection $\{\mathscr{O}_{X_{R_n}}\}_{n\in\mathbb{N}}$ of structure sheaves also consists of the sequence of deformations of ω_{X_0}. We have $\omega_{X_R/R} \cong \omega_{X_{R_i}/R_i} \otimes_{R_i} R$ by [Har77, Proposition II8.10] and the construction of X_S. One can apply [Lie06, Proposition 2.2.1] to find an isomorphism $\omega_{X_{R_i}/R_i} \cong \mathscr{O}_{X_{R_i}}$

for sufficiently large i with respect to the partial order of I. We obtain $\omega_{X_S/S} \cong \mathcal{O}_{X_S}$, again by [Har77, Proposition II8.10] and the construction of X_S. □

Remark 5.1. Assuming $\mathbf{k} = \mathbb{C}$, one can show the previous lemma without using the deformation theory of perfect complexes as follows. The fact that deformations of Calabi–Yau manifolds are Calabi–Yau is well-known. Let $X_{\mathbf{k}}$ be a very general fiber of X_S. Choose a subfield $L \subset \mathbf{k}$ over which $X_{\mathbf{k}}$ is defined, i.e., we are given a L-variety X_L such that the base change $X_L \times_L \mathbf{k}$ is isomorphic to $X_{\mathbf{k}}$. From the proof of [Via13, Lemma 2.1] there exists an isomorphism $\mathbf{k} \to \bar{Q}$ fixing L such that the base change $X_{\mathbf{k}} \times_{\mathbf{k}} \bar{Q}$ is isomorphic to $X_{\bar{Q}}$. Note that $X_{\mathbf{k}}$ and $X_{\bar{Q}}$ are isomorphic as a scheme but not as a variety, since the induced isomorphism of the schemes does not lie over a fixed base.

5.3. **The derived equivalence.** Suppose that X_0 is derived-equivalent to another Calabi–Yau manifold X'_0. Recall that there are effectivizations X_R, X'_R over the same regular affine $\mathbf{k}$-scheme $\operatorname{Spec} R$. We have the pullback diagram

$$\begin{tikzcd}
X_{\bar{K}} \times X'_{\bar{K}} \arrow[r, "\bar{i}''"] \arrow[d, "\pi_{\bar{K}} \times \pi'_{\bar{K}}"'] & X_K \times X'_K \arrow[r, "i''"] \arrow[d, "\pi_K \times \pi'_K"'] & X_R \times_R X'_R \arrow[d, "\pi_R \times_R \pi'_R"] \\
\operatorname{Spec} \bar{K} \arrow[r] & \operatorname{Spec} K \arrow[r] & \operatorname{Spec} R.
\end{tikzcd}$$

Let $\mathcal{E}_0$ be a Fourier–Mukai kernel. By [Mor23, Proposition 3.3, Corollary 4.2] one can deform $\mathcal{E}_0$ to a Fourier–Mukai kernel $\mathcal{E}$ on $X_R \times_R X'_R$. Applying Proposition 4.1 and Lemma 4.4, we obtain the derived equivalence of the geometric generic fibers

$$\Phi_{(i'' \circ \bar{i}'')^* \mathcal{E}} \colon D^b(X_{\bar{K}}) \to D^b(X'_{\bar{K}}).$$

Recall that there are smooth projective versal deformations X_S, X'_S of X_0, X'_0 over the same nonsingular affine $\mathbf{k}$-variety $\operatorname{Spec} S$. We have the following pullback diagram

$$\begin{tikzcd}
X_{\bar{Q}} \times X'_{\bar{Q}} \arrow[r, "\bar{j}''"] \arrow[d, "\pi_{\bar{Q}} \times \pi'_{\bar{Q}}"'] & X_Q \times X'_Q \arrow[r, "j''"] \arrow[d, "\pi_Q \times \pi'_Q"'] & X_S \times_S X'_S \arrow[d, "\pi_S \times_S \pi'_S"] \\
\operatorname{Spec} \bar{Q} \arrow[r] & \operatorname{Spec} Q \arrow[r] & \operatorname{Spec} S.
\end{tikzcd}$$

By [Mor23, Proposition 3.3] one can deform $\mathcal{E}_0$ to a perfect complex $\mathcal{E}_S$ on $X_S \times_S X'_S$. After possible shrinking of $\operatorname{Spec} S$, the relative integral functor $\Phi_{\mathcal{E}_S}$ is an equivalence [Mor23, Theorem 1.1]. Applying Proposition 4.1 and Lemma 4.4, we obtain the derived equivalence of the geometric generic fibers

$$\Phi_{(j'' \circ \bar{j}'')^* \mathcal{E}} \colon D^b(X_{\bar{Q}}) \to D^b(X'_{\bar{Q}}).$$

5.4. **Nonbirationality.** Let X be a smooth proper variety over an algebraically closed field. Recall that the Néron–Severi group $\operatorname{NS} X$ is the quotient of the Picard group $\operatorname{Pic} X$ by the subgroup $\operatorname{Pic}^0 X$ of isomorphism classes of line bundles which are algebraically equivalent to 0. The group $\operatorname{NS} X$ is a finitely generated with rank $\rho(X)$ called the Picard number. We denote by $\operatorname{NS}_{tor} X$ the subgroup of torsion elements, which is known to be a birational invariant.

Lemma 5.3. *If* $\operatorname{NS}_{tor} X_0, \operatorname{NS}_{tor} X'_0$ *are nonisomorphic, then* $X_{\bar{K}}, X'_{\bar{K}}$ *are nonbirational.*

Proof. By [MP12, Proposition 3.6] there is an injection

$$\mathrm{sp}_{\bar{K},\mathbf{k}} \colon \mathrm{NS}\, \mathrm{X}_{\bar{K}} \to \mathrm{NS}\, \mathrm{X}_0$$

whose cokernel is torsion free. In particular, $\mathrm{sp}_{\bar{K},\mathbf{k}}$ is bijective on torsion subgroups. Then $\operatorname{NS}_{tor} X_{\bar{K}}, \operatorname{NS}_{tor} X'_{\bar{K}}$ cannot be isomorphic. □

The same argument yields

Lemma 5.4. *If* $\mathrm{NS}_{tor} X_0, \mathrm{NS}_{tor} X'_0$ *are nonisomorphic, then* $X_{\bar{Q}}, X'_{\bar{Q}}$ *are nonbirational.*

If $\rho(X_0) = \rho(X'_0) = 1$, then X_0, X'_0 are birational if and only if they are isomorphic [BC09, Section 0.5]. Indeed, birational Calabi–Yau manifolds are connected by a sequence of flops [Kaw08]. However, no such flops are possible on neither X_0 nor X'_0, since by $\rho(X_0) = \rho(X'_0) = 1$ all their nonzero nef divisors are ample.

Lemma 5.5. *Assume that* $\rho(X_0) = \rho(X'_0) = 1$ *and* $\deg(X_0) \neq \deg(X'_0)$*. Then* $X_{\bar{K}}, X'_{\bar{K}}$ *are nonbirational. Here, the degree is defined with respect to the unique ample generator of the Picard group.*

Proof. By [MP12, Proposition 3.6] we have $\rho(X_{\bar{K}}) \leq \rho(X'_0) = 1$. There is an ample divisor H on $X_{\bar{K}}$, as it is a projective variety of positive dimension. It follows that H is neither torsion nor algebraically equivalent to 0. Indeed, torsion divisors are numerically trivial and one of the two numerically effective divisors is ample if and only if so is the other. Two algebraically equivalent divisors share the degree. Hence we obtain $\rho(X_{\bar{K}}) = 1$.

Recall that $\deg(X_0)$ is the highest order coefficient of the Hibert polynomial of X_0 multiplied with $(\dim X_0)!$. Since π_R is flat projective, we have $\deg(X_0) = \deg(X_K)$. Let $S(X_K)$ be the homogeneous coordinate ring of X_K and P_{X_K} the Hilbert polynomial of X_K. By definition $P_{X_K}(l)$ are given by $\dim_K S(X_K)_l$ for sufficiently large integers $l \in \mathbb{Z}$. Since $X_{\bar{K}}$ is irreducible, $\dim_K S(X_K)_l$ is stable under the base change $X_{\bar{K}} \to X_K$ along algebraic extension $K \subset \bar{K}$. Thus we obtain $\deg(X_K) = \deg(X_{\bar{K}})$ and $\deg(X_{\bar{K}}) \neq \deg(X'_{\bar{K}})$. □

Lemma 5.6. *Assume that* $\rho(X_0) = \rho(X'_0) = 1$ *and* $\deg(X_0) \neq \deg(X'_0)$*. Then* $X_{\bar{Q}}, X'_{\bar{Q}}$ *are nonbirational. Here, the degree is defined with respect to the unique ample generator of the Picard group.*

Proof. The same argument as above works also here. Assuming that $\mathbf{k} = \mathbb{C}$, we show $\rho(X_{\bar{Q}}) = 1$ in another way. By [Ser56] the morphism $\pi_S : X_S \to \operatorname{Spec} S$ corresponds to a proper submersion of complex manifolds $(\pi_S)_h \colon (X_S)_h \to (\operatorname{Spec} S)_h$. Ehresmann lemma tells us that $(\pi_S)_h$ gives a locally trivial fibration of real manifolds [Ehr52]. In particular, all the fibers of $(\pi_S)_h$ share the differential type and $\mathrm{H}^2((X_s)_h, \mathbb{Z})$ is independent from closed points $s \in \operatorname{Spec} S$. On the other hand, we have $\mathrm{NS}\, X_s \cong \operatorname{Pic} X_s \cong \mathrm{H}^2((X_s)_h, \mathbb{Z})$, as X_s are Calabi–Yau threefolds. By [MP12, Theorem 1.1], we obtain $\rho(X_{\bar{Q}}) = \rho(X_0) = 1$. □

In summary, we obtain

Theorem 5.7. *Let* X_0, X'_0 *be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then the geometric generic fibers* $X_{\bar{K}}, X'_{\bar{K}}$ *of proper effectivizations and that* $X_{\bar{Q}}, X'_{\bar{Q}}$ *of smooth projective versal deformations are respectively derived-equivalent Calabi–Yau manifolds. If, in addition, we have either* $\mathrm{NS}_{tor} X_0 \neq \mathrm{NS}_{tor} X'_0$*, or* $\rho(X_0) = \rho(X'_0) = 1$ *and* $\deg(X_0) \neq \deg(X'_0)$*, then they are respectively nonbirational.*

Remark 5.2. Consider base changes $\pi_B \colon X_B \to B, \pi'_B \colon X'_B \to B$ of π_S, π'_S to a smooth connected $\mathbf{k}$-curve B containing the point s_0. The derived equivalence of X_S, X'_S induces that of X_B, X'_B. If X_0, X'_0 are nonbirational, then by [KT19, Theorem 1.1] the generic fibers of π_B, π'_B must be nonbirational. Hence the function fields of X_B, X'_B are nonisomorphic. Since any field extension gives a faithfully flat module over the original field, also the geometric generic fibers must be nonbirational.

5.5. **Geometric generic Gross–Popescu pair.** In this subsection, we temporarily assume that $\mathbf{k} = \mathbb{C}$. Now, we will consider the case of the Gross–Popescu pair

$$X_0 = V^1_{8,y}, \ X'_0 = V^1_{8,y}/G$$

where $V^1_{8,y}$ is one of the two small resolutions of $V_{8,y}$ and $G = \mathbb{Z}_8 \times \mathbb{Z}_8$ freely acts on $V^1_{8,y}$. Here, $V_{8,y}$ is a 3-dimensional complete intersection in $\mathbb{P}^8$ of four hypersurfaces parametrized by a general point $y \in \mathbb{P}^2$. They are derived-equivalent Calabi–Yau threefolds with $h^{1,1}(X_0) = h^{1,2}(X_0) = 2$ [GP01a]. Since X_0 is simply-connected [GP01b, Theorem 1.4], we have $\mathrm{H}^i(X_0, \mathscr{O}_{X_0}) = 0, \ i = 1, 2$. Although the fundamental group of X'_0 is given by $G \neq 0$, we also have $\mathrm{H}^i(X'_0, \mathscr{O}_{X'_0}) = 0, \ i = 1, 2$ by [PS97, Corollary B]. Then $\mathrm{NS}_{tor} X_0 \neq \mathrm{NS}_{tor} X'_0$, or in this case equivalently $\mathrm{Pic}_{tor} X_0 \neq \mathrm{Pic}_{tor} X'_0$. Indeed, from the exponential sequence it follows $\mathrm{Pic}\, X_0 = \mathrm{H}^2((X_0)_h, \mathbb{Z})$. The torsion part of $\mathrm{H}^2((X_0)_h, \mathbb{Z})$ is $\mathrm{Tor}_1(\mathrm{H}_1((X_0)_h, \mathbb{Z}), \mathbb{C}) = 0$ by the universal coefficient theorem and Van Kampen's theorem. On the other hand, the torsion part of $\mathrm{H}^2((X'_0)_h, \mathbb{Z})$ is $\mathrm{Tor}_1(\mathrm{H}_1((X'_0)_h, \mathbb{Z}), \mathbb{C}) = G$. Thus one can apply Theorem 5.7 to see that $X_{\bar{K}}, X'_{\bar{K}}$ and $X_{\bar{Q}}, X'_{\bar{Q}}$ are respectively nonbirational derived-equivalent Calabi–Yau threefolds.

5.6. **Geometric generic Pfaffian–Grassmannian pair.** Next, we will consider the case of the Pfaffian-Grassmannian pair

$$X_0 = \mathrm{Gr}(2, V_7) \cap \mathbb{P}(W), \ X'_0 = \mathrm{Pf}(4, V_7) \cap \mathbb{P}(W^\perp)$$

where V_7 is a 7-dimensional $\mathbf{k}$-vector space, W is a 14-dimensional general quotient vector space of $\wedge^2 V_7 \twoheadrightarrow W$, and $W^\perp = \mathrm{Coker}(W^\vee \hookrightarrow \wedge^2 V_7^\vee)$. They are derived-equivalent Calabi–Yau threefolds with $\rho(X_0) = \rho(X'_0) = 1$ and $\deg(X_0) \neq \deg(X'_0)$ [BC09, Kuz]. One can apply Theorem 5.7 to see that $X_{\bar{K}}, X'_{\bar{K}}$ and $X_{\bar{Q}}, X'_{\bar{Q}}$ are respectively nonbirational derived-equivalent Calabi–Yau threefolds. Similarly, one obtains another example from Reye congruence and double quintic symmetroid Calabi–Yau threefolds [HT16].

We will study $X_{\bar{Q}}, X'_{\bar{Q}}$ slightly further. Assume that $\mathbf{k}$ is a universal domain. Let $L \subset \mathbf{k}$ be a subfield over which X_0, X'_0 are defined, i.e., we are given L-varieties X_L, X'_L such that $X_L \times_L \mathbf{k} \cong X_0, \ X'_L \times_L \mathbf{k} \cong X'_0$. From the proof of [Via13, Lemma 2.1] there exists an isomorphism $\mathbf{k} \to \bar{Q}$ fixing L such that the base change of very general fibers X_s, X'_s of X_S, X'_S are respectively isomorphic to $X_{\bar{Q}}, X'_{\bar{Q}}$. Note that by [Mor23, Lemma 5.1] and [KK16, Corollary 6.3], general fibers of their smooth projective versal deformations X_S, X'_S are isomorphic to X_0, X'_0. The isomorphisms $X_s \cong X_0, \ X'_s \cong X'_0$ of $\mathbf{k}$-varieties induces that

$$X_0 \cong X_{\bar{Q}}, \ X'_0 \cong X'_{\bar{Q}}$$

of schemes.

There is another Fourier–Mukai partner called IMOU varieties [IMOU, Kuz18], consisting of derived-equivalent Calabi–Yau threefolds Y_0, Y'_0 which are deformation equivalent to X_0, X'_0 respectively [KK16, IIM19]. Extending L if necessary, we may assume that also Y_0, Y'_0 are defined over L. Then either of them fails to be isomorphic to $X_{\bar{Q}}, X'_{\bar{Q}}$ as a scheme, otherwise we would have $X_L \cong Y_L$ and $X'_L \cong Y'_L$. Hence Y_0, Y'_0 cannot be isomorphic $X_{\bar{Q}}, X'_{\bar{Q}}$ at the same time even as a scheme. Thus $X_{\bar{Q}}, X'_{\bar{Q}}$ provide a new example of Fourier–Mukai partners. They are isomorphic to X_0, X'_0 as a scheme, but have different structures from both the Pfaffian–Grassmannian pair and IMOU varieties as a variety.

Remark 5.3. General fibers of smooth projective versal deformations of Y_0, Y'_0 are isomorphic to X_0, X'_0 as a $\mathbf{k}$-varieties. This implies that one can obtain $X_{\bar{Q}}, X'_{\bar{Q}}$ starting from IMOU varieties. Moreover, one sees that $X_{\bar{Q}}, X'_{\bar{Q}}$ are nonbirational, otherwise X_0, X'_0 must be birational.

As in this case, when deformations of Fourier–Mukai partners are well-understood, one can deduce nonbirationality of the geometric generic fibers immediately. For instance, GPK^3 threefolds [BCP20] are isomorphic to general fibers of smooth projective versal deformations of Kapustka–Rampazzo varieties [KR19] as a **k**-varieties [IIM19, Proposition 4.7]. Then by the same argument the geometric generic fibers are nonbirational and one obtains another Fourier–Mukai partner. Note that for Pfaffian-Grassmannian pair and GPK^3 threefolds, the derived equivalences of the geometric generic fibers stems from birationality of noncompact Calabi–Yau manifolds connected by simple K-flops [Ued19, Mor21].

Remark 5.4. In this case $X_{\bar{Q}}, X'_{\bar{Q}}$ are also $\mathbb{L}$-equivalent. Indeed, the isomorphism $\mathbf{k} \to \bar{Q}$ induces that $K_0(\mathrm{Var}_{\mathbf{k}}) \to K_0(\mathrm{Var}_{\bar{Q}})$ of Grothendieck rings mapping the relation in [Mar16, Theorem 1.1] to

$$([X_{\bar{Q}}] - [X'_{\bar{Q}}]) \cdot \mathbb{L}^6_{\bar{Q}} = 0.$$

6. Specialization of derived equivalence

As advertised, for flat proper families of **k**-varieties over a common base we study the induced derived equivalence from their generic to general fibers. The key is the ability of Corollary 2.6 to lift Fourier–Mukai kernels along the projection. First, although it is not strictly necessary for our purpose, we extend Corollary 2.6 to nonaffine base case for flat proper families of **k**-varieties. It suffices to show that, when restricted to general fibers, the relative integral functor defined by the lift admits fully faithful left adjoints. As in the proof of [Mor23, Theorem 1.1], we show that the associated counit morphism is a natural isomorphism. However, since in general the generic finer is not a subscheme of a family, we have to adapt the proof as follows. Shrinking the base, we remove torsion parts with respect to the base from a fixed strong generator and its relevant Hom-sets. Then we invoke some basic categorical results to show that the value of the counit morphism on the trimmed strong generator is an isomorphism, which implies that the restriction of the counit morphism is a natural isomorphism.

6.1. **Lifts of Fourier–Mukai kernels.** Let $\pi\colon X \to S$ be a flat proper morphism of **k**-varieties. Since S is integral, the function field $K = k(S)$ is given by local ring $\mathscr{O}_{X,\xi}$ and it coincides with the field of fractions $Q(R)$ for any affine open **k**-subvariety $U = \operatorname{Spec} R$ [Har77, Exercise II3.6]. Hence we have the following pullback diagram

$$\begin{tikzcd} X_\xi \arrow[r, "\bar{\iota}_\xi"] \arrow[d, "\pi_\xi"'] & X \arrow[d, "\pi"] \\ \operatorname{Spec} K \arrow[r, "\iota_\xi"'] & S \end{tikzcd}$$

where ι_ξ is the canonical morphism.

Definition 6.1. The *categorical generic fiber* of $\pi\colon X \to S$ is the Verdier quotient

$$D^b(X)/\operatorname{Ker}(\bar{\iota}^*_\xi),$$

where $\operatorname{Ker}(\bar{\iota}^*_\xi)$ is the kernel [SP, Tag 05RF] of the exact functor $\bar{\iota}^*_\xi\colon D^b(X) \to D^b(X_\xi)$.

Recall that for a smooth separated family $\pi_R\colon X_R \to \operatorname{Spec} R$ over a nonsingular affine **k**-variety, by Corollary 2.6 there exists a $Q(R)$-linear exact equivalence

$$D^b(X_R)/D^b_0(X_R) \simeq D^b(X_\xi),$$

where $D^b_0(X_R)$ is the full triangulated subcategory spanned by complexes with R-torsion cohomology. The above definition is an extension of this local description in the following sense.

Theorem 6.1. *Let $\pi\colon X \to S$ be a flat proper morphism of* $\mathbf{k}$*-varieties. Then there exists a K-linear exact equivalence*

$$D^b(X)/\operatorname{Ker}(\overline{\iota}_\xi^*) \simeq D^b(X_\xi).$$

Proof. Let $[\overline{\iota}_\xi^*]\colon D^b(X)/\operatorname{Ker}(\overline{\iota}_\xi^*) \to D^b(X_\xi)$ be the unique functor which makes the diagram

$$\begin{tikzcd} D^b(X) \arrow[r, "\overline{\iota}_\xi^*"] \arrow[dr, "Q"'] & D^b(X_\xi) \\ & D^b(X)/\operatorname{Ker}(\overline{\iota}_\xi^*) \arrow[u, "{[\overline{\iota}_\xi^*]}"'] \end{tikzcd} \tag{6.1}$$

commute, where $Q\colon D^b(X) \to D^b(X)/\operatorname{Ker}(\overline{\iota}_\xi^*)$ is the quotient functor. Take any affine open subset $U = \operatorname{Spec} R \subset S$. Let $\pi_U\colon X_U \to U$ and $\pi_Z\colon X_Z \to Z$ be the base changes to U and its complement $Z = S \setminus U$ respectively. We have

$$\operatorname{coh}(X_U) \simeq \operatorname{coh}(X)/\operatorname{coh}_Z(X)$$

where the right hand side is the Serre quotient by the Serre subcategory $\operatorname{coh}_Z(X) \subset \operatorname{coh}(X)$ of sheaves supported on X_Z. Passing to the derived category, via [Miy91, Theorem 3.2] we obtain U-linear exact equivalence

$$D^b(X_U) \simeq D^b(X)/D^b_Z(X)$$

where $D^b_Z(X) \subset D^b(X)$ is the full S-linear triangulated subcategory with cohomology supported on X_Z. Since $D^b_Z(X)$ is contained in $\operatorname{Ker}(\overline{\iota}_\xi^*)$, the commutative diagram (6.1) extends to

$$\begin{tikzcd} D^b(X) \arrow[r, "\overline{\iota}_\xi^*"] \arrow[d, "\overline{\iota}_U^*"'] \arrow[dr, "Q"] & D^b(X_\xi) \\ D^b(X_U) \arrow[r, "{[Q]}"'] & D^b(X)/\operatorname{Ker}(\overline{\iota}_\xi^*). \arrow[u, "{[\overline{\iota}_\xi^*]}"'] \end{tikzcd}$$

On the other hand, the inclusion $D^b_Z(X) \subset \operatorname{Ker}(\overline{\iota}_\xi^*)$ induces a commutative diagram

$$\begin{tikzcd} D^b(X) \arrow[r, "\overline{\iota}_\xi^*"] \arrow[d, "\overline{\iota}_U^*"'] & D^b(X_\xi) \\ D^b(X_U) \arrow[ur, "Q_R"'] & \end{tikzcd} \tag{6.2}$$

where $Q_R\colon D^b(X_U) = D^b(X_R) \to D^b(X_\xi) \simeq D^b(X_R)/D^b_0(X_R)$ is the quotient functor. Note that shrinking U if necessary, we may assume that π_R is smooth in order to apply Corollary 2.6. Indeed, by [Har77, Theorem I5.3] the singular locus of X_R is a proper closed subset, whose image under the flat proper morphism π_R is proper closed subset of $\operatorname{Spec} R$. Changing the base to its complement, we may assume that X_R is nonsingular. Then one can apply [Har77, Corollary III10.7] to find an open subset of $\operatorname{Spec} R$ over which the restriction of π becomes smooth. Since $D^b_0(X_R)$ is contained in $\operatorname{Ker}([Q])$, the commutative diagram (6.2) extends to

$$\begin{tikzcd} D^b(X) \arrow[r, "\overline{\iota}_\xi^*"] \arrow[d, "\overline{\iota}_U^*"'] & D^b(X_\xi) \arrow[d, "{[[Q]]}"] \\ D^b(X_U) \arrow[r, "{[Q]}"'] \arrow[ur, "Q_R"'] & D^b(X)/\operatorname{Ker}(\overline{\iota}_\xi^*) \end{tikzcd}$$

with unique $[[Q]]$. Thus we obtain a commutative diagram

$$\begin{tikzcd} D^b(X) \arrow[r, "Q"] \arrow[dr, "Q"'] & D^b(X)/\operatorname{Ker}(\bar{\iota}_\xi^*) \\ & D^b(X)/\operatorname{Ker}(\bar{\iota}_\xi^*). \arrow[u, "{[[Q]]\circ[\bar{\iota}_\xi^*]}"'] \end{tikzcd}$$

By universality of Verdier quotient, the composition $[[Q]] \circ [\bar{\iota}_\xi^*]$ is natural isomorphic to the identity functor. Hence $[\bar{\iota}_\xi^*]$ is an equivalence. □

Remark 6.1. The above theorem is a direct consequence of [Miy91, Theorem 3.2] and the K-linear equivalence

$$\operatorname{coh}(X)/\operatorname{Ker}(\bar{\iota}_\xi^*) \simeq \operatorname{coh}(X_\xi),$$

which can be deduced by the similar argument. Here, we use the same symbol $\operatorname{Ker}(\bar{\iota}_\xi^*)$ to denote the kernel [SP, Tag 02MR] of the exact functor $\bar{\iota}_\xi^*$ of abelian categories. In particular, Theorem 2.5 also extends to nonaffine base case for flat proper families of $\mathbf{k}$-varieties.

Corollary 6.2. *Let $\pi\colon X \to S$ be a flat proper morphism of $\mathbf{k}$-varieties. Then any object of $D^b(X_\xi)$ can be lifted to that of $D^b(X)$ along the projection Q.*

6.2. **Basic categorical results.**

Lemma 6.3. *Let $\mathscr{C}, \mathscr{D}$ be small categories and $F\colon \mathscr{C} \to \mathscr{D}, G\colon \mathscr{D} \to \mathscr{C}$ functors with $F \dashv G$. Assume that there exists an object $D \in \mathscr{D}$ such that the canonical maps*

$$\operatorname{Hom}(D, D) \to \operatorname{Hom}(G(D), G(D)),\ \operatorname{Hom}(D, FG(D) \to \operatorname{Hom}(G(D), GFG(D))$$

are bijective. Then the counit morphism $\epsilon\colon FG \Rightarrow 1_{\mathscr{D}}$ induces an isomorphism $\epsilon_D\colon FG(D) \to D$.

Proof. Let $\alpha_D, \alpha_{FG(D)}$ be the compositions

$$\operatorname{Hom}(D, D) \to \operatorname{Hom}(G(D), G(D)) \to \operatorname{Hom}(FG(D), D),$$
$$\operatorname{Hom}(D, FG(D) \to \operatorname{Hom}(G(D), GFG(D)) \to \operatorname{Hom}(FG(D), FG(D))$$

of the canonical maps. By assumption and definition of adjoint functors $\alpha_D, \alpha_{FG(D)}$ are bijective. We denote $\epsilon_D = \alpha_D(1_D)$ by f and $\alpha_{FG(D)}^{-1}(1_{FG(D)})$ by g. Consider the diagrams

$$\begin{tikzcd} \operatorname{Hom}(D, D) \arrow[r, "\alpha_D"] \arrow[d, "g\circ"'] & \operatorname{Hom}(FG(D), D) \arrow[d, "g\circ"] \\ \operatorname{Hom}(D, FG(D)) \arrow[r, "\alpha_{FG(D)}"'] & \operatorname{Hom}(FG(D), FGD), \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Hom}(D, FG(D)) \arrow[r, "\alpha_{FG(D)}"] \arrow[d, "f\circ"'] & \operatorname{Hom}(FG(D), FGD) \arrow[d, "f\circ"] \\ \operatorname{Hom}(D, D) \arrow[r, "\alpha_D"'] & \operatorname{Hom}(FG(D), D), \end{tikzcd}$$

which are commutative by definition of functors and naturality of adjoints. As expressions of the images of $1_D, g$ we respectively obtain

$$1_{FG(D)} = \alpha_{FG(D)}(\alpha_{FG(D)}^{-1}(1_{FG(D)})) = \alpha_{FG(D)}(g) = g(\alpha_D(1_D)) = gf,$$
$$\alpha_D(fg) = f(\alpha_{FG(D)}(g)) = f(\alpha_{FG(D)}(\alpha_{FG(D)}^{-1}(1_{FG(D)}))) = f.$$

From the second line it follows $fg = \alpha_D^{-1}(f) = 1_D$. Hence f is an isomorphism. □

Remark 6.2. The above proof is just an adaptation of the proof of the fact that G is fully faithful if and only if ϵ is a natural isomorphism.

Similarly, one can prove the dual statement.

Lemma 6.4. *Let $\mathscr{C}, \mathscr{D}$ be small categories and $F: \mathscr{C} \to \mathscr{D}, G: \mathscr{D} \to \mathscr{C}$ functors with $F \dashv G$. Assume that there exists an object $C \in \mathscr{C}$ such that the canonical maps*

$$\mathrm{Hom}(C, C) \to \mathrm{Hom}(F(D), F(D)),\ \mathrm{Hom}(GF(C), C) \to \mathrm{Hom}(FGF(C), F(C))$$

are bijective. Then the unit morphism $\eta: 1_{\mathscr{C}} \Rightarrow GF$ induces an isomorphism $\eta_C: C \to GF(C)$.

Remark 6.3. Again, the proof is just an adaptation of the proof of the fact that F is fully faithful if and only if η is a natural isomorphism.

6.3. **Removal of torsion supports.**

Lemma 6.5. *Let $\pi_R: X \to \operatorname{Spec} R, \pi'_R: X' \to \operatorname{Spec} R$ be a smooth proper morphisms to a nonsingular affine* **k***-variety. Assume that their generic fibers X_K, X'_K are derived-equivalent. Let $\Phi_{\mathcal{E}_K}: D^b(X_K) \to D^b(X'_K)$ be a Fourier–Mukai functor giving the equivalence with kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$. Fix a lift $\mathcal{E} \in D^b(X \times_R X')$ of $\mathcal{E}_K$ along the projection*

$$D^b(X \times_R X') \to D^b(X \times_R X')/D^b_0(X \times_R X') \simeq D^b(X_K \times X'_K)$$

and a strong generator E of $D^b(X)$. Then there exists an affine open subset $U \subset \operatorname{Spec} R$ over which the restriction

$$\Phi_U = \Phi_{\mathcal{E}_U}: D^b(X_U) \to D^b(X'_U)$$

induces bijections

$$\begin{aligned} &\mathrm{Hom}(E_U, E_U) \to \mathrm{Hom}(\Phi_U(E_U), \Phi_U(E_U)), \\ &\mathrm{Hom}(E_U, \Phi_U^L\Phi_U(E_U)) \to \mathrm{Hom}(\Phi_U(E_U), \Phi_U\Phi_U^L\Phi_U(E_U)), \end{aligned} \tag{6.3}$$

where $\Phi_U^L: D^b(X'_U) \to D^b(X_U)$ is the left adjoint to Φ_U.

Proof. By [BV03, Theorem 2.1.2, Lemma 3.4.1] the base change E_K of E along the canonical inclusion $R \hookrightarrow K$ is a strong generator of $D^b(X_K)$, which can not be trivial. Consider the support of the R-torsion part of E, i.e., the union $\bigcup_i \operatorname{supp} \mathcal{H}^i(E)_{tors}$ of the supports of the R-torsion parts $\mathcal{H}^i(E)_{tors}$ of $\mathcal{H}^i(E)$. Each $\mathcal{H}^i(E)_{tors}$ is a coherent sheaf on X, as every submodule of a finitely presented module over a noetherian ring is finitely presented. Since the union is finite, $\bigcup_i \operatorname{supp} \mathcal{H}^i(E)_{tors}$ is a closed subset of X. Its complement must contain the generic point of X, otherwise E_K is trivial. Let $U \subset \operatorname{Spec} R$ be the image of the complement under π_R, which is a nonempty open subset. By construction over U the restriction $E_U = E|_{\pi_U^{-1}(U)}$ is $\mathscr{O}_U(U)$-torsion free. Since $\mathrm{Hom}(E_U, E_U)$ is coherent as an $\mathscr{O}_U(U)$-module by Lemma 6.6 below, shrinking U if necessary, we may assume that it is $\mathscr{O}_U(U)$-torsion free. The tensor product $\Phi_U(E_U) \otimes_{\mathscr{O}_U(U)} K$ can not be trivial, otherwise it does not map to an object quasi-isomorphic to E_K under Φ_K^{-1}. Hence by the same argument one finds an affine open subset $V \subset \operatorname{Spec} R$ over which the restriction $\Phi_V(E_V) \cong \Phi_{\mathcal{E}}(E)|_{\pi'^{-1}_V(V)}$ and $\mathrm{Hom}(\Phi_V(E_V), \Phi_V(E_V))$ are $\mathscr{O}_V(V)$-torsion free. The intersection $U \cap V \subset \operatorname{Spec} R$ is a nonempty open subset, as $\operatorname{Spec} R$ is integral. Now, we replace $U \cap V$ with U.

Consider a map

$$\mathrm{Hom}(E_U, E_U) \xrightarrow{\Phi_U} \mathrm{Hom}(\Phi_U(U), \Phi_U(E_U)) \tag{6.4}$$

of $\mathscr{O}_U(U)$-modules. By assumption and Lemma 2.2 tensoring K yields an exact sequence

$$0 \to \mathrm{Hom}(E_U, E_U) \otimes_{\mathscr{O}_U(U)} K \xrightarrow{\Phi_U \otimes_{\mathscr{O}_U(U)} K} \mathrm{Hom}(\Phi_U(U), \Phi_U(E_U)) \otimes_{\mathscr{O}_U(U)} K \to 0. \tag{6.5}$$

Hence the map Φ_U in (6.4) is injective, as $\mathrm{Hom}(E_U, E_U)$ is $\mathscr{O}_U(U)$-torsion free. The associated sheaf with the cokernel

$$\mathrm{Hom}(\Phi_U(E_U), \Phi_U(E_U)) / \mathrm{Hom}(E_U, E_U)$$

might have nontrivial $\mathscr{O}_U(U)$-torsion part. However, since by Lemma 6.6 below it is coherent, one finds an affine open subset $W \subset U$ to which the restriction of the associated sheaf is $\mathscr{O}_W(W)$-torsion free. Now, we replace W with U. Then the exactness of (6.5) implies that of (6.4). Hence the map Φ_U in (6.4) is bijective. Shrinking U if necessary, by the same argument we conclude that

$$\operatorname{Hom}(E_U, \Phi_U^L \Phi_U(E_U)) \xrightarrow{\Phi_U} \operatorname{Hom}(\Phi_U(E_U), \Phi_U \Phi_U^L \Phi_U(E_U))$$

is also bijective. □

Remark 6.4. For our purpose, we do not need the generator E to be strong. Nevertheless, we put the adjective "strong" as there always exists a strong generator of $D^b(X)$ under the assumption.

Remark 6.5. Note that the base change

$$(\operatorname{Hom}(\Phi_U(E_U), \Phi_U(E_U))/\operatorname{Hom}(E_U, E_U)) \otimes_{\mathscr{O}_U(U)} \mathscr{O}_W(W) \tag{6.6}$$

is isomorphic to the cokernel of the sequence

$$0 \to \operatorname{Hom}(E_W, E_W) \xrightarrow{\Phi_W} \operatorname{Hom}(\Phi_W(W), \Phi_W(E_W)) \to 0.$$

Indeed, (6.6) is isomorphic to

$$\operatorname{Hom}(\Phi_U(E_U), \Phi_U(E_U)) \otimes_{\mathscr{O}_U(U)} \mathscr{O}_W(W)/\operatorname{Hom}(E_U, E_U) \otimes_{\mathscr{O}_U(U)} \mathscr{O}_W(W)$$

as the pullback by an open immersion is exact. Consider the pullback diagrams

$$\begin{tikzcd} X_W \arrow[r, hook, "\bar{\iota}"] \arrow[d, "\pi_W"'] & X_U \arrow[d, "\pi_U"] \\ W \arrow[r, hook, "\iota"'] & U, \end{tikzcd} \qquad \begin{tikzcd} X'_W \arrow[r, hook, "\bar{\iota}'"] \arrow[d, "\pi'_W"'] & X'_U \arrow[d, "\pi'_U"] \\ W \arrow[r, hook, "\iota"'] & U. \end{tikzcd}$$

By the derived flat base change we have

$$\iota^* R\operatorname{Hom}^\bullet(E_U, E_U) \cong \iota^* R\pi_{U*} R\underline{\operatorname{Hom}}^\bullet(E_U, E_U) \cong R\operatorname{Hom}^\bullet(E_W, E_W),$$

$$\iota'^* R\operatorname{Hom}^\bullet(\Phi_U(E_U), \Phi_U(E_U)) \cong \iota'^* R\pi'_{U*} R\underline{\operatorname{Hom}}^\bullet(\Phi_U(E_U), \Phi_U(E_U)) \cong R\operatorname{Hom}^\bullet(\Phi_W(E_W), \Phi_W(E_W)).$$

Passing to the 0-th cohomology of complexes, we obtain

$$\operatorname{Hom}(E_U, E_U) \otimes_{\mathscr{O}_U(U)} \mathscr{O}_W(W) \cong \operatorname{Hom}(E_W, E_W),$$

$$\operatorname{Hom}(\Phi_U(E_U), \Phi_U(E_U)) \otimes_{\mathscr{O}_U(U)} \mathscr{O}_W(W) \cong \operatorname{Hom}(\Phi_W(E_W), \Phi_W(E_W))$$

as ι^*, ι'^* are exact.

Lemma 6.6. *Let $\pi_R \colon X \to \operatorname{Spec} R$ be a smooth proper morphism to a nonsingular affine* **k**-*variety. Then for any object $E, F \in D^b(X)$ the R-module* $\operatorname{Hom}(E, F) = \operatorname{Ext}^0_X(E, F)$ *is coherent.*

Proof. Consider the spectral sequences

$$E_2^{p,q} = \operatorname{Ext}^p_X(E, \mathcal{H}^q(F)) \Rightarrow \operatorname{Ext}^{p+q}_X(E, F),$$

$$E_2^{p,q} = \operatorname{Ext}^p_X(\mathcal{H}^{-q}(E), F) \Rightarrow \operatorname{Ext}^{p+q}_X(E, F),$$

$$E_2^{p,q} = H^p(X, \underline{\operatorname{Ext}}^q_X(E, F)) \Rightarrow \operatorname{Ext}^{p+q}_X(E, F),$$

from [Huy06, Example 2.70, Compatibilities(v)]. Applying the first two, we may assume that E, F are coherent sheaves on X. Then $\underline{\operatorname{Ext}}^q_X(E, F)$ is coherent. Since π_R is proper, $R^p\pi_{R*}\underline{\operatorname{Ext}}^q_X(E, F)$ is also coherent, which is isomorphic to the associated sheaf on $\operatorname{Spec} R$ with $H^p(X, \underline{\operatorname{Ext}}^q_X(E, F))$ [Har77, Proposition III8.5]. In the decreasing filtration of $\operatorname{Ext}^0_X(E, F)$, the smallest nontrivial submodule is isomorphic to $E_\infty^{l,-l}$ for some $l \in \mathbb{Z}$, which is a coherent R-module. Ascending the filtration, one sees that $\operatorname{Ext}^0_X(E, F)$ is coherent by two out of three principle. □

Similarly, one can prove the dual statement.

Lemma 6.7. *Let $\pi_R\colon X \to \operatorname{Spec} R, \pi'_R\colon X' \to \operatorname{Spec} R$ be a smooth proper morphisms to a nonsingular affine* **k***-variety. Assume that their generic fibers X_K, X'_K are derived-equivalent. Let $\Phi_{\mathcal{E}_K}\colon D^b(X_K) \to D^b(X'_K)$ be a Fourier–Mukai functor giving the equivalence with kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$. Fix a lift $\mathcal{E} \in D^b(X \times_R X')$ of $\mathcal{E}_K$ along the projection*

$$D^b(X \times_R X') \to D^b(X \times_R X')/D^b_0(X \times_R X') \simeq D^b(X_K \times X'_K)$$

and a strong generator E' of $D^b(X')$. Then there exists an affine open subset $U \subset \operatorname{Spec} R$ over which the restriction

$$\Phi^L_U = \Phi^L_{\mathcal{E}_U}\colon D^b(X'_U) \to D^b(X_U)$$

induces bijections

$$\begin{gathered}\operatorname{Hom}(E'_U, E'_U) \to \operatorname{Hom}(\Phi^L_U(E'_U), \Phi^L_U(E'_U)),\\ \operatorname{Hom}(\Phi_U\Phi^L_U(E'_U), E'_U) \to \operatorname{Hom}(\Phi^L_U\Phi_U\Phi^L_U(E'_U), \Phi^L_U(E'_U)),\end{gathered} \tag{6.7}$$

where $\Phi_U\colon D^b(X_U) \to D^b(X'_U)$ is the right adjoint to Φ^L_U.

6.4. **Specialization.**

Theorem 6.8. *Let $\pi_R\colon X \to \operatorname{Spec} R, \pi'_R\colon X' \to \operatorname{Spec} R$ be smooth proper morphisms to a nonsingular affine* **k***-variety. Assume that their generic fibers X_K, X'_K are derived-equivalent. Let $\Phi_{\mathcal{E}_K}\colon D^b(X_K) \to D^b(X'_K)$ be a Fourier–Mukai functor giving the equivalence with kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$. Fix a lift $\mathcal{E} \in D^b(X \times_R X')$ of $\mathcal{E}_K$ along the projection*

$$D^b(X \times_R X') \to D^b(X \times_R X')/D^b_0(X \times_R X') \simeq D^b(X_K \times X'_K).$$

Then there exists an affine open subset $U \subset \operatorname{Spec} R$ over which the restriction

$$\Phi_U = \Phi_{\mathcal{E}|_{\mathrm{pr}_1^{-1}\circ\pi_U^{-1}(U)}}\colon D^b(X_U) \to D^b(X'_U)$$

become an $\mathcal{O}_U(U)$-linear exact equivalence. In particular, over U any pair of closed fibers are derived-equivalent.

Proof. The proof is an adaptation of the argument in the proof of [Mor23, Theorem 1.1]. Fix a strong generator E of $D^b(X)$. The counit morphism $\epsilon\colon \Phi^L_{\mathcal{E}} \circ \Phi_{\mathcal{E}} \to \mathrm{id}_{D^b(X)}$ gives a distinguished triangle

$$\Phi^L_{\mathcal{E}} \circ \Phi_{\mathcal{E}}(E) \xrightarrow{\epsilon_E} E \to F := \operatorname{Cone}(\epsilon(E)). \tag{6.8}$$

Over any open subset $U \subset \operatorname{Spec} R$, (6.8) restricts to a distinguished triangle

$$\Phi^L_U \circ \Phi_U(E_U) \xrightarrow{\epsilon_{E_U}} E_U \to F_U.$$

Note that the restriction of the counit morphism is the counit morphism. Choose U so that we have the bijections (6.3) from Lemma 6.5. Then by Lemma 6.3 the counit morphism ϵ_{E_U} on E_U is an isomorphism. Since E_U is a strong generator of $D^b(X_U)$ by [BV03, Theorem 2.1.2, Lemma 3.4.1], this implies that Φ_U is fully faithful. Recall that a triangulated category is strongly finitely generated if there exist an object E_U and a nonnegative integer k such that every object can be obtained from E_U by taking isomorphisms, finite direct sums, direct summands, shifts, and not more than k times cones. Now, we may assume that E_U has no nontrivial direct summands, as Φ_U and Φ^L_U commute with direct sums on $D^b(X_U)$ by [BV03, Corollary 3.3.4]. Since ϵ_{E_U} is an isomorphism, one inductively sees that over U the counit morphism on any object is an isomorphism.

Fix a strong generator E' of $D^b(X')$. The unit morphism $\eta\colon \mathrm{id}_{D^b(X')} \to \Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}}^L$ gives a distinguished triangle

$$E' \xrightarrow{\eta_{E'}} \Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}}^L(E') \to F' := \mathrm{Cone}(\eta_{E'}). \tag{6.9}$$

Over any open subset $U \subset \operatorname{Spec} R$, (6.9) restricts to a distinguished triangle

$$E'_U \xrightarrow{\eta_{E'_U}} \Phi_U \circ \Phi_U(E'_U) \to F'_U.$$

Note that the restriction of the unit morphism is the unit morphism. Choose U so that we have the bijections (6.7) from Lemma 6.7. Then by Lemma 6.4 the unit morphism $\eta_{E'_U}$ on E'_U is an isomorphism. Since E'_U is a strong generator of $D^b(X')$ by [BV03, Theorem 2.1.2, Lemma 3.4.1], this implies that Φ_U^L is fully faithful. Shrinking U if necessary, we may assume that over U both Φ_U and Φ_U^L are fully faithful. Then Φ_U is an equivalence, as a fully faithful functor which admits a fully faithful left adjoint is an equivalence. □

Remark 6.6. If $\Phi_{\mathcal{E}}$ induces the derived equivalence of a single pair of closed fibers, then there exists a Zariski open subset $U \subset \operatorname{Spec} R$ such that the base changes X_U, X'_U are derived-equivalent. This follows from the proof of [Mor23, Theorem 1.1], which exploits the fact that the restriction of the counit morphism $\epsilon_E\colon \Phi_{\mathcal{E}}^L \circ \Phi_{\mathcal{E}}(E) \to E$ to any closed fiber is the counit morphism for each object $E \in D^b(X)$. However, it does not work for the generic fiber. In general, the generic fiber is not a subscheme of X_R, while any closed fiber can naturally be regarded as a subscheme of X_R via the reduced induced structure on the image of the closed immersion.

Corollary 6.9. *Let $\pi\colon X \to S, \pi'\colon X' \to S$ be flat proper morphisms of* $\mathbf{k}$*-varieties. Assume that their generic fibers X_K, X'_K are derived-equivalent. Then there exists an open subset $U \subset S$ to which base changes X_U, X'_U become derived-equivalent. In particular, over U any pair of closed fibers are derived-equivalent.*

Proof. By [Har77, Theorem I5.3] the singular locus of X, X' are proper closed subsets, whose images under flat proper morphisms π, π' are proper closed subsets of S. Changing the base to the complement of their union, we may assume that X, X' are nonsingular. Then one can apply [Har77, Corollary III10.7] to find an open subset of S over which the restrictions of π, π' become smooth. Hence we may assume further that π, π' are smooth.

Let $\Phi_{\mathcal{E}_K}\colon D^b(X_K) \to D^b(X'_K)$ be a Fourier–Mukai functor giving the equivalence with kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$. Fix a lift $\mathcal{E} \in D^b(X \times_S X')$ of $\mathcal{E}_K$ along the projection

$$D^b(X \times_S X') \to D^b(X \times_S X')/D^b_0(X \times_S X') \simeq D^b(X_K \times X'_K).$$

Take an affine open cover $\bigcup_{i=1}^N \operatorname{Spec} R_i$ of S. One can apply Theorem 6.8 to find open subsets $U_i \subset \operatorname{Spec} R_i$ over which the restrictions

$$\Phi_{U_i} = \Phi_{\mathcal{E}_{U_i}}\colon D^b(X_{U_i}) \to D^b(X'_{U_i})$$

become $\mathscr{O}_{U_i}(U_i)$-linear exact equivalences. Let $V = \bigcup_{i=1}^N U_i$ be their union, which is an open $\mathbf{k}$-subvariety of S. Consider the restriction

$$\Phi_V = \Phi_{\mathcal{E}_V}\colon D^b(X_V) \to D^b(X'_V)$$

over V. Since its restriction to any pair of closed fibers over V defines an equivalence, Φ_V is an equivalence by [HLS09, Proposition 2.15]. □

References

[Art69] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math-Paris. 36, 23-58 (1969).

[BC09] L. Borisov and A. Căldăraru, *The Pfaffian–Grassmannian derived equivalence*, J. Algebraic. Geom. 18(2), 201-222 (2009).

[BCP20] L. Borisov, A. Căldăraru, and A. Perry, *Intersections of two Grassmannians in* $\mathbb{P}^9$, J. Reine. Angew. Math. 2020(760), 133-162 (2020).

[BFN10] D. Ben-zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. 23(4), 909-966 (2010).

[BV03] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and non-commutative geometry*, Mosc. Math. J. 3(1), 1-36 (2003).

[CP21] J. Calabrese and R. Pirisi, *Gabriel's theorem and birational geometry*, Proc. Amer. Math. Soc. 149(3), 907-922 (2021).

[Coh] L. Cohn, *Differential graded categories are k-linear stable infinity categories*, arXiv:1308.2587

[Ehr52] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré differéntiable*, Séminaire Bourbaki : années 1948/49-1949/50-1950/51, exposés 1-49, Séminaire Bourbaki no. 1(1952).

[Gab62] P. Gabriel, *Des catégories abéliennes*, Bull. Math. Soc. France. 90, 323-448 (1962).

[GD61] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: III. Étude cohomologique des faisceaux cohérent, première partie*, Publ. Math-Paris. 11, 5-167 (1961).

[GD66] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie*, Publ. Math-Paris. 28, 5-255 (1966).

[GP01a] M. Gross and S. Popescu, *Calabi–Yau threefolds and moduli of abelian surfaces I*, Compositio Math. 127(2), 169-228 (2001).

[GP01b] M. Gross and S. Pavanelli, *A Calabi–Yau threefold with Brauer–Group* $(\mathbb{Z}/8\mathbb{Z})^2$, Proc. Amer. Math. Soc. 136(1), 1-9 (2001).

[Har77] R Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, (1977), ISBN: 0-387-90244-9.

[HLS09] D Hernández Ruipérez, A.C. López Martín, and D. Sánchez Gómez *Relative integral functors for singular fibrations and singular partners*, J. Eur. Math. Soc. 11, 597-625 (2009).

[HMS09] D. Huybrechts, E. Macrì, and P. Stellari, *Derived equivalences of K3 surfaces and orientation*, Duke Math. 149(3), 461-507 (2009).

[HMS11] D. Huybrechts, E. Macrì, and P. Stellari, *Formal deformations and their categorical general fibre*, Comment. Math. Helv. 86(1), 41-71 (2011).

[HT16] S. Hosono and H. Takagi, *Double quintic symmetroids, Reye congruences, and their derived equivalence*, J. Differ. Geom. 104(3), 443–497, (2016).

[Huy06] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Science Publications, (2006) ISBN: 978-0-19-929686-6.

[IIM19] D. Inoue, A. Ito, and M. Miura, *Complete intersection Calabi–Yau manifolds with respect to homogeneous vector bundles on Grassmannians*, Math. Z. 292(1-2), 677-703 (2019).

[IMOU] A. Ito, M. Miura, S. Okawa, and K. Ueda, *Calabi–Yau complete intersections in exceptional Grassmannians*, arXiv:1606.04076

[Kaw08] Y. Kawamata, *Flops connect minimal models*, Publ. RIMS, Kyoto Univ. 44, 419-423 (2008).

[KK16] G. Kapustka and M. Kapustka, *Calabi–Yau threefolds in* $\mathbb{P}^6$, Ann. Mat. Pur. Appl. 195(2), 529-556 (2016).

[KR19] M. Kapustka and M. Rampazzo, *Torelli problem for Calabi–Yau threefolds with GLSM description*, Commun. Num. Theory. Phys. 13(4), 725-761 (2019).

[KT19] M. Kontsevich and Y. Tschinkel, *Specialization of birational types*, Invent. Math. 217, 415-432 (2019).

[Kuz] A. Kuznetsov, *Homological projective duality for Grassmannians of lines*, arXiv:math/0610957

[Kuz18] A. Kuznetsov, *Derived equivalence of Ito–Miura–Okawa–Ueda Calabi–Yau* 3*-folds*, J. Math. Soc. Jap. 70(3), 1007-1013 (2018).

[Lie06] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15, 175-206 (2006).

[Mar16] N. Martin, *The class of the affine line is a zero divisor in the Grothendieck ring: An improvement*, Comptes Rendus Math. 354(9), 936-939 (2016).

[Miy91] J. Miyachi, *Localization of triangulated categories and derived categories*, J Algebra. 141(2), 463-483 (1991).

[Mor21] H. Morimura, *Derived equivalences for the flops of type C^2 and A^4_G via mutation of semiorthogonal decomposition*, Algebras Represent. Theory. https://doi.org/10.1007/s10468-021-10036-y

[Mor23] H. Morimura, *Algebraic deformations and Fourier–Mukai transforms for Calabi–Yau manifolds*, Proc. Amer. Math. Soc. 151, 29-43 (2023)

[MP12] D. Maulik and B. Poonen, *Néron–Severi groups under specialization*, Duke Math. J. 161(11), 2167-2206 (2012).

[Ola] N. Olander, *Orlov's theorem in the smooth proper case*, arXiv:2006.15173.

[Orl97] D. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. 84, 1361-1381 (1997).

[PS97] M. Popa and C. Schnell, *Derived invariance of the number of holomporphic* 1*-forms and vector fields*, Ann. Sci. Éc. Supér, Série 4 44(3), 527-536 (1997).

[Ray74] M. Raynaud, *Géométrie analytique rigide d'eaprès Tate, Kiehl*, Mémoires de la S.M.F. 39-40, 319-327 (1974).

[Sch] C. Schnell, *The fundamental group is not a derived invariant*, arXiv:1112.3586

[Ser56] J. P. Serre, *Géométrie algébrique et géométrie analytique*, Annales de l'Institut Fourier, Tome 6 (1956).

[SP] The Stacks Project authors, *Stacks project*, https://stacks.math.columbia.edu.

[Ued19] K. Ueda, *G_2-Grassmannians and derived equivalences*, Manuscripta Math. 159(3-4), 549-559 (2019).

[Via13] C. Vial, *Algebraic cycles and fibrations*, Documenta Math. 18, 1521-1553 (2013).

VERSAL DG DEFORMATION OF CALABI–YAU MANIFOLDS

HAYATO MORIMURA

Abstract. We prove the equivalence of the deformation theory for a higher dimensional Calabi–Yau manifold and that for its dg category of perfect complexes by giving a natural isomorphism of the deformation functors. As a consequence, the dg category of perfect complexes on a versal deformation of the original manifold provides a versal Morita deformation of its dg category of perfect complexes. Besides the classical uniqueness up to étale neiborhood of the base, we prove another sort of uniqueness of versal Morita deformations.

1. Introduction

The derived category of coherent sheaves on an algebraic variety is an intensively studied invariant which carries rich information about geometric properties of the variety. For instance, given a smooth projective variety either of whose canonical or anticanonical bundle is ample, one can reconstruct the variety from its derived category [BO01]. The condition guarantees the absence of nontrivial autoequivalences of the derived category. Such autoequivalences often stem from the derived equivalence of nonisomorphic, sometimes even nonbirational Calabi–Yau manifolds. According to the homological mirror symmetry conjecture by Kontsevich, derived-equivalent Calabi–Yau manifolds should share their mirror partner. Usually, the homological mirror symmetry is considered for families of Kähler manifolds.

A goal of this paper is to study the relationship between deformations and the derived category of a higher dimensional Calabi–Yau manifold. There seems to be a consensus among some experts that deforming an algebraic variety and its derived category are essentially the same. Philosophically, it is reasonable since their Hochschild cohomology, which in general is known to control deformations of a mathematical object, are isomorphic. However, before [LV06b] we were not given the correct framework to study deformations of even linear nor abelian categories. To every second Hochschild cocycle on a smooth projective variety, Toda associated the category of twisted coherent sheaves on the corresponding noncommutative scheme over the ring of dual numbers [Tod09]. In [DLL17] Dinh–Liu–Lowen showed that Toda's construction indeed yields flat abelian first order deformations of the category of coherent sheaves on the variety in the sense of [LV06b]. We fill the gap between this point and the conclusion stated below more precisely.

Let X_0 be a Calabi–Yau manifold of dimension more than two in the strict sense, i.e., a smooth projective $\mathbf{k}$-variety with $\omega_{X_0} \cong \mathscr{O}_{X_0}$ and $H^i(\mathscr{O}_{X_0}) = 0$ for $0 < i < \dim X_0$. We denote by $\mathrm{Perf}_{dg}(X_0)$ the dg category of perfect complexes on X_0. The deformation functor

$$\mathrm{Def}_{X_0} \colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$$

sends each local artinian $\mathbf{k}$-algebra $A \in \mathrm{Art}_{\mathbf{k}}$ with residue field $\mathbf{k}$ to the set of equivalence classes of A-deformations of X_0 and each morphism $B \to A$ in $\mathrm{Art}_{\mathbf{k}}$ to the map $\mathrm{Def}_{X_0}(B) \to \mathrm{Def}_{X_0}(A)$ induced by the base change. Consider another deformation functor

$$\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)} \colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$$

which sends each $A \in \mathrm{Art}_{\mathbf{k}}$ to the set of isomorphism classes of Morita A-deformations of $\mathrm{Perf}_{dg}(X_0)$ and each morphism $B \to A$ in $\mathrm{Art}_{\mathbf{k}}$ to the map $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(B) \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A)$

induced by the derived dg functor $-\otimes_B^L A$. Our first main result claims that the deformation theory for X_0 is equivalent to that for $\mathrm{Perf}_{dg}(X_0)$ in the following sense.

Theorem 1.1. (Theorem 7.1) *There is a natural isomorphism*

$$\zeta\colon \mathrm{Def}_{X_0} \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$$

of deformation functors.

In particular, Morita deformations of $\mathrm{Perf}_{dg}(X_0)$ is controlled by the Kodaira–Spencer differential graded Lie algebra. To obtain ζ we need to consider certain maximal partial curved dg deformations of $\mathrm{Perf}_{dg}(X_0)$. Curved dg deformations of a dg category is a special case of curved A_∞-deformations of an A_∞-category. Let $(\mathfrak{a}, \mu)$ be a dg category over $\mathbf{R} \in \mathrm{Art}_{\mathbf{k}}$ with a square zero extension

$$0 \to \mathbf{I} \to \mathbf{S} \to \mathbf{R} \to 0.$$

Choose generators $\epsilon = (\epsilon_1, \dots, \epsilon_l)$ of $\mathbf{I}$ regarded as a free $\mathbf{R}$-module of rank l. By [Low08, Theorem 4.11] there is a bijection

$$H^2\mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathrm{Def}^{cdg}_{\mathfrak{a}}(\mathbf{S}),\ \phi \mapsto \mathfrak{a}_\phi = (\mathfrak{a}[\epsilon], \mu + \phi\epsilon) \tag{1.1}$$

where ϕ is a Hochschild cocycle. In other words, curved dg $\mathbf{S}$-deformations of $\mathfrak{a}$ are classified by the direct sum of the second Hochschild cohomology.

Assume that $\mathfrak{a}$ is an $\mathbf{R}$-linear category. We denote by $\mathrm{Com}^+(\mathfrak{a})$ the dg category of bounded below complexes of $\mathfrak{a}$-objects. Then by [Low08, Theorem 4.8] the characteristic morphism

$$\chi_{\mathfrak{a}}^{\oplus l}\colon H^\bullet\mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathfrak{Z}^\bullet K^+(\mathfrak{a})^{\oplus l}$$

maps $\phi \in Z^2\mathbf{C}(\mathfrak{a})^{\oplus l}$ to the obstructions against lifting objects of $K^+(\mathfrak{a})$ to $K^+(\mathfrak{a}_\phi)$ along the functor $\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -)$. In particular, for each $C \in K^+(\mathfrak{a})$ there exists a lift to $K^+(\mathfrak{a}_\phi)$ if and only if $\chi_{\mathfrak{a}}^{\oplus l}(\phi)_C = 0$. The characteristic morphism $\chi_{\mathfrak{a}}^{\oplus l}$ is induced by a B_∞-section

$$\mathrm{embr}_\delta\colon \mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathbf{C}(\mathrm{Com}^+(\mathfrak{a}))^{\oplus l}$$

of the canonical projection, which is a quasi-isomorphism of B_∞-algebras [Low08, Theorem 3.22]. Hence (1.1) induces another bijection

$$\begin{gathered} H^2\mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathrm{Def}^{cdg}_{\mathrm{Com}^+(\mathfrak{a})}(\mathbf{S}), \\ \phi \mapsto \mathrm{Com}^+(\mathfrak{a})_{\mathrm{embr}_\delta(\phi)} = (\mathrm{Com}^+(\mathfrak{a})[\epsilon], \mathrm{embr}_\delta(\mu) + \mathrm{embr}_\delta(\phi)\epsilon). \end{gathered} \tag{1.2}$$

From the proof of [Low08, Theorem 4.8] it follows that $\chi_{\mathfrak{a}}^{\oplus l}(\phi)_C = 0$ if and only if the curvature element $(\mathrm{embr}_\delta(\mu + \phi\epsilon))_{0,C}$ vanishes for each $C \in \mathrm{Com}^+(\mathfrak{a})$. Hence any full dg subcategory of $\mathrm{Com}^+(\mathfrak{a})$ spanned by object C with $\chi_{\mathfrak{a}}^{\oplus l}(\phi)_C = 0$ dg deforms along the restriction of $\mathrm{embr}_\delta(\phi)$. Note that the restrictions of $\mathrm{embr}_\delta(\mu)+\mathrm{embr}_\delta(\phi)\epsilon$ and $\mathrm{embr}_\delta(\mu+\phi\epsilon)$ to such a full dg subcategory coincide up to coboundary.

Let X be an $\mathbf{R}$-deformation of X_0 and X_ϕ its deformation along a cocycle $\phi \in HH^2(X)^{\oplus l} = H^1(\mathscr{T}_{X/\mathbf{R}})^{\oplus l}$. The above argument can be adapted to our setting so that for each $E \in \mathrm{Perf}_{dg}(X)$ the curvature element vanishes if and only if there exists a lift of $E \in \mathrm{Perf}(X)$ to $\mathrm{Perf}(X_\phi)$. With a little more effort one can apply [KL09, Proposition 3.12] to obtain

Theorem 1.2. (Theorem 6.11) *There is a bijection*

$$\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X)}(\mathbf{S}) \to \mathrm{Def}^{cdg}_{\mathrm{Perf}_{dg}(X)}(\mathbf{S})$$

between the set of isomorphism classes of Morita $\mathbf{S}$*-deformations and that of curved dg* $\mathbf{S}$*-deformations of* $\mathrm{Perf}_{dg}(X)$.

In particular, giving curved dg $\mathbf{S}$-deformations of $\mathrm{Perf}_{dg}(X)$ is equivalent to giving its Morita $\mathbf{S}$-deformations. Consider the dg category $\mathrm{Perf}_{dg}(X_\phi)$ of perfect complexes on X_ϕ. It defines a Morita $\mathbf{S}$-deformation of $\mathrm{Perf}_{dg}(X)$. Let

$$\mathfrak{m}(\phi) = \mathrm{Perf}_{dg}(X_\phi) \otimes^L_{\mathbf{S}} \mathbf{R}$$

be the image of the derived base change. Then any h-flat resolution $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ defines a dg deformation of $\mathfrak{m}(\phi)$. There is an isomorphism

$$HH^2(X)^{\oplus l} \cong H^2\mathbf{C}(\mathrm{Perf}_{dg}(X))^{\oplus l}$$

induced by the B_∞-section

$$\mathrm{embr}_\delta \colon \mathbf{C}(\mathrm{Inj}(\mathrm{Qch}(X)))^{\oplus l} \to \mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X))))^{\oplus l},$$

where $\mathrm{Inj}(\mathrm{Qch}(X)) \subset \mathrm{Qch}(X)$ is the full $\mathbf{R}$-linear subcategory of injective objects. We denote by $\mathrm{embr}_\delta(\phi)$ the image of ϕ under the isomorphism, which defines another dg $\mathbf{S}$-deformation $\mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}$ of $\mathfrak{m}(\phi)$ along $\mathrm{embr}_\delta(\phi)$. Deformations and taking the dg category of perfect complexes intertwine in the following sense.

Theorem 1.3. (Theorem 6.12) *There is an isomorphism*

$$\overline{\mathrm{Perf}_{dg}(X_\phi)} \simeq \mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}$$

of dg $\mathbf{S}$-deformations of $\mathfrak{m}(\phi)$. In particular, the Morita $\mathbf{S}$-deformation $\mathrm{Perf}_{dg}(X_\phi)$ defines a maximal partial dg $\mathbf{S}$-deformaiton of $\mathrm{Perf}_{dg}(X)$ along $\mathrm{embr}_\delta(\phi)$.

This is the key to prove Theorem 1.1. Unwinding Toda's construction, from [DLL17, Theorem 5.12] we obtain an equivalence $\mathrm{Qch}(X)_\phi \simeq \mathrm{Qch}(X_\phi)$ of Grothendieck abelian categories, where $\mathrm{Qch}(X)_\phi$ is the flat abelian $\mathbf{S}$-deformation of $\mathrm{Qch}(X)$ along $\phi \in Z^2\mathbf{C}_{ab}(\mathrm{Qch}(X))^{\oplus l}$. Here, we use the same symbol ϕ to denote the image under the isomorphism

$$HH^2(X)^{\oplus l} \cong H^2\mathbf{C}_{ab}(\mathrm{Qch}(X))^{\oplus l}.$$

Via the induced equivalence

$$D_{dg}(\mathrm{Qch}(X_\phi)) \simeq D_{dg}(\mathrm{Qch}(X)_\phi)$$

we regard $\mathrm{Perf}_{dg}(X_\phi)$ as the full dg subcategory of compact objects of $D_{dg}(\mathrm{Qch}(X)_\phi)$. Based on the idea in the proof of [Low08, Theorem 4.15], we compare the dg structure on $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ with that on $\mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}$.

Working with Morita deformations of $\mathrm{Perf}_{dg}(X_0)$, by [Coh, Corollary 5.7] we may apply [BFN10, Theorem 1.2] to obtain reductions. In particular, given a deformation $(X_B, i_B) \in \mathrm{Def}_{X_0}(B)$ and a morphism $B \to A$ in $\mathrm{Art}_{\mathbf{k}}$, there is a Morita equivalence

$$\mathrm{Perf}_{dg}(X_B) \otimes^L_B A \simeq_{mo} \mathrm{Perf}_{dg}(X_B) \otimes^L_B \mathrm{Perf}_{dg}(A) \simeq_{mo} \mathrm{Perf}_{dg}(X_A)$$

of A-linear dg categories, where $- \otimes^L_B -$ is the derived pointwise tensor product of dg categories. Further application of [BFN10, Theorem 1.2] shows that any universal formal family for $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ is effective. If we ignore the set theoretical issues, the deformation functor $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ can naturally be extended to a functor defined on the category $\mathrm{Alg}^{aug}(\mathbf{k})$ of augmented noetherian $\mathbf{k}$-algebras. Although we do not know whether it would be locally of finite presentation (colimit preserving), one can always construct a versal Morita deformation via geometric realization in the following sense.

Corollary 1.4. (Corollary 7.4) *Any effective universal formal family for $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ is algebraizable. In particular, an algebraization is given by $\mathrm{Perf}_{dg}(X_S)$ where $(\mathrm{Spec}\, S, s, X_S)$ is a versal deformation of X_0.*

The versal Morita deformation $\mathrm{Perf}_{dg}(X_S)$ may be regarded as a family of Morita deformations of $\mathrm{Perf}_{dg}(X_0)$. More generally, for such a family determined by an enough nice S-scheme X_S we introduce its generic fiber as follows.

Definition 1.5. *Let X_S be a smooth separated scheme over a noetherian connected regular affine* $\mathbf{k}$*-scheme* $\operatorname{Spec} S$ *whose closed points are* $\mathbf{k}$*-rational. Then the* dg categorical generic fiber *of* $\mathrm{Perf}_{dg}(X_S)$ *is the Drinfeld quotient*

$$\mathrm{Perf}_{dg}(X_S)/\,\mathrm{Perf}_{dg}(X_S)_0,$$

where $\mathrm{Perf}_{dg}(X_S)_0 \subset \mathrm{Perf}_{dg}(X_S)$ *is the full dg subcategory of perfect complexes with S-torsion cohomology.*

We impose a technical assumption on S to include also the case where S is a formal power series ring. The Drinfeld quotient is a natural dg enhancement of the categorical generic fiber introduced in [Morb], which is in turn based on the categorical general fiber by Huybrechts–Macrì–Stellari [HMS11]. Taking the generic fiber and the dg category of perfect complexes intertwine in the following sense.

Proposition 1.6. (Proposition 7.5) *Let X_S be a smooth separated scheme over a noetherian connected regular affine* $\mathbf{k}$*-scheme* $\operatorname{Spec} S$ *whose closed points are* $\mathbf{k}$*-rational. Then there is a quasi-equivalence*

$$\mathrm{Perf}_{dg}(X_S)/\,\mathrm{Perf}_{dg}(X_S)_0 \simeq_{qeq} \mathrm{Perf}_{dg}(X_{Q(S)})$$

where $Q(S)$ is the quotient field of S and $X_{Q(S)}$ is the generic fiber of X_S.

Another goal of this paper is to show the uniqueness of versal Morita deformations with respect to geometric realizations. Recall that up to étale neighborhood of the base versal deformations of X_0 are unique. Namely, if $(\operatorname{Spec} S, s, X_S)$ $(\operatorname{Spec} S', s', X_{S'})$ are two versal deformations of X_0, then there is another versal deformation $(\operatorname{Spec} S'', s'', X_{S''})$ such that $(\operatorname{Spec} S'', s'')$ is an étale neighborhood of s, s' in $\operatorname{Spec} S, \operatorname{Spec} S'$ respectively and $X_{S''}$ is the pullback along the corresponding étale morphisms. The deformation functor Def_{X_0} has an effective universal formal family (R, ξ), where R is a regular complete local noetherian $\mathbf{k}$-algebra. Choose an isomorphism $R \cong \mathbf{k}[\![t_1, \dots, t_d]\!]$ and let $T = \mathbf{k}[t_1, \dots, t_d]$ with $d = \dim_{\mathbf{k}} \mathrm{H}^1(\mathscr{T}_{X_0})$. There is a filtered inductive system $\{R_i\}_{i \in I}$ of finitely generated T-subalgebras of R whose colimit is R. Then $(\operatorname{Spec} S, s)$ is an étale neighborhood of t in $\operatorname{Spec} T$ with t corresponding to the maximal ideal $(t_1, \dots, t_d) \subset T$, and X_S is the pullback of a deformation X_{R_j} of X_0 along a first order approximation $R_j \to S$ of $R_j \hookrightarrow R$ for sufficiently large $j \in I$. Hence the ambiguity of X_S stems from the choice of $j \in I$, besides the choice of étale neighborhoods.

In [Mora] the author constructed smooth projective versal deformations X_S, X'_S of X_0, X'_0 over a common nonsingular affine variety $\operatorname{Spec} S$, while deforming simultaneously the Fourier–Mukai kernel connecting deformations of X_0, X'_0. By Corollary 1.4 we have two versal Morita deformations $\mathrm{Perf}_{dg}(X_S), \mathrm{Perf}_{dg}(X'_S)$ of $\mathrm{Perf}_{dg}(X_0)$. Theorem 1.1 together with the construction of versal deformations suggests that $\mathrm{Perf}_{dg}(X_S), \mathrm{Perf}_{dg}(X'_S)$ should be determined only by quasi-equivalent universal formal families and the same sufficiently large index $j \in I$. From this observation we arrive at our second main result.

Theorem 1.7. (Theorem 8.3) *Let X_0, X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two and $\mathcal{P}_0 \in D^b(X_0 \times_{\mathbf{k}} X'_0)$ the Fourier–Mukai kernels. Then there exists an index $j \in I$ such that for all $k \geq j$ the integral functors*

$$\Phi_{\mathcal{P}_k} \colon \mathrm{Perf}(X_{R_k}) \to \mathrm{Perf}(X'_{R_k})$$

defined by deformations $\mathcal{P}_k$ of $\mathcal{P}_0$ are equivalences of triangulated categories of perfect complexes. In particular, the dg categories $\mathrm{Perf}_{dg}(X_{R_k}), \mathrm{Perf}_{dg}(X'_{R_k})$ *of perfect complexes are quasi-equivalent.*

Theorem 1.7 tells us that, given two algebraic Morita deformations $\mathrm{Perf}_{dg}(X_{R_k}), \mathrm{Perf}_{dg}(X'_{R_k})$ geometrically realized by algebraic deformations X_{R_k}, X'_{R_k} of two derived-equivalent higher dimensional Calabi–Yau manifolds X_0, X'_0, if X_{R_k}, X'_{R_k} are enough close to effectivizations X_R, X'_R then $\mathrm{Perf}_{dg}(X_{R_k}), \mathrm{Perf}_{dg}(X'_{R_k})$ are Morita equivalent. The base change along the homomorphism $R_k \to S$ yields Morita equivalent versal Morita deformations $\mathrm{Perf}_{dg}(X_S), \mathrm{Perf}_{dg}(X'_S)$. In other words, up to Morita equivalence the versal Morita deformation $\mathrm{Perf}_{dg}(X_S)$ does not depend on the choice of geometric realizations in the following sense.

Corollary 1.8. (Corollary 8.4) *Let X_0, X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two and X_S, X'_S their smooth projective versal deformations over a common nonsingular affine* $\mathbf{k}$*-variety* $\mathrm{Spec}\, S$*. Let $\{R_i\}_{i\in I}$ be a filtered inductive system of finitely generated T-subalgebras $R_i \subset R$ whose colimit is R with*

$$T = \mathbf{k}[t_1, \dots, t_d],\ R \cong \mathbf{k}[\![t_1, \dots, t_d]\!],\ d = \dim_{\mathbf{k}} H^1(\mathscr{T}_{X_0}).$$

Assume that X_S, X'_S correspond to a first order approximation $R_j \to S$ of $R_j \hookrightarrow R$ for sufficiently large $j \in I$. Then X_S, X'_S are derived-equivalent. In particular, the dg categories $\mathrm{Perf}_{dg}(X_S), \mathrm{Perf}_{dg}(X'_S)$ *of perfect complexes are quasi-equivalent.*

The uniqueness result also holds for the dg categorical generic fiber. Corollary 1.8 slightly improves [Mora, Theorem 1.1], which extends the derived equivalence from special to general fibers. Here, the advantage is that we do not have to shrink the base $\mathrm{Spec}\, S$ as long as the construction passes enough close to effectivizations. In particular, beginning with a pair of general fibers, one obtains the derived equivalence of special fibers contained in the versal deformations. Hence the above corollary partially provides a method for the opposite direction, i.e., how to extend the derived equivalence from general to special fibers.

Notations and conventions. *We work over an algebraically closed field* $\mathbf{k}$ *of characteristic* 0 *throughout this paper. For an augmented* $\mathbf{k}$*-algebra A by $\mathfrak{m}_A$ we denote its augmentation ideal. All higher dimensional Calabi–Yau manifolds we treat are smooth projective* $\mathbf{k}$*-varieties X_0 of dimension more than two with $\omega_{X_0} \cong \mathscr{O}_{X_0}$ and $H^i(\mathscr{O}_{X_0}) = 0$ for $0 < i < \dim X_0$.*

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2. Hochschild cohomology of relatively smooth proper schemes

In this section, we review various kinds of complexes whose cohomology controls deformations of associated mathematical objects, mainly following the exposition from [DLL17, Section 2, 3]. We always assume that all algebras have units, morphism of algebras preserve units, and modules are unital. In the sequel, we fix a local artinian $\mathbf{k}$-algebra $\mathbf{R}$ with residue field $\mathbf{k}$ and its square zero extension

$$0 \to \mathbf{I} \to \mathbf{S} \to \mathbf{R} \to 0,$$

and choose generators $\epsilon = (\epsilon_1, \dots, \epsilon_l)$ of $\mathbf{I}$ regarded as a free $\mathbf{R}$-module of rank l. For smooth proper $\mathbf{R}$-schemes, we explain the correspondence between its relative Hochschild cohomology and cohomology of the Gerstenharber–Shack complex associated with its restricted structure sheaf.

2.1. **Relative Hochschild cohomology of schemes.** Let X be a smooth proper $\mathbf{R}$-scheme. We denote by $\Delta_{\mathbf{R}}\colon X_{\mathbf{R}} \hookrightarrow X \times_{\mathbf{R}} X$ the relative diagonal embedding. The *relative Hochschild cohomology* is defined as the graded $\mathbf{R}$-algebra

$$HH^\bullet(X/\mathbf{R}) = \mathrm{Ext}^\bullet_{X\times_{\mathbf{R}}X}(\mathscr{O}_{\Delta_{\mathbf{R}}}, \mathscr{O}_{\Delta_{\mathbf{R}}}) \cong \mathrm{Ext}^\bullet_X(\Delta^*_{\mathbf{R}}\mathscr{O}_{\Delta_{\mathbf{R}}}, \mathscr{O}_X).$$

Here, the multiplication in $HH^\bullet(X/\mathbf{R})$ is given by the composition in $D^b(X \times_{\mathbf{R}} X)$. Then the natural map $\mathbf{R} \to \mathrm{End}_{X\times_{\mathbf{R}}X}(\mathscr{O}_{\Delta_{\mathbf{R}}})$ induces the $\mathbf{R}$-algebra structure. There is a quasi-isomorphism

$$\Delta^*_{\mathbf{R}}\mathscr{O}_{\Delta_{\mathbf{R}}} \cong \bigoplus_i \Omega^i_{X/\mathbf{R}}[i]$$

called the *relative Hochschild–Kostant–Rosenberg isomorphism*, which induces an isomorphism

$$\mathrm{Ext}^\bullet_X(\Delta^*_{\mathbf{R}}\mathscr{O}_{\Delta_{\mathbf{R}}}, \mathscr{O}_X) \to \mathrm{Ext}^\bullet_X(\bigoplus_i \Omega^i_{X/\mathbf{R}}[i], \mathscr{O}_X) \to \bigoplus_i H^{\bullet-i}(X, \wedge^i \mathscr{T}_{X/\mathbf{R}})$$

where $\mathscr{T}_{X/\mathbf{R}}$ is the relative tangent sheaf and $\Omega_{X/\mathbf{R}}$ is its dual. We also call the compositions

$$\mathrm{I}^{\mathrm{HKR}}_{X/\mathbf{R}}\colon\ \mathrm{Ext}^n_{X\times_{\mathbf{R}}X}(\mathscr{O}_{\Delta_{\mathbf{R}}}, \mathscr{O}_{\Delta_{\mathbf{R}}}) = HH^n(X/\mathbf{R}) \to HT^n(X/\mathbf{R}) = \bigoplus_{p+q=n} H^p(X, \wedge^q \mathscr{T}_{X/\mathbf{R}})$$

the relative Hochschild–Kostant–Rosenberg isomorphisms.

2.2. **Hochschild cohomology of algebras.** Let $A = (A, m)$ be an $\mathbf{R}$-algebra and M an A-bimodule. The *Hochschild complex* $\mathbf{C}(A, M)$ has $\mathbf{C}^n(A, M) = \mathrm{Hom}_{\mathbf{R}}(A^{\otimes n}, M)$ as its n-th term and $d^n_{Hoch}\colon \mathbf{C}^n(A, M) \to \mathbf{C}^{n+1}(A, M)$, called the *Hochschild differential*, as its differential which is given by

$$\begin{aligned} d^n_{Hoch}(\phi)(a_n, a_{n-1}, \dots, a_0) &= a_n\phi(a_{n-1}, \dots, a_0) \\ &\quad + \sum_{i=0}^{n-1}(-1)^{i+1}\phi(a_n, \dots, a_{n-i}a_{n-i-1}, \dots, a_0) \\ &\quad + (-1)^{n+1}\phi(a_n, \dots, a_1)a_0. \end{aligned}$$

A cochain $\phi \in \mathbf{C}^n(A, M)$ is *normalized* if $\phi(a_{n-1}, \dots, a_0) = 0$ whenever $a_i = 1$ for some $0 \le i \le n-1$. The normalized cochains form a subcomplex $\bar{\mathbf{C}}(A, M)$ quasi-isomorphic to $\mathbf{C}(A, M)$ via the inclusion. When $M = A$, we call $\mathbf{C}(A) = \mathbf{C}(A, A)$ the *Hochschild complex* and $H^n\mathbf{C}(A)$ the n-th *Hochschild cohomology* of A. Note that the multiplication m on A belongs to $\mathbf{C}^2(A)$.

The direct sum of the second normalized Hochschild cohomology of A classifies $\mathbf{S}$-deformations of A up to equivalence. Recall that an $\mathbf{S}$-*deformation* of A is an $\mathbf{S}$-algebra $(\bar{A}, \bar{m}) = (A[\epsilon] = A \otimes_{\mathbf{R}} \mathbf{S}, m + \mathbf{m}\epsilon)$ with $\mathbf{m} \in \mathbf{C}^2(A)^{\oplus l}$ such that the unit of $\bar{A}$ is the same as that of A. Two deformations $(\bar{A}, \bar{m}), (\bar{A}', \bar{m}')$ are *equivalent* if there is an isomorphism of the form $1 + \mathbf{g}\epsilon\colon \bar{A} \to \bar{A}'$ with $\mathbf{g} \in \mathbf{C}^1(A)^{\oplus l}$. We denote by $\mathrm{Def}^{alg}_A(\mathbf{S})$ the set of equivalence classes of $\mathbf{S}$-deformations of A. It is known that there is a bijection

$$H^2\bar{\mathbf{C}}(A)^{\oplus l} \to \mathrm{Def}^{alg}_A(\mathbf{S}),\ \mathbf{m} \mapsto (A[\epsilon], m + \mathbf{m}\epsilon),\ \mathbf{m} \in Z^2\bar{\mathbf{C}}(A)^{\oplus l}.$$

2.3. **Simplicial cohomology of presheaves.** Let $\mathfrak{U}$ be a small category and $\mathcal{N}(\mathfrak{U})$ its simplicial nerve. We write

$$\sigma = (d\sigma = U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} \cdots \xrightarrow{u_p} U_p \xrightarrow{u_{p+1}} U_{p+1} = c\sigma)$$

for a $(p+1)$-simplex $\sigma \in \mathcal{N}_{p+1}(\mathfrak{U})$. Let $(\mathscr{F}, f), (\mathscr{G}, g)$ be presheaves of $\mathbf{R}$-modules with restriction maps $f^u\colon \mathscr{F}(U) \to \mathscr{F}(V), g^u\colon \mathscr{G}(U) \to \mathscr{G}(V)$ for $u\colon V \to U$ in $\mathfrak{U}$. We write f^σ for the map $f^{u_{p+1}\dots u_2 u_1}\colon \mathscr{F}(U_{p+1}) \to \mathscr{F}(U_0)$. Consider a complex whose p-th term is

$$\mathbf{C}^p_{simp}(\mathscr{G}, \mathscr{F}) = \prod_{\tau \in \mathcal{N}_p(\mathfrak{U})} \mathrm{Hom}_{\mathbf{R}}(\mathscr{G}(c\tau), \mathscr{F}(d\tau)).$$

and whose differential d^p_{simp} is defined as follows. Recall that we have the maps

$$\partial_i\colon \mathcal{N}_{p+1}(\mathfrak{U}) \to \mathcal{N}_p(\mathfrak{U}),\ \sigma \mapsto \partial_i\sigma,$$

for $i = 0, 1, \dots, p+1$ given by

$$\begin{aligned}\partial_i\sigma &= (U_0 \xrightarrow{u_1} \cdots U_{i-1} \xrightarrow{u_{i+1}u_i} U_{i+1} \xrightarrow{u_{i+1}} \cdots \xrightarrow{u_p} U_p \xrightarrow{u_{p+1}} U_{p+1}),\ i \neq 0, p+1,\\ \partial_0\sigma &= (U_1 \xrightarrow{u_2} U_2 \xrightarrow{u_3} \cdots \xrightarrow{u_p} U_p \xrightarrow{u_{p+1}} U_{p+1}),\\ \partial_{p+1}\sigma &= (U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} \cdots \xrightarrow{u_p} U_p).\end{aligned}$$

Each ∂_i induces a map

$$d_i\colon \mathbf{C}^p_{simp}(\mathscr{G}, \mathscr{F}) \to \mathbf{C}^{p+1}_{simp}(\mathscr{G}, \mathscr{F}),\ \phi = (\phi^\tau)_\tau \mapsto d_i\phi = ((d_i\phi)^\sigma)_\sigma$$

given by

$$\begin{aligned}(d_i\phi)^\sigma &= \phi^{\partial_i\sigma},\ i \neq 0, p+1,\\ (d_0\phi)^\sigma &= f^{u_1} \circ \phi^{\partial_0\sigma},\\ (d_{p+1}\phi)^\sigma &= \phi^{\partial_{p+1}\sigma} \circ g^{u_{p+1}}.\end{aligned}$$

Then one defines

$$d_{simp} = \sum_{i=0}^{p+1} (-1)^i d_i\colon \mathbf{C}^p_{simp}(\mathscr{G}, \mathscr{F}) \to \mathbf{C}^{p+1}_{simp}(\mathscr{G}, \mathscr{F}).$$

When $\mathscr{G}$ is the constant presheaf $\mathbf{R}$, we call $H^p(\mathfrak{U}, \mathscr{F}) = H^p\mathbf{C}_{simp}(\mathscr{F}) = H^p\mathbf{C}_{simp}(\mathbf{R}, \mathscr{F})$ the *simplicial presheaf cohomology* of $\mathscr{F}$. A $(p+1)$-simplex $\sigma \in \mathcal{N}_{p+1}(\mathfrak{U})$ is *degenerate* if $u_i = 1_{U_i}$ for some $1 \leq i \leq p+1$. A p-cochain $\phi = (\phi^\tau)_\tau \in \mathbf{C}^p(\mathscr{G}, \mathscr{F})$ is *reduced* if $\phi^\tau = 0$ whenever τ is degenerate. All 0-cochains are reduced by convention. The reduced cohains are preserved by d_{simp} and form a subcomplex $\mathbf{C}'_{simp}(\mathscr{G}, \mathscr{F})$, which is quasi-isomorphic to $\mathbf{C}_{simp}(\mathscr{G}, \mathscr{F})$ by [DLL17, Proposition 2.9].

The direct sum of the first reduced simplicial presheaf cohomology of $\mathscr{F}$ classifies $\mathbf{S}$-deformations of $\mathscr{F}$ up to equivalence. Recall that an $\mathbf{S}$-*deformation* of $\mathscr{F}$ is a presheaf of $\mathbf{S}$-modules $(\bar{\mathscr{F}}, \bar{f}) = (\mathscr{F}[\epsilon], f + \mathbf{f}\epsilon)$ with $\mathbf{f} \in \mathbf{C}^1_{simp}(\mathscr{F}, \mathscr{F})^{\oplus l}$. Two deformations $(\bar{\mathscr{F}}, \bar{f}), (\bar{\mathscr{F}}', \bar{f}')$ are *equivalent* if there is an isomorphism of the form $1 + \mathbf{g}\epsilon$ with $\mathbf{g} \in \mathbf{C}^0_{simp}(\mathscr{F}, \mathscr{F})^{\oplus l}$. We denote by $\mathrm{Def}^{psh}_{\mathscr{F}}(\mathbf{S})$ the set of equivalence classes of $\mathbf{S}$-deformations of $\mathscr{F}$.

Lemma 2.1. ([DLL17, Proposition 2.11]) *Let $(\mathscr{F}, f)$ be a presheaf of $\mathbf{R}$-modules. Then there is a bijection*

$$H^1\mathbf{C}'_{simp}(\mathscr{F}, \mathscr{F})^{\oplus l} \to \mathrm{Def}^{psh}_{\mathscr{F}}(\mathbf{S}),\ \phi \mapsto (\mathscr{F}[\epsilon], f + \mathbf{f}\epsilon),\ \mathbf{f} \in Z^1\mathbf{C}'_{simp}(\mathscr{F}, \mathscr{F})^{\oplus l}.$$

Another cocycle $\mathbf{f}' \in Z^1\mathbf{C}'_{simp}(\mathscr{F}, \mathscr{F})^{\oplus l}$ maps to an equivalent deformation if and only if there is an element $\mathbf{g} \in \mathbf{C}'^0_{simp}(\mathscr{F}, \mathscr{F})$ satisfying $\mathbf{f}' - \mathbf{f} = d_{simp}(\mathbf{g})$.

2.4. **Gerstenharber–Schack complexes.** Let $\mathfrak{U}$ be a small category and $(\mathscr{A}, m, f)$ a presheaf of $\mathbf{R}$-algebras on $\mathfrak{U}$. The *Gerstenharber–Schack complex* $\mathbf{C}_{GS}(\mathscr{A})$ introduced in [GS88] is the total complex of the double complex whose (p, q)-term for $p, q \geq 0$ is

$$\mathbf{C}_{GS}^{p,q}(\mathscr{A}) = \prod_{\tau \in \mathcal{N}_p(\mathfrak{U})} \mathrm{Hom}_{\mathbf{R}}(\mathscr{A}(c\tau)^{\otimes q}, \mathscr{A}(d\tau)),$$

where we regard $\mathscr{A}(d\tau)$ as an $\mathscr{A}(c\tau)$-bimodule via $f^\tau \colon \mathscr{A}(c\tau) \to \mathscr{A}(d\tau)$. When q is fixed, we have

$$\mathbf{C}_{GS}^{\bullet,q}(\mathscr{A}) = \mathbf{C}_{simp}(\mathscr{A}^{\otimes q}, \mathscr{A})$$

endowed with the simplicial differential d_{simp} horizontally. When p is fixed, we have

$$\mathbf{C}_{GS}^{p,\bullet} = \prod_{\tau \in \mathcal{N}_p(\mathfrak{U})} \mathbf{C}\left(\mathscr{A}(c\tau), \mathscr{A}(d\tau)\right)$$

endowed with the product Hochschild differential d_{Hoch} vertically. The differential

$$d_{GS}^n(\mathscr{A}) \colon \mathbf{C}_{GS}^n(\mathscr{A}) \to \mathbf{C}_{GS}^{n+1}(\mathscr{A})$$

is defined as $d_{GS}^n = (-1)^{n+1} d_{simp} + d_{Hoch}$.

A cochain $\phi = (\phi^\tau)_\tau \in \mathbf{C}_{GS}^{p,q}(\mathscr{A})$ is *normalized* if ϕ^τ is normalized for each p-simplex τ, and it is *reduced* if $\phi^\tau = 0$ whenever τ is degenerate. The normalized cochains form a subcomplex $\bar{\mathbf{C}}_{GS}(\mathscr{A})$ of $\mathbf{C}_{GS}(\mathscr{A})$ called the *normalized Hochschild complex* of $\mathscr{A}$, and the normalized reduced cochains form a subcomplex $\bar{\mathbf{C}}'_{GS}(\mathscr{A})$ of $\bar{\mathbf{C}}_{GS}(\mathscr{A})$ called the *normalized reduced Hochschild complex* of $\mathscr{A}$. These three complexes are quasi-isomorphic via the inclusions. Eliminating the bottom row from $\mathbf{C}_{GS}(\mathscr{A})$, one obtains a subcomplex $\mathbf{C}_{tGS}(\mathscr{A})$ called the *truncated Hochschild complex*. There is a short exact sequence

$$0 \to \mathbf{C}_{tGS}(\mathscr{A}) \to \mathbf{C}_{GS}(\mathscr{A}) \to \mathbf{C}_{simp}(\mathscr{A}) \to 0.$$

Since $\mathbf{R}$ is commutative, one can apply [DLL17, Proposition 2.14] to see that the sequence splits and we have

$$\mathbf{C}_{GS}(\mathscr{A}) = \mathbf{C}_{tGS}(\mathscr{A}) \oplus \mathbf{C}_{simp}(\mathscr{A}).$$

Similarly, we have

$$\bar{\mathbf{C}}'_{GS}(\mathscr{A}) = \bar{\mathbf{C}}'_{tGS}(\mathscr{A}) \oplus \mathbf{C}'_{simp}(\mathscr{A}).$$

The direct sum of the second normalized reduced Gerstenharber–Schack cohomology of $\mathscr{A}$ classifies twisted $\mathbf{S}$-deformations of $\mathscr{A}$ up to equivalence. Recall that a *twisted presheaf* $\mathscr{A} = (\mathscr{A}, m, f, c, z)$ of $\mathbf{R}$-algebras on $\mathfrak{U}$ consists of the following data:

- for each $U \in \mathfrak{U}$ an $\mathbf{R}$-algebra $(\mathscr{A}(U), m^U)$,
- for each $u \colon V \to U$ in $\mathfrak{U}$ a homomorphism of $\mathbf{R}$-algebras $f^u \colon \mathscr{A}(U) \to \mathscr{A}(V)$,
- for each pair $u \colon V \to U$, $v \colon W \to V$ in $\mathfrak{U}$ an invertible element $c^{u,v} \in \mathscr{A}(W)$ satisfying for any $a \in \mathscr{A}(U)$
$$c^{u,v} f^v(f^u(a)) = f^{uv}(a) c^{u,v}.$$
- for each $U \in \mathfrak{U}$ an invertible element $z^U \in \mathscr{A}(U)$ satisfying for any $a \in \mathscr{A}(U)$
$$z^U a = f^{1_U}(a) z^U.$$

Moreover, these data must satisfy

$$c^{u,vw} c^{v,w} = c^{uv,w} f^w(c^{u,v}),$$
$$c^{u,1_V} z^V = 1, \quad c^{1_U,u} f^u(z^U) = 1$$

for each triple $u\colon V \to U$, $v\colon W \to V$, $w\colon T \to W$ in $\mathfrak{U}$. When $c^{u,v}, z^U$ are central for all u, v and U, we call $\mathscr{A}$ a *twisted presheaf with central twists* and denote by $|\mathscr{A}| = (\mathscr{A}, m, f)$ the underlying ordinary presheaf.

For twisted sheaves $\mathscr{A} = (\mathscr{A}, m, f, c, z), \mathscr{A}' = (\mathscr{A}', m', f', c', z')$ of $\mathbf{R}$-algebras on $\mathfrak{U}$, a *morphism* $(g, h)\colon \mathscr{A} \to \mathscr{A}'$ consists of the following data:

- for each $U \in \mathfrak{U}$ a homomorphism of $\mathbf{R}$-algebras $g^U\colon \mathscr{A}(U) \to \mathscr{A}'(U)$,
- for each $u\colon V \to U$ in $\mathfrak{U}$ an invertible element $h^u \in \mathscr{A}'(V)$.

Moreover, these data must satisfy

$$m'^V(g^V f^u(a), h^u) = m'^V(h^u, f'^u(g^U(a))),$$
$$m'^W(h^{uv}, c'^{u,v}) = m'^W(g^W(c^{u,v}), h^v, f'^v(h^u)),$$
$$m'^U(h^{1_U}, z'^U) = g^U(z^U)$$

for all u, v and $a \in \mathscr{A}(U)$. Morphisms can be composed and the identity $1_{\mathscr{A}}$ is given by $g^U = 1_{\mathscr{A}(U)}$ and $h^u = 1 \in \mathscr{A}(V)$. When g^U are isomorphisms of $\mathbf{R}$-algebras for all U, we call (g, h) an *isomorphism*. Any twisted presheaf $(\mathscr{A}, m, f, c, z)$ is isomorphic to the one of the form $(\mathscr{A}', m', f', c', 1)$.

Let $\mathscr{A} = (\mathscr{A}, m, f, c)$ be a twisted presheaf of $\mathbf{R}$-algebras on $\mathfrak{U}$. A *twisted* $\mathbf{S}$*-deformation* of $\mathscr{A}$ is a twisted presheaf

$$\overline{\mathscr{A}} = (\overline{\mathscr{A}}, \overline{m}, \overline{f}, \overline{c}) = (\mathscr{A}[\epsilon], m + \mathbf{m}\epsilon, f + \mathbf{f}\epsilon, c + \mathbf{c}\epsilon)$$

of $\mathbf{S}$-algebras such that $(\overline{\mathscr{A}}(U), \overline{m}^U)$ is an $\mathbf{S}$-deformation of $(\mathscr{A}(U), m^U)$ for each $U \in \mathfrak{U}$ with

$$(\mathbf{m}, \mathbf{f}, \mathbf{c}) \in \mathbf{C}^2_{GS}(\mathscr{A})^{\oplus l} = \mathbf{C}^{0,2}_{GS}(\mathscr{A})^{\oplus l} \oplus \mathbf{C}^{1,1}_{GS}(\mathscr{A})^{\oplus l} \oplus \mathbf{C}^{2,0}_{GS}(\mathscr{A})^{\oplus l}.$$

Two twisted deformations $(\overline{\mathscr{A}}, \overline{m}, \overline{f}, \overline{c}), (\overline{\mathscr{A}'}, \overline{m'}, \overline{f'}, \overline{c'})$ are *equivalent* if there is an isomorphism of the form $(1 + \mathbf{g}\epsilon, 1 + \mathbf{h}\epsilon)$ with

$$(\mathbf{g}, \mathbf{h}) \in \mathbf{C}^1_{GS}(\mathscr{A})^{\oplus l} = \mathbf{C}^{0,1}_{GS}(\mathscr{A})^{\oplus l} \oplus \mathbf{C}^{1,0}_{GS}(\mathscr{A})^{\oplus l}.$$

We denote by $\mathrm{Def}^{tw}_{\mathscr{A}}(\mathbf{S})$ the set of equivalence classes of twisted $\mathbf{S}$-deformations of $\mathscr{A}$.

When $c = 1$, A *presheaf* $\mathbf{S}$*-deformation* of $\mathscr{A}$ is a twisted $\mathbf{S}$-deformation with $\mathbf{c} = 0$. Two presheaf deformations $(\overline{\mathscr{A}}, \overline{m}, \overline{f}), (\overline{\mathscr{A}'}, \overline{m'}, \overline{f'})$ are *equivalent* if there is an isomorphism of the form $1 + \mathbf{g}\epsilon$ with $\mathbf{g} \in \mathbf{C}^{0,1}_{GS}(\mathscr{A})^{\oplus l}$. We denote by $\mathrm{Def}^{psh}_{\mathscr{A}}(\mathbf{S})$ the set of equivalence classes of presheaf $\mathbf{S}$-deformations of $\mathscr{A}$.

Lemma 2.2. ([DLL17, Theorem 2.21]) *Let* $(\mathscr{A}, m, f)$ *be a presheaf of* $\mathbf{R}$*-algebras on* $\mathfrak{U}$*. Then there is a bijection*

$$H^2\bar{\mathbf{C}}'_{GS}(\mathscr{A})^{\oplus l} \to \mathrm{Def}^{tw}_{\mathscr{A}}(\mathbf{S}),\ (\mathbf{m}, \mathbf{f}, \mathbf{c}) \mapsto (\mathscr{A}[\epsilon], m + \mathbf{m}\epsilon, f + \mathbf{f}\epsilon, c + \mathbf{c}\epsilon),\ (\mathbf{m}, \mathbf{f}, \mathbf{c}) \in Z^2\bar{\mathbf{C}}'_{GS}(\mathscr{A})^{\oplus l}.$$

Another cocycle $(\mathbf{m}', \mathbf{f}', \mathbf{c}') \in Z^2\bar{\mathbf{C}}'_{GS}(\mathscr{A})^{\oplus l}$ *maps to an equivalent deformation if and only if there is an element* $(\mathbf{g}, \mathbf{h}) \in \bar{\mathbf{C}}'^1_{GS}(\mathscr{A})^{\oplus l}$ *satisfying* $(\mathbf{m}', \mathbf{f}', \mathbf{c}') - (\mathbf{m}, \mathbf{f}, \mathbf{c}) = d_{GS}(\mathbf{g}, \mathbf{h})$*. In particular, there is a bijection*

$$H^2\bar{\mathbf{C}}'_{tGS}(\mathscr{A})^{\oplus l} \to \mathrm{Def}^{psh}_{\mathscr{A}}(\mathbf{S}),\ (\mathbf{m}, \mathbf{f}) \mapsto (\mathscr{A}[\epsilon], m + \mathbf{m}\epsilon, f + \mathbf{f}\epsilon).$$

2.5. **Hodge decomposition.** In the group algebra $\mathbb{Q}S_n$ of the n-th symmetric group S_n, there is a collection of pairwise orthogonal idempotents $e_n(r)$ for $1 \leq r \leq n$ such that $\sum_{r=1}^n e_n(r) = 1$ [GS87, Theorem 1.2]. Put $e_n(0) = 0$, $e_0(0) = 1 \in \mathbb{Q}$, and $e_n(r) = 0$ for $r > n$. Let A be a commutative $\mathbf{R}$-algebra and M a symmetric A-bimodule. The subcomplex $\mathbf{C}(A, M)_r \subset \mathbf{C}(A, M)$ whose n-th term is $\mathbf{C}(A, M)e_n(r)$ gives rise to a Hodge decomposition

$$\mathbf{C}(A, M) = \bigoplus_{r \in \mathbb{N}} \mathbf{C}(A, M)_r.$$

Assume that $\mathscr{A}$ is a presheaf of commutative $\mathbf{R}$-algebras. Then the Hodge decomposition

$$\mathrm{Hom}(\mathscr{A}(c\tau)^{\otimes q}, \mathscr{A}(d\tau)) = \bigoplus_{r=0}^{q} \mathrm{Hom}(\mathscr{A}(c\tau)^{\otimes q}, \mathscr{A}(d\tau))_r$$

induces a decomposition of the double complex $\mathbf{C}_{GS}^{n-q,q}(\mathscr{A})$ preserved by d_{Hoch} and d_{simp}. Hence one obtains a *Hodge decomposition*

$$\mathbf{C}_{GS}(\mathscr{A}) = \bigoplus_{r\in\mathbb{N}} \mathbf{C}_{GS}(\mathscr{A})_r.$$

Taking cohomology yields a decomposition for $H^\bullet\mathbf{C}_{GS}(\mathscr{A})$.

Assume further the following.

- The restriction map $f^u\colon \mathscr{A}(U) \to \mathscr{A}(V)$ is a flat epimorphism of rings for each $u\colon V \to U$.
- The algebra $\mathscr{A}(U)$ is essentially of finite type and smooth $\mathbf{R}$-algebra for each U.

Recall that a homomorphism of rings is called an epimorphism if it is an epimorphism in the category of noncommutative rings. For instance, every surjective homomorphism of commutative rings is an epimorphism. Then one obtains the *presheaf of differential* $\Omega_{\mathscr{A}}\colon \mathfrak{U}^{op} \to \mathrm{Mod}(\mathscr{A})$ with $\Omega_{\mathscr{A}}(U) = \Omega_{\mathscr{A}(U)/\mathbf{R}}$. Since we have a canonical isomorphism $\mathscr{A}(V)\otimes_{\mathscr{A}(U)}\Omega_{\mathscr{A}(U)} \cong \Omega_{\mathscr{A}(V)}$ by the first additional assumption, the induced restriction maps $\mathscr{T}_{\mathscr{A}(U)/\mathbf{R}} \to \mathscr{T}_{\mathscr{A}(V)/\mathbf{R}}$ yield the *tangent presheaf* $\mathscr{T}_{\mathscr{A}}\colon \mathfrak{U}^{op} \to \mathrm{Mod}(\mathscr{A})$ with $\mathscr{T}_{\mathscr{A}}(U) = \mathscr{T}_{\mathscr{A}(U)/\mathbf{R}}$. From the second additional assumption it follows that antisymmetrizations $\wedge^n\mathscr{T}_{\mathscr{A}(U)} \to H^n\mathbf{C}(\mathscr{A}(U))$ are isomorphisms.

Lemma 2.3. ([DLL17, Theorem 3.3]) *Let $\mathfrak{U}$ be a small category and $\mathscr{A}\colon \mathfrak{U}^{op} \to \mathrm{CAlg}(\mathbf{R})$ a presheaf of commutative algebras. Assume that the algebra $\mathscr{A}(U)$ is essentially of finite type and smooth $\mathbf{R}$-algebra for each U. Assume further that the restriction map $f^u\colon \mathscr{A}(U) \to \mathscr{A}(V)$ is a flat epimorphism of rings for each $u\colon V \to U$. Then there is a canonical bijection*

$$H^n\mathbf{C}_{GS}(\mathscr{A}) = \bigoplus_{r=0}^{n} H^n\mathbf{C}_{GS}(\mathscr{A})_r \cong \bigoplus_{p+q=n} H^p(\mathfrak{U}, \wedge^q\mathscr{T}_{\mathscr{A}}). \tag{2.1}$$

From the proof, one sees that any Gerstenharber–Shack cohomology class $\mathfrak{c}_{GS}$ is represented by a normalized reduced *decomposable* cocycle $\theta_{0,n}, \theta_{1,n-1}, \dots, \theta_{n,0}$ in the sense that $\theta_{n-r,r}$ are reduced and belong to $\bar{\mathbf{C}}^{n-r,r}(\mathscr{A})_r$. Each $\theta_{n-r,r} = (\theta^{\tau}_{n-r,r})_{\tau\in\mathcal{N}_{n-r}(\mathfrak{U})}$ lifts to a unique simplicial cocycle $\Theta_{n-r,r} = (\Theta^{\tau}_{n-r,r})_{\tau\in\mathcal{N}_{n-r}(\mathfrak{U})} \in \mathbf{C}'^{n-r}_{simp}(\wedge^r\mathscr{T}_{\mathscr{A}})$. The image of $\mathfrak{c}_{GS}$ under the bijection is the cohomology class $\mathfrak{c}_{simp}$ represented by $\Theta_{0,n}, \Theta_{1,n-1}, \dots, \Theta_{n,0}$.

2.6. **Comparison with relative Hochschild cohomology.** We describe the relationship between simplicial cohomology and Čech cohomology for a presheaf $\mathscr{F}\colon \mathfrak{U}^{op} \to \mathrm{Mod}(\mathbf{R})$ in the case where $\mathfrak{U}$ is a poset with binary meets. We use the symbol $\cap$ to denote meets in $\mathfrak{U}$. For a p-sequence $\tau = (U^\tau_0, U^\tau_1, \dots, U^\tau_p) \in \mathfrak{U}^{p+1}$ we denote by $\cap\tau$ the meet of all coordinates of τ. The Čech complex $\check{\mathbf{C}}(\mathscr{F})$ of $\mathscr{F}$ has

$$\check{\mathbf{C}}^p(\mathscr{F}) = \prod_{\tau\in\mathfrak{U}^{p+1}} \mathscr{F}(\cap\tau)$$

as the p-th term with the usual differentials. A Čech cochain $\psi = (\psi^\tau)_\tau$ is *alternating* if $\psi^\tau = 0$ whenever two coordinates of τ are equal, and $\psi^{\tau s} = (-1)^{\mathrm{sign}(s)}\psi^\tau$ for any permutation s of the set $\{0, 1, \dots, p\}$. Here, we regard τ as a set theoretic map $\{0, 1, \dots, p\} \to \mathfrak{U}$. The alternating Čech cochains form a subcomplex $\check{\mathbf{C}}'(\mathscr{F})$ which is quasi-isomorphic to $\check{\mathbf{C}}(\mathscr{F})$ via the inclusion.

To a p-sequence τ, one associates a p-simplex

$$\bar{\tau} = (d\bar{\tau} = \cap^p_{j=0}U^\tau_j \to \cap^p_{j=1}U^\tau_j \to \cdots \to \cap^p_{j=p-1}U^\tau_j \to U^\tau_p = c\bar{\tau}).$$

Conversely, any p-simplex μ can be regarded as a p-sequence $\tilde{\mu}$ by forgetting the inclusions. Define a map $\delta_i \colon \mathfrak{U}^{p+1} \to \mathfrak{U}^p$ for $i = 1, \ldots, p$ as

$$\delta_i \tau = (U_0^\tau, \ldots, U_{i-2}^\tau, U_{i-1}^\tau \cap U_i^\tau, U_{i+1}^\tau, \ldots, U_p^\tau).$$

There are morphisms $\iota \colon \mathbf{C}'_{simp} \to \check{\mathbf{C}}'(\mathscr{F})$, $\pi \colon \check{\mathbf{C}}'(\mathscr{F}) \to \mathbf{C}'_{simp}$ of complexes defined as

$$\iota(\phi)^\tau = \sum_{s \in \mathfrak{S}_{p+1}} (-1)^{\mathrm{sign}(s)} \phi^{\overline{\tau s}}, \ \phi \in \mathbf{C}'_{simp}(\mathscr{F}), \ \tau \in \mathfrak{U}^{p+1},$$

$$\pi(\psi)^\mu = \psi^{\tilde{\mu}}, \ \psi \in \check{\mathbf{C}}'(\mathscr{F}), \ \mu \in \mathcal{N}_p(\mathfrak{U}),$$

which induce mutually inverse isomorphisms between $H^\bullet(\mathfrak{U}, \mathscr{F})$ and $\check{H}^\bullet(\mathfrak{U}, \mathscr{F})$ [DLL17, Lemma 3.9].

Now, for a smooth proper $\mathbf{R}$-scheme X we give an alternative description of the relative Hochschild cohomology. As explained above, we have $HH^\bullet(X/\mathbf{R}) \cong HT^\bullet(X/\mathbf{R})$. Choose a finite affine open cover $\mathfrak{U}$ closed under intersections. By definition $\mathfrak{U}$ is semi-separating, i.e., $\mathfrak{U}$ is closed under finite intersections. For every quasi-coherent sheaf $\mathscr{F}$ on X, one can apply [DLL17, Lemma 3.9] and Leray's theorem [Har77, Theorem 4.5] to obtain

$$H^\bullet(\mathfrak{U}, \mathscr{F}|_{\mathfrak{U}}) \cong \check{H}^\bullet(\mathfrak{U}, \mathscr{F}|_{\mathfrak{U}}) \cong \check{H}^\bullet(\mathfrak{U}, \mathscr{F}) \cong H^\bullet(X, \mathscr{F}) \tag{2.2}$$

with $\mathscr{F}|_{\mathfrak{U}}$ regarded as a presheaf on $\mathfrak{U}$. Since X is smooth over $\mathbf{R}$ and open immersions $V \hookrightarrow U$ in $\mathfrak{U}$ define flat epimorphisms $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$, combining (2.1) with (2.2), we obtain

Lemma 2.4. ([DLL17, Corollary 3.4]) *Let X be a smooth proper $\mathbf{R}$-scheme with a finite affine open cover $\mathfrak{U}$ closed under intersections. Let $\mathscr{O}_X|_{\mathfrak{U}}$, $\mathscr{T}_{X/\mathbf{R}}|_{\mathfrak{U}}$ be the restrictions of $\mathscr{O}_X$, $\mathscr{T}_{X/\mathbf{R}}$ to $\mathfrak{U}$ respectively. Then there are canonical isomorphisms*

$$H^n\mathbf{C}_{GS}(\mathscr{O}_X|_{\mathfrak{U}}) = \bigoplus_{r=0}^{n} H^n\mathbf{C}_{GS}(\mathscr{O}_X|_{\mathfrak{U}})_r \cong \bigoplus_{p+q=n} H^p(\mathfrak{U}, \wedge^q \mathscr{T}_{X/\mathbf{R}}|_{\mathfrak{U}}) \cong HH^n(X/\mathbf{R}), \tag{2.3}$$

where the first isomorphism respects the Hodge decomposition.

3. Deformations of relatively smooth proper schemes

In this section, we review the classical deformation theory of schemes. The main reference is [Har10]. We explain how deformations of smooth proper $\mathbf{k}$-varieties extend to Toda's construction [Tod09], which can be adapted to deformations of relatively smooth proper schemes along square zero extensions in a straightforward way. When the original scheme is a deformation of a higher dimensional Calabi–Yau manifold, Toda's construction gives the category of quasi-coherent sheaves on deformations of the Calabi–Yau manifold.

3.1. Deformations of schemes.

Let X be a $\mathbf{k}$-scheme and A a local artinian $\mathbf{k}$-algebra with residue field $\mathbf{k}$. An *A-deformation* of X is a pair (X_A, i_A), where X_A is a scheme flat over A and $i_A \colon X \hookrightarrow X_A$ is a closed immersion such that the induced map $X \to X_A \times_A \mathbf{k}$ is an isomorphism. Two deformations $(X_A, i_A), (X'_A, i'_A)$ are *equivalent* if there is an A-isomorphism $X_A \to X'_A$ compatible with i_A, i'_A. The deformation functor

$$\mathrm{Def}_X \colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$$

sends each $A \in \mathrm{Art}_{\mathbf{k}}$ to the set of equivalence classes of A-deformations of X.

Assume that X is projective over $\mathbf{k}$. Then Def_X satisfies Schlessinger's criterion and there exists a *miniversal formal family* (R, ξ) for Def_X, where R is a complete local noetherian $\mathbf{k}$-algebra with residue field $\mathbf{k}$, and $\xi = \{\xi_n\}_n$ belongs to the limit

$$\widehat{\mathrm{Def}}_X(R) = \varprojlim \mathrm{Def}_X(R/\mathfrak{m}_R^n)$$

of the inverse system

$$\cdots \to \mathrm{Def}_X(R/\mathfrak{m}_R^{n+2}) \to \mathrm{Def}_X(R/\mathfrak{m}_R^{n+1}) \to \mathrm{Def}_X(R/\mathfrak{m}_R^{n}) \to \cdots$$

induced by the natural quotient maps $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$. The formal family ξ corresponds to a natural transformation

$$h_R = \mathrm{Hom}_{\mathbf{k}\text{-alg}}(R, -) \to \mathrm{Def}_X,$$

which sends each $g \in h_R(A)$ factorizing through $R \to R/\mathfrak{m}_R^{n+1} \xrightarrow{g_n} A$ to $\mathrm{Def}_X(g_n)(\xi_n)$.

Let X_n be the schemes which define ξ_n. There is a noetherian formal scheme $\mathscr{X}$ over R such that $X_n \cong \mathscr{X} \times_R R/\mathfrak{m}_R^{n+1}$ for each n. By abuse of notation, we use the same symbol ξ to denote $\mathscr{X}$. Thus any scheme which defines an equivalence class $[X_A, i_A]$ can be obtained as the pullback of ξ along some morphism of noetherian formal schemes $\operatorname{Spec} A \to \operatorname{Spf} R$. If X has no infinitesimal automorphisms which restrict to the identity of X, then every equivalence class $[X_A, i_A]$ becomes just a deformation (X_A, i_A) and we have a natural isomorphism $h_R \cong \mathrm{Def}_X$. In this case, we call Def_X *prorepresentable* and (R, ξ) a *universal formal family* for Def_X.

3.2. **Algebraization.** Let X be a projective $\mathbf{k}$-variety. We call a miniversal formal family (R, ξ) for Def_X *effective* when there exists a scheme X_R flat and of finite type over R whose formal completion along the closed fiber X is isomorphic to ξ. By [GD61, Theorem III5.4.5] the family (R, ξ) is effective if deformations of any invertible sheaf on X are unobstructed. Note that this is the case, for instance, if we have $H^2(\mathscr{O}_X) = 0$. From the proof, one sees that X_R is projective over R. We will call such X_R an *effectivization* of ξ.

The deformation functor Def_X can naturally be extended to a functor defined on the category $\mathrm{Alg}^{aug}(\mathbf{k})$ of augmented noetherian $\mathbf{k}$-algebras. By abuse of notation, we use the same symbol Def_X to denote the extended functor, which sends each $(\mathbf{P}, \mathfrak{m}_{\mathbf{P}}) \in \mathrm{Alg}^{aug}(\mathbf{k})$ to the set of equivalence classes of deformations over $(\mathbf{P}, \mathfrak{m}_{\mathbf{P}})$. Since the functor Def_X is locally of finite presentation, by [Art69b, Theorem 1.6] the miniversal formal family is *algebraizable*, i.e., there exists a triple (S, s, X_S) where S is an algebraic $\mathbf{k}$-scheme with a distinguished closed point $s \in S$, and X_S is a flat and of finite type S-scheme whose formal completion along the closed fiber X over s is isomorphic to ξ. We call the scheme X_S a *versal deformation* over S. When there exists a versal deformation, we say that the miniversal formal family (R, ξ) is *algebraizable*.

3.3. **Deformations of higher dimensional Calabi–Yau manifolds.** Here, we focus on a special case where several interesting results hold. Let X_0 be a Calabi–Yau manifold of dimension more than two. Then the deformation functor Def_{X_0} has an effective universal formal family (R, ξ). Since deformations of Calabi–Yau manifolds are unobstructed, the complete local noetherian ring R is regular and we have

$$R \cong \mathbf{k}[\![t_1, \ldots, t_d]\!]$$

with $d = \dim_{\mathbf{k}} H^1(\mathscr{T}_{X_0})$. Every A-deformation of X_0 is smooth projective over A, as we have

Lemma 3.1. ([Mora, Lemma 2.4]) *The effectivization* X_R *for* (R, ξ) *is regular and the morphism* $\pi_R \colon X_R \to \operatorname{Spec} R$ *is smooth of relative dimension* $\dim X_0$.

Now, we briefly recall the construction of X_S. Consider the extended functor

$$\mathrm{Def}_{X_0} \colon \mathrm{Alg}^{aug}(\mathbf{k}) \to \mathrm{Set}\,.$$

Fix an isomorphism $R \cong \mathbf{k}[\![t_1, \ldots, t_d]\!]$. Let $T = \mathbf{k}[t_1, \ldots, t_d]$ and $t \in \operatorname{Spec} T$ be the closed point corresponding to maximal ideal $(t_1, \ldots, t_d)$. There is a filtered inductive system $\{R_i\}_{i \in I}$ of finitely generated T-subalgebras of R whose colimit is R. Since Def_{X_0} is locally of finite presentation, $[X_R, i_R]$ is the image of some element $\zeta_i \in \mathrm{Def}_{X_0}((R_i, \mathfrak{m}_{R_i}))$ by the canonical map

$\mathrm{Def}_{X_0}((R_i, \mathfrak{m}_{R_i})) \to \mathrm{Def}_{X_0}(R)$. By [Art69a, Corollary 2.1] there exists an étale neighborhood $\operatorname{Spec} S$ of t in $\operatorname{Spec} T$ with first order approximation $\varphi\colon R_i \to S$ of $R_i \hookrightarrow R$. Let $[X_S, i_S]$ be the image of ζ_i by the map $\mathrm{Def}_{X_0}(\varphi)$. From miniversality of (R, ξ), it follows that the formal completion of X_S along the closed fiber X_0 over $s \in \operatorname{Spec} S$ is isomorphic to ξ, where s is the distinguished closed point mapping to t. By construction, $\operatorname{Spec} S$ is a nonsingular affine $\mathbf{k}$-variety and X_S is flat of finite type over S. Exploiting inherited smoothness and projectivity of X_R by terms in the projective system $\{X_{R_i}\}_{i\in I}$ for sufficiently large indices, one can show

Lemma 3.2. ([Mora, Lemma 2.3]) *Let X_0 be a Calabi–Yau manifold of dimension more than two. Then there exists a nonsingular affine* $\mathbf{k}$*-variety* $\operatorname{Spec} S$ *with a versal deformation X_S which is smooth projective of relative dimension* $\dim X_0$ *over S.*

3.4. T^i **functors.** Let $A \to B$ be a ring homomorphism and M a B-module. Define the groups $T^i(B/A, M)$ for $i = 0, 1, 2$ as the i-th cohomology of the complex $\mathrm{Hom}_B(L_\bullet, M)$, where

$$L_\bullet = L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$$

is the *cotangent complex*. When the ring homomorphism $A \to B$ is a surjection with kernel J, $L_\bullet$ is given as follows. Choose a free A-module P and a surjection $j\colon P \to J$ with kernel Q. We have two short exaxt sequences

$$0 \to J \to A \to B \to 0,\ 0 \to Q \to P \xrightarrow{j} J \to 0.$$

Let P_0 be the submodule of P generated by all relations of the form $j(a)b - j(b)a$ for $a, b \in P$. From $j(P_0) = 0$ it follows $P_0 \subset Q$. Take $L_2 = Q/P_0$, $L_1 = P \otimes_A B$, and $L_0 = 0$. Note that L_2 is a B-module. Indeed, for $a \in J$ there is an element $a' \in P$ such that $a = j(a')$. Then we have $ax \equiv j(x)a' \equiv 0$ modulo P_0 for $x \in Q$. The differential $d_2\colon L_2 \to L_1$ is the map induced by the inclusion $Q \to P$ and $d_1 = 0$. By [Har10, Lemma 3.2] the B-modules $T^i(B/A, M)$ do not depend on the choice of P up to isomorphism.

Lemma 3.3. ([Har10, Theorem 3.4]) *Let $A \to B$ be a homomorphism of rings. Then*

$$T^i(B/A, -)\colon \mathrm{Mod}(B) \to \mathrm{Mod}(B),\ i = 0, 1, 2$$

define covariant additive functor.

The construction of T^i functors is compatible with localization and one obtains sheaves $\mathscr{T}^i(X/Y, \mathscr{F})$, $i = 0, 1, 2$ for any morphism of $\mathbf{k}$-schemes $f\colon X \to Y$ and any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ [Har10, Exercise 3.5]. The sections of $\mathscr{T}^i(X/Y, \mathscr{F})$ over $U = \operatorname{Spec} B \subset f^{-1}(V)$ give $T^i(B/A, M)$, where $V = \operatorname{Spec} A \subset Y$ and $\mathscr{F}|_U = \tilde{M}$ for some B-module M.

3.5. **Infinitesimal extension of schemes.** Let X be a scheme of finite type over $\mathbf{R}$ and $\mathscr{F}$ a coherent sheaf on X. An *infinitesimal extension* of X by $\mathscr{F}$ is a pair $(Y, \mathfrak{I})$, where Y is a scheme of finite type over $\mathbf{S}$ and $\mathfrak{I} \subset \mathscr{O}_Y$ is an ideal sheaf such that $\mathfrak{I}^2 = 0$, $(Y, \mathscr{O}_Y/\mathfrak{I}) \cong (X, \mathscr{O}_X)$, and $\mathfrak{I} \cong \mathscr{F}$ as an $\mathscr{O}_X$-module. Two infinitesimal extensions $(Y, \mathfrak{I}), (Y', \mathfrak{I}')$ are *equivalent* if there is an isomorphism $\mathscr{O}_Y \to \mathscr{O}_{Y'}$ which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathscr{O}_Y \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{O}_{Y'} \arrow[r] & \mathscr{O}_X \arrow[r] & 0
\end{tikzcd}$$

commute. The trivial extension is a sheaf $\mathscr{O}_X \oplus \mathscr{F}$ of abelian group endowed with the ring structure by

$$(a, f) \cdot (a', f') = (aa', af' + a'f).$$

Assume that X is smooth proper over $\mathbf{R}$. Recall that for the square zero extension

$$0 \to \mathbf{I} \to \mathbf{S} \to \mathbf{R} \to 0 \tag{3.1}$$

we have $\mathbf{I} \cong \mathbf{R}^{\oplus l}$ as an $\mathbf{R}$-module. Note that given an infinitesimal extension $(Y, \mathfrak{I})$ of X by $\mathscr{O}_X^{\oplus l}$, Y is flat over $\mathbf{S}$ since $\mathscr{O}_X$ is flat over $\mathbf{R}$ and $\mathscr{O}_X^{\oplus l} \to \mathscr{O}_Y$ is injective [Har10, Proposition 2.2]. Below, we collect fundamental results necessary to describe the relationship between deformations and extensions of schemes.

Lemma 3.4. ([Har10, Exercise 4.7]) *Let X be a smooth $\mathbf{R}$-scheme and $g\colon Y \to X$ a morphism from an affine $\mathbf{R}$-scheme Y to X, and $i_{\mathbf{S}}\colon Y \hookrightarrow Y'$ an $\mathbf{S}$-deformation of Y. Then g lifts to a morphism $h\colon Y' \to X$ such that $h \circ i_{\mathbf{S}} = g$.*

Lemma 3.5. ([Har10, Proposition 3.6, Exercise 5.2]) *Let $A \to B$ be a homomorphism of rings, M a B-module, and B' an extension of B by M. Then the automorphism group of B' is given by*

$$T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M).$$

Lemma 3.6. ([Har10, Theorem 5.1]) *Let $A \to B$ be a homomorphism of rings and M a B-module. Then there is a bijection between the set of equivalence classes of B by M and the group $T^1(B/A, M)$. The trivial extension corresponds to the zero element.*

Lemma 3.7. ([Har10, Theorem 4.11]) *Let $f\colon X \to Y$ be an of finite type morphism of noetherian $\mathbf{k}$-schemes. Then f is smooth if and only if it is flat and $\mathscr{T}^1(X/Y, \mathscr{F}) = 0$ for every coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

Now, we are ready to show relevant results to our setting.

Lemma 3.8. *Let X be a smooth separated $\mathbf{R}$-scheme. Then every $\mathbf{S}$-deformation (Y, j) of X is locally trivial.*

Proof. Since Y is flat over $\mathbf{S}$, (3.1) induces a short exact sequence

$$0 \to \mathscr{O}_X^{\oplus l} \to \mathscr{O}_Y \to \mathscr{O}_X \to 0,$$

which defines equivalence classes of infinitesimal extensions of coordinate rings on affine open subschemes of X. Let $i_{\mathbf{S}}\colon \operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ be the induced deformation of any affine open subscheme. Since $\operatorname{Spec} B$ is smooth over $\mathbf{R}$, by Lemma 3.4 the identity $\operatorname{Spec} B \to \operatorname{Spec} B$ lifts to a morphism $h\colon \operatorname{Spec} A \to \operatorname{Spec} B$ such that $h \circ i_{\mathbf{S}} = \mathrm{id}$. The lift h induces a morphism $\operatorname{Spec} A \to \operatorname{Spec} B \times_{\mathbf{R}} \mathbf{S}$ of schemes flat of finite type over $\mathbf{S}$. Now, one can apply [Har10, Exercise 4.2] to see that the induced morphism is an isomorphism. □

Proposition 3.9. *Let X be a smooth separated $\mathbf{R}$-scheme. Then there is a bijection*

$$\mathrm{Def}_X(\mathbf{S}) \cong H^1(X, \mathscr{T}_{X/\mathbf{R}})^{\oplus l},$$

where $\mathscr{T}_{X/\mathbf{R}}$ is the relative tangent sheaf on X.

Proof. Let (Y, j) be an $\mathbf{S}$-deformation of X. Take an affine open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X. By Lemma 3.8 we may assume that the induced deformations $U_i \hookrightarrow V_i \subset Y$ are trivial. Choose isomorphisms $\varphi_i\colon U_i \times_{\mathbf{R}} \mathbf{S} \to V_i$ and write φ_{ij} for the composition $\varphi_j^{-1} \circ \varphi_i$ on $U_{ij} \times_{\mathbf{R}} \mathbf{S}$, where the intersections $U_{ij} = U_i \cap U_j$ are again affine as X is separated over $\mathbf{R}$. Let $\operatorname{Spec} B = U_{ij}$ and $\operatorname{Spec} A = U_{ij} \times_{\mathbf{R}} \mathbf{S}$. According to Lemma 3.5, the set of automorphisms of extensions $\tilde{A}$ of $\tilde{B}$ by $\tilde{B}^{\oplus l}$ bijectively corresponds to $T^0(B/\mathbf{R}, B^{\oplus l}) \cong \mathrm{Hom}(\Omega_{B/\mathbf{R}}, B)^{\oplus l}$. Then $\{\varphi_{ij}\}_{i,j \in I}$ define a collection $\{\theta_{ij}\}_{i,j \in I}$ of sections $\theta_{ij} \in H^0(U_{ij}, \mathscr{T}_{X/\mathbf{R}})^{\oplus l}$ on U_{ij}. One checks $\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$ and $\{\theta_{ij}\}_{i,j \in I}$ is a Čech 1-cocycle with respect to $\mathfrak{U}$. Another choice of isomorphisms $\varphi_i'\colon U_i \times_{\mathbf{R}} \mathbf{S} \to V_i$ yields a collection $\{\varphi_{ij}'\}_{i,j \in I}$ of automorphisms such that $\varphi_{ij}' = (\varphi_j^{-1} \circ \varphi_j')^{-1} \circ \varphi_{ij} \circ (\varphi_i^{-1} \circ \varphi_i')$. It follows $\theta'_{ij} = \theta_{ij} + \alpha_i - \alpha_j$ for some sections $\alpha_i \in H^0(U_i, \mathscr{T}_{X/\mathbf{R}})^{\oplus l}$. Thus we obtain a well defined assignment

$$\mathrm{Def}_X(\mathbf{S}) \to H^1(X, \mathscr{T}_{X/\mathbf{R}})^{\oplus l},\ [Y, j] \mapsto \{\theta_{ij}\}_{i,j\in I},$$

as $\{\theta_{ij}\}_{i,j\in I}$ does not depends on $\mathfrak{U}$.

Conversely, an element of $H^1(X, \mathscr{T}_{X/\mathbf{R}})^{\oplus l}$ can be represented by Čech 2-cocycle $\{\theta_{ij}\}_{i,j\in I}$ with respect to $\mathfrak{U}$. As explained above, the cocycle define automorphisms of the trivial deformations $U_{ij} \times_{\mathbf{R}} \mathbf{S}$, which glue to yield a global deformation (Y', j') of X. Clearly, this construction gives the inverse assignment. □

Corollary 3.10. *There is a canonical bijection between* $\mathrm{Def}_X(\mathbf{S})$ *and the set of equivalence classes of infinitesimal extensions of* X *by* $\mathscr{O}_X^{\oplus l}$.

Proof. By Lemma 3.6 and Lemma 3.7 any extension of X by $\mathscr{O}_X^{\oplus l}$ is locally trivial. Then due to Lemma 3.5 the claim follows from the same argument as in the proof of Proposition 3.9. □

3.6. **Toda's construction.** Let X_0 be a smooth projective $\mathbf{k}$-variety. In [Tod09] Toda constructed the category of $\tilde{\alpha}$-twisted sheaves on the noncommutative scheme $(X_0, \mathscr{O}_{X_0}^{(\beta,\gamma)})$ over the ring of dual numbers for each $[\phi_0] \in HT^2(X_0)$ represented by a cocycle

$$(\alpha_0, \beta_0, \gamma_0) \in H^2(\mathscr{O}_{X_0}) \oplus H^1(\mathscr{T}_{X_0}) \oplus H^0(\wedge^2 \mathscr{T}_{X_0}).$$

Here, we apply his idea to a smooth proper $\mathbf{R}$-scheme X and $[\phi] \in HT^2(X/\mathbf{R})^{\oplus l}$ represented by

$$(\alpha, \beta, \gamma) = \big((\alpha^1, \ldots, \alpha^l), (\beta^1, \ldots, \beta^l), (\gamma^1, \ldots, \gamma^l)\big) \in H^2(\mathscr{O}_X)^{\oplus l} \oplus H^1(\mathscr{T}_{X/\mathbf{R}})^{\oplus l} \oplus H^0(\wedge^2 \mathscr{T}_{X/\mathbf{R}})^{\oplus l}.$$

Take a finite affine open cover $\mathfrak{U} = \{U_i\}_{i=1}^N$ of X and let $\mathfrak{U} \times_{\mathbf{R}} \mathbf{S} = \{U_i \times_{\mathbf{R}} \mathbf{S}\}_{i=1}^N$. Consider the extension of X by $\mathscr{O}_X^{\oplus l}$ whose equivalence class corresponds to β, giving rise to an classical $\mathbf{S}$-deformation X_β of X by Corollary 3.10. We modify the multiplication on $\mathscr{O}_X \oplus C(\mathfrak{U}, \mathscr{O}_X^{\oplus l})$ as

$$\begin{aligned}&(a, \{b_i^1\}, \ldots, \{b_i^l\}) *_\gamma (c, \{d_i^1\}, \ldots, \{d_i^l\})\\ =&(ac, \{ad_i^1 + b_i^1 c + \gamma_i^1(a, c)\}, \ldots, \{ad_i^l + b_i^l c + \gamma_i^l(a, c)\}),\end{aligned}$$

where $\gamma^j \colon \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$ are regarded as bidifferential operators. We denote by $X_{(\beta,\gamma)} = (X_\beta, \mathscr{O}_{X_\beta}^\gamma)$ the resulting noncommutative $\mathbf{S}$-scheme. By the standard argument, one sees that up to isomorphism the scheme does not depend on the choice of $\mathfrak{U}$ and Čech representative of γ. From α one obtains an element

$$\tilde{\alpha} = \{1 - \alpha^1_{i_0 i_1 i_2}\epsilon_1 - \cdots - \alpha^l_{i_0 i_1 i_2}\epsilon_l\}_{i_0 i_1 i_2} \in \mathbf{C}^2(X_\beta, Z(\mathscr{O}_{X_\beta}^\gamma)^*),$$

which is a cocycle. Then $\tilde{\alpha}$-twisted sheaves on $X_{(\beta,\gamma)}$ form a category $\mathrm{Mod}(X_{(\beta,\gamma)}, \tilde{\alpha})$. By the similar argument to [Căl00, Lemma 1.2.3, 1.2.8], one sees that up to equivalence the category does not depend on the choice of $\mathfrak{U}$ and Čech representative of α. We denote by $\mathrm{Qch}(X, \phi)$ the full abelian subcategory spanned by $\tilde{\alpha}$-twisted quasi-coherent sheaves.

Assume that X is an $\mathbf{R}$-deformation of a higher dimensional Calabi–Yau manifold. Then we have

$$HT^2(X/\mathbf{R}) = H^1(\mathscr{T}_{X/\mathbf{R}}).$$

In this case, Toda's construction yields nothing but the category of quasi-coherent sheaves on the $\mathbf{S}$-deformation of X along ϕ.

Proposition 3.11. *Let* X_0 *be a Calabi–Yau manifold with* $\dim X_0 > 2$ *and* X *an* $\mathbf{R}$*-deformation of* X_0*. Then for every cocycle* $\phi \in HT^2(X/\mathbf{R})^{\oplus l} = H^1(\mathscr{T}_{X/\mathbf{R}})^{\oplus l}$ *we have*

$$\mathrm{Qch}(X, \phi) = \mathrm{Qch}(X_\phi),$$

where X_ϕ *is the* $\mathbf{S}$*-deformation of* X *along* ϕ.

In this section, we review the deformation theory of linear and abelian categories developed by Lowen and van den Bergh in [LV06b], introducing the fundamental notion of flatness. As explained there, when considering only flat nilpotent deformations over a certain class of rings, one avoids any set theoretic issue by choosing sufficiently large universe. Moreover, both linear and abelian deformations reduce to strict linear deformations without affecting the deformation theory up to equivalence. Along square zero extensions, flat deformations of linear and abelian categories are controlled by the second Hochschild cohomology of the corresponding linear categories.

4.1. **Universes.** First, we need to extend the Zermelo–Fraenkel axioms of the set theory to avoid foundational issues in the deformation theory of categories. One solution is the theory of universes introduced by Grothendieck with the axiom of choice and the universe axiom. A *universe* is a set $\mathcal{U}$ with the following properties:

- if $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$,
- if $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$,
- if $x \in \mathcal{U}$ then the powerset $\mathcal{P}(x)$ of x is in $\mathcal{U}$,
- if $(x_i)_{i\in I}$ is a family of objects of $\mathcal{U}$ indexed by an element of $\mathcal{U}$ then $\bigcup_{i\in I} x_i \in \mathcal{U}$,
- if $U \in U$ and $f\colon U \to \mathcal{U}$ is a function then $\{f(x) \mid x \in U\} \in \mathcal{U}$.

A universe $\mathcal{U}$ containing $\mathbb{N}$ is a model for the Zermelo–Fraenkel axioms of the set theory with the axiom of choice. Since the known nonempty universe only contains finite sets, the *universe axiom* is added, which imposes every set to be an element of a universe.

Consider the category $\mathcal{U} - \mathrm{Set}$ whose objects are elements of $\mathcal{U}$ and whose morphisms are ordinary maps between sets in $\mathcal{U}$. The category $\mathcal{U} - \mathrm{Cat}$ consists of categories whose objects and morphisms respectively form sets being an element of $\mathcal{U}$. Similarly, by requiring the underlying sets to belong to $\mathcal{U}$, we obtain categories with a structure such as abelian groups and rings. We call a category *$\mathcal{U}$-small* when its objects and morphisms respectively form sets with the same cardinality as an element of $\mathcal{U}$, and *essentially $\mathcal{U}$-small* when it is equivalent to a $\mathcal{U}$-small category. A *$\mathcal{U}$-category* is a category whose Hom-sets have the the same cardinality as an element of $\mathcal{U}$. The axiom of choice allows us to replace a $\mathcal{U}$–category $\mathscr{C}$ by an equivalent category $\mathscr{C}'$ with $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{C}')$ and $\mathscr{C}'(C, D) \in \mathcal{U}$ for all $C, D \in \mathrm{Ob}(\mathscr{C}')$. When $\mathscr{C}$ is abelian with a generator, we call $\mathscr{C}$ *$\mathcal{U}$-Grothendieck*. Every $\mathcal{U}$-Grothendieck category $\mathscr{C}$ admits $\mathcal{U}$-small colimits and $\mathcal{U}$-small filtered colimits are exact in $\mathscr{C}$.

Throughout the paper, we work with a fixed universe $\mathcal{U}$ containing $\mathbb{N}$. All the notion based on universes will be with respect to $\mathcal{U}$ and all the related symbols will be tacitly prefixed by $\mathcal{U}$. By taking $\mathcal{U}$ sufficiently large, we may assume all categories to be small. Unless otherwise specified, we will be free from any issue caused by the choice of universes.

4.2. **Flatness.** The notion of flatness for abelian categories was introduced in [LV06b]. For a while, we temporarily drop the assumption on $\mathbf{R}$ and $\mathbf{S}$ imposed at the beginning of Section 2. Let $\mathbf{R}$ be a commutative ring. An $\mathbf{R}$-*linear category* is a category $\mathfrak{a}$ enriched over the abelian category $\mathrm{Mod}(\mathbf{R})$ of $\mathbf{R}$-modules. Namely, $\mathfrak{a}$ is a pre-additive category together with a ring map $\rho\colon \mathbf{R} \to \mathrm{Nat}(1_{\mathfrak{a}}, 1_{\mathfrak{a}})$ inducing a ring map $\rho_A\colon \mathbf{R} \to \mathfrak{a}(A, A)$ for each $A \in \mathfrak{a}$ and an action of $\mathbf{R}$ on each Hom-set.

Assume that $\mathbf{R}$ is coherent, i.e., any finitely generated ideal is finitely presented as an $\mathbf{R}$-module. Typical examples are given by noetherian rings. We denote by $\mathrm{mod}(\mathbf{R})$ the full abelian subcategory of finitely presented $\mathbf{R}$-modules. Let $\mathscr{C}$ be an $\mathbf{R}$-linear abelian category. We call an object $C \in \mathscr{C}$ *flat* if the natural finite colimit preserving functor $(-) \otimes_{\mathbf{R}} C\colon \mathrm{mod}(\mathbf{R}) \to \mathscr{C}$

is exact, and *coflat* if the natural finite limit preserving functor $\mathrm{Hom}_{\mathbf{R}}(-, C)\colon \mathrm{mod}(\mathbf{R}) \to \mathscr{C}$ is exact.

An $\mathbf{R}$-linear category $\mathfrak{a}$ is *flat* if its Hom-sets are flat $\mathbf{R}$-modules. Namely, the functors $- \otimes_{\mathbf{R}} \mathfrak{a}(A, A')\colon \mathrm{mod}(\mathbf{R}) \to \mathrm{Mod}(\mathbf{R})$ are exact for all $A, A' \in \mathfrak{a}$. An $\mathbf{R}$-linear abelian category $\mathscr{C}$ is *flat* if for each $Y \in \mathrm{mod}(\mathbf{R})$ the functor $\mathrm{Tor}_1^{\mathbf{R}}(Y, -)\colon \mathscr{C} \to \mathscr{C}$ is co-effaceble, i.e., for each $C \in \mathscr{C}$ there is an epimorphism $f\colon C' \to C$ with $\mathrm{Tor}_1^{\mathbf{R}}(Y, f) = 0$ [LV06b, Proposition 3.1]. Here, $\mathrm{Tor}_i^{\mathbf{R}}(Y, -)$ is the left derived functor of the finite colimit preserving functor $Y \otimes_{\mathbf{R}} (-)\colon \mathscr{C} \to \mathscr{C}$. The flatness has the following characterizations [LV06b, Proposition 3.3, 3.4, 3.6, 3.7].

- $\mathscr{C}$ is flat if and only if $\mathscr{C}^{op}$ is flat.
- $\mathscr{C}$ is flat if and only if all injectives in $\mathscr{C}$ are coflat.
- $\mathscr{C}$ is flat if and only if $\mathrm{Ind}(\mathscr{C})$ is flat.
- $\mathfrak{a}$ is flat if and only if the abelian category $\mathrm{Mod}(\mathfrak{a})$ is flat.

Here, $\mathrm{Ind}(\mathscr{C})$ is the Ind-completion of $\mathscr{C}$, i.e., the full subcategory of $\mathrm{Mod}(\mathscr{C})$ consisting of left exact functors, where $\mathrm{Mod}(\mathscr{C})$ is the category of covariant additive functors from $\mathscr{C}$ to the category Ab of abelian groups. Note that we are assuming all categories to be small in our fixed universe $\mathcal{U}$.

4.3. **Base change.** We fix a homomorphism $\theta\colon \mathbf{S} \to \mathbf{R}$ of commutative rings. For an $\mathbf{R}$-module M, by $\overline{M}$ we denote M regarded as an $\mathbf{S}$-module via θ. Let $\mathfrak{a}$ be an $\mathbf{R}$-linear category. We have the category $\overline{\mathfrak{a}}$ with $\mathrm{Ob}(\overline{\mathfrak{a}}) = \mathrm{Ob}(\mathfrak{a})$ and $\overline{\mathfrak{a}}(A, A') = \overline{\mathfrak{a}(A, A')}$. For an $\mathbf{S}$-linear category $\mathfrak{b}$, we denote by $\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R}$ the $\mathbf{R}$-linear category with $\mathrm{Ob}(\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R}) = \mathrm{Ob}(\mathfrak{b})$ and $(\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R})(B, B') = \mathfrak{b}(B, B') \otimes_{\mathbf{S}} \mathbf{R}$. The functor $(-) \otimes_{\mathbf{S}} \mathbf{R}$ is left adjoint to $\overline{(-)}$ in the sense that there is a natural isomorphism

$$\overline{\mathrm{Add}(\mathbf{R})(\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R}, \mathfrak{a})} \simeq \mathrm{Add}(\mathbf{S})(\mathfrak{b}, \overline{\mathfrak{a}})$$

of $\mathbf{S}$-linear categories, where $\mathrm{Add}(\mathbf{S})$ is the category of $\mathbf{S}$-linear functors.

Let $(\mathfrak{b}, \rho)$ be an $\mathbf{S}$-linear category. We have the category $\mathfrak{b}_{\mathbf{R}}$ of $\mathbf{R}$-linear objects whose objects are pairs (B, φ) where $B \in \mathfrak{b}$ and $\varphi\colon \mathbf{R} \to \mathfrak{b}(B, B)$ is a ring map with $\varphi \circ \theta = \rho_B$, and whose morphisms are those of $\mathfrak{b}$ compatible with the ring maps. An object $B \in \mathfrak{b}$ belongs to $\mathfrak{b}_{\mathbf{R}}$ if and only if 1_B is annihilated by the kernel of θ. Taking $\mathbf{R}$-linear objects defines a functor $(-)_{\mathbf{R}}\colon \mathfrak{b} \to \mathfrak{b}_{\mathbf{R}}$, which is right adjoint to $\overline{(-)}$ in the sense that there is a natural isomorphism

$$\overline{\mathrm{Add}(\mathbf{R})(\mathfrak{a}, \mathfrak{b}_{\mathbf{R}})} \simeq \mathrm{Add}(\mathbf{S})(\overline{\mathfrak{a}}, \mathfrak{b})$$

of $\mathbf{S}$-linear categories. If $\mathscr{D}$ is an $\mathbf{S}$-linear abelian category, then $\mathscr{D}_{\mathbf{R}}$ is also abelian and by [LV06b, Proposition 4.2] the forgetful functor $\mathscr{D}_{\mathbf{R}} \to \mathscr{D}$ is exact. From $(\mathrm{Mod}(\mathbf{S}))_{\mathbf{R}} \simeq \mathrm{Mod}(\mathbf{R})$, it follows

$$\overline{\mathrm{Add}(\mathbf{R})(\mathfrak{a}, \mathrm{Mod}(\mathbf{R}))} \simeq \mathrm{Mod}(\mathfrak{a})$$

for any $\mathbf{R}$-linear category $\mathfrak{a}$.

Lemma 4.1. ([LV06b, Proposition 4.4(1)]) *Let* $\mathfrak{b}$ *be an* $\mathbf{S}$*-linear category. Then there is an equivalence* $\mathrm{Mod}(\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R}) \to \mathrm{Mod}(\mathfrak{b})_{\mathbf{R}}$ *of* $\mathbf{R}$*-linear categories which makes the diagram*

$$\begin{tikzcd}
\mathrm{Mod}(\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Mod}(\mathfrak{b})_{\mathbf{R}} \arrow[d] \\
\mathrm{Mod}(\mathfrak{b}) \arrow[r, "\mathrm{id}"] & \mathrm{Mod}(\mathfrak{b})
\end{tikzcd}$$

commutes, where the left vertical arrow is the dual to $\mathfrak{b} \to \mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R}$ *and the right vertical arrow is the forgetful functor.*

4.4. **Deformations of linear categories.** Let $\mathfrak{a}$ be an $\mathbf{R}$-linear category. A *linear* $\mathbf{S}$*-deformation* of $\mathfrak{a}$ is an $\mathbf{S}$-linear category $\mathfrak{b}$ together with an $\mathbf{S}$-linear functor $\mathfrak{b} \to \overline{\mathfrak{a}}$ inducing an equivalence $\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R} \to \mathfrak{a}$. Two deformations $f: \mathfrak{b} \to \overline{\mathfrak{a}},\ f': \mathfrak{b}' \to \overline{\mathfrak{a}}$ are *equivalent* if there is an equivalence $\Phi: \mathfrak{b} \to \mathfrak{b}'$ of $\mathbf{S}$-linear categories such that $f' \circ \Phi$ is natural isomorphic to f. When $\mathfrak{b}$ is flat over $\mathbf{R}$, we call the deformation $\mathfrak{b}$ *flat*. We denote by $\mathrm{Def}^{lin}_{\mathfrak{a}}(\mathbf{S})$ the set of equivalence classes of flat linear $\mathbf{S}$-deformations of $\mathfrak{a}$. The notation will be justified below with respect to the choice of universe. When $\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R} \to \mathfrak{a}$ is an isomorphism, we call the deformation $\mathfrak{b}$ *strict*. Two strict linear deformations $f: \mathfrak{b} \to \overline{\mathfrak{a}},\ f': \mathfrak{b}' \to \overline{\mathfrak{a}}$ are *equivalent* if there is an isomorphism $\Phi: \mathfrak{b} \to \mathfrak{b}'$ of $\mathbf{S}$-linear categories such that $f' \circ \Phi = f$. We denote by $\mathrm{Def}^{s-lin}_{\mathfrak{a}}(\mathbf{S})$ the set of equivalence classes of strict flat linear $\mathbf{S}$-deformations of $\mathfrak{a}$. Also this notation will be justified below.

4.5. **Deformations of abelian categories.** Let $\mathscr{C}$ be an $\mathbf{R}$-linear abelian category. An *abelian* $\mathbf{S}$*-deformation* of $\mathscr{C}$ is an $\mathbf{S}$-linear abelian category $\mathscr{D}$ together with an $\mathbf{S}$-linear functor $\overline{\mathscr{C}} \to \mathscr{D}$ inducing an equivalence $\mathscr{C} \to \mathscr{D}_{\mathbf{R}}$. When $\mathscr{D}$ is flat over $\mathbf{R}$, we call the deformation $\mathscr{D}$ flat. Two deformations $g: \overline{\mathscr{C}} \to \mathscr{D}, g': \overline{\mathscr{C}} \to \mathscr{D}'$ are *equivalent* if there is an equivalence $\Psi: \mathscr{D} \to \mathscr{D}'$ of $\mathbf{S}$-linear abelian categories such that $\Psi \circ g'$ is natural isomorphic to g. We denote by $\mathrm{Def}^{ab}_{\mathscr{C}}(\mathbf{S})$ the set of equivalence classes of flat abelian $\mathbf{S}$-deformations of $\mathscr{C}$. The notation will be justified below with respect to the choice of universe. When $\mathscr{C} \to \mathscr{D}_{\mathbf{R}}$ is an isomorphism, we call the deformation $\mathscr{D}$ *strict*. Two strict abelian deformations $g: \overline{\mathscr{C}} \to \mathscr{D}, g': \overline{\mathscr{C}} \to \mathscr{D}'$ are *equivalent* if there is an isomorphism $\Psi: \mathscr{D} \to \mathscr{D}'$ of $\mathbf{S}$-linear abelian categories such that $\Psi \circ g' = g$. We denote by $\mathrm{Def}^{s-ab}_{\mathscr{C}}(\mathbf{S})$ the set of equivalence classes of strict flat abelian $\mathbf{S}$-deformations of $\mathscr{C}$. Also this notation will be justified below.

Assume that $\theta: \mathbf{S} \to \mathbf{R}$ is a homomorphism of coherent commutative rings with $\mathbf{R}$ being finitely presented as an $\mathbf{S}$-module. Then the bifunctors

$$(-) \otimes_{\mathbf{S}} (-): \mathscr{D} \times \mathrm{mod}(\mathbf{S}) \to \mathscr{D},\ \mathrm{Hom}_{\mathbf{S}}(-,-): \ \mathrm{mod}(\mathbf{S}) \times \mathscr{D} \to \mathscr{D}$$

yield respectively left and right adjoint

$$(-) \otimes_{\mathbf{S}} \mathbf{R}: \mathscr{D} \to \mathscr{D}_{\mathbf{R}} \simeq \mathscr{C},\ \mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -): \mathscr{D} \to \mathscr{D}_{\mathbf{R}} \simeq \mathscr{C}$$

to the natural inclusion functor $\mathscr{C} \simeq \mathscr{D}_{\mathbf{R}} \hookrightarrow \mathscr{D}$ [LV06b, Proposition 4.3]. They agree with the adjoints in the Section 4.3.

4.6. **Flat nilpotent deformations of categories.** Assume further that θ is surjective. Then for an $\mathbf{S}$-linear abelian category $\mathscr{D}$ the forgetful functor $\mathscr{D}_{\mathbf{R}} \to \mathscr{D}$ is fully faithful. When the kernel $I = \ker \theta$ is nilpotent, we call both linear and abelian $\mathbf{S}$-deformations *nilpotent*. From now on, we restrict our attention to flat nilpotent deformations. The following properties of $\mathbf{R}$-linear category $\mathfrak{a}$ and $\mathbf{R}$-linear abelian category $\mathscr{C}$ are respectively preserved under flat nilpotent linear and abelian deformations [LV06b, Proposition 6.7, 6.9, Theorem 6.16, 6.29, 6.36].

- $\mathfrak{a}, \mathscr{C}$ are essentially small.
- $\mathscr{C}$ has enough injectives.
- $\mathscr{C}$ is a Grothendieck category.
- $\mathscr{C}$ is a locally coherent Grothendieck category.

Here, we call $\mathscr{C}$ *locally coherent Grothendieck* when it is Grothendieck and generated by a small abelian subcategory of finitely presented objects.

4.7. **Deformation pseudofunctors.** In order to be careful about our choices of universes, we temporarily make them explicit in the notation. Let $\mathcal{U}$ be a universe containing the field $\mathbf{k}$. We denote by $\mathcal{U} - \mathrm{Rng}^0$ the category whose objects are coherent commutative $\mathcal{U}$-rings and whose morphisms are surjective ring maps with finitely generated nilpotent kernels. We are interested in the category $\mathcal{U} - \mathrm{Rng}^0 / \mathbf{k}$. Fix some other universe $\mathcal{W}$. A *deformation pseudofunctor* is a pseudofunctor $D\colon \mathcal{U} - \mathrm{Rng}^0 /\mathbf{k} \to \mathcal{W} - \mathrm{Gd}$. Two deformation pseudofunctors D, D' are *equivalent* if there is a pseudonatural transformation $\mu\colon D \to D'$ such that for each $\mathbf{R} \in \mathcal{U} - \mathrm{Rng}^0 /\mathbf{k}$ we have an equivalence $D(\mathbf{R}) \to D'(\mathbf{R})$ of categories. For any enlargement $\mathcal{U}'$ of $\mathcal{U}$, the canonical functor

$$\mathcal{U} - \mathrm{Rng}^0 /\mathbf{k} \to \mathcal{U}' - \mathrm{Rng}^0 /\mathbf{k}$$

is an equivalence of categories [LV06b, Proposition 8.1]. Thus the deformation pseudofunctor is independent of the choice of $\mathcal{U}$ up to equivalence.

Let $\mathfrak{a}$ be a flat $\mathbf{k}$-linear $\mathcal{U}$-category and $\mathscr{C}$ a flat $\mathbf{k}$-linear abelian $\mathcal{U}$-category. Fix a universe $\mathcal{V}$ such that $\mathfrak{a}, \mathscr{C}$ are essentially $\mathcal{V}$-small and $\mathcal{U} \in \mathcal{V}$. For $\mathbf{R} \in \mathcal{U} - \mathrm{Rng}^0 /\mathbf{k}$ we consider the groupoid $\mathcal{V} - \mathrm{def}^{lin}_{\mathfrak{a}}(\mathbf{R})$ whose objects are flat linear $\mathbf{R}$-deformations of $\mathfrak{a}$ belonging to $\mathcal{V}$, and whose morphisms are equivalences of deformations up to natural isomorphism of functors. Also we consider the groupoid $\mathcal{V} - \mathrm{def}^{ab}_{\mathscr{C}}(\mathbf{R})$ whose objects are flat abelian $\mathbf{R}$-deformations of $\mathscr{C}$ belonging to $\mathcal{V}$, and whose morphisms are equivalences of deformations up to natural isomorphism of functors. Enlarging the universe $\mathcal{W}$ if necessary, we may assume that $\mathcal{V} \in \mathcal{W}$ and we obtain deformation pseudofunctors

$$\mathcal{V} - \mathrm{def}^{lin}_{\mathfrak{a}}, \mathcal{V} - \mathrm{def}^{ab}_{\mathscr{C}}\colon \mathcal{U} - \mathrm{Rng}^0 /\mathbf{k} \to \mathcal{W} - \mathrm{Gd}\,.$$

The universe $\mathcal{W}$ is a purely technical device which guarantees $\mathcal{V}-\mathrm{def}^{lin}_{\mathfrak{a}}, \mathcal{V}-\mathrm{def}^{ab}_{\mathscr{C}}$ taking values in categories. Moreover, whether two deformation pseudofunctors are equivalent is preserved under enlargement of $\mathcal{W}$. On the other hand, by [LV06b, Proposition 8.3] for any enlargement $\mathcal{V}' \in \mathcal{W}$ of $\mathcal{V}$, the canonical pseudonatural transformations

$$\mathcal{V} - \mathrm{def}^{lin}_{\mathfrak{a}} \to \mathcal{V}' - \mathrm{def}^{lin}_{\mathfrak{a}},\ \mathcal{V} - \mathrm{def}^{ab}_{\mathscr{C}} \to \mathcal{V}' - \mathrm{def}^{ab}_{\mathscr{C}}$$

define equivalences of deformation pseudofunctors.

In summary, as long as we consider flat nilpotent deformations, the choice of universe does not affect deformation pseudofunctors up to equivalence. Thus we simply write $\mathrm{def}^{lin}_{\mathfrak{a}}, \mathrm{def}^{ab}_{\mathscr{C}}$ for deformation pseudofunctors. Since they have small skeletons [LV06b, Theorem 8.4, 8.5], we also write $\mathrm{Def}^{lin}_{\mathfrak{a}}, \mathrm{Def}^{ab}_{\mathscr{C}}$ for deformation functors

$$\mathcal{V} - \mathrm{Def}^{lin}_{\mathfrak{a}}, \mathcal{V} - \mathrm{Def}^{ab}_{\mathscr{C}}\colon \mathcal{U} - \mathrm{Rng}^0 /\mathbf{k} \to \mathcal{W} - \mathrm{Set}$$

which take values in sets.

Finally, we collect relevant results on deformations of linear and abelian categories.

Lemma 4.2. ([LV06b, Theorem 8.16]) *Let* $\mathbf{S} \to \mathbf{R}$ *be a morphism in* $\mathcal{U} - \mathrm{Rng}^0 /\mathbf{k}$ *and* $\mathfrak{a}$ *an essentially small flat* $\mathbf{R}$*-linear category. Then there is a bijection*

$$\mathrm{Def}^{lin}_{\mathfrak{a}}(\mathbf{S}) \to \mathrm{Def}^{ab}_{\mathrm{Mod}(\mathfrak{a})}(\mathbf{S}),\ \mathfrak{b} \mapsto \mathrm{Mod}(\mathfrak{b}).$$

In particular, deformations of a module category are module categories.

Lemma 4.3. ([LV06b, Theorem 8.17]) *Let* $\mathbf{S} \to \mathbf{R}$ *be a morphism in* $\mathcal{U} - \mathrm{Rng}^0 /\mathbf{k}$ *and* $\mathscr{C}$ *an essentially small flat* $\mathbf{R}$*-linear abelian category with enough injectives. Then there is a bijection*

$$\mathrm{Def}^{lin}_{\mathrm{Inj}(\mathscr{C})}(\mathbf{S}) \to \mathrm{Def}^{ab}_{\mathscr{C}}(\mathbf{S}),\ \mathfrak{j} \mapsto (\mathrm{mod}(\mathfrak{j}))^{op}.$$

Lemma 4.4. ([LV06b, Proposition B.3]) *Let* $\mathbf{S} \to \mathbf{R}$ *be a morphism in* $\mathcal{U} - \mathrm{Rng}^0 /\mathbf{k}$ *and* $\mathfrak{a}$ *an essentially small flat* $\mathbf{R}$*-linear category. Then the map*

$$\mathrm{Def}^{s-lin}_{\mathfrak{a}}(\mathbf{S}) \to \mathrm{Def}^{lin}_{\mathfrak{a}}(\mathbf{S})$$

induced by the canonical pseudofunctor

$$\mathrm{def}^{s-lin}_{\mathfrak{a}} \to \mathrm{def}^{lin}_{\mathfrak{a}}$$

is bijective.

Now, let again $\mathbf{R}$ be the fixed local artinian $\mathbf{k}$-algebra with residue field $\mathbf{k}$, the square zero extension

$$0 \to \mathbf{I} \to \mathbf{S} \to \mathbf{R} \to 0,$$

and the chosen generators $\epsilon = (\epsilon_1, \dots, \epsilon_l)$ of $\mathbf{I}$ regarded as a free $\mathbf{R}$-module of rank l.

Lemma 4.5. ([Low08, Proposition 4.2]) *Let $(\mathfrak{a}, m)$ be a flat $\mathbf{R}$-linear category with compositions m. Then there is a bijection*

$$H^2\mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathrm{Def}^{lin}_{\mathfrak{a}}(\mathbf{S}),\ \phi \mapsto (\mathfrak{a}[\epsilon], m + \phi\epsilon),\ \phi \in Z^2\mathbf{C}(\mathfrak{a})^{\oplus l}. \tag{4.1}$$

Another cocycle $\phi' \in Z^2\mathbf{C}(\mathfrak{a})^{\oplus l}$ maps to an isomorphic linear deformation if and only if there is an element $h \in \mathbf{C}^1(\mathfrak{a})$ satisfying $\phi' - \phi = d_m(h)$.

Lemma 4.6. ([LV06a, Theorem 3.1]) *Let $\mathscr{C}$ be a flat $\mathbf{R}$-linear abelian category. Then there is a bijection*

$$H^2\mathbf{C}_{ab}(\mathscr{C})^{\oplus l} \to \mathrm{Def}^{ab}_{\mathscr{C}}(\mathbf{S}). \tag{4.2}$$

Here, $\mathbf{C}(\mathfrak{a})$ is the Hochschild object associated with $\mathfrak{a}$. The compositions m is an element of

$$\prod_{A_0,A_1,A_2 \in \mathfrak{a}} [\mathfrak{a}(A_1, A_2) \otimes_{\mathbf{R}} \mathfrak{a}(A_0, A_1), \mathfrak{a}(A_0, A_2)]^0$$

rather than the ring map ρ defining the $\mathbf{R}$-linear structure. For an $\mathbf{R}$-linear abelian category $\mathscr{C}$, the associated Hochschild object is defined as $\mathbf{C}_{ab}(\mathscr{C}) = \mathbf{C}_{sh}(\mathrm{Ind}(\mathrm{Inj}(\mathscr{C})))$, where $\mathbf{C}_{sh}(\mathrm{Ind}(\mathrm{Inj}(\mathscr{C})))$ is the Shukla complex associated with $\mathrm{Ind}(\mathrm{Inj}(\mathscr{C}))$. Note that we have $H^*\mathbf{C}_{sh}(\mathfrak{a}) = H^*\mathbf{C}(\mathfrak{a})$. We will review the definitions in Section 6.

4.8. **Examples.** Let X be a smooth proper $\mathbf{R}$-scheme. Since it is noetherian, the category $\mathrm{Qch}(X)$ of quasi-coherent sheaves on X has enough injectives. We denote by $\mathfrak{i} = \mathrm{Inj}(\mathrm{Qch}(X))$ the full $\mathbf{R}$-linear subcategory of injective objects. Since X is flat separated, by [DLL17, Proposition 4.28, 4.30(2)] the $\mathbf{R}$-linear abelian category $\mathrm{Qch}(X)$ is flat. From [LV06b, Proposition 2.9(6)] it follows that the $\mathbf{R}$-linear category $\mathfrak{i}$ is flat. Then by Lemma 4.5 and Lemma 4.6 or Lemma 4.3 both flat linear $\mathbf{S}$-deformations of $\mathfrak{i}$ and flat abelian $\mathbf{S}$-deformations of $\mathrm{Qch}(X)$ are classified by $H^2\mathbf{C}(\mathfrak{i})^{\oplus l}$.

5. The category of quasi-coherent sheaves

In this section, we review an alternative description of Toda's construction in terms of the descent category of the category of twisted quasi-coherent presheaves over the restricted structure sheaf, following the exposition from [DLL17, Section 4,5]. It follows that, for square zero extension of relatively smooth $\mathbf{R}$-schemes, Toda's construction coincides with the deformation of the category of quasi-coherent sheaves along the corresponding Hochschild cocycle. As a consequence, deforming the category of the quasi-coherent sheaves is equivalent to deforming the complex structure for higher dimensional Calabi–Yau manifolds. In particular, deformations of the category of quasi-coherent sheaves are given by the category of quasi-coherent sheaves on deformations.

5.1. **Descent categories.** Let $\mathfrak{U}$ be a small category and $\mathrm{Cat}(\mathbf{R})$ the category of small $\mathbf{R}$-linear categories and $\mathbf{R}$-linear functors. A *prestack* $\mathscr{A}$ is a pseudofunctor $\mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{R})$ consists of the following data:

- for each $U \in \mathfrak{U}$ an $\mathbf{R}$-linear category $\mathscr{A}(U)$,
- for each $u\colon V \to U$ in $\mathfrak{U}$ an $\mathbf{R}$-linear functor $f^u\colon \mathscr{A}(U) \to \mathscr{A}(V)$,
- for each pair $u\colon V \to U,\ v\colon W \to V$ in $\mathfrak{U}$ a natural isomorphism $c^{u,v}\colon f^v f^u \to f^{uv}$,
- for each $U \in \mathfrak{U}$ a natural isomorphism $z^U\colon 1_{\mathscr{A}(U)} \to f^{1_U}$.

Moreover, these data must satisfy

$$c^{u,vw}(c^{v,w}\circ f^u) = c^{uv,w}(f^w\circ c^{u,v}),$$
$$c^{u,1_V}(z^V\circ f^u) = 1,\ \ c^{1_U,u}(f^u\circ z^U) = 1$$

for each triple $u\colon V\to U,\ v\colon W\to V,\ w\colon T\to W$ in $\mathfrak{U}$. With $\mathscr{A}(U)$ regarded as one-objected categories, a twisted presheaf $\mathscr{A}$ of $\mathbf{R}$-algebras provides an example of a prestack of $\mathbf{R}$-linear category.

For prestacks $\mathscr{A} = (\mathscr{A}, m, f, c, z), \mathscr{A}' = (\mathscr{A}', m', f', c', z')$ of $\mathbf{R}$-linear category on $\mathfrak{U}$, a *morphism* $(g,h)\colon \mathscr{A}\to\mathscr{A}'$ is a pseudonatural transformation which consists of the following data:

- for each $U\in\mathfrak{U}$ an $\mathbf{R}$-linear functor $g^U\colon \mathscr{A}(U)\to\mathscr{A}'(U)$,
- for each $u\colon V\to U$ in $\mathfrak{U}$ a natural isomorphism $h^u\colon f'^u g^U\to g^V f^u$.

Moreover, these data must satisfy

$$h^{uv}(c'^{u,v}\circ g^U) = (g^W\circ c^{u,v})(h^v\circ f^u)(f'^v\circ h^u),$$
$$h^{1_U}(z'^U\circ g^U) = g^U\circ z^U.$$

for each pair $u\colon V\to U,\ v\colon W\to V$ in $\mathfrak{U}$. When $\mathscr{A}$ is a twisted presheaf of $\mathbf{R}$-algebras, morphisms of twisted presheaves of $\mathbf{R}$-algebras coincide with morphisms of prestacks.

A *pre-descent datum* in a prestack $\mathscr{A}$ is a collection $(A_U)_U$ of objects $A_U\in\mathscr{A}(U)$ with a morphism $\varphi_u\colon f^uA_U\to A_V$ in $\mathscr{A}(V)$ for each $u\colon V\to U$ in $\mathfrak{U}$, which satisfies

$$\varphi_v f^v\varphi_u = \varphi_{uv}c^{u,v,A_U}$$

given an additional $v\colon W\to V$ in $\mathfrak{U}$. A *morphism* of pre-descent data $g\colon (A_U)_U\to(A'_U)_U$ is a collection $(g_U)_U$ of compatible morphisms $g_U\colon A_U\to A'_U$. Pre-descent data and their morphisms form a category $\mathrm{PDes}(\mathscr{A})$ equipped with a canonical functor

$$\pi_V\colon\ \mathrm{PDes}(\mathscr{A})\to\mathscr{A}(V),\ (A_U)_U\mapsto A_V.$$

When all φ_u are isomorphisms, $(A_U)_U$ is called a *descent datum* and we denote by $\mathrm{Des}(\mathscr{A})$ the full subcategory of descent data. Given limits and colimits in each $\mathscr{A}(U)$ preserved by all $f^u\colon\mathscr{A}(U)\to\mathscr{A}(V)$, there exist ones in $\mathrm{Des}(\mathscr{A})$ preserved by all $\pi_V\colon \mathrm{Des}(\mathscr{A})\to\mathscr{A}(V)$ [DLL17, Proposition 4.5(3)]. In particular, if each category $\mathscr{A}(U)$ is abelian and all f^u are exact, then $\mathrm{Des}(\mathscr{A})$ is abelian and π_V are exact.

5.2. **The category of quasi-coherent modules over a prestack.** The *category of right quasi-coherent modules* over a prestack $\mathscr{A}$ is defined as

$$\mathrm{Qch}(\mathscr{A}) = \mathrm{Qch}^r(\mathscr{A}) = \mathrm{Des}(\mathrm{Mod}_{\mathscr{A}}),$$

where $\mathrm{Mod}_{\mathscr{A}}$ is the *associated prestack* with a prestack $\mathscr{A}$ given by

$$\mathrm{Mod}_{\mathscr{A}} = \mathrm{Mod}^r_{\mathscr{A}}\colon \mathfrak{U}^{op}\to\mathrm{Cat}(\mathbf{R}),\ U\mapsto\mathrm{Mod}_{\mathscr{A}}(U) = \mathrm{Mod}(\mathscr{A}(U)),$$

whose restriction functor

$$-\otimes_u\mathscr{A}(V)\colon\ \mathrm{Mod}(\mathscr{A}(U))\to\mathrm{Mod}(\mathscr{A}(V))$$

is the unique colimit preserving functor extending $f^u\colon\mathscr{A}(U)\to\mathscr{A}(V)$. The functor sends each $F\in\mathrm{Mod}(\mathscr{A}(U))$ to an $\mathbf{R}$-linear functor $F\otimes_u\mathscr{A}(V)\colon\mathscr{A}(V)^{op}\to\mathrm{Mod}(\mathbf{R})$ such that

$$F\otimes_u\mathscr{A}(V)(B) = \bigoplus_{A\in\mathscr{A}(U)} F(A)\otimes_{\mathbf{R}}\mathscr{A}(V)(B, f^uA)/\sim$$

for each $B\in\mathscr{A}(V)$. Here, $\sim$ denotes the equivalence relation defined as

$$F(a)(x)\otimes y\sim x\otimes f^u(a)y$$

for $x \in F(A'),\ y \in \mathscr{A}(V)(B, f^u A)$, and $a\colon A \to A'$ in $\mathscr{A}(U)$.

In the case where $F = \mathscr{A}(U)(-, A')$ for some $A' \in \mathscr{A}(U)$, f^u induces an isomorphism

$$\theta^u_{A'}\colon \mathscr{A}(U)(-, A') \otimes_u \mathscr{A}(V) \to \mathscr{A}(V)(-, f^u A').$$

If $u = 1_U$, then $z^U\colon 1_{\mathscr{A}(U)} \to f^{1_U}$ induces an isomorphism

$$\mathrm{Mod}(z)^U\colon 1_{\mathrm{Mod}_{\mathscr{A}(U)}} \to - \otimes_{1_U} \mathscr{A}(U).$$

Since we have

$$F \otimes_u \mathscr{A}(V) \otimes_v \mathscr{A}(W)(C) = \bigoplus_{A \in \mathscr{A}(U), B \in \mathscr{A}(V)} F(A) \otimes_{\mathbf{R}} \mathscr{A}(V)(B, f^u A) \otimes_{\mathbf{R}} \mathscr{A}(W)(C, f^v B)/\sim,$$

$\theta^v_{f^u A}$ and $c^{u,v}$ induce another isomorphism

$$\mathrm{Mod}(c^{u,v})\colon\ - \otimes_u \mathscr{A}(V) \otimes_v \mathscr{A}(W) \to - \otimes_{uv} \mathscr{A}(W).$$

When $\mathscr{A}$ is a twisted presheaf of $\mathbf{R}$-algebras, $\mathrm{Mod}(\mathscr{A}(U))$ coincides with the category of right $\mathscr{A}(U)$-modules whose restriction functor is the ordinary tensor product and $\mathrm{Mod}(c)^{u,v}$, $\mathrm{Mod}(z)^U$ are respectively given by

$$\mathrm{Mod}(c)^{u,v}_M\colon M \otimes_u \mathscr{A}(V) \otimes_v \mathscr{A}(W) \to M \otimes_{uv} \mathscr{A}(W),\ m \otimes a \otimes b \mapsto m \otimes c^{u,v} f^v(a) b,$$

$$\mathrm{Mod}(z)^U_M\colon M \to M \otimes_{1_U} \mathscr{A}(U),\ m \mapsto m \otimes z^U$$

for any right $\mathscr{A}(U)$-module M.

5.3. **The category of twisted quasi-coherent presheaves over a twisted presheaf.** Let $\mathscr{A}$ be a presheaf of $\mathbf{R}$-algebras on $\mathfrak{U}$. We denote by $\mathrm{Pr}(\mathscr{A}|_U)$ the category of presheaves of right $\mathscr{A}|_U$-modules on $\mathfrak{U}/U$, where $\mathscr{A}|_U$ is the induced presheaf on $\mathfrak{U}/U$ with $\mathscr{A}|_U(V \to U) = \mathscr{A}(V)$ for $U \in \mathfrak{U}$ and $u\colon V \to U$ in $\mathfrak{U}$. Each $u\colon V \to U$ in $\mathfrak{U}$ induces a functor $u^*_{\mathrm{Pr}}\colon\ \mathrm{Pr}(\mathscr{A}|_U) \to \mathrm{Pr}(\mathscr{A}|_V)$. Since we have $v^*_{\mathrm{Pr}} u^*_{\mathrm{Pr}} = (uv)^*_{\mathrm{Pr}}$ and $(1_U)^*_{\mathrm{Pr}} = 1_{\mathrm{Pr}(\mathscr{A}|_U)}$ given an additional $v\colon W \to V$ in $\mathfrak{U}$, the assignments $U \mapsto \mathrm{Pr}(\mathscr{A}|_U)$ and $u \mapsto u^*_{\mathrm{Pr}}$ define a functor

$$\mathrm{Pr}(\mathscr{A})\colon \mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{R}).$$

Let M be a right $\mathscr{A}(U)$-module. Then $\tilde{M}(u) := M \otimes_u \mathscr{A}(V) = M \otimes_{\mathscr{A}(U)} \mathscr{A}(V)$ is a right $\mathscr{A}(V)$-module with $\mathscr{A}(V)$ regarded as a left $\mathscr{A}(U)$-module via f^u. Suppose that $u'\colon V' \to U$ satisfies $uv' = u'$ for $v'\colon V' \to V$. We have the right $\mathscr{A}(U)$-module homomorphism $1_M \otimes f^{v'}\colon \tilde{M}(u) \to \tilde{M}(u')$. The assignments $u \mapsto \tilde{M}(u)$ and $f^{v'} \mapsto 1_M \otimes f^{v'}$ define a presheaf $\tilde{M}$ of right $\mathscr{A}(U)$-modules on $\mathfrak{U}/U$. Any $\mathscr{A}(U)$-module homomorphism $g\colon M \to N$ induces a natural transform $\tilde{g} = \{\tilde{g}^u := g \otimes 1_{\mathscr{A}(V)}\}_u$. Thus the assignments $M \mapsto \tilde{M}$ and $g \mapsto \tilde{g}$ define a functor

$$Q^U\colon\ \mathrm{Mod}(\mathscr{A}(U)) \to \mathrm{Pr}(\mathscr{A}|_U). \tag{5.1}$$

We have the canonical isomorphism

$$\mathrm{can}^{u,v}_M\colon M \otimes_u \mathscr{A}(V) \otimes_v \mathscr{A}(W)) \to M \otimes_{uv} \mathscr{A}(W),\ m \otimes a \otimes b \mapsto m \otimes f^v(a) b.$$

By [DLL17, Lemma 4.10] the functor Q^U is fully faithful and there is a natural isomorphism

$$\tau^u\colon u^*_{\mathrm{Pr}} Q^U \to Q^V(- \otimes_u \mathscr{A}(V)) \tag{5.2}$$

induced by $(\mathrm{can}^{u,v}_M)^{-1}$. A *quasi-coherent presheaf* over $\mathscr{A}_U$ is defined as the essential image of some $\mathscr{A}(U)$-module M by Q^U. We denote by $\mathrm{QPr}(\mathscr{A}|_U)$ the category of quasi-coherent presheaves over $\mathscr{A}|_U$.

When $\mathscr{A}$ is a twisted presheaf with central twists c, one can adapt $\mathrm{Mod}(c)^{u,v}$ to $\mathrm{Pr}(c)^{u,v}$ as follows. For $\mathscr{F} \in \mathrm{Pr}(|\mathscr{A}\|_U)$ and $w\colon T \to W$ in $\mathfrak{U}/W$ the central invertible element $f^w(c^{u,v})$ in $\mathscr{A}(T)$ gives an automorphism

$$f^w(c^{u,v})_r\colon \mathscr{F}(uvw) \to \mathscr{F}(uvw),\ m \mapsto m f^w(c^{u,v})$$

inducing an isomorphism

$$\Pr(c)^{u,v}_{\mathscr{F}} \colon v^*_{\Pr} u^*_{\Pr}(\mathscr{F}) \to (uv)^*_{\Pr}(\mathscr{F})$$

in $\Pr(|\mathscr{A}||_U)$. Since we have

$$\Pr(c)^{u,vw} \Pr(c)^{v,w} = \Pr(c)^{uv,w} w^*_{\Pr}(\Pr(c)^{u,v}),$$

the assignments $U \mapsto \Pr(|\mathscr{A}||_U)$ and $u \mapsto u^*_{\Pr}$ define a prestack

$$\Pr_{\mathscr{A}} \colon \mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{R})$$

whose twist functor is given by $\Pr(c)$ and z is given by the identity. Restricting to the essential images $\mathrm{QPr}(|\mathscr{A}||_U)$, we obtain another prestack $\mathrm{QPr}_{\mathscr{A}}$. The *category of right twisted quasi-coherent presheaves* over a twisted presheaf $\mathscr{A} \colon \mathfrak{U}^{op} \to \mathrm{Alg}(\mathbf{R})$ with central twists is defined as

$$\mathrm{QPr}(\mathscr{A}) = \mathrm{Des}(\mathrm{QPr}_{\mathscr{A}}).$$

Lemma 5.1. ([DLL17, Theorem 4.12]) *Let $\mathscr{A} \colon \mathfrak{U}^{op} \to \mathrm{Alg}(\mathbf{R})$ be a twisted presheaf with central twists. Then $Q = (Q^U, \tau^u)_{U,u}$ defines an equivalence*

$$\mathrm{Qch}(\mathscr{A}) \simeq \mathrm{QPr}(\mathscr{A})$$

of $\mathbf{R}$-linear categories, where Q^U, τ^u are given by (5.1)*,* (5.2) *respectively.*

5.4. **Deformations of the restricted structure sheaves.** Let $\mathscr{A} \colon \mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{R})$ be a flat prestack. Recall that $\mathscr{A}$ is *flat* if $\mathbf{R}$-modules $\mathscr{A}(U)(A, A')$ are flat for all $U \in \mathfrak{U}$ and $A, A' \in \mathscr{A}(U)$. An $\mathbf{S}$-*deformation* of $\mathscr{A}$ is a flat prestack $\mathscr{B} \colon \mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{S})$ together with an equivalence of prestacks $\mathscr{B} \otimes_{\mathbf{S}} \mathbf{R} \to \mathscr{A}$, i.e., for each U there is a morphism of prestacks inducing an equivalence $\mathscr{B}(U) \otimes_{\mathbf{S}} \mathbf{R} \to \mathscr{A}(U)$ of $\mathbf{R}$-linear categories [DLL17, Proposition 4.7]. Two deformations $\mathscr{B}, \mathscr{B}'$ are *equivalent* if there is an equivalence $\mathscr{B} \to \mathscr{B}'$ of prestacks compatible with equivalences $\mathscr{B} \otimes_{\mathbf{S}} \mathbf{R} \to \mathscr{A}$, $\mathscr{B}' \otimes_{\mathbf{S}} \mathbf{R} \to \mathscr{A}$. We denote by $\mathrm{Def}^{tw}_{\mathscr{A}}(\mathbf{S})$ the set of equivalence classes of $\mathbf{S}$-deformations of $\mathscr{A}$. When $\mathscr{B} \otimes_{\mathbf{S}} \mathbf{R} \to \mathscr{A}$ is an isomorphism of prestacks, i.e., for each U there is a morphism of prestacks inducing an isomorphism $\mathscr{B}(U) \otimes_{\mathbf{S}} \mathbf{R} \to \mathscr{A}(U)$ of $\mathbf{R}$-linear categories, we call the deformation $\mathscr{B}$ *strict*. Two strict deformations $\mathscr{B}, \mathscr{B}'$ are *equivalent* if there is an isomorphism $\mathscr{B} \to \mathscr{B}'$ of prestacks inducing the identity on $\mathscr{A}$. We denote by $\mathrm{Def}^{s-tw}_{\mathscr{A}}(\mathbf{S})$ the set of equivalence classes of strict $\mathbf{S}$-deformations of $\mathscr{A}$. Recall that twised presheaves of $\mathbf{R}$-algebras can be regarded as a prestack. Due to the lemma below, as long as we consider equivalence classes of twisted deformations of flat presheaves, we may restrict our attention to strict twisted deformations.

Lemma 5.2. ([DLL17, Proposition 5.9]) *Let $(\mathscr{A}, m, f, c, z)$ be a flat prestack of $\mathbf{R}$-linear categories on $\mathfrak{U}$. Then the canonical map*

$$\mathrm{Def}^{s-tw}_{\mathscr{A}}(\mathbf{S}) \to \mathrm{Def}^{tw}_{\mathscr{A}}(\mathbf{S})$$

is bijective.

Let $\mathfrak{U}$ be a finite poset with binary meets. Then any prestack on $\mathfrak{U}$ is *quasi-compact* since $\mathfrak{U}$ is finite. A prestack $\mathscr{A} \colon \mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{R})$ is *right semi-separated* if the associated prestack $\mathrm{Mod}_{\mathscr{A}}$ is of affine localizations. Namely, for all $U, V, W \in U$ with $v \colon V \to U, w \colon W \to U$ in $\mathfrak{U}$ and the pullback diagram

$$\begin{tikzcd} V \cap W \arrow[r, "\bar{w}"] \arrow[d, "\bar{v}"'] & V \arrow[d, "v"] \\ W \arrow[r, "w"] & U \end{tikzcd}$$

the following conditions are satisfied.

- The category $\mathrm{Mod}_{\mathscr{A}}(U)$ is Grothendieck abelian.
- The functor $v^*\colon \mathrm{Mod}_{\mathscr{A}}(U) \to \mathrm{Mod}_{\mathscr{A}}(V)$ is exact.
- The functor v^* admits a fully faithful exact right adjoint $v_*\colon \mathrm{Mod}_{\mathscr{A}}(V) \to \mathrm{Mod}_{\mathscr{A}}(U)$.
- There are natural isomorphisms

$$(v_*v^*)(w_*w^*) \cong (v\bar{w})_*(v\bar{w})^* \cong (w_*w^*)(v_*v^*).$$

A presheaf $\mathscr{A}\colon \mathfrak{U}^{op} \to \mathrm{Alg}(\mathbf{R})$ is *right semi-separated* if so is $\mathscr{A}$ with $\mathscr{A}(U)$ regarded as one-objected categories. Every right semi-separated prestack is *geometric*, i.e., the restriction functor

$$-\otimes_u \mathscr{A}(V)\colon \mathrm{Mod}(\mathscr{A}(U)) \to \mathrm{Mod}(\mathscr{A}(V))$$

is exact. Note that for any geometric prestack $\mathscr{A}\colon \mathfrak{U}^{op} \to \mathrm{Cat}(\mathbf{R})$ on a small category $\mathrm{Qch}(\mathscr{A})$ is a Grothendieck abelian category [DLL17, Theorem 4.14].

Let X be a smooth proper $\mathbf{R}$-scheme. Choose a finite affine open cover $\mathfrak{U}$ closed under intersections. We denote by $\mathscr{O}_X|_{\mathfrak{U}}$ the restricted structure sheaf to $\mathfrak{U}$. Since $U \cap V$ is affine as X is separated, we have isomorphisms of $\mathscr{O}_X(U)$-modules

$$\mathscr{O}_X(V) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(W) \cong \mathscr{O}_X(U \cap V) \cong \mathscr{O}_X(W) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(V)$$

for all $U, V, W \in U$ with $V, W \subset U$. Since pushforwards along open immersions $V \hookrightarrow U$ of affine schemes are fully faithful, by [DLL17, Lemma 3.1] the restriction maps $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ are flat epimorphism of rings. Then one can apply [DLL17, Proposition 4.28] to see that the presheaf $\mathscr{O}_X|_{\mathfrak{U}}\colon \mathfrak{U}^{op} \to \mathrm{Alg}(\mathbf{R})$ is right semi-separated. Since $\mathscr{O}_X(U)$ are flat $\mathbf{R}$-modules, the category

$$\mathrm{Qch}(\mathscr{O}_X|_{\mathfrak{U}}) \simeq \mathrm{QPr}(\mathscr{O}_X|_{\mathfrak{U}}) \simeq \mathrm{Qch}(X)$$

is flat over $\mathbf{R}$ and Grothendieck abelian [DLL17, Proposition 4.30].

Lemma 5.3. ([DLL17, Theorem 5.10]) *Let X be a smooth proper $\mathbf{R}$-scheme with a finite affine open cover $\mathfrak{U}$ closed under intersections. Then every twisted $\mathbf{S}$-deformations of the restricted structure sheaf $\mathscr{O}_X|_{\mathfrak{U}}$ is a quasi-compact semi-separated presheaf on $\mathfrak{U}$ and there is a bijection*

$$\mathrm{Def}^{tw}_{\mathscr{O}_X|_{\mathfrak{U}}}(\mathbf{S}) \to \mathrm{Def}^{ab}_{\mathrm{Qch}(X)}(\mathbf{S}),\ (\mathscr{O}_X|_{\mathfrak{U}})_\phi \mapsto \mathrm{Qch}((\mathscr{O}_X|_{\mathfrak{U}})_\phi),$$

where $\phi \in H^2\mathbf{C}_{GS}(\mathscr{O}_X|_{\mathfrak{U}})^{\oplus l}$ is a cocycle and $(\mathscr{O}_X|_{\mathfrak{U}})_\phi$ is the twisted $\mathbf{S}$-deformation of $\mathscr{O}_X|_{\mathfrak{U}}$ along ϕ. In particular, the category of right quasi-coherent modules over a twisted deformation of $\mathscr{O}_X|_{\mathfrak{U}}$ is given by an abelian deformation of the category $\mathrm{Qch}(X)$ of quasi-coherent sheaves.

5.5. **Toda's construction revisited.** Let $\mathfrak{U}$ be a small category and $(\mathscr{A}, m, f)$ a presheaf of $\mathbf{R}$-algebras on $\mathfrak{U}$. The *simplicial complex of presheaves* associated with $\mathscr{A}$ is the complex $(\mathscr{A}^\bullet, \varphi^\bullet)$ defined as follows. Consider the presheaf of algebras $\mathscr{A}^n = (\mathscr{A}^n, m^n, f^n)$ for $n \geq 0$ given by

$$\mathscr{A}^n(U) = \prod_{\tau \in \mathcal{N}_n(\mathfrak{U}/U)} \mathscr{A}|_U(\tau)$$

endowed with the product algebra structure $m^{n,U}$. Here, $\tau \in \mathcal{N}_n(\mathfrak{U}/U)$ is identified with the object $d\tau \to U \in \mathfrak{U}/U$ by composing all morphisms of τ, and the restriction map

$$f^{n,U}\colon \mathscr{A}^n(U) \to \mathscr{A}^n(V),\ (a^\tau)_\tau \mapsto (a^{u\sigma})_\sigma$$

is induced by the natural map $\mathcal{N}_n(\mathfrak{U}/V) \to \mathcal{N}_n(\mathfrak{U}/U)$, $\sigma \to u\sigma$. Define morphisms of presheaves $\varphi^n\colon \mathscr{A}^n \to \mathscr{A}^{n+1}$ as

$$\varphi^{n,U}\colon \prod_{\tau \in \mathcal{N}_n(\mathfrak{U}/U)} \mathscr{A}|_U(\tau) \to \prod_{\sigma \in \mathcal{N}_{n+1}(\mathfrak{U}/U)} \mathscr{A}|_U(\sigma),\ (a^\tau)_\tau \mapsto \left(f^{u_1^\sigma}(a^{\partial_0\sigma}) + \sum_{i=1}^{n+1}(-1)^i a^{\partial_i\sigma}\right)_\sigma$$

which specialize to

$$\varphi^{0,U}\colon \prod_{u\colon V\to U}\mathscr{A}(V)\to\prod_{u\colon V\to U, v\colon W\to V}\mathscr{A}(W),\ (a^u)_u\mapsto f^v(a^u)-a^{uv}.$$

Then one obtains the complex $(\mathscr{A}^\bullet,\varphi^\bullet)$ with $\ker(\varphi^0)\cong\mathscr{A}$ [DLL17, Lemma 2.12].

Using a part of the simplicial complex of presheaves, one can give an alternative description of Toda's construction. Since $\mathbf{R}$ is commutative, by [DLL17, Proposition 2.14] every normalized reduced cocycle

$$\phi=(\mathbf{m},\mathbf{f},\mathbf{c})=(m_1,\ldots,m_l,f_1,\ldots,f_l,c_1,\ldots,c_l)\in\bar{\mathbf{C}}'^{0,2}_{GS}(\mathscr{A})^{\oplus l}\oplus\bar{\mathbf{C}}'^{1,1}_{GS}(\mathscr{A})^{\oplus l}\oplus\bar{\mathbf{C}}'^{2,0}_{GS}(\mathscr{A})^{\oplus l}.$$

admits a weak decomposition

$$(\mathbf{m},\mathbf{f},\mathbf{c})=(\mathbf{m},\mathbf{f},0)+(0,0,\mathbf{c})\in\bar{\mathbf{C}}'^{2}_{tGS}(\mathscr{A})^{\oplus l}\oplus\bar{\mathbf{C}}'^{2}_{simp}(\mathscr{A})^{\oplus l}.$$

From [DLL17, Proposition 2.24] it follows that the twisted $\mathbf{S}$-deformation $\mathscr{A}_\phi$ of $\mathscr{A}$ along ϕ has central twists and the underlying presheaf $|\mathscr{A}_\phi|$ is the presheaf $\mathbf{S}$-deformation of $\mathscr{A}$ along $|\phi|=(\mathbf{m},\mathbf{f},0)$. Consider the morphism $F\colon\mathscr{A}\oplus(\mathscr{A}^0)^{\oplus l}\to(\mathscr{A}^1)^{\oplus l}$ of presheaves defined as

$$F^U\colon\mathscr{A}(U)\oplus\prod_{u\colon V\to U}\mathscr{A}^0(V)^{\oplus l}\to\prod_{u\colon V\to U, v\colon W\to V}\mathscr{A}^1(W)^{\oplus l},$$
$$(a,(b_1^u,\ldots,b_l^u)_u)\mapsto(f_1^vu^*(a)+v^*(b_1^u)-b_1^{uv},\ldots,f_l^vu^*(a)+v^*(b_l^u)-b_l^{uv})_{u,v},$$

where we denote f^u by u^* for clarity. Define the multiplication on $\mathscr{A}\oplus(\mathscr{A}^0)^{\oplus l}$ as

$$\begin{aligned}&(a,(b_1^u,\ldots,b_l^u)_u)\cdot(a',(b_1'^u,\ldots,b_l'^u)_u)\\=&(aa',(u^*(a)b_1'^u+b_1^uu^*(a')+m_1(u^*(a),u^*(a')),\ldots,u^*(a)b_l'^u+b_l^uu^*(a')+m_l(u^*(a),u^*(a')))_u).\end{aligned}$$

With the scalar given by

$$(\lambda+\kappa_1\epsilon_1+\ldots+\kappa_l\epsilon_l)(a,(b_1^u,\ldots,b_l^u)_u)=(\lambda a,(\kappa_1u^*(a)+\lambda b_1^u,\ldots,\kappa_lu^*(a)+\lambda b_l^u)_u),$$

$\mathscr{A}\oplus(\mathscr{A}^0)^{\oplus l}$ becomes an $\mathbf{S}$-algebra. Then the morphism $G\colon|\mathscr{A}_\phi|\to\mathscr{A}\oplus(\mathscr{A}^0)^{\oplus l}$ of presheaves of $\mathbf{S}$-algebras defined as

$$G^U\colon|\mathscr{A}_\phi|(U)\to\mathscr{A}(U)\oplus\mathscr{A}^0(U)^{\oplus l},$$
$$a+b_1^u\epsilon_1+\cdots+b_l^u\epsilon_l\mapsto(a,(f_1^u(a)+u^*(b_1),\ldots,f_l^u(a)+u^*(b_l))_u)$$

yields an exact sequence

$$0\to|\mathscr{A}_\phi|\xrightarrow{G}\mathscr{A}\oplus(\mathscr{A}^0)^{\oplus l}\xrightarrow{F}(\mathscr{A}^1)^{\oplus l}.$$

Consider the case where $\mathscr{A}$ is the restricted structure sheaf $\mathscr{O}_X|_{\mathfrak{U}}$ of a smooth proper $\mathbf{R}$-scheme X. Fix a finite affine open cover $\mathfrak{U}$ of X closed under intersections. As explained above, $\mathscr{O}_X|_{\mathfrak{U}}$ gives a quasi-compact right semi-separated presheaf of $\mathbf{R}$-algebras. Since $\mathscr{O}_X(U)$ is smooth $\mathbf{R}$-algebra for each $U\in\mathfrak{U}$, we may assume further that $\phi=(\mathbf{m},\mathbf{f},\mathbf{c})$ is decomposable. We use the same symbol ϕ to denote the cocycle

$$(\alpha,\beta,\gamma)\in H^2(\mathscr{O}_X)\oplus H^1(\mathscr{T}_{X/\mathbf{R}})\oplus H^0(\wedge^2\mathscr{T}_{X/\mathbf{R}})$$

which is the image of $(\mathbf{m},\mathbf{f},\mathbf{c})$ under the bijection (2.3). Then ϕ defines the $\mathbf{S}$-linear abelian category $\mathrm{Qch}(X,\phi)$ obtained by Toda's construction.

Lemma 5.4. ([DLL17, Theorem 5.12]) *For a smooth proper $\mathbf{R}$-scheme X with a finite affine open cover $\mathfrak{U}$ closed under intersections, let $(\mathscr{O}_X|_{\mathfrak{U}})_\phi$ be the twisted $\mathbf{S}$-deformation of the restricted structure sheaf $\mathscr{O}_X|_{\mathfrak{U}}$ along a normalized reduced decomposable cocycle*

$$\phi=(\mathbf{m},\mathbf{f},\mathbf{c})\in\bar{\mathbf{C}}'^{0,2}_{GS}(\mathscr{O}_X|_{\mathfrak{U}})^{\oplus l}\oplus\bar{\mathbf{C}}'^{1,1}_{GS}(\mathscr{O}_X|_{\mathfrak{U}})^{\oplus l}\oplus\bar{\mathbf{C}}'^{2,0}_{GS}(\mathscr{O}_X|_{\mathfrak{U}})^{\oplus l},$$

which maps to a cocycle

$$(\alpha, \beta, \gamma) \in H^2(\mathscr{O}_X) \oplus H^1(\mathscr{T}_{X/\mathbf{R}}) \oplus H^0(\wedge^2 \mathscr{T}_{X/\mathbf{R}})$$

under the bijection (2.3)*. Then there is an equivalence*

$$\mathrm{Qch}((\mathscr{O}_X|_{\mathfrak{U}})_\phi) \simeq \mathrm{Qch}(X, \phi)$$

of $\mathbf{S}$*-linear Grothendieck abelian categories, where* $\mathrm{Qch}(X, \phi)$ *is the abelian category obtained by Toda's construction from* $\mathrm{Qch}(X)$ *along* (α, β, γ)*.*

By (2.3), Lemma 2.2, and Lemma 5.3 we obtain a bijection

$$HT^2(X/\mathbf{R})^{\oplus l} \cong HH^2(X/\mathbf{R})^{\oplus l} \cong HH^2_{ab}(\mathrm{Qch}(X))^{\oplus l}. \tag{5.3}$$

Let $\mathrm{Qch}(X)_\phi$ the flat abelian $\mathbf{S}$-deformation of $\mathrm{Qch}(X)$ along the image of (α, β, γ) under (5.3). Combining Lemma 5.3 and Lemma 5.4, we obtain

Proposition 5.5. *For a smooth proper* $\mathbf{R}$*-scheme X, let* $\mathrm{Qch}(X)_\phi$ *be the flat abelian* $\mathbf{S}$*-deformation of* $\mathrm{Qch}(X)$ *and* $\mathrm{Qch}(X, \phi)$ *the abelian category obtained by Toda's construction from* $\mathrm{Qch}(X)$ *corresponding to* $[\phi] \in HH^2(X/\mathbf{R})^{\oplus l}$ *via the isomorphism* (5.3)*. Then there is an equivalence*

$$\mathrm{Qch}(X)_\phi \simeq \mathrm{Qch}(X, \phi)$$

of $\mathbf{S}$*-linear Grothendieck abelian categories.*

Now, we return to our setting. Let X_0 be a Calabi–Yau manifold with $\dim X_0 > 2$ and $(X, i_{\mathbf{R}})$ an $\mathbf{R}$-deformation of X_0. Since we have

$$HT^2(X/\mathbf{R}) = H^2(\mathscr{O}_X/\mathbf{R}) \oplus H^1(\mathscr{T}_X/\mathbf{R}) \oplus H^0(\wedge^2 \mathscr{T}_X/\mathbf{R}) \cong H^1(\mathscr{T}_X/\mathbf{R}),$$

every cocycle $\phi \in HH^2(X/\mathbf{R})^{\oplus l}$ defines an $\mathbf{S}$-deformation $(X_\phi, i_{\mathbf{S}})$ of $(X, i_{\mathbf{R}})$. By Proposition 3.11 we have $\mathrm{Qch}(X, \phi) \simeq \mathrm{Qch}(X_\phi)$. Along square zero extensions, deforming Calabi–Yau manifolds and taking the category of quasi-coherent sheaves are compatible in the following sense.

Corollary 5.6. *Let* X_0 *be a Calabi–Yau manifold with* $\dim X_0 > 2$*,* $(X, i_{\mathbf{R}})$ *an* $\mathbf{R}$*-deformation of* X_0*, and* $(X_\phi, i_{\mathbf{S}})$ *the* $\mathbf{S}$*-deformation of* $(X, i_{\mathbf{R}})$ *corresponding to* $[\phi] \in HH^2(X/\mathbf{R})^{\oplus l}$*. Then there is an equivalence*

$$\mathrm{Qch}(X)_\phi \simeq \mathrm{Qch}(X_\phi)$$

of $\mathbf{S}$*-linear Grothendieck abelian categories, where* $\mathrm{Qch}(X)_\phi$ *is the flat abelian* $\mathbf{S}$*-deformation of* $\mathrm{Qch}(X)$ *corresponding to* $[\phi]$ *via the isomorphism* (5.3)*.*

Remark 5.1. Since we have $HT^2(X_\phi/\mathbf{S}) \cong H^1(\mathscr{T}_{X_\phi}/\mathbf{S})$ by Calabi–Yau condition and the finite affine open cover $\mathfrak{U} = \{U_i\}_{i=1}^N$ of X closed under intersections canonically lifts to the locally trivial deformation $\mathfrak{U} \times_{\mathbf{R}} \mathbf{S} = \{U_i \times_{\mathbf{R}} \mathbf{S}\}_{i=1}^N$, one may iteratively use Corollary 5.6 along a sequence of square zero extensions.

6. Deformations of the dg category of perfect complexes

In this section, we review the deformation theory of dg category following the exposition from [Low08] and [KL09]. Based on the ideas thereof, for a higher dimensional Calabi–Yau manifold we prove the compatibility of deformations with taking the dg category of perfect complexes. Namely, the dg category of perfect complexes on a deformation is Morita equivalent to the corresponding dg deformation of a certain full dg category determined by the direction of the deformation.

6.1. **Curved A_∞-categories.** In the sequel, by a *quiver* we will mean a $\mathbb{Z}$-graded quiver. We choose shift functors Σ^k on the category $G(\mathbf{R})$ of $\mathbb{Z}$-graded $\mathbf{R}$-modules. Let $\mathfrak{a}$ be an $\mathbf{R}$-linear quiver. Namely, $\mathfrak{a}$ consists of a set $\mathrm{Ob}(\mathfrak{a})$ of objects and a $\mathbb{Z}$-graded $\mathbf{R}$-module $\mathfrak{a}(A, A')$ for each pair $A, A' \in \mathrm{Ob}(\mathfrak{a})$. The category of quivers with a fixed set of objects admits a tensor product

$$\mathfrak{a} \otimes \mathfrak{b}(A, A') = \bigoplus_{A''} \mathfrak{a}(A'', A') \otimes_{\mathbf{R}} \mathfrak{b}(A, A'')$$

and an internal hom

$$[\mathfrak{a}, \mathfrak{b}](A, A') = [\mathfrak{a}(A, A'), \mathfrak{b}(A, A')].$$

Morphisms of degree k are elements of $[\mathfrak{a}, \mathfrak{b}]^k = \prod_{A,A'} [\mathfrak{a}, \mathfrak{b}](A, A')^k$.

The *tensor cocategory* $T(\mathfrak{a})$ of $\mathfrak{a}$ is the quiver

$$T(\mathfrak{a}) = \bigoplus_{n \geq 0} \mathfrak{a}^{\otimes n}$$

equipped with the comultiplication which separates tensors. There is a natural brace algebra structure on $[T(\mathfrak{a}), \mathfrak{a}] = \prod_{n \geq 0} [T(\mathfrak{a}), \mathfrak{a}]_n$, where

$$[T(\mathfrak{a}), \mathfrak{a}]_n = [\mathfrak{a}^{\otimes n}, \mathfrak{a}] = \prod_{A_0, \ldots, A_n \in \mathfrak{a}} [\mathfrak{a}(A_{n-1}, A_n) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathfrak{a}(A_0, A_1), \mathfrak{a}(A_0, A_n)].$$

It is given by the operations

$$[T(\mathfrak{a}), \mathfrak{a}]_n \otimes_{\mathbf{R}} [T(\mathfrak{a}), \mathfrak{a}]_{n_1} \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} [T(\mathfrak{a}), \mathfrak{a}]_{n_i} \to [T(\mathfrak{a}), \mathfrak{a}]_{n-i+n_1+\cdots+n_i},$$
$$(\phi, \phi_1, \ldots, \phi_i) \mapsto \phi\{\phi_1, \ldots, \phi_i\}$$

with

$$\phi\{\phi_1, \ldots, \phi_i\} = \sum \phi(1 \otimes \cdots \otimes \phi_1 \otimes 1 \otimes \cdots \otimes \phi_i \otimes 1 \otimes \cdots \otimes 1)$$

satisfying

$$\phi\{\phi_1, \ldots, \phi_i\}\{\psi_1, \ldots, \psi_j\} = \sum (-1)^\alpha \phi\{\psi_1, \ldots, \phi_1\{\psi_{m_1}, \ldots\}, \psi_{n_1}, \ldots, \phi_i\{\psi_{m_i} \ldots\}, \psi_{n_i}, \ldots, \psi_j\},$$

where $\alpha = \sum_{k=1}^{i} |\phi_k| \sum_{l=1}^{m_k - 1} |\psi_l|$. We denote by $B\mathfrak{a}$ the Bar cocategory $T(\Sigma\mathfrak{a})$ and by $\mathbf{C}_{br}(\mathfrak{a})$ the brace algebra $[B\mathfrak{a}, \Sigma\mathfrak{a}]$. The *associated Hochschild object* is defined as $\mathbf{C}(\mathfrak{a}) = \Sigma^{-1}\mathbf{C}_{br}(\mathfrak{a})$. By [Low08, Proposition 2.2] the tensor coalgebra $T(\mathbf{C}_{br}(\mathfrak{a})) = B\mathbf{C}(\mathfrak{a})$ becomes a graded bialgebra with the associative multiplication defined by the composition.

A *curved A_∞-structure* on $\mathfrak{a}$ is an element $b \in \mathbf{C}^1_{br}(\mathfrak{a})$ satisfying $b\{b\} = 0$. The pair $(\mathfrak{a}, b)$ is called a *curved A_∞-category*. When the defining morphisms $b_n \colon \Sigma\mathfrak{a}^{\otimes n} \to \Sigma\mathfrak{a}$ vanish for $n \geq 3$, we call $(\mathfrak{a}, b)$ a *cdg category*. The *curvature elements* of $(\mathfrak{a}, b)$ is the morphism b_0. When it vanishes, we drop "curved" and "c" from the notation.

Definition 6.1. ([Low08, Definition 2.5]) *For curved A_∞-categories* $(\mathfrak{a}, b), (\mathfrak{a}', b')$ *with* $\mathrm{Ob}(\mathfrak{a}) = \mathrm{Ob}(\mathfrak{a}')$ *a* morphism *is a fixed object morphism of quivers $f \colon B\mathfrak{a} \to B\mathfrak{a}'$, which is determined by morphisms $f_n \colon (\Sigma\mathfrak{a})^{\otimes n} \to \Sigma\mathfrak{a}'$ and respects the comultiplications and the curved A_∞-structures.*

6.2. **Hochschild complexes of curved A_∞-categories.** The *associated Lie bracket* with the brace algebra $\mathbf{C}_{br}(\mathfrak{a})$ is defined as

$$\langle \phi, \psi \rangle = \phi\{\psi\} - (-1)^{|\phi||\psi|} \psi\{\phi\}.$$

Via an isomorphism

$$\mathbf{C}_{br}(\mathfrak{a}) \cong \mathrm{Coder}(B\mathfrak{a}, B\mathfrak{a})$$

of $\mathbb{Z}$-graded $\mathbf{R}$-modules to coderivations between cocategories, it corresponds to the commutator of coderivations. For a curved A_∞-structure b on $\mathfrak{a}$ the *Hochschild differential* on $\mathbf{C}_{br}(\mathfrak{a})$ is defined as

$$d_b = \langle b, - \rangle \in [\mathbf{C}_{br}(\mathfrak{a}), \mathbf{C}_{br}(\mathfrak{a})]^1,\ \phi \mapsto \langle b, \phi \rangle.$$

In particular, $\mathbf{C}_{br}(\mathfrak{a})$ can be regarded as a dg Lie algebra. Then $\mathbf{C}(\mathfrak{a})$ is known to be isomorphic to the classical Hochschild complex of $\mathfrak{a}$, whose definition we will review later. Since b naturally belongs to $B\mathbf{C}(\mathfrak{a})^1$, it induces a differential

$$D_b = [b, -] \in [B\mathbf{C}(\mathfrak{a}), B\mathbf{C}(\mathfrak{a})]^1,\ \phi \mapsto [b, \phi],$$

where $[-,-]$ is the commutator of the multiplication given by [Low08, Proposition 2.2]. As D_b belongs to $\mathrm{Coder}(B\mathbf{C}(\mathfrak{a}), B\mathbf{C}(\mathfrak{a}))$, it defines a curved A_∞- structure on $\mathbf{C}(\mathfrak{a})$. The differential D_b gives a dg bialgebra structure on $B\mathbf{C}(\mathfrak{a})$ and $\mathbf{C}(\mathfrak{a}), \mathbf{C}_{br}(\mathfrak{a})$ become B_∞-algebras [GJ, Definition 5.2].

We will use the same symbol $\mathbf{C}(\mathfrak{a})$ to denote the bigraded object with

$$\mathbf{C}^{p,q}(\mathfrak{a}) = \prod_{A_0,\dots,A_q \in \mathfrak{a}} [\mathfrak{a}(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathfrak{a}(A_0, A_1), \mathfrak{a}(A_0, A_q)]^p.$$

An element $\phi \in \mathbf{C}^{p,q}(\mathfrak{a})$ is said to have the *degree* $|\phi| = p$, the *arity* $\mathrm{ar}(\phi) = q$, and the *Hochschild degree* $\deg(\phi) = n = p + q$. The total complex of Hochschild degree n is defined as $\mathbf{C}^n(\mathfrak{a}) = \prod_{p+q=n} \mathbf{C}^{p,q}(\mathfrak{a})$. Via the canonical isomorphisms

$$\begin{aligned}&\Sigma^{1-q}[\mathfrak{a}(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathfrak{a}(A_0, A_1), \mathfrak{a}(A_0, A_q)]\\ &\to [\Sigma\mathfrak{a}(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \Sigma\mathfrak{a}(A_0, A_1), \Sigma\mathfrak{a}(A_0, A_q)],\end{aligned}$$

the B_∞-structure on $\mathbf{C}_{br}(\mathfrak{a})$ is translated in terms of $\mathfrak{a}$. For instance, the operation

$$\mathrm{dot}\colon \mathbf{C}_{br}(\mathfrak{a})_q \otimes \mathbf{C}_{br}(\mathfrak{a})_s \to \mathbf{C}_{br}(\mathfrak{a})_{q+s-1},\ (\phi, \psi) \mapsto \phi\{\psi\}$$

induces the classical "dot product"

$$\bullet\colon \mathbf{C}^{p,q}(\mathfrak{a}) \otimes \mathbf{C}^{r,s}(\mathfrak{a}) \to \mathbf{C}^{p+r,q+s-1}(\mathfrak{a})$$

on $\mathbf{C}(\mathfrak{a})$ given by

$$\phi \bullet \psi = \sum_{k=0}^{q-1} (-1)^\beta \phi(1^{\otimes q-k-1} \otimes \psi \otimes 1^{\otimes k}),$$

where $\beta = (\deg(\phi) + k + 1)(\mathrm{ar}(\psi) + 1)$. We also call the bigraded object $\mathbf{C}(\mathfrak{a})$ the Hochschild complex of $\mathfrak{a}$ and its elements Hochschild cochains. In the sequel, curved A_∞-structure on $\mathfrak{a}$ will often be translated into an element of $\mathbf{C}^2(\mathfrak{a})$ without further comments.

6.3. **Curved dg category of precomplexes.** Let $\mathfrak{a}$ be an $\mathbf{R}$-linear category. Consider the category $\mathrm{PCom}(\mathfrak{a})$ of precomplexes of $\mathfrak{a}$-objects. A *precomplex* of $\mathfrak{a}$-objects is a $\mathbb{Z}$-graded $\mathfrak{a}$-objects C with $C^i \in \mathfrak{a}$ together with a *predifferential*, a $\mathbb{Z}$-graded $\mathfrak{a}$-morphism $\delta_C\colon C \to C$ of degree 1. For any $C, C' \in \mathrm{PCom}(\mathfrak{a})$ the Hom-set $\mathrm{PCom}(\mathfrak{a})(C, C')$ is a $\mathbb{Z}$-graded $\mathbf{R}$-module with

$$\mathrm{PCom}(\mathfrak{a})(C, C')^k = \prod_{i \in \mathbb{Z}} \mathfrak{a}(C^i, C'^{i+k}).$$

The cdg structure $\mu \in \mathbf{C}(\mathfrak{a})^2$ on $\mathrm{PCom}(\mathfrak{a})$ consists of the compositions m, differentials d, and curvature elements c, where

$$m = \mu_2 \in \prod_{C_0,C_1,C_2 \in \mathrm{PCom}(\mathfrak{a})} [\mathrm{PCom}(\mathfrak{a})(C_1,C_2) \otimes_{\mathbf{R}} \mathrm{PCom}(\mathfrak{a})(C_0,C_1), \mathrm{PCom}(\mathfrak{a})(C_0,C_2)]^0,$$
$$d = \mu_1 \in \prod_{C_0,C_1 \in \mathrm{PCom}(\mathfrak{a})} [\mathrm{PCom}(\mathfrak{a})(C_0,C_1), \mathrm{PCom}(\mathfrak{a})(C_0,C_1)]^1,$$
$$c = \mu_0 \in \prod_{C \in \mathrm{PCom}(\mathfrak{a})} \mathrm{PCom}(\mathfrak{a})(C,C)^2$$

are given by

$$m(g,f)_i = (gf)_i = g_{i+|f|} f_i \colon C_0^i \to C_2^{i+|f|+|g|},$$
$$d(f) = \delta_{C_1} f - (-1)^{|f|} f \delta_{C_0},$$
$$c_C = -\delta_C^2$$

for morphisms $f\colon C_0 \to C_1, g\colon C_1 \to C_2$ in $\mathrm{PCom}(\mathfrak{a})$. One can check that m, d, and c satisfy

$$d(c) = 0,$$
$$d^2 = -m(c \otimes 1 - 1 \otimes c),$$
$$dm = m(d \otimes 1 + 1 \otimes d),$$
$$m(m \otimes 1) = m(1 \otimes m).$$

We denote by $\mathrm{Com}(\mathfrak{a})$ the full dg subcategoy of complexes of $\mathfrak{a}$-objects, where δ_C become differentials.

Here, we demonstrate how the cdg structure is translated. The differential

$$d_b = \langle b, - \rangle = \langle \Sigma c + d + \Sigma^{-1} m, - \rangle \in [\mathbf{C}_{br}(\mathrm{PCom}(\mathfrak{a})), \mathbf{C}_{br}(\mathrm{PCom}(\mathfrak{a}))]^1$$

on $\mathbf{C}_{br}(\mathrm{PCom}(\mathfrak{a}))$ sends $\Sigma^{1-q}\phi \in \mathbf{C}_{br}(\mathrm{PCom}(\mathfrak{a}))$ with $\phi \in \mathbf{C}^{p,q}(\mathrm{PCom}(\mathfrak{a}))$ to

$$\mathrm{dot}(\Sigma c + d + \sigma^{-1} m, \Sigma^{1-q}\phi) - (-1)^{1-q+p} \mathrm{dot}(\Sigma^{1-q}\phi, \Sigma c + d + \Sigma^{-1} m).$$

In terms of $\mathbf{C}(\mathfrak{a})$ the image corresponds to $[c + d + m, \phi]$, where

$$[c, \phi] = \sum_{k=0}^{q-1} (-1)^{k+1} \phi(1^{\otimes q-k-1} \otimes c \otimes 1^{\otimes k}),$$
$$[d, \phi] = (-1)^{\mathrm{ar}(\phi)+1} d(\phi) + \sum_{k=0}^{q-1} (-1)^{\deg(\phi)} \phi(1^{\otimes q-k-1} \otimes c \otimes 1^{\otimes k}),$$
$$[m, \phi] = m(\phi \otimes 1) + (-1)^{\mathrm{ar}(\phi)+1} m(1 \otimes \phi) + \sum_{k=0}^{q-1} (-1)^{k+1} \phi(1^{\otimes q-k-1} \otimes c \otimes 1^{\otimes k}).$$

6.4. **Curved dg deformations of dg categories.** Assume that $\mathfrak{a}$ is an $\mathbf{R}$-linear cdg category. A *cdg* $\mathbf{S}$*-deformation* of $\mathfrak{a}$ is an $\mathbf{S}$-linear cdg structure on an $\mathbf{S}$-linear quiver $\mathfrak{b}$ together with an isomorphism $\mathfrak{b} \to \mathfrak{a}[\epsilon] = \mathfrak{a} \otimes_{\mathbf{R}} \mathbf{S}$ of $\mathbf{S}$-linear quivers whose reduction $\mathfrak{b} \otimes_{\mathbf{S}} \mathbf{R} \to \mathfrak{a}$ induces an isomorphism of cdg categories. Two cdg deformations $\mathfrak{b}, \mathfrak{b}'$ are *isomorphic* if there is an isomorphism $\mathfrak{b} \to \mathfrak{b}'$ of cdg categories inducing the identity on $\mathfrak{a}$. We denote by $\mathrm{Def}_{\mathfrak{a}}^{cdg}(\mathbf{S})$ the set of isomorphism classes of cdg $\mathbf{S}$-deformations of $\mathfrak{a}$.

Theorem 6.2. ([Low08, Theorem 4.11]) *Let* $(\mathfrak{a}, \mu)$ *be an* $\mathbf{R}$*-linear cdg category. Then there is a bijection*

(6.1) $$H^2\mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathrm{Def}_{\mathfrak{a}}^{cdg}(\mathbf{S}),\ \phi \mapsto (\mathfrak{a}[\epsilon], \mu + \phi\epsilon),\ \phi \in Z^2\mathbf{C}(\mathfrak{a})^{\oplus l}.$$

Another cocycle $\phi' \in Z^2\mathbf{C}(\mathfrak{a})^{\oplus l}$ maps to an isomorphic cdg deformation if and only if there is an element $h \in \mathbf{C}^1(\mathfrak{a})$ satisfying $\phi' - \phi = d_\mu(h)$.

A *partial cdg* **S***-deformation* of $\mathfrak{a}$ is a cdg **S**-deformation of some full cdg subcategory $\mathfrak{a}'$. Two partial cdg deformations $\mathfrak{b}, \mathfrak{b}'$ are *isomorphic* if there is an isomorphism $\mathfrak{b} \to \mathfrak{b}'$ of cdg categories inducing the identity on $\mathfrak{a}'$. A *morphism* of partial cdg deformations $\mathfrak{b}, \mathfrak{b}'$ is an isomorphism of cdg deformations between $\mathfrak{b}$ and a full cdg subcategory of $\mathfrak{b}'$. When every morphism of $\mathfrak{b} \to \mathfrak{b}'$ of partial cdg deformations is an isomorphism, we call $\mathfrak{b}$ *maximal*. We denote by $\mathrm{Def}^{p-cdg}_{\mathfrak{a}}(\mathbf{S})$ the set of morphism classes of partial cdg **S**-deformations of $\mathfrak{a}$ and by $\mathrm{Def}^{mp-cdg}_{\mathfrak{a}}(\mathbf{S})$ the set of isomorphism classes of maximal partial cdg **S**-deformations of $\mathfrak{a}$.

Assume further that $\mathfrak{a}$ is a dg category. For $\phi \in Z^2\mathbf{C}(\mathfrak{a})$ the $[\phi] - \infty$ *part* of $\mathfrak{a}$ is the full dg subcategory $\mathfrak{a}_{[\phi]-\infty}$ spanned by objects $A \in \mathfrak{a}$ satisfying

$$H^2(\pi_0)([\phi])_A = 0 \in H^2(\mathfrak{a}(A, A)),$$

where $\pi_0 \colon \mathbf{C}(\mathfrak{a}) \to \mathbf{C}(\mathfrak{a})_0$ is the projection onto the *zero part*

$$\mathbf{C}(\mathfrak{a})_0 = \Sigma^{-1}[T(\Sigma\mathfrak{a}), \Sigma\mathfrak{a}]_0 = \prod_{A \in \mathfrak{a}} \mathfrak{a}(A, A).$$

For a cdg **S**-deformation $\mathfrak{b} = (\mathfrak{a}[\epsilon], \mu + \phi\epsilon)$ of $\mathfrak{a}$, the ∞*-part* $\mathfrak{b}_\infty$ is the full cdg subcategory spanned by objects $B \in \mathfrak{b}$ satisfying

$$(\mu + \phi\epsilon)_{0,B} = 0 \in \mathfrak{b}(B, B)^2.$$

It is a partial dg deformation of $\mathfrak{a}$ and a dg deformation of $\mathfrak{a}_{[\phi]-\infty}$. More explicitly, if we restrict ϕ to $\mathfrak{a}_{[\phi]-\infty}$, then ϕ_0 becomes a coboundary and there is an element

$$h \in \prod_{A \in \mathfrak{a}_{[\phi]-\infty}} \mathfrak{a}(A, A)^1 \subset \mathbf{C}^1(\mathfrak{a})$$

with $d_\mu(h) = (\phi|_{\mathfrak{a}_{[\phi]-\infty}})_0$. Thus the cocycle $\phi|_{\mathfrak{a}_{[\phi]-\infty}} - d_\mu(h)$ has trivial curvature elements.

Proposition 6.3. ([Low08, Proposition 4.14]) *Let $(\mathfrak{a}, \mu)$ be an* **R***-linear dg category. Then there is a map*

$$H^2\mathbf{C}(\mathfrak{a})^{\oplus l} \to \mathrm{Def}^{p-dg}_{\mathfrak{a}}(\mathbf{S}),\ \phi \mapsto (\mathfrak{a}_{[\phi]-\infty}[\epsilon], \mu + (\phi|_{\mathfrak{a}_{[\phi]-\infty}} - d_\mu(h))\epsilon),\ \phi \in Z^2\mathbf{C}(\mathfrak{a})^{\oplus l}. \tag{6.2}$$

6.5. **The characteristic morphism for dg categories.** Let $(\mathfrak{a}, \mu = d + m)$ be an **R**-linear dg category. Consider the ∞-part $\mathrm{Tw}(\mathfrak{a}) = \mathrm{Tw}_{\mathrm{ilnil}}(\mathfrak{a})_\infty$ of the cdg category of locally nilpotent twisted objects over $\mathfrak{a}$ from [Low08, Proposition 3.6]. It is known to be a dg enhancement of the derived category $\mathbf{D}(\mathfrak{a})$ of right dg modules over $\mathfrak{a}$. Each object of $\mathrm{Tw}(\mathfrak{a})$ is given by a pair (M, δ_M) of $M \in \mathrm{Free}(\mathfrak{a})$ and $\delta_M \in \mathrm{Free}(\mathfrak{a})(M, M)^1$. The collection $\{\delta_M\}_{M \in \mathrm{Tw}(\mathfrak{a})}$ canonically determines an element $\delta \in \mathbf{C}^1(\mathrm{Tw}(\mathfrak{a}))$. Here, $\mathrm{Free}(\mathfrak{a})$ is a quiver whose objects are formal expressions $M = \oplus_{i \in I}\Sigma^{n_i}A_i$ with $A_i \in \mathfrak{a}$, $n_i \in \mathbb{Z}$, and I an arbitrary index set. For any $M, M' \in \mathrm{Free}(\mathfrak{a})$ the Hom-set is

$$\mathrm{Free}(\mathfrak{a})(M, M') = \prod_{i \in I} \oplus_{i' \in I'}\Sigma^{n_{i'} - n_i}\mathfrak{a}(A_i, A'_{i'}).$$

Theorem 6.4. ([Low08, Theorem 3.19]) *Let $\mathfrak{a}$ be a dg category and $\mathrm{Tw}(\mathfrak{a})$ the ∞-part of the cdg category of locally nilpotent twisted objects from* [Low08, Proposition 3.6] *with its canonical Hochschild cochain $\delta \in \mathbf{C}^1(\mathrm{Tw}(\mathfrak{a}))$. Then the canonical projection $\pi \colon \mathbf{C}(\mathrm{Tw}(\mathfrak{a})) \to \mathbf{C}(\mathfrak{a})$ has a B_∞-section*

$$\mathrm{embr}_\delta \colon \mathbf{C}(\mathfrak{a}) \to \mathbf{C}(\mathrm{Tw}(\mathfrak{a})), \phi \mapsto \sum_{i=0}^{\infty} \phi\{\ \delta^{\otimes i}\}, \tag{6.3}$$

which is an inverse in $\mathrm{Ho}(B_\infty)$ *of* B_∞*-algebras. In particular, both* π *and* embr_δ *are quasi-isomorphisms.*

Consider the projection onto the zero part

$$\pi_0\colon \mathbf{C}(\mathrm{Tw}(\mathfrak{a})) \to \mathbf{C}(\mathrm{Tw}(\mathfrak{a}))_0 = \prod_{(M,\delta_M)\in \mathrm{Tw}(\mathfrak{a})} \mathrm{Tw}(\mathfrak{a})((M,\delta_M),(M,\delta_M)). \tag{6.4}$$

Since $\mathrm{Tw}(\mathfrak{a})$ is uncurved, π_0 induces a morphism of dg algebras [Low08, Proposition 2.7]. Composing (6.3) and (6.4), one obtains the *characteristic dg morphism*

$$\mathbf{C}(\mathfrak{a}) \to \prod_{(M,\delta_M)\in \mathrm{Tw}(\mathfrak{a})} \mathrm{Tw}(\mathfrak{a})((M,\delta_M),(M,\delta_M)). \tag{6.5}$$

The full dg subcategory $\mathrm{tw}(\mathfrak{a}) = \mathrm{tw}_{\mathrm{ilnil}}(\mathfrak{a})_\infty \subset \mathrm{Tw}(\mathfrak{a})$ spanned by objects (M,δ_M) with

$$d\{\delta_M\} + m\{\delta_M,\delta_M\} = 0$$

is equivalent to the classical dg category of twisted complexes over $\mathfrak{a}$ [Low08, Example 3.16]. It is known to be a dg enhancement of the smallest triangulated subcategory of $\mathbf{D}(\mathfrak{a})$. We denote by $\mathrm{Tw}(\mathfrak{a})^c$ the full dg subcategory of $\mathrm{Tw}(\mathfrak{a})$ spanned by compact objects. It is known to be a dg enhancement of the idempotent completion of the homotopy category of $\mathrm{tw}(\mathfrak{a})$.

Assume that $\mathfrak{a}$ is triangulated, i.e., pretriangulated and closed under homotopy direct summands. Then the homotopy category of $\mathrm{Tw}(\mathfrak{a})^c$ get identified with that of $\mathrm{tw}(\mathfrak{a})$ by assumption and that of $\mathfrak{a}$ via the Yoneda embedding. When restricted to $\mathrm{Tw}(\mathfrak{a})^c$, on cohomology (6.5) induces the characteristic morphism

$$\chi_{\mathfrak{a},A}\colon H^\bullet\mathbf{C}(\mathfrak{a}) \to \mathfrak{Z}^\bullet(\mathrm{Tw}(\mathfrak{a})^c) = \prod_{A\in\mathfrak{a}} H^\bullet(\mathfrak{a}(A,A)),$$

where we identify each object $A\in\mathfrak{a}$ with its image under the Yoneda embedding. Here,

$$\mathfrak{Z}(\mathrm{Tw}(\mathfrak{a})^c) = \mathrm{Hom}(1_{\mathrm{Tw}(\mathfrak{a})^c}, 1_{\mathrm{Tw}(\mathfrak{a})^c})$$

is the *center* of the graded category $\mathrm{Tw}(\mathfrak{a})^c$, where Hom denotes the graded $\mathbf{R}$-module of graded natural transformations [Low08, Remark 4.6].

6.6. **The characteristic morphism for linear categories.** In this subsection, although it is not strictly necessary to deduce our main results, for completeness we review the relationship between the characteristic morphism and the obstruction against lifting objects of derived categories. Besides [Low08, Theorem 4.8] and [Low08, Corollary 4.9], we include their dual statements due to Lowen.

Let $\mathfrak{i}$ be an $\mathbf{R}$-linear category. The canonical projection

$$\pi\colon \mathbf{C}(\mathrm{PCom}(\mathfrak{i})) \to \mathbf{C}(\mathfrak{i})$$

has a B_∞-section

$$\mathrm{embr}_\delta\colon \mathbf{C}(\mathfrak{i}) \to \mathbf{C}(\mathrm{PCom}(\mathfrak{i})), \tag{6.6}$$

whose restriction to the full dg subcategory $\mathrm{Com}^+(\mathfrak{i})$ of bounded below complexes of $\mathfrak{i}$-objects is an inverse in the homotopy category $\mathrm{Ho}(B_\infty)$ of B_∞-algebras [Low08, Theorem 3.21]. Consider the projection onto the zero part

$$\pi_0\colon \mathbf{C}(\mathrm{Com}(\mathfrak{i})) \to \mathbf{C}(\mathrm{Com}(\mathfrak{i}))_0 = \prod_{C\in\mathrm{Com}(\mathfrak{i})} \mathrm{Com}(\mathfrak{i})(C,C). \tag{6.7}$$

Since $\mathrm{Com}(\mathfrak{i})$ is uncurved, π_0 induces a morphism of dg algebras [Low08, Proposition 2.7]. Composing (6.6) and (6.7), one obtains the *characteristic dg morphism*

$$\mathbf{C}(\mathfrak{i}) \to \prod_{C\in\mathrm{Com}(\mathfrak{i})} \mathrm{Com}(\mathfrak{i})(C, C).$$

Note that $\mathrm{Tw}(\mathfrak{i})$ is precisely $\mathrm{Com}(\mathfrak{i})$ [Low08, Section 3.5]. On cohomology it induces the *characteristic morphism*

$$\chi_{\mathfrak{i}} : H^\bullet\mathbf{C}(\mathfrak{i}) \to \mathfrak{Z}^\bullet(K(\mathfrak{i}))$$

for a linear category $\mathfrak{i}$. Here, $\mathfrak{Z}(K(\mathfrak{i}))$ is the *graded center* of $K(\mathfrak{i})$, i.e., the center

$$\mathfrak{Z}(\mathrm{Com}(\mathfrak{i})) = \mathrm{Hom}(1_{\mathrm{Com}(\mathfrak{i})}, 1_{\mathrm{Com}(\mathfrak{i})})$$

of the graded category $\mathrm{Com}(\mathfrak{i})$. The characteristic morphism can be interpreted in terms of deformations of categories.

Theorem 6.5. ([Low08, Theorem 4.8]) *Let $\mathfrak{i}$ be an $\mathbf{R}$-linear category and $\mathfrak{i}_\phi$ its $\mathbf{S}$-deformation along $\phi \in Z^2\mathbf{C}(\mathfrak{i})^{\oplus l}$. Then for each $C \in K(\mathfrak{i})$ the element $\chi_{\mathfrak{i}}^{\oplus l}(\phi)_C \in K(\mathfrak{i})(C, C)[2]^{\oplus l}$ is the obstruction against lifting C to an object of $K(\mathfrak{i}_\phi)$ along* $\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -) : K(\mathfrak{i}_\phi) \to K(\mathfrak{i})$.

Remark 6.1. The notation in [Low08, Section 4.4] is quite confusing. It was confirmed by Wendy Lowen that ϕ-deforming C in the statement of [Low08, Theorem 4.8] means precisely lifting C along $\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -)$.

Let $\mathscr{C}$ be an $\mathbf{R}$-linear abelian category with enough injectives. Assume that $\mathfrak{i}$ is the full linear subcategory $\mathrm{Inj}(\mathscr{C})$ of injective objects. Taking cohomology of (6.7) restricted to $\mathrm{Com}^+(\mathfrak{i})$ and composing with the isomorphism $HH^\bullet_{ab}(\mathscr{C}) \cong H\mathbf{C}^\bullet(\mathrm{Com}^+(\mathfrak{i}))$ induced by (6.6), one obtains the *characteristic morphism*

$$\chi_{\mathscr{C}} : HH^\bullet_{ab}(\mathscr{C}) \to \mathfrak{Z}^\bullet(D^+(\mathscr{C})) \tag{6.8}$$

for an abelian category $\mathscr{C}$. Here, we use the isomorphism $HH^\bullet_{ab}(\mathscr{C}) \cong H\mathbf{C}^\bullet(\mathfrak{i})$ obtained from [LV06b, Theorem 6.6]. Note that the graded center $\mathfrak{Z}(D^+(\mathscr{C}))$ of $D^+(\mathscr{C}) \simeq K^+(\mathfrak{i})$ is given by the center $\mathfrak{Z}(\mathrm{Com}^+(\mathfrak{i}))$.

Corollary 6.6. ([Low08, Corollary 4.9]) *Let $\mathscr{C}$ be an $\mathbf{R}$-linear abelian category with enough injectives and $\mathscr{C}_\phi$ its abelian $\mathbf{S}$-deformation along a cocycle $\phi \in HH^2_{ab}(\mathscr{C})^{\oplus l}$. Then for each $C \in D^+(\mathscr{C})$ the element $\chi_{\mathscr{C}}^{\oplus l}(\phi)_C \in \mathrm{Ext}^2_{\mathscr{C}}(C, C)^{\oplus l}$ is the obstruction against lifting C to an object of $D^+(\mathscr{C}_\phi)$ along $R\,\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -) : D^+(\mathscr{C}_\phi) \to D^+(\mathscr{C})$.*

Remark 6.2. Similarly to the previous remark, deforming C in the statement of [Low08, Corollary 4.9] means precisely lifting C along $R\,\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -)$.

As mentioned in [Low05, Introduction], dualizing the relevant arguments, one obtains similar results for lifting objects along the functors $- \otimes_{\mathbf{S}} \mathbf{R}$ and $- \otimes^L_{\mathbf{S}} \mathbf{R}$. We thank Wendy Lowen for explaining the next two results below to the author.

Theorem 6.7. *Let $\mathfrak{p}$ be an $\mathbf{R}$-linear category and $\mathfrak{p}_\phi$ its $\mathbf{S}$-deformation along $\phi \in Z^2\mathbf{C}(\mathfrak{p})^{\oplus l}$. Then for each $C \in K(\mathfrak{p})$ the element $\chi_{\mathfrak{p}}^{\oplus l}(\phi)_C \in K(\mathfrak{p})(C, C)[2]^{\oplus l}$ is the obstruction against lifting C to an object of $K(\mathfrak{p}_\phi)$ along $- \otimes_{\mathbf{S}} \mathbf{R} : K(\mathfrak{p}_\phi) \to K(\mathfrak{p})$.*

Let $\mathscr{C}$ be an $\mathbf{R}$-linear abelian category with enough projectives. Assume that $\mathfrak{p}$ is the full linear subcategory $\mathrm{Proj}(\mathscr{C})$ of projective objects. Taking cohomology of (6.7) restricted to $\mathrm{Com}^-(\mathfrak{p})$ and composing with the isomorphism $HH^\bullet_{ab}(\mathscr{C}) \cong H\mathbf{C}^\bullet(\mathrm{Com}^-(\mathfrak{p}))$ induced by (6.6), one obtains the characteristic morphism

$$\chi_{\mathscr{C}} : HH^\bullet_{ab}(\mathscr{C}) \to \mathfrak{Z}^\bullet(D^-(\mathscr{C}))$$

dual to (6.8). Here, we use the isomorphism $HH^\bullet_{ab}(\mathscr{C}) \cong H\mathbf{C}^\bullet(\mathfrak{p})$ obtained from the dual statement of [LV06b, Theorem 6.6]. Note that the graded center $\mathfrak{Z}(D^-(\mathscr{C}))$ of $D^-(\mathscr{C}) \simeq K^-(\mathfrak{p})$ is given by the center $\mathfrak{Z}(\mathrm{Com}^-(\mathfrak{p}))$.

Corollary 6.8. *Let $\mathscr{C}$ be an $\mathbf{R}$-linear abelian category with enough projectives and $\mathscr{C}_\phi$ its abelian $\mathbf{S}$-deformation along a cocycle $\phi \in HH^2_{ab}(\mathscr{C})^{\oplus l}$. Then for each $C \in D^-(\mathscr{C})$ the element $\chi^{\oplus l}_{\mathscr{C}}(\phi)_C \in \mathrm{Ext}^2_{\mathscr{C}}(C, C)^{\oplus l}$ is the obstruction against lifting C to an object of $D^-(\mathscr{C}_\phi)$ along $- \otimes^L_{\mathbf{S}} \mathbf{R}\colon D^-(\mathscr{C}_\phi) \to D^-(\mathscr{C})$.*

6.7. **Maximal partial dg deformations of the dg category of bounded below complexes.** In this subsection, for a linear category we explain that the maximal partial deformation of the dg category of bounded below complexes along a given Hochschild cocycle is precisely the dg deformation of the full dg subcategory spanned by all objects whose lifts become curvature free with respect to the Hochschild cocycle. Similar observation for the dg category of perfect complexes will be crucial later.

Consider the map

$$\rho'\colon H^2\mathbf{C}(\mathfrak{i})^{\oplus l} \to \mathrm{Def}^{p-dg}_{\mathrm{Com}^+(\mathfrak{i})}(\mathbf{S}),\ \phi \mapsto (\mathrm{Com}^+(\mathfrak{i})_{\mathrm{embr}_\delta(\phi)})_\infty \tag{6.9}$$

obtained from (6.2) and (6.6). The partial dg deformation $(\mathrm{Com}^+(\mathfrak{i})_{\mathrm{embr}_\delta(\phi)})_\infty$ of $\mathrm{Com}^+(\mathfrak{a})$ coincides with a dg deformation $(\mathrm{Com}^+(\mathfrak{i})_{[\mathrm{embr}_\delta(\phi)]-\infty})_{\mathrm{embr}_\delta(\phi)}$ of $\mathrm{Com}^+(\mathfrak{i})_{[\mathrm{embr}_\delta(\phi)]-\infty}$, where the cdg structure $\mathrm{embr}_\delta(\phi)$ restricted to the $[\mathrm{embr}_\delta(\phi)] - \infty$ part.

Theorem 6.9. ([Low08, Theorem 4.15(iii)]) *Let $\mathfrak{i}$ be an $\mathbf{R}$-linear category and $\mathfrak{i}_\phi$ its $\mathbf{S}$-deformation along $\phi \in Z^2\mathbf{C}(\mathfrak{i})^{\oplus l}$. Then, for every collection of complexes $\Gamma = \{\bar{C}\}_{C \in \mathrm{Com}^+(\mathfrak{i})_{[\mathrm{embr}_\delta(\phi)]-\infty}}$ with $\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, \bar{C}) = C$, the full dg subcategory $\mathrm{Com}^+_\Gamma(\mathfrak{i}_\phi) \subset \mathrm{Com}^+(\mathfrak{i}_\phi)$ spanned by Γ is a maximal partial dg $\mathbf{S}$-deformation of $\mathrm{Com}^+(\mathfrak{i})$ representing $\rho'(\phi)$.*

From the proof, one sees that $\mathrm{Com}^+_\Gamma(\mathfrak{i}_\phi)$ is a dg deformation of $\mathrm{Com}^+(\mathfrak{i})_{[\mathrm{embr}(\phi)]-\infty}$. According to [Low08, Example 4.13], an object $C \in \mathrm{Com}^+(\mathfrak{i})$ belongs to $\mathrm{Com}^+(\mathfrak{i})_{[\mathrm{embr}(\phi)]-\infty}$ if and only if

$$\chi^{\oplus l}_{\mathfrak{i}}(\phi)_C = 0 \in K(\mathfrak{i})(C, C[2])^{\oplus l}.$$

Since $\chi^{\oplus l}_{\mathfrak{i}}(\phi)_C$ is the obstruction against lifting C to an object of $\mathrm{Com}^+(\mathfrak{i}_\phi)$ along $\mathrm{Hom}_{\mathbf{S}}(\mathbf{R}, -)$, we may take a collection of lifts of unobstructed complexes in $\mathrm{Com}^+(\mathfrak{i})$ as Γ. Clearly, any dg subcategory of $\mathrm{Com}^+(\mathfrak{i})_{[\mathrm{embr}(\phi)]-\infty}$ dg deforms along the restriction of $\mathrm{embr}_\delta(\phi)$.

6.8. **Hochschild cohomology of the dg category of perfect complexes.** We review the definition of the classical Hochschild complex of dg categories. Assume that $\mathfrak{a}$ is a small $\mathbf{R}$-cofibrant dg category, i.e., all Hom-sets are cofibrant in the dg category $\mathrm{Mod}_{dg}(\mathbf{R}) = \mathrm{Com}(\mathrm{Mod}(\mathbf{R}))$ of complexes of $\mathbf{R}$-modules. Recall that $N \in \mathrm{Mod}_{dg}(\mathbf{R})$ is cofibrant if its terms are projective. For an $\mathfrak{a}$-bimodule $M\colon \mathfrak{a}^{op} \otimes \mathfrak{a} \to \mathrm{Mod}_{dg}(\mathbf{R})$, the *Hochschild complex* $\mathbf{C}(\mathfrak{a}, M)$ of $\mathfrak{a}$ with coefficients in M is the total complex of the double complex whose q-th columns are given by

$$\prod_{A_0,\dots,A_q \in \mathfrak{a}} \mathrm{Hom}(\mathfrak{a}(A_{q-1}, A_q) \otimes_{\mathbf{R}} \cdots \otimes_{\mathbf{R}} \mathfrak{a}(A_1, A_0), M(A_0, A_q))$$

with horizontal differentials d^q_{Hoch}. When $M = \mathfrak{a}$ we call $\mathbf{C}(\mathfrak{a}) = \mathbf{C}(\mathfrak{a}, \mathfrak{a})$ the *Hochschild complex* of $\mathfrak{a}$. The Hochschild complex satisfies a "limited functoriality" property. Namely, if $j\colon \mathfrak{a} \hookrightarrow \mathfrak{b}$ is a fully faithful dg functor to a small $\mathbf{R}$-cofibrant dg category $\mathfrak{b}$, then there is an associated map between Hochschild complexes

$$j^*\colon \mathbf{C}(\mathfrak{b}) \to \mathbf{C}(\mathfrak{a})$$

given by restriction. As mentioned above, the Hochschild complex is isomorphic to the associated Hochschild object $\mathbf{C}(\mathfrak{a}) = \Sigma^{-1}\mathbf{C}_{br}(\mathfrak{a})$. In particular, the Hochschild complex $\mathbf{C}(\mathfrak{a})$ has a B_∞-algebra structure compatible with the map j^*.

The definition of the Hochschild complex was modified by Shukla and Quillen [Shu61, Qui70] to general dg categories. Now, we drop the assumption on $\mathfrak{a}$ to be $\mathbf{R}$-cofibrant and fix a good $\mathbf{R}$-cofibrant resolution $\bar{\mathfrak{a}} \to \mathfrak{a}$, which is a quasi-equivalence with an $\mathbf{R}$-cofibrant dg category $\bar{\mathfrak{a}}$ inducing surjection of Hom-sets in the graded category [LV06a, Proposition-Definition 2.3.2]. The *Shukla complex* of $\mathfrak{a}$ with coefficient M is defined as

$$\mathbf{C}_{sh}(\mathfrak{a}, M) = \mathbf{C}(\bar{\mathfrak{a}}, M).$$

According to [LV06a, Section 4.2], which in turn is attributed to [Kel], the assignment

$$\mathbf{C}_{sh} \colon \mathfrak{a} \mapsto \mathbf{C}_{sh}(\mathfrak{a})$$

defines up to canonical natural isomorphism a contravariant functor on a suitable category of small dg categories with values in $\mathrm{Ho}(B_\infty)$. In particular, $\mathbf{C}_{sh}(\mathfrak{a})$ does not depend on the choice of good $\mathbf{R}$-cofibrant resolutions of $\mathfrak{a}$ up to canonical isomorphism. The functor $\mathbf{C}_{sh}$ satisfies some extended "limited functoriality". Let $j \colon \mathfrak{a} \hookrightarrow \mathfrak{b}$ be a fully faithful dg functor to a small dg category $\mathfrak{b}$ with a good $\mathbf{R}$-cofibrant resolution $\bar{\mathfrak{b}} \to \mathfrak{b}$. One may restrict the resolution to a good $\mathbf{R}$-cofibrant resolution $\bar{\mathfrak{a}} \to \mathfrak{a}$ of $\mathfrak{a}$. Then the restriction along the extended fully faithful dg functor $\bar{\mathfrak{a}} \hookrightarrow \bar{\mathfrak{b}}$ defines a morphism of Shukla complexes

$$\mathbf{C}_{sh}(\mathfrak{b}) \to \mathbf{C}_{sh}(\mathfrak{a})$$

still denoted by j^*. In the sequel, we write $\mathbf{C}(\mathfrak{a})$ for $\mathbf{C}_{sh}(\mathfrak{a})$.

Now, we return to our setting. Let X_0 be a Calabi–Yau manifold with $\dim X_0 > 2$. We denote by $D_{dg}(\mathrm{Qch}(X_0))$ the dg category of unbounded complexes of quasi-coherent sheaves on X_0. In our setting, the full dg subcategory $\mathrm{Perf}_{dg}(X_0)$ of compact objects consists of perfect complexes on X_0. The canonical embedding $\mathrm{Perf}_{dg}(X_0) \hookrightarrow D_{dg}(\mathrm{Qch}(X_0))$ factorizes through the dg category $D^+_{dg}(\mathrm{Qch}(X_0))$ of bounded below complexes of quasi-coherent sheaves on X_0. Let $(X, i_{\mathbf{R}})$ be an $\mathbf{R}$-deformation of X_0 and $\mathfrak{i} = \mathrm{Inj}(\mathrm{Qch}(X))$. As explained above, X is smooth projective over $\mathbf{R}$. The Hochschild cohomology of $\mathrm{Perf}_{dg}(X)$ can be expressed in terms of $\mathfrak{i}$.

Lemma 6.10. *There is an isomorphism*

$$HC^\bullet(\mathrm{Com}^+(\mathfrak{i})) \to HC^\bullet(\mathrm{Perf}_{dg}(X)). \tag{6.10}$$

Proof. Consider the quasi-fully faithful functor

$$\mathrm{Perf}_{dg}(X) \hookrightarrow D^+_{dg}(\mathrm{Qch}(X)) \to \mathrm{Com}^+(\mathfrak{i}), \tag{6.11}$$

where the first functor is the canonical embedding and the second functor is a quasi-equivalence induced by the canonical equivalence

$$D^+(\mathrm{Qch}(X)) \to K^+(\mathfrak{i})$$

of their homotopy categories [CNS, Theorem A]. The functor (6.11) induces a morphism

$$\mathbf{C}(\mathrm{Com}^+(\mathfrak{i})) \to \mathbf{C}(D^+_{dg}(\mathrm{Qch}(X))) \to \mathbf{C}(\mathrm{Perf}_{dg}(X)) \tag{6.12}$$

of B_∞-algebra, which in turn induces a morphism

$$H^\bullet\mathbf{C}(\mathrm{Com}^+(\mathfrak{i})) \to H^\bullet\mathbf{C}(D^+_{dg}(\mathrm{Qch}(X))) \to H^\bullet\mathbf{C}(\mathrm{Perf}_{dg}(X)) \tag{6.13}$$

of Hochschild cohomology. Since quasi-equivalences preserve Hochschld cohomology, the first arrow in (6.13) is an isomorphism.

It remains to show that the second arrow in (6.13) is an isomorphism. We claim that the restriction

$$\mathbf{C}(D_{dg}(\mathrm{Qch}(X))) \to \mathbf{C}(\mathrm{Perf}_{dg}(X))$$

is an isomorphism in $\mathrm{Ho}(B_\infty)$. Fix a good $\mathbf{R}$-cofibrant resolution

$$\overline{D_{dg}(\mathrm{Qch}(X))} \to D_{dg}(\mathrm{Qch}(X)).$$

Via the canonical embedding $\mathrm{Perf}_{dg}(X) \hookrightarrow D_{dg}(\mathrm{Qch}(X))$ it induces a good $\mathbf{R}$-cofibrant resolution $\overline{\mathrm{Perf}_{dg}(X)} \to \mathrm{Perf}_{dg}(X)$ and the fully faithful embedding

$$\bar{j} \colon \overline{\mathrm{Perf}_{dg}(X)} \hookrightarrow \overline{D_{dg}(\mathrm{Qch}(X))}.$$

We denote by j the induced fully faithful functor on the homotopy categories. One can apply [Por10, Theorem 1.2] to see that the functor

$$D(\mathrm{Qch}(X)) \to \mathrm{Mod}(\mathrm{Perf}(X)),\ F \mapsto \mathrm{Hom}_{D(\mathrm{Qch}(X))}(j(-), F)$$

lifts to a localization $D(\mathrm{Qch}(X)) \to \mathbf{D}(\mathrm{Mod}_{dg}(\mathrm{Perf}_{dg}(X))$, where $\mathbf{D}(\mathrm{Mod}_{dg}(\mathrm{Perf}_{dg}(X))$ is the derived category of right dg modules over the dg category $\mathrm{Mod}_{dg}(\mathrm{Perf}_{dg}(X))$ of right dg modules over $\mathrm{Perf}_{dg}(X)$. In particular, the lift is fully faithful. Then the claim follows from [DL13, Proposition 5.1]. Similarly, one can show that the restriction

$$\mathbf{C}(D_{dg}(\mathrm{Qch}(X))) \to \mathbf{C}(D^+_{dg}(\mathrm{Qch}(X)))$$

is an isomorphism in $\mathrm{Ho}(B_\infty)$. Hence the restriction

$$\mathbf{C}(D^+_{dg}(\mathrm{Qch}(X))) \to \mathbf{C}(\mathrm{Perf}_{dg}(X))$$

is an isomorphism in $\mathrm{Ho}(B_\infty)$, which induces the desired isomorphism on Hochschild cohomology. □

6.9. **Morita deformations of the dg category of perfect complexes.** Let $\mathrm{dgCat}_{\mathbf{R}}$ be the category of small $\mathbf{R}$-linear dg categories and dg functors. The category $\mathrm{dgCat}_{\mathbf{R}}$ has two model structures, so called the Dwyer–Kan model structure and the Morita model structure, constructed by Tabuada respectively in [Tab05a] and [Tab05b]. On the Dwyer–Kan model structure, weak equivalences are given by quasi-equivalences of dg categories. On the Morita model structure, weak equivalences are given by Morita morphisms. Recall that a dg functor in $\mathrm{dgCat}_{\mathbf{R}}$ is a *Morita morphism* if it induces an derived equivalence. Also recall that for each object $\mathfrak{a} \in \mathrm{dgCat}_{\mathbf{R}}$ the derived category $\mathbf{D}(\mathfrak{a})$ of right dg modules over $\mathfrak{a}$ is defined as the Verdier quotient

$$[\mathrm{Mod}_{dg}(\mathfrak{a})]/[\mathrm{Acycl}(\mathfrak{a})]$$

of the homotopy category of the dg category $\mathrm{Mod}_{dg}(\mathfrak{a})$ of right dg modules over $\mathfrak{a}$ by the homotopy category of the full dg subcategory $\mathrm{Acycl}(\mathfrak{a})$ of acyclic right dg modules.

We denote by $\mathrm{Ho}_{\mathbf{R}}$ the localization of $\mathrm{dgCat}_{\mathbf{R}}$ by weak equivalences in the Dwyer–Kan model structure and by $\mathrm{Hmo}_{\mathbf{R}}$ the localization of $\mathrm{dgCat}_{\mathbf{R}}$ by weak equivalences in the Morita model structure. Passing to $\mathrm{Ho}_{\mathbf{R}}$, two quasi-equivalent $\mathbf{R}$-linear dg categories $\mathfrak{a}, \mathfrak{b}$ get identified. Passing to $\mathrm{Hmo}_{\mathbf{R}}$, two Morita equivalent $\mathbf{R}$-linear dg categories $\mathfrak{a}, \mathfrak{b}$ get identified. Recall that two $\mathbf{R}$-linear dg categories $\mathfrak{a}, \mathfrak{b}$ are said to be *Morita equivalent* if they are connected by a Morita morphism. By [Töe, Proposition 7] or [Töe, Exercise 28] for $\mathbf{R}$-linear triangulated dg categories Morita equivalences coincide with quasi-equivalences. The dg category $\mathrm{Perf}_{dg}(X)$ is triangulated. Namely, it is pretriangulated and closed under homotopy direct summands.

Let $\mathfrak{a}$ be a small $\mathbf{R}$-linear dg category. A *Morita* $\mathbf{S}$*-deformation* of $\mathfrak{a}$ is an $\mathbf{S}$-linear dg category $\mathfrak{b}$ together with a Morita equivalence $\mathfrak{b} \otimes^L_{\mathbf{S}} \mathbf{R} \to \mathfrak{a}$, where $- \otimes^L_{\mathbf{S}} \mathbf{R} \colon \mathrm{Hmo}_{\mathbf{S}} \to \mathrm{Hmo}_{\mathbf{R}}$ is the derived functor of the base change $- \otimes_{\mathbf{S}} \mathbf{R} \colon \mathrm{dgCat}_{\mathbf{S}} \to \mathrm{dgCat}_{\mathbf{R}}$. Two Morita deformations $\mathfrak{b}, \mathfrak{b}'$ are *isomorphic* if there is a Morita equivalence $\mathfrak{b} \to \mathfrak{b}'$ inducing the identity on $\mathfrak{a}$. We denote by $\mathrm{Def}^{mo}_{\mathfrak{a}}(\mathbf{S})$ the set of isomorphism classes of Morita $\mathbf{S}$-deformations of $\mathfrak{a}$. By [KL09, Proposition 3.3] there is a canonical map

$$\mathrm{Def}^{mo}_{\mathfrak{a}}(\mathbf{S}) \to H^2\mathbf{C}(\mathfrak{a})^{\oplus l} \tag{6.14}$$

defined as follows. Any Morita $\mathbf{S}$-deformation $\mathfrak{b}$ of a small $\mathbf{R}$-linear dg category $\mathfrak{a}$ can be represented by a h-flat resolution $\bar{\mathfrak{b}} \to \mathfrak{b}$, which defines a dg $\mathbf{S}$-deformation of $\bar{\mathfrak{b}} \otimes_{\mathbf{S}} \mathbf{R}$. Let $\phi_{\bar{\mathfrak{b}}} \in Z^2\mathbf{C}(\bar{\mathfrak{b}} \otimes_{\mathbf{S}} \mathbf{R})^{\oplus l}$ be a Hochschild cocycle representing $\bar{\mathfrak{b}}$ via the bijection (6.1). The map (6.14) sends $\mathfrak{b}$ to the image $\phi_{\mathfrak{b}}$ of $\phi_{\bar{\mathfrak{b}}}$ under the isomorphism $H^2\mathbf{C}(\bar{\mathfrak{b}} \otimes_{\mathbf{S}} \mathbf{R})^{\oplus l} \to H^2\mathbf{C}(\mathfrak{a})^{\oplus l}$ induced by the Morita equivalence $\bar{\mathfrak{b}} \otimes_{\mathbf{S}} \mathbf{R} \to \mathfrak{a}$.

Theorem 6.11. *The composition*

$$\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X)}(\mathbf{S}) \to \mathrm{Def}^{cdg}_{\mathrm{Perf}_{dg}(X)}(\mathbf{S}) \tag{6.15}$$

of (6.14) *with the inverse of* (6.1) *is bijective.*

Proof. To show the injectivity, by [KL09, Proposition 3.7] it suffices to check that $\mathrm{Perf}_{dg}(X)$ has bounded above cohomology, i.e., the dg module $\mathrm{Perf}_{dg}(X)(E, F)$ has bounded above cohomology for all $E, F \in \mathrm{Perf}_{dg}(X)$. Consider the spectral sequence

$$E_2^{p,q} = H^p(X, \underline{\mathrm{Ext}}^q_X(E, F)) \Rightarrow \mathrm{Ext}^{p+q}_X(E, F).$$

Since we have $\underline{\mathrm{Ext}}^q_X(E, F) \cong \mathcal{H}^q(E^\vee \otimes_{\mathscr{O}_X} F)$, the cohomology

$$\mathcal{H}^{p+q}(\mathrm{Perf}_{dg}(X)(E, F)) \cong \mathrm{Ext}^{p+q}_X(E, F)$$

vanishes whenever p, q are sufficiently large.

To show the surjectivity, consider the characteristic dg morphism

$$\mathbf{C}(\mathrm{Perf}_{dg}(X)) \to \prod_{(M,\delta_M)\in \mathrm{Tw}(\mathrm{Perf}_{dg}(X))} \mathrm{Tw}(\mathrm{Perf}_{dg}(X))((M, \delta_M), (M, \delta_M)) \tag{6.16}$$

for the dg category $\mathrm{Perf}_{dg}(X)$. Here, $\mathrm{Tw}(\mathrm{Perf}_{dg}(X)) = \mathrm{Tw}_{\mathrm{ilnil}}(\mathrm{Perf}_{dg}(X))_\infty$ is the ∞-part of the cdg category of locally nilpotent twisted object over $\mathrm{Perf}_{dg}(X)$. It is a dg enhancement of the derived category $\mathbf{D}(\mathrm{Perf}_{dg}(X))$ of right dg modules over $\mathrm{Perf}_{dg}(X)$. We denote by $\mathrm{Tw}(\mathrm{Perf}_{dg}(X))^c$ the full dg subcategory of $\mathrm{Tw}(\mathrm{Perf}_{dg}(X))$ spanned by compact objects. Its homotopy category $\mathbf{D}(\mathrm{Perf}_{dg}(X))^c$, the full triangulated subcategory of $\mathbf{D}(\mathrm{Perf}_{dg}(X))$ spanned by compact objects, get identified with $\mathrm{Perf}(X)$ via the Yoneda embedding as $\mathrm{Perf}_{dg}(X)$ is triangulated. When restricted to $\mathrm{Tw}(\mathrm{Perf}_{dg}(X))^c$, on cohomology (6.16) induces the characteristic morphism

$$\chi_{\mathrm{Perf}_{dg}(X),E} \colon H^\bullet\mathbf{C}(\mathrm{Perf}_{dg}(X)) \to \mathrm{Ext}^\bullet_X(E, E),$$

where we identify each object $E \in \mathrm{Perf}_{dg}(X)$ with its image under the Yoneda embedding.

By Lemma 6.10 any element of $H^2\mathbf{C}(\mathrm{Perf}_{dg}(X))^{\oplus l}$ can be represented by the image $\mathrm{embr}_\delta(\phi)$ of $\phi \in Z^2\mathbf{C}(\mathfrak{i})^{\oplus l}$ under the B_∞-section (6.6). We use the same symbol to denote the image under the direct sum of (6.10). Since $\mathrm{Perf}_{dg}(X)$ has bounded above cohomology, by [KL09, Proposition 3.12] the map (6.15) is surjective if there exists a full dg subcategory $\mathfrak{m}(\phi) \subset \mathrm{Perf}_{dg}(X)$ which is Morita equivalent to $\mathrm{Perf}_{dg}(X)$ such that $\chi^{\oplus l}_{\mathrm{Perf}_{dg}(X),E}(\mathrm{embr}_\delta(\phi))_E = 0$ for any $E \in \mathfrak{m}(\phi)$ and cocycle $\mathrm{embr}_\delta(\phi) \in Z^2\mathbf{C}(\mathrm{Perf}_{dg}(X))^{\oplus l}$. If the possibly curved dg $\mathbf{S}$-deformation $\mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}$ along $\mathrm{embr}_\delta(\phi)$ is uncurved, then by definition $\chi^{\oplus l}_{\mathrm{Perf}_{dg}(X),E}(\mathrm{embr}_\delta(\phi))_E$ must vanish for any $E \in \mathfrak{m}(\phi)$. Thus it suffices to construct a Morita $\mathbf{S}$-deformation of $\mathrm{Perf}_{dg}(X)$ along each $\mathrm{embr}_\delta(\phi)$.

For a cocycle $\phi \in HH^2_{ab}(\mathrm{Qch}(X))^{\oplus l}$ we denote by $(X_\phi, i_{\mathbf{S}})$ the $\mathbf{S}$-deformation of $(X, i_{\mathbf{R}})$ along ϕ via the bijection (5.3). Let $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ be the h-flat resolution of $\mathrm{Perf}_{dg}(X_\phi)$ from [CNS, Proposition 3.10]. Then the dg category $\mathfrak{m}(\phi) = \overline{\mathrm{Perf}_{dg}(X_\phi)} \otimes_{\mathbf{S}} \mathbf{R}$ is a full dg subcategory of $\mathrm{Perf}_{dg}(X)$ with a Morita equivalence

$$\mathfrak{m}(\phi) \simeq_{mo} \mathrm{Perf}_{dg}(X_\phi) \otimes^L_{\mathbf{S}} \mathbf{R} \simeq_{mo} \mathrm{Perf}_{dg}(X)$$

from [BFN10, Theorem 1.2]. Hence $\mathrm{Perf}_{dg}(X_\phi)$ defines a Morita $\mathbf{S}$-deformation of $\mathrm{Perf}_{dg}(X)$. As explained in the proof of Theorem 6.12 below, the direction of the deformation coincides with $\mathrm{embr}_\delta(\phi)$ up to coboundary. □

Remark 6.3. The canonical equivalence

$$\mathrm{Perf}_\infty(X_\phi) \otimes_{\mathrm{Perf}_\infty(\mathbf{S})} \mathrm{Perf}_\infty(\mathbf{R}) \simeq_\infty \mathrm{Perf}_\infty(X)$$

of corresponding ∞-categories from [BFN10, Theorem 1.2] translates via [Coh, Corollary 5.7] into a Morita equivalence

$$\mathrm{Perf}_{dg}(X_\phi) \otimes^L_{\mathrm{Perf}_{dg}(\mathbf{S})} \mathrm{Perf}_{dg}(\mathbf{R}) \simeq_{mo} \mathrm{Perf}_{dg}(X) \tag{6.17}$$

of dg categories, where $-\otimes^L-\colon\ \mathrm{Hmo}_{\mathbf{S}} \times \mathrm{Hmo}_{\mathbf{S}} \to \mathrm{Hmo}_{\mathbf{S}}$ is the derived pointwise tensor product of dg categories. The left hand side of (6.17) is a triangulated dg category split-generated by objects of the form $E_\phi \boxtimes M$ for $E_\phi \in \mathrm{Perf}_{dg}(X_\phi)$ and $M \in \mathrm{Perf}_{dg}(\mathbf{R})$, which maps to $E \otimes_{\mathscr{O}_X} \pi^*_{\mathbf{R}} M \in \mathrm{Perf}_{dg}(X)$ via the Morita equivalence. Here, $E = E_\phi \otimes_{\mathbf{S}} \mathbf{R}$ and $\pi_{\mathbf{R}}\colon X \to \mathrm{Spec}\,\mathbf{R}$ is the structure morphism. Hence we obtain a Morita equivalence

$$\mathrm{Perf}_{dg}(X_\phi) \otimes^L_{\mathbf{S}} \mathbf{R} \simeq_{mo} \mathrm{Perf}_{dg}(X_\phi) \otimes^L_{\mathrm{Perf}_{dg}(\mathbf{S})} \mathrm{Perf}_{dg}(\mathbf{R}) \simeq_{mo} \mathrm{Perf}_{dg}(X)$$

used in the above proof.

Remark 6.4. Alternatively, the surjectivity of the map (6.15) can be shown in terms of obstruction theory as follows. Let $\mathfrak{p} = \mathrm{Proj}(\mathrm{Pro}(\mathrm{Qch}(X)))$ be the full linear subcategory of projective objects of the Pro-completion $\mathrm{Pro}(\mathrm{Qch}(X))$ of $\mathrm{Qch}(X)$. Consider the quasi-fully faithful functor

$$\mathrm{Perf}_{dg}(X) \hookrightarrow D^-_{dg}(\mathrm{Pro}(\mathrm{Qch}(X))) \to \mathrm{Com}^-(\mathfrak{p}), \tag{6.18}$$

where the first functor is the canonical embedding and the second functor is a quasi-equivalence induced by the canonical equivalence

$$D^-(\mathrm{Pro}(\mathrm{Qch}(X))) \to K^-(\mathfrak{p})$$

of their homotopy categories. The functor (6.18) induces a morphism

$$\mathbf{C}(\mathrm{Com}^-(\mathfrak{p})) \to \mathbf{C}(D^-_{dg}(\mathrm{Pro}(\mathrm{Qch}(X)))) \to \mathbf{C}(\mathrm{Perf}_{dg}(X)) \tag{6.19}$$

of B_∞-algebra, which in turn induces a morphism

$$H^\bullet\mathbf{C}(\mathrm{Com}^-(\mathfrak{p})) \to H^\bullet\mathbf{C}(D^-_{dg}(\mathrm{Pro}(\mathrm{Qch}(X)))) \to H^\bullet\mathbf{C}(\mathrm{Perf}_{dg}(X)) \tag{6.20}$$

of Hochschild cohomology. Since quasi-equivalences preserve Hochschld cohomology, the first arrow in (6.20) is an isomorphism. From the proof of Lemma 6.10 it follows that the second arrow is also an isomorphism.

One can apply [KL09, Proposition 2.3] to obtain a commutative diagram

$$\begin{tikzcd}
H^\bullet\mathbf{C}(\mathrm{Com}^-(\mathfrak{p})) \arrow[r, "\chi_{\mathrm{Com}^-(\mathfrak{p}),E}"] \arrow[d] & \mathrm{Ext}^\bullet_X(E,E) \arrow[d] \\
H^\bullet\mathbf{C}(D^-_{dg}(\mathrm{Pro}(\mathrm{Qch}(X)))) \arrow[r, "\chi_{D^-_{dg}(\mathrm{Pro}(\mathrm{Qch}(X))),E}"] \arrow[d] & \mathrm{Ext}^\bullet_X(E,E) \arrow[d] \\
H^\bullet\mathbf{C}(\mathrm{Perf}_{dg}(X)) \arrow[r, "\chi_{\mathrm{Perf}_{dg}(X),E}"] & \mathrm{Ext}^\bullet_X(E,E)
\end{tikzcd} \tag{6.21}$$

for each object $E \in \mathrm{Perf}_{dg}(X)$, whose horizontal arrows are the characteristic morphisms and whose left vertical arrows are (6.20). Applying [KL09, Proposition 2.3] to the quasi-fully faithful functor,

$$\mathfrak{p} \to \mathrm{Com}^-(\mathfrak{p}) \tag{6.22}$$

we obtain another commutative diagram

$$\begin{tikzcd} H^\bullet \mathbf{C}(\mathrm{Com}^-(\mathfrak{p})) \arrow[r, "\chi_{\mathrm{Com}^-(\mathfrak{p}),E}"] \arrow[d] & \mathrm{Ext}^\bullet_X(E,E) \arrow[d] \\ H^\bullet \mathbf{C}(\mathfrak{p}) \arrow[r, "\chi_{\mathfrak{p},E}"] & \mathrm{Ext}^\bullet_X(E,E) \end{tikzcd} \tag{6.23}$$

whose horizontal arrows are the characteristic morphisms and whose vertical arrows are the canonical isomorphisms, as the morphism $\mathbf{C}(\mathrm{Com}^-(\mathfrak{p})) \to \mathbf{C}(\mathfrak{p})$ induced by the quasi-fully faithful functor (6.22) coincides with the canonical projection and its inverse in $\mathrm{Ho}(B_\infty)$ is the B_∞-section $\mathrm{embr}_\delta \colon \mathbf{C}(\mathfrak{p}) \to \mathbf{C}(\mathrm{Com}^-(\mathfrak{p}))$, which induces an isomorphism on cohomology. Note that $\mathrm{Tw}(\mathfrak{p})$ is precisely $\mathrm{Com}(\mathfrak{p})$.

The composition of the opposite vertical arrow in (6.23) and the vertical arrows in (6.21) coincides with the induced map by $\mathrm{embr}_\delta \colon \mathbf{C}(\mathfrak{p}) \to \mathbf{C}(\mathrm{Com}^-(\mathfrak{p}))$ up to coboundary. By Corollary 6.8 the image of $\phi \in H^2\mathbf{C}(\mathfrak{p}) \cong H^2\mathbf{C}_{ab}(\mathrm{Pro}(\mathrm{Qch}(X))) \cong H^2\mathbf{C}_{ab}(\mathrm{Qch}(X))$ under $\chi^{\oplus l}_{\mathfrak{p},E}$ is the obstruction against deforming E to an object of $D^-(\mathrm{Qch}(X_\phi))$. As each object $E \in \mathfrak{m}(\phi)$ lifts to an object of $\mathrm{Perf}(X_\phi)$, we obtain $\chi^{\oplus l}_{\mathfrak{p}}(\phi)_E = 0$. Chasing the diagrams (6.21) and (6.23), we obtain $\chi^{\oplus l}_{\mathrm{Perf}_{dg}(X)}(\mathrm{embr}_\delta(\phi))_E = 0$, where $\mathrm{embr}_\delta(\phi)$ denotes the image of ϕ under the bijection $H^\bullet\mathbf{C}(\mathfrak{p}) \to H^\bullet\mathbf{C}(\mathrm{Perf}_{dg}(X))$ induced by the $\mathrm{embr}_\delta \colon \mathbf{C}(\mathfrak{p}) \to \mathbf{C}(\mathrm{Com}^-(\mathfrak{p}))$. Thus the surjectivity of (6.15) follows from [KL09, Proposition 3.12].

6.10. **Maximal partial dg deformations of the dg category of perfect complexes.** As explained above, the category $\mathrm{Perf}_{dg}(X_\phi)$ defines a Morita $\mathbf{S}$-deformation of $\mathrm{Perf}_{dg}(X)$. The h-flat resolution $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ of $\mathrm{Perf}_{dg}(X_\phi)$ from [CNS, Proposition 3.10] defines a dg $\mathbf{S}$-deformation of $\mathfrak{m}(\phi) = \overline{\mathrm{Perf}_{dg}(X_\phi)} \otimes_{\mathbf{S}} \mathbf{R}$. On the other hand, $\mathfrak{m}(\phi)$ admits a dg deformation

$$\mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)} = (\mathfrak{m}(\phi)[\epsilon] = \mathfrak{m}(\phi) \otimes_{\mathbf{R}} \mathbf{S}, \mathrm{embr}_\delta(m) + \mathrm{embr}_\delta(\phi)\epsilon)\,,$$

where $\mathrm{embr}_\delta(m) \in Z^2\mathbf{C}(\mathfrak{m}(\phi))$ and $\mathrm{embr}_\delta(\phi) \in Z^2\mathbf{C}(\mathfrak{m}(\phi))^{\oplus l}$ are respectively the images under the B_∞-section (6.6) and its direct sum of the compositions m in $\mathfrak{i}$ and the cocycle $\phi \in Z\mathbf{C}^2(\mathfrak{i})^{\oplus l}$ corresponding to $\phi \in HH^2_{ab}(\mathrm{Qch}(X))^{\oplus l}$ via the isomorphism obtained from [LV06b, Theorem 6.6]. Here, as above we use the same symbol to denote the images under the compositions of the bijections induced by the quasi-equivalences with (6.10) and its direct sum respectively. Also, we use the same symbol to denote the images under the morphism $\mathbf{C}(\mathrm{Perf}_{dg}(X)) \to \mathbf{C}(\mathfrak{m}(\phi))$ of B_∞-algebras induced by the Morita equivalence $\mathfrak{m}(\phi) \to \mathrm{Perf}_{dg}(X)$.

Theorem 6.12. *There is an isomorphism*

$$\overline{\mathrm{Perf}_{dg}(X_\phi)} \simeq \mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}$$

of dg $\mathbf{S}$*-deformations of* $\mathfrak{m}(\phi)$*. In particular, the Morita* $\mathbf{S}$*-deformation* $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ *defines a maximal partial dg* $\mathbf{S}$*-deformaiton of* $\mathrm{Perf}_{dg}(X)$ *along* $\mathrm{embr}_\delta(\phi)$*.*

Proof. Since both dg deformations share their underlying quiver $\mathfrak{m}(\phi)[\epsilon]$, it suffices to show the coincidence of their dg structures up to coboundary. The dg structure on $\mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}$ is $\mathrm{embr}_\delta(m) + \mathrm{embr}_\delta(\phi)\epsilon$. Let $\overline{D^+_{dg}(\mathrm{Qch}(X_\phi))}$ be the h-flat resolution of $D^+_{dg}(\mathrm{Qch}(X_\phi))$ from [CNS, Proposition 3.10]. There is a canonical dg functor

$$\overline{\mathrm{Perf}_{dg}(X_\phi)} \hookrightarrow \overline{D^+_{dg}(\mathrm{Qch}(X_\phi))}$$

extending the canonical embedding $\mathrm{Perf}_{dg}(X_\phi) \hookrightarrow D^+_{dg}(\mathrm{Qch}(X_\phi))$. By [Low08, Proposition 2.6] the dg structure on $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ is the restriction of that on $\overline{D^+_{dg}(\mathrm{Qch}(X_\phi))}$.

We compute the dg structure on $\overline{\mathrm{Perf}_{dg}(X_\phi)}$. Consider the quasi-fully faithful functor

$$\mathrm{Perf}_{dg}(X_\phi) \hookrightarrow D^+_{dg}(\mathrm{Qch}(X_\phi)) \to \mathrm{Com}^+(\mathrm{i}_\phi) \tag{6.24}$$

where the first functor is the canonical embedding and the second functor is a quasi-equivalence induced by the canonical equivalence

$$D^+(\mathrm{Qch}(X_\phi)) \to K^+(\mathrm{i}_\phi) \tag{6.25}$$

by [CNS, Theorem A]. The functor (6.24) canonically extends to that

$$\overline{\mathrm{Perf}_{dg}(X_\phi)} \hookrightarrow \overline{D^+_{dg}(\mathrm{Qch}(X_\phi))} \to \overline{\mathrm{Com}^+(\mathrm{i}_\phi)} \tag{6.26}$$

of the h-flat resolutions from [CNS, Proposition 3.10]. It induces a morphism

$$\mathbf{C}(\overline{\mathrm{Com}^+(\mathrm{i}_\phi)}) \to \mathbf{C}(\overline{D^+_{dg}(\mathrm{Qch}(X_\phi))}) \to \mathbf{C}(\overline{\mathrm{Perf}_{dg}(X_\phi)}) \tag{6.27}$$

of B_∞-algebras, which in turn induces an isomorphism

$$H^\bullet\mathbf{C}(\overline{\mathrm{Com}^+(\mathrm{i}_\phi)}) \to H^\bullet\mathbf{C}(D^+_{dg}(\overline{\mathrm{Qch}(X_\phi)})) \to H^\bullet\mathbf{C}(\overline{\mathrm{Perf}_{dg}(X_\phi)}) \tag{6.28}$$

of Hochschild cohomology by Lemma 6.10.

Recall that $\delta \in \mathbf{C}^1(\mathrm{Com}^+(\mathrm{i}))$ is the differentials of objects in $\mathrm{Com}^+(\mathrm{i})$. Let $\delta + \delta'\epsilon \in \mathbf{C}^1(\mathrm{Com}^+(\mathrm{i}_\phi))$ be the differentials of objects in $\mathrm{Com}^+(\mathrm{i}_\phi)$ with $\delta' = (\delta'_1, \dots, \delta'_l) \in \mathbf{C}^1(\mathrm{Com}(\mathrm{i}))^{\oplus l}$. Then the dg structure on $\mathrm{Com}^+(\mathrm{i}_\phi)$ is

$$\mathrm{embr}_{\delta+\delta'\epsilon}(m + \phi\epsilon) = (m + \phi\epsilon)\{\delta + \delta'\epsilon, \delta + \delta'\epsilon\} + (m + \phi\epsilon)\{\delta + \delta'\epsilon\} + (m + \phi\epsilon).$$

We use the same symbol to denote the images under the composition of (6.27) with the morphism $\mathbf{C}(\mathrm{Com}^+(\mathrm{i}_\phi)) \to \mathbf{C}(\overline{\mathrm{Com}^+(\mathrm{i}_\phi)})$ induced by the h-flat resolution. Note that $\delta \in \mathbf{C}^1(\overline{\mathrm{Perf}_{dg}(X)})$ is the differentials of objects in $\overline{\mathrm{Perf}_{dg}(X)}$ and $\delta + \delta'\epsilon \in \mathbf{C}^1(\overline{\mathrm{Perf}_{dg}(X_\phi)})$ is the differentials of objects in $\overline{\mathrm{Perf}_{dg}(X_\phi)}$ with $\delta' = (\delta'_1, \dots, \delta'_l) \in \mathbf{C}^1(\overline{\mathrm{Perf}_{dg}(X)})^{\oplus l}$, as (6.27) is a morphism of B_∞-algebras induced by the canonical equivalence (6.25). One can apply the same argument as in the proof of [Low08, Theorem 4.15] to obtain

$$\mathrm{embr}_{\delta+\delta'\epsilon}(m + \phi\epsilon) = \mathrm{embr}_\delta(m) + \mathrm{embr}_\delta(\phi)\epsilon + d_{\mathrm{embr}_\delta(m)}(\delta'_1)\epsilon_1 + \cdots + d_{\mathrm{embr}_\delta(m)}(\delta'_l)\epsilon_l$$

on $\overline{\mathrm{Perf}_{dg}(X_\phi)}$. Note that up to coboundary the image of

$$d_{\mathrm{embr}_\delta(m)}(\delta'_1)\epsilon_1 + \cdots + d_{\mathrm{embr}_\delta(m)}(\delta'_l)\epsilon_l \in \mathbf{C}(\overline{\mathrm{Com}^+(\mathrm{i}_\phi)})^{\oplus l}$$

under (6.27) coincide with the image of

$$\delta'_1\epsilon_1 + \cdots + \delta'_l\epsilon_l \in \mathbf{C}(\mathrm{Perf}_{dg}(X_\phi))^{\oplus l}$$

under Hochschild differential $d_{\mathrm{embr}_\delta(m)}$ on $\mathbf{C}(\mathrm{Perf}_{dg}(X))$. Hence we obtain an isomorphism

$$\begin{aligned}\overline{\mathrm{Perf}_{dg}(X_\phi)} &= (\mathfrak{m}(\phi)[\epsilon], \mathrm{embr}_{\delta+\delta'\epsilon}(m + \phi\epsilon)) \\ &\simeq (\mathfrak{m}(\phi)[\epsilon], \mathrm{embr}_\delta(m) + \mathrm{embr}_\delta(\phi)\epsilon) \\ &= \mathfrak{m}(\phi)_{\mathrm{embr}_\delta(\phi)}.\end{aligned}$$

of dg deformaitons of $\mathfrak{m}(\phi)$ from (6.1). □

Remark 6.5. From the above theorem it follows that the image of $\mathrm{Perf}_{dg}(X_\phi)$ under the map (6.14) is represented by $\mathrm{embr}_\delta(\phi)$.

Remark 6.6. In general, $\mathfrak{m}(\phi)$ is strictly smaller than $\mathrm{Perf}_{dg}(X)$. For instance, let X_0 be a quintic 3-fold of Fermat type. By [AK91, Proposition 2.1] for any general first order deformation X_1 of X_0, there is no line in X_0 which lifts to a closed subvariety of X_1. Hence deformations of any perfect complex quasi-isomorphic to the pushforward of the structure sheaf of a line in X_0 is obstructed. This example was informed to the author by Yukinobu Toda.

7. Deformations of higher dimensional Calabi–Yau manifolds revisited

Now, we are ready to prove our first main result. Consider the functor

$$\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}\colon \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$$

which sends each $A \in \mathrm{Art}_{\mathbf{k}}$ to the set of isomorphism classes of Morita A-deformations of $\mathrm{Perf}_{dg}(X_0)$ and each morphism $B \to A$ in $\mathrm{Art}_{\mathbf{k}}$ to the map $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(B) \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A)$ induced by $- \otimes^L_{\mathbf{S}} \mathbf{R}$. The deformation theory for X_0 is equivalent to that for $\mathrm{Perf}_{dg}(X_0)$ in the following sense.

Theorem 7.1. *There is a natural isomorphism*

$$\zeta\colon \mathrm{Def}_{X_0} \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)} \tag{7.1}$$

of deformation functors.

Proof. We show that the assignment

$$(X_A, i_A) \mapsto (\mathrm{Perf}_{dg}(X_A), i_A^*)$$

for each $A \in \mathrm{Art}_{\mathbf{k}}$ defines a natural transformation. Here, we use the same symbol i_A^* to denote both the derived pullback functor

$$i_A^*\colon \mathrm{Perf}_{dg}(X_A) \to \mathrm{Perf}_{dg}(X_0)$$

and the induced Morita equivalence

$$\mathrm{Perf}_{dg}(X_A) \otimes^L_A \mathbf{k} \simeq_{mo} \mathrm{Perf}_{dg}(X_0).$$

The surjection $A \to \mathbf{k}$ factorizes through a sequence

$$A = A_m \to A_{m-1} \to \cdots \to A_1 \to \mathbf{k}$$

of small extensions. Pullback of X_A yields a sequence

$$(X_{A_m}, i_{A_m}) \mapsto (X_{A_{m-1}}, i_{A_{m-1}}) \mapsto \cdots \mapsto (X_{A_1}, i_{A_1}) \mapsto X_0$$

of deformations of X_0. Let $\phi_{A_1} \in HH^2(X_0) = \mathrm{H}^1(\mathscr{T}_{X_0})$ be the cocycle representing (X_{A_1}, i_{A_1}).

By Theorem 6.12 the Morita deformation $\mathrm{Perf}_{dg}(X_{A_1})$ of $\mathrm{Perf}_{dg}(X)$ corresponds to $\mathrm{embr}_{\delta_0}(\phi_{A_1})$ via the bijection (6.15). Here, $\mathrm{embr}_{\delta_0}(\phi_{A_1})$ denotes the image under the composition of

$$H^2\mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X_0)))) \cong H^2\mathbf{C}(\mathrm{Perf}_{dg}(X_0))$$

with the induced isomorphism

$$HH^2(X_0) \cong H^2\mathbf{C}(\mathrm{Inj}(\mathrm{Qch}(X_0))) \cong H^2\mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X_0))))$$

by the B_∞-section of the canonical projection $\mathbf{C}(\mathrm{Inj}(\mathrm{Qch}(X_0))) \to \mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X_0))))$. Induction yields a sequence

$$[\mathrm{Perf}_{dg}(X_{A_m}), i^*_{A_m}] \mapsto [\mathrm{Perf}_{dg}(X_{A_{m-1}}), i^*_{A_{m-1}}] \mapsto \cdots \mapsto [\mathrm{Perf}_{dg}(X_{A_1}), i^*_{A_1}] \mapsto \mathrm{Perf}_{dg}(X_0)$$

of Morita deformations of $\mathrm{Perf}_{dg}(X_0)$. In particular, $(\mathrm{Perf}_{dg}(X_{A_m}), i^*_{A_m})$ is a Morita deformation of $\mathrm{Perf}_{dg}(X_0)$ corresponding to the collection $\{\mathrm{embr}_{\delta_{n-1}}(\phi_{A_n})\}_{n=1}^m$. Here, each $\mathrm{embr}_{\delta_{n-1}}(\phi_{A_n})$ denotes the image of $\phi_{A_n} \in H^1(\mathscr{T}_{X_{A_{n-1}}}/A_{n-1})^{\oplus l_{n-1}}$, where l_{n-1} is the rank of the kernel of square zero extension $A_n \to A_{n-1}$ as a free A_{n-1}-module, under the composition of

$$H^2\mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X_{A_{n-1}}))))^{\oplus l_{n-1}} \cong H^2\mathbf{C}(\mathrm{Perf}_{dg}(X_{A_{n-1}}))^{\oplus l_{n-1}}$$

with the induced isomorphism

$$HH^2(X_{A_{n-1}}/A_{n-1})^{\oplus l_{n-1}} \cong H^2\mathbf{C}(\mathrm{Inj}(\mathrm{Qch}(X_{A_{n-1}})))^{\oplus l_{n-1}} \cong H^2\mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X_{A_{n-1}}))))^{\oplus l_{n-1}}$$

by the B_∞-section of the canonical projection $\mathbf{C}(\mathrm{Inj}(\mathrm{Qch}(X_{A_{n-1}}))) \to \mathbf{C}(\mathrm{Com}^+(\mathrm{Inj}(\mathrm{Qch}(X_{A_{n-1}}))))$.

It follows that the assignment defines a map

$$\zeta_A\colon \mathrm{Def}_{X_0}(A) \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A),\ (X_A, i_A) \mapsto [\mathrm{Perf}_{dg}(X_A), i_A^*].$$

For each morphism $f\colon B \to A$ in $\mathrm{Art}_{\mathbf{k}}$ the diagram

$$\begin{tikzcd}
\mathrm{Def}_{X_0}(B) \arrow[r] \arrow[d, "\mathrm{Def}_{X_0}(f)"'] & \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(B) \arrow[d, "\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(f)"] \\
\mathrm{Def}_{X_0}(A) \arrow[r] & \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A)
\end{tikzcd} \tag{7.2}$$

commutes. To see this, we may assume that f is a square zero extension $B = \mathbf{S}$ of $A = \mathbf{R}$. Then for any $(X_{\mathbf{S}}, i_{\mathbf{S}}) \in \mathrm{Def}_{X_0}(\mathbf{S})$ with $X_{\mathbf{S}} \times_{\mathbf{S}} \mathbf{R} \cong X_{\mathbf{R}}$ we have $X_{\mathbf{S}} \cong (X_{\mathbf{R}})_\phi$ for some cocycle $\phi \in H^1(\mathscr{T}_{X_{\mathbf{R}}}/\mathbf{R})^{\oplus l}$. We already know that $(\mathrm{Perf}_{dg}(X_{\mathbf{S}}), i^*_{\mathbf{S}})$ is the Morita deformation of $\mathrm{Perf}_{dg}(X_{\mathbf{R}})$. Since $(X_{\mathbf{S}}, i_{\mathbf{S}})$ maps to $(X_{\mathbf{R}}, i_{\mathbf{R}})$, the derived pullback functor $i^*_{\mathbf{S}}$ factorizes through $i^*_{\mathbf{R}}$. Thus the assignments $\{\zeta_A\}_{A\in\mathrm{Art}_{\mathbf{k}}}$ defines a natural transformation $\zeta\colon \mathrm{Def}_{X_0} \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$.

It remains to show that ζ_A is bijective for each $A \in \mathrm{Art}_{\mathbf{k}}$. We will proceed by induction. Now, assume that ζ_{A_i} are bijective for all $1 \le i \le n$. In order to show the surjectivity of $\zeta_{A_{n+1}}$, take any element $[\mathfrak{a}_{A_{n+1}}, u^*_{A_{n+1}}] \in \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A_{n+1})$. By the assumption of induction, the reduction $[\mathfrak{a}_{A_n}, u^*_{A_n}] \in \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A_n)$ is equal to $[\mathrm{Perf}_{dg}(Y_{A_n}), j^*_{A_n}]$ for some $(Y_{A_n}, j_{A_n}) \in \mathrm{Def}_{X_0}(A_n)$. Combining Theorem 6.11 with Theorem 6.12, one sees that the Morita A_{n+1}-deformation $(\mathfrak{a}_{A_{n+1}}, u^*_{A_{n+1}})$ of $\mathrm{Perf}_{dg}(Y_{A_n})$ is represented by $\mathrm{embr}_{\delta_n}(\phi_{A_{n+1}})$ for some cocycle $\phi_{A_{n+1}} \in H^1(\mathscr{T}_{Y_{A_n}/A_n})^{\oplus l_n}$. Then we have

$$[\mathfrak{a}_{A_{n+1}}, u^*_{A_{n+1}}] = [\mathrm{Perf}_{dg}(Y_{A_n,\phi_{A_{n+1}}}), j^*_{A_{n+1}}].$$

In order to show the injectivity, suppose that we have

$$[\mathrm{Perf}_{dg}(X_{A_{n+1}}), i^*_{A_{n+1}}] = [\mathrm{Perf}_{dg}(Y_{A_{n+1}}), j^*_{A_{n+1}}], \tag{7.3}$$

i.e., there is a Morita equivalence $\mathrm{Perf}_{dg}(X_{A_{n+1}}) \simeq_{mo} \mathrm{Perf}_{dg}(Y_{A_{n+1}})$ reducing to the identity on $\mathrm{Perf}_{dg}(X_0)$. Combining the above argument with the commutative diagram (7.2), we have

$$[\mathrm{Perf}_{dg}(X_{A_{n+1}}), i^*_{A_{n+1}}] = [\mathrm{Perf}_{dg}(X_{A_n,\phi_{A_n}}), i^*_{A_{n+1}}],\ [\mathrm{Perf}_{dg}(Y_{A_{n+1}}), j^*_{A_{n+1}}] = [\mathrm{Perf}_{dg}(Y_{A_n,\psi_{A_n}}), j^*_{A_{n+1}}]$$

for some elements

$$(X_{A_n}, i_{A_n}), (Y_{A_n}, j_{A_n}) \in \mathrm{Def}_{X_0}(A_n)$$

and cocycles

$$\phi_{A_n} \in H^1(\mathscr{T}_{X_{A_n}/A_n})^{\oplus l_n},\ \psi_{A_n} \in H^1(\mathscr{T}_{Y_{A_n}/A_n})^{\oplus l_n}.$$

Applying $- \otimes^L_{A_{n+1}} A_n$, we obtain a Morita equivalence $\mathrm{Perf}_{dg}(X_{A_n}) \simeq_{mo} \mathrm{Perf}_{dg}(Y_{A_n})$ reducing to the identity on $\mathrm{Perf}_{dg}(X_0)$. By the assumption of induction there is an isomorphim $X_{A_n} \cong Y_{A_n}$ reducing to the identity on X_0. Then (7.3) implies $[\mathrm{embr}_{\delta_{n-1}}(\phi_{A_n})] = [\mathrm{embr}_{\delta_{n-1}}(\psi_{A_n})]$, which in turn implies $[\phi_{A_n}] = [\psi_{A_n}]$. Thus we obtain an isomorphism $X_{A_{n+1}} \cong Y_{A_{n+1}}$ reducing to the identity on X_0. □

Remark 7.1. Consider the functor

$$\widetilde{\mathrm{Def}}^{mo}_{\mathrm{Perf}_{dg}(X_0)}\colon\ \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$$

which sends each $A \in \mathrm{Art}_{\mathbf{k}}$ to the set of isomorphism classes of Morita A-deformations of $\mathrm{Perf}_{dg}(X_0)$ and each morphism $B \to A$ in $\mathrm{Art}_{\mathbf{k}}$ to the map $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(B) \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(A)$ induced by the derived pointwise tensor product with $\mathrm{Perf}_{dg}(A)$ over $\mathrm{Perf}_{dg}(B)$. Based on Remark 6.3, one can rewrite the proof of Theorem 7.1 in terms of $\widetilde{\mathrm{Def}}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ to obtain a natural isomorphism

$$\tilde{\zeta}\colon\ \mathrm{Def}_{X_0} \to \widetilde{\mathrm{Def}}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$$

of deformation functors. In the sequel, we will identify the deformation functors

$$\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}, \widetilde{\mathrm{Def}}^{mo}_{\mathrm{Perf}_{dg}(X_0)}\colon\ \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$$

without further comments.

Remark 7.2. Theorem 7.1 tells us that infinitesimal deformations of $\mathrm{Perf}_{dg}(X_0)$ is controlled by the Kodaira–Spencer differential graded Lie algebra KS_{X_0} of X_0. Consider the functor $\mathrm{Def}_{\mathrm{KS}_{X_0}}\colon\ \mathrm{Art}_{\mathbf{k}} \to \mathrm{Set}$ defined as

$$\mathrm{Def}_{\mathrm{KS}_{X_0}}(A) = \frac{\mathrm{MC}_{\mathrm{KS}_{X_0}}(A)}{\textit{gauge equivalence}} \tag{7.4}$$

for each $A \in \mathrm{Art}_{\mathbf{k}}$, where

$$\mathrm{MC}_{\mathrm{KS}_{X_0}}(A) = \Big\{x \in \mathrm{KS}^1_{X_0} \otimes_{\mathbf{k}} \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0\Big\}. \tag{7.5}$$

Recall that given a differential graded Lie algebra L and a commutative $\mathbf{k}$-algebra $\mathfrak{m}$ there exists a natural structure of differential graded Lie algebra on the tensor product $L \otimes_{\mathbf{k}} \mathfrak{m}$ given by

$$d(x \otimes_{\mathbf{k}} r) = dx \otimes_{\mathbf{k}} r,\ [x \otimes_{\mathbf{k}} r, y \otimes_{\mathbf{k}} s] = [x,y] \otimes_{\mathbf{k}} rs,\ x, y \in L,\ r, s \in \mathfrak{m}.$$

For every surjection $A \to \mathbf{k}[t]/t^2$ in $\mathrm{Art}_{\mathbf{k}}$ the set $\mathrm{Def}_{\mathrm{KS}_{X_0}}(A)$ consists of solutions of the extended Maurer–Cartan equation to $\mathfrak{m}_A$. Giving higher order deformations of X_0 is equivalent to giving solutions of the extended Maurer–Cartan equation. Indeed, we have $\mathrm{Def}_{\mathrm{KS}_{X_0}} \simeq \mathrm{Def}_{X_0}$ by [Man09, Example 2.3].

Corollary 7.2. *The functor* $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ *is prorepresented by* R*.*

Proof. This follows immediately as Def_{X_0} is prorepresented by R. □

Corollary 7.3. *The functor* $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ *has an effective universal formal family.*

Proof. Let $(R, \tilde{\xi})$ be a universal formal family for $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$, where $\tilde{\xi} = \{\tilde{\xi}_n\}_n$ belongs to the limit

$$\widehat{\mathrm{Def}}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(R) = \varprojlim \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(R/\mathfrak{m}^n_R)$$

of the inverse system

$$\cdots \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(R/\mathfrak{m}^{n+2}_R) \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(R/\mathfrak{m}^{n+1}_R) \to \mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}(R/\mathfrak{m}^{n}_R) \to \cdots$$

induced by the natural quotient maps $R/\mathfrak{m}^{n+1}_R \to R/\mathfrak{m}^n_R$. Recall that for the universal formal family (R, ξ) there is a noetherian formal scheme $\mathscr{X}$ over R such that $X_n \cong \mathscr{X} \times_R R/\mathfrak{m}^{n+1}_R$ for each n, where (X_n, i_n) are R_n-deformations of X_0 defining ξ_n. By [GD61, Theorem III5.4.5] there exists a scheme X_R flat projective over R whose formal completion along the closed fiber X_0 is isomorphic to $\mathscr{X}$. From the proof of Theorem 7.1 it follows that $(\mathrm{Perf}_{dg}(X_n), i^*_n)$ defines $\tilde{\xi}_n$. Then by [BFN10, Theorem 1.2] the R-linear dg category $\mathrm{Perf}_{dg}(X_R)$ yields the compatible system $\{\tilde{\xi}_n\}_n$ via reduction along the natural quotient maps $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$, which means $\tilde{\xi}$ is effective. $\square$

Remark 7.3. Recall that the Dwyer–Kan model structure on $\mathrm{dgCat}_{\mathbf{k}}$ has a natural simplicial enrichment [Töe, Section 5]. We denote by $\mathrm{dgCat}_{\mathbf{k}}^{\infty}$ the underlying ∞-category. There is a notion of limits in ∞-categories that behaves similarly to the classical one [Lur09, Chapter 4]. As $\mathrm{dgCat}_{\mathbf{k}}^{\infty}$ is the underlying ∞-category of a simplicial model category, it admits limits [Lur09, Corollary 4.2.4.8]. Hence we obtain a limit

$$\widehat{\mathrm{Perf}}_{dg}(X_R) = \varprojlim \mathrm{Perf}_{dg}(X_n)$$

of the inverse system

$$\cdots \to \mathrm{Perf}_{dg}(X_{n+2}) \to \mathrm{Perf}_{dg}(X_{n+1}) \to \mathrm{Perf}_{dg}(X_n) \to \cdots$$

of small $\mathbf{k}$-linear dg categories induced by the natural quotient maps $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$.

We claim that the limit is quasi-equivalent to $\mathrm{Perf}_{dg}(\mathscr{X})$. By [GD61, Corollary 5.1.3] the canonical map

$$\mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F}) \to \mathrm{Hom}_{\mathscr{X}}(\hat{\mathscr{E}}, \hat{\mathscr{F}})$$

defined by taking the formal completion of each morphism along the closed fiber is an isomorphism for all coherent sheaves $\mathscr{E}, \mathscr{F}$ on X_R. In particular, we may write

$$\widehat{\mathrm{Hom}_{X_R}(\mathscr{E}, \mathscr{F})} = \mathrm{Hom}_{\mathscr{X}}(\hat{\mathscr{E}}, \hat{\mathscr{F}}).$$

Since X_R is projective over a complete local noetherian ring R, by [GD61, Corollary III5.1.6] the functor

$$\mathrm{coh}(X_R) \to \mathrm{coh}(\mathscr{X}),$$

which sends each coherent sheaf $\mathscr{F}$ on X_R to its formal completion $\hat{\mathscr{F}}$ along the closed fiber is an equivalence of abelian categories. We obtain the induced derived equivalence

$$\mathrm{Perf}(X_R) \simeq D^b(X_R) \simeq D^b(\mathscr{X}) \simeq \mathrm{Perf}(\mathscr{X}).$$

Hence for $E, F \in \mathrm{Perf}_{dg}(X_R)$ with formal completions $\hat{E}, \hat{F} \in \mathrm{Perf}_{dg}(\mathscr{X})$ we may write

$$\widehat{\mathrm{Ext}^i_{X_R}(E, F)} = \mathrm{Ext}^i_{\mathscr{X}}(\hat{E}, \hat{F}).$$

Now, one sees that the objects and morphisms in $\mathrm{Perf}(\mathscr{X})$ satisfy universality with respect to the induced inverse system on homotopy categories. Thus the dg functor

$$\mathrm{Perf}_{dg}(\mathscr{X}) \to \widehat{\mathrm{Perf}}_{dg}(X_R)$$

uniquely determined by universality of the limit is a quasi-equivalence. Namely, the formal completion of $\mathrm{Perf}_{dg}(X_R)$ is quasi-equivalent to $\mathrm{Perf}_{dg}(\mathscr{X})$.

Corollary 7.4. *Any effective universal formal family for* $\mathrm{Def}^{mo}_{\mathrm{Perf}_{dg}(X_0)}$ *is algebraizable. In particular, an algebraization is given by* $\mathrm{Perf}_{dg}(X_S)$ *where* X_S *is a versal deformation of* X_0.

Proof. Consider the triple $(\operatorname{Spec} S, s, \mathrm{Perf}_{dg}(X_S))$. Since the reduction of X_S along the natural quotient maps $S/\mathfrak{m}_S^{n+1} \to S/\mathfrak{m}_S^n$ yields a compatible system isomorphic to ξ, the reduction of $\mathrm{Perf}_{dg}(X_S)$ yields a compatible system isomorphic to $\tilde{\xi}$. Thus $(\operatorname{Spec} S, s, \mathrm{Perf}_{dg}(X_S))$ gives a versal Morita deformation of $\mathrm{Perf}_{dg}(X_0)$. $\square$

Proposition 7.5. *There is a quasi-equivalence*

$$\mathrm{Perf}_{dg}(X_S)/\mathrm{Perf}_{dg}(X_S)_0 \simeq_{qeq} \mathrm{Perf}_{dg}(X_{Q(S)})$$

where $Q(S)$ *is the quotient field of* S *and* $X_{Q(S)}$ *is the generic fiber of* X_S.

Proof. By [Dri04, Theorem 3.4] and [Morb, Theorem 1.1] we have an equivalence

$$[\mathrm{Perf}_{dg}(X_S)/\,\mathrm{Perf}_{dg}(X_S)_0] \simeq \mathrm{Perf}(X_S)/\,\mathrm{Perf}(X_S)_0 \simeq \mathrm{Perf}(X_{Q(S)})$$

of idempotent complete triangulated categories, where the middle category is the Verdier quotient by the full triangulated subcategory $\mathrm{Perf}(X_S)_0 \subset \mathrm{Perf}(X_S)$ of perfect complexes with S-torsion cohomology. Then the claim follows from [CNS, Theorem B]. $\square$

Remark 7.4. From the proof one sees that the dg categorical generic fiber is a natural dg enhancement of the categorical generic fiber introduced in [Morb], which is in turn based on the categorical general fiber by Huybrechts–Macrì–Stellari [HMS11].

8. Independence from geometric realizations

Due to Corollary 7.4, a versal Morita deformation of $\mathrm{Perf}_{dg}(X_0)$ is given by $\mathrm{Perf}_{dg}(X_S)$ where X_S is a versal deformatiion of X_0. Suppose that there is another Calabi–Yau manifold X'_0 derived-equivalent to X_0. Since by [CNS, Theorem B] dg enhancements of

$$\mathrm{Perf}(X_0) \simeq D^b(X_0) \simeq D^b(X'_0) \simeq \mathrm{Perf}(X'_0)$$

are unique, we obtain a quasi-equivalence

$$\mathrm{Perf}_{dg}(X_0) \to \mathrm{Perf}_{dg}(X'_0).$$

Hence $\mathrm{Perf}_{dg}(X_S)$ gives also a versal Morita deformation of $\mathrm{Perf}_{dg}(X'_0)$.

By Lemma 3.2 we may assume X_S to be smooth projective over S. Then one finds a smooth projective versal deformation X'_S over the same base. The construction requires the deformation theory of Fourier–Mukai kernels, which we briefly review below. It passes through effectivizations, i.e., there are effectivizations X_R, X'_R of X_0, X'_0 over the same regular affine scheme $\operatorname{Spec} R$. Applying [Mora, Corollary 4.2] and [CNS, Theorem B], we obtain a quasi-equivalence

$$\mathrm{Perf}_{dg}(X_R) \simeq_{qeq} \mathrm{Perf}_{dg}(X'_R). \tag{8.1}$$

Unwinding the construction of versal deformations recalled in Section 3.3, one sees that, up to equivalence of deformations, the ambiguity of X_S essentially stems from the choice of indices $i \in I$ of the filtered inductive system $\{R_i\}_{i\in I}$, where R_i are finitely generated T-subalgebras of R whose colimit is R. The versal deformations X_S, X'_S over the same base are obtained by choosing the same sufficiently large index. From this observation combined with Theorem 7.1 and the quasi-equivalence (8.1), it is natural to expect that the versal Morita deformations $\mathrm{Perf}_{dg}(X_S), \mathrm{Perf}_{dg}(X'_S)$ become quasi-equivalent close to effectivizations. In this section, we prove our second main result which yields the quasi-equivalence as a corollary.

8.1. **Deformations of Fourier–Mukai kernels.** Suppose that the derived equivalence of X_0, X'_0 is given by a Fourier–Mukai kernel $\mathcal{P}_0 \in D^b(X_0 \times X'_0)$. In order to define a relative integral functor from $D^b(X_S)$ to $D^b(X'_S)$, we deform $\mathcal{P}_0$ to a perfect complex $\mathcal{P}_S$ on $X_S \times_S X'_S$. Here, for a deformation $[X_{\mathbf{P}}, i_{\mathbf{P}}] \in \mathrm{Def}_X((\mathbf{P}, \mathfrak{m}_{\mathbf{P}}))$ of a $\mathbf{k}$-scheme X, by a deformation of $E \in \mathrm{Perf}(X)$ over $(\mathbf{P}, \mathfrak{m}_{\mathbf{P}})$ we mean a pair $(E_{\mathbf{P}}, u_{\mathbf{P}})$, where $E_{\mathbf{P}} \in \mathrm{Perf}(X_{\mathbf{P}})$ and $u_{\mathbf{P}} \colon E_{\mathbf{P}} \otimes^L_{\mathbf{P}} \mathbf{k} \to E$ is an isomorphism. Two deformations $(E_{\mathbf{P}}, u_{\mathbf{P}})$ and $(F_{\mathbf{P}}, v_{\mathbf{P}})$ are equivalent if there is an isomorphism $E_{\mathbf{P}} \to F_{\mathbf{P}}$ reducing to an isomorphism of E.

The R_n-deformations X_n, X'_n of X_0, X'_0 and their fiber product $X_n \times_{R_n} X'_n$ form the diagram

$$\begin{tikzcd} & X_n \times_{R_n} X'_n \arrow[dl, "q_n"'] \arrow[dr, "p_n"] & \\ X_n & & X'_n \end{tikzcd}$$

with the natural projections q_n and p_n. For any perfect complex $\mathcal{P}_n$ on $X_n \times_{R_n} X'_n$, the relative integral functor

$$\Phi_{\mathcal{P}_n}(-) = Rp_{n*}\left(\mathcal{P}_n \otimes^L q_n^*(-)\right)$$

sends each object of $D^b(X_n)$ to $D^b(X'_n)$. Due to the Grothendieck–Verdier duality the functor $\Phi_{\mathcal{P}_n}$ admits the right adjoint $\Phi^R_{\mathcal{P}_n} = \Phi_{(\mathcal{P}_n)_R}$ with kernel $(\mathcal{P}_n)_R = \mathcal{P}_n^\vee \otimes p_n^*\omega_{\pi'_n}[\dim X_0]$, where $\omega_{\pi'_n}$ is the determinant of the relative cotangent sheaf associated with the natural projection $\pi'_n \colon X'_n \to \operatorname{Spec} R_n$.

Lemma 8.1. ([Mora, Lemma 3.1, 3.2]) *Assume that $\Phi_{\mathcal{P}_n}$ is an equivalence. Then for any thickening $X_n \hookrightarrow X_{n+1}$ there exist a thickening $X'_n \hookrightarrow X'_{n+1}$ and a perfect complex $\mathcal{P}_{n+1}$ on $X_{n+1} \times_{R_{n+1}} X'_{n+1}$ with an isomorphism $\mathcal{P}_{n+1} \otimes^L_{R_{n+1}} R_n \cong \mathcal{P}_n$ such that the integral functor $\Phi_{\mathcal{P}_{n+1}} \colon D^b(X_{n+1}) \to D^b(X'_{n+1})$ is an equivalence.*

Iterative application of Lemma 8.1 allows us to deform the Fourier–Mukai kernel $\mathcal{P}_0 \in D^b(X_0 \times X'_0)$ to some Fourier–Mukai kernel $\mathcal{P}_n \in \operatorname{Perf}(X_n \times_{R_n} X'_n)$ for arbitrary order n. We obtain a system of deformations $\mathcal{P}_n \in \operatorname{Perf}(X_n \times_{R_n} X'_n)$ of $\mathcal{P}_0$ with compatible isomorphisms $\mathcal{P}_{n+1} \otimes^L_{R_{n+1}} R_n \to \mathcal{P}_n$. According to [Lie06, Proposition 3.6.1] there exists an effectivization, i.e., a perfect complex $\mathcal{P}_R$ on $X_R \times_R X'_R$ with compatible isomorphisms $\mathcal{P}_R \otimes^L_R R_n \to \mathcal{P}_n$. Recall that to algebrize $\mathscr{X}$ we used a filtered inductive system $\{R_i\}_{i \in I}$ of finitely generated T-subalgebras of R whose colimit is R. Taking an index i sufficiently large, one finds smooth projective R_i-deformations X_{R_i}, X'_{R_i} of X_0, X'_0 whose pullback along the canonical homomorphism $R_i \hookrightarrow R$ are X_R, X'_R. Since we have $X_R \times_R X'_R \cong \left(X_{R_i} \times_{R_i} X'_{R_i}\right) \times_{R_i} R$, by [Lie06, Proposition 2.2.1] there exists a perfect complex $\mathcal{P}_{R_i}$ on $X_{R_i} \times_{R_i} X'_{R_i}$ with an isomorphism $\mathcal{P}_{R_i} \otimes^L_{R_i} R \to \mathcal{P}_R$. Finally, the derived pullback $\mathcal{P}_S \in \operatorname{Perf}(X_S \times_S X'_S)$ along $R_i \to S$ yields a deformation of $\mathcal{P}_0$.

Lemma 8.2. ([Mora, Proposition 3.3]) *Let $\mathcal{P}_0$ be a Fourier–Mukai kernel defining the derived equivalence of Calabi–Yau manifolds X_0, X'_0 of dimension more than two. Then there exists a perfect complex $\mathcal{P}_S$ on the fiber product $X_S \times_S X'_S$ of smooth projective versal deformations with an isomorphism $\mathcal{P}_S \otimes^L_S \mathbf{k} \to \mathcal{P}_0$.*

8.2. **Inherited equivalences.** The schemes X_{R_i}, X'_{R_i}, and their fiber product $X_{R_i} \times_{R_i} X'_{R_i}$ together with the pullbacks along T-algebra homomorphisms $R_i \to R_j \to R$ for $i \leq j$ form the commutative diagram

$$\begin{tikzcd}
& X_R \times_R X'_R \arrow[dl, "q"'] \arrow[d, "f''_j"] \arrow[dr, "p"] & \\
X_R \arrow[d, "f_j"] & X_{R_j} \times_{R_j} X'_{R_j} \arrow[dl, "q_j"'] \arrow[d, "f''_{ij}"] \arrow[dr, "p_j"] & X'_R \arrow[d, "f'_j"] \\
X_{R_j} \arrow[d, "f_{ij}"] & X_{R_i} \times_{R_i} X'_{R_i} \arrow[dl, "q_i"'] \arrow[dr, "p_i"] & X'_{R_j} \arrow[d, "f'_{ij}"] \\
X_{R_i} & & X'_{R_i},
\end{tikzcd}$$

where q_i, p_i are smooth projective of relative dimension $\dim X_0$. Given a collection $\{\mathcal{P}_i\}_{i \in I}$ with $\mathcal{P}_i \in \operatorname{Perf}(X_{R_i} \times_{R_i} X'_{R_i})$ satisfying $\mathcal{P}_j \cong \mathcal{P}_{R_i} \otimes^L_{R_i} R_j$ and $\mathcal{P}_R \cong \mathcal{P}_{R_j} \otimes^L_{R_j} R$ for all $i \leq j$, consider the relative integral functors

$$\Phi_{\mathcal{P}_i} = Rp_{i*}\left(\mathcal{P}_i \otimes^L q_i^*(-)\right) \colon D^b(X_{R_i}) \to D^b(X'_{R_i}).$$

Since p_i is projective and $\mathcal{P}_i$ is of finite homological dimension, i.e., $\mathcal{P}_i \otimes^L q_i^* F_{R_i}$ are bounded for each object $F_{R_i} \in D^b(X_{R_i})$, one can apply [LST13, Lemma 1.8] to see that $\Phi_{\mathcal{P}_i}$ send perfect complexes to perfect complexes. We use the same symbol to denote the restricted functor.

Theorem 8.3. *There exists an index $j \in I$ such that for all $k \geq j$ the functors*

$$\Phi_{\mathcal{P}_k}\colon\ \mathrm{Perf}(X_{R_k}) \to \mathrm{Perf}(X'_{R_k})$$

are equivalences of triangulated categories of perfect complexes. In particular, the dg categories $\mathrm{Perf}_{dg}(X_{R_k}), \mathrm{Perf}_{dg}(X'_{R_k})$ of perfect complexes are quasi-equivalent.

Proof. Under the assumption one always finds deformations $[X_{R_j}, i_{R_j}], [X'_{R_j}, i'_{R_j}]$ smooth projective over $(R_j, \mathfrak{m}_{R_j})$ for sufficiently large index $j \in I$. Moreover, the pullbacks along $R_j \to R$ and $R_j \to S$ yield respectively effectivizations X_R, X'_R of universal formal families ξ, ξ' and versal deformations $(\operatorname{Spec} S, s, X_S), (\operatorname{Spec} S, s, X'_S)$ of X_0, X'_0. Recall that $(\operatorname{Spec} S, s)$ is an étale neighborhood of t in $\operatorname{Spec} T$ with t corresponding to the maximal ideal $(t_1, \ldots, t_d) \subset T$, and the formal completions of X_S, X'_S along the closed fibers over s are isomorphic to $\hat{X}_R, \hat{X}'_R$. In summary, we have the pullback diagrams

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d, "\pi"] & X_S \arrow[r, "f_{jS}"] \arrow[d, "\pi_S"] & X_{R_j} \arrow[d, "\pi_{R_j}"] & X_R \arrow[l, "f_j"'] \arrow[d, "\pi_R"] \\
\operatorname{Spec} \mathbf{k} \arrow[r, hook] & \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R_j & \operatorname{Spec} R \arrow[l] \\
X'_0 \arrow[r, hook] \arrow[u, "\pi'"] & X'_S \arrow[r, "f'_{jS}"] \arrow[u, "\pi'_S"] & X'_{R_j} \arrow[u, "\pi'_{R_j}"] & X'_R. \arrow[l, "f'_j"'] \arrow[u, "\pi'_R"]
\end{tikzcd}$$

Let $\mathcal{P}_0 \in D^b(X_0 \times_{\mathbf{k}} X'_0)$ be a Fourier–Mukai kernel defining the derived equivalence. As explained above, one can deform $\mathcal{P}_0$ to a perfect complex $\mathcal{P}_j \in D^b(X_{R_j} \times_{R_j} X'_{R_j})$. Due to the Grothendieck–Verdier duality the functor $\Phi_{\mathcal{P}_j}$ admits a left adjoint $\Phi^L_{\mathcal{P}_j} = \Phi_{(\mathcal{P}_i)_L}$ with kernel $(\mathcal{P}_j)_L = \mathcal{P}^\vee_{R_j} \otimes p_j^* \omega_{\pi'_{R_j}}[\dim X_0]$. By [BV03, Corollary 3.1.2] the category $\mathrm{Perf}(X_{R_j})$ is generated by some single object E_{R_j}. Namely, each object $F_{R_j} \in \mathrm{Perf}(X_{R_j})$ can be obtained from E_{R_j} by taking isomorphisms, finite direct sums, direct summands, shifts, and bounded number of cones. The counit morphism $\eta_j\colon \Phi^L_{\mathcal{P}_j} \circ \Phi_{\mathcal{P}_j} \to \mathrm{id}_{\mathrm{Perf}(X_{R_j})}$ gives the distinguished triangle

$$\Phi^L_{\mathcal{P}_j} \circ \Phi_{\mathcal{P}_j}(E_{R_j}) \xrightarrow{\eta_j(E_{R_j})} E_{R_j} \to C(E_{R_j}) := \mathrm{Cone}(\eta_j(E_{R_j})).$$

For sufficiently large $k \geq j$ we will show that $\eta_k(E_{R_k})$ is an isomorphism and then $\Phi_{\mathcal{P}_k}$ is fully faithful. Similarly, one can show that $\Phi^L_{\mathcal{P}_k}$ is also fully faithful. Thus $\Phi_{\mathcal{P}_k}$ is an equivalence, as it is a fully faithful functor admitting a fully faithful left adjoint.

Pullback along $R_j \subset R_k$ yields

$$\Phi^L_{\mathcal{P}_k} \circ \Phi_{\mathcal{P}_k}(E_{R_k}) \xrightarrow{f^*_{jk}\eta_j(E_{R_j})} E_{R_k} \to f^*_{jk} C(E_{R_j}) \tag{8.2}$$

with $E_k = f^*_{jk} E_j$ and $\mathcal{P}_k = (f_{jk} \times f'_{jk})^* \mathcal{P}_j$. Further pullback along $R_k \subset R$ yields

$$\Phi^L_{\mathcal{P}_R} \circ \Phi_{\mathcal{P}_R}(E_R) \xrightarrow{f^*_j \eta_j(E_{R_j})} E_R \to f^*_j C(E_{R_j}),$$

where $f_j\colon X_R \to X_{R_j}$ satisfies $f_{jk} \circ f_k = f_j$. Restriction to the closed fiber X_0 yields

$$\Phi^L_{\mathcal{P}_0} \circ \Phi_{\mathcal{P}_0}(E_R|_{X_0}) \xrightarrow{\eta_j(E_{R_j}|_{X_0})} E_R|_{X_0} \to (f^*_j C(E_{R_j}))|_{X_0}.$$

Note that since $f_j^{-1}(X_0) = X_0$ and the restriction of the counit morphism is the counit morphism, we have $(f_j^*\eta(E_{R_j}))|_{X_0} = \eta(E_{R_j}|_{X_0})$. Each term in the above distinguished triangle is perfect so that we may consider the restriction to the closed fiber. Since $\Phi_{\mathcal{P}_0}$ is an equivalence, $\eta_j(E_{R_j}|_{X_0})$ is an isomorphism and we obtain a quasi-isomorphism $f_j^*C(E_{R_j})|_{X_0} \cong 0$. Then the support of $f_j^*C(E_{R_j})$ is a proper closed subscheme of X_R which does not contain any closed point of X_R. Thus the quasi-isomorphism extends to $f_j^*C(E_{R_j}) \cong 0$. From [Lie06, Proposition 2.2.1] it follows $f_{jk}^*C(E_{R_j}) \cong 0$ when $k \in I$ is sufficiently large.

Take any closed point $u \in \operatorname{Spec} R_j$ whose inverse image by $g_{jk}\colon \operatorname{Spec} R_k \to \operatorname{Spec} R_j$ is not empty. We have the pullback diagrams

$$\begin{tikzcd}
X_{R_j} \arrow[d, "\pi_{R_j}"'] & X_u \arrow[l, hook'] \arrow[d] & f_{jk}^{-1}(X_u) \arrow[l, "f_{u,jk}"'] \arrow[r, hook] \arrow[d] & X_{R_k} \arrow[d, "\pi_{R_k}"] \\
\operatorname{Spec} R_j & \operatorname{Spec} \mathbf{k} \arrow[l, hook'] & g_{jk}^{-1}(u) \arrow[l, "g_{u,jk}"'] \arrow[r, hook] & \operatorname{Spec} R_k.
\end{tikzcd}$$

Note that $f_{u,jk}$ is surjective by construction and flat as $g_{u,jk}$ is flat. The restriction of (8.2) to $f_{jk}^{-1}(X_u)$ yields

$$\Phi^L_{\mathcal{P}_{u,jk}} \circ \Phi_{\mathcal{P}_{u,jk}}(E_{R_k}|_{f_{jk}^{-1}(X_u)}) \xrightarrow{f_{u,jk}^*\eta_j(E_{R_j}|_{X_u})} E_{R_k}|_{f_{jk}^{-1}(X_u)} \to f_{u,jk}^*C(E_{R_j}|_{X_u}) \cong 0,$$

where $\mathcal{P}_{u,jk} = \mathcal{P}_k|_{f_{jk}^{-1}(X_u)\times_{g_{jk}^{-1}(u)} f_{jk}^{-1}(X'_u)}$. It follows $C(E_{R_j}|_{X_u}) \cong 0$ and $\eta_j(E_{R_j}|_{X_u})$ is an isomorphism.

By [BV03, Lemma 3.4.1] the restriction $E_{R_j}|_{X_u}$ is a generator of $\operatorname{Perf}(X_u)$. Then each object $F_u \in \operatorname{Perf}(X_u)$ can be obtained from $E_{R_j}|_{X_u}$ by taking isomorphisms, finite direct sums, direct summands, shifts, and bounded number of cones. We may assume that $E_{R_j}|_{X_u}$ has no nontrivial direct summands, as $\Phi^L_{\mathcal{P}_{u,j}}$ and $\Phi_{\mathcal{P}_{u,j}}$ commute with direct sums on $\operatorname{Perf}(X'_u)$ and $\operatorname{Perf}(X_u)$ respectively with $\mathcal{P}_{u,j} = \mathcal{P}_j|_{X_u\times X'_u}$ [BV03, Corollary 3.3.4]. One inductively sees that the counit morphism $\Phi^L_{\mathcal{P}_{u,j}} \circ \Phi_{\mathcal{P}_{u,j}}(F_u) \to F_u$ is an isomorphism. In other words, the restriction $\Phi_{\mathcal{P}_{u,j}}$ of $\Phi_{\mathcal{P}_j}$ to X_u is fully faithful. Similarly, the restriction $\Phi^L_{\mathcal{P}_{u,j}}$ of $\Phi^L_{\mathcal{P}_j}$ to X'_u is also fully faithful. Thus $\Phi_{\mathcal{P}_{u,j}}$ is an equivalence.

Since X_u is a smooth projective $\mathbf{k}$-variety,

$$\Phi^L_{\mathcal{P}_{u,j}} \circ \Phi_{\mathcal{P}_{u,j}} \cong \mathrm{id}_{\operatorname{Perf}(X_u)},\ \Phi_{\mathcal{P}_{u,j}} \circ \Phi^L_{\mathcal{P}_{u,j}} \cong \mathrm{id}_{\operatorname{Perf}(X'_u)}$$

imply

$$\mathcal{P}_{u,j} * (\mathcal{P}_{u,j})_L \cong \mathcal{O}_{\Delta_{u,j}},\ (\mathcal{P}_{u,j})_L * \mathcal{P}_{u,j} \cong \mathcal{O}_{\Delta'_{u,j}}$$

where

$$\Delta_{u,j}\colon X_u \hookrightarrow X_u \times X_u,\ \Delta'_{u,j}\colon X'_u \hookrightarrow X'_u \times X'_u$$

are the diagonal embeddings. Pullback by $f_{u,jk}$ yields

$$\mathcal{P}_{u,jk} * (\mathcal{P}_{u,jk})_L \cong \mathcal{O}_{\Delta_{u,jk}},\ (\mathcal{P}_{u,jk})_L * \mathcal{P}_{u,jk} \cong \mathcal{O}_{\Delta'_{u,jk}}$$

where

$$\Delta_{u,jk}\colon f_{jk}^{-1}(X_u) \hookrightarrow f_{jk}^{-1}(X_u) \times_{g_{jk}^{-1}(u)} f_{jk}^{-1}(X_u),\ \Delta'_{u,jk}\colon (f'_{jk})^{-1}(X'_u) \hookrightarrow (f'_{jk})^{-1}(X'_u) \times_{g_{jk}^{-1}(u)} (f'_{jk})^{-1}(X'_u)$$

are the relative diagonal embeddings. Thus $\Phi_{\mathcal{P}_{u,jk}}$ is an equivalence. Since X_{R_k} is covered by the collection $\{f_{jk}^{-1}(X_u)\}_u$ with u running through all the closed points of $\operatorname{Spec} R_j$, from [LST13, Proposition 1.3] it follows that $\Phi_{\mathcal{P}_k}$ is an equivalence. By the same argument, we conclude that $\Phi_{\mathcal{P}_l}$ are equivalences for all $l \geq k$. Applying [CNS, Theorem B], we obtain a quasi-equivalence $\operatorname{Perf}_{dg}(X_{R_l}) \simeq_{qeq} \operatorname{Perf}_{dg}(X'_{R_l})$ for all $l \geq k$. □

Corollary 8.4. *Let X_0, X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two and X_S, X'_S their smooth projective versal deformations over a common nonsingular affine* $\mathbf{k}$*-variety* $\operatorname{Spec} S$. *Assume that X_S, X'_S correspond to a first order approximation $R_j \to S$ of $R_j \hookrightarrow R$ for sufficiently large $j \in I$. Then X_S, X'_S are derived-equivalent. In particular, the dg categories* $\mathrm{Perf}_{dg}(X_S), \mathrm{Perf}_{dg}(X'_S)$ *of perfect complexes are quasi-equivalent.*

Proof. By assumption one can apply Theorem 8.3 to find an index $j \in I$ such that X_S, X'_S are the pullbacks of smooth projective families X_{R_j}, X'_{R_j} over R_j satisfying $\mathrm{Perf}_{dg}(X_{R_j}) \simeq_{qeq} \mathrm{Perf}_{dg}(X'_{R_j})$. Consider the distinguished triangle

$$\Phi^L_{\mathcal{P}_j} \circ \Phi_{\mathcal{P}_j}(E_{R_j}) \xrightarrow{\eta_j(E_{R_j})} E_{R_j} \to C(\eta_j(E_{R_j})) \cong 0.$$

Applying the same argument in the above proof to $X_S \to X_{R_j}$ instead of $X_{R_k} \to X_{R_j}$, one sees that $\Phi_{\mathcal{P}_S} : D^b(X_S) \to D^b(X'_S)$ is an equivalence with $\mathcal{P}_S = (f_{jS} \times f'_{jS})^* \mathcal{P}_j$. □

Proposition 8.5. *Let X_0, X'_0 be derived-equivalent Calabi–Yau manifolds of dimension more than two and X_S, X'_S smooth projective versal deformations over a common nonsingular affine variety* $\operatorname{Spec} S$. *Then the dg categorical generic fibers are quasi-equivalent.*

Proof. We have

$$\mathrm{Perf}_{dg}(X_S)/\,\mathrm{Perf}_{dg}(X_S)_0 \simeq_{qeq} \mathrm{Perf}_{dg}(X_{Q(S)}) \simeq_{qeq} \mathrm{Perf}_{dg}(X'_{Q(S)}) \simeq_{qeq} \mathrm{Perf}_{dg}(X_S)/\,\mathrm{Perf}_{dg}(X_S)_0,$$

where the first and the tirhd quasi-equivalences follow from Proposition 7.5. The second quasi-equivalence follows from the above corollary, [Mora, Theorem 1.1], [Morb, Corollary 4.2], and [CNS, Theorem B]. □

References

[AK91] A. Albano and S. Katz, *Lines on the Fermat quintic threefold and the infinitesimal generalized Hodge conjecture*, Trans. Amer. Math. Soc. 324(1), 353-368 (1991).

[Art69a] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math-Paris. 36, 23-58 (1969).

[Art69b] M. Artin, *Algebraization of formal moduli: I*, Global Analysis (Papers in Honor of K. Kodaira), University of Tokyo Press, 21-71 1969, ISBN:978-1-4008-7123-0.

[BO01] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compos. Math. J. 125, 327-344 (2001).

[BV03] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3(1), 1-36 (2003).

[BFN10] D. Ben-zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Am. Math. Soc. 23(4), 909-966 (2010).

[Coh] L. Cohn, *Differential graded categories are k-linear stable infinity categories*, arXiv:1308.2587

[Căl00] A. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University (2000).

[CNS] A. Canonaco, A. Neeman and P. Stellari, *Uniqueness of enhancements for derived and geometric categories*, arXiv:2101.04404

[DL13] O. D. Deken and W. Lowen *On deformations of triangulated models*, Adv. Math 243, 330-374 (2013).

[DLL17] H. Dinh Van, L. Liu and W. Lowen, *Non-commutative deformations and quasi-coherent modules*, Sel. Math. New Ser. 23, 1061–1119 (2017).

[Dri04] V. Drinfeld, *DG quotients of DG categories*, J. Algebra 272(2), 643-691 (2004).

[GD61] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: III. Étude cohomologique des faisceaux cohérent, première partie.*, Publ. Math-Paris. 11, 5-167 (1961).

[GJ] E. Getzler and J.D.S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, arXiv:hep-th/9403055

[GS87] M. Gerstenhaber and S. D. Schack, *A Hodge-type decomposition for commutative algebra cohomology*, J. Pure. Appl. Algebra. 48(3) 229-247 (1987).

[GS88] M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and Deformation theory*, Deformation Theory of Algebras and Structures and Applications. NATO ASI Series (Series C: Mathematical and Physical Sciences), 247, Springer, https://doi.org/10.1007/978-94-009-3057-5_2

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, 1977, ISBN: 0-387-90244-9.

[Har10] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, 257, Springer-Verlag, 2010, ISBN: 978-1-4419-1596-2.

[HMS11] D. Huybrechts, E. Macrì, and P. Stellari, *Formal deformations and their categorical general fibre*, Comment. Math. Helv. 86(1), 41-71 (2011).

[Kel] B. Keller, *Derived invariance of higher structures on the Hochschild complexes*, http://www.math.jussieu.fr/~keller/publ/dih.dvi

[KL09] B. Keller and W. Lowen, *On hochschild cohomology and morita deformations*, Int. Math. Res. Not. 17, 3221–3235 (2009)

[Lie06] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15, 175-206 (2006).

[Low05] W. Lowen, *Obstruction theory for objects in abelian and derived categories*, Comm. Algebra. 33(9), 3195-3223 (2005).

[Low08] W. Lowen, *Hochschild cohomology, the characteristic morphism and derived deformations*, Compos. Math. 144(6), 1557-1580 (2008).

[LV06a] W. Lowen and M. Van den Bergh, *Hochschild cohomology of abelian categories and ringed spaces*, Adv. Math. 198(1), 172-221 (2006).

[LV06b] W. Lowen and M. Van den Bergh, *Deformation theory of abelian categories*, Trans. Amer. Math. Soc. 358(12), 5441-5483 (2006).

[LST13] A. C. López Martín, D. Sánchez Gómez, and C. Tejero Prieto, *Relative Fourier–Mukai transforms for Weierstrass fibrations, abelian schemes and Fano fibrations*, Math. Proc. Cambridge. 155(1), 129-153 (2013).

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, 170, Princeton University Press, 2009, ISBN: 978-0-6911-4049-0

[Man09] M. Manetti, *Differential graded Lie algebras and formal deformation theory*, Proceedings of Symposia in Pure Mathematics, 80(2), American Mathematical Society, Providence, RI, 2009. ISBN: 978-0-8218-4703-9

[Mora] H. Morimura, *Algebraic deformations and Fourier–Mukai transforms for Calabi–Yau manifolds*, to appear in Proc. Amer. Math. Soc.

[Morb] H. Morimura, *Categorical generic fiber*, arXiv:2111.00239

[Por10] M. Porta, *The Popescu–Gabriel theorem for triangulated categories*, Adv. Math. 225(3), 1669-1715 (2010).

[Qui70] D. Quillen, *On the (co-)homology of commutative rings*, Applications of categorical algebra (Proc. Sympos. Pure. Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 65-87 (1970)

[Shu61] U. Shukla, *Cohomologie des algèbres associatives*, Ann. Sci. École Norm. Sup. 78(3), 163-209 (1961).

[STZ14] N. Sibilla, D. Treumann, and E. Zaslow *Ribbon Graphs and Mirror Symmetry I*, Selecta Mathematica (N.S.) 20(4), 979-1002 (2014).

[Tab05a] G. Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, Comptes Rendus de lAcadadémie de Sciences de Paris. 340, 15–19 (2005).

[Tab05b] G. Tabuada, *Invariants additifs de DG-catégories*, Int. Math. Res. Not. 53, 3309-3339 (2005).

[Tod09] Y. Toda, *Deformations and Fourier–Mukai transforms*, J. Differ. Geom. 81(1), 197-224 (2009).

[Töe] B. Töen, *Lectures on DG-categories*, http://www.math.univ-toulouse.fr/~toen/

SISSA, via Bonomea 265, 34136 Trieste, Italy
Email address: `hmorimur@sissa.it`

REDUCED TATE–SHAFAREVICH GROUP

HAYATO MORIMURA

Abstract. We prove a sort of reconstruction theorem for generic elliptic Calabi–Yau 3-folds. As an application, we give a method to construct a family of pairs of derived-equivalent Calabi–Yau 3-folds whose general fibers are nobirational if they are nonisomorphic.

1. Introduction

For a normal integral excellent scheme S and an abelian variety E over the function filed $k(S)$, the Tate–Shafarevich group $Ш_S(E)$ was introduced by Dolgachev and Gross in [DG94] as a subset of the Weil–Châtelet group. When S is a complex surface, they gave its geometric interpretation. Namely, any element of $Ш_S(E)$ can be obtained as the generic fiber of an elliptic fibration $f\colon X \to S$ with possible isolated multiple fibers.

Given a smooth elliptic fibration $\pi\colon J \to S$ with section, we have $Ш_S(J_\eta) \cong \mathrm{Br}'(J)/\mathrm{Br}'(S)$ and $Ш_S(J_\eta)$ bijectively corresponds to the set of smooth elliptic fibrations f with relative Jacobian π. In fact, f can be obtained from π and a representative $\alpha \in \mathrm{Br}'(J)$. Via the description of Brauer classes as gerbes, α gives the gluing data for enough refined étale cover $\{J_i = J \times_S U_i\}$ to yield X [Căl00, Section 4.4].

Categorically, this amounts to an S-linear exact equivalence

$$D^b(X) \simeq D^b(J, \alpha)$$

established by Căldăraru in [Căl02] for *generic elliptic* 3*-folds* [Căl00, Definition 6.1.6]. On the other hand, Antieau–Krashen–Ward showed that if there is an S-linear exact equivalence

$$D^b(J, \alpha) \simeq D^b(J, \beta),$$

then we have $\beta = \alpha^d$ for some $d \in \mathbb{Z}$ coprime to $\mathrm{ord}([\alpha])$ in $Ш_S(J_\eta)$ [AKW17, Theorem 1.5].

Define an equivalence relation $\sim$ in $\mathrm{Br}'(J)$ as

$$\alpha \sim \beta \Leftrightarrow \beta = \alpha^d \text{ for some d coprime to } \mathrm{ord}([\alpha]),$$

descending to that in $Ш_S(J_\eta)$. The following lemma, which is a straightforward consequence of the arguments in [Căl00, Chapter 4, 6], shows that $Ш_S(J_\eta)/\sim$ classifies up to S-linear exact equivalence derived categories of smooth elliptic 3-folds with relative Jacobian π.

Lemma 1.1 (Lemma 4.3)**.** *Let $f\colon X \to S, g\colon Y \to S$ be smooth elliptic* 3*-folds with relative Jacobian $\pi\colon J \to S$. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then g is a* coprime twisted power *of f in the sense of Definition 2.14.*

Recall that the morphism g in the above statement is isomorphic to the relative moduli space of stable sheaves of rank 1, degree d on the fibers $X_s, s \in S$ of f with respect to a fixed relative ample line bundle $\mathscr{O}_{X/S}(1)$. The moduli problem is fine if and only if d is coprime to $\mathrm{ord}([\alpha])$. It is natural to seek a similar reconstruction result for more general elliptic fibrations. Unwinding the arguments in [Căl00, Chapter 4, 6], one can easily extend Lemma 1.1 to

SISSA, via Bonomea 265, 34136 Trieste, Italy
E-mail address: hmorimur@sissa.it.

Proposition 1.2 (Proposition 4.5)**.** *Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic 3-folds with relative Jacobian $\pi\colon J \to S$. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then g is an* almost coprime twisted power *of f in the sense of Definition 2.14.*

Presumably, this is the best possible reconstruction result for generic elliptic 3-folds. Indeed, if the smooth parts $f^\circ\colon X^\circ \to S^\circ, g^\circ\colon Y^\circ \to S^\circ$ are represented by $\alpha^\circ, \beta^\circ \in \mathrm{Br}'(J^\circ)$, for any analytic small resolution $\rho\colon \bar{J} \to J$ we have the S-linear exact equivalences

$$D^b(X) \simeq D^b(\bar{J}, \bar{\alpha}),\ D^b(Y) \simeq D^b(\bar{J}, \bar{\beta})$$

where $\bar{\alpha}, \bar{\beta} \in \mathrm{Br}_{\mathrm{an}}(\bar{J})$ denote the Brauer classes canonically determined by $\alpha^\circ, \beta^\circ$. Moreover, each $\rho_i\colon \bar{J}_i \to J_i$ over U_i with only one node can be either of the two possible resolutions, which is the source of the differences between X, Y and $\bar{J}$. Hence the right hand sides should not recover $X \setminus X^\circ, Y \setminus Y^\circ$.

Our proof of Proposition 1.2 works without assuming $\dim X = 3$, the condition responsible in [Căl02, Theorem 5.1] for construction of the pseudo-universal sheaf giving rise to the Fourier–Mukai kernel. Hence, unless $\dim X = 3$ one might not have any S-linear exact equivalence

$$D^b(\bar{J}, \bar{\alpha}) \simeq D^b(\bar{J}, \bar{\beta}),$$

which would be input for [AKW17, Theorem 1.5] to obtain $\bar{\beta} = \bar{\alpha}^d$ for some $d \in \mathbb{Z}$ coprime to $\mathrm{ord}(\bar{\alpha})$ in $\mathrm{Br}(\bar{J})$.

The reconstruction problem will be more interesting when X is Calabi–Yau, as the derived category of a Calabi–Yau manifold might have nontrivial autoequivalences. Moreover, most of Calabi–Yau 3-folds admit elliptic fibrations. Some of them are connected with their mirrors admitting dual fibrations via relative Fourier–Mukai transforms [HLS09, HM02], which can be seen as S-duality in string theory [Don98, DP08, DP12]. In this case, we strengthen Proposition 1.2 as follows.

Theorem 1.3 (Theorem 5.2, Corollary 5.3)**.** *Let $f\colon X \to S$ be a generic elliptic Calabi–Yau 3-fold. Then any flat projective family $g\colon Y \to S$ with an S-linear exact equivalence $D^b(X) \simeq D^b(Y)$ is an almost coprime twisted power of f.*

Here, we could remove the assumption from Proposition 1.2 on f, g to share the relative Jacobian. First, we reconstruct closed fibers based on [LT17]. The key is [AKW17, Lemma 2.4], which tells us that the generic fibers X_η, Y_η share the Jacobian. We use the triviality of ω_X, ω_Y to guarantee that the base changes of the relative Jacobians π_X, π_Y along enough refined étale morphisms coincide.

Due to [Wil94, Wil98] and [Mor23, Theorem 1.1], the morphisms f, g in the above statement deform to families of elliptic Calabi–Yau 3-folds $\mathbf{f}\colon \mathbf{X} \to \mathbf{S}, \mathbf{g}\colon \mathbf{Y} \to \mathbf{S}$ over a smooth affine $\mathbb{C}$-variety $\operatorname{Spec} B$. Their general fibers $\mathbf{f}_b, \mathbf{g}_b$ are smooth elliptic fibrations. Recently, in [Morb] the author used the following application of Theorem 1.3 to construct families of nonbirational derived-equivalent Calabi–Yau 3-folds.

Theorem 1.4 (Theorem 6.5, Corollary 6.6)**.** *Any general fiber $\mathbf{g}_b$ is a coprime twisted power of $\mathbf{f}_b$. In particular, $\mathbf{X}_b, \mathbf{Y}_b$ are $\mathbf{S}_b$-linear derived-equivalent. Moreover, if $\mathbf{X}_b, \mathbf{Y}_b$ are nonisomorphic, then they are nonbirational.*

Note that we call g a *coprime twisted power* of f if it is isomorphic to the relative moduli space of stable sheaves of rank 1 with a suitable degree. It would be interesting to find such a moduli structure via our result in, for instance, the third pair of Fourier–Mukai partners constructed by Inoue [Ino22] and their deformations, whose derived equivalence follows from homological projective duality for categorical joins developed by Kuznetsov and Perry [KP21].

Notations and conventions. Throughout the paper, all $\mathbb{C}$-varieties are integral separated scheme of finite type over $\mathbb{C}$. Via Serre's GAGA theorem we sometimes go from algebraic to analytic categories for proper $\mathbb{C}$-varieties. For any morphism $f\colon X \to S$ of smooth $\mathbb{C}$-varieties, we consider the canonical S-linear structure on $D^b(X) \simeq \mathrm{perf}(X)$ given by the action

$$\mathrm{perf}(X) \times \mathrm{perf}(S) \to \mathrm{perf}(X),\ (F, G) \mapsto F \otimes_{\mathscr{O}_X} f^*G.$$

Then for morphisms $f\colon X \to S, g\colon Y \to S$ of smooth $\mathbb{C}$-varieties, an exact functor $\Phi\colon D^b(X) \to D^b(Y)$ is S-linear if Φ respects the S-linear structure, i.e., we have functorial isomorphisms

$$\Phi(F \otimes_{\mathscr{O}_X} f^*G) \cong \Phi(F) \otimes_{\mathscr{O}_Y} g^*G,\ F \in \mathrm{perf}(X), G \in \mathrm{perf}(S).$$

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2. Review on generic elliptic 3-folds

Definition 2.1 ([DG94, Definition 2.1])**.** An *elliptic fibration* $f\colon X \to S$ is a projective morphism of $\mathbb{C}$-schemes whose generic fiber X_η is a genus one regular $k(S)$-curve and all fibers are geometrically connected. The *discriminant locus* Δ_f of f is the closed subset of points $s \in S$ over which the fiber X_s is not regular. We denote by $f^\circ\colon X^\circ \to S^\circ$ the smooth part, i.e., the restriction $f_U\colon X_U \to S_U$ of f over $U = S \setminus \Delta_f$. A fiber X_s over a point $s \in S$ is *multiple* if f is not smooth at any $x \in X_s$. A *section* of f is a morphism $\sigma_X\colon S \to X$ satisfying $f \circ \sigma_X = \mathrm{id}$. A *multisection* of f is a closed subscheme to which the restriction of f becomes a finite morphism.

Definition 2.2 ([Căl00, Definition 6.1.6])**.** A *generic elliptic* 3*-fold* $f\colon X \to S$ is an elliptic fibration from a smooth 3-fold X to a smooth surface S over $\mathbb{C}$ satisfying:

(1) f is flat.
(2) f does not have multiple fibers.
(3) f admits a multisection.
(4) The discriminant locus $\Delta_f \subset S$ is an integral curve in S with only nodes and cusps as singularities.
(5) The fiber over a general point of Δ_f is a rational curve with one node.

We call f a *generic elliptic Calabi–Yau* 3*-fold* if in addition X is a Calabi–Yau in the strict sence, i.e., we have $\omega_X \cong \mathscr{O}_X$ and $H^1(X, \mathscr{O}_X) = 0$.

Remark 2.3. The conditions (4), (5) are null for smooth elliptic 3-folds. We will regard them as a special case of generic elliptic 3-folds.

Lemma 2.4 ([Căl00, Theorem 6.1.9])**.** *Let $f\colon X \to S$ be a generic elliptic* 3*-fold. Then over any closed point $s \in S$ the fiber X_s is one of the following:*

- *a smooth elliptic curve when $s \in S \setminus \Delta_f$;*
- *a rational curve with one node when $s \in \Delta_f$ is a smooth point;*
- *two copies of $\mathbb{P}^1$ intersecting transversely at two points when $s \in \Delta_f$ is a node;*
- *a rational curve with one cusp when $s \in \Delta_f$ is a cusp.*

Definition 2.5 ([Căl00, Definition 6.4.1])**.** Let $f\colon X \to S$ be a flat elliptic fibration of $\mathbb{C}$-varieties. Fix a relatively ample line bundle $\mathscr{O}_{X/S}(1)$ of f and a closed point $s \in S$. Let P be the Hilbert polynomial of $\mathscr{O}_{X_s}$ on X_s with respect to the polarization given by $\mathscr{O}_{X/S}(1)|_{X_s}$. Consider the relative moduli space $M_{X/S}(P) \to S$ of semistable sheaves of Hilbert polynomial P on the fibers of f. By the universal property of $M_{X/S}(P) \to S$ there exists a natural section $S \to M_{X/S}(P)$ which sends s to the point $[\mathscr{O}_{X_s}]$ representing $\mathscr{O}_{X_s}$. Let J be the unique component of M which contains the image of this section. The *relative Jacobian* of f is the restriction $\pi\colon J \to S$ of the morphism $M_{X/S}(P) \to S$ to J.

Definition 2.6 ([Căl00, Notation 6.6.3])**.** Let $f\colon X \to S$ be a generic elliptic 3-fold. Fix a relatively ample line bundle $\mathcal{O}_{X/S}(1)$. Let $M^d_{X/S}(P) \to S$ be the relative moduli space of semistable sheaves of rank 1, degree d on the fibers of f. Let X^d be the union of the components of $M^d_{X/S}(P)$ which contains a point corresponding to a stable line bundle on a fiber of f. The *d-th twisted power* of f is the restriction $f^d\colon X^d \to S$ of the morphism $M^d_{X/S}(P) \to S$ to X^d.

Remark 2.7. The relative Jacobian $\pi\colon J \to S$ is a flat elliptic fibration with section. It has the same discriminant locus Δ_f as f. The restriction $J \times_S S^\circ \to S^\circ$ coincides with the relative Jacobian π° for the smooth part f°. Similarly, the restriction $X^d \times_S S^\circ \to S^\circ$ coincides with the d-th twisted power $f^{\circ d}$ of f°.

Definition 2.8 ([Sil08, Section X.3])**.** A *torsor* or *principal homogeneous space* for $J_\eta/k(S)$ is a smooth curve $C/k(S)$ together with a simply transitive algebraic group action of J_η on C defined over $k(S)$. It is *trivial* if $C(K) \neq \emptyset$. Two torsors $C/k(S), C'/k(S)$ are *equivalent* if there is a $k(S)$-isomorphism $\theta\colon C \to C'$ compatible with the J_η-action. The *Weil–Châtelet group* $\mathrm{WC}(J_\eta/k(S))$ is the set of equivalence classes of principal homogeneous spaces for $J_\eta/k(S)$.

Definition 2.9 ([DG94, Section 1])**.** The *Tate–Shafarevich group* $Ш_S(J_\eta)$ is defined as the subset $\bigcap_{s\in S} \mathrm{Ker}(loc_{\bar{s}}) \subset \mathrm{WC}(J_\eta/k(S))$ for natural specialization maps

$$loc_{\bar{s}}\colon\ \mathrm{WC}(J_\eta/k(S)) \to \mathrm{WC}(J_\eta(\bar{s})/k(\bar{s})),\ C \mapsto C(\bar{s}) = C \times_{k(s)} k(\bar{s}).$$

Remark 2.10. There is a standard cohomological interpretation of $Ш_S(J_\eta)$. Consider the exact sequence

$$0 \to H^1_{\text{ét}}(S, \iota_* J_\eta) \to H^1_{\text{ét}}(\eta, J_\eta) \to H^0_{\text{ét}}(S, R^1\iota_* J_\eta) \to H^2_{\text{ét}}(S, \iota_* J_\eta) \to H^2_{\text{ét}}(\eta, J_\eta)$$

where $\iota\colon\ \operatorname{Spec} k(S) \to S$ denotes the canonical morphism. For any $s \in S$ we have $(R^1\iota_* J_\eta)_{\bar{s}} \cong H^1_{\text{ét}}(\eta_{\bar{s}}, J_\eta(\bar{s}))$ and the natural homomorphism

$$H^0_{\text{ét}}(S, R^1\iota_* J_\eta) \to \prod_{s\in S} (R^1\iota_* J_\eta)_{\bar{s}}$$

is injective. Since the composition $H^1_{\text{ét}}(\eta, J_\eta) \to H^0_{\text{ét}}(S, R^1\iota_* J_\eta) \to (R^1\iota_* J_\eta)_{\bar{s}}$ coincides with $loc_{\bar{s}}$, one obtains an exact sequence

$$0 \to H^1_{\text{ét}}(S, \iota_* J_\eta) \to \mathrm{WC}(J_\eta/k(S)) \to \prod_{s\in S} \mathrm{WC}(J_\eta(\bar{s})/k(\bar{s}))$$

and $Ш_S(J_\eta) = H^1_{\text{ét}}(S, \iota_* J_\eta)$.

Remark 2.11. An element $C/k(S) \in \mathrm{WC}(J_\eta/k(S))$ maps to 0 in $\mathrm{WC}(J_\eta(\bar{s})/k(\bar{s}))$ if and only if there exists an irreducible étale neighborhood $U \to S$ of s such that $J_\eta \times_{k(S)} k(U)$ has a rational point over $k(U)$. Indeed, if the image of $C/k(S)$ in $(R^1\iota_* J_\eta)_{\bar{s}}$ is 0, then there exists an irreducible étale neighborhood $U \to S$ of s such that the image of $C/k(S)$ in $H^1_{\text{ét}}(k(U), J_\eta \times_{k(S)} k(U))$ is 0. Hence $Ш_S(J_\eta)$ consists of equivalence classes of étale locally trivial principal homogeneous spaces. In fact, there is a bijective correspondence between smooth elliptic fibrations $f\colon X \to S$ with relative Jacobian $\pi\colon J \to S$ and elements in $Ш_S(J_\eta)$ [Căl00, Section 4.4].

Definition 2.12 ([Căl00, Definition 1.1.3, 1.1.7])**.** For a $\mathbb{C}$-scheme X the *cohomological Brauer group* $\mathrm{Br}'(X)$ is defined as $H^2_{\text{ét}}(X, \mathcal{O}^*_X)$. For a complex analytic space X the *cohomological Brauer group* $\mathrm{Br}'_{\text{an}}(X)$ is defined as $H^2_{\text{an}}(X, \mathcal{O}^*_X)$.

Theorem 2.13 ([DG94, Corollary 1.17])**.** *Let $\pi\colon J \to S$ be a smooth elliptic fibration of smooth $\mathbb{C}$-varieties with section. Then we have*

$$Ш_S(J_\eta) \cong \mathrm{Coker}(\mathrm{Br}'(S) \to \mathrm{Br}'(J))$$

where the map $\mathrm{Br}'(S) \to \mathrm{Br}'(J)$ is given by the pullback.

Definition 2.14. Let $f\colon X \to S$ be a generic elliptic fibration with relative Jacobian $\pi\colon J \to S$ and $\alpha^\circ \in \mathrm{Br}'(J^\circ)$ a representative of $Ш_{S^\circ}(J_\eta)$ for f°. We call a generic elliptic fibration $g\colon Y \to S$ a *coprime twisted power* of f if g is isomorphic to f^d for some $d \in \mathbb{Z}$ coprime to the order $\mathrm{ord}([\alpha^\circ])$ in $Ш_{S^\circ}(J_\eta)$. We call $g\colon Y \to S$ an *almost coprime twisted power* of f if g° is isomorphic to $f^{\circ d}$ for some $d \in \mathbb{Z}$ coprime to $\mathrm{ord}([\alpha^\circ])$ and there exists an analytic open cover $\{U_i\}$ of S such that $Y \times_S U_i, X^k \times_S U_i$ are isomorphic as an analytic space over U_i.

3. Reconstruction of fibers

Lemma 3.1. *Let $f\colon X \to S, g\colon Y \to S$ be flat morphisms of $\mathbb{C}$-varieties with Y, S smooth over $\mathbb{C}$ and $f_Z\colon X_Z \to Z, g_Z\colon Y_Z \to Z$ their base changes to $Z = S \setminus U$ for any open subset $U \subset S$. Then every S-linear exact functor $\Phi\colon D^b(X) \to D^b(Y)$ restricts to*

$$\Phi_Z\colon D^b_{\mathrm{coh}_{X_Z}(X)}(X) \to D^b_{\mathrm{coh}_{Y_Z}(Y)}(Y),$$

where $D^b_{\mathrm{coh}_{X_Z}(X)}(X) \subset D^b(X), D^b_{\mathrm{coh}_{Y_Z}(Y)}(Y) \subset D^b(Y)$ denote the full S-linear triangulated subcategories with cohomology supported on X_Z, Y_Z respectively.

Proof. For each $F \in D^b(X)$ we have

$$\Phi(\bigoplus_i \mathcal{H}^i(F)[-i]) \cong \bigoplus_i \Phi(\mathcal{H}^i(F))[-i]$$

as Φ is exact. Consider the pullback diagrams

$$\begin{tikzcd} X_Z \arrow[r, hook, "\bar\iota_X"] \arrow[d, "f_Z"'] & X \arrow[d, "f"] \\ Z \arrow[r, hook, "\iota"'] & S, \end{tikzcd} \qquad \begin{tikzcd} Y_Z \arrow[r, hook, "\bar\iota_Y"] \arrow[d, "g_Z"'] & Y \arrow[d, "g"] \\ Z \arrow[r, hook, "\iota"'] & S. \end{tikzcd}$$

If F is supported on X_Z then we have

$$\begin{aligned} \Phi(\mathcal{H}^i(F)) &\cong \Phi(\mathcal{H}^i(F) \otimes_{\mathcal{O}_X} \bar\iota_{X*}\mathcal{O}_{X_Z}) \\ &\cong \Phi(\mathcal{H}^i(F) \otimes_{\mathcal{O}_X} f^*\iota_*\mathcal{O}_Z) \\ &\cong \Phi(\mathcal{H}^i(F)) \otimes_{\mathcal{O}_Y} g^*\iota_*\mathcal{O}_Z \\ &\cong \Phi(\mathcal{H}^i(F)) \otimes_{\mathcal{O}_Y} \bar\iota_{Y*}\mathcal{O}_{Y_Z}, \end{aligned}$$

where the first, the second, the third, and the forth isomorphisms respectively follow from F being supported on X_Z, the isomorphism $\bar\iota_{X*}f_Z^*\mathcal{O}_Z \cong f^*\iota_*\mathcal{O}_Z$, S-linearity of Φ, and the isomorphism $\bar\iota_{Y*}g_Z^*\mathcal{O}_Z \cong g^*\iota_*\mathcal{O}_Z$. Since Y is smooth over $\mathbb{C}$, replacing it with a quasi-isomorphic object if necessary, we may assume that $\Phi(\mathcal{H}^i(F))$ are perfect. Then the last term becomes $\bar\iota_{Y*}(\Phi(\mathcal{H}^i(F))|_{Y_Z})$ by the projection formula. Note that the functor $\bar\iota_{Y*}\colon \mathrm{coh}(Y_Z) \to \mathrm{coh}(Y)$ of abelian categories is exact as $\bar\iota_Y$ is affine. Hence $\Phi(\mathcal{H}^i(F))$ is supported on Y_Z, which completes the proof. □

Lemma 3.2. *Let $f\colon X \to S, g\colon Y \to S$ be flat morphisms of smooth $\mathbb{C}$-varieties. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then over any closed point $s \in S$ the fibers X_s, Y_s are derived-equivalent.*

Proof. Take an affine open subset $U = \operatorname{Spec} R \subset S$. First, we show that Φ induces an R-linear exact equivalence

$$\Phi_U\colon D^b(X_U) \to D^b(Y_U),$$

where $f_U\colon X_U \to U, g_U\colon Y_U \to U$ denote the base changes to U. For their complements $X_Z = X \setminus X_U, Y_Z = Y \setminus Y_U$ we have

$$\mathrm{coh}(X_U) \simeq \mathrm{coh}(X)/\,\mathrm{coh}_{X_Z}(X),\ \mathrm{coh}(Y_U) \simeq \mathrm{coh}(Y)/\,\mathrm{coh}_{Y_Z}(Y),$$

where the right hand sides denote the quotients by the Serre subcategories

$$\mathrm{coh}_{X_Z}(X) \subset \mathrm{coh}(X),\ \mathrm{coh}_{Y_Z}(X) \subset \mathrm{coh}(Y)$$

of coherent sheaves supported on X_Z, Y_Z respectively. Passing to their derived categories, via [Miy91, Theorem 3.2] we obtain

$$D^b(X_U) \simeq D^b(X)/D^b_{\mathrm{coh}_{X_Z}(X)}(X),\ D^b(Y_U) \simeq D^b(Y)/D^b_{\mathrm{coh}_{Y_Z}(Y)}(Y).$$

Since Y, S are smooth over $\mathbb{C}$, one can apply Lemma 3.1 to obtain the induced equivalence

$$\Phi_U\colon D^b(X_U) \simeq D^b(X)/D^b_{\mathrm{coh}_{X_Z}(X)}(X) \to D^b(Y_U) \simeq D^b(Y)/D^b_{\mathrm{coh}_{Y_Z}(Y)}(Y)$$

by universality of Verdier quotients.

Next, we show that Φ_U induces an equivalence $\Phi_s\colon\ \mathrm{perf}(X_s) \to \mathrm{perf}(Y_s)$ of categories of perfect complexes on the closed fibers X_s, Y_s. Consider dg enhancements $\mathrm{Perf}(X), \mathrm{Perf}(Y)$ of $\mathrm{perf}(X) \simeq D^b(X), \mathrm{perf}(Y) \simeq D^b(Y)$, which are unique up to quasi-equivalence [CS18, Proposition 6.10]. Here, we use the assumption on X to be smooth over $\mathbb{C}$. Since f, g are flat, one can apply [BFN10, Theorem 1.2] and [Töe, Exercise 32] to the pullback diagrams

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X_U \arrow[d] \\ \operatorname{Spec} k(s) \arrow[r] & \operatorname{Spec} R, \end{tikzcd} \qquad \begin{tikzcd} Y_s \arrow[r] \arrow[d] & Y_U \arrow[d] \\ \operatorname{Spec} k(s) \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

to obtain a quasi-equivalence $\mathrm{Perf}(X_s) \to \mathrm{Perf}(Y_s)$. Passing to their homotopy categories, we obtain the induced equivalence

$$\Phi_s\colon\ \mathrm{perf}(X_s) \to \mathrm{perf}(Y_s).$$

Now, the claim follows from [CS18, Proposition 7.4]. □

Proposition 3.3. *Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic 3-folds. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then over any closed point $s \in S$ the fibers X_s, Y_s are isomorphic.*

Proof. By Lemma 3.2 the closed fibers X_s, Y_s are derived-equivalent. Then the claim follows from Lemma 2.4 and [LT17, Theorem 7.4(2)]. □

Remark 3.4. The above proposition holds for flat morphisms of smooth $\mathbb{C}$-varieties whose fibers are reduced Kodaira curves. Under the assumption in the statement, one could obtain the first line of the proof by [Orl97, Section 2], S-linearity and [HLS09, Proposition 2.15].

4. Reconstruction of fibrations

Lemma 4.1. *Let $f\colon X \to S, g\colon Y \to S$ be flat separated morphisms of smooth $\mathbb{C}$-varieties. Then every S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$ induces a $k(S)$-linear exact equivalence $\Phi_{k(S)}\colon D^b(X_\eta) \to D^b(Y_\eta)$ on the generic fibers X_η, Y_η.*

Proof. By the same argument as Lemma 3.2 and [Har77, Cororallry III10.7], we may assume that f, g are smooth over $S = \operatorname{Spec} R$. One can apply [Mora, Theorem 1.1] to obtain $k(S)$-linear exact equivalences

$$D^b(X_\eta) \to D^b(X)/D^b_0(X),\ D^b(Y_\eta) \to D^b(Y)/D^b_0(Y).$$

to the Verdier quotients by the full S-linear triangulated subcategories spanned by complexes with coherent R-torsion cohomology. Since Φ is S-linear, universality of Verdier quotients induces the desired equivalence. $\square$

Corollary 4.2. *Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic 3-folds. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then their generic fibers X_η, Y_η share the Jacobian $J_\eta/k(S)$.*

Proof. The generic fibers X_η, Y_η are derived-equivalent by Lemma 4.1. Now, the claim follows from [AKW17, Lemma 2.4]. $\square$

Lemma 4.3. *Let $f\colon X \to S, g\colon Y \to S$ be smooth elliptic 3-folds with relative Jacobian $\pi\colon J \to S$. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then g is a coprime twisted power of f.*

Proof. Let $\alpha,\beta \in \mathrm{Br}'(J)$ be representatives of $\text{Ш}_S(J_\eta) \cong \mathrm{Br}'(J)/\mathrm{Br}'(S)$ for f, g. There is an étale cover $\{U_i\}$ of S such that to each U_i the base changes f_i, g_i admit sections

$$\sigma_{X,i}\colon U_i \to X_i = X \times_S U_i,\ \sigma_{Y,i}\colon U_i \to Y_i = Y \times_S U_i$$

and α,β give the gluing data for $\{J_i\}$ to yield X, Y respectively [Căl00, Chapter 4.2, 4.4]. The injection $\mathrm{Br}'(J) \to \mathrm{Br}'(J_\eta)$ defined by the base change $J_\eta \to J$ to $k(S)$ sends α,β to $\alpha_\eta,\beta_\eta \in \mathrm{Br}'(J_\eta)$. Then the induced étale cover $\{U_i \times_S k(S)\}$ of $\operatorname{Spec} k(S)$ represents both α_η and β_η.

Since by assumption and Lemma 4.1 the generic fibers X_η, Y_η are derived-equivalent, one can apply [AKW17, Lemma 2.4, Theorem 2.5] to see that Y_η is isomorphic to the moduli space of degree d line bundles on X_η for some $d \in \mathbb{Z}$ coprime to the order $\mathrm{ord}([\alpha])$. Hence both α_η^d and β_η yield g_η via gluing the étale cover $\{J_i \times_{U_i} k(U_i)\}$ of J_η. Since the map $\mathrm{Br}'(J) \to \mathrm{Br}'(J_\eta)$ is injective, both α^d and β yield g via gluing the étale cover $\{J_i\}$ of J. As explained in [Căl00, Section 4.5], the Brauer class α^d represents f^d. Thus g is isomorphic to f^d. $\square$

Remark 4.4. One could have used [AKW17, Theorem 1.5] to obtain $\beta = \alpha^d$, provided the inherited Fourier–Mukai transform $D^b(J_\eta, \alpha_\eta) \simeq D^b(J_\eta, \beta_\eta)$ from $D^b(J, \alpha) \simeq D^b(J, \beta)$ for derived categories of twisted coherent sheaves.

Proposition 4.5. *Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic 3-folds with relative Jacobian $\pi\colon J \to S$. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then g is an almost coprime twisted power of f.*

Proof. Let $\alpha^\circ,\beta^\circ \in \mathrm{Br}'(J^\circ)$ be representatives of $\text{Ш}_{S^\circ}(J_\eta) \cong \mathrm{Br}'(J^\circ)/\mathrm{Br}'(S^\circ)$ for the smooth parts f°, g°. By [Căl00, Theorem 6.5.1] there exist unique extensions $\alpha,\beta \in \mathrm{Br}'_{\mathrm{an}}(J)$ of α°,β°. Let $\bar\alpha,\bar\beta \in \mathrm{Br}'_{\mathrm{an}}(\bar J)$ be their images under the map $\mathrm{Br}'_{\mathrm{an}}(J) \to \mathrm{Br}'_{\mathrm{an}}(\bar J)$ induced by the pullback along any analytic small resolution $\rho\colon \bar J \to J$ of singularities. Take an analytic open cover $\{U_i\}$ of S representing both $\bar\alpha$ and $\bar\beta$ such that each U_i contains at most one node of $\Delta_f = \Delta_g$. Then X_i, Y_i are isomorphic as an analytic space over U_i [Căl00, Theorem 6.4.6]. By the same argument as Lemma 3.2, there is an S°-linear exact equivalence $D^b(X^\circ) \to D^b(Y^\circ)$. One can apply Lemma 4.3 to find some $d \in \mathbb{Z}$ coprime to $\mathrm{ord}([\alpha^\circ])$ such that g° is isomorphic to $f^{\circ d}$. As explained in [Căl00, Section 6.6], the base changes X_i^d is U_i-isomorphic to X_i. $\square$

Remark 4.6. In the above proof, we have not used the assumption on X to be 3-dimensional. However, this is crucial for Lemma 2.4 on the classification of the fibers, which in turn is responsible for the construction of local universal sheaves $\mathscr{U}_i$ on $V_i = X_i \times_{U_i} \bar J_i$ [Căl00, Theorem 6.4.2]. The collection defines the $\mathrm{pr}_2^* \bar\alpha^{-1}$-twisted pseudo-universal sheaf $\mathscr{U}_{\bar\alpha} = (\{\mathscr{U}_i\}, \{\varphi_{ij}\})$ on $X \times_S \bar J$, where $\varphi_{ij}\colon \mathscr{U}_j|_{V_{ij}} \to \mathscr{U}_i|_{V_{ij}}$ denotes an isomorphism on each $V_{ij} = V_i \cap V_j$. Hence without the assumption one might not have the equivalence

$$\Phi_{\iota_{S*}\mathscr{U}_{\bar\alpha}}\colon D^b(X) \to D^b(\bar J, \bar\alpha) \tag{4.1}$$

from [Căl00, Theorem 5.1], where $\iota_S\colon X \times_S \bar{J} \hookrightarrow X \times \bar{J}$ denotes the closed immersion.

Remark 4.7. For the convenience of readers we add an explanation to the above remark. The Brauer class $\bar{\alpha}$ appeared in (4.1) gives the obstruction against the collection $\{\mathscr{U}_{\bar{\alpha}}\}$ to glue to yield a universal sheaf. The discrepancy between the composition $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki}$ and the identity on each $V_{ijk} = V_i \cap V_j \cap V_k$ defines precisely the $\mathrm{pr}_2^* \bar{\alpha}^{-1}$-twisting of $\mathscr{U}_{\bar{\alpha}}$. By [Căl00, Theorem 4.4.1] the image $[\alpha^\circ]$ of α° under the projection $\mathrm{Br}'(J^\circ) \to \mathrm{Br}'(J^\circ)/\,\mathrm{Br}'(S^\circ)$ corresponds to f° via Ogg–Shafarevich theory. Namely, $[\alpha^\circ]$ gives the gluing data for $\{J_i^\circ\}$ to yield X° via the description of Brauer classes as gerbes. Hence up to elements of $\mathrm{Br}'(S^\circ)$ the obstruction $\bar{\alpha}$ can be interpreted geometrically, which was tacitly used to obtain the equivalence (4.1).

5. Calabi–Yau case

Lemma 5.1. *Let $f\colon X \to S$ be a projective morphism of smooth $\mathbb{C}$-varieties with $\omega_X \cong \mathscr{O}_X$ such that over any closed point $s \in S$ the fiber X_s is connected. Then the relative canonical sheaf $\omega_{X/S}$ is invertible and isomorphic to $f^*\omega_S^{-1}$. In particular, we have $f_*\omega_{X/S} \cong \omega_S^{-1}$.*

Proof. Consider the first fundamental exact sequence

$$f^*\Omega_S \xrightarrow{u} \Omega_X \to \Omega_{X/S} \to 0. \tag{5.1}$$

If u is injective, then we obtain $\omega_{X/S} \cong f^*\omega_S^{-1}$, which is invertible. Since X, S are smooth over $\mathbb{C}$, for any $x \in X$ there is an open neighborhood $V \subset X$ on which u can be expressed as a homomorphism $\mathscr{O}_V^{\oplus \dim S} \to \mathscr{O}_V^{\oplus \dim X}$ of free $\mathscr{O}_V$-modules. As X is integral, it suffices to check the injectivity on the generic point. However, (5.1) restricts to yield the short exact sequence

$$0 \to f_U^*\Omega_U \xrightarrow{u|_{X_U}} \Omega_{X_U} \to \Omega_{X_U/U} \to 0.$$

on the smooth part $f_U\colon X_U \to U$ of f for some open subset $U \subset S$ [Har77, Corollary III10.7]. The second claim follows from the projection formula. Indeed, since any closed fiber X_s is connected, by [Har77, Corollary III 12.9] the coherent sheaf $f_*\mathscr{O}_X$ is invertible and isomorphic to $\mathscr{O}_S$. □

Theorem 5.2. *Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic Calabi–Yau 3-folds. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then f, g share the relative Jacobian $\pi\colon J \to S$.*

Proof. Since by Proposition 3.3 any closed fibers X_s, Y_s over $s \in S$ are isomorphic, we have $\Delta_f = \Delta_g$. Let $\pi_X\colon J_X \to S, \pi_Y\colon J_Y \to S$ be the relative Jacobians of f, g. For any analytic small resolutions $\rho_X\colon \bar{J}_X \to J_X, \rho_Y\colon \bar{J}_Y \to J_Y$ of singularities, we denote by $\bar{\pi}_X, \bar{\pi}_Y$ the compositions $\pi_X \circ \rho_X, \pi_Y \circ \rho_Y$ respectively. Take an analytic open cover $\{U_i\}$ of S such that each U_i contains at most one node of $\Delta_f = \Delta_g$ and to each U_i the base changes $f_i, g_i, \bar{\pi}_{X,i}, \bar{\pi}_{Y,i}$ admit sections

$$\sigma_{X,i}\colon U_i \to X_i,\ \sigma_{Y,i}\colon U_i \to Y_i,\ \bar{\sigma}_{X,i}\colon U_i \to \bar{J}_{X,i} = \bar{J}_X \times_S U_i,\ \bar{\sigma}_{Y,i}\colon U_i \to \bar{J}_{Y,i} = \bar{J}_Y \times_S U_i.$$

Applying [Nak87, Theorem 2.1] or [DG94, Theorem 2.3], we obtain birational morphisms

$$\varpi_{X,i}\colon \bar{J}_{X,i} \to W(\mathscr{L}_i, a_i, b_i),\ \varpi_{Y,i}\colon \bar{J}_{Y,i} \to W(\mathscr{M}_i, c_i, d_i),$$

where $W(\mathscr{L}_i, a_i, b_i), W(\mathscr{M}_i, c_i, d_i)$ denote the Weierstrass fibrations associated with line bundles $\mathscr{L}_i, \mathscr{M}_i$ on U_i, global sections a_i, c_i of $\mathscr{L}_i^{\otimes 4}, \mathscr{M}_i^{\otimes 4}$ and b_i, d_i of $\mathscr{L}_i^{\otimes 6}, \mathscr{M}_i^{\otimes 6}$ such that $4a_i^3 + 27b_i^2, 4c_i^3 + 27d_i^2$ are nonzero global sections of $\mathscr{L}_i^{\otimes 12}, \mathscr{M}_i^{\otimes 12}$. Since $\varpi_{X,i}, \varpi_{Y,i}$ are the morphisms which contract all components of fibers not intersecting $\bar{\sigma}_{X,i}(U_i), \bar{\sigma}_{Y,i}(U_i)$ respectively, the fibrations $W(\mathscr{L}_i, a_i, b_i) \to S, W(\mathscr{M}_i, c_i, d_i) \to S$ coincide with the base changes $\pi_{X,i}, \pi_{Y,i}$ of the relative Jacobians to U_i. See also [Căl02, Theorem 6.4.3].

From Lemma 5.1 it follows

$$\mathscr{L}_i \cong \bar{\pi}_{X,i*}\omega_{\bar{J}_{X,i}/U_i} \cong \omega_{U_i}^{-1} \cong \bar{\pi}_{Y,i*}\omega_{\bar{J}_{Y,i}/U_i} \cong \mathscr{M}_i,$$

as $X_i, \bar{J}_{X,i}$ and $Y_i, \bar{J}_{Y,i}$ are isomorphic as an analytic space over U_i. Since by Corollary 4.2 the generic fibers $J_{X,\eta}, J_{Y,\eta}$ are isomorphic, we obtain U_i-isomorphisms

$$W(\mathscr{L}_i, a_i, b_i) \cong W(\mathscr{M}_i, c_i, d_i), \tag{5.2}$$

as two Weierstrass fibrations $W(\mathscr{L}_i, a_i, b_i), W(\mathscr{L}_i, c_i, d_i)$ must coincide whenever they share the generic fiber. The construction of a Weierstrass fibration is functorial. Hence for each $U_{ij} = U_i \times_S U_j$ we have the U_{ij}-isomorphism

$$W(\mathscr{L}_i|_{U_{ij}}, a_i|_{U_{ij}}, b_i|_{U_{ij}}) \cong W(\mathscr{L}_i, a_i, b_i)|_{U_{ij}}.$$

Thus (5.2) canonically glue to yield an S-isomorphism $J_X \to J_Y$. □

Corollary 5.3. *Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic Calabi–Yau 3-folds. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then g is an almost coprime twisted power of f.*

Proof. The claim follows immediately from Proposition 4.5 and Theorem 5.2. □

Remark 5.4. It sufficed to assume either X or Y is Calabi–Yau. Without loss of generality we may assume that X is Calabi–Yau. The triviality of the canonical bundle of Y follows from the uniqueness of Serre functors. The vanishing of $H^1(Y, \mathscr{O}_Y)$ follows from $H^2(X, \mathscr{O}_X) = H^2(Y, \mathscr{O}_Y)$ and Serre duality. Note that the S-linear exact equivalence is in particular $\mathbb{C}$-linear exact equivalence and hence naturally isomorphic to the Fourier–Mukai transform Φ_P with $P \in D^b(X \times Y)$ unique up to isomorphism [Orl97, Section 2]. One obtains the induced isometry

$$H^2(X, \mathscr{O}_X) \oplus H^4(X, \mathscr{O}_X) \cong H^2(Y, \mathscr{O}_Y) \oplus H^4(Y, \mathscr{O}_Y)$$

from [Căl00, Corollary 3.1.13, 3.1.14].

Remark 5.5. If X is Calabi–Yau, then $\bar{J}$ is also Calabi–Yau. Again, the triviality of the canonical bundle of $\bar{J}$ follows from the uniqueness of Serre functors. By [CR11, Corollary 3.2.10] the higher direct image $R^j\rho_*\mathscr{O}_{\bar{J}}$ vanishes for each $j > 0$. Hence we obtain $R\bar{\pi}_*\mathscr{O}_{\bar{J}} = \mathscr{O}_S$ and Leray spectral sequence implies $H^1(\bar{J}, \mathscr{O}_{\bar{J}}) = H^1(S, \mathscr{O}_S)$. Since X is Calabi–Yau the base S must be either a rational or Enriques surface by [Gro94, Proposition 2.3]. In both cases $H^1(S, \mathscr{O}_S)$ vanishes. See also [Gra91, Proposition 2.2] for a more general result.

6. Deformations of almost coprime twisted powers

Let $f\colon X \to S, g\colon Y \to S$ be generic elliptic Calabi–Yau 3-folds. Assume that there exists an S-linear exact equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then by [Orl97, Section 2] and S-linearity Φ is naturally isomorphic to a relative Fourier–Mukai transform

$$\bar{\Phi}_{\bar{P}/S} = Rp_{S*}(\bar{P} \otimes q_S^*(-))\colon D^b(X) \to D^b(Y)$$

with kernel $\bar{P} \in \operatorname{perf}(X \times_S Y)$ where $q_S : X \times_S Y \to X, p_S : X \times_S Y \to Y$ denote the projections. Note that we have $\Phi_P = \bar{\Phi}_{\bar{P}/S}$ for the pushforward $P = \tau_{S*}\bar{P}$ along the closed immersion $\tau_S : X \times_S Y \hookrightarrow X \times Y$. By [Mor23, Theorem 1.1] there exists a smooth affine $\mathbb{C}$-variety $\operatorname{Spec} B$ over which we have smooth projective versal deformations $\mathbf{X}, \mathbf{Y}$ of X, Y and a deformation $\mathbf{P} \in \operatorname{perf}(\mathbf{X} \times_B \mathbf{Y})$ of P defining a relative Fourier–Mukai transform

$$\bar{\Phi}_{\mathbf{P}/B} = Rp_{B*}(\mathbf{P} \otimes q_B^*(-))\colon D^b(\mathbf{X}) \to D^b(\mathbf{Y}),$$

where $q_B\colon X \times_S Y \to X, p_B\colon X \times_S Y \to Y$ denote the projections.

Lemma 6.1. *Up to taking étale neighborhood of* $\operatorname{Spec} B$*, also the base* S *and morphisms* f, g *deform to give families of elliptic fibrations* $\mathbf{f}\colon \mathbf{X} \to \mathbf{S}$, $\mathbf{g}\colon \mathbf{Y} \to \mathbf{S}$ *over* $\operatorname{Spec} B$.

Proof. The claim immediately follows from [Wil94, Wil98]. □

Lemma 6.2. *Up to shrinking* $\operatorname{Spec} B$*, over any closed point* $b \in \operatorname{Spec} B$ *the fibers* $\mathbf{f}_b\colon \mathbf{X}_b \to \mathbf{S}_b, \mathbf{g}_b\colon \mathbf{Y}_b \to \mathbf{S}_b$ *are smooth elliptic Calabi–Yau* 3*-folds. In particular, they are generic smooth elliptic Calabi–Yau* 3*-folds.*

Proof. By construction $\mathbf{f}_b, \mathbf{g}_b$ are elliptic Calabi–Yau 3-folds. We check that general fibers of $\mathbf{f}, \mathbf{g}$ satisfy the conditions $(1), \dots, (5)$ in Definition 2.2. By [Har77, Corollary III10.7], after shrinking $\mathbf{S}$, the morphisms $\mathbf{f}, \mathbf{g}$ become smooth. Under the structure morphism, which is flat, the open subset of $\mathbf{S}$ maps to a nonempty open subset of $\operatorname{Spec} B$. Hence general fibers $\mathbf{f}_b, \mathbf{g}_b$ are smooth elliptic fibrations and satisfy all the conditions but (3), which follows from the well known fact that any elliptic Calabi–Yau manifold admits a multisection. □

Remark 6.3. Presumably, if the initial fibers $\mathbf{f}_{b_0}, \mathbf{g}_{b_0}$ are nonsmooth, then over a sufficiently small open neighborhood of $b_0 \in \operatorname{Spec} B$ the fibers $\mathbf{f}_b, \mathbf{g}_b$ are also nonsmooth. Here, we do not pursue this as anyway we need smooth ellitpic fibrations to construct the desired family.

Lemma 6.4. *The kernel* $\mathbf{P} \in \operatorname{perf}(\mathbf{X} \times_B \mathbf{Y})$ *is supported on* $\mathbf{X} \times_{\mathbf{S}} \mathbf{Y}$.

Proof. Let $R \cong \mathbb{C}[\![t_1, \dots, t_{\dim_{\mathbb{C}} \mathrm{H}^1(X, \mathcal{T}_X)}]\!]$ be the formal power series ring which prorepresents the deformation functors $\operatorname{Def}_X, \operatorname{Def}_Y$. By [GD61, Theorem III5.4.5] there exist effectivizations $\mathcal{X}, \mathcal{Y}$ of universal formal families. Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be the filtered inductive system used to algebrize $\mathcal{X}, \mathcal{Y}$. It is a compatible system of finitely generated $\mathbb{C}[t_1, \dots, t_{\dim_{\mathbb{C}} \mathrm{H}^1(X, \mathcal{T}_X)}]$-subalgebras of R whose colimit is R. Let $\mathcal{X}_{R_\lambda}, \mathcal{Y}_{R_\lambda}$ be the R_λ-deformations of X, Y used to algebrize $\mathcal{X}, \mathcal{Y}$. Their pullbacks along the canonical homomorphism $R_\lambda \to R$ are isomorphic to $\mathcal{X}, \mathcal{Y}$. Then $\mathbf{X}, \mathbf{Y}$ are the pullbacks of $\mathcal{X}_{R_\lambda}, \mathcal{Y}_{R_\lambda}$ along some homomorphism $R_\lambda \to B$. In summary, we have the commutative diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, hook] \arrow[d] & \mathcal{X} \times_S \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{X}_{R_\lambda} \times_{S_{R_\lambda}} \mathcal{Y}_{R_\lambda} \arrow[d] & \mathbf{X} \times_{\mathbf{S}} \mathbf{Y} \arrow[l] \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r, hook] & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} R_\lambda & \operatorname{Spec} B \arrow[l] \\
X \times Y \arrow[r, hook] \arrow[u] & \mathcal{X} \times_R \mathcal{Y} \arrow[r] \arrow[u] & \mathcal{X}_{R_\lambda} \times_{R_\lambda} \mathcal{Y}_{R_\lambda} \arrow[u] & \mathbf{X} \times_B \mathbf{Y}. \arrow[l] \arrow[u]
\end{tikzcd}$$

Note that the upper vertical arrows are flat projective, while the lower vertical arrows are smooth projective for sufficiently large $\lambda \in \Lambda$.

By [Lie06, Proposition 3.6.1] there exists an effectivization $\mathcal{P} \in \operatorname{perf}(\mathcal{X} \times_R \mathcal{Y})$ of a formal R-deformation of P. Let $\mathcal{P}_{R_\lambda} \in \operatorname{perf}(\mathcal{X}_{R_\lambda} \times_{R_\lambda} \mathcal{Y}_{R_\lambda})$ be the perfect complex used to algebrize $\mathcal{P}$ [Lie06, Proposition 2.2.1]. Its derived pullback $\mathcal{P}_{R_\lambda} \otimes^L_{R_\lambda} R$ is isomorphic to $\mathcal{P}$. Then $\mathbf{P}$ is the derived pullback of $\mathcal{P}_{R_\lambda}$ along the homomorphism $R_\lambda \to B$ used to algebrize $\mathcal{X}, \mathcal{Y}$. Regarding $X \times Y$ as a closed subscheme of $\mathcal{X} \times_R \mathcal{Y}$, by [Huy06, Lemma 3.29] we have

$$\operatorname{supp}(\mathcal{P}) \cap (X \times Y) = \operatorname{supp}(\mathcal{P}|_{X \times Y}) = \operatorname{supp}(P) = X \times_S Y \tag{6.1}$$

and $\operatorname{supp}(\mathcal{P}) \subset \mathcal{X} \times_R \mathcal{Y}$ is a proper closed subset. Since the structure morphism $\mathcal{X} \times_R \mathcal{Y} \to \operatorname{Spec} R$ is flat proper, it sends $\operatorname{supp}(\mathcal{P})$ to the closed point which implies $\operatorname{supp}(\mathcal{P}) \subset X \times Y$. From (6.1) it follows $\operatorname{supp}(\mathcal{P}) \subset X \times_S Y$. In particular, the restriction $\mathcal{P}|_{\mathcal{U}}$ to $\mathcal{U} = \mathcal{X} \times_R \mathcal{Y} \setminus \mathcal{X} \times_S \mathcal{Y}$ is acyclic. Consider the collection $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of complements $\mathcal{U}_\lambda = \mathcal{X}_{R_\lambda} \times_{R_\lambda} \mathcal{Y}_{R_\lambda} \setminus \mathcal{X}_{R_\lambda} \times_{S_{R_\lambda}} \mathcal{Y}_{R_\lambda}$, which are flat separated R_λ-schemes of finite presentation. For $\lambda' \in \Lambda$ with $\lambda' > \lambda$ we have $\mathcal{U}_\lambda \cong \mathcal{U}_{\lambda'} \times_{R_{\lambda'}} R_\lambda$ by construction. Now, one can apply [Lie06, Proposition 2.2.1] to see that

$\mathcal{P}_\lambda|_{\mathcal{U}_\lambda}$ is acyclic for sufficiently large λ. Thus the restriction $\mathbf{P}|_\mathbf{U}$ to $\mathbf{U} = \mathbf{X}\times_B\mathbf{Y}\setminus\mathbf{X}\times_\mathbf{S}\mathbf{Y}$ becomes acyclic after replacing λ if necessary, which completes the proof. $\square$

Remark 6.5. While the natural projection $\mathrm{supp}(P) \to X$ is surjective [Huy06, Lemma 6.4], by $\mathrm{supp}(\mathcal{P}) \subset X \times Y \subsetneq \mathcal{X} \times_R \mathcal{Y}$ the natural projection $\mathrm{supp}(\mathcal{P}) \to \mathcal{X}$ cannot be surjective. This is not a contradiction, as [Huy06, Lemma 6.4] is a statement for $\mathbb{C}$-varieties. Indeed, the proof does not work in our setting as there is no closed point in the complement of X in $\mathcal{X}$.

Theorem 6.6. *Up to taking étale neighborhood of* Spec B*, there exists $d(b) \in \mathbb{Z}$ for each closed point $b \in$* Spec B *such that $\mathbf{g}_b$ is a coprime $d(b)$-th twisted power of $\mathbf{f}_b$.*

Proof. The claim immediately follows from Corollary 5.3, Lemma 6.2 and Lemma 6.4. $\square$

Corollary 6.7. *If general fibers $\mathbf{X}_b, \mathbf{Y}_b$ are nonisomorphic, then they are nonbirational.*

Proof. By Theorem 6.6 general fibers $\mathbf{f}_b, \mathbf{g}_b$ are smooth and $\mathbf{g}_b$ is isomorphic to $\mathbf{f}_b^{d(b)}$ for some $d(b) \in \mathbb{Z}$. Suppose that the generic fibers $\mathbf{f}_{b,\eta}, \mathbf{f}_{b,\eta}^{d(b)}$ are isomorphic. Then $\mathbf{f}_b, \mathbf{f}_b^{d(b)}$ must be isomorphic. It follows that $\mathbf{X}_b, \mathbf{Y}_b$ is isomorphic, which contradicts the assumption. Hence $\mathbf{X}_{b,\eta}, \mathbf{Y}_{b,\eta}$ are nonisomorphic. By [Sil08, Corollary 2.4.1] they are nonbirational and their function fields $k(\mathbf{X}_{b,\eta}), k(\mathbf{Y}_{b,\eta})$ are nonisomorphic. Then the function fields $k(\mathbf{X}_b), k(\mathbf{Y}_b)$ of $\mathbf{X}_b, \mathbf{Y}_b$ must be nonisomorphic, as they are respectively isomorphic to $k(\mathbf{X}_{b,\eta}), k(\mathbf{Y}_{b,\eta})$. $\square$

References

[AKW17] B. Antieau, D. Krashen, and M. Ward, *Derived categories of torsors for abelian schemes*, Adv. Math. 306(10), 1-23 (2017).

[BFN10] D. Ben-zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. 23(4), 909-966 (2010).

[Căl00] A. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University (2000).

[Căl02] A. Căldăraru, *Derived categories of twisted sheaves on elliptic threefolds*, J. Reine Angew. Math. 544, 161-179 (2002).

[CR11] A. Chatzistamatiou and K. Rülling, *Higher direct images of the structure sheaf in positive characteristic*, Algebra Number Theory, 5(6) 693–775 (2011).

[CS18] A. Canonaco and P. Stellari, *Uniqueness of dg enhancements for the derived category of a Grothendieck category*, J. Eur. Math 20(11), 2607–2641 (2018).

[DG94] I. Dolgachev and M. Gross, *Elliptic three-folds I: Ogg–Shafarevich theory*, J. Alg. Geom. 3, 39-80 (1994).

[Don98] R. Donagi, *Taniguchi lectures on principal bundles on elliptic fibrations* in Integral systems and algebraic geometry (Kobe / Kyoto, 1997), World Science publishing, (1998).

[DP08] R. Donagi and T. Pantev, *Torus fibrations, gerbes, and duality*, Mem. Amer. Math. Soc. 193(901) (2008).

[DP12] R. Donagi and T. Pantev, *Langrands duality for Hitchin systems*, Invent. Math. 189, 653-735 (2012).

[GD61] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: III. Étude cohomologique des faisceaux cohérent, première partie.*, Publ. Math-Paris. 11, 5-167 (1961).

[Gra91] A. Grassi, *On minimal models of elliptic threefolds*, Math. Ann. 290(2), 287-302 (1991).

[Gro94] M. Gross, *Finiteness theorems for elliptic Calabi-Yau threefolds*, Duke. Math. 74(2), 271-299 (1994).

[Har77] R Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, (1977).

[HLS09] D. Hernández Ruipérez, A.C. López Martín and F. Sancho de Salas, *Relative integral functors for singular fibrations and singular partners*, J. Eur. Math. Soc. 11, 597-625 (2009).

[HM02] D. Hernández Ruipérez and J. M. Muñoz Porras, *stable sheaves on elliptic fibrations*, J. Geom. Phys. 43, 163-183 (2002).

[Huy06] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Science Publications, (2006).

[Ino22] D. Inoue, *Calabi–Yau* 3*-folds from projective joins of del Pezzo manifolds*, Adv. Math. 400, https://doi.org/10.1016/j.aim.2022.108243

[KP21] A. Kuznetsov and A. Perry, *Categorical joins*, J. Amer. Math. Soc. 34(2), 505-564 (2021).

[Lie06] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. 15, 175-206 (2006).

[LT17] A.C. López Martín and C. Tejero Prieto, *Derived equivalence and Kodaira fibers*, J. Geom. Phys. 122, 69-79 (2017).
[Miy91] J. Miyachi, *Localization of triangulated categories and derived categories*, J. Algebra. 141(2), 463-483 (1991).
[Mora] H. Morimura, *Categorical generic fiber*, arXiv:2111.00239v3
[Morb] H. Morimura, *Totaro–Vial lemma in F-theory*, arXiv: 2301.12751v4
[Mor23] H. Morimura, *Algebraic deformations and Fourier–Mukai transforms for Calabi–Yau manifolds*, Proc. Amer. Math. Soc. 151(1), 29-43 (2023).
[Nak87] N. Nakayama, *On Weierstrass models*, Algebraic geometry and commutative algebra in honor of Masayoshi Nagata. 405-431 (1991).
[Orl97] D. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. 84, 1361-1381 (1997).
[Sil08] J. H. Silverman, *The Arithmetics of Elliptic curves*, Graduate Texts in Mathematics, 106, Springer-Verlag, (2008).
[SP] The Stacks Project authors, *Stacks project*, https://stacks.math.columbia.edu.
[Töe] B. Töen, *Lectures on DG-categories*, http://www.math.univ-toulouse.fr/˜toen/
[Wil94] P.M.H. Wilson, *The existence of elliptic fibre space structures on Calabi–Yau threefolds*, Math. Ann. 300(4), 693-703 (1994).
[Wil98] P.M.H. Wilson, *The existence of elliptic fibre space structures on Calabi–Yau threefolds, II*, Math. Proc Cambridge Philos. Soc. 123(2), 259-262 (1998).

TOTARO–VIAL LEMMA IN F-THEORY

HAYATO MORIMURA

Abstract. For each pair of elliptic Calabi–Yau 3-folds in [KSS, Table 19], we prove that they are $\mathbb{P}^2$-linear derived-equivalent. Except one self-dual pair, each yields two families of smooth elliptic fibrations over a common base whose general fibers are nonbirational derived-equivalent.

1. Introduction

Among Calabi–Yau manifolds, there are two classes of considerable interest for both algebraic geometers and string theorists. One consists of *Fourier–Mukai partners*, pairs of nonbirational derived-equivalent Calabi–Yau 3-folds. The other consists of *elliptic Calabi–Yau manifolds*, those which admit elliptic fibrations.

Recently, Knapp–Scheidegger–Schimannek constructed 12 pairs of Calabi–Yau 3-folds admitting elliptic fibrations over $\mathbb{P}^2$ with 5-section. Although not smooth, they are flat and have no multiple fibers. Moreover, all reducible fibers are isolated and of type I_2. The idea was to consider fiberwise homological projective duality [Kuz06] for Grassmannian $\mathrm{Gr}(2, V_5)$ of 2-planes in $V_5 \cong \mathbb{C}^5$ and its dual $\mathrm{Gr}(2, V_5^\vee)$ with respective Plücker embeddings into $\mathbb{P}(\wedge^2 V_5)$ and $\mathbb{P}(\wedge^2 V_5^\vee)$.

Definition 1.1. Let $A_i, B_i, i = 1, \dots, 12$ be one of the 12 pairs of elliptic Calabi–Yau 3-folds over $\mathbb{P}^2$ labeled as i_a, i_b in [KSS, Table 19]. We call A_i, B_i *type i KSS varieties*. We denote by f_i, g_i the elliptic fibrations $A_i \to \mathbb{P}^2, B_i \to \mathbb{P}^2$ induced by the canonical projections.

Based on F-theoretical observations, Knapp–Scheidegger–Schimannek raised

Conjecture 1.2 ([KSS]). *The elliptic fibrations f_i, g_i share the relative Jacobian $\pi_i \colon J_i \to \mathbb{P}^2$.*

By construction it is natural to ask

Conjecture 1.3 ([KSS]). *Type i KSS varieties A_i, B_i are derived-equivalent.*

For $i = 11$ the statement is trivial, as A_{11}, B_{11} are isomorphic. For $i = 1, 2$ the statement follows from [KSS, Remarks 2.3.3, 2.4.3] and [Ino22, Proposition 3.5]. Explicitly, Y_2, Y_1 and X_2, X_1 in [Ino22] which admit elliptic fibrations over $\mathbb{P}^2$ are respectively isomorphic to A_1, A_2 and B_1, B_2.

In this paper, we give affirmative answers to these conjectures by proving

Theorem 1.4. *The elliptic fibrations f_i, g_i are mutually an* almost coprime twisted power *of the other in the sense of* [Mor, Definition 2.14].

In particular, the smooth parts f_i°, g_i° of f_i, g_i are respectively isomorphic to the relative moduli spaces of stable sheaves of rank 1 and degree k, l on the fibers of g_i°, f_i°. Here, $k, l \in \mathbb{Z}$ are respectively coprime to the fiber degree of g_i°, f_i°. Then one can apply [Căl02, Theorem 5.1, 6.1] to obtain a $\mathbb{P}^2$-linear Fourier–Mukai transform.

Realizing KSS varieties as two different geometric phases of non-abelian gauged linear sigma models, they also raised

SISSA, via Bonomea 265, 34136 Trieste, Italy
E-mail address: hmorimur@sissa.it.
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Conjecture 1.5 ([KSS]). *For $i \neq 11$ type i KSS varieties A_i, B_i are nonbirational.*

For $i = 1, 2$ the statement follows from [Ino22, Theorem 3.6]. If f_i, g_i were smooth, by Theorem 1.4 and [KSS, Table 19] the morphism g_i would be a nonisomorphic *coprime twisted power* of f_i in the sense of [Mor, Definition 2.14], Then the argument in the proof of [Căl07, Proposition 2.2] implies their generic fibers being nonisomorphic. Here, we obtain nonbirational pairs through deformation method based on [Mor23, Theorem 1.1].

Corollary 1.6. *Let $\mathbf{f}_i\colon \mathbf{A}_i \to \mathbf{S}, \mathbf{g}_i\colon \mathbf{B}_i \to \mathbf{S}$ be the deformations of f_i, g_i over a smooth affine $\mathbb{C}$-variety* $\operatorname{Spec} T$ *from* [Mor, Theorem 6.5]. *Their general fibers $\mathbf{A}_{i,t}, \mathbf{B}_{i,t}$ are nonbirational.*

By [Mor, Lemma 4.3] general fibers $\mathbf{f}_{i,t}, \mathbf{g}_{i,t}$ are mutually a coprime twisted power of the other. As far as we know, combined with [KSS], this provides the first systematic construction of multiple pairs of (familes of) Fourier–Mukai partners. Our arguments should work also for elliptic fibrations with higher multisections.

Some computations in [KSS] were carried out assuming

Conjecture 1.7 ([KSS]). *The Tate–Shafarevich group $\text{Ш}_{\mathbb{P}^2}(J_{i,\eta})$ is isomorphic to $\mathbb{Z}_5$.*

If f_i, g_i were smooth, one could adapt the argument as in [DG94, Example 1.18] to prove the conjecture, after checking either $\mathrm{Br}'(A_i)$ or $\mathrm{Br}'(B_i)$ vanishes. Here, we could only identify the Tate–Shafarevich group associated with the above deformations.

Proposition 1.8. *Let $\pi_i(b)\colon J_i(b) \to \mathbf{S}_b$ be the relative Jacobian of general fibers $\mathbf{f}_{i,t}, \mathbf{g}_{i,t}$. Then we have $\text{Ш}_{\mathbf{S}_b}(J(b)_{i,\eta}) \cong \mathbb{Z}_{\delta'(b)_{i,\eta}}$, where $\delta'(b)_{i,\eta}$ is the minimal positive degree of an element of* $\operatorname{Pic}((\mathbf{A}_{i,t})_{\bar{\eta}})^{G_{\overline{k(S)}/k(S)}}$ *dividing the positive generator $\delta(b)_{i,\eta}$ of the image of the degree map* $\operatorname{Pic}((\mathbf{A}_{i,t})_{\eta}) \to \mathbb{Z}$ *from* [DG94, Definition 1.6].

Toward Theorem 1.4 one needs to prove that the generic fibers $A_{i,\eta}, B_{i,\eta}$ of f_i, g_i share the Jacobian $J_{i,\eta}/k(\mathbb{P}^2)$. In our setting, this follows from [AKW17, Lemma 2.4], as the fiberwise homological projective duality induces the derived equivalence

$$D^b(A_{i,\eta}) \simeq D^b(B_{i,\eta}).$$

Similarly, one obtains derived equivalences of general fibers of f_i, g_i, which implies their being isomorphic [LT17, Theorem 7.4]. Then one might seek to invoke instead [DG94, Lemma 5.5]. Alternatively, one would try to apply [Căl00, Proposition 4.2.2], since A_i, B_i are Calabi–Yau.

However, their proofs seem incomplete. The former proof ended showing $A_{i,\eta}, B_{i,\eta}$ to be twists of the geometric generic fiber $J_{A_i,\bar{\eta}} \cong A_{i,\bar{\eta}} \cong B_{i,\bar{\eta}} \cong J_{B_i,\bar{\eta}}$. In general, not all twists come from a torsors [Sil08, Proposition 5.3]. Namely, the $\bar{K}$-isomorphism $J_{A_i,\eta} \to J_{B_i,\eta}$ might be any isomorphism fixing the origin. The latter proof ended showing the relative Jacobians to be isomorphic to minimal Weierstrass models $W(\mathscr{L}, a, b), W(\mathscr{L}, a', b')$, which become isomorphic over the base if and only if their generic fibers are isomorphic. See also Remark 4.2.

Originally, we also aimed to improve the above situation, which might be useful for studying broader classes of elliptic fibrations, exploiting the following lemma found by Totaro and proved by Vial.

Lemma 1.9 ([Via13, Lemma 2.1]). *Let Ω be a universal domain and $f\colon X \to S$ a morphism of Ω-varieties. Then there exists an intersection $U \subset S$ of countably many nonempty Zariski open subsets, for each closed point s of which one finds an isomorphism $\varphi_s\colon \overline{k(S)} \to \Omega$ satisfying $X_s \cong X_{\bar{\eta}} \times_{\overline{k(S)}} \varphi_s(\overline{k(S)})$. In particular, any very general fiber X_s is isomorphic to $X_{\bar{\eta}}$ as a scheme.*

Although rather surprising, this lemma seems not to be known so widely. Despite our failure to achieve the additional goal in earlier version of this paper, we believe that it is worthwhile to share the following question.

Question 1.10. Let $f\colon X \to S, g\colon Y \to S$ be elliptic fibrations without section between $\mathbb{C}$-varieties. Assume that their very general fibers are isomorphic. Then under which additional conditions the generic fibers X_η, Y_η of f, g share the Jacobian $J_\eta/k(S)$?

Theorem 1.4 holds also for *type* $1,2$ *Inoue varieties* X_1, Y_1 and X_2, Y_2, the Fourier–Mukai partners admitting elliptic fibrations with 5-section constructed by Inoue [Ino22]. In particular, we give a partial affirmative answer to the question raised in [Ino22, Remark 2.12]. This will be completed if the remaining *type* 3 *Inoue varieties* X_3, Y_3 have only irreducible fibers except isolated of type I_2, which presumably can be checked by the same method as in [KSS, Section 5, 6]. Note that Y_3 is isomorphic to the first in [KSS, Miscellaneous examples] and admits only one elliptic fibration over $\mathbb{P}^1 \times \mathbb{P}^1$ [Ino22, Remark 3.13].

By construction the derived equivalence

$$\Phi_{\mathcal{P}_j}\colon D^b(X_j) \xrightarrow{\sim} D^b(Y_j),\ \mathcal{P}_j \in D^b(X_j \times Y_j),\ j = 1,2,3$$

of X_j, Y_j follows from homological projective duality for categorical joins developed in [KP21]. If $\Phi_{\mathcal{P}_j}$ are S_j-linear, then by [Mor, Corollary 4.3] we would obtain an alternative proof of Theorem 1.4 for Inoue varieties. Hence it is interesting to see whether the Fourier–Mukai kernels $\mathcal{P}_j$ are supported on the fiber products $X_j \times_{S_j} Y_j$ so that $\Phi_{\mathcal{P}_j}$ become S_j-linear.

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2. Review on Ogg–Shafarevich theory over the complex number field

2.1. The Weil–Châtelet group.

Definition 2.1 ([Sil08, Section X.2])**.** Let E be an elliptic curve over a filed K of characteristic 0. The *isomorphism group* $\mathrm{Isom}(E)$ of E is the group of $\bar{K}$-isomorphism from E to E. The *automorphism group* $\mathrm{Aut}(E)$ of E is the $G_{\bar{K}/K}$-invariant subgroup of $\mathrm{Isom}(E)$ whose elements preserve the origin of E. We use the same symbol E to denote the elliptic curve and its *translation group*, the $G_{\bar{K}/K}$-invariant subgroup of $\mathrm{Isom}(E)$ of translations. For a point $p \in E$ we denote by τ_p the corresponding translation.

Lemma 2.2 ([Sil08, Proposition X5.1])**.** *There is a bijection of pointed sets*

$$E \times \mathrm{Aut}(E) \to \mathrm{Isom}(E),\ (p, \alpha) \mapsto \tau_p \circ \alpha,$$

identifying $\mathrm{Isom}(E)$ *with the product* $E \times \mathrm{Aut}(E)$ *twisted by the natural action of* $\mathrm{Aut}(E)$ *on* E.

Definition 2.3 ([Sil08, Section X.2])**.** A *twist* of E/K is a smooth curve C/K which is isomorphic to E/K over $\bar{K}$. Two twists $C/K, C'/K$ are *equivalent* if they are K-isomorphic. We denote by $\mathrm{Twist}(E/K)$ the set of equivalence classes of twists of E/K.

Lemma 2.4 ([Sil08, Theorem X2.2])**.** *There is a canonical bijection of pointed sets*

$$H^1_{ét}(G_{\bar{K}/K}, \mathrm{Isom}(E)) \to \mathrm{Twist}(E/K).$$

Definition 2.5 ([Sil08, Section X.3])**.** A *torsor* or *principal homogeneous space* for E/K is a smooth curve C/K together with a simply transitive algebraic group action of E on C defined over K. It is *trivial* if $C(K) \neq \emptyset$. Two torsors $C/K, C'/K$ are *equivalent* if there is a K-isomorphism $\theta\colon C \to C'$ compatible with the E-action. The *Weil–Châtelet group* $\mathrm{WC}(E/K)$ is the set of equivalence classes of principal homogeneous spaces for E/K.

Theorem 2.6 ([Sil08, Theorem X3.6]). *There is a canonical bijection of pointed sets*

$$\mathrm{WC}(E/K) \to H^1_{ét}(G_{\bar{K}/K}, E).$$

In particular, the image of $H^1_{ét}(G_{\bar{K}/K}, E)$ under the inclusion induced by $E \subset \mathrm{Isom}(E)$ gives a natural group structure to $\mathrm{WC}(E/K) \subset \mathrm{Twist}(E/K)$.

Theorem 2.7 ([Sil08, Proposition X5.3]). *The inclusion* $\mathrm{Aut}(E) \subset \mathrm{Isom}(E)$ *induces*

$$H^1_{ét}(G_{\bar{K}/K}, \mathrm{Aut}(E)) \subset H^1_{ét}(G_{\bar{K}/K}, \mathrm{Isom}(E)).$$

Let $\mathrm{Twist}((E, O)/K)$ *be the image of* $H^1_{ét}(G_{\bar{K}/K}, \mathrm{Aut}(E))$ *regarded as a subset of* $\mathrm{Twist}(E/K)$. *If* $C/K \in \mathrm{Twist}((E, O)/K)$ *then* $C(K) \neq \emptyset$. *Conversely, if E'/K is an elliptic curve isomorphic to E/K over $\bar{K}$, then E'/K represents an element of* $\mathrm{Twist}((E, O)/K)$.

Remark 2.8. In general, $C/K \in \mathrm{Twist}((E, O)/K)$ is not K-isomorphic to E/K. By [Sil08, Proposition 5.4] the group $\mathrm{Twist}((E, O)/K)$ is canonically isomorphic to $K^*/(K^*)^n$ where n becomes equal to $2, 4$ or 6 depending on the j-invariant of E/K. Hence the elements of $K^*/(K^*)^n$ correspond to the twists of E/K which do not come from principal homogeneous spaces.

2.2. **The Tate–Shafarevich group.**

Definition 2.9 ([DG94, Section 1]). Let S be a normal integral excellent scheme. We denote by $\eta = \mathrm{Spec}\, k(S)$ its generic point. Let E be an elliptic curve over $k(S)$. The *Tate–Shafarevich group* $\text{Ш}_S(E)$ is the subset of $\mathrm{WC}(E/k(S))$ of equivalence classes of étale locally trivial principal homogeneous spaces for $E/k(S)$.

Definition 2.10 ([Căl00, Definition 1.1.3, 1.1.7]). Let X be a scheme. The *cohomological Brauer group* $\mathrm{Br}'(X)$ of X is defined as $H^2_{\text{ét}}(X, \mathscr{O}^*_X)$. The *Brauer group* $\mathrm{Br}(X)$ of X is the group of isomorphism classes of Azumaya algebras on X modulo equivalence relation. Here, the group structure is given by tensor products. Two Azumaya algebras $\mathscr{A}, \mathscr{A}'$ on X are *equivalent* if there exists a locally free sheaf $\mathscr{E}$ satisfying

$$\mathscr{A} \otimes \underline{\mathrm{End}}(\mathscr{E}) \cong \mathscr{A}' \otimes \underline{\mathrm{End}}(\mathscr{E}).$$

Theorem 2.11 ([Căl00, Theorem 1.1.4]). *For a smooth $\mathbb{C}$-scheme X the cohomological Brauer group* $\mathrm{Br}'(X)$ *is torsion. Let X^h be the associated analytic space. Then we have*

$$\mathrm{Br}'(X) = \mathrm{Br}'_{an}(X^h) = H^2_{an}(X^h, \mathscr{O}^*_X) = H^2_{an}(X^h, \mathscr{O}^*_X)_{tors}.$$

Theorem 2.12 ([SP, Tag 0A2J], [Jon]). *If X is quasicompact or connected, then* $\mathrm{Br}(X)$ *is torsion and there is a canonical injection* $\mathrm{Br}(X) \to \mathrm{Br}'(X)$. *If X is quasiprojective over $\mathbb{C}$, then* $\mathrm{Br}(X)$ *surjects onto* $\mathrm{Br}'(X)_{tors}$. *In particular, if X is smooth quasiprojective over $\mathbb{C}$, then we have*

$$\mathrm{Br}(X) = \mathrm{Br}'(X) = H^2(X, \mathscr{O}^*_X)_{tors}.$$

Theorem 2.13 ([DG94, Corollary 1.17]). *Let $f\colon X \to S$ be a flat elliptic fibration of smooth $\mathbb{C}$-varieties with section whose generic fiber is isomorphic to E. Let $S^{(1)}$ be the set of points in S with* $\dim \mathscr{O}_{S,s} = 1$. *If the fiber X_s is geometrically integral over each $s \in S^{(1)}$, then we have*

$$\text{Ш}_S(E) \cong \mathrm{Coker}(\mathrm{Br}'(S) \to \mathrm{Br}'(X))$$

where the map $\mathrm{Br}'(S) \to \mathrm{Br}'(X)$ *is given by the pullback.*

2.3. **Minimal Weierstrass fibrations.**

Definition 2.14 ([DG94, Definition 2.1], [Căl00, Definition 6.1.5])**.** An *elliptic fibration* $f\colon X \to S$ is a projective morphism of $\mathbb{C}$-schemes whose generic fiber X_η is a genus one regular $k(S)$-curve and all fibers are geometrically connected. The *discriminant locus* Δ_f of f is the closed subset of points $s \in S$ over which the fiber X_s is not regular. A fiber X_s over a point $s \in S$ is *multiple* if f is not smooth at any $x \in X_s$. A *section* of f is a morphism $\sigma_X\colon S \to X$ satisfying $f \circ \sigma_X = \mathrm{id}$. An *n-section* of f is a closed subscheme to which the restriction of f becomes a finite morphism of degree n.

Remark 2.15. Each component of a multiple fiber must be either of dimension more than one or of dimension one and nonreduced at all points. If an elliptic fibration $f\colon X \to S$ has no multiple fibers, then étale locally it admits a section.

Example 2.16. Let $\mathscr{L}$ be a line bundle on S. Take global sections a of $\mathscr{L}^{\otimes 4}$ and b of $\mathscr{L}^{\otimes 6}$ such that $4a^3 + 27b^2$ is a nonzero global section of $\mathscr{L}^{\otimes 12}$. Consider the projective bundle $\mathbf{P}_{\mathscr{L}} = \mathbb{P}_S(\mathscr{O}_S \oplus \mathscr{L}^{\otimes -2} \oplus \mathscr{L}^{\otimes -3})$. We denote by $\mathscr{O}_{\mathbf{P}_{\mathscr{L}}/S}(1)$ the line bundle corresponding to the relative hyperplane class. Let $W(\mathscr{L}, a, b) \subset \mathbf{P}_{\mathscr{L}}$ be the closed subscheme defined by the equation $Y^2Z = X^3 + aXZ^2 + bZ^3$, where X, Y and Z are respectively given by the global sections of $\mathscr{O}_{\mathbf{P}_{\mathscr{L}}/S}(1) \otimes \mathscr{L}^{\otimes 2}$, $\mathscr{O}_{\mathbf{P}_{\mathscr{L}}/S}(1) \otimes \mathscr{L}^{\otimes 3}$ and $\mathscr{O}_{\mathbf{P}_{\mathscr{L}}/S}(1)$ which corresponds to the natural injections of $\mathscr{L}^{\otimes -2}$, $\mathscr{L}^{\otimes -3}$ and $\mathscr{O}_S$ into $\mathscr{O}_S \oplus \mathscr{L}^{\otimes -2} \oplus \mathscr{L}^{\otimes -3}$. The canonical projection induces a flat elliptic fibration $\pi_{\mathscr{L}}\colon W(\mathscr{L}, a, b) \to S$ admitting a section $\sigma_{\mathscr{L}}\colon S \to W(\mathscr{L}, a, b)$, called a *Weierstrass fibration*.

Remark 2.17. All fibers of $\pi_{\mathscr{L}}$ are irreducible plane cubic curves. The discriminant locus of $\pi_{\mathscr{L}}$ is the support of the Cartier divisor defined by $4a^3 + 27b^2$. The construction of a Weierstrass fibrations is functorial. Namely, we have

$$W(h^*\mathscr{L}, h^*(a), h^*(b)) \cong W(\mathscr{L}, a, b) \times_S S'$$

for any morphism $h\colon S' \to S$ of $\mathbb{C}$-schemes. If S is smooth, then $\sigma_{\mathscr{L}}(S)$ lies in the smooth locus of $W(\mathscr{L}, a, b)$.

Lemma 2.18 ([Nak87, Theorem 2.1], [DG94, Theorem 2.3])**.** *Let $f\colon X \to S$ be an elliptic fibration of smooth $\mathbb{C}$-varieties admitting a section $\sigma_X\colon S \to X$. Then there exists a birational S-morphism from X to $W(\mathscr{L}, a, b)$ contracting all components of fibers which do not intersect $\sigma_X(S)$. Moreover, $\mathscr{L}$ is isomorphic to all of $\sigma_X^*(\Omega_{X/S})$, $f_*\omega_{X/S}$, $(R^1 f_*\mathscr{O}_X)^{-1}$ and $\mathscr{O}_{\sigma_X(S)}(-\sigma_X(S))$ when they are invertible.*

Lemma 2.19 ([DG94, Proposition 2.4])**.** *Let E be an elliptic curve over K with a rational point $\xi \in E(K)$. For any smooth $\mathbb{C}$-scheme S with $k(S) \cong K$ there exists a Weierstrass fibration $\pi_{\mathscr{L}}\colon W(\mathscr{L}, a, b) \to S$ whose generic fiber is isomorphic to E. The closure of ξ is $\sigma_{\mathscr{L}}(S)$.*

Definition 2.20 ([DG94, Definition 2.6])**.** A Weierstrass fibration $W(\mathscr{L}, a, b)$ is *minimal* if there is no effective divisor D such that $\mathrm{div}(a) \geq 4D, \mathrm{div}(b) \geq 6D$.

Remark 2.21. Every Weierstrass fibration is birational to a minimal Weierstrass fibration [DG94, Proposition 2.5]. Its discriminant locus is not smaller than that of the minimal Weierstrass fibration.

Definition 2.22 ([DG94, Definition 2.11])**.** A projective morphism $f\colon X \to S$ is *relatively minimal* if X is $\mathbb{Q}$-factorial and has only terminal singularities, and if $C \subset X$ is any irreducible curve mapping to a point in S, then $K_X.C \geq 0$ for the canonical divisor K_X of X.

Lemma 2.23 ([DG94, Proposition 2.16])**.** *Let $f\colon X \to S$ be a relatively minimal elliptic fibration admitting a section. Assume that $f_*\omega_{X/S}$ is invertible. Then $W(f_*\omega_{X/S}, a, b)$ from Lemma 2.18 gives a minimal Weierstrass fibration.*

Lemma 2.24 ([DG94, Proposition 2.17]). *Let $f\colon X \to S$ be a relatively minimal elliptic fibration and $\pi_{\mathscr{L}}\colon W(\mathscr{L}, a, b) \to S$ a minimal Weierstrass fibration whose generic fiber is isomorphic to E. If the Jacobian of X_η is E, then the discriminant loci $\Delta_f, \Delta_{\pi_{\mathscr{L}}}$ coincide.*

2.4. **The relative Jacobian.**

Definition 2.25 ([Căl00, Definition 4.1.1, 4.2.1, 4.5.1]). A *smooth elliptic fibration* $f\colon X \to S$ is a smooth projective morphism of smooth $\mathbb{C}$-schemes whose fiber X_s over any point $s \in S$ is a genus one regular $k(s)$-curve. Fix a relatively ample line bundle $\mathscr{O}_{X/S}(1)$ for a smooth elliptic fibration $f\colon X \to S$. The *relative Jacobian* $\pi\colon J \to S$ is the relative moduli space of stable sheaves of rank 1, degree 0 on the fibers of f. The *k-th twisted power* $f^k\colon X^k \to S$ of f is the relative moduli space of stable sheaves of rank 1, degree $k \in \mathbb{Z}$ on the fibers of f.

Theorem 2.26 ([Căl00, Theorem 4.5.2]). *Fix a relatively ample line bundle $\mathscr{O}_{X/S}(1)$ for a smooth elliptic fibration $f\colon X \to S$. Any k-th twisted power $f^k\colon X^k \to S$ of f is a smooth elliptic fibration which has the same relative Jacobian $\pi\colon J \to S$ as f. If $\alpha \in Ш_S(J_\eta)$ is the element representing f, then f^k is represented by α^k.*

Definition 2.27 ([Căl00, Definition 6.4.1]). Let $f\colon X \to S$ be a flat elliptic fibration of $\mathbb{C}$-varieties. Fix a relatively ample line bundle $\mathscr{O}_{X/S}(1)$ of f and a closed point $s \in S$. Let P be the Hilbert polynomial of $\mathscr{O}_{X_s}$ on X_s with respect to the polarization given by $\mathscr{O}_{X/S}(1)|_{X_s}$. Consider the relative moduli space $M_{X/S}(P) \to S$ of semistable sheaves of Hilbert polynomial P on the fibers of f. By the universal property of $M_{X/S}(P) \to S$ there exists a natural section $S \to M_{X/S}(P)$ which sends s to the point $[\mathscr{O}_{X_s}]$ representing $\mathscr{O}_{X_s}$. Let J be the unique component of M which contains the image of this section. The *relative Jacobian* of f is the restriction $\pi\colon J \to S$ of the morphism $M_{X/S}(P) \to S$ to J.

Remark 2.28. The relative Jacobian $\pi\colon J \to S$ is a flat elliptic fibration with section whose discriminant locus Δ_π equals Δ_f. The restriction over the complement $S \setminus \Delta_f$ coincides with the relative Jacobian for a smooth elliptic fibration. Similarly, the restriction over $S \setminus \Delta_f$ of f^d coincides with the d-th twisted power of the smooth part of f.

Definition 2.29 ([Căl00, Notation 6.6.3]). Let $f\colon X \to S$ be a flat elliptic fibration with n-section without multiple fibers. Assume that all reducible fibers of f are isolated and of type I_2. Fix a relatively ample line bundle $\mathscr{O}_{X/S}(1)$. Let $M^k_{X/S}(P) \to S$ be the relative moduli space of semistable sheaves of rank 1, degree k on the fibers of f. Let X^k be the union of the components of $M^k_{X/S}(P)$ which contains a point corresponding to a stable line bundle on a fiber of f. The *k-th twisted power* of f is the restriction $f^k\colon X^k \to S$ of the morphism $M^k_{X/S}(P) \to S$ to X^k.

3. Review on KSS varieties

3.1. **Grassmannian side.** Let $\mathscr{F}^\vee$ be a globally generated vector bundle of rank 5 on $\mathbb{P}^2$ and $\mathbf{G} = \mathrm{Gr}_{\mathbb{P}^2}(2, \mathscr{F})$ the Grassmannian bundle whose fiber over any point $x \in \mathbb{P}^2$ is the Grassmannian $\mathrm{Gr}(2, \mathrm{tot}(\mathscr{F})_x)$ of 2-planes in the $k(x)$-vector space $\mathrm{tot}(\mathscr{F})_x$. We denote by $\mathscr{O}_{\mathbf{G}/\mathbb{P}^2}(1)$ the line bundle corresponding to the relative hyperplane class and by $\pi_{\mathbf{G}}$ the canonical projection. Let $\mathscr{E}^\vee$ be a globally generated homogeneous vector bundle of rank 5 on $\mathbf{G}$ and $s \in H^0(\mathbf{G}, \mathscr{E}^\vee)$ a general section. By the generalized Bertini theorem, the zero locus $A = Z(s)$ is a smooth projective 3-fold. If in addition $\omega_{\mathbf{G}} \cong \det^{-1} \mathscr{E}^\vee$ then ω_A becomes trivial. Setting $\mathscr{F}^\vee = F$ and $\mathscr{E}^\vee = \mathscr{O}_{\mathbf{G}/\mathbb{P}^2}(1) \otimes \pi^*_{\mathbf{G}} E'$ for F, E' in [KSS, Table 2], one obtains Calabi–Yau 3-folds A. We will put subscript i on $\mathscr{F}^\vee, \mathscr{E}^\vee, F, E', \mathbf{G}$ and A to specify which row we are dealing with.

Lemma 3.1. *The 3-fold A_i is Calabi–Yau in the strict sense, i.e., we have $H^1(A_i, \mathscr{O}_{A_i}) = 0$ in addition to $\omega_{A_i} \cong \mathscr{O}_{A_i}$.*

Proof. Concatenating Koszul resolution of the ideal sheaf $\mathfrak{I}_{A_i}$ of A_i and the short exact sequence $0 \to \mathfrak{I}_{A_i} \to \mathscr{O}_{\mathbf{G}_i} \to \mathscr{O}_{A_i} \to 0$, we obtain an exact sequence

$$0 \to \wedge^5 \mathscr{E}_i \to \wedge^4 \mathscr{E}_i \to \cdots \to \mathscr{E}_i \to \mathscr{O}_{\mathbf{G}_i} \to \mathscr{O}_{A_i} \to 0.$$

Due to the spectral sequences

$$H^q(\mathbf{G}_i, \wedge^p \mathscr{E}_i) \Rightarrow H^{q-p}(A_i, \mathscr{O}_{A_i}),$$

it suffices to show the vanishing of $H^{p+1}(\mathbf{G}_i, \wedge^p \mathscr{E}_i)$ for $0 \leq p \leq 5$, which follows from Leray spectral sequence

$$H^s(\mathbb{P}^2, \wedge^p E_i'^{\vee} \otimes R^r \pi_{\mathbf{G}_i *} \mathscr{O}_{\mathbf{G}_i/\mathbb{P}^2}(-p)) \Rightarrow H^{r+s}(\mathbf{G}_i, \wedge^p \mathscr{E}_i).$$

□

3.2. **Pfaffian side.** Let $\mathscr{E}^{\vee}$ be a globally generated vector bundle of rank 5 on $\mathbb{P}^2$ and $\mathbf{P} = \mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$ the projective bundle. We denote by $\mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(1)$ the line bundle corresponding to the relative hyperplane class and by $\pi_{\mathbf{P}}$ the canonical projection. Let $\mathscr{F}^{\vee}$ be a globally generated vector bundle of rank 5 on $\mathbf{P}$ and $\phi\colon \mathscr{F} \to \mathscr{F}^{\vee} \otimes \mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(1)$ a skew-symmetric morphism corresponding to $s_{\phi} \in H^0(\mathbf{P}, \wedge^2 \mathscr{F}^{\vee} \otimes \mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(1))$. We denote by B the first nontrivial degeneracy locus

$$D_3(\phi) = \{x \in \mathbf{P} | \operatorname{rank} \phi(x) \leq 3\} = \{x \in \mathbf{P} | \operatorname{rank} \phi(x) \leq 2\} = D_2(\phi)$$

of ϕ. Since $\operatorname{Pic}(\mathbf{P})$ has no torsion, one can apply the first lemma in [Oko94, Section 3] to obtain an exact sequence

$$0 \to \det \mathscr{F}^{\vee} \otimes \mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(-2) \xrightarrow{(\frac{1}{2}\wedge^2\phi)^T} \mathscr{F} \xrightarrow{\phi} \mathscr{F}^{\vee} \otimes \mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(1) \xrightarrow{\frac{1}{2}\wedge^2\phi} \mathfrak{I}_B \otimes \det \mathscr{F}^{\vee} \otimes \mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(3) \to 0,$$

where $\mathfrak{I}_B$ denotes the ideal sheaf of B. From the first proposition in [Oko94, Section 3], it follows that B is a smooth projective 3-fold, as $\wedge^2 \mathscr{F}^{\vee} \otimes \mathscr{O}_{\mathbf{P}/\mathbb{P}^2}(1)$ is globally generated. If in addition $(\det \mathscr{F})^{\otimes 2} \cong (\det \mathscr{E})(3)$ then ω_B becomes trivial by the second lemma in [Oko94, Section 3]. Setting $\mathscr{E}^{\vee} = E'$ and $\mathscr{F}^{\vee} = \pi_{\mathbf{P}}^* F$ for E', F in [KSS, Table 2], one obtains Calabi–Yau 3-folds B. We will put subscript i on $\mathscr{E}^{\vee}, \mathscr{F}^{\vee}, E', F, \mathbf{P}$ and B to specify which row we are dealing with.

Lemma 3.2. *The* 3*-fold* B_i *is Calabi–Yau in the strict sense, i.e., we have* $H^1(B_i, \mathscr{O}_{B_i}) = 0$ *in addition to* $\omega_{B_i} \cong \mathscr{O}_{B_i}$.

Proof. Concatenating the locally free resolution of $\mathfrak{I}_{B_i}$ from the second lemma in [Oko94, Section 3] and the short exact sequence $0 \to \mathfrak{I}_{B_i} \to \mathscr{O}_{\mathbf{P}_i} \to \mathscr{O}_{B_i} \to 0$, we obtain

$$0 \to \mathscr{L}_{0,i} \to \mathscr{F}_{0,i} \to \mathscr{F}_{0,i}^{\vee} \otimes \mathscr{L}_{0,i} \to \mathscr{O}_{\mathbf{P}_i} \to \mathscr{O}_{B_i} \to 0$$

where $\mathscr{L}_{0,i} = (\det \mathscr{F}_i)^{\otimes 2} \otimes \mathscr{O}_{\mathbf{P}_i/\mathbb{P}^2}(-5)$ and $\mathscr{F}_{0,i} = \mathscr{F}_i \otimes \det \mathscr{F}_i \otimes \mathscr{O}_{\mathbf{P}_i/\mathbb{P}^2}(-3)$. The vanishing of the first and the second cohomology of $\mathscr{L}_{0,i}, \mathscr{F}_{0,i}, \mathscr{F}_{0,i}^{\vee} \otimes \mathscr{L}_{0,i}$ and $\mathscr{O}_{\mathbf{P}_i}$ follows from Leray spectral sequence. □

3.3. **Elliptic fibrations over** $\mathbb{P}^2$**.** According to [KSS, Section 2.3, 2.4], one can apply the main theorem in [Ogu93] to see that $\pi_{\mathbf{G}_i}, \pi_{\mathbf{P}_i}$ respectively restrict to elliptic fibrations $f_i\colon A_i \to \mathbb{P}^2, g_i\colon B_i \to \mathbb{P}^2$ with 5-sections. They are flat and have no multiple fibers. Moreover, all reducible fibers of f_i, g_i are isolated and of type I_2 [KSS, Section 5, 6].

Remark 3.3. The existence of type I_2 fibers [KSS, Section 5, 6] implies that the morphisms f_i, g_i are not smooth. In [KSS] the authors called f_i, g_i smooth genus one fibrations, apparently because A_i, B_i are smooth. If this is the case, then smoothness follows automatically from the above constructions, despite the comment on usage of Higgs transitions in [KSS, Introduction].

Lemma 3.4. *The generic fibers of f_i, g_i are derived-equivalent. In particular, they share the Jacobian $J_i/k(\mathbb{P}^2)$.*

Proof. The fiber of f_i over a point $x \in \mathbb{P}^2$ is given by

$$\mathrm{Gr}(2, V_5) \times_{\mathbb{P}(\wedge^2 V_5)} \mathbb{P}(\mathrm{tot}(E_i'^{\vee})_x^{\perp})$$

with identifications

$$\mathrm{tot}(F_i^{\vee})_x \cong V_5,\ \mathrm{tot}(\mathscr{O}_{\mathbf{G}_i/\mathbb{P}^2}(1))_x \otimes \pi^*_{\mathbf{G}_i} \mathrm{tot}(E_i'^{\vee})_x^{\perp} \cong \mathrm{tot}(E_i'^{\vee})_x^{\perp},$$

where $\mathrm{tot}(E_i'^{\vee})_x^{\perp} \subset \wedge^2 \mathrm{tot}(F_i^{\vee})_x$ denotes the orthogonal subspace to a fixed inclusion $\mathrm{tot}(E_i'^{\vee})_x \subset \wedge^2 \mathrm{tot}(F_i)_x$. Observe from the explicit description of $s \in H^0(\mathbf{G}_i, E_i)$ as in [KSS, Section 2.3] that s defines a 5-dimensional quotient $k(x)$-vector space of $\wedge^2 \mathrm{tot}(F_i^{\vee})_x$ whose complement is $\mathrm{tot}(E_i'^{\vee})_x^{\perp}$. Then the fiber of g_i over general $x \in \mathbb{P}^2$ is given by

$$\mathrm{Gr}(2, V_5^{\vee}) \times_{\mathbb{P}(\wedge^2 V_5^{\vee})} \mathbb{P}(\mathrm{tot}(E_i'^{\vee})_x).$$

Note that the subvariety of $\mathbb{P}(\wedge^2 V_5^{\vee})$ defined by the 4×4 Pfaffians of a general 5×5 skew-symmetric matrix is isomorphic to $\mathrm{Gr}(2, V_5^{\vee})$. One can apply [Kuz07, Theorem 1.1] to obtain a derived equivalence of the generic fibers. See [KP21, Theorem 2.24] for the same statement over more general base. Now, the claim follows from [AKW17, Lemma 2.4]. □

Remark 3.5. In [KSS, Section 2.5] the authors claimed that the above fiberwise orthogonal description globalizes to that of A_i and B_i. For their global description, one needs $E_i'^{\vee}$ to be a subbundle of $\wedge^2 F_i$ up to twisting by line bundles. However, each F_i is a direct sum of line bundles on $\mathbb{P}^2$ and a subbundle of any line bundle is either 0 or itself. Then most of $E_i'^{\vee}$ in [KSS, Table 2] cannot be a subbundle of $\wedge^2 F_i$ no matter how twisted.

Corollary 3.6. *Over any closed point $x \in \mathbb{P}^2$ the fibers of f_i, g_i are isomorphic.*

Proof. Using Lemma 3.4, we will obtain $\mathbb{P}^2$-linear Fourier–Mukai transforms $D^b(A_i) \to D^b(B_i)$ in Section 5. Then the claim follows from [Mor, Proposition 3.3]. □

4. Common relative Jacobian

4.1. A sufficient condition.

Proposition 4.1. *Let $f\colon X \to S, g\colon Y \to S$ be flat elliptic fibrations between smooth $\mathbb{C}$-varieties without multiple fibers. Assume that the following conditions hold:*

(1) *The generic fibers of f, g share the Jacobian $J_\eta/k(S)$.*
(2) *There exist resolutions of singularities*

$$\rho_X\colon \bar{J}_X \to J_X,\ \rho_Y\colon \bar{J}_Y \to J_Y$$

such that $\bar{\pi}_X = \pi_X \circ \rho_X, \bar{\pi}_Y = \pi_Y \circ \rho_Y$ give relatively minimal elliptic fibrations and $\bar{\pi}_{X*}\omega_{\bar{J}_X/S}, \bar{\pi}_{Y*}\omega_{\bar{J}_Y/S}$ are isomorphic invertible sheaves.

Then f, g share the relative Jacobian $\pi\colon J \to S$.

Proof. The condition (2) tells us that $\bar{\pi}_X$ gives a relatively minimal elliptic fibration admitting a section. Since in addition $\bar{\pi}_{X*}\omega_{\bar{J}_X/S}$ is invertible, one can apply Lemma 2.23 to see that the Weierstrass fibration $W(\bar{\pi}_{X*}\omega_{\bar{J}_X/S}, a, b) \to S$ from Lemma 2.18 is minimal. Similarly, one obtains another minimal Weierstrass fibration $W(\bar{\pi}_{Y*}\omega_{\bar{J}_Y/S}, a', b') \to S$ associated with π_Y. Moreover, there is an S-isomorphism

$$W(\bar{\pi}_{Y*}\omega_{\bar{J}_Y/S}, a', b') \cong W(\bar{\pi}_{X*}\omega_{\bar{J}_X/S}, a', b').$$

It is well known that two Weierstrass fibrations $W(\bar{\pi}_{X*}\omega_{\bar{J}_X/S}, a, b), W(\bar{\pi}_{X*}\omega_{\bar{J}_X/S}, a', b')$ with isomorphic generic fibers must coincide. Now, the claim follows from the condition (1). □

Remark 4.2. In earlier version of this paper, the condition (1) required only very general fibers to be isomorphic. However, the following example informed by an anonymous referee implies that our original proof was wrong. This example also implies that [DG94, Lemma 5.5] cannot be true. Consider any elliptic fibration $f\colon X \to S$. Suppose that S has a nontrivial double covering $T \to S$. Let Y be the quotient of $X_T = X \times_S T$ by $\mathbb{Z}_2$, where the action is given by involution on T and negation on the fibers. The generic fiber of $g\colon Y \to S$, so called *quadratic twist*, is not isomorphic to that of f. On the other hand, over any closed point $s \in S$ the fibers of f, g are isomorphic.

4.2. **Answer to Conjecture 1.2.**

Corollary 4.3. *Type i KSS varieties A_i, B_i share the relative Jacobian $\pi_i\colon J_i \to \mathbb{P}^2$.*

Proof. We check that f_i, g_i satisfy the conditions in Proposition 4.1. The condition (1) follows from Lemma 3.4. As for the condition (2), take any analytic small resolutions of singularities ρ_{A_i}, ρ_{B_i}. One can show that $\bar{J}_{A_i}, \bar{J}_{B_i}$ are analytic Calabi–Yau 3-folds in the strict sense and we have $\bar{\pi}_{A_i *}\omega_{\bar{J}_{A_i}/\mathbb{P}^2} \cong \bar{\pi}_{B_i *}\omega_{\bar{J}_{B_i}/\mathbb{P}^2} \cong \omega_{\mathbb{P}^2}^{-1}$. This follows for instance from the same argument as in [Mor, Section 4] based on [Căl00, Section 6], which we will briefly review in Section 5. □

5. Derived equivalence

Remark 5.1. In this section, we invoke some results from [Căl00]. As mentioned above, the proof of [Căl00, Proposition 4.2.2] seems incomplete. However, it is used only once in the proof of [Căl00, Theorem 4.5.2] to show that the k-th twisted power f^k of a smooth elliptic fibration $f\colon X \to S$ has the same relative Jacobian as f. There the usage of [Căl00, Proposition 4.2.2] is not crucial, as the claim immediately follows from [Căl00, Proposition 4.2.3]. Moreover, the author explicitly constructed this S-isomorphism $J_X \to J_{X^k}$ in terms of the cut-and-reglue procedure in [Căl00, Section 4.5].

5.1. **The proof of Theorem 1.4.**

Lemma 5.2. *For any analytic small resolution of singularities*

$$\rho_{A_i}\colon \bar{J}_{A_i} \to J_{A_i},\ \rho_{B_i}\colon \bar{J}_{B_i} \to J_{B_i}$$

of the relative Jacobians of type i KSS varieties A_i, B_i, there exists an analytic open cover $\{U_j\}$ of $\mathbb{P}^2$ such that $A_{i,U_j} = A_i \times_{\mathbb{P}^2} U_j, B_{i,U_j} = B_i \times_{\mathbb{P}^2} U_j$ are respectively isomorphic to $\bar{J}_{A_i,U_j} = \bar{J}_{A_i} \times_{\mathbb{P}^2} U_j, \bar{J}_{B_i,U_j} = \bar{J}_{B_i} \times_{\mathbb{P}^2} U_j$ as an analytic space over U_j.

Proof. This is a straightforward adaptation of [Căl00, Theorem 6.4.6]. Since f_i, g_i have no multiple fibers, analytic locally they admit sections [Căl00, Theorem 6.1.8]. Moreover, as all their reducible fibers are isolated and of type I_2, over sufficiently small analytic open subset U_j there is at most one type I_2 fiber of f_i, g_i. For each component $C \cong \mathbb{P}^1$ of type I_2 fiber, the normal bundles $\mathscr{N}_{C/A_i}, \mathscr{N}_{C/B_i}$ are isomorphic to $\mathscr{O}_C(-1) \oplus \mathscr{O}_C(-1)$ [Căl00, Theorem 6.1.9]. It is well known that contraction of such a curve as C in a 3-fold yields an ordinary double point. Hence ρ_{A_i}, ρ_{B_i} resolve the ordinary double points.

By [Căl00, Proposition 6.4.2, Theorem 6.4.3] there exist sheaves $\mathscr{U}_{A_i,U_j}, \mathscr{U}_{B_i,U_j}$ on $A_{i,U_j} \times_{U_j} A_{i,U_j}, B_{i,U_j} \times_{U_j} B_{i,U_j}$ flat over the second factors, which by universality of $J_{A_i,U_j}/U_j, J_{B_i,U_j}/U_j$ induce surjective morphisms $A_{i,U_j} \to J_{A_i,U_j}, B_{i,U_j} \to J_{B_i,U_j}$. These are at most contraction of one of the two components of type I_2 fiber which does not intersect the local section. Note that such morphisms coincide with the morphisms from Lemma 2.18. Hence $A_{i,U_j} \to J_{i,U_j}, B_{i,U_j} \to J_{i,U_j}$ might differ respectively from $\rho_{A_i,U_j}\colon \bar{J}_{A_i,U_j} \to J_{A_i,U_j}, \rho_{B_i,U_j}\colon \bar{J}_{B_i,U_j} \to J_{B_i,U_j}$ up to whether the contracted component intersect the local sections. Switching components amounts to performing flops to $A_{i,U_j} \to J_{i,U_j}, B_{i,U_j} \to J_{i,U_j}$. □

Remark 5.3. As we use some results in this section to prove Corollary 4.3, here we do not assume that f_i, g_i share the relative Jacobian $\pi_i \colon J_i \to \mathbb{P}^2$.

Remark 5.4. The minimal Weierstrass fibrations for f_i, g_i coincide with their relative Jacobians.

Lemma 5.5 ([Căl00, Theorem 3.3.2])**.** *Let $f \colon X \to S$ be a flat projective morphism. Fix a relatively ample line bundle $\mathcal{O}_{X/S}(1)$. Let $M_{X/S}(P) \to S$ be the relative moduli space of semistable sheaves with a fixed Hilbert polynomial P on the fibers of f. Then there exists a local universal sheaf $\mathcal{U}_j$ on each $X \times_S U_j$ for some open cover $\{U_j\}$ of $M_{X/S}(P)$. Moreover, there exist an element $\alpha \in \mathrm{Br}'(M_{X/S}(P))$ and isomorphisms $\varphi_{jk} \colon \mathcal{U}_k|_{U_j \times_S U_k} \to \mathcal{U}_j|_{U_j \times_S U_k}$ which make $(\{\mathcal{U}_j\}, \{\varphi_{jk}\})$ into a $\mathrm{pr}_2^* \alpha$-twisted sheaf. The element α, called the* obstruction *to the existence of a universal sheaf on $X \times_S M_{X/S}(P)$, depends only on f, $\mathcal{O}_{X/S}(1)$ and P.*

Lemma 5.6 ([Căl00, Theorem 3.3.4])**.** *Let $X \to S, M \to S$ be morphisms between proper $\mathbb{C}$-schemes. Assume that $X \to S$ is projective and M is integral. Assume further that for $\alpha \in \mathrm{Br}'(M)$ there exists a coherent $\mathrm{pr}_2^* \alpha$-twisted sheaf $\mathcal{U}$ on $X \times_S M$ which is flat over M. Then α belongs to $\mathrm{Br}(M)$.*

Let $V_i = \mathbb{P}^2 \setminus \Delta_{f_i}, W_i = \mathbb{P}^2 \setminus \Delta_{g_i}$ be the complements of the discriminant loci of f_i, g_i. By Theorem 2.13 and $\mathrm{Br}'(\mathbb{P}^2) = 0$ the smooth elliptic fibrations f_{i,V_i}, g_{i,W_i} represent some elements $\alpha_i \in \mathrm{Br}'(J_{A_i,V_i}), \beta_i \in \mathrm{Br}'(J_{B_i,W_i})$. Moreover, α_i, β_i respectively coincide with the obstructions to the existence of a universal sheaf on $A_{i,V_i} \times_{V_i} J_{A_i,V_i}, B_{i,W_i} \times_{V_i} J_{B_i,W_i}$.

Lemma 5.7. *There exist unique extensions*

$$\alpha'_i \in H^2_{an}(J_{A_i}, \mathcal{O}^*_{J_{A_i}}),\ \beta'_i \in H^2_{an}(J_{B_i}, \mathcal{O}^*_{J_{B_i}})$$

of α_i, β_i. For any analytic small resolutions of singularities ρ_{A_i}, ρ_{B_i}, let $\bar{\alpha}_i = \rho^*_{A_i}\alpha'_i, \bar{\beta}_i = \rho^*_{B_i}\beta'_i$. Then $\bar{\alpha}_i, \bar{\beta}_i$ respectively belong to $\mathrm{Br}(\bar{J}_{A_i}), \mathrm{Br}(\bar{J}_{B_i})$ and there exist $\mathrm{pr}_2^* \bar{\alpha}_i, \mathrm{pr}_2^* \bar{\beta}_i$-twisted sheaves $\bar{\mathcal{U}}_{A_i}, \bar{\mathcal{U}}_{B_i}$ on $A_i \times_{\mathbb{P}^2} \bar{J}_{A_i}, B_i \times_{\mathbb{P}^2} \bar{J}_{B_i}$ whose restrictions over V_i, W_i are isomorphic to the $\mathrm{pr}_2^* \alpha_i, \mathrm{pr}_2^* \beta_i$-twisted sheaves from Lemma 5.5.

Proof. This is a straightforward adaptation of [Căl00, Theorem 6.5.1]. Let

$$\varphi_j \colon A_{i,U_j} \to \bar{J}_{A_i,U_j},\ \psi_j \colon B_{i,U_j} \to \bar{J}_{B_i,U_j}$$

be isomorphisms of analytic spaces from Lemma 5.2. As explained above, there exist sheaves $\mathcal{U}_{A_i,U_j}, \mathcal{U}_{B_i,U_j}$ on $A_{i,U_j} \times_{U_j} A_{i,U_j}, B_{i,U_j} \times_{U_j} B_{i,U_j}$ flat over the second factors. We write $\bar{\mathcal{U}}_{A_i,U_j}, \bar{\mathcal{U}}_{B_i,U_j}$ for their pullbacks by $\mathrm{id} \times_{U_j} \varphi_j^{-1}, \mathrm{id} \times_{U_j} \psi_j^{-1}$. Restricted over the intersection with V_i, W_i, the pullbacks $\bar{\mathcal{U}}_{A_i,U_j}, \bar{\mathcal{U}}_{B_i,U_j}$ become local universal sheaves. The collections $\{\bar{\mathcal{U}}_{A_i,U_j}\}, \{\bar{\mathcal{U}}_{B_i,U_j}\}$ together with isomorphisms on double intersections form $\mathrm{pr}_2^* \alpha_i, \mathrm{pr}_2^* \beta_i$-twisted sheaves. Since by construction any double intersections

$$\bar{J}_{A_i,U_j} \cap \bar{J}_{A_i,U_k} = J_{A_i,U_j} \cap J_{A_i,U_k},\ \bar{J}_{B_i,U_j} \cap \bar{J}_{B_i,U_k} = J_{B_i,U_j} \cap J_{B_i,U_k}$$

for $j \neq k$ do not contain type I_2 fibers, all the claim but $\bar{\alpha}_i \in \mathrm{Br}(\bar{J}_{A_i}), \bar{\beta}_i \in \mathrm{Br}(\bar{J}_{B_i})$, which follows from Lemma 5.6, are obvious. □

Proposition 5.8 ([Căl02, Theorem 5.1])**.** *The relative integral transforms*

$$\bar{\Phi}_{\bar{\mathcal{U}}_{A_i}} \colon D^b(A_i) \to D^b(\bar{J}_{A_i}, \bar{\alpha}_i),\ \bar{\Phi}_{\bar{\mathcal{U}}_{B_i}} \colon D^b(B_i) \to D^b(\bar{J}_{B_i}, \bar{\beta}_i)$$

with kernel $\bar{\mathcal{U}}_{A_i}, \bar{\mathcal{U}}_{B_i}$ are equivalences.

Remark 5.9. By Proposition 5.8 and uniqueness of Serre functors, the canonical bundle of $\bar{J}_{A_i}, \bar{J}_{B_i}$ are trivial. Moreover, $\bar{J}_{A_i}, \bar{J}_{B_i}$ are analytic Calabi–Yau 3-folds in the strict sense [Mor, Remark 4.5]. From Lemma 5.2 and the same argument as in [Mor, Theorem 4.2] it follows $\bar{\pi}_{A_i *}\omega_{\bar{J}_{A_i}/\mathbb{P}^2} \cong \bar{\pi}_{B_i *}\omega_{\bar{J}_{B_i}/\mathbb{P}^2} \cong \omega_{\mathbb{P}^2}^{-1}$. Since our argument in this section is independent of that in Section 4, so is the proof of Corollary 4.3 and we conclude that π_{A_i}, π_{B_i} coincide.

Now, one can prove Theorem 1.4 as follows. By Corollary 4.3 the restrictions of f_i, g_i over $V_i = \mathbb{P}^2 \setminus \Delta_{\pi_i}$ represents some elements $\alpha_i, \beta_i \in \mathrm{Br}(J_{i,V_i})$. We use the same symbol to denote their images under the injection $\mathrm{Br}'(J_{i,V_i}) \to \mathrm{Br}'(J_{i,\eta})$ induced by the pullback along the canonical morphism $J_{i,\eta} \to J_{i,V_i}$. Since by Lemma 3.4 the generic fibers of f_i, g_i are derived-equivalent, one can apply [AKW17, Lemma 2.4, Theorem 2.5] to obtain $\beta_i = \alpha_i^k$ for some $k \in \mathbb{Z}$ coprime to the order $\mathrm{ord}([\alpha_i])$ in $Ш_{V_i}(J_{i,\eta}) \cong \mathrm{Br}'(J_{i,V_i}) / \mathrm{Br}'(V_i) = \mathrm{Br}'(J_{i,V_i})$. Then g_{i,V_i} is isomorphic to f^k_{i,V_i} by [Mor, Lemma 4.3] and B_{i,U_j} become isomorphic to A^k_{i,U_j} as an analytic space after refining the cover $\{U_j\}$ by [Căl00, Theorem 6.4.6].

5.2. Answer to Conjecture 1.3.

Corollary 5.10. *There exists a $\mathbb{P}^2$-linear Fourier–Mukai transform $\Phi_i \colon D^b(A_i) \xrightarrow{\sim} D^b(B_i)$.*

Proof. By Theorem 1.4 we may assume that g_{i,V_i} is isomorphic to f^k_{i,V_i}. Then g_{i,V_i} represents $\alpha_i^k \in \mathrm{Br}'(J_{i,V_i})$. Applying Proposition 5.8, we obtain $\mathbb{P}^2$-linear equivalences

$$D^b(A_i) \to D^b(\bar{J}_i, \bar{\alpha}_i),\ D^b(B_i) \to D^b(\bar{J}_i, \bar{\alpha}_i^k).$$

Then the claim follows from the $\mathbb{P}^2$-linear equivalence [Căl02, Theorem 6.1]

$$D^b(\bar{J}_i, \bar{\alpha}_i) \to D^b(\bar{J}_i, \bar{\alpha}_i^k).$$

□

6. Nonbirationality of deformations

6.1. **The proof of Corollary 1.6.** Recall that the deformations $\mathbf{f}_i \colon \mathbf{A}_i \to \mathbf{S}, \mathbf{g}_i \colon \mathbf{B}_i \to \mathbf{S}$ over $\operatorname{Spec} T$ are obtained from Theorem 1.4 and [Mor, Theorem 6.5]. General fibers $\mathbf{f}_{i,t}, \mathbf{g}_{i,t}$ are smooth elliptic fibrations and mutually a coprime twisted power of the other in the sense of [Mor, Definition 2.14]. For $i \neq 11$ general fibers $\mathbf{A}_{i,t}, \mathbf{B}_{i,t}$ are nonisomorphic, as they have distinct Betti numbers b_1 [KSS, Table 19], which depend only on diffeomorphism type. Since by [Har77, Corollary III10.7] the restriction of the flat projective morphisms $\mathbf{f}_i, \mathbf{g}_i$ over some open subset of $\operatorname{Spec} T$ become smooth, one can apply Ehresmann fibration theorem to see that pairs of these distinct Betti numbers b_1 does not depend on general $t \in \operatorname{Spec} T$. Now, the claim follows from [Mor, Corollary 6.6].

7. Toward identification of the Tate–Shafarevich group

7.1. Vanishing of the Brauer group.

Lemma 7.1. *The cohomological Brauer group* $\mathrm{Br}'(A_i)$ *of* A_i *vanishes.*

Proof. Since by Lemma 3.1 the 3-fold A_i is Calabi–Yau in the strict sense, we have $\mathrm{Br}'(A_i) \cong H^3(A_i, \mathbb{Z})_{tors}$. Each generator of the general section $s_i \in H^0(\mathbf{G}_i, \mathscr{E}_i^\vee)$ defines an ample divisor by construction and [Har77, Exercise II7.5(a)]. Applying [Fuj80, Theorem C] iteratively to the divisors, one sees that $H_2(A_i, \mathbb{Z})$ is torsion free. Then the first term in the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_2(A_i, \mathbb{Z}), \mathbb{Z}) \to H^3(A_i, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_3(A_i, \mathbb{Z}), \mathbb{Z}) \to 0$$

from the universal coefficient theorem vanishes. Since $\mathrm{Hom}_{\mathbb{Z}}(H_3(A_i, \mathbb{Z}), \mathbb{Z})$ is always torsion free, we obtain $H^3(A_i, \mathbb{Z})_{tors} = 0$. □

7.2. **Computation of the Tate–Shafarevich group.**

Proposition 7.2. *Let $f\colon X \to S$ be a smooth elliptic fibration between smooth $\mathbb{C}$-varieties without sections. Assume that the following conditions hold:*

(1) $H^3_{ét}(S, \mu_{l,S}) = 0$ *for some* $l \in \mathbb{N}$.
(2) *The number* δ_η *from* [DG94, Definition 1.6] *is prime.*
(3) $\mathrm{Br}'(X) = 0$.

Then the Tate–Shafarevich group $Ш_S(J_{X,\eta})$ *is isomorphic to* $\mathbb{Z}_{\delta_\eta}$.

Proof. This is an adaptation of the arguments in [DG94, Example 1.18]. Since f is flat with X, S smooth over $\mathbb{C}$, from [DG94, Proposition 1.16] we obtain exact sequences

$$\begin{gathered} 0 \to \mathbb{Z}_{\delta'_\eta} \to Ш_S(J_{X,\eta}) \to H^1_{\text{ét}}(S, \iota_*\iota^*R^1f_*\mathscr{O}_X^*) \to 0, \\ H^1_{\text{ét}}(S, R^1f_*\mathscr{O}_X^*) \to H^1_{\text{ét}}(S, \iota_*\iota^*R^1f_*\mathscr{O}_X^*) \to H^2_{\text{ét}}(S, \mathscr{E}). \end{gathered} \tag{7.1}$$

Here, δ'_η is a certain positive integer with $\delta'_\eta | \delta_\eta$ from [DG94, Definition 1.6], $\iota\colon \operatorname{Spec} k(S) \to S$ denotes the canonical morphism and $\mathscr{E}$ is a certain sheaf on S from [DG94, Definition 1.8]. If we have $H^1_{\text{ét}}(S, \iota_*\iota^*R^1f_*\mathscr{O}_X^*) = 0$, then from the condition (2) it follows $Ш_S(J_{X,\eta}) = \mathbb{Z}_{\delta_\eta}$.

Consider the exact sequence

$$\mathrm{Br}'(X) \to H^1_{\text{ét}}(S, R^1f_*\mathscr{O}_X^*) \to H^3_{\text{ét}}(S, \mathscr{O}_S^*)$$

from [DG94, Corollary 1.5], where the first term vanishes by the condition (3). The Kummer sequence $0 \to \mu_{l,S} \to \mathscr{O}_S^* \to \mathscr{O}_S^* \to 0$ induces an exact sequence

$$H^3_{\text{ét}}(S, \mu_{l,S}) \to H^3_{\text{ét}}(S, \mathscr{O}_S^*) \to H^3_{\text{ét}}(S, \mathscr{O}_S^*).$$

The term $H^3_{\text{ét}}(S, \mathscr{O}_S^*)$ is torsion by [DG94, Proposition 1.2]. From the condition (1) it follows $H^1_{\text{ét}}(S, R^1f_*\mathscr{O}_X^*) = H^3_{\text{ét}}(S, \mathscr{O}_S^*) = 0$. Since f is smooth, one can apply [DG94, Proposition 1.12, 1.13] to obtain $H^2_{\text{ét}}(S, \mathscr{E}) = 0$, which implies $H^1_{\text{ét}}(S, \iota_*\iota^*R^1f_*\mathscr{O}_X^*) = 0$ due to the second exact sequence in (7.1). □

7.3. **The proof of Proposition 1.8.** We check that $\mathbf{f}_{i,t}$ satisfies the conditions (1), (3) in Proposition 7.2. Due to [AGV73, Theorem 4.4], when coefficients are finite, étale cohomology of a smooth $\mathbb{C}$-scheme coincides with singular cohomology of its analytification. In particular, we have $H^3_{\text{ét}}(\mathbb{P}^2, \mu_{l,\mathbb{P}^2}) \cong H^3(\mathbb{P}^2, \mathbb{Z}/l\mathbb{Z}) = 0$ for any $l \in \mathbb{N}$ by the universal coefficient theorem. Since the family $\mathbf{S} \to \operatorname{Spec} B$ is smooth proper, one can apply Ehresmann fibration theorem to obtain $H^3_{\text{ét}}(\mathbf{S}_b, \mu_{l,\mathbf{S}_b}) = H^3(\mathbf{S}_b, \mathbb{Z}/l\mathbb{Z}) = H^3(\mathbb{P}^2, \mathbb{Z}/l\mathbb{Z}) = 0$ for any $l \in \mathbb{N}$. From the same argument and Lemma 7.1 it follows $\mathrm{Br}'(\mathbf{A}_{i,t}) = 0$, as we have $\mathrm{Br}'(\mathbf{A}_{i,t}) \cong H^3(\mathbf{A}_{i,t}, \mathbb{Z})_{tors}$.

8. Inoue varieties as almost coprime twisted powers

8.1. **Review on Inoue varieties.** Let $M_1 = \mathrm{Gr}(2, V_5)$ be a Grassmannian of 2-planes in $V_5 \cong \mathbb{C}^5$ and $M_2 = \mathbb{P}_{S_i}(\mathscr{E}_i)$ a rank r_i projective bundle over a del Pezzo surface S_i satisfying the following conditions:

(i) $\mathscr{E}_i^\vee$ is globally generated.
(ii) $\dim_{\mathbb{C}} \varphi_{L_2}(\mathbb{P}_{S_i}(\mathscr{E}_i)) \geq r$ where φ_{L_2} denotes the morphism defined by the line bundle $\mathscr{O}_{M_2/S_i}(L_2)$ corresponding to the relative hyperplane class L_2 of $\pi_{\mathscr{E}_i, S_i}\colon M_2 \to S_i$.
(iii) $\det \mathscr{E}_i \cong \omega_{S_i}$.

We denote by Σ_1 and Σ_2 the image of M_1 and M_2 under the Plücker embedding and the morphism defined by the relative hyperplane class respectively.

Lemma 8.1 ([Ino22, Proposition 3.1]). *Let $\mathbb{P}_{M_1,M_2} = \mathbb{P}_{M_1\times M_2}(\mathscr{O}(-L_1)\oplus\mathscr{O}(-L_2))$ be the resolved join of M_1 and M_2, where L_1 denotes the Schubert divisor class of M_1 and L_2 is the relative hyperplane class of M_2. Then a general complete intersection X of r_i+5 relative hyperplanes in $\mathbb{P}_{M_1,M_2}$ is a Calabi–Yau 3-fold in the strict sense.*

Remark 8.2. The image of $\mathbb{P}_{M_1,M_2}$ under the morphism φ_H coincides with the projective join $\mathrm{Join}(\Sigma_1,\Sigma_2)$ of Σ_1 and Σ_2, where H denotes the relative hyperplane class of $\pi_{M_1,M_2}\colon \mathbb{P}_{M_1,M_2}\to M_1\times M_2$. In general, $\mathrm{Join}(\Sigma_1,\Sigma_2)$ is singular along the disjoint union $\Sigma_1\sqcup\Sigma_2$. The morphism φ_H gives a resolution of $\mathrm{Join}(\Sigma_1,\Sigma_2)$. In particular, the restriction of φ_H to any enough general complete intersection X becomes an isomorphism.

Let $\mathscr{E}_i^{\perp}$ be the orthogonal locally free sheaf of $\mathscr{E}_i$. Namely, we have a short exact sequence

$$0\to\mathscr{E}_i^{\perp}\to H^0(S_i,\mathscr{E}_i^{\vee})\otimes\mathscr{O}_{S_i}\to\mathscr{E}_i^{\vee}\to 0.$$

We denote by r_i' and L_2' the rank of $\mathscr{E}_i^{\perp}$ and the relative hyperplane class of $\pi_{\mathscr{E}_i^{\perp},S_i}\colon \mathbb{P}_{S_i}(\mathscr{E}_i^{\perp})\to S_i$ respectively. Assume the following additional conditions:

(iv) $\dim_{\varphi_{L_2'}}(\mathbb{P}_{S_i}(\mathscr{E}_i^{\perp}))\geq r'$.

(v) $H^1(S_i,\mathscr{E}_i)=0$.

Then $(\mathscr{E}_i^{\perp})^{\vee}$ is globally generated and $\det\mathscr{E}_i^{\perp}\cong\omega_{S_i}$.

Corollary 8.3. *Let $\mathbb{P}_{M_1',M_2'}=\mathbb{P}_{M_1'\times M_2'}(\mathscr{O}(-L_1')\oplus\mathscr{O}(-L_2'))$ be the resolved join of $M_1'=\mathrm{Gr}(2,V_5^{\vee})$ and $M_2'=\mathbb{P}_S(\mathscr{E}_i^{\perp})$, where L_1' denotes the Schubert divisor class of M_1' and L_2' is the relative hyperplane class of M_2'. Then a general complete intersection Y of $r_i'+5$ relative hyperplanes in $\mathbb{P}_{M_1',M_2'}$ is a Calabi–Yau 3-fold in the strict sense.*

Consider the cases where $M_2=N_i=\mathbb{P}_{S_i}(\mathscr{E}_i)$, $M_2'=N_i'=\mathbb{P}_{S_i}(\mathscr{E}_i^{\perp})$ are one of the following:

(1) $N_1=\mathbb{P}_{\mathbb{P}^2}(\mathscr{O}_{\mathbb{P}^2}(-1)^{\oplus 3})=\mathbb{P}^2\times\mathbb{P}^2,\ N_1'=\mathbb{P}_{\mathbb{P}^2}(\mathscr{K}_1^{\oplus 3})$,

(2) $N_2=\mathbb{P}_{\mathbb{P}^2}(\mathscr{O}_{\mathbb{P}^2}(-2)\oplus\mathscr{O}_{\mathbb{P}^2}(-1))=\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^3,\ N_2'=\mathbb{P}_{\mathbb{P}^2}(\mathscr{K}_2\oplus\mathscr{K}_1)$,

(3) $N_3=\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)^{\oplus 2})=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1,\ N_3'=\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathscr{K}_{1,1}^{\oplus 2})$.

Here, $\mathscr{K}_{1,1},\mathscr{K}_j$ for $j=1,2$ denote respectively the kernel of the surjections

$$H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))\otimes\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}\to\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1),\ H^0(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(j))\otimes\mathscr{O}_{\mathbb{P}^2}\to\mathscr{O}_{\mathbb{P}^2}(j).$$

We write V_{N_i} for $H^0(N_i,\mathscr{O}(L_2))^{\vee}$. Let $W_i\subset\wedge^2V_5\oplus V_{N_i}$ be general codimension r_i+5 linear subspaces and $W_i^{\perp}\subset\wedge^2V_5^{\vee}\oplus V_{N_i}^{\vee}$ their orthogonal subspaces. By Lemma 8.1 and Corollary 8.3 the complete intersections

$$X_i=\mathbb{P}_{M_1,N_i}\times_{\mathbb{P}(\wedge^2V_5\oplus V_{N_i})}\mathbb{P}(W_i),\ Y_i=\mathbb{P}_{M_1',N_i'}\times_{\mathbb{P}(\wedge^2V_5^{\vee}\oplus V_{N_i}^{\vee})}\mathbb{P}(W_i^{\perp})$$

of $r_i+5, r_i'+5$ relative hyperplanes in $\mathbb{P}_{M_1,N_i},\mathbb{P}_{M_1',N_i'}$ are Calabi–Yau 3-folds in the strict sense. For $i=1,2,3$ we call X_i,Y_i *type i Inoue varieties*.

Theorem 8.4 ([Ino22, Proposition 3.5, Theorem 3.6]). *Type i Inoue varieties X_i,Y_i are nonbirational derived-equivalent.*

8.2. **Elliptic fibrations of Inoue varieties.** For the rest of the paper, we discuss an alternative proof of the derived equivalence of X_i,Y_i via Theorem 1.4. Consider the compositions

$$\varpi_{X_1}\colon X_1\hookrightarrow\mathbb{P}_{M_1,N_1}\to M_1\times N_1\to N_1=\mathbb{P}^2\times\mathbb{P}^2\xrightarrow{\mathrm{pr}_1}\mathbb{P}^2,$$

$$\varpi_{X_2}\colon X_2\hookrightarrow\mathbb{P}_{M_2,N_2}\to M_2\times N_2\to N_2=\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_2)\xrightarrow{\pi_{\mathscr{E}_2,\mathbb{P}^2}}\mathbb{P}^2,$$

$$\varpi_{X_3}\colon X_3\hookrightarrow\mathbb{P}_{M_3,N_3}\to M_3\times N_3\to N_3=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\xrightarrow{\mathrm{pr}_{1,2}}\mathbb{P}^1\times\mathbb{P}^1$$

which are shown to be elliptic fibrations [Ino22, Lemma 3.7]. Similarly, Y_i admit elliptic fibrations ϖ_{Y_i} over $\mathbb{P}^2$ for $i=1,2$ and $\mathbb{P}^1\times\mathbb{P}^1$ for $i=3$ [Ino22, Lemma 3.11, Remark 3.13].

Lemma 8.5. *The generic fibers of $\varpi_{X_i}, \varpi_{Y_i}$ for $i = 1, 2, 3$ are derived-equivalent. In particular, they share the Jacobian $J_i'/k(\mathbb{P}^2)$.*

Proof. The fibers of $\varpi_{X_1}, \varpi_{Y_1}$ over a point $x \in \mathbb{P}^2$ are given by

$$\mathrm{Join}(\mathrm{Gr}(2, V_5), \{x\} \times \mathbb{P}^2) \cap \mathbb{P}(W),\ \mathrm{Join}(\mathrm{Gr}(2, V_5^\vee), \{x\} \times \mathbb{P}^2) \cap \mathbb{P}(W^\perp).$$

Hence general fibers are linear sections of $\mathrm{Gr}(2, V_5), \mathrm{Gr}(2, V_5^\vee)$ of codimension 5. By definition of $W, W^\perp$, they respectively coincide with

$$\mathrm{Gr}(2, V_5) \times_{\mathbb{P}(\wedge^2 V_5)} \mathbb{P}(W_x),\ \mathrm{Gr}(2, V_5^\vee) \times_{\mathbb{P}(\wedge^2 V_5^\vee)} \mathbb{P}(W_x^\perp)$$

for some 5-dimensional subspace $W_x \in \wedge^2 V_5$ and its orthogonal subspace $W_x^\perp \in \wedge^2 V_5$. We have similar dual descriptions of the fibers also for $i = 2, 3$. Now, the claim follows from the same argument as in Lemma 3.4. □

Corollary 8.6. *Over any closed point $x \in \mathbb{P}^2$ the fibers of $\varpi_{X_i}, \varpi_{Y_i}$ for $i = 1, 2$ are isomorphic.*

Proof. The proof of Corollary 3.6 carries over. According to [KSS, Remarks 2.3.3, 2.4.3] the Calabi–Yau 3-folds Y_1, Y_2 and X_1, X_2 are respectively isomorphic to A_2, A_1 and B_2, B_1. By [Ino22, Lemma 3.11, Remark 3.13] the Calabi–Yau 3-folds X_1, X_2 admit only one elliptic fibration. Recall that all reducible fibers of g_2, g_1 are of type I_2. □

8.3. **The proof of Theorem 1.4 for Inoue varieties.**

Lemma 8.7. *Type i Inoue varieties $X_i, Y_i, i = 1, 2$ share the relative Jacobian $\varpi_i \colon J_i' \to \mathbb{P}^2$.*

Proof. The proof of Corollary 4.3 carries over, since the generic fibers of $\varpi_{X_i}, \varpi_{Y_i}$ share the Jacobian and all reducible fibers of $\varpi_{X_i}, \varpi_{Y_i}$ are isolated and of type I_2. □

Remark 8.8. According to [KSS, Remark 2.3.3] the Calabi–Yau 3-fold Y_3 is isomorphic to the first in [KSS, Miscellaneous examples]. The elliptic fibration from it must be isomorphic to ϖ_{Y_3}, as Y_3 admit only one elliptic fibration [Ino22, Remark 3.13]. By Lemma 8.5 the generic fibers of $\varpi_{X_3}, \varpi_{Y_3}$ share the Jacobian. If all reducible fibers of $\varpi_{X_3}, \varpi_{Y_3}$ are isolated and of type I_2, then from Proposition 4.1 it follows that $\varpi_{X_3}, \varpi_{Y_3}$ share the relative Jacobian $\varpi_3 \colon J_3' \to \mathbb{P}^2$.

Theorem 8.9. *The elliptic fibrations $\varpi_{X_i}, \varpi_{Y_i}$ for $i = 1, 2$ are mutually an almost coprime twisted power of the other in the sense of* [Mor, Definition 2.14].

Proof. Provided Lemma 8.7, the proof of Theorem 1.4 carries over. □

Remark 8.10. Suppose that all reducible fibers of $\varpi_{X_3}, \varpi_{Y_3}$ are isolated and of type I_2. Then one similarly shows Theorem 8.9, since we have $\mathrm{Br}'(\mathbb{P}^1 \times \mathbb{P}^1) = 0$ by the standard purity theorem for the cohomological Brauer group, as $\mathbb{P}^1 \times \mathbb{P}^1$ is rational.

Now, from the same arguments as in Corollary 5.10 we obtain

Corollary 8.11. *For $i = 1, 2$ type i Inoue varieties X_i, Y_i are $\mathbb{P}^2$-linear derived-equivalent.*

References

[AGV73] M. Artin, A. Grothendieck and J. Verdier, *Théorie des Topos et Cohomologie Etale des Schémas. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4)*, Lecture Notes in Mathematics, 305, Springer-Verlag, (1973).

[AKW17] B. Antieau, D. Krashen, and M. Ward, *Derived categories of torsors for abelian schemes*, Adv. Math. 306(10), 1-23 (2017).

[Căl00] A. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University (2000).

[Căl02] A. Căldăraru, *Derived categories of twisted sheaves on elliptic threefolds*, J. Reine Angew. Math. 544, 161-179 (2002).

[Căl07] A. Căldăraru, *Non-birational Calabi–Yau threefolds that are derived equivalent*, Int. J. Math. 18(5), 491-504 (2007).

[DG94] I. Dolgachev and M. Gross, *Elliptic three-folds I: Ogg–Shafarevich theory*, J. Alg. Geom. 3, 39-80 (1994).

[Fis08] T. Fisher, *The invariants of a genus one curve*, Proc. Lond. Math. Soc. 97(3), 753–782 (2008).

[Fuj80] T. Fujita, *On the hyperplane section principle of Lefschetz*, J. Math. Soc. Japan. 32(1), 153-169 (1980).

[GD66] A. Grothendieck and J. Dieudonné, *Elements de géométrie algebriqué: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie*, Publ. Math-Paris. 28, 5-255 (1966).

[Har77] R Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, (1977).

[Har10] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, 257, Springer-Verlag, (2010).

[HLS09] D. Hernández Ruipérez, A.C. López Martín and F. Sancho de Salas, *Relative integral functors for singular fibrations and singular partners*, J. Eur. Math. Soc. 11, 597-625 (2009).

[Ino22] D. Inoue, *Calabi–Yau 3-folds from projective joins of del Pezzo manifolds*, Adv. Math. 400, https://doi.org/10.1016/j.aim.2022.108243

[Jon] J. Jong, *A result of Gabber*, https://www.math.columbia.edu/ dejong/papers/2-gabber.pdf.

[KP21] A. Kuznetsov and A. Perry, *Categorical joins*, J. Amer. Math. Soc. 34(2), 505-564 (2021).

[Kuz] A. Kuznetsov, *Homological projective duality for Grassmannians of lines*, arXiv:math/0610957

[Kuz06] A. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Math. 70(3), 447-547 (2006).

[Kuz07] A. Kuznetsov, *Homological projective duality*, Publ. Math. Inst. Hautes Études Sci. 105, 157-220 (2007).

[KSS] J. Knapp, E. Scheidegger and T. Schimannek, *On genus one fibered Calabi-Yau threefolds with 5-sections*, arXiv:2107.05647

[LT17] A.C. López Martín and C. Tejero Prieto, *Derived equivalence and Kodaira fibers*, J. Geom. Phys. 122, 69-79 (2017).

[Mor] H. Morimura, *Reduced Tate–Shafarevich group*, arXiv:2205.11268v2

[Mor23] H. Morimura, *Algebraic deformations and Fourier–Mukai transforms for Calabi–Yau manifolds*, Proc. Amer. Math. Soc. 151(1), 29-43 (2023).

[Muk81] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its applicationto Picard sheaves*, Nagoya Math.J., 81, 153-175 (1981).

[Nak87] N. Nakayama, *On Weierstrass models*, Algebraic geometry and commutative algebra in honor of Masayoshi Nagata. 405-431 (1991).

[Ogu93] K. Oguiso, *On algebraic fiber space structures on a Calabi–Yau 3-fold*, Int. J. Math. 4(3), 439-465 (1993).

[Oko94] C. Okonek, *Notes on varieties of codimension 3 in $\mathbb{P}^N$*, Manuscr. Math. 84, 421–442 (1994).

[Sil08] J. H. Silverman, *The Arithmetics of Elliptic curves*, Graduate Texts in Mathematics, 106, Springer-Verlag, (2008).

[SP] The Stacks Project authors, *Stacks project*, https://stacks.math.columbia.edu.

[Via13] C. Vial, *Algebraic cycles and fibrations*, Documenta Math. 18, 1521-1553 (2013).

HOMOLOGICAL MIRROR SYMMETRY FOR COMPLETE INTERSECTIONS IN ALGEBRAIC TORI

HAYATO MORIMURA, NICOLÒ SIBILLA, AND PENG ZHOU

ABSTRACT. We prove one direction of homological mirror symmetry for complete intersections in algebraic tori, in all dimensions. The mirror geometry is not a space but a LG model, i.e. a pair given by a space and a regular function. We show that the Fukaya category of the complete intersection is equivalent to the category of matrix factorizations of the LG pair. Our approach yields new results also in the hypersurface setting, which was treated earlier by Gammage and Shende. Our argument depends on breaking down the complete intersection into smaller more manageable pieces, i.e. finite covers of products of higher dimensional pairs-of-pants, thus implementing a program first suggested by Seidel.

1. INTRODUCTION

Mirror symmetry is a mysterious duality discovered by string theorists in the '80-s. It asserts that string theory backgrounds should come in pairs (called *mirror partners*) that, despite having different geometric properties, give rise to the same physics. To the untrained eye mirror partners might look nothing alike, but string theory predicts the existence of an intricate dictionary allowing to transfer geometric information across between them. Roughly, complex geometric information on a space is encoded as symplectic data on its mirror partner, and vice versa. Since the early '90-s, mathematicians have made various attempts to distill the geometric meaning of mirror symmetry. Homological Mirror Symmetry (HMS) is one of the most influential mathematical formulations of mirror symmetry. It posits that mirror symmetry is, at bottom, an equivalence of categories. HMS was first proposed by Kontsevich in 1994 and it is still, thirty years on, the focus on intense research. It is fundamental, in the sense that it is expected to encompass most other mathematical formulations of mirror symmetry.

According to HMS if X and X' are mirror partners the *derived category* of coherent sheaves of X should be equivalent to the *Fukaya category* of X', and vice versa. The derived category is a repository of algebraic information. The objects living inside it include, for instance, vector bundles and the structure sheaves of subvarieties of X. The Fukaya category is a highly sophisticated symplectic invariant, which captures the quantum intersection theory of Lagrangian submanifolds of X'. The original formulation of HMS requires the mirror partners X and X' to be compact Calabi–Yau (CY) varieties, and under these assumptions it has been established in many important cases, starting with [PZ98] [Sei03] [She15]. However HMS has also been generalized to wider non-proper and non-CY settings. This requires readjusting the nature of the objects involved in the equivalence. In this article we contribute to this line of research by studying HMS for a particularly interesting class of non-compact symplectic manifolds.

We prove one direction of HMS for complete intersections in algebraic tori, in all dimensions. We adopt the formulation of HMS for complete intersections proposed in [AAK16], see also [GKR17]. We remark that the other direction of HMS for complete intersections was proved in [AA]. The mirror geometry is not a space but a LG model, i.e. a pair given by a space and a regular function. Our proof follows by implementing an algorithm that allows us to break down the complexity of complicated symplectic manifolds into small computable pieces, and that ultimately goes back to ideas of Seidel [Sei10]. The key input is given by recent advances in the study of the Fukaya category, which reveal its hidden local nature, at least for non-compact symplectic manifolds. We will briefly review this story in the next section.

1.1. **Fukaya categories and locality.** The Fukaya category of a symplectic manifold M was introduced by Fukaya [Fuk93]. It is a highly non-trivial symplectic invariant. Providing adequate foundations for the theory in the general setting is delicate, and this has been accomplished only relatively recently by Fukaya and his collaborators [FOOO1, FOOO2]. In fact, the Fukaya category is not quite a category: it is an A_∞-category. In an A_∞-category the composition of morphisms is associative only up to homotopy, and homotopies are themselves just the first layer in an infinite tower of higher composition laws. Roughly, the objects of the Fukaya category of M are Lagrangian submanifolds, while the Hom space between two Lagrangians is the linear span of their intersection points. The actual picture is much more complicated: for instance, as Lagrangians intersect in finitely many points only under transversality assumptions, all these data are well defined only up to appropriate choices of perturbations.

The higher A_∞-operations in the Fukaya category are controlled by counts of pseudoholomorphic disks with Lagrangian boundary conditions. This is the source of some of the biggest challenges in the theory. In particular, pseudo-holomorphic disks are non-local in nature, so higher operations in the Fukaya category depend on the global geometry of the manifold. This is in sharp contrast with the derived category of coherent sheaves, that satisfies descent with respect to the analytic topology and most other Grothendieck topologies commonly used in algebraic geometry. Around 2010 however, two new paradigms emerged suggesting that under favourable assumptions the Fukaya category should also exhibit a good local-to-global behaviour. The computational payoffs would be tremendous, as complicated global computations would be reduced to more manageable local ones. The first of these approaches breaks down a *Liouville manifold* into pieces called *Liouville sectors*; while the second, which originated with Seidel [Sei10], relies on the availability of higher dimensional *pants decompositions*. These two point of views are subtly different, and rely on somewhat distinct sets of assumptions. As we will show, in the setting of symplectic submanifolds of $(\mathbb{C}^*)^N$ they turn out to give compatible pictures of the locality of the Fukaya category. In fact, this is one of the key inputs in our argument. Before explaining our results in greater detail, let us briefly explain these two stories.

The Fukaya category was long expected to be a kind of quantization of the symplectic manifold. A precise proposal was made in the influential paper of Kapustin–Witten [KW07], where the authors model the Fukaya category of a holomorphic cotangent bundle in terms of $\mathfrak{D}$-modules over the base. Motivated by this and by earlier work of Fukaya, Nadler–Zaslow show that the (infinitesimally wrapped) Fukaya category of a cotangent bundle T^*X is equivalent to the category of contructible sheaves over X (which is assumed to be an analytic manifold) [NZ09]. Via microlocalization, the category of constructible sheaves sheafifies over T^*X. This implies, in particular, that the Fukaya category of cotangent bundles displays suprisingly good local-to-global properties. An extension of this picture to *Weinstein manifolds* was later proposed by Kontsevich [Kon09]. Cotangent bundles are exact: the standard symplectic form admits a primitive, called a Liouville 1-form. Weinstein manifolds are a class of exact symplectic manifolds satisfying some extra regularity assumptions on the Liouville1-form. Weinstein manifolds retract to an exact Lagrangian core, called the *skeleton* which is a kind of generalized zero section with singularities. Kontsevich conjectured that the wrapped Fukaya category localizes on the skeleton. That is, it defines a (co)sheaf of categories whose global sections recover the wrapped Fukaya category, and whose local sections are in many cases easily computable. This line of research has been intensely pursued in the last ten years, and we now have a robust theory of the local behaviour of the Fukaya category in this setting. The state-of-the-art is provided by a series of works by Ganatra–Pardon–Shende [GPS1, GPS2, GPS3], one of whose main results is a complete descent package for the Fukaya category relative to a class of covers of Weinstein manifolds called Weinstein sectors.

This point of view has had numerous applications to HMS, starting from [Kon09] [STZ14] [DK18]. In [GS1], Gammage–Shende use this framework to prove HMS for hypersurfaces in $(\mathbb{C}^*)^N$. In this paper we study HMS for all complete intersections in $(\mathbb{C}^*)^N$, but our approach differs from Gammage–Shende already in the hypersurface case, and yields more general results. Our methods combine sectorial descent and a different locality with respect to pants decompositions, that was first suggested by Seidel, and that we explain next.

Pants decompositions have long played a central role in understanding the topology of complex curves. Higher dimensional pants decompositions were studied by Mikhalkin in [Mik04]. A higher dimensional pair-of-pants is the complement of $N + 2$ generic hyperplanes in $\mathbb{P}^N$. Mikhalkin proves that hypersurfaces in $(\mathbb{C}^*)^N$ admit a higher dimensional pants decomposition. Mikhalkin's result is formulated in purely topological terms but, as he points out, it can be upgraded so as to be compatible with the natural symplectic structures. Mikhalkin's work has had many applications in HMS, and it plays for instance an important role in [She15]. Higher dimensional pants decompositions exist in more general settings, and sometimes also for compact varieties: for instance, hypersurfaces in abelian varieties admit pants decompositions. Seidel suggested that when such a decomposition exists, pairs-of-pants should provide the building blocks of the global Floer theory of the manifold. In particular, the Fukaya category should be expressible as a limit of the Fukaya category of the pairs-of-pants making up the decomposition. It is important to stress that this provides a very different kind of local-to-global principle for the Fukaya category. Pants are very different from the Liouville/Weinstein sectors underpinning the locality on the skeleton which we have described above.

Remarkably the locality of the Fukaya category with respect to pants decomposition is expected to match neatly, under mirror symmetry, Zariski descent on the mirror category. This opens the way to implement divide-and-conquer algorithms in HMS, reducing a difficult global mirror symmetry statement to a much more computable local one. Up to now, there have been only a few attempts to implement rigorously Seidel's proposal. One instance was the beautiful paper of Lee [Lee16] that proves Hori-Vafa HMS for curves in $(\mathbb{C}^*)^2$. The same result was proved independently, and with very different methods by Pascaleff and the second author in [PS1], with follow-ups in the compact setting in [PS2] [PS3].

Since [PS1] serves as the blueprint for some of the key arguments in this paper, it is worthwhile to review its main ideas here. The critical point is exactly the interplay between the two regimes of locality. In [PS1] the authors work within the framework of microlocal sheaves on skeleta. Thus, sectorial descent is built in their underlying theory. They show that sectorial descent, supplemented with a local calculation, implies the seemingly very different Seidel type localization on pants. Proving this involves setting up a recursion which builds up the Riemann surface in a step-by-step fashion, by gluing together the pants making up the pants decomposition. Crucially, at each step the Weinstein structure of the surface is modified so as to be adapted to the gluing. Geometrically, this means deforming the skeleton in such a way that some of its components are pushed towards the portions on the boundary along which the gluing is taking place.

The vertices of the diagram implementing Seidel's locality are the Fukaya categories of the pairs-of-pants and of their intersections. The latter, in the surface case, are isomorphic to symplectic annuli. The arrows are Viterbo restrictions. The claim is that the limit of this diagram of categories is equivalent to the wrapped Fukaya category. Geometrically, the Seidel type localization is a mechanism that allows to glue skeleta along a common closed subskeleton, on the condition that the latter lies in a separating contact hypersurface. So it can be rephrased as a kind of descent for the wrapped Fukaya category with respect to *closed covers* of the skeleton, subject to appropriate assumptions. Sectorial descent, on the other hand, captures a more straightforward descent of the wrapped Fukaya category with respect to *open covers* of the skeleton.

In this paper, we adapt this strategy to the higher dimensional case. We recover Gammage–Shende's result on HMS for hypersurfaces in $(\mathbb{C}^*)^N$ in a more general form, as we remove all assumptions on the triangulation of the associated Newton polytope. Our methods extend to give a proof of HMS for complete intersections in $(\mathbb{C}^*)^N$. We remark that extending the approach of Gammage–Shende to complete intersections is unfeasible with current technology. Their argument requires controlling the global skeleton of the Weinstein manifolds, which is not known with current methods in the case of complete intersections. Our approach bypasses this delicate issue, as it depends on constructing only smaller local pieces of the skeleton near the place where the gluing is taking place. As such it provides an algorithm for proving HMS that has potential to be applicable in more general settings beyond the one we consider in this article.

We explain our main results and the structure of the paper next.

1.2. **Main results. Hypersurfaces.** Let $\mathbb{T} = M^\vee_{\mathbb{R}/\mathbb{Z}} = M^\vee_{\mathbb{R}}/M^\vee$ be a real $(d+1)$-dimensional torus with cocharacter lattice $M^\vee$. Let M be the character lattice of T. We denote by $\mathbb{T}_{\mathbb{C}}$ the associated complex torus. Let $H \subset \mathbb{T}^\vee_{\mathbb{C}}$ be a hypersurface cut out by the Laurent polynomial

$$W\colon \mathbb{T}^\vee_{\mathbb{C}} \to \mathbb{C},\ x \mapsto \sum_{\alpha\in A} c_\alpha x^\alpha$$

where $A \subset M^\vee$ is a finite set of monomials. The hypersurface H is a closed subvariety of $\mathbb{T}^\vee_{\mathbb{C}}$ and is thereofore naturally Weinstein. An *adapted triangulation* $\mathcal{T}$ of the convex hull $\mathrm{Conv}(A)$ of A is, by definition, a triangulation arising as the corner locus of a convex piecewise linear function. These data determine a tropical hypersurface Π in $M_{\mathbb{R}}$, the *tropicalization* of H, and equip H with a higher-dimensional pair-of-pants decomposition. As an abstract topological space, Π is a homeomorphic to the dual intersection complex of the pair-of-pants decomposition of H.

The mirror of H is a $(d+2)$-dimensional toric LG model. Let Y be the noncompact toric variety associated with the fan

$$\Sigma_Y = \mathrm{Cone}(-\mathcal{T} \times \{1\}) \subset M^\vee_{\mathbb{R}} \times \mathbb{R}$$

Via the usual toric dictionary, the map of fans induced by the projection $M^\vee_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ determines a regular funtion $W_Y : Y \to \mathbb{C}$. On this side of the mirror correspondence, the tropicalization Π of H is naturally identified with the image of the singular locus of $W_Y^{-1}(0)$ under the moment map.

Theorem A. *There is an equivalence of categories*

$$\mathrm{Fuk}(H) \simeq \mathrm{MF}(Y, W_Y)$$

Let us outline the argument. Both side of the equivalence are local in nature. The key is that, as we explained, the manifold H carries a pants decomposition, and therefore its wrapped Fukaya category can be built out of the local Fukaya categories of the individual pants P. Although our proof of this fact is remarkably simple, it relies in a crucial way on the machinery developed by Ganatra–Pardon–Shende. The locality of $\mathrm{MF}(Y, W_Y)$ is straightforward. Indeed, the category of matrix factorizations satisfies Zariski descent. As Y is smooth, it has a canonical toric open cover by affine spaces. Further the restriction of W_Y on each patch coincides, up to coordinates change, with the standard superpotential

$$y_1 \cdots y_{d+2} : \mathbb{A}^{d+2} \to \mathbb{A}^1$$

Our argument involves two steps. The first consists in establishing the local equivalence

$$\mathrm{Fuk}(P) \simeq \mathrm{MF}(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2}) \tag{1.1}$$

This was proved by Nadler in [Nad] using an alternative model for the B-side category. Recall that, by a theorem of Orlov, there is an equivalence

$$\mathrm{MF}(\mathbb{A}^{d+2}, y_1 \dots y_{d+2}) \simeq \mathrm{Coh}(\{y_1 \cdots y_{d+1} = 0\})_{\mathbb{Z}_2}$$

where the latter is the $\mathbb{Z}_2$-folding of the ordinary category of coherent sheaves. Nadler describes a family of Weinstein structures on P depending on the choice of a *leg* of P. The corresponding skeleton has non-trivial intersections with all the legs of P, except with the chosen one. Nadler computes the Fukaya category in terms of microlocal sheaves on the skeleton, and proves in this way that $\mathrm{Fuk}(P)$ is equivalent to $\mathrm{Coh}(\{y_1 \cdots y_{d+1} = 0\})_{\mathbb{Z}_2}$. The choice of skeleton collapses the natural $\mathfrak{S}_{d+2}$-symmetry of the pair-of-pants to the smaller $\mathfrak{S}_{d+1}$-symmetry of the skeleton. This matches the $\mathfrak{S}_{d+1}$-action on $\mathrm{Coh}(\{y_1 \cdots y_{d+1} = 0\})_{\mathbb{Z}_2}$ by permutation of coordinates, which is also the residue of the richer $\mathfrak{S}_{d+2}$-symmetry of

$$\mathrm{MF}(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2})$$

For our purposes, we need to restore the complete $\mathfrak{S}_{d+2}$-symmetry of the problem which remains hidden in Nadler's formulation. The locality of the two mirror categories is neatly encoded in the combinatorics of the tropicalization Π. Both $\mathrm{Fuk}(H)$ and $\mathrm{MF}(Y, W_Y)$ define in a natural way two constructible sheaves of categories over Π, where Π is equipped with its natural stratification. The final globalization step consists in noticing that as the local sections and local restrictions of these two sheaves match, their global sections must also be equivalent. This is the content of Theorem A.

1.3. **Main results. Complete intersections.** Let us describe next the complete intersection setting. The underlying toric geometry is a simple extension of the ideas entering in the hypersurface setting, so we will give a somewhat abbreviated treatment of this story and refer the reader to the main text for full details. We keep the notations from the previous section.

Consider hypersurfaces $H_1, \dots, H_r \subset \mathbb{T}_{\mathbb{C}}$ in general position, cut out by Laurent polynomials

$$W_1, \dots, W_r \colon \mathbb{T}^{\vee}_{\mathbb{C}} \to \mathbb{C},\ x \mapsto \sum_{\alpha \in A_i} c_\alpha x^\alpha$$

By the genericity assumption, they meet transversely in a subvariety of dimension $n - r$

$$\mathbf{H} = H_1 \cap \cdots \cap H_r \subset \mathbb{T}^{\vee}_{\mathbb{C}}$$

The subvariety $\mathbf{H}$ carries a natural Weinstein structure. We form a total superpotential $W_{\mathbf{H}}$ by adding an extra factor $\mathbb{C}^r$ with coordinates $u_1, \dots, u_r$

$$W_{\mathbf{H}} = u_1 W_1 + \cdots + u_r W_r \colon \mathbb{T}^{\vee}_{\mathbb{C}} \times \mathbb{C}^r \to \mathbb{C}$$

Note that $\mathbf{H}$ can be recovered as the critical locus of $W_{\mathbf{H}}$.

The Newton polytope of $W_{\mathbf{H}}$ is the convex hull of

$$\mathbf{A} = \bigcup_i -A_i \times \{e_i\} \subset M^{\vee}_{\mathbb{R}} \times \mathbb{R}^r$$

where $e_1, \dots, e_r$ is the standard basis of $\mathbb{R}^r$. A choice of adapted triangulations of the convex polytopes $\mathrm{Conv}(A_i) \subset M^{\vee}_{\mathbb{R}}$ determines a triangulation $\mathbf{T}$ of $\mathrm{Conv}(\mathbf{A})$. Following [AAK16], the mirror of $\mathbf{H}$ is the higher dimensional LG model determined by $\mathbf{T}$. More precisely, let

$$\Sigma_{\mathbf{Y}} \subset M^{\vee}_{\mathbb{R}} \times \mathbb{R}^r$$

be the fan corresponding to $\mathbf{T}$, and let $\mathbf{Y}$ be the noncompact $(d+r+1)$-dimensional toric variety associated with $\Sigma_{\mathbf{Y}}$. The fan $\Sigma_{\mathbf{Y}}$ admits r projections to $\mathbb{R}$, and the sum of the corresponding monomials induces a regular function $W_{\mathbf{Y}}$ on $\mathbf{Y}$. The mirror of $\mathbf{H}$ is the LG model $(\mathbf{Y}, W_{\mathbf{Y}})$.

Theorem B. *There is an equivalence of categories*

$$\mathrm{Fuk}(\mathbf{H}) \simeq \mathrm{MF}(\mathbf{Y}, W_{\mathbf{Y}})$$

The proof strategy follows the pattern of the hypersurface case, but there are some features which are specific to the complete intersection setting which are worth highlighting. Strictly speaking, complete intersections do not admit a higher dimensional pair-of-pants decomposition. Rather, generic intersections of pants are locally isomorphic to (finite covers of) products of lower dimensional pairs-of-pants. The appearance of finite covers cannot be avoided, however it is easily controlled. For clarity, in this introduction, we shall ignore this issue. Via the Künneth formula for the wrapped Fukaya category, the Fukaya category of $\mathbf{H}$ is thus locally equivalent to the tensor product of the Fukaya categories of the factors, i.e. lower dimensional pairs-of-pants. This is matched, on the B-side, by the Zariski local behaviour of $\mathrm{MF}(\mathbf{Y}, W_{\mathbf{Y}})$.

On each affine toric open subset of $\mathbf{Y}$, the superpotential $W_{\mathbf{Y}}$ can be written as a sum of monomials. Preygel's Thom-Sebastiani theorem implies that, locally, the category of matrix factorizations factors as a tensor product of matrix factorizations of lower dimensional superpotentials. Thus, in the complete intersection case, the local HMS equivalence is just a tensor product of the fundamental local equivalences

$$\mathrm{Fuk}(P) \simeq \mathrm{MF}(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2})$$

underpinning the hypersurface case. The globalization step follows along exactly parallel lines as in the hypersurface case.

As we have already remarked, our methods allow us to give a description of the Fukaya category of $\mathbf{H}$ bypassing the difficult task of describing a global skeleton. The explicit calculation of the skeleton is, in contrast, a key input in the approach of Gammage–Shende in the hypersurface case. We obviate the absence of a computable model of the global skeleton by setting up a recursion that builds the complicated global symplectic geometry of $\mathbf{H}$ out of simple pieces amenable to computation: products of lower dimensional pants, and their finite covers. This is a mild generalization of the set-up originally envisioned by Seidel in terms of pants decompositions. This allows us to get away with building skeleta, or rather Weinstein structures, for these local pieces only. This extra flexibility crucially relies on the invariance of the wrapped Fukaya category under Liouville homotopy, which allows us to engineer Weinstein structures with good properties near a boundary where local pieces are glued together.

Our methods and results open the way to several potential directions for future investigations. We limit ourselves to mention one, which we intend to pursue in future work. In this paper we espouse the viewpoint on HMS for complete intersections proposed in [AAK16]. There is however another important model for mirror symmetry for complete intersections in toric ambient varieties, which was developed by Batyrev–Borisov in [BB96]. That framework encompasses both the non-compact regime where the ambient manifold is a torus (which is the setting we work in this article), and its toric compactifications. In future work we will explore in which ways our methods can be used to obtain Batyrev–Borisov type HMS for complete intersections.

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understanding of the Fukaya category and mirror symmetry. We want thank James for his generosity in sharing his ideas, and for his encouragement and interest in this project.

2. Review on HMS for pairs of pants

In this section, we review HMS for pairs of pants established by Nadler in [Nad]. This gives local equivalences which we glue to yiled HMS for hypersurfaces in an algebraic torus. Also, thorough understanding of such local equivalences plays an important role when working in the complete intersection setting.

2.1. **Tailored pants.** Let $T^{d+1} = (\mathbb{R}/2\pi\mathbb{Z})^{d+1}$ be the real torus with coordinates $\theta = (\theta_1, \dots, \theta_{d+1})$. Fix the usual identification $T^*T^{d+1} \cong T^{d+1} \times \mathbb{R}^{d+1}$ with canonical coordinates (θ, ξ) for $\xi = (\xi_1, \dots, \xi_{d+1})$. The symplectic manifold T^*T^{d+1} carries the standard Liouville structure

$$\alpha_{d+1} = \sum_{i=1}^{d+1} \xi_i d\theta_i, \ \omega_{d+1} = \sum_{i=1}^{d+1} d\xi_i \wedge d\theta_i$$

whose skeleton is the zero section $T^{d+1} \subset T^*T^{d+1}$. The self-action of T^{d+1} lifts to a Hamiltonian action on T^*T^{d+1} with the moment map

$$\mu_{d+1} \colon T^*T^{d+1} \to \mathrm{Lie}(T^{d+1})^\vee \cong \mathbb{R}^{d+1}, \ (\theta, \xi) \mapsto \xi.$$

Taking its squared length, one obtains a Weinstein manifold $(T^*T^{d+1}, \alpha_{d+1}, |\mu_{d+1}|^2)$. Note that the function $|\mu_{d+1}|^2$ is Morse–Bott.

Fix the identification $T^*T^{d+1} \cong T^{d+1}_{\mathbb{C}} = (\mathbb{C}^*)^{d+1}$ with coordinates $x = (x_1, \dots, x_{d+1})$ via $x_i = e^{\xi_i + \sqrt{-1}\theta_i}$. Then μ_{d+1} transports to the log projection

$$\mathrm{Log}_{d+1} \colon T^{d+1}_{\mathbb{C}} \to \mathbb{R}^{d+1}, \ x \mapsto (\log|x_1|, \dots, \log|x_{d+1}|).$$

Definition 2.1. The d-dimensional *standard pair of pants* is a complex hypersurface

$$P_d = \{1 + x_1 + \cdots + x_{d+1} = 0\} \subset T^{d+1}_{\mathbb{C}}.$$

We regard P_d as an exact symplectic manifold equipped with the restricted standard Liouville structure. Via the open embedding $T^{d+1}_{\mathbb{C}} \hookrightarrow \mathbb{P}^{d+1}_{\mathbb{C}} = \mathrm{Proj}\, \mathbb{C}[x_0, x_1, \dots, x_{d+1}]$ the pants P_d maps to the complement of

$$\bigcup_{i=0}^{d+1} (\{x_0 + x_1 + \cdots + x_{d+1} = 0\} \cap \{x_i = 0\})$$

in the hyperplane $\{x_0 + x_1 + \cdots + x_{d+1} = 0\} \subset \mathbb{P}^{d+1}_{\mathbb{C}}$. Hence the symmetric group $\mathfrak{S}_{d+2}$ naturally acts on P_d by permutation of the homogeneous coordinates.

Lemma 2.2 ([Mik04, Proposition 4.6])**.** *There is a $\mathfrak{S}_{d+2}$-equivariant isotopy of Liouville submanifolds of $T^{d+1}_{\mathbb{C}}$ from P_d to $(\tilde{P}_d, \alpha_{\tilde{P}_d} = \alpha_{d+1}|_{\tilde{P}_d})$ with the following properties:*

- *The isotopy is constant inside $\mathrm{Log}^{-1}_{d+1}(\Delta_d(R))$ for some constant $0 \ll R$ with*

$$\Delta_d(R) = \{\xi \in \mathbb{R}^{d+1} \mid -R \leq \xi_1, \dots, -R \leq \xi_{d+1}, \sum_{i=1}^{d+1} \xi_i \leq R\}.$$

- *We have the inductive compatibility*

$$L_{d,d+1}(K) = \tilde{P}_d \cap T^{d+1}_{\mathbb{C},d+1}(K) \cong \tilde{P}_{d-1} \times \mathbb{C}^*_{d+1}(K)$$

for some constant $0 \ll K$, where

$$\mathbb{C}^*_{d+1}(K) = \{x_{d+1} \in \mathbb{C}^* \mid \log|x_{d+1}| < -K\}, \ T^{d+1}_{\mathbb{C},d+1}(K) = \{x \in T^{d+1}_{\mathbb{C}} \mid \log|x_{d+1}| < -K\}.$$

Note that $\tilde{P}_d$ coincides with P_d inside $\mathrm{Log}_{d+1}^{-1}(\Delta_d(R))$. The $\mathfrak{S}_{d+2}$-action implies similar compatibilities in other directions. Namely, we have the inductive compatibility

$$L_{d,i}(K) = \tilde{P}_d \cap T^{d+1}_{\mathbb{C},i}(K) \cong \tilde{P}_{d-1} \times \mathbb{C}^*_i(K),$$

for some constant $0 \ll K$ and fixed $i = 1, \ldots, d$, where

$$\mathbb{C}^*_i(K) = \{x_i \in \mathbb{C}^* \mid \log|x_i| < -K\},\ T^{d+1}_{\mathbb{C},i}(K) = \{x \in T^{d+1}_{\mathbb{C}} \mid \log|x_i| < -K\}.$$

Definition 2.3. We call the Liouville manifold $(\tilde{P}_d, \alpha_{\tilde{P}_d})$ the d-dimensional *tailored pants*. We call the open Liouville submanifold $L_{d,i}(K)$ the *i-th leg* of $(\tilde{P}_d, \alpha_{\tilde{P}_d})$. The $(d+2)$*-th leg* of $(\tilde{P}_d, \alpha_{\tilde{P}_d})$ is the remaining open Liouville submanifold with respect to the $\mathfrak{S}_{d+2}$-action, which corresponds to the positive diagonal direction in $\mathbb{R}^{d+1}$.

In order to provide $\tilde{P}_d$ with particularly simple skeleton, Nadler broke symmetry and applied a natural isotopy to its Liouville structure. Consider the translated Liouville structure

$$\alpha^l_{d+1} = \sum_{i=1}^{d+1} (\xi_i + l) d\theta_i, \omega^l_{d+1} = \sum_{i=1}^{d+1} d(\xi_i + l) \wedge d\theta_i = \omega_{d+1} \tag{2.1}$$

on $T^{d+1}_{\mathbb{C}}$ for some constant $0 \ll l$.

Definition 2.4. We call the Weinstein structure given by a triple

$$(\tilde{P}_d, \beta_{\tilde{P}_d} = \alpha^l_{d+1}|_{\tilde{P}_d}, \Sigma_{i=1}^{d+1}(\log|x_i| + l)^2)$$

Nadler's Weinstein structure. We write $\mathrm{Core}(\tilde{P}_d)$ for its skeleton. We call the $(d+2)$-th leg of $\tilde{P}_d$ equipped with Nadler's Weinstein structure the *final leg*.

Remark 2.5. All the legs but the final remain symmetric under the $\mathfrak{S}_{d+2}$-action.

For a proper subset $I \subsetneq \{1, \ldots, d+1\}$ let

$$\Delta_I(l) = \{x \in \tilde{P}_d \cap T^{d+1}_{\mathbb{R}<0} \mid \log|x_i| = -l \text{ for } i \in I,\ \log|x_j| > -l \text{ for } j \in I^c\}$$

be the relatively open subsimplex of the closed simplex

$$\Delta_d(l) = \{x \in \tilde{P}_d \cap T^{d+1}_{\mathbb{R}<0} \mid \log|x_i| \geq -l\}.$$

We denote by $\delta_I(l)$ the barycenter

$$\{x \in \tilde{P}_d \cap T^{d+1}_{\mathbb{R}<0} \mid \log|x_i| = -l \text{ for } i \in I,\ \log|x_j| = \log|x_{j'}| > -l \text{ for } j, j' \in I^c\}$$

of the subsimplex $\Delta_I(l)$.

Lemma 2.6 ([Nad, Theorem 5.13])**.** *For a subset $I \subset \{1, \ldots, d+1\}$ let $T^I \subset T^{d+1}$ be the subtorus defined by $\theta_i = 0, i \in I^c$. Then we have*

$$\mathrm{Core}(\tilde{P}_d) = \bigcup_{I \subsetneq \{1,\ldots,d+1\}} T^I \cdot \Delta_I(l).$$

Proof. The original proof uses induction on d. The case $d = 0$ is obvious. When $d = 1$, a nonempty proper subset $I \subsetneq \{1, \ldots, d+1\}$ is either $\{1\}$ or $\{2\}$. On $\delta_i(l) = \Delta_i(l)$ the Liouville form $\beta_{\tilde{P}_1}$ vanishes and their stable manifolds are isomorphic to $T^i \cdot \delta_{\{i\}}(l) = T^i \cdot \Delta_{\{i\}}(l)$. From [Mik04, Corollary 4.4, 4.5, Proposition 4.6] it follows that the critical locus $\mathrm{Crit}(\Sigma_{i=1}^{d+1}(\log|x_i| + l)^2)$ coincides with $\tilde{P}_d \cap T^{d+1}_{\mathbb{R}_{<0}}$. In particular, since $\tilde{P}_d$ is Weinstein, $\tilde{P}_d \cap T^{d+1}_{\mathbb{R}_{<0}}$ contains the zero locus $Z(\beta_{\tilde{P}_d})$. The negative real points $\tilde{P}_d \cap T^{d+1}_{\mathbb{R}_{<0}}$ is a Lagrangian as $d\theta_i$ vanishes there and the Liouville flow on $\tilde{P}_d \cap T^{d+1}_{\mathbb{R}_{<0}}$ attracts the points to a point $\delta_\emptyset(l)$. Hence, aside $T^i \cdot \Delta_{\{i\}}(l)$, only the stable manifold $T^\emptyset \cdot \Delta_\emptyset(l)$ of $T^\emptyset \cdot \delta_\emptyset(l)$ contributes to $\mathrm{Core}(\tilde{P}_1)$. For general d, combine the same argument and the inductive compatibility from Lemma 2.2. $\square$

2.2. **Microlocal interpretation.** Next, we review the geometry of $\mathrm{Core}(\tilde{P}_d)$. The action of the diagonal circle $T^1_\Delta \subset T^{d+1}$ by translation lifts to a Hamiltonian action with the moment map

$$\mu_\Delta \colon T^*T^{d+1} \to \mathbb{R},\ (\theta, \xi) \mapsto \sum_{i=1}^{d+1} \xi_i.$$

Distinguishing the final coordinate θ_{d+1} on T^{d+1}, we identify the quotient $\mathbb{T}^d = T^{d+1}/T^1_\Delta$ with T^d via $[\theta] \mapsto (\theta_1 - \theta_{d+1}, \ldots, \theta_d - \theta_{d+1})$. Denoting by $\mathfrak{t}^*_d$ the dual of $\mathrm{Lie}(\mathbb{T}^d) = \{\xi \in \mathbb{R}^{d+1} | \sum_{i=1}^{d+1} \xi_i = 0\}$, we identify $T^*\mathbb{T}^d$ with $\mathbb{T}^d \times \mathfrak{t}^*_d$. The product conic Lagrangian

$$\Lambda_{d+1} = (\Lambda_1)^{d+1} \subset \mu_\Delta^{-1}(\mathbb{R}_{\geq 0}) \subset T^*T^{d+1}$$

is transverse to $\mu_\Delta^{-1}(\chi)$ for $\chi > 0$, where

$$\Lambda_1 = \{(\theta, 0) \mid \theta \in T^1\} \cup \{(0, \xi) | \xi \in \mathbb{R}_{\geq 0}\} \subset T^1 \times \mathbb{R} \cong T^*T^1.$$

Consider the twisted Hamiltonian reduction correspondence

$$T^*T^{d+1} \overset{q_\chi}{\hookleftarrow} \mu_\Delta^{-1}(\chi) = \{(\theta, \xi) \in T^*T^{d+1} | \sum_{i=1}^{d+1} \xi_i = \chi\} \overset{p_\chi}{\twoheadrightarrow} T^*\mathbb{T}^d$$

where q_χ is the canonical inclusion and p_χ is the translated projection

$$p_\chi((\theta, \xi)) = ([\theta], \xi_1 - \hat{\chi}, \ldots, \xi_{d+1} - \hat{\chi}),\ \hat{\chi} = \chi/(d+1).$$

For a proper subset $I \subsetneq \{1, \ldots, d+1\}$ let

$$\tilde{\Xi}_d(\chi) = \{\xi \in \mathbb{R}^{d+1}_{\geq 0} | \xi_1, \ldots, \xi_{d+1} \geq 0, \sum_{i=1}^{d+1} \xi_i = \chi\}$$

be the closed subsimplex. The map p_χ restricts to an isomorphism

$$\mu_\Delta^{-1}(\chi) \cap \Lambda_{d+1} = \mu_{d+1}^{-1}(\tilde{\Xi}_d(\chi)) \to \mathfrak{L}_d := p_\chi(q_\chi^{-1}(\Lambda_{d+1})).$$

Let $\tilde{\Xi}_I(\chi) = \tilde{\Xi}_d(\chi) \cap \sigma_I$ be the relatively open subsimplex with

$$\sigma_I = \{\xi \in \mathbb{R}^{d+1}_{\geq 0} | \xi_i = 0 \text{ for } i \in I,\ \xi_j > 0 \text{ for } j \in I^c\}.$$

From $\mu_{d+1}(\Lambda_{d+1}) = (\mathbb{R}_{\geq 0})^{d+1}$ it follows $\mu_{d+1}^{-1}(\xi) \cap \Lambda_{d+1} \cong T^I$ for $\xi \in \sigma_I$ and p_χ restricts to an isomorphism $\bigcup_{I \subsetneq \{1,\ldots,d+1\}} T^I \times \tilde{\Xi}_I(\chi) \to \mathfrak{L}_d$. Hence we obtain an identification of subspaces

$$\mathfrak{L}_d \cong \bigcup_{I \subsetneq \{1,\ldots,d+1\}} \mathbb{T}^I \times \Xi_I(\chi) \subset \mathbb{T}^d \times \mathfrak{t}^*_d \cong T^*\mathbb{T}^d,$$

where $\mathbb{T}^I \subset \mathbb{T}^d$ are isomorphic images of T^I under the quotient and

$$\Xi_I(\chi) = \{\xi \in \mathbb{R}^{d+1}_{\geq 0} | \xi_i = -\hat{\chi} \text{ for } i \in I,\ \xi_j > -\hat{\chi} \text{ for } j \in I^c, \sum_{i=1}^{d+1} \xi_i = 0\}$$

are isomorphic images of $\tilde{\Xi}_I(\chi)$ under translation by $\hat{\chi}$.

The symplectic geometry of a certain open neighborhood of $\mathrm{Core}(\tilde{P}_d)$ in $\tilde{P}_d$ is equivalent to that of the associated open neighborhood of $\mathfrak{L}_d$ in $T^*\mathbb{T}^d$.

Lemma 2.7 ([Nad, Theorem 5.13]). *There is an open neighborhood $U_d \subset \tilde{P}_d$ of* $\mathrm{Core}(\tilde{P}_d)$ *with an open symplectic embedding $\mathfrak{j} \colon U_d \hookrightarrow T^*\mathbb{T}^d$ which makes the diagram*

$$\begin{tikzcd} U_d \arrow[r, "\sim"] & \mathfrak{U}_d := \mathfrak{j}(U_d) \\ \mathrm{Core}(\tilde{P}_d) \arrow[u, hook] \arrow[r, "\sim"] & \mathfrak{L}_d \arrow[u, hook] \end{tikzcd}$$

commute, where the vertical arrows are the canonical inclusions.

Proof. Let U°_d be a sufficiently small open neighborhood of $\Delta_d(l)$ in $\tilde{P}_d$. By the inductive compatibility from Lemma 2.2, near $\Delta_I(l)$ for $\emptyset \neq I \subsetneq \{1, \ldots, d+1\}$ the functions $\log|x_i|, i \in I$ define a local coisotropic foliation of U°_d. If $I' \subsetneq \{1, \ldots, d+1\}$ contains I, then the foliation defined by $\log|x_i|, i \in I'$ refines that defined by $\log|x_i|, i \in I$. Let $\mathfrak{U}^\circ_d$ be a sufficiently small open neighborhood of $\Xi_d(\chi)$ in $T\mathbb{T}^d$. Near $\Xi_I(\chi)$ for $\emptyset \neq I \subsetneq \{1, \ldots, d+1\}$ the functions $\xi_i, i \in I$ define a local coisotropic foliation of $\mathfrak{U}^\circ_d$. If $I' \subsetneq \{1, \ldots, d+1\}$ contains I, then the foliation defined by $\xi_i, i \in I'$ refines that defined by $\xi_i, i \in I$.

The neighborhoods U°_d and $\mathfrak{U}^\circ_d$ are symplectomorphic to the cotangent bundles of their Lagrangians $U^\circ_d \cap T^{d+1}_{\mathbb{R}<0}$ and $\mathfrak{U}^\circ_d \cap \mathfrak{t}^*_d$. Hence one finds a symplectomorphism $\mathfrak{j}^\circ \colon U^\circ_d \to \mathfrak{U}^\circ_d$ restricting to a diffeomorphism $U^\circ_d \cap T^{d+1}_{\mathbb{R}<0} \to \mathfrak{U}^\circ_d \cap \mathfrak{t}^*_d$ and an isomorphism $\Delta_d(l) \to \Xi_d(\chi)$, which is compatible with the above local coisotropic foliations. Choose a sufficiently small open neighborhood U°_I of $\Delta_I(l)$ in U°_d for each $I \subsetneq \{1, \ldots, d+1\}$. We denote by $\mathfrak{U}^\circ_I$ the open neighborhood $\mathfrak{j}^\circ(U^\circ_I)$ of $\Xi_I(\chi)$. Then

$$U_d = \bigcup_{I \subsetneq \{1,\ldots,d+1\}} T^I \cdot U^\circ_I, \ \mathfrak{U}_d = \bigcup_{I \subsetneq \{1,\ldots,d+1\}} \mathbb{T}^I \cdot \mathfrak{U}^\circ_I$$

are respectively open neighborhoods of $\mathrm{Core}(\tilde{P}_d), \mathfrak{L}_d$. Since the matched local coisotropic foliations correspond to the moment maps for the Hamiltonian actions of $T^I, \mathbb{T}^I$, the symplectomorphism $\mathfrak{j}^\circ$ canonically extends to $\mathfrak{j} \colon U_d \to \mathfrak{U}_d$. □

On $\mathfrak{U}_d$ there are two Liouville forms $\alpha_{T^*\mathbb{T}^d}|_{\mathfrak{U}_d}$ and $\beta_{d+1} = (\mathfrak{j}^{-1})^*\beta_{\tilde{P}_d}$. When $\hat{\chi} = \chi/(d+1) \in \mathbb{Z}$ the function

$$\mu_\Delta^{-1}(\chi) \cap \Lambda_{d+1} \to T^1, \ (\theta, \xi) \mapsto \sum_{i=1}^{d+1} (\xi_i + \hat{\chi})\theta_i$$

is invariant under the T^1_Δ-action and descends to an integral structure $f \colon \mathfrak{L}_d \to T^1$ [Nad, Definition 5.17(1)]. By [Nad, Remark 5.18(1)] the graph $\Gamma_{\mathfrak{L}_d, -f}$ of $-f$ gives a Legendrian lift of $\mathfrak{L}_d$ to the circular contactification

$$(N_d, \lambda_d) = (\mathfrak{U}_d \times T^1, \alpha_{T^*\mathbb{T}^d}|_{\mathfrak{U}_d} + dt).$$

Since we have $\beta_{d+1}|_{\mathfrak{L}_d} = 0$, the Lagrangian $\mathfrak{L}_d$ is exact [Nad, Definition 5.17(2)]. By [Nad, Definition 5.18(2)] the zero section $\mathfrak{L}_d \times \{0\}$ gives a Legendrian lift of $\mathfrak{L}_d$ to the circular contactification

$$(N'_d, \lambda'_d) = (\mathfrak{U}_d \times T^1, \beta_{d+1} + dt).$$

The contact geometry of (N_d, λ_d) near $\Gamma_{\mathfrak{L}_d, -f}$ is equivalent to that of (N'_d, λ'_d) near $\mathfrak{L}_d \times \{0\}$.

Lemma 2.8 ([Nad, Section 5.3])**.** *There is a contactomorphism*

$$G \colon (N_d, \lambda_d) \to (N'_d, \lambda'_d), \ (([\theta], \xi), t) \mapsto (([\theta], \xi), t + g([\theta], \xi))$$

which makes the diagram

$$\begin{tikzcd} N_d \arrow[r, "\sim"] & N'_d \\ \Gamma_{\mathfrak{L}_d, -f} \arrow[u, hook] \arrow[r, "\sim"] & \mathfrak{L}_d \times \{0\} \arrow[u, hook] \end{tikzcd}$$

commute, where the vertical arrows are the canonical inclusions.

Proof. The difference $\alpha_{T^*\mathbb{T}^d}|_{\mathfrak{U}_d} - \beta_{d+1}$ is closed and integral, as we have $\beta_{d+1}|_{\mathfrak{L}_d} = 0$. Since the inclusion $\mathfrak{L}_d \subset \mathfrak{U}_d$ is a homotopy equivalence, there is a unique function $g\colon \mathfrak{U}_d \to T^1$ such that $dg = \alpha_{T^*\mathbb{T}^d}|_{\mathfrak{U}_d} - \beta_{d+1}$ with normalization $g|_{\mathfrak{L}_d} = f$. Then one obtains the desired map from [Nad, Remark 5.16]. □

We denote by $\Omega^\infty_{d+1} \subset S^*T^{d+1} = (T^*T^{d+1} \setminus T^{d+1})/\mathbb{R}_{>0}$ and Λ^∞_{d+1} the spherical projectivizations of the open conic subset $\Omega_{d+1} = \mu_\Delta^{-1}(\mathbb{R}_{>0}) \subset T^*T^{d+1}$ and the Lagrangian Λ_{d+1}. The projection $\Omega_{d+1} \to \Omega^\infty_{d+1} = \Omega_{d+1}/\mathbb{R}_{>0}$ induces a contactomorphism from $\mu_\Delta^{-1}(\chi)$ to Ω^∞_{d+1}. Let $\delta\colon T^{d+1} \to T^1$ be the diagonal character. By [Nad, Lemma 5.19] the map

$$(p_\chi, \delta)\colon \Omega^\infty_{d+1} \cong \mu_\Delta^{-1}(\chi) \to T^*\mathbb{T}^d \times T^1,\ (\theta, \xi) \mapsto (([\theta], \xi_1 - \hat{\chi}, \ldots, \xi_{d+1} - \hat{\chi}), \sum_{i=1}^{d+1} \theta_i)$$

defines a finite contact cover for $\chi = d + 1$. The cover is trivializable over (N_d, λ_d) with a canonical section $s\colon N_d \to \Omega^\infty_{d+1}$ satisfying $s(\Gamma_{\mathfrak{L}_d,-f}) = \Lambda^\infty_{d+1}$.

The contact geometry of (N_d, λ_d) near $\Gamma_{\mathfrak{L}_d,-f}$ is equivalent to that of Ω^∞_{d+1} near Λ^∞_{d+1}.

Lemma 2.9 ([Nad, Lemma 5.19])**.** *There is an open contactomorphism $s\colon (N_d, \lambda_d) \to \Omega^\infty_{d+1}$ which makes the diagram*

$$\begin{tikzcd} N_d \arrow[r, "\sim"] & s(N_d) \\ \Gamma_{\mathfrak{L}_d,-f} \arrow[u, hook] \arrow[r, "\sim"] & \Lambda^\infty_{d+1} \arrow[u, hook] \end{tikzcd}$$

commute, where the vertical arrows are the canonical inclusions.

Consider the symplectization $\tilde{P}_d \times T^1 \times \mathbb{R}$ of the circular contactification $(\tilde{P}_d \times T^1, \beta_{d+1} + dt)$ whose Liouville form is given by $e^t(\beta_{d+1} + dt)$. The skeleton $\mathrm{Core}(\tilde{P}_d) \subset \tilde{P}_d$ lifts to the Legendrian submanifold $\mathrm{Core}(\tilde{P}_d) \times \{0\} \subset \tilde{P}_d \times T^1$, which in turn lifts to a conic Lagrangian $\mathrm{Core}(\tilde{P}_d) \times \{0\} \times \mathbb{R} \subset \tilde{P}_d \times T^1 \times \mathbb{R}$ along the canonical projections. Note that the contact geometry of a cooriented contact manifold is equivalent to the conic symplectic geometry of its symplectization. In particular, taking the inverse image under the canonical projection induces a bijection from subspaces of the contact manifold to conic subspaces of its symplectization.

The symplectic geometry of the open neighborhood $U_d \times T^1 \times \mathbb{R}$ of $\mathrm{Core}(\tilde{P}_d) \times \{0\} \times \mathbb{R}$ in $\tilde{P}_d \times T^1 \times \mathbb{R}$ is equivalent to that of the open neighborhood $s(N_d) \times \mathbb{R}$ of $\Lambda_{d+1} \cap \Omega_{d+1}$ in Ω_{d+1}.

Theorem 2.10 ([Nad, Theorem 5.23])**.** *There is an open symplectomorphism $U_d \times T^1 \times \mathbb{R} \hookrightarrow \Omega_{d+1}$ which makes the diagram*

$$\begin{tikzcd} U_d \times T^1 \times \mathbb{R} \arrow[r, hook] & \Omega_{d+1} \\ \mathrm{Core}(\tilde{P}_d) \times \{0\} \times \mathbb{R} \arrow[u, hook] \arrow[r, "\sim"] & \Lambda_{d+1} \cap \Omega_{d+1} \arrow[u, hook] \end{tikzcd}$$

commute, where the vertical arrows are the canonical inclusions.

Proof. The restriction of $\mathfrak{j}$ from Lemma 2.7 induces a symplectomorphism

$$U_d \times T^1 \times \mathbb{R} \to \mathfrak{U}_d \times T^1 \times \mathbb{R}$$

which sends $\mathrm{Core}(\tilde{P}_d) \times \{0\} \times \mathbb{R}$ to $\mathfrak{L}_d \times \{0\} \times \mathbb{R}$. The inverse of the contactomorphism G from Lemma 2.8 induces a symplectomorphism

$$(N'_d, \lambda'_d) \times \mathbb{R} = \mathfrak{U}_d \times T^1 \times \mathbb{R} \to \mathfrak{U}_d \times T^1 \times \mathbb{R} = (N_d, \lambda_d) \times \mathbb{R}$$

which sends $\mathfrak{L}_d \times \{0\} \times \mathbb{R}$ to $\Gamma_{\mathfrak{L}_d,-f} \times \mathbb{R}$. The contactomorphism s from Lemma 2.9 induces a symplectomorphism

$$(N_d, \lambda_d) \times \mathbb{R} = \mathfrak{U}_d \times T^1 \times \mathbb{R} \to s(\mathfrak{U}_d \times T^1) \times \mathbb{R} = s(N_d) \times \mathbb{R}$$

which sends $\Gamma_{\mathfrak{L}_d,-f} \times \mathbb{R}$ to $\Lambda_{d+1} \cap \Omega_{d+1}$. Note that the symplectization of Ω_{d+1}^{∞} is isomorphic to Ω_{d+1}, as Ω_{d+1} does not intersect the zero section T^{d+1}. □

2.3. *A***-side category.** Let Z be a real analytic manifold over $\mathbb{C}$. We denote by $\mathrm{Sh}^{\diamond}(Z)$ the cocomplete dg category of large constructible sheaves on Z. Recall that a *large constructible sheaf* $\mathscr{F}$ on Z is a complex of $\mathbb{C}$-vector space on Z for which there exists a Whitney stratification $\mathcal{S} = \{Z_\alpha\}$ of Z such that $\mathcal{H}^i(\mathscr{F}|_{Z_\alpha})$ are locally constant for all i. We denote by $\mathrm{Sh}^{\diamond}_{\mathcal{S}}(Z) \subset \mathrm{Sh}^{\diamond}(Z)$ the full dg subcategory of such sheaves, called large $\mathcal{S}$-constructible sheaves.

Fix a point $(z, \xi) \in T^*Z$. Let $B \subset Z$ be a sufficiently small open ball around $z \in Z$ and $f\colon B \to \mathbb{R}$ a *compatible test function*, i.e., a smooth function with $f(z) = 0$ and $df|_z = \xi$. Consider the *vanishing cycle functor*

$$\phi_f\colon \mathrm{Sh}^{\diamond}(Z) \to \mathrm{Mod}(\mathbb{C}), \ \mathscr{F} \mapsto \Gamma_{f \geq 0}(B, \mathscr{F}|_B).$$

Definition 2.11. The *microsupport* $\mathrm{ss}(\mathscr{F}) \subset T^*Z$ of $\mathscr{F} \in \mathrm{Sh}^{\diamond}(Z)$ is the largest closed subset with $\phi_f(\mathscr{F}) \cong 0$ for any $(z, \xi) \in T^*Z \setminus \mathrm{ss}(\mathscr{F})$ and its compatible test function f.

Definition 2.12. For a conic Lagrangian $\Lambda \subset T^*Z$, we denote by $\mathrm{Sh}^{\diamond}_{\Lambda}(Z) \subset \mathrm{Sh}^{\diamond}(Z)$ the full dg subcategory of large constructible sheaves with microsupport in Λ.

Given a closed embedding $\Lambda \subset \Lambda'$ of conic Lagrangians, there is a full embedding $\mathrm{Sh}^{\diamond}_{\Lambda}(Z) \hookrightarrow \mathrm{Sh}^{\diamond}_{\Lambda'}(Z)$. Note that the microsupport is a closed conic Lagrangian.

Remark 2.13. Let $-\Lambda \subset T^*Z$ be the antipodal conic Lagrangian and ω_Z the Verdier dualizing complex. There is an involutive equivalence

$$\mathbf{D}_Z\colon \mathrm{Sh}^{\diamond}_{\Lambda}(Z)^{op} \to \mathrm{Sh}^{\diamond}_{-\Lambda}(Z), \ \mathscr{F} \mapsto \underline{\mathrm{Hom}}(\mathscr{F}, \omega_Z)$$

defined by Verdier duality.

Definition 2.14. For a closed conic Lagrangian $\Lambda \subset T^*Z$ and an open conic subspace $\Omega \subset T^*Z$, we define the dg category $\mu\mathrm{Sh}^{\diamond}_{\Lambda}(\Omega)$ of large microlocal sheaves on Ω supported along Λ as the Verdier localization

$$\mu\mathrm{Sh}^{\diamond}_{\Lambda}(\Omega) = \mathrm{Sh}^{\diamond}_{\Lambda \cup (T^*Z \setminus \Omega)}(Z) / \mathrm{Sh}^{\diamond}_{T^*Z \setminus \Omega}(Z).$$

Given an inclusion $\Omega \subset \Omega'$ of open conic subspaces of T^*Z, there is a restriction functor

$$\rho_{\Omega \subset \Omega'}\colon \mu\mathrm{Sh}^{\diamond}_{\Lambda}(\Omega') \to \mu\mathrm{Sh}^{\diamond}_{\Lambda}(\Omega).$$

Lemma 2.15. *The assignments* $\Omega \mapsto \mu\mathrm{Sh}^{\diamond}_{\Lambda}(\Omega)$ *and* $(\Omega \subset \Omega') \mapsto \rho_{\Omega \subset \Omega'}$ *assemble into a sheaf of dg categories supported along* Λ*. Moreover, there exists a Whitney stratification of* Λ *the restriction of* $\mu\mathrm{Sh}^{\diamond}_{\Lambda}$ *to whose strata are locally constant.*

Remark 2.16. One can check that $\mu\mathrm{Sh}^{\diamond}_{\Lambda}$ is conic, i.e., invariant under the cotangent scaling of T^*Z. Since the intersection of the microsupport and Z coincides with the support, $\mu\mathrm{Sh}^{\diamond}_{\Lambda}$ is the pushforward of a sheaf supported on Λ, which we also denote by $\mu\mathrm{Sh}^{\diamond}_{\Lambda}$.

Given a closed embedding $\Lambda \subset \Lambda'$ of conic Lagrangians, there is a full embedding

$$i_{\Lambda \subset \Lambda'}\colon \mu\mathrm{Sh}^{\diamond}_{\Lambda} \hookrightarrow \mu\mathrm{Sh}^{\diamond}_{\Lambda'}.$$

Remark 2.17. For the antipodal conic Lagrangian $-\Lambda \subset T^*Z$ and the antipodal open conic subspace $-\Omega \subset T^*Z$, there is an involutive equivalence

$$\mathbf{D}_Z\colon\ \mu\mathrm{Sh}^{\diamond}_{\Lambda}(\Omega)^{op} \to \mu\mathrm{Sh}^{\diamond}_{-\Lambda}(-\Omega)$$

induced by Verdier duality.

Definition 2.18. For a closed conic Lagrangian $\Lambda \subset T^*Z$ and an open conic subspace $\Omega \subset T^*Z$, the *category of wrapped microlocal sheaves* on Ω supported along Λ is the full dg category $\mu\mathrm{Sh}_{\Lambda}(Z) \subset \mu\mathrm{Sh}^{\diamond}_{\Lambda}(Z)$ of compact objects.

The restriction functor $\rho_{\Omega\subset\Omega'}$ preserves products. Hence it admits a left adjoint which preserves coproducts. Thus the restriction to compact objects yields a corestriction functor

$$\rho^{l}_{\Omega\subset\Omega'}\colon\ \mu\mathrm{Sh}_{\Lambda}(\Omega) \to \mu\mathrm{Sh}_{\Lambda}(\Omega').$$

Lemma 2.19 ([Nad, Proposition 3.16]). *The assignments* $\Omega \mapsto \mu\mathrm{Sh}_{\Lambda}(\Omega)$ *and* $(\Omega \subset \Omega') \mapsto \rho^{l}_{\Omega\subset\Omega'}$ *assemble into a cosheaf of dg categories supported along* Λ. *Moreover, there exists a Whitney stratification of* Λ *the restriction of* $\mu\mathrm{Sh}_{\Lambda}$ *to whose strata are locally constant.*

The full embedding $i_{\Lambda\subset\Lambda'}$ preserves products. Hence it admits a left adjoint which preserves coproducts. The restriction to compact objects yields a Verdier localization

$$i^{l}_{\Lambda\subset\Lambda'}\colon\ \mu\mathrm{Sh}_{\Lambda'} \to \mu\mathrm{Sh}_{\Lambda}\,.$$

Definition 2.20. For a Liouville manifold H, we denote by $\mathrm{Fuk}(H)$ the ind-completion of the wrapped Fukaya category of H.

Lemma 2.21 ([GPS3, Theorem 1.4]). *Let H be a real analytic Weinstein manifold. For any stable polarization of H, there is an equivalence*

$$\mathrm{Fuk}(H)^{op} \simeq \mu\mathrm{Sh}^{\diamond}_{\mathrm{Core}(H)}(\mathrm{Core}(H)).$$

Remark 2.22 ([GPS3, Remark 1.2]). Due to the involutive equivalence from Remark 2.17, one could equivalently negate $\mathrm{Core}(H)$ rather than passing to the opposite category of $\mathrm{Fuk}(H)$.

2.4. ***B*-side category.** For any stable dg category $\mathscr{C}$ we denote by $\mathscr{C}_{\mathbb{Z}_2}$ its folding, i.e., $\mathscr{C}_{\mathbb{Z}_2}$ is the stable envelope of the $\mathbb{Z}_2$-dg category with the same objects as $\mathscr{C}$ whose morphism complex for $c_1, c_2 \in \mathscr{C}_{\mathbb{Z}_2}$ is given by

$$\mathrm{Hom}^0_{\mathscr{C}_{\mathbb{Z}_2}}(c_1,c_2) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^{2n}_{\mathscr{C}}(c_1,c_2),\ \mathrm{Hom}^1_{\mathscr{C}_{\mathbb{Z}_2}}(c_1,c_2) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^{2n+1}_{\mathscr{C}}(c_1,c_2).$$

For any stable $\mathbb{Z}_2$-dg category $\mathscr{C}$ we denote by $\mathscr{C}_{2\mathbb{Z}}$ its unfurling, i.e., $\mathscr{C}_{2\mathbb{Z}}$ is the 2-periodic dg category with the same objects as $\mathscr{C}$ whose morphism complex for $c_1, c_2 \in \mathscr{C}_{2\mathbb{Z}}$ is given by

$$\mathrm{Hom}^n_{\mathscr{C}_{2\mathbb{Z}}}(c_1,c_2) = \mathrm{Hom}^{\bar{n}}_{\mathscr{C}}(c_1,c_2),\ n \in \mathbb{Z} \mapsto \bar{n} \in \mathbb{Z}_2.$$

The folding and unfurling define equivalences between stable $\mathbb{Z}_2$-dg categories and 2-periodic dg categories.

Let $\mathbb{A}^{d+2} = \mathrm{Spec}\, R_{d+2}$ for $R_{d+2} = \mathbb{C}[y_1,\dots,y_{d+2}]$ and $W_{d+2} = y_1\cdots y_{d+2} \in R_{d+2}$.

Definition 2.23. A matrix factorization for the pair $(\mathbb{A}^{d+2}, W_{d+2})$ is given by the diagram

$$V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^0$$

where $V^0 \oplus V^1$ is a $\mathbb{Z}_2$-graded free R_{d+2}-modules of finite rank and $d_0 \in \mathrm{Hom}(V^0,V^1), d_1 \in \mathrm{Hom}(V^1,V^0)$ satisfy $d_1 d_0 = W_{d+2}\cdot \mathrm{id}_{V^0}, d_0 d_1 = W_{d+2}\cdot\mathrm{id}_{V^1}$. We denote by $\mathrm{MF}(\mathbb{A}^{d+2}, W_{d+2})$ the $\mathbb{Z}_2$-dg category of the matrix factorizations for $(\mathbb{A}^{d+2}, W_{d+2})$ with obvious morphisms.

Let $\underline{\mathscr{O}}^i_{d+1}$ be the matrix factrization

$$R_{d+2} \xrightarrow{W^i_{d+2}} R_{d+2} \xrightarrow{y_i} R_{d+2}$$

with $W^i_{d+2} = W_{d+2}/y_i$ for $i = 1, \ldots, d+2$.

Lemma 2.24 ([Nad, Proposition 2.1])**.** *The $\mathbb{Z}_2$-dg category* $\mathrm{MF}(\mathbb{A}^{d+2}, W_{d+2})$ *is split-generated by $\{\underline{\mathscr{O}}^i_{d+1}\}_{i=1}^{d+1}$. There are isomorphisms of $\mathbb{Z}_2$-graded $\mathbb{C}$-modules*

$$H^*(\mathrm{Hom}(\underline{\mathscr{O}}^i_{d+1}, \underline{\mathscr{O}}^i_{d+1})) \cong R_{d+2}/(y_i, W^i_{d+2}),\ 1 \le i \le d+2,$$
$$H^*(\mathrm{Hom}(\underline{\mathscr{O}}^i_{d+1}, \underline{\mathscr{O}}^j_{d+1})) \cong R_{d+2}/(y_i, y_j)[-1],\ 1 \le i \ne j \le d+2.$$

Remark 2.25. The matrix factorization $\underline{\mathscr{O}}^{d+2}_{d+1}$ belongs to the stable envelope of $\{\underline{\mathscr{O}}^i_{d+1}\}_{i=1}^{d+1}$

Let $Y_d = \operatorname{Spec} R_{d+1}/(W_{d+1})$ be the union of the coordinate hyperplanes $Y^i_d = \operatorname{Spec} R_{d+1}/(y_i)$, $1 \le i \le d+1$. We denote by $\mathscr{O}^i_d$ the structure sheaf of Y^i_d.

Lemma 2.26 ([Nad, Proposition 2.2])**.** *The dg category* $\mathrm{Coh}(Y_d)$ *is generated by $\{\mathscr{O}^i_d\}_{i=1}^{d+1}$. There are isomorphisms of $\mathbb{Z}$-graded $\mathbb{C}$-modules*

$$H^*(\mathrm{Hom}(\mathscr{O}^i_d, \mathscr{O}^i_d)) \cong R_{d+1}[u]/(y_i, uW^i_{d+1}),\ 1 \le i \le d+1,$$
$$H^*(\mathrm{Hom}(\mathscr{O}^i_d, \mathscr{O}^j_d)) \cong R_{d+1}/(y_i, y_j)[-1],\ 1 \le i \ne j \le d+1,$$

where u is a variable of cohomological degree 2

Lemma 2.27 ([Orl04, Theorem 3.7])**.** *Let $D_{sing}(Y_d) = \mathrm{Coh}(Y_d)/\mathrm{Perf}(Y_d)$ be the 2-periodic dg quotient category of singularities. Then there is an equivalence*

$$\mathrm{MF}(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2}) \to D_{sing}(Y_d),$$
$$(V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^0) \mapsto \mathrm{Coker}(d_1).$$

Lemma 2.28 ([Nad, Proposition 2.3])**.** *Let $\pi_{d+1,d}\colon Y_{d+1} \to Y_d$ be the natural projection. Then the pullback functor $\pi^*_{d+1,d}\colon \mathrm{Coh}(Y_d) \to \mathrm{Coh}(Y_{d+1})$ induces an equivalence*

$$\mathrm{Coh}(Y_d)_{\mathbb{Z}_2} \simeq \mathrm{MF}(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2})$$

which sends $\mathscr{O}^i_d$ to $\underline{\mathscr{O}}^i_{d+1}$ for $1 \le i \le d+1$ and u to y_{d+2}.

2.5. **Homological mirror symmetry.** Let $\mathcal{I}_{d+1}$ be the category whose objects are subsets $I \subset \{1, \ldots, d+1\}$ and whose morphisms are given by inclusions. We denote by $\mathcal{I}^\circ_{d+1}$ the full subcategory of proper subsets. For $I \in \mathcal{I}^\circ_{d+1}$ we define Λ_I as the product conic Lagrangian $(\Lambda_1)^I \subset (T^*T^1)^I$. Consider the hyperbolic restriction

$$\eta_{I\subset I'} = (p_{I\subset I'})_*(q_{I\subset I'})^!\colon\ \mathrm{Sh}^\diamond_{\Lambda_{I'}}(T^{I'}) \to \mathrm{Sh}^\diamond_{\Lambda_I}(T^I)$$

where $p_{I\subset I'}\colon T^I \times [0, \frac{1}{2})^{I'\setminus I} \to T^I$ is the projection and $q_{I\subset I'}\colon T^I \times [0, \frac{1}{2})^{I'\setminus I} \to T^{I'}$ is the canonical inclusion. Note that $\eta_{I\subset I'}$ is the product of hyperbolic restrictions in the coordinate directions indexed by $I' \setminus I$ and the identity in the coordinate directions indexed by I. We denote by η_I the hyperbolic restriction with $I' = \{1, \ldots, d+1\}$.

Lemma 2.29 ([Nad, Lemma 5.25, 5.26])**.** *There is an equivalence*

$$\mathrm{Sh}^\diamond_{\Lambda_{d+1}}(T^{d+1}) \simeq \mathrm{IndCoh}(\mathbb{A}^{d+1})$$

which makes the diagram

$$\begin{tikzcd}
\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}(T^{d+1}) \arrow[r, "\sim"] \arrow[d, "\eta_I"'] & \mathrm{IndCoh}(\mathbb{A}^{d+1}) \arrow[d, "\iota_I^*"] \\
\mathrm{Sh}^{\diamond}_{\Lambda_I}(T^I) \arrow[r, "\sim"] & \mathrm{IndCoh}(\mathbb{A}^I)
\end{tikzcd}$$

commute, where

$$\iota_I\colon \mathbb{A}^I = \operatorname{Spec}\mathbb{C}[y_i \mid i \in I] \hookrightarrow \mathbb{A}^{d+1} = \operatorname{Spec}\mathbb{C}[y_1, \dots, y_{d+1}]$$

is the canonical inclusion of the subvariety defined by $y_j = 0$ for $j \in I^c$.

For $I \in \mathcal{I}^{\circ}_{d+1}$ we define Ω_I as the open conic subset

$$\Omega_I = \{(\theta, \xi) \in T^*T^{d+1} | \Sigma_{i=1}^{d+1} \xi_i > 0, \xi_j \neq 0 \text{ for } j \in I^c\} \subset \Omega_{d+1}.$$

The collection $\{\Omega_I\}_{I \in \mathcal{I}^{\circ}_{d+1}}$ forms an open conic cover of Ω_{d+1} satisfying $\Omega_{I \cap I'} = \Omega_I \cap \Omega_{I'}$. Note that we have $\Omega_I \subset \Omega_{I'}$ whenever $I \subset I'$. Let ** DG be the category of cocomplete dg categories and functors which preserve colimits and compact objects. Consider a functor

$$\mu\mathrm{Sh}^{\diamond}\colon (\mathcal{I}^{\circ}_{d+1})^{op} \to {}^{**}\,\mathrm{DG},\ I \mapsto \mu\mathrm{Sh}^{\diamond}_{\Lambda_I}(\Omega_I) = \mu\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}(\Omega_I)$$

which sends inclusions $I \subset I'$ to the restriction functors $\rho_{I \subset I'}$ along the inclusions $\Omega_I \subset \Omega_{I'}$. We denote by ρ_I the restriction functor with $I' = \{1, \dots, d+1\}$. As $\mu\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}$ forms a sheaf, the canonical functor

$$\mu\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}(\Omega_{d+1}) \to \lim_{I \in (\mathcal{I}^{\circ}_{d+1})^{op}} \mu\mathrm{Sh}^{\diamond}_{\Lambda_I}(\Omega_I)$$

is an equivalence.

Theorem 2.30 ([Nad, Theorem 5.27])**.** *There is an equivalence*

$$\mu\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}(\Omega_{d+1}) \simeq \mathrm{IndCoh}(Y_d) = \lim_{I \in (\mathcal{I}^{\circ}_{d+1})^{op}} \mathrm{IndCoh}(\mathbb{A}^I)$$

which makes the diagram

$$\begin{tikzcd}
\mu\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}(\Omega_{d+1}) \arrow[r, "\sim"] \arrow[d, "\rho_I"'] & \mathrm{IndCoh}(Y_d) \arrow[d, "\tau_I"] \\
\mu\mathrm{Sh}^{\diamond}_{\Lambda_I}(\Omega_I) \arrow[r, "\sim"] & \mathrm{IndCoh}(\mathbb{A}^I)
\end{tikzcd}$$

commute, where τ_I is the canonical functor.

Proof. There is a natural isomorphism $\mu\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}} \to \mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}$ induced by η_I. Indeed, η_I factors through the microlocalization

$$\mathrm{Sh}^{\diamond}_{\Lambda_{d+1}}(T^{d+1}) \to \mu\mathrm{Sh}^{\diamond}_{\Lambda_I}(\Omega_I) \xrightarrow{\tilde{\eta}_I} \mathrm{Sh}^{\diamond}_{\Lambda_I}(T^I)$$

and for inclusions $I \subset I'$ the diagrams

$$\begin{tikzcd}
\mu\mathrm{Sh}^{\diamond}_{\Lambda_{I'}}(\Omega_{I'}) \arrow[r, "\tilde{\eta}_{I'}"] \arrow[d, "\rho_{I \subset I'}"'] & \mathrm{Sh}^{\diamond}_{\Lambda_{I'}}(T^{I'}) \arrow[d, "\eta_{I \subset I'}"] \\
\mu\mathrm{Sh}^{\diamond}_{\Lambda_I}(\Omega_I) \arrow[r, "\tilde{\eta}_I"] & \mathrm{Sh}^{\diamond}_{\Lambda_I}(T^I)
\end{tikzcd}$$

commute. Note that the hyperbolic restriction in the coordinate direction indexed by $j \in I^c$ vanishes on sheaves whose microsupport does not intersect the locus $\{\xi_j > 0\} \subset T^*T^{d+1}$. For each $I \in (\mathcal{I}^\circ_{d+1})^{op}$ the functor $\tilde{\eta}_I$ is an equivalence, since it admits an inverse induced by the pushforward

$$\mathrm{Sh}^\diamond_{\Lambda_I}(T^I) \to \mathrm{Sh}^\diamond_{\Lambda_{d+1}}(T^{d+1})$$

along the inclusion $T^I \hookrightarrow T^{d+1}$. □

Corollary 2.31 ([Nad, Corollary 5.28])**.** *There is an equivalence*

$$\mathrm{Fuk}(\tilde{P}_d)_{\mathbb{Z}_2} \simeq \mu\mathrm{Sh}^\diamond_{\Lambda_{d+1}}(\Omega_{d+1})_{\mathbb{Z}_2} \simeq \mathrm{MF}^\infty(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2})$$

where $\mathrm{MF}^\infty(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2})$ *is the ind-completion of* $\mathrm{MF}(\mathbb{A}^{d+2}, y_1 \cdots y_{d+2})$*.*

Proof. The first equivalence follows from Lemma 2.21. Taking compact objects of the equivalence from Theorem 2.30 and passing to $\mathbb{Z}_2$-folding, one obtains the second equivalence from Lemma 2.28. □

3. Critical loci of Landau–Ginzburg models for very affine hypersufaces

In this section, following [AAK16, Section 3], we realize the mirror pair for very affine hypersuface as critical loci of associated Landau–Ginzburg models. They give rise to fibrations over the tropical hypersurface equipped with the canonical stratification.

3.1. **Very affine hypersurfaces.** Let $\mathbb{T} = M^\vee_{\mathbb{R}/\mathbb{Z}} = M^\vee_{\mathbb{R}}/M^\vee$ be a real $(d+1)$-dimensional torus with cocharacter lattice $M^\vee$. We denote by $\mathbb{T}_{\mathbb{C}} = M^\vee_{\mathbb{C}^*}$ the associated complex torus. Taking its dual, one obtains the complex torus $\mathbb{T}^\vee_{\mathbb{C}} = M_{\mathbb{C}^*}$ associated with $\mathbb{T}^\vee = M_{\mathbb{R}/\mathbb{Z}} = M_{\mathbb{R}}/M$ whose cocharacter lattice is M. We choose an inner product to identify $T\mathbb{T}^\vee$ with $T^*\mathbb{T}^\vee$ and regard $\mathbb{T}^\vee_{\mathbb{C}} \cong T\mathbb{T}^\vee \cong T^*\mathbb{T}^\vee$ as an exact symplectic manifold equipped with the standard Liouville structure.

Definition 3.1. Let $\mathcal{T}$ be a triangulation of a lattice polytope $\Delta^\vee \subset M^\vee_{\mathbb{R}}$. We call $\mathcal{T}$ *adapted* if there is a convex piecewise function $\rho\colon \Delta^\vee \to \mathbb{R}$ whose corner locus is $\mathcal{T}$. We call $\mathcal{T}$ *unimodular* if each cell is congruent to the standard $(d+1)$-simplex Δ_{d+1} under the $\mathrm{GL}(d+1,\mathbb{Z})$-action.

For a latiice polytope $\Delta^\vee \subset M^\vee_{\mathbb{R}}$, choose an adapted unimodular triangulation $\mathcal{T}$. We denote by A the set of vertices of $\mathcal{T}$. In other words, $\mathcal{T}$ is the convex hull $\mathrm{Conv}(A)$ of A. The convex piecewise function $\rho\colon \Delta^\vee \to \mathbb{R}$ defines a Laurent polynomial

$$W_t\colon \mathbb{T}^\vee_{\mathbb{C}} \to \mathbb{C},\ x \mapsto \sum_{\alpha \in A} c_\alpha t^{-\rho(\alpha)} x^\alpha \tag{3.1}$$

in coordinates $x = (x_1, \ldots, x_{d+1})$ on $\mathbb{T}^\vee_{\mathbb{C}}$, where $c_\alpha \in \mathbb{C}^*$ are arbitrary constants and $t \gg 0$ is a tropicalization parameter.

Definition 3.2. For sufficiently general $t \gg 0$ we call the hypersurface $H_t = W_t^{-1}(0)$ *very affine*.

Since t is sufficiently general, a very affine hypersurface H_t is smooth. Due to the above choice of inner product, we may regard H_t as a Liouville submanifold of $\mathbb{T}^\vee_{\mathbb{C}}$.

Definition 3.3. The *amoeba* Π_t of H_t is its image under $\mathrm{Log}_{d+1}\colon \mathbb{T}^\vee_{\mathbb{C}} \to \mathbb{R}^{d+1}$.

Definition 3.4. The *tropical hypersurface* Π_Σ associated with H_t is the hypersurface defined by the *tropical polynomial*

$$\varphi\colon M_{\mathbb{R}} \to \mathbb{R},\ \varphi(m) = \max\{\langle m, n\rangle - \rho(n) \mid n \in \Delta^\vee\}.$$

Namely, Π_Σ is the set of points where the maximum is achieved more than once.

According to [Mik04, Corollary 6.4], when $t \to \infty$ the rescaled amoeba $\Pi_t/\log t$ converges to Π_Σ. It is known that Π_Σ is a deformation retract of Π_t for $t \gg 0$. Combinatorially, Π_Σ is the dual cell complex of $\mathcal{T}$. In particular, the set of connected components of $\mathbb{R}^{d+1} \setminus \Pi_\Sigma$ bijectively corresponds to A according to which $\alpha \in A$ achieves the maximum of $\langle m, \alpha\rangle - \rho(\alpha)$ for $m \in \mathbb{R}^{d+1} \setminus \Pi_\Sigma$. Note that $\mathbb{R}^{d+1} \setminus \Pi_t$ for $t \gg 0$ has the same combinatrics as $\mathbb{R}^{d+1} \setminus \Pi_\Sigma$.

Remark 3.5. Each connected component of $\mathbb{R}^{d+1}\setminus\Pi_\Sigma$ is the locus where the monomial $c_\alpha t^{-\rho(\alpha)} x^\alpha$ becomes dominant.

In the sequel, we will fix a general $t \gg 0$ and drop t from the notation.

3.2. **Landau–Ginzburg A-models for very affine hypersurfaces.** For $X = \mathbb{T}^\vee_\mathbb{C} \times \mathbb{C}$ with coordinates $(x, u) = (x_1, \dots, x_{d+1}, u)$, consider a Laurent polynomial

$$W_X \colon X \to \mathbb{C},\ (x, u) \mapsto uW(x)$$

where W is the Laurent polynomial (3.1).

Definition 3.6. Let $H \subset \mathbb{T}^\vee_\mathbb{C}$ be a very affine hypersurface defined by the Laurent polynomial W from (3.1). We call the pair (X, W_X) the *Landau–Ginzburg A-model* for H.

Definition 3.7. The *Newton polytope* $\Delta^\vee_X$ of W_X is the convex hull

$$\mathrm{Conv}(0, -\Delta^\vee \times \{1\}) \subset M^\vee_\mathbb{R} \times \mathbb{R}.$$

Remark 3.8. The polytope $\Delta^\vee_X$ admits an adapted unimodular star-shaped triangulation $\tilde{\mathcal{T}}$ canonically induced by $\mathcal{T}$. Recall that a triangulation of $\Delta^\vee_X$ is *star-shaped* if all of its simplices not contained in the boundary $\partial\Delta^\vee_X$ share a common vertex 0 [GS1, Definition 3.3.1].

Lemma 3.9. *The critical locus* $\mathrm{Crit}(W_X)$ *is given by* $\{u = 0\} \cap \{W = 0\} \subset X$.

Proof. Express the tangent map dW_X of W_X as a vector (udW, W). Since $H \subset \mathbb{T}^\vee_\mathbb{C}$ is smooth, dW nowhere vanishes. Hence $\mathrm{rank}(dW_X) = 0$ if and only if $u = 0$ and $W = 0$. □

Remark 3.10. By Lemma 3.9 the projection $\mathrm{pr}_1 \colon X = \mathbb{T}^\vee_\mathbb{C} \times \mathbb{C} \to \mathbb{T}^\vee_\mathbb{C}$ preserves $\mathrm{Crit}(W_X)$. Let $\mathrm{ret} \colon \Pi \to \Pi_\Sigma$ be the continuous map induced by the retraction. Then the composition

$$f \colon H \cong \mathrm{Crit}(W_X) \hookrightarrow X \xrightarrow{\mathrm{pr}_1} \mathbb{T}^\vee_\mathbb{C} \xrightarrow{\mathrm{Log}_{d+1}} \Pi \xrightarrow{\mathrm{ret}} \Pi_\Sigma \tag{3.2}$$

gives the fibration from [Mik04, Theorem 1']. Recall that k-th intersections of legs of a pair of pants has torus factor of dimension k. Away from lower dimensional strata, the fiber over a point in a k-stratum contains a real k-torus in the torus factor.

3.3. **Landau–Ginzburg B-models for very affine hypersurfaces.** Let Y be the noncompact $(d + 2)$-dimensional toric variety associated with the fan

$$\Sigma_Y = \mathrm{Cone}(-\mathcal{T} \times \{1\}) \subset M^\vee_\mathbb{R} \times \mathbb{R}.$$

The primitive ray generators of Σ_Y are the vectors of the form $(-\alpha, 1)$ with $\alpha \in A$. Such vectors span a smooth cone of Σ_Y if and only if α span a cell of $\mathcal{T}$.

Dually, Y is associated with the noncompact moment polytope

$$\Delta_Y = \{(m, u) \in M_\mathbb{R} \times \mathbb{R} \mid u \geq \varphi(m)\}.$$

The facets of Δ_Y correspond to the maximal domains of linearity of φ. Hence the irreducible toric divisors of Y bijectively correspond to the connected components of $\mathbb{R}^{d+1} \setminus \Pi_\Sigma$. In particular, the combinatrics of toric strata of Y can be read off Π_Σ.

Remark 3.11. The noncompact polytope Δ_Y is homeomorphic to the image of Y under the composition

$$Y \to (Y)_{\geq 0} \to M_{\mathbb{R}} \times \mathbb{R} \tag{3.3}$$

of the map induced by retraction to the nonnegative real points with the restriction of negated algebraic moment map.

Lemma 3.12. *Let $q\colon M_{\mathbb{R}} \times \mathbb{R} \to M_{\mathbb{R}}$ be the natural projection. Then under q the union of facets of Δ_Y homeomorphically maps to $M_{\mathbb{R}}$. Moreover, the union of codimension 2 faces of Δ_Y homeomorphically maps to Π_{Σ}.*

Proof. By construction of Σ_Y under q each facet of Δ_Y homeomorphically maps to the maximal domain of linearity of φ corresponding to the same $\alpha \in A$. Any codimension 2 face of Δ_Y can be obtained as the intersection of two distinct facets. Hence q restricted to the union of codimension 2 faces of Δ_Y gives an injection to Π_{Σ}. This is also surjective, as each full dimensional face of Π_{Σ} is adjacent to exactly two maximal domains of linearity. □

For each $\alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in A$ let $Y_\alpha = (\mathbb{C}^*)^{d+1} \times \mathbb{C}$ with coordinates $y_\alpha = (y_{\alpha,1}, \ldots, y_{\alpha,d+1}, v_\alpha)$, where $y_{\alpha,1}, \ldots, y_{\alpha,d+1}, v_\alpha$ are the monomials with weights

$$\eta_1 = (-1, 0, \ldots, 0, -\alpha_1), \ldots, \eta_{d+1} = (0, \ldots, 0, -1, -\alpha_{d+1}), \eta_{d+2} = (0, \ldots, 0, 1) \in M \times \mathbb{Z}.$$

Their pairing with the monomial with weight $(-\alpha, 1) \in M^\vee \times \mathbb{Z}$ yields $0, \ldots, 0, 1$ respectively.

Lemma 3.13. *The complex algebraic variety Y_α is the affine open subset of Y associated with the ray spanned by $(-\alpha, 1) \in M^\vee \times \mathbb{Z}$.*

Proof. Suppose that $\sigma \in \Sigma_Y(1)$ is the cone associated with the affine open subset $Y_\alpha \subset Y$. We have

$$\operatorname{div}(y_{\alpha,i}^{\pm 1}) = \sum_{\xi \in \sigma(1)} \langle \pm\eta_i, u_\xi \rangle D_\xi, \ \operatorname{div}(v_\alpha) = \sum_{\xi \in \sigma(1)} \langle \eta_{d+2}, u_\xi \rangle D_\xi,$$

where u_ξ are primitive ray generators of ξ and $D_\xi = \overline{O(\xi)}$ are the closures of the orbits corresponding to ξ. Since $y_{\alpha,1}^{\pm 1}, \ldots, y_{\alpha,d+1}^{\pm 1}$ never vanish on Y_α, pairing of η_i with the primitive ray generators in σ must yield 0 for $1 \leq i \leq d+1$. On the other hand, pairing of η_{d+2} with the primitive ray generators of σ must yield 1. □

Due to the above lemma, Y_α covers the open stratum of Y and the open stratum of the irreducible toric divisor corresponding to α. If $\alpha, \beta \in A$ are connected by an edge in $\mathcal{T}$, then we glue Y_α to Y_β with the coordinate transformations

$$y_{\alpha,i} = v_\beta^{\beta_i - \alpha_i} y_{\beta,i}, \ v_\alpha = v_\beta, \ 1 \leq i \leq d+1.$$

Thus the coordinate charts $\{Y_\alpha\}_{\alpha \in A}$ cover the complement in Y of the codimension more than 1 strata.

We may write v for v_α as it does not depend on the choice of $\alpha \in A$. Since the weight $(0, \ldots, 0, 1)$ pairs nonnegatively with the primitive ray generators of Σ_Y, the monomial v defines a regular function on Y, which we denote by W_Y.

Definition 3.14. Let $H \subset \mathbb{T}^\vee_{\mathbb{C}}$ be a very affine hypersurface defined by the Laurent polynomial W from (3.1). We call the pair (Y, W_Y) the *Landau–Ginzburg B-model* for H.

Remark 3.15. The pair (Y, W_Y) is a conjectural SYZ mirror to H [AAK16, Theorem 1.6].

The critical locus $\operatorname{Crit}(W_Y)$ is the preimage of the codimension 2 strata of Δ_Y under (3.3). One can check this locally in each affine chart, which is isomorphic to $\mathbb{C}^{d+2}$ as $\mathcal{T}$ is unimodular.

Lemma 3.16. *The critical locus* $\mathrm{Crit}(W_Y)$ *is given by* $\bigcup_{\alpha\in A}\overline{Y}_\alpha\setminus Y_\alpha\subset Y$.

Proof. For each $\alpha\in A$ the intersection $\mathrm{Crit}(W_Y)\cap Y_\alpha$ is empty. Indeed, when restricted to Y_α, the tangent map dW_Y of W_Y is expressed as a vector whose last factor is 1. Hence $dW_Y|_{Y_\alpha}$ is surjective and we obtain

$$\mathrm{Crit}(W_Y)\subset Y\setminus\bigcup_{\alpha\in A}Y_\alpha=\bigcup_{\alpha\in A}\overline{Y}_\alpha\setminus Y_\alpha.$$

Take any point $y\in\overline{Y}_\alpha\setminus Y_\alpha$. Suppose that there is a vertex $\alpha'\in A$ connected with α by an edge in $\mathcal{T}$ such that $y\in\overline{Y}_{\alpha'}\setminus Y_{\alpha'}$. Let $\sigma\in\Sigma_Y$ be the cone generated by two rays $\xi_\alpha=\mathrm{Cone}(-\alpha,1),\xi_{\alpha'}=\mathrm{Cone}(-\alpha',1)$. Then we have

$$\mathrm{div}(v)=\langle\eta_{d+2},u_{\xi_\alpha}\rangle D_{\xi_\alpha}+\langle\eta_{d+2},u_{\xi_{\alpha'}}\rangle D_{\xi_{\alpha'}}$$

on the associated affine open subset $\mathrm{Spec}\,\mathbb{C}[\sigma^\vee\cap(M\times\mathbb{Z})]\cong(\mathbb{C}^*)^d\times\mathbb{C}^2$ of Y. Hence the restriction of dW_Y vanishes on $D_{\xi_\alpha}\cap D_{\xi'_{\alpha'}}$, which is

$$((\overline{Y}_\alpha\setminus Y_\alpha)\cap(\overline{Y}_{\alpha'}\setminus Y_{\alpha'}))|_{\mathrm{Spec}\,\mathbb{C}[\sigma^\vee\cap(M\times\mathbb{Z})]}.$$

The union of such intersections for all $\alpha'\in A$ is $\overline{Y}_\alpha\setminus Y_\alpha$. Applying the same argument to the other cases, we obtain

$$\bigcup_{\alpha\in A}\overline{Y}_\alpha\setminus Y_\alpha\subset\mathrm{Crit}(W_Y).$$

□

Remark 3.17. Since the map (3.3) sends each k-th intersection of $D_{\xi_\alpha},\alpha\in A$ to a codimension k face of Δ_Y, by Lemma 3.16 it sends $\mathrm{Crit}(W_Y)$ to the union of codimension 2 faces. On the other hand, by Lemma 3.12 the map $q\colon M_\mathbb{R}\times\mathbb{R}\to M_\mathbb{R}$ homeomorphically sends the union of codimension 2 faces of Δ_Y to Π_Σ. Hence the composition

$$g\colon\ \mathrm{Crit}(W_Y)\hookrightarrow Y\xrightarrow{(3.3)}\Delta_Y\xrightarrow{q}\Pi_\Sigma\tag{3.4}$$

gives a fibration. The fiber over a point in a k-stratum is a real k-torus [CLS11, Prop 12.2.3(b)].

4. Constructible sheaves of categories

In this section, we define two constructible sheaves of categories over the tropical hypersurface $\Pi_\Sigma\subset\mathbb{R}^{d+1}$ with the canonical stratification and a certain topology generated by the vertices. In the sequel, by a *Liouville manifold* we mean a Liouville manifold of finite type, i.e., the completion of some Liouville domain. By a *Weinstein manifold* we mean a Liouville manifold together with a Morse–Bott function constant on the cylindrical ends for which the Liouville vector field is gradient-like.

Definition 4.1. We introduce into Π_Σ with the canonical stratification a topology defined as follows. To a vertex $v\in\Pi_\Sigma$ we define the associated open subset U_v as the union of all strata adjacent to v, which is homeomorphic to a d-dimensional tropical pants. To an edge $e\subset\Pi_\Sigma$ connecting two vertices v_1,v_2 we define the associated open subset U_e as the intersection $U_{v_1}\cap U_{v_2}$. Similarly, to each k-stratum $S^{(k)}\subset\Pi_\Sigma$ adjacent to l vertices $v_1,\dots,v_l$ we define the associated open subset $U_{S^{(k)}}$ as the intersection $U_{v_1}\cap\cdots\cap U_{v_l}$. A general open subset U is of the form $U_{S^{(k_1)}}\cup\cdots\cup U_{S^{(k_m)}}$ for some strata $S^{(k_1)},\dots,S^{(k_m)}\subset\Pi_\Sigma$.

4.1. **A-side partially presheaves of categories for very affine hypersurfaces.** Fix a pants decomposition of $H \subset \mathbb{T}^\vee_{\mathbb{C}} \cong T^*T^{d+1}$ [Mik04, Theorem 1'].

Definition 4.2. The *A-side partially presheaf* $\mathcal{F}^{pre}_A$ of categories for H is a collection

$$\{\mathcal{F}^{pre}_A(U_{S^{(k)}}), R^A_{S^{(k)},S^{(l)}}\}$$

of sections and restriction functors defined on connected open subsets of Π_Σ as follows:

- The section over $U_{S^{(k)}}$ is given by $\mathbb{Z}_2$-folding of the ind-completion of the wrapped Fukaya category
$$\mathcal{F}^{pre}_A(U_{S^{(k)}}) = \mathrm{Fuk}(H_{S^{(k)}})_{\mathbb{Z}_2}$$
where $H_{S^{(k)}}$ is the inverse image of suitably shrunk $U_{S^{(k)}}$ under (3.2). In other words, $H_{S^{(k)}}$ is symplectomorphic to the intersection of k legs of $\tilde{P}_d$.
- Along an inclusion $U_{S^{(l)}} \hookrightarrow U_{S^{(k)}}$ the restriction functor is given by $\mathbb{Z}_2$-folding of the ind-completion of the Viterbo restriction [GPS2, Section 11.1]
$$R^A_{S^{(k)},S^{(l)}} = (V^A_{S^{(k)},S^{(l)}})_{\mathbb{Z}_2}\colon \mathrm{Fuk}(H_{S^{(k)}})_{\mathbb{Z}_2} \to \mathrm{Fuk}(H_{S^{(l)}})_{\mathbb{Z}_2}.$$

We will show that $\mathcal{F}^{pre}_A$ is well defined. First, since our Liouville manifolds are of finite type, the section is unique up to canonical equivalence. In particular, the sections are well defined. The following is a special case of [GPS2, Lemma 3.4].

Lemma 4.3. *Let $\lambda_{S^{(k)}}, \lambda'_{S^{(k)}}$ be the completions of two Liouville forms on a Liouville domain $[H_{S^{(k)}}]$ completing to $H_{S^{(k)}}$. Then there is a canonical equivalence*

$$\mathrm{Fuk}(H_{S^{(k)}}, \lambda_{U^{(k)}_S}) = \mathrm{Fuk}(H_{S^{(k)}}, \lambda'_{U^{(k)}_S}).$$

Proof. Our argument is essentially the same as [Jef22, Lemma 2]. Since the space of Liouville forms for a compact symplectic manifold-with-boundary is convex, any two Liouville forms on $[H_{S^{(k)}}]$ are canonically homotopic and the homotopy completes to that for $\lambda_{S^{(k)}}, \lambda'_{S^{(k)}}$. Then one can apply [CE12, Proposition 11.8] to obtain a strictly exact symplectomorphism $\psi\colon H_{S^{(k)}} \to H_{S^{(k)}}$. By definition it satisfies $\psi^*\lambda' - \lambda = df$ for some compactly supported function $f\colon H_{S^{(k)}} \to \mathbb{R}$. In particular, ψ defines a trivial inclusion of open Liouville sectors in the sense of [GPS2, Definition 3.3]. Then one can apply [GPS2, Lemma 3.4] to see that the pushforward functor from [GPS1, Section 3.6] gives the canonical equivalence. □

In the sequel, we drop Liouville structures from the notation. Since by [Mik04, Remark 5.2] each piece of the pants decomposition can be made symplectomorphic to $\tilde{P}_d$, we obtain

Corollary 4.4. *For each vertex $v \in \Pi_\Sigma$ the section $\mathcal{F}^{pre}_A(U_v)$ is given by* $\mathrm{Fuk}(\tilde{P}_d)_{\mathbb{Z}_2}$.

Let $\tilde{P}_{S^{(k)}}$ be the intersection of k-legs of $\tilde{P}_d$ mapping onto $U_{S^{(k)}}$ under Log_{d+1}.

Corollary 4.5. *For each $U_{S^{(k)}} = \cap^l_{i=1} U_{v_i}$ the section $\mathcal{F}^{pre}_A(U_{S^{(k)}})$ is given by* $\mathrm{Fuk}(\tilde{P}_{S^{(k)}})_{\mathbb{Z}_2}$.

Next, along an inclusion $U_{S^{(l)}} \hookrightarrow U_{S^{(k)}}$ the restriction functor comes from a certain quotient functor. In particular, the restriction functors are well defined.

Lemma 4.6. *Along an inclusion $U_{S^{(l)}} \hookrightarrow U_{S^{(k)}}$ the Viterbo restriction functor is given by the quotient by the cocores of $\tilde{P}_{S^{(k)}}$ not in $\tilde{P}_{S^{(l)}}$. Here, we regagrd $[\tilde{P}_{S^{(l)}}]$ as a Weinstein subdomain of $\tilde{P}_{S^{(k)}}$ with respect to Nadler's Weinstein structure.*

Proof. As the other cases can be proved similarly, we restrict ourselves to the case where $S^{(k)} = U_v$ and $S^{(l)} = U_e$ for some edge e connecting v with v'. By Corollary 4.5 the section $\mathcal{F}^{pre}_A(U_e)$ is given by $\mathrm{Fuk}(\tilde{P}_e)_{\mathbb{Z}_2}$. Permuting legs by $\mathfrak{S}_{d+2}$-action if necessary, we may assume that $\tilde{P}_e$ does not correspond to the final leg of $\tilde{P}_v$. Since both $[\tilde{P}_e]$ and the cobordism $[\tilde{P}_v] \setminus [\tilde{P}_e]^\circ$ are Weinstein, one can apply [GPS2, Proposition 11.2] to see that the Viterbo restriction coincides with the quotient by the cocores of $\tilde{P}_v$ not in $\tilde{P}_e$. □

4.2. *A***-side constructible sheaves of categories for very affine hypersurfaces.** Since U_v for all $v \in \mathrm{Vert}(\Pi_\Sigma)$ form a subbase of the topology of Π_Σ, we may pass to its sheafification.

Definition 4.7. The *A-side constructible sheaf of categories* for H is the sheafification

$$\mathcal{F}_A \colon \mathrm{Open}(\Pi_\Sigma)^{op} \to {}^{**}\,\mathrm{DG},$$

where $\mathrm{Open}(\Pi_\Sigma)$ is the category of open subsets of Π_Σ with respect to the topology defined in Definition 4.1.

Remark 4.8. In general, the existence of sheafification might be delicate because of size issues. However, this is not the case in our setting as ${}^{**}\,\mathrm{DG}$ has small limits and colimits, and the topology on Π_Σ has finite cardinality.

We will show that the global section is given by the wrapped Fukaya category of H. In our proof, the following two lemmas play key roles.

Lemma 4.9. *Let $\mathscr{C}$ be a stable presentable dg category and $\mathscr{A}, \mathscr{B}$ its full presentable dg subcategories such that*

$$\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A) = 0$$

for any $A \in \mathscr{A}, B \in \mathscr{B}$. Then there is a fiber product

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathscr{C}/\mathscr{B} \arrow[d] \\ \mathscr{C}/\mathscr{A} \arrow[r] & \mathscr{C}/\langle \mathscr{A}, \mathscr{B} \rangle. \end{tikzcd}$$

Proof. Since we have the pushouts

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ 0 \arrow[r] & \mathscr{C}/\mathscr{A}, \end{tikzcd} \qquad \begin{tikzcd} \mathscr{B} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ 0 \arrow[r] & \mathscr{C}/\mathscr{B}, \end{tikzcd}$$

the Verdier localizations $\mathscr{C} \to \mathscr{C}/\mathscr{A}, \mathscr{C} \to \mathscr{C}/\mathscr{B}$ admit right adjoints as well as the inclusions $\mathscr{A} \hookrightarrow \mathscr{C}, \mathscr{B} \hookrightarrow \mathscr{C}$. Hence we obtain two semiorthogonal decompositions of $\mathscr{C}$ by $\mathscr{A}, \mathscr{A}^\perp$ and by $\mathscr{B}, \mathscr{B}^\perp$, which respectively yield cofiber sequences

$$C_{\mathscr{A}} \to C \to C_{\mathscr{A}^\perp},\ C_{\mathscr{B}} \to C \to C_{\mathscr{B}^\perp},\ C_{\mathscr{A} \oplus \mathscr{B}} \to C \to C_{(\mathscr{A} \oplus \mathscr{B})^\perp}$$

for any object $C \in \mathscr{C}$. Note that the full dg subcategory $\langle \mathscr{A}, \mathscr{B} \rangle \subset \mathscr{C}$ is equivalent to $\mathscr{A} \oplus \mathscr{B}$, as $\mathscr{A}, \mathscr{B}$ are mutually orthogonal. Here, the morphism $C_{\mathscr{A} \oplus \mathscr{B}} \to C$ in the last cofiber sequence is the direct sum of that $C_{\mathscr{A}} \to C, C_{\mathscr{B}} \to C$ in the first two. Since $\mathscr{A}, \mathscr{B}$ are mutually orthogonal, its cone can be computed by taking the cone of $C_{\mathscr{B}} \to C$ followed by taking the cone of $C_{\mathscr{A}} \to C_{\mathscr{B}^\perp}$. Hence we obtain a cofiber sequence

$$C_{\mathscr{A}} \to C_{\mathscr{B}^\perp} \to C_{(\mathscr{A} \oplus \mathscr{B})^\perp}.$$

The conclusion is equivalent to there being a fiber product

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(C_1, C_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{C}/\mathscr{B}}(C_1, C_2) \arrow[d] \\ \mathrm{Hom}_{\mathscr{C}/\mathscr{A}}(C_1, C_2) \arrow[r] & \mathrm{Hom}_{\mathscr{C}/\langle \mathscr{A}, \mathscr{B} \rangle}(C_1, C_2) \end{tikzcd} \tag{4.1}$$

of morphism complexes for any $C_1, C_2 \in \mathscr{C}$. Note that it suffices to check the latter when $C = C_1 = C_2$. Since $C_{\mathscr{A}^\perp}, C_{\mathscr{B}^\perp}, C_{(\mathscr{A}\oplus\mathscr{B})^\perp}$ are the images of C under the right adjoints of the Verdier localizations $\mathscr{C} \to \mathscr{C}/\mathscr{A}, \mathscr{C} \to \mathscr{C}/\mathscr{B}, \mathscr{C} \to \mathscr{C}/\langle\mathscr{A}, \mathscr{B}\rangle$, one can rewrite (4.1) as

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(C, C) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{C}}(C, C_{\mathscr{B}^\perp}) \arrow[d] \\ \mathrm{Hom}_{\mathscr{C}}(C, C_{\mathscr{A}^\perp}) \arrow[r] & \mathrm{Hom}_{\mathscr{C}}(C, C_{(\mathscr{A}\oplus\mathscr{B})^\perp}). \end{tikzcd}$$

Now, as the functor $\mathrm{Hom}_{\mathscr{C}}(C, -)$ preserves fiber products, it suffices to show that

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_{\mathscr{B}^\perp} \arrow[d] \\ C_{\mathscr{A}^\perp} \arrow[r] & C_{(\mathscr{A}\oplus\mathscr{B})^\perp} \end{tikzcd} \tag{4.2}$$

is a fiber product in $\mathscr{C}$. Consider the diagram

$$\begin{tikzcd} C_{\mathscr{A}} \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C_{\mathscr{B}^\perp} \arrow[d] \\ 0 \arrow[r] & C_{\mathscr{A}^\perp} \arrow[r] & C_{(\mathscr{A}\oplus\mathscr{B})^\perp}. \end{tikzcd} \tag{4.3}$$

Since both the left and the outer squares are pushouts, the right square is also a pushout. It follows that (4.2) is a fiber product, as for stable dg categories any fiber product is a bicartesian. □

Lemma 4.10. *Let W be a d-dimensional Weinstein manifold with a fixed pants decomposition. Consider the gluing $W \cup \tilde{P}_d$ of Weinstein manifolds W with $\tilde{P}_d$ along a union of l legs of $\tilde{P}_d$ for $1 \le l < d+2$ with restricted Nadler's Weinstein structures. Then we have*

$$\mathrm{Hom}_{\mathrm{Fuk}(W\cup\tilde{P}_d)}(L_1, L_2) = \mathrm{Hom}_{\mathrm{Fuk}(W\cup\tilde{P}_d)}(L_2, L_1) = 0$$

for cocores L_1, L_2 of $W \cup \tilde{P}_d$ respectively not in $\tilde{P}_d, W$.

Proof. As the other cases can be proved similarly, we restrict ourselves to the simplest case. Consider the gluing $\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d$ of two d-dimensional pairs of pants along a leg C, where $\tilde{P}^1_d, \tilde{P}^2_d$ are equipped with Nadler's Weinstein structures. Here, we choose their final legs different from C. Then $[\tilde{P}^1_d], [\tilde{P}^2_d]$ are Weinstein subdomains of $[\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d]$. Since the cobordisms $[\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d] \setminus [\tilde{P}^1_d]^\circ$, $[\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d] \setminus [\tilde{P}^2_d]^\circ$ are also Weinstein, by [GPS2, Proposition 11.2] the Viterbo restriction functors

$$\mathrm{Fuk}^\omega(\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d) \to \mathrm{Fuk}^\omega(\tilde{P}^1_d),\ \mathrm{Fuk}^\omega(\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d) \to \mathrm{Fuk}^\omega(\tilde{P}^2_d)$$

are the quotients by the cocores respectively not in $\tilde{P}^1_d, \tilde{P}^2_d$. Here, $(-)^\omega$ denotes taking compact objects.

Let L_1, L_2 be cocores of $\tilde{P}^1_d \cup_{\mathrm{Core}(C)} \tilde{P}^2_d$ respectively not in $\tilde{P}^2_d, \tilde{P}^1_d$. We claim that the generating set of Floer complex $CF^\bullet(L_1, L_2)$ is empty. Suppose that the time 1 trajectory $\phi^1_{\mathrm{Ham}}(L_1)$ of L_1 under the Hamiltonian flow ϕ_{Ham} intersects L_2. Then one finds a point $p \in L_1$ which needs to be pushed from the initial position through C to reach L_2. Let $0 < t < 1$ be the minimum time such that $\phi^t_{\mathrm{Ham}}(p) \in \partial\bar{C}$ and $\phi^{t+\epsilon}_{\mathrm{Ham}}(p) \in C \setminus \partial\bar{C}$ for $0 < \epsilon \ll 1$. By construction the Hamiltonian flow on the gluing region is orthogonal to the Liouville flow on C. Recall that on cylindrical ends the wrapping is defined by the Hamiltonian flow which is orthogonal to the Liouville flow and we glue $\tilde{P}^1_d, \tilde{P}^2_d$ along their skeleta inside C. Consider its restriction to the first factors in the product decompositions

$$C \cong \mathbb{C}^*_z \times \tilde{P}_{d-1},\ C \cong \mathbb{C}^*_{\bar{z}} \times \tilde{P}_{d-1}$$

of C respectively as a leg of $\tilde{P}^1_d, \tilde{P}^2_d$. The restricted Liouville flow to $\mathbb{C}^*_z, \mathbb{C}^*_{\bar{z}}$ is parallel to their radial coordinate directions. Hence $\phi^{t+\epsilon}_{\mathrm{Ham}}(p)$ would never exit C to reach L_2, leading to contradiction. □

Theorem 4.11. *The canonical functor*

$$\mathrm{Fuk}(H)_{\mathbb{Z}_2} \to \mathcal{F}_A(\Pi_\Sigma) = \lim\left(\prod_{v\in\mathrm{Vert}(\Pi_\Sigma)} \mathcal{F}_A(U_v) \to \prod_{e\in\mathrm{Edge}(\Pi_\Sigma)} \mathcal{F}_A(U_e) \to \cdots\right)$$

is an equivalence.

Proof. We begin with a vertex $v^0_1 \in \Pi_\Sigma$ to which some free edge is adjacent. Here, by a free edge we mean an edge not connecting two vertices. There are in total $l_1 < d+2$ vertices $v^1_1, \dots, v^1_{l_1} \in \Pi_\Sigma$ connected with v^0_1 by single edges $e^1_{11}, \dots, e^1_{1l_1}$. Let $v^k_1, \dots, v^k_{l_k}$ be the vertices of Π_Σ connected with v^0_1 by at least k edges. Each v^k_i is connected with at least one vertex v^{k-1}_j by a single edge e^k_{ji} for some $1 \le j \le l_{k-1}$. We will compute the section of $\mathcal{F}_A$ over the union

$$U_{v^0_1} \cup \bigcup_{i_1=1}^{l_1} U_{v^1_{i_1}} \cup \cdots \cup \bigcup_{i_k=1}^{l_k} U_{v^k_{i_k}}.$$

Consider Nadler's Weinstein structures transported to $H_{v^0_1}, H_{v^1_1}$ whose final legs correspond to different edges from e^1_{11}. Then $H_{v^0_1}, H_{v^1_1}$ glue along their Weinstein submanifold $H_{e^1_{11}}$ to yield a Weinstein manifold $H_{v^0_1} \cup H_{v^1_1}$. Note that the pants decomposition from [Mik04, Theorem 1'] is nothing but the gluing of the closures of tailored pants along their boundaries. By [Mik04, Remark 5.2] this gluing is compatible with the natural symplectic structures. To the product of their boundaries with a sufficiently small open interval, one can transport Nadler's Weinstein structure restricted to $H_{e^1_{11}}$ via radial deformation. By Lemma 4.9 and Lemma 4.10 we obtain a canonical equivalence

$$\mathrm{Fuk}(H_{v^0_1} \cup H_{v^1_1})_{\mathbb{Z}_2} = \mathcal{F}_A(U_{v^0_1} \cup U_{v^1_1}).$$

Consider Nadler's Weinstein structures transported to $H_{v^0_1}, H_{v^1_1}, H_{v^1_2}$ whose final legs correspond to different edges from e^1_{11}, e^1_{12} and possibly existing edge e^{11}_{12} connecting v^1_1 with v^1_2. Note that each of $H_{v^0_1}, H_{v^1_1}, H_{v^1_2}$ has at least one free leg which is not involved in this gluing. Then $H_{v^0_1}, H_{v^1_1}, H_{v^1_2}$ glue along their Weinstein submanifolds $H_{e^1_{11}}, H_{e^1_{12}}, H_{e^{11}_{12}}$ to yield a Weinstein manifold $H_{v^0_1} \cup H_{v^1_1} \cup H_{v^1_2}$. By Lemma 4.9 and Lemma 4.10 we obtain a canonical equivalence

$$\mathrm{Fuk}(H_{v^0_1} \cup H_{v^1_1} \cup H_{v^1_2})_{\mathbb{Z}_2} = \mathcal{F}_A(U_{v^0_1} \cup U_{v^1_1} \cup U_{v^1_2}).$$

Iteratively, we obtain a canonical equivalence

$$\mathrm{Fuk}(H_{v^0_1} \cup \bigcup_{i=1}^{l_1} H_{v^1_i})_{\mathbb{Z}_2} = \mathcal{F}_A(U_{v^0_1} \cup \bigcup_{i=1}^{l_1} U_{v^1_i}).$$

Suppose that the canonical functor

$$\mathrm{Fuk}(H_{v^0_1} \cup \bigcup_{i_1=1}^{l_1} H_{v^1_{i_1}} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v^{k-1}_{i_{k-1}}})_{\mathbb{Z}_2} \to \mathcal{F}_A(U_{v^0_1} \cup \bigcup_{i_1=1}^{l_1} U_{v^1_{i_1}} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} U_{v^{k-1}_{i_{k-1}}})$$

is an equivalence. When v_i^k is connected with only one vertex v_j^{k-1}, consider Nadler's Weinstein structures transported to $H_{v_j^{k-1}}, H_{v_i^k}$ whose final legs correspond to different edges from e_{ji}^k. The former extends to give another Weinstein structure on $H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v_{i_{k-1}}^{k-1}}$ as follows. First, recall the translated Liouville structure α_{d+1}^{-l} on $T_{\mathbb{C}}^{d+1}$ from (2.1). Note that we negate l here. Restrict it to $\tilde{P}_d$ and then transport to all the other legs but the final compatibly with the outward Liouville vector field from $H_{v_j^{k-1}}$. To the final leg, transport its slight modification. Namely, push the zero locus of the Liouville vector field associated with α_{d+1}^{-l} far away along the positive diagonal direction. Then the closure of $\tilde{P}_d$ becomes a Weinstein cobordisim. We extend this along the direction towards cylindrical ends of $H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v_{i_{k-1}}^{k-1}}$. Note that the extension might not be canonical.

In any case, the result is a Liouville manifold of finite type and Liouville homotopic to the standard Liouville structure, which can be canonically upgraded to Weinstein. Then $H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v_{i_{k-1}}^{k-1}}, H_{v_i^k}$ glue along their Weinstein submanifold $H_{e_{ji}^k}$ to yield a Weinstein manifold $H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v_{i_{k-1}}^{k-1}} \cup H_{v_i^k}$. Here, we use the same symbol to denote the exact symplectic manifolds with modified Weinstein structure. By Lemma 4.9 and Lemma 4.10 we obtain a canonical equivalence

$$\mathrm{Fuk}(H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v_{i_{k-1}}^{k-1}} \cup H_{v_i^k})_{\mathbb{Z}_2} = \mathcal{F}_A(U_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} U_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} U_{v_{i_{k-1}}^{k-1}} \cup U_{v_i^k}).$$

When v_i^k is connected with more than one vertices $v_{j_1}^{k-1}, \ldots, v_{j_l}^{k-1}$ by single edges, consider Nadler's Weinstein structures transported to $H_{v_{j_1}^{k-1}}, \ldots, H_{v_{j_l}^{k-1}}, H_{v_i^k}$ whose final legs correspond to different edges from the ones connecting any two of $v_{j_1}^{k-1}, \ldots, v_{j_l}^{k-1}, v_i^k$. The Weinstein structure on $H_{v_{j_1}^{k-1}} \cup \cdots \cup H_{v_{j_l}^{k-1}}$ extends to give another Weinstein structure on $H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_{k-1}=1}^{l_{k-1}} H_{v_{i_{k-1}}^{k-1}}$ in a similar way as above. Iteratively, we obtain a canonical equivalence

$$\mathrm{Fuk}(H_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} H_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_k=1}^{l_k} H_{v_{i_k}^k})_{\mathbb{Z}_2} = \mathcal{F}_A(U_{v_1^0} \cup \bigcup_{i_1=1}^{l_1} U_{v_{i_1}^1} \cup \cdots \cup \bigcup_{i_k=1}^{l_k} U_{v_{i_k}^k}).$$

□

4.3. ***B*-side constructible sheaves of categories for very affine hypersurfaces.** Recall that Π_Σ is the dual cell complex of $\mathcal{T}$, which we assume to be an adapted unimodular triangulation of the convex lattice polytope $\Delta^\vee \subset M_{\mathbb{R}}^\vee$. In particular, each vertex $v \in \Pi_\Sigma$ bijectively corresponds to a cell congruent to a standard simplex under the $\mathrm{GL}(d+1, \mathbb{Z})$-action. The cell in turn bijectively corresponds to a cone $\sigma_v \in \Sigma_Y$, which defines an affine open subvariety $Y_v = \operatorname{Spec} \mathbb{C}[\sigma_v^\vee \cap (M \times \mathbb{Z})] \subset Y$ isomorphic to $\mathbb{A}^{d+2}$. Introduce coordinates $(y_{e_1^v}, \ldots, y_{e_{d+2}^v})$ on Y_v, where e_i^v stand for edges adjacent to v and dual to facets $\sigma_{e_i^v}$ of σ_v so that $Y_{e_i^v} = \operatorname{Spec} \mathbb{C}[\sigma_{e_i^v}^\vee \cap (M \times \mathbb{Z})] \subset Y_v$ are the open subvarieties defined by $y_{e_i^v} \neq 0$.

Definition 4.12. The *B-side constructible sheaf* $\mathcal{F}_B$ of categories for H is a collection

$$\{\mathcal{F}_B(U_{S^{(k)}}), R^B_{S^{(k)}, S^{(l)}}\}$$

of sections and restriction functors defined on connected open subsets of Π_Σ as follows:

- The section over $U_{S^{(k)}}$ is given by the ind-completion of the category of matrix factorizations

$$\mathcal{F}_B(U_{S^{(k)}}) = \mathrm{MF}^\infty(Y_v|_{\bigcap_{m=1}^k \{y_{e_{i_m}^v} \neq 0\}}, y_{e_1^v} \cdots y_{e_{d+2}^v}),$$

where $e^v_{i_1},\dots,e^v_{i_k}$ are the edges adjacent to v determining the k-stratum $S^{(k)}$.

- Along an inclusion $U_{S^{(l)}} \hookrightarrow U_{S^{(k)}}$ the restriction functor is given by the canonical restriction functor

$$R^B_{S^{(k)},S^{(l)}}\colon\ \mathrm{MF}^\infty(Y_v|_{\bigcap_{m=1}^k\{y_{e^v_{i_m}}\neq 0\}}, y_{e^v_1}\cdots y_{e^v_{d+2}}) \to \mathrm{MF}^\infty(Y_v|_{\bigcap_{n=1}^l\{y_{e^v_{j_n}}\neq 0\}}, y_{e^v_1}\cdots y_{e^v_{d+2}}),$$

where $e^v_{i_1},\dots,e^v_{i_k}$ and $e^v_{j_1},\dots,e^v_{j_l}$ are edges adjacent to v respectively determining $S^{(k)}$ and $S^{(l)}$.

Note that $\mathcal{F}_B$ defines a sheaf on Π_Σ without passing to sheafification, since U_v for all $v\in\mathrm{Vert}(\Pi_\Sigma)$ form a subbase of the topology of Π_Σ and MF^∞ is a sheaf on Y [Pre, Proposition A.3.1].

Lemma 4.13. *There is a canonical equivalence*

$$\mathcal{F}_B(\Pi_\Sigma) = \mathrm{MF}^\infty(Y, W_Y)$$

compatible with restrictions. In particular, for every open subset $U\subset\Pi_\Sigma$ it makes the diagram

$$\begin{tikzcd}
\mathcal{F}_B(\Pi_\Sigma) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{F}_B(U) \arrow[d, "\simeq"] \\
\mathrm{MF}^\infty(Y, W_Y) \arrow[r] & \mathrm{MF}^\infty(Y_U, W_Y)
\end{tikzcd}$$

commute where horizontal arrows are the restrictions. Here, $Y_U\subset Y$ is the open subset mapping to U under the composition of (3.3) *with $q\colon M_\mathbb{R}\times\mathbb{R}\to M_\mathbb{R}$.*

Proof. Take an affine open cover $\bigcup_{v\in\mathrm{Vert}(\Pi_\Sigma)} Y_v$ of Y. By definition of $\mathcal{F}_B$ we have

$$\mathcal{F}_B(\Pi_\Sigma) = \lim\left(\prod_{v\in\mathrm{Vert}(\Pi_\Sigma)}\mathcal{F}_B(U_v)\to\prod_{e\in\mathrm{Edge}(\Pi_\Sigma)}\mathcal{F}_B(U_e)\to\cdots\right).$$

If e is an edge connecting two vertices v, v', then the associated restriction functors are

$$R^B_{v,e}\colon\ \mathrm{MF}^\infty(Y_v, y_{e^v_1}\cdots y_{e^v_{d+2}}) \to \mathrm{MF}^\infty(Y_v|_{\{y_{e^v_i}\neq 0\}}, y_{e^v_1}\cdots y_{e^v_{d+2}}),$$
$$R^B_{v',e}\colon\ \mathrm{MF}^\infty(Y_{v'}, y_{e^{v'}_1}\cdots y_{e^{v'}_{d+2}}) \to \mathrm{MF}^\infty(Y_{v'}|_{\{y_{e^{v'}_j}\neq 0\}}, y_{e^{v'}_1}\cdots y_{e^{v'}_{d+2}}).$$

Recall that Π_Σ encodes all the combinatorial information to recover both H and Y from pieces. In particular, it gives a coordinate transformation

$$(y_{e^{v'}_1},\cdots,y_{e^{v'}_{d+2}})\to(y_{e^v_1},\cdots,y_{e^v_{d+2}})$$

on Y_e which defines a gluing datum

$$\mathrm{MF}^\infty(Y_v|_{\{y_{e^v_i}\neq 0\}}, y_{e^v_1}\cdots y_{e^v_{d+2}}) \simeq \mathrm{MF}^\infty(Y_{v'}|_{\{y_{e^{v'}_j}\neq 0\}}, y_{e^{v'}_1}\cdots y_{e^{v'}_{d+2}}).$$

As Y_v form an open cover of Y, such a gluing datum is compatible with further restrictions. Hence $\mathcal{F}_B(\Pi_\Sigma)$ is also the limit of the diagram

$$\prod_{v\in\mathrm{Vert}(\Pi_\Sigma)}\mathrm{MF}^\infty(Y_v, y_{e^v_1}\cdots y_{e^v_{d+2}})\to\prod_{e\in\mathrm{Edge}(\Pi_\Sigma)}\mathrm{MF}^\infty(Y_v|_{\{y_e\neq 0\}}, y_{e^v_1}\cdots y_{e^v_{d+2}})\to\cdots.$$

Since the sheaf MF^∞ on Y satisfies Zariski descent [Pre, Proposition A.3.1], it coincides with $\mathrm{MF}^\infty(Y,W_Y)$ and the compatible equivalences on pieces glue to yield a canonical equivalence $\mathcal{F}_B(\Pi_\Sigma)=\mathrm{MF}^\infty(Y,W_Y)$. By the same argument, we obtain a canonical equivalence $\mathcal{F}_B(U)=\mathrm{MF}^\infty(Y_U,W_Y)$ for every open subset $U\subset\Pi_\Sigma$. Since U_v for all $v\in\mathrm{Vert}(\Pi_\Sigma)$ form a subbase, these equivalences respect restrictions. □

5. Isomorphism of the constructible sheaves

In this section, we give a proof of HMS for very affine hypersurfaces by gluing HMS for pairs of pants established in [Nad]. When gluing such equivalences, the combinatorial duality over Π_Σ from Section 3 plays a crucial role.

5.1. **Local equivalences.** Passing to the category of matrix factorizations might be delicate because of Knörrer periodicity. First, we show how to lift Nadler's equivalence, i.e., the equivalence from Corollary 2.31 to the category of matrix factorizations preserving the compatibility with restrictions.

Lemma 5.1. *Let $[H_U]$ be a Weinstein subdomain of $[H]$. Then the skeleton $\mathrm{Core}(H_U)$ of H_U is a closed subset of the skeleton $\mathrm{Core}(H)$ of H.*

Proof. Replacing $[H_U]$ with its radial deformation if necessary, we may assume that $\mathrm{Core}(H_U)$ is disjoint from $\partial[H_U] = \partial_\infty H_U$. Indeed, skeleta of Liouville manifolds of finite type are maximal compact subsets conic with respect to Liouville vector fields. Since any point in $\partial_\infty H_U$ escapes to infinity along the Liouville vector flow, it never contributes to $\mathrm{Core}(H_U)$. We may assume further that $\mathrm{Core}(H)$ is contained in $[H]$. Then $\mathrm{Core}(H) \cap ([H] \setminus [H_U]^\circ)$ is an open subset of $\mathrm{Core}(H)$. Since, up to deformation, $\mathrm{Core}(H_U)$ coincides with the complement of $\mathrm{Core}(H) \cap ([H] \setminus [H_U]^\circ)$ in $\mathrm{Core}(H)$, it is a closed subset of $\mathrm{Core}(H)$. $\square$

Lemma 5.2. *For each vertex $v \in \Pi_\Sigma$ there is an equivalence*

$$\mathcal{F}_A(U_v) \simeq \mathrm{IndCoh}(Y_{v,d})_{\mathbb{Z}_2},\ Y_{v,d} = \{y_{e_1^v} \cdots y_{e_{d+1}^v} = 0\} \subset \mathbb{A}^{d+1}$$

compatible with restrictions. In particular, for every open subset $U_{S^{(k)}} \subset U_v$ determined by edges $e_{i_1}^v, \ldots, e_{i_k}^v$ adjacent to v, we have the commutative diagram

$$\begin{tikzcd} \mathcal{F}_A(U_v) \arrow[r, "R^A_{v,S^{(k)}}"] \arrow[d, "\simeq"'] & \mathcal{F}_A(U_{S^{(k)}}) \arrow[d, "\simeq"] \\ \mathrm{IndCoh}(Y_{v,d})_{\mathbb{Z}_2} \arrow[r] & \mathrm{IndCoh}(Y_{v,d}|_{\bigcap_{j=1}^k \{y_{e_{i_j}^v} \neq 0\}})_{\mathbb{Z}_2} \end{tikzcd}$$

where the lower horizontal arrow is the canonical restriction functor.

Proof. From the argument in the proof of Theorem 2.30 we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Coh}(\bigcup_{j=1}^k \mathbb{A}^{\{i_j\}^c}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Coh}(Y_{v,d}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Coh}(Y_{v,d}|_{\bigcap_{j=1}^k \{y_{e_{i_j}^v} \neq 0\}}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ \mu\mathrm{Sh}(\bigcup_{j=1}^k \Lambda^\infty_{\{i_j\}^c}) \arrow[r] & \mu\mathrm{Sh}(\Lambda^\infty_{d+1}) \arrow[r] & \mu\mathrm{Sh}(\Lambda^\infty_{d+1}) / \mu\mathrm{Sh}(\bigcup_{j=1}^k \Lambda^\infty_{\{i_j\}^c}) \arrow[r] & 0 \end{tikzcd} \tag{5.1}$$

where the horizontal arrows form exact sequences. Here, the upper left horizontal arrows is the canonical functor to the colimit and the lower left horizontal arrows is left adjoint to the restriction.

Recall that $H_{S^{(k)}}$ is isomorphic to the intersection of legs $L_{d,i_1}(K), \ldots, L_{d,i_k}(K) \subset \tilde{P}_d$. We write $\Lambda_{S^{(k)}}$ for the isomorphic image of $\mathrm{Core}(H_{S^{(k)}}) \times \{0\} \times \mathbb{R}$ under the symplectomorphism $U_d \times T^1 \times \mathbb{R} \hookrightarrow \Omega_{d+1}$ from Lemma 2.10. Let $\tilde{P}_I \subset T^*T^I$ be the tailored pants for $I = \{i_1, \ldots, i_k\}$. We write $\mathfrak{L}_I$ for the isomorphic image of $\mathrm{Core}(\tilde{P}_I)$ under the symplectomorphism from Lemma 2.10. Unwinding the proof of Lemma 2.7 and Lemma 2.10, one sees that

$$\mathrm{Core}(H_{S^{(k)}}) \cong T^{I^c} \times \mathrm{Core}(\tilde{P}_I) \cong \mathbb{T}^{I^c} \times \mathfrak{L}_I \cong T^{I^c} \times s(\Gamma_{\mathfrak{L}_I, -f|_{\mathfrak{L}_I}}) \cong T^{I^c} \times \Lambda_I^\infty \subset \Lambda^\infty_{d+1}$$

and $\Lambda^\infty_{S^{(k)}} \subset \Lambda^\infty_{d+1}$ is defined as $\bigcap_{j=1}^k \{\xi_{i_j} = 0\}$. Hence we obtain

$$\bigcup_{j=1}^k \Lambda^\infty_{\{i_j\}^c} = \Lambda^\infty_{d+1} \setminus \Lambda^\infty_{S^{(k)}}, \ \mu\mathrm{Sh}(\bigcup_{j=1}^k \Lambda^\infty_{\{i_j\}^c}) = \mu\mathrm{Sh}(\Lambda^\infty_{d+1} \setminus \Lambda^\infty_{S^{(k)}}).$$

From the argument in the end of [GS1, Section 4] we obtain another commutative diagram

$$\begin{tikzcd} \mu\mathrm{Sh}^\diamond(\Lambda^\infty_{d+1} \setminus \Lambda^\infty_{S^{(k)}}) \arrow[r] \arrow[d, equal] & \mu\mathrm{Sh}^\diamond(\Lambda^\infty_{d+1}) \arrow[r] \arrow[d, equal] & \mu\mathrm{Sh}^\diamond(\Lambda^\infty_{S^{(k)}}) \arrow[r] \arrow[d, equal] & 0 \\ \langle \text{Cocores of } [H_v] \setminus [H_{S^{(k)}}]^\circ \rangle \arrow[r] & \mathrm{Fuk}(H_v) \arrow[r] & \mathrm{Fuk}(H_{S^{(k)}}) \arrow[r] & 0 \end{tikzcd} \tag{5.2}$$

where the horizontal arrows form exact sequences and the middle and right vertical arrows are the equivalences from [GPS3, Theorem 1.4]. Here, the upper left horizontal arrow is left adjoint to the restriction to $\Lambda^\infty_{d+1} \setminus \Lambda^\infty_{S^{(k)}}$ which is open by Lemma 5.1, and the lower right horizontal arrow is the ind-completion of the Viterbo restriction functor from [GPS2, Proposition 11.2]. Hence we may concatenate (5.1) and (5.2) to obtain a commutative diagram

$$\begin{tikzcd} \mathrm{IndCoh}(\bigcup_{j=1}^k \mathbb{A}^{\{i_j\}^c}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{IndCoh}(Y_{v,d}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{IndCoh}(Y_{v,d}|_{\bigcap_{j=1}^k \{y_{e^v_{i_j}} \neq 0\}}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ \langle \text{Cocores of } [H_v] \setminus [H_{S^{(k)}}]^\circ \rangle \arrow[r] & \mathrm{Fuk}(H_v) \arrow[r] & \mathrm{Fuk}(H_{S^{(k)}}) \arrow[r] & 0. \end{tikzcd} \tag{5.3}$$

□

Remark 5.3. Recall that Nadler broke the symmetry of $\tilde{P}_d$ so that the final leg attracts the Liouville vector flow while remaining the other legs symmetric. In particular, each of the other legs defines a Weinstein submanifold, which is isomorphic to a product of $\mathbb{C}^*$ and a 1-dimensional lower tailored pants. Via the combinatorial duality incorporated into the definition of $\mathcal{F}_B$, the Viterbo restriction to such a Weinstein submanifold associated with the j-th leg corresponds to the restriction to the open subset defined by $y_{e^v_j} \neq 0$ for $j = 1, \ldots, d+1$.

Consider the natural projection

$$Y_{v,d+1} = \{y_{e^v_1} \cdots y_{e^v_{d+1}} y_{e^v_{d+2}} = 0\} \subset Y_v \cong \mathbb{A}^{d+2} \to Y^i_{v,d+1} = \{y_{e^v_1} \cdots \widehat{y_{e^v_i}} \cdots y_{e^v_{d+2}} = 0\} \subset \mathbb{A}^{d+1}$$

of the union of the coordinate hyperplanes. By Lemma 2.28 the pullback induces an equivalence

$$\mathrm{Coh}(Y^i_{v,d+1})_{\mathbb{Z}_2} \simeq \mathrm{MF}(\mathbb{A}^{d+2}, y_{e^v_1} \cdots y_{e^v_{d+1}} y_{e^v_{d+2}}). \tag{5.4}$$

Theorem 5.4. *For each vertex $v \in \Pi_\Sigma$ there is an equivalence*

$$\mathcal{F}_A(U_v) \simeq \mathcal{F}_B(U_v)$$

compatible with restrictions. In particular, for every open subset $U_{S^{(k)}} \subset U_v$ determined by edges $e^v_{i_1}, \ldots, e^v_{i_k}$ adjacent to v, we have the commutative diagram

$$\begin{tikzcd} \mathcal{F}_A(U_v) \arrow[r, "R^A_{v,S^{(k)}}"] \arrow[d, "\simeq"'] & \mathcal{F}_A(U_{S^{(k)}}) \arrow[d, "\simeq"] \\ \mathcal{F}_B(U_v) \arrow[r, "R^B_{v,S^{(k)}}"'] & \mathcal{F}_B(U_{S^{(k)}}). \end{tikzcd}$$

Proof. On each vertex $v \in \Pi_\Sigma$ we have $d+2$ Nadler's equivalences

$$\varphi_i^v \colon \operatorname{Fuk}(H_v)_{\mathbb{Z}_2} \to \operatorname{IndCoh}(Y^i_{v,d+1})_{\mathbb{Z}_2} \to \operatorname{MF}^\infty(y_{e_1^v} \dots y_{e_{d+2}^v})$$

depending on the choice of the final leg. Here, we use the symbol $\operatorname{MF}^\infty(y_{e_1^v} \dots y_{e_{d+2}^v})$ to denote $\operatorname{MF}^\infty(\mathbb{A}^{d+2}, y_{e_1^v} \cdots y_{e_{d+1}^v} y_{e_{d+2}^v})$ for brevity. The equivalence φ_{d+2}^v is compatible with restrictions along edges except e_{d+2}^v. Passing to φ_i^v for $i = 1, \dots, d+1$, we obtain restrictions

$$\psi_i^v \circ \varphi_i^v|_{e_{d+2}^v} \colon \operatorname{Fuk}(H_{e_{d+2}^v})_{\mathbb{Z}_2} \to \operatorname{IndCoh}(Y^i_{v,d+1}|_{\{y_{e_{d+2}^v} \neq 0\}})_{\mathbb{Z}_2} \to \operatorname{MF}^\infty(y_{e_1^v} \cdots y_{e_{d+2}^v}|_{\{y_{e_{d+2}^v} \neq 0\}})$$

along e_{d+2}^v, where ψ_i^v are the autoequivalences $\operatorname{MF}^\infty(y_{e_1^v} \cdots y_{e_{d+2}^v}) \to \operatorname{MF}^\infty(y_{e_1^v} \cdots y_{e_{d+2}^v})$ induced by shuffling the coordinates corresponding to the final legs. By Lemma 4.3 and the argument in the proof of Theorem 2.30 the additional restrictions $\psi_i^v \circ \varphi_i^v|_{e_{d+2}^v}$ do not depend on the choice of i. Namely, we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Fuk}(H_{e_{d+2}^v})_{\mathbb{Z}_2} \arrow[r, "\psi_j^v \circ \varphi_j^v|_{e_{d+2}^v}"] \arrow[d, equal] & \operatorname{MF}^\infty(y_{e_1^v} \cdots y_{e_{d+2}^v}|_{\{y_{e_{d+2}^v} \neq 0\}}) \arrow[d, equal] \\
\operatorname{Fuk}(H_{e_{d+2}^v})_{\mathbb{Z}_2} \arrow[r, "\psi_i^v \circ \varphi_i^v|_{e_{d+2}^v}"] & \operatorname{MF}^\infty(y_{e_1^v} \cdots y_{e_{d+2}^v}|_{\{y_{e_{d+2}^v} \neq 0\}}).
\end{tikzcd}$$

By the same argument, one sees that φ_{d+2}^v is compatible with further restrictions and defines the desired equivalence. $\square$

5.2. **Gluing equivalences.** Finally, we glue the local equivalence from Theorem 5.4 on each vertex $v \in \Pi_\Sigma$ to obtain a global equivalence which is compatible with restrictions.

Theorem 5.5. *There is an equivalence*

$$\mathcal{F}_A(\Pi_\Sigma) \simeq \mathcal{F}_B(\Pi_\Sigma)$$

compatible with restrictions. In particular, for every open subset $U \subset \Pi_\Sigma$ we have the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_A(\Pi_\Sigma) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{F}_A(U) \arrow[d, "\simeq"] \\
\mathcal{F}_B(\Pi_\Sigma) \arrow[r] & \mathcal{F}_B(U)
\end{tikzcd}$$

where the horizontal arrows are the restrictions.

Proof. Choose an integer $1 \leq k(v) \leq d+2$ for a vertex $v \in \Pi_\Sigma$ to fix a Nadler's equivalence

$$\varphi_{k(v)}^v \colon \operatorname{Fuk}(H_v)_{\mathbb{Z}_2} \to \operatorname{IndCoh}(Y^{k(v)}_{v,d+1})_{\mathbb{Z}_2} \to \operatorname{MF}^\infty(y_{e_1^v}, \dots, y_{e_{d+2}^v}).$$

Suppose that v is connected with another vertex v' by an edge e. Then Σ_Y gives the correspondences

$$y_{e_i^v} \leftrightarrow y_{e_{i'}^{v'}}, \ i = 1, \dots, d+2 \tag{5.5}$$

of the coordinates on the intersection $Y_e = Y_v \cap Y_{v'}$. Hence there is a unique integer $1 \leq k(v') \leq d+2$ such that $\varphi^{v'}_{k(v')}$ is compatible with $\varphi^{v}_{k(v)}$. Namely, we have the commutative diagram

$$\begin{tikzcd}
\varphi^{v'}_{k(v')}\colon \mathrm{Fuk}(H_{v'})_{\mathbb{Z}_2} \arrow[r] \arrow[d, "R^A_{v',e^{v'}_{i'}}"] & \mathrm{IndCoh}(Y^{k(v')}_{v',d+1})_{\mathbb{Z}_2} \arrow[r] & \mathrm{MF}^\infty(y_{e^{v'}_1}\cdots y_{e^{v'}_{d+2}}) \arrow[d, "R^B_{v',e^{v'}_{i'}}"] \\
\varphi^{v'}_{k(v')}|_{e^{v'}_{i'}}\colon \mathrm{Fuk}(H_{e^{v'}_{i'}})_{\mathbb{Z}_2} \arrow[r] \arrow[d, equal] & \mathrm{IndCoh}(Y^{k(v')}_{v',d+1}|_{\{y_{e^{v'}_{i'}}\neq 0\}})_{\mathbb{Z}_2} \arrow[r] & \mathrm{MF}^\infty(y_{e^{v'}_1}\dots y_{e^{v'}_{d+2}}|_{\{y_{e^{v'}_{i'}}\neq 0\}}) \arrow[d, "\phi^{vv'}_{ii'}"] \\
\varphi^{v}_{k(v)}|_{e^{v}_{i}}\colon \mathrm{Fuk}(H_{e^{v}_{i}})_{\mathbb{Z}_2} \arrow[r] & \mathrm{IndCoh}(Y^{k(v)}_{v,d+1}|_{\{y_{e^{v}_{i}}\neq 0\}})_{\mathbb{Z}_2} \arrow[r] & \mathrm{MF}^\infty(y_{e^{v}_1}\dots y_{e^{v}_{d+2}}|_{\{y_{e^{v}_{i}}\neq 0\}}) \\
\varphi^{v}_{k(v)}\colon \mathrm{Fuk}(H_{U_v})_{\mathbb{Z}_2} \arrow[r] \arrow[u, "R^A_{v,e^{v}_{i}}"] & \mathrm{IndCoh}(Y^{k(v)}_{v,d+1})_{\mathbb{Z}_2} \arrow[r] & \mathrm{MF}^\infty(y_{e^{v}_1}\cdots y_{e^{v}_{d+2}}) \arrow[u, "R^B_{v,e^{v}_{i}}"]
\end{tikzcd} \tag{5.6}$$

where $\phi^{vv'}_{ii'}$ is the canonical equivalence induced by (5.5). Clearly, it yields the commutative diagrams for further restrictions.

Suppose further that v' is connected with another vertex v'' by an edge e'. Then Σ_Y gives the correspondences

$$y_{e^{v'}_{i'}} \leftrightarrow y_{e^{v''}_{i''}},\ i' = 1, \dots, d+2 \tag{5.7}$$

of the coordinates on the intersection $Y_{e'} = Y_{v'} \cap Y_{v''}$. Hence there is a unique integer $1 \leq k(v'') \leq d+2$ such that $\varphi^{v''}_{k(v'')}$ is compatible with $\varphi^{v'}_{k(v')}$. Namely, we have the same commutative diagram as (5.6). On the intersection $Y_v \cap Y_{v'} \cap Y_{v''}$, the fan Σ_Y also gives the correspondences

$$y_{e^{v}_{i}} \leftrightarrow y_{e^{v''}_{i'''}},\ i = 1, \dots, d+2 \tag{5.8}$$

of the coordinates. Hence there is a unique integer $1 \leq k'(v'') \leq d+2$ such that $\varphi^{v''}_{k'(v'')}$ is compatible with $\varphi^{v}_{k(v)}$. Since the affine pieces $Y_v, Y_{v'}, Y_{v''}$ glue to yield an open subset of Y, the correspondences (5.8) are compatible with (5.5) and (5.7). Namely, we have $k(v'') = k'(v'')$ and $y_{e^{v''}_{i''}} = y_{e^{v''}_{i'''}}$ for $i = 1, \dots, d+2$. Hence Nadler's equivalences $\varphi^{v}_{k(v)}, \varphi^{v'}_{k(v')}, \varphi^{v''}_{k(v'')}$ glue to yield an equivalence

$$\varphi^{v}_{k(v)} \cup \varphi^{v'}_{k(v')} \cup \varphi^{v''}_{k(v'')}\colon \mathrm{Fuk}(H_v \cup H_{v'} \cup H_{v''}) \to \mathrm{MF}^\infty(Y_v \cup Y_{v'} \cup Y_{v''}, W_Y).$$

Iteratively, we obtain a compatible system $\{\varphi^{v}_{k(v)}\}_{v \in \mathrm{Vert}(\Pi_\Sigma)}$ of Nadler's equivalences whose gluing

$$\bigcup_{v \in \mathrm{Vert}(\Pi_\Sigma)} \varphi^{v}_{k(v)}\colon \mathcal{F}_A(\Pi_\Sigma) \to \mathcal{F}_B(\Pi_\Sigma)$$

gives the desired equivalence. $\square$

6. Critical loci of Landau–Ginzburg models for complete intersections

In this section, following [AAK16, Section 10], we realize the mirror pair for a complete intersection of very affine hypersufaces as critical loci of associated Landau–Ginzburg models. They give rise to fibrations over the complete intersection of the tropical hypersurfaces equipped with the canonical stratification.

6.1. **Landau–Ginzburg A-model for complete intersections.** Let $H_1, \ldots, H_r \subset \mathbb{T}^\vee_\mathbb{C}$ be very affine hypersurfaces in general position defined by Laurent polynomials

$$W_i\colon \mathbb{T}^\vee_\mathbb{C} \to \mathbb{C},\ x \mapsto \sum_{\alpha^i \in A_i} c_{\alpha^i} t^{-\rho_i(\alpha^i)} x^{\alpha^i},\ i = 1, \ldots, r. \tag{6.1}$$

Here,

- $c_{\alpha^i} \in \mathbb{C}^*$ are arbitrary constants,
- $t \gg 0$ is a sufficiently general tropical parameter,
- ρ_i are convex piecewise linear functions on convex lattice polytopes $\Delta^\vee_i$ whose corner loci give adapted unimodular triangulations $\mathcal{T}_i$ of $\Delta^\vee_i$, and
- $A_i \subset M^\vee$ are the set of vertices of $\mathcal{T}_i$.

We denote by $\mathbf{H}$ the complete intersection $H_1 \cap \cdots \cap H_r \subset \mathbb{T}^\vee_\mathbb{C}$. For $\mathbf{X} = \mathbb{T}^\vee_\mathbb{C} \times \mathbb{C}^r$ with coordinates $(x, u) = (x_1, \ldots, x_{d+1}, u_1, \ldots, u_r)$, consider a Laurent polynomial

$$W_\mathbf{X}\colon \mathbf{X} \to \mathbb{C},\ (x, u) \mapsto \sum_{i=1}^r u_i W_i(x).$$

Definition 6.1. Let $\mathbf{H} \subset \mathbb{T}^\vee_\mathbb{C}$ be the complete intersection of the very affine hypersurfaces $H_1, \ldots, H_r$ defined by the Laurent polynomials $W_1, \ldots, W_r$ from (6.1). We call the pair $(\mathbf{X}, W_\mathbf{X})$ the *Landau–Ginzburg A-model* for $\mathbf{H}$.

Definition 6.2. The *Newton polytope* $\Delta^\vee_\mathbf{X}$ of $W_\mathbf{X}$ is the convex hull

$$\mathrm{Conv}(0, -\Delta^\vee_1 \times e_1, \ldots, -\Delta^\vee_r \times e_r) \subset M^\vee_\mathbb{R} \times \mathbb{R}^r$$

where $e_1, \ldots, e_r \in \mathbb{R}^r$ are the standard basis vectors.

Remark 6.3. The polytope $\Delta^\vee_\mathbf{X}$ admits an adapted star-shaped triangulation $\mathbf{T}$ canonically induced by $\rho_1, \ldots, \rho_r$. However, it might not be unimodular.

Lemma 6.4. *The critical locus* $\mathrm{Crit}(W_\mathbf{X})$ *is given by* $\bigcap_{i=1}^r \{u_i = 0\} \cap \bigcap_{i=1}^r \{W_i = 0\} \subset \mathbf{X}$.

Proof. Express the tangent map $dW_\mathbf{X}$ of $W_\mathbf{X}$ as a $(1, 2r)$-matrix

$$(u_1 dW_1, \ldots, u_r dW_r, W_1, \ldots, W_r).$$

Since $H_1, \ldots, H_r \subset \mathbb{T}^\vee_\mathbb{C}$ are in general position, we may assume that they intersect transversely. Then dW_i nowhere vanish. Hence $\mathrm{rank}(dW_\mathbf{X}) = 0$ if and only if $u_1 = \cdots = u_r = 0$ and $W_1 = \cdots = W_r = 0$. □

Remark 6.5. By Lemma 6.4 the projection $\mathrm{pr}_1\colon \mathbf{X} = \mathbb{T}^\vee_\mathbb{C} \times \mathbb{C}^r \to \mathbb{T}^\vee_\mathbb{C}$ preserves $\mathrm{Crit}(W_\mathbf{X})$ and the inclusions $\mathbf{H} \subset H_i \hookrightarrow \mathrm{Crit}(W_\mathbf{X})$. Now, we may assume that the tropical hypersurfaces $\Pi_{\Sigma_1}, \ldots, \Pi_{\Sigma_r} \subset \mathbb{R}^{d+1}$ intersect transversely, as $H_1, \ldots, H_r \subset \mathbb{T}^\vee_\mathbb{C}$ are in general position. We may assume further that $H_1, \ldots, H_r$ intersect along their legs. Let $\mathrm{ret}_i\colon \Pi_i \to \Pi_{\Sigma_i}$ be the continuous maps induced by the retractions. Then the composition

$$\mathbf{f}\colon \mathbf{H} \cong \mathrm{Crit}(W_\mathbf{X}) \hookrightarrow \mathbf{X} \xrightarrow{\mathrm{pr}_1} \mathbb{T}^\vee_\mathbb{C} \xrightarrow{\mathrm{Log}_{d+1}} \bigcap_{i=1}^r \Pi_i \xrightarrow{\mathrm{ret}_r \circ \cdots \circ \mathrm{ret}_1} \bigcap_{i=1}^r \Pi_{\Sigma_i} \tag{6.2}$$

gives a fibration. Away from lower dimensional strata, the fiber over a point in the intersection of r top dimensional strata one from each Π_{Σ_i} is a real $(d - r + 1)$-torus.

6.2. **Landau–Ginzburg B-model for complete intersections.** Recall that $\mathcal{T}_i$ are the chosen adapted unimodular triangulations of $\Delta_i^\vee \subset M_\mathbb{R}^\vee$ obtained as the corner loci of the convex piecewise linear functions $\rho_i \colon \Delta_i^\vee \to \mathbb{R}$. Recall also from Remark 6.3 that $\mathbf{T}$ is the adapted star-shaped triangulation of $\Delta_\mathbf{X}^\vee \subset M_\mathbb{R}^\vee \times \mathbb{R}^r$ canonically induced by $\rho_1, \dots, \rho_r$. Let $\Sigma_\mathbf{Y} \subset M_\mathbb{R}^\vee \times \mathbb{R}^r$ be the fan corresponding to $\mathbf{T}$ and $\mathbf{Y}$ the noncompact $(d+r+1)$-dimensional toric variety associated with $\Sigma_\mathbf{Y}$. The primitive ray generators of $\Sigma_\mathbf{Y}$ are the vectors of the form $(-\alpha^{i,j}, e_i)$ with $\alpha^{i,j} \in A_i$. Such vectors span a smooth cone of $\Sigma_\mathbf{Y}$ if $-\alpha^{i,j}$ span a cell of $\mathcal{T}_i$ for fixed i.

Remark 6.6. Unlike the case $r = 1$, there might be nonsmooth cones in $\Sigma_\mathbf{Y}$ as $\mathbf{T}$ is not necessarily unimodular. Indeed, consider the case where $d = 2, r = 2$ and the two defining polynomials are $x_1 + x_2 + x_3, x_1^2 x_2 + x_3$. Then the Newton polytope $\Delta_\mathbf{X}^\vee$ is a 5-dimensional simplex which has twice volume of the unit simplex. As $x_1 + x_2 + x_3, x_1^2 x_2 + x_3$ cannot be further divided, there is no room for subdivision of $\Delta_\mathbf{X}^\vee$. Possible nonsmooth cones would contain at least two rays belonging to distinct subfans of the form

$$\Sigma_i = \mathbb{R}_{\geq 0} \cdot (-\mathcal{T}_i \times \{e_i\}) \subset \Sigma_\mathbf{Y}.$$

Dually, $\mathbf{Y}$ is associated with the noncompact moment polytope

$$\Delta_\mathbf{Y} = \{(m, u_1, \dots, u_r) \in M_\mathbb{R} \times \mathbb{R}^r \mid u_i \geq \varphi_i(m),\ 1 \leq i \leq r\}.$$

The facets of $\Delta_\mathbf{Y}$ correspond to the maximal domains of linearity of $\varphi_1, \dots, \varphi_r$. We denote by $\mathbf{A}$ the set of connected components of $\mathbb{R}^{d+1} \setminus \bigcup_{i=1}^r \Pi_{\Sigma_i}$ and index each component by the tuple

$$\vec{\alpha} = (\alpha^{1,j_1}, \dots, \alpha^{r,j_r}) \in M^\vee \times \cdots \times M^\vee$$

of vertices.

Remark 6.7. The noncompact polytope $\Delta_\mathbf{Y}$ is homeomorphic to the image of $\mathbf{Y}$ under the composition

$$\mathbf{Y} \to (\mathbf{Y})_{\geq 0} \to M_\mathbb{R} \times \mathbb{R}^r \tag{6.3}$$

of the map induced by retraction to the nonnegative real points with the restriction of the negated algebraic moment map.

Lemma 6.8. *Let $\mathbf{q} \colon M_\mathbb{R} \times \mathbb{R}^r \to M_\mathbb{R}$ be the natural projection. Then under $\mathbf{q}$ the union of intersections of r facets of $\Delta_\mathbf{Y}$ one from each $\{u_i \geq \varphi_i(m)\}$ homeomorphically maps to $M_\mathbb{R}$. Moreover, the union of intersections of r codimension 2 faces of $\Delta_\mathbf{Y}$ one from each $\{u_i \geq \varphi_i(m)\}$ homeomorphically maps to $\bigcap_{i=1}^r \Pi_{\Sigma_i}$.*

Proof. By construction of $\Sigma_\mathbf{Y}$ under $\mathbf{q}$ the intersection of r facets of $\Delta_\mathbf{Y}$ one from each $\{u_i \geq \varphi_i(m)\}$ homeomorphically maps to the intersection of r maximal domains of linearity one from each φ_i corresponding to the same $\alpha^{i,j_i} \in A_i$. When $\vec{\alpha} = (\alpha^{1,j_1}, \dots, \alpha^{r,j_r})$ runs through $\mathbf{A}$, the closure of the latter covers $M_\mathbb{R}$. Then the second statement follows from the same argument as in the proof of Lemma 3.12. □

For each $\vec{\alpha} \in \mathbf{A}$ let $\mathbf{Y}_{\vec{\alpha}} = \mathbb{T}_\mathbb{C} \times \mathbb{C}^r$ with coordinates $y_{\vec{\alpha}} = (y_{\vec{\alpha},1}, \dots, y_{\vec{\alpha},d+1}, v_{\vec{\alpha},1}, \dots, v_{\vec{\alpha},r})$, where $y_{\vec{\alpha},1}, \dots, y_{\vec{\alpha},d+1}$ are the monomials with weights

$$\eta_1 = (-1, \dots, 0, -\alpha_1^{1,j_1}, \dots, -\alpha_1^{r,j_r}), \dots, \eta_{d+1} = (0, \dots, -1, -\alpha_{d+1}^{1,j_1}, \dots, -\alpha_{d+1}^{r,j_r}) \in M \times \mathbb{Z}^r$$

and $v_{\vec{\alpha},1}, \dots, v_{\vec{\alpha},r}$ are the monomials with weights

$$\eta_{d+2} = (0, \dots, 0, 1, \dots, 0), \dots, \eta_{d+r+1} = (0, \dots, 0, 0, \dots, 1) \in M \times \mathbb{Z}^r.$$

Pairing of the former monomials with the monomials with weight

$$u_{\xi_1} = (-\alpha^{1,j_1}, e_1), \dots, u_{\xi_r} = (-\alpha^{r,j_r}, e_r) \in M^\vee \times \mathbb{Z}^r$$

yield 0 while that of the latter monomials yield 1.

Lemma 6.9. *The complex algebraic variety $\mathbf{Y}_{\vec{\alpha}}$ is the affine open subset of $\mathbf{Y}$ associated with the cone spanned by $u_{\xi_1}, \dots, u_{\xi_r} \in M^\vee \times \mathbb{Z}^r$.*

Proof. Suppose that $\sigma \in \Sigma_{\mathbf{Y}}(r)$ is the cone associated with the affine open subset $\mathbf{Y}_{\vec{\alpha}} \subset \mathbf{Y}$. We have

$$\operatorname{div}(y_{\vec{\alpha},i}^{\pm 1}) = \sum_{\xi \in \sigma(1)} \langle \pm \eta_i, u_\xi \rangle D_\xi, \ \operatorname{div}(v_{\vec{\alpha},j}) = \sum_{\xi \in \sigma(1)} \langle \eta_{d+j+1}, u_\xi \rangle D_\xi$$

for $1 \leq i \leq d+1$ and $1 \leq j \leq r$, where u_ξ are the primitive ray generators of ξ and $D_\xi = \overline{O(\xi)}$ are the closures of the orbits corresponding to ξ. Since $y_{\vec{\alpha},1}^{\pm 1}, \dots, y_{\vec{\alpha},d+1}^{\pm 1}$ never vanish on $\mathbf{Y}_{\vec{\alpha}}$, pairing of η_i with the the primitive ray generators in σ must yield 0 for $1 \leq i \leq d+1$. On the other hand, pairing of η_{d+j+1} with u_{ξ_k} must yield δ_{jk} for $1 \leq j, k \leq d+1$. □

Due to the above lemma, $\mathbf{Y}_{\vec{\alpha}}$ covers the open stratum of $\mathbf{Y}$ and the open strata of the irreducible toric divisors corresponding to $\alpha^{1,j_1}, \dots, \alpha^{r,j_r}$. If $\alpha^{i,j_i}, \beta^{i,k_i}$ are connected by an edge in $\mathcal{T}_i$ for some $1 \leq i \leq d+1$, then we glue $\mathbf{Y}_{\vec{\alpha}}$ to $\mathbf{Y}_{\vec{\beta}}$ with the coordinate transformations

$$y_{\vec{\alpha},l} = v_{\vec{\beta},i}^{\beta_l^{i,k_i} - \alpha_l^{i,j_i}} y_{\vec{\beta},l}, \ v_{\vec{\alpha},i} = v_{\vec{\beta},i}, \ 1 \leq l \leq d+1.$$

Thus the coordinate charts $\{\mathbf{Y}_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathbf{A}}$ cover the complement in $\mathbf{Y}$ of the codimension more than 1 strata.

We may write v_i for $v_{\vec{\alpha},i}$ as they do not depend on the choice of $\vec{\alpha} \in \mathbf{A}$. Since the weights

$$(0, \dots, 0, 1, \dots, 0), \dots, (0, \dots, 0, 0, \dots, 1) \in M \times \mathbb{Z}^r$$

pair nonnegatively with the primitive ray generators of $\Sigma_{\mathbf{Y}}$, the polynomial $v_1 + \cdots + v_r$ defines a regular function on $\mathbf{Y}$, which we denote by $W_{\mathbf{Y}}$.

Definition 6.10. Let $\mathbf{H} \subset \mathbb{T}_{\mathbb{C}}^\vee$ be the complete intersection of very affine hypersurfaces $H_1, \dots, H_r$ defined by the Laurent polynomials $W_1, \dots, W_r$ from (6.1). We call the pair $(\mathbf{Y}, W_{\mathbf{Y}})$ the *Landau–Ginzburg B-model* for $\mathbf{H}$.

Remark 6.11. The pair $(\mathbf{Y}, W_{\mathbf{Y}})$ is a conjectural SYZ mirror to $\mathbf{H}$ [AAK16, Theorem 1.6].

Lemma 6.12. *The critical locus* $\operatorname{Crit}(W_{\mathbf{Y}})$ *is given by* $\bigcap_{i=1}^r \operatorname{Crit}(v_i)$.

Proof. Since we have $\bigcap_{i=1}^r \operatorname{Crit}(v_i) \subset \operatorname{Crit}(W_{\mathbf{Y}})$, it remains to show the opposite inclusion. For $y \in \operatorname{Crit}(W_{\mathbf{Y}})$ there are r rays $\xi_{i,j_i} = \operatorname{Cone}(-\alpha^{i,j_i} \times e_i) \in \Sigma_{\mathbf{Y}}$ one from each Σ_i such that $y \in \bigcap_{i=1}^r D_{\xi_{i,j_i}}$ where $D_{\xi_{i,j_i}} = \overline{O(\xi_{i,j_i})}$ are the closures of the orbits corresponding to ξ_{i,j_i}. Indeed, we have

$$\operatorname{Crit}(W_{\mathbf{Y}}) \subset \mathbf{Y} \setminus (\mathbb{T}_{\mathbb{C}} \times (\mathbb{C}^*)^r) = \bigcup_{\xi \in \Sigma_{\mathbf{Y}}(1)} D_\xi = \bigcup_{i=1}^r \bigcup_{\xi_i \in \Sigma_i(1)} D_{\xi_i}.$$

If $y \notin \bigcup_{\xi_j \in \Sigma_j(1)} D_{\xi_j}$ for some $1 \leq j \leq r$, then there is a neighborhood $y \in U \subset \mathbf{Y}$ such that $U \cap \bigcup_{\xi_j \in \Sigma_j(1)} D_{\xi_j} = \emptyset$ and $\operatorname{Crit}(W_{\mathbf{Y}}|_U) = \emptyset$, as v_j never vanishes on U. Hence y belongs to at least one $D_{\xi_{i,j_i}}$ for each i and we obtain

$$\operatorname{Crit}(W_{\mathbf{Y}}) \subset \bigcup_{\vec{\alpha} \in \mathbf{A}} \bigcap_{i=1}^r D_{\xi_{i,j_i}}.$$

From the proof of Lemma 6.8 it follows that under $\mathbf{q}$ the intersection of r facets of $\Delta_{\mathbf{Y}}$ corresponding to $D_{\xi_{1,j_1}}, \dots, D_{\xi_{r,j_r}}$ maps to the closure $\bar{C}_{\vec{\alpha}}$ of the connected component $C_{\vec{\alpha}} \subset M_{\mathbb{R}} \setminus \bigcup_{i=1}^r \Pi_{\Sigma_i}$ indexed by $\vec{\alpha} = (\alpha^{1,j_1}, \dots, \alpha^{r,j_r})$. By Lemma 6.9 under (6.3) $\mathbf{Y}_{\vec{\alpha}}$ maps to the intersection of r facets of $\Delta_{\mathbf{Y}}$ corresponding to $D_{\xi_{1,j_1}},\dots,D_{\xi_{r,j_r}}$. As explained above, on $\mathbf{Y}_{\vec{\alpha}}$ we are given the coordinates

$$(w_1^{-1}v_1^{-\alpha_1^{1,j_1}}\cdots v_r^{-\alpha_1^{r,jr}},\dots,w_r^{-1}v_1^{-\alpha_{d+1}^{1,j_1}}\cdots v_r^{-\alpha_{d+1}^{r,jr}},v_1,\dots,v_r)$$

which implies $\mathrm{Crit}(W_{\mathbf{Y}})|_{\mathbf{Y}_{\vec{\alpha}}}=\emptyset$. Hence we obtain

$$\mathrm{Crit}(W_{\mathbf{Y}})\subset\bigcup_{\vec{\alpha}\in\mathbf{A}}\left(\bigcap_{i=1}^{r}D_{\xi_{i,j_i}}\setminus\mathbf{Y}_{\vec{\alpha}}\right)=\bigcap_{i=1}^{r}\mathrm{Crit}(v_i)$$

as we have $\bigcup_{\alpha^{i,j_i}\in A_i}(D_{\xi_{i,j_i}}\setminus(Y_{\alpha^i}\times\mathbb{C}^{r-1}))=\mathrm{Crit}(v_i)$ by Lemma 3.16. □

Remark 6.13. Since the map (6.3) sends each k-th intersection of $D_{\xi_{i,j_i}},\alpha^{i,j_i}\in\mathbf{A}$ to a codimension k face of $\Delta_{\mathbf{Y}}$, by Lemma 6.12 it sends $\mathrm{Crit}(W_{\mathbf{Y}})$ to the union of codimension $2r$ faces. On the other hand, by Lemma 6.8 the map $\mathbf{q}\colon M_{\mathbb{R}}\times\mathbb{R}^r\to M_{\mathbb{R}}$ homeomorphically sends the union of intersections of r codimension 2 faces of $\Delta_{\mathbf{Y}}$ one from each $\{u_i\geq\varphi_i(m)\}$ to $\bigcap_{i=1}^r\Pi_{\Sigma_i}$. Hence the composition

$$\mathbf{g}\colon\ \mathrm{Crit}(W_{\mathbf{Y}})\hookrightarrow\mathbf{Y}\xrightarrow{(6.3)}\Delta_{\mathbf{Y}}\xrightarrow{\mathbf{q}}\bigcap_{i=1}^{r}\Pi_{\Sigma_i}\tag{6.4}$$

gives a fibration. The fiber over a point in the intersection of r top dimensional strata one from each Π_{Σ_i} is a real $(d-r+1)$-torus [CLS11, Prop 12.2.3(b)].

7. Equivariantization and de-equivariantization

In this section, following [She22, Section 4], we review the last piece of our proof, i.e., equivariantization and de-equivaiantization of presentable dg categories with certain group actions. The fact that, they give mutually inverse equivalences of the categories we will consider, enables us to deduce our main result for nonunimodular case from unimodular case.

7.1. **Equivariantization.** Let $G\subset(\mathbb{C}^*)^N$ be any subgroup. Assume that G acts on a presentable dg category $\mathscr{C}$. Namely, there is a monoidal functor $G\to\mathrm{End}(\mathscr{C})$. Then $\mathscr{C}$ becomes a module over the monoidal category $(\mathrm{Qcoh}(G),\star)$, where $\star$ is the convolution product induced by the multiplication on G. Let $(\mathrm{Qcoh}(BG),\otimes)$ be the monoidal category of G-representations. Taking G-invariants defines a functor

$$\mathscr{C}\mapsto\mathscr{C}^G=\mathrm{Hom}_{\mathrm{Qcoh}(G)}(\mathrm{Mod}(\mathbb{C}),\mathscr{C})$$

from the category of $(\mathrm{Qcoh}(G),\star)$-modules to the category of $(\mathrm{Qcoh}(BG),\otimes)$-modules, called *$G$-equivariantization*. Here, the action on $\mathrm{Mod}(\mathbb{C})$ is trivial.

7.2. **De-equivariantization.** We denote by $G^\vee$ the character group $\mathrm{Hom}(G,\mathbb{C}^*)$ of G. Assume that $G^\vee$ acts on a presentable dg category $\mathscr{D}$. This is the same as an action of the monoidal category of $G^\vee$-graded $\mathbb{C}$-modules, which in turn is equivalent to $(\mathrm{Qcoh}(BG),\otimes)$. Hence $\mathscr{D}$ becomes a module over $(\mathrm{Qcoh}(BG),\otimes)$. Taking G-coinvariants defines a functor

$$\mathscr{D}\mapsto\mathscr{D}_{BG}=\mathrm{Mod}(\mathbb{C})\otimes_{\mathrm{Qcoh}(BG)}\mathscr{D}$$

from the category of $(\mathrm{Qcoh}(BG),\otimes)$-modules to the category of $(\mathrm{Qcoh}(G),\star)$-modules, called *G-de-equivariantization*. Also here, the action on $\mathrm{Mod}(\mathbb{C})$ is trivial.

7.3. **Mutually inverse equivalences.** For a presentable dg category $\mathscr{D}$ with a $G^\vee$-action, taking its $G^\vee$-invariants is equivalent to taking its G-coinvariants. Since equivariantization and de-equivariantization give mutually inverse equivalences, one obtains

Lemma 7.1 ([She15, Lemma 8]). *Let $G \subset (\mathbb{C}^*)^N$ be any subgroup and $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$. Then G-equivariantization and $G^\vee$-equivariantization give mutually inverse equivalences between the category of presentable dg categories with a G-action and that with a $G^\vee$-action.*

7.4. **Quotient construction of toric varieties.** Let Y_Σ be the toric variety associated with a fan $\Sigma \subset M^\vee_\mathbb{R}$. Assume that Y_Σ has no torus factors, i.e., $M^\vee_\mathbb{R}$ is spanned by primitive ray generators u_ρ for all $\rho \in \Sigma(1)$ [CLS11, Proposition 3.3.9]. Then [CLS11, Theorem 4.1.3] gives the short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho \to \mathrm{Cl}(Y_\Sigma) \to 0,$$

where $m \in M$ maps to $\mathrm{div}(\chi^m) = \Sigma_{\rho\in\Sigma(1)}\langle m, u_\rho\rangle D_\rho$ and $\mathrm{Cl}(Y_\Sigma)$ is the divisor class group. Applying $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$, one obtains another short exact sequence

$$1 \to G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(Y_\Sigma), \mathbb{C}^*) \to \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\Sigma(1)} \to \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \cong M^\vee_\mathbb{C} \to 1.$$

Lemma 7.2 ([CLS11, Lemma 5.1.1]). *The subgroup $G \subset (\mathbb{C}^*)^{\Sigma(1)}$ is isomorphic to a product of an algebraic torus and a finite abelian group. More explicitly, we have*

$$G = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} | \prod_{\rho\in\Sigma(1)} t_\rho^{\langle m,u_\rho\rangle} = 1 \textit{ for all } m \in M\}.$$

Let $S = \mathbb{C}[y_\rho \mid \rho \in \Sigma(1)]$ be the total coordinate ring of Y_Σ. Then we have $\mathbb{C}^{\Sigma(1)} = \mathrm{Spec}\, S$ and the irrelevant ideal is defined as

$$B(\Sigma) = \langle y^{\hat\sigma} \mid \sigma \in \Sigma\rangle = \langle y^{\hat\sigma} \mid \sigma \in \Sigma_{\max}\rangle \subset S,\ y^{\hat\sigma} = \prod_{\rho\notin\sigma(1)} y_\rho.$$

We denote by $Z(\Sigma)$ the zero locus $V(B(\Sigma)) \subset \mathbb{C}^{\Sigma(1)}$ of $B(\Sigma)$. Via inclusion $G \subset (\mathbb{C}^*)^{\Sigma(1)}$ the canonical action of $(\mathbb{C}^*)^{\Sigma(1)}$ on $\mathbb{C}^{\Sigma(1)}$ induces a G-action on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$.

Lemma 7.3 ([CLS11, Proposition 5.1.9, Theorem 5.1.11]). *There is a toric morphism*

$$\pi\colon \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to Y_\Sigma$$

which is constant on G-orbit and gives an isomorphism

$$Y_\Sigma \cong (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))//G.$$

Namely, π is an almost geometric quotient for the G-action. It is a geometric quotient if and only if Σ is simplicial.

Now, we drop the assumption that Y_Σ has no torus factors. Then the primitive ray generators $u_\rho, \rho \in \Sigma(1)$ span a proper subspace $(M^\vee_\mathbb{R})' \subsetneq M^\vee_\mathbb{R}$. Pick the complement $(M^\vee)''$ of $(M^\vee)' = (M^\vee_\mathbb{R})' \cap M^\vee$ so that $M^\vee = (M^\vee)' \oplus (M^\vee)''$. The cones of Σ defines a fan $\Sigma' \subset (M^\vee_\mathbb{R})'$. Note that we have $\Sigma'(1) = \Sigma(1)$ and $B(\Sigma') = B(\Sigma) \subset S$. We denote by G' the subgroup $\mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(Y_{\Sigma'}), \mathbb{C}^*) \subset (\mathbb{C}^*)^{\Sigma'(1)}$. Since $Y_{\Sigma'}$ has no torus factors, from the above argument it follows

$$Y_\Sigma \cong Y_{\Sigma'} \times (M^\vee_\mathbb{C})'' \cong ((\mathbb{C}^{\Sigma'(1)} \setminus Z(\Sigma'))//G') \times (\mathbb{C}^*)^r.$$

7.5. **Model case.** Let Y_Σ be a simplicial affine toric variety associated with a fan $\Sigma \subset M^\vee_{\mathbb{R}}$. Then Y_Σ has no torus factors. Since Σ contains only one top dimensional cone, by definition of $B(\Sigma)$ and Lemma 7.3 we obtain $Z(\Sigma) = \emptyset$ and

$$Y_\Sigma \cong \mathbb{A}^{d+r+1}/G, \ d+r+1 = \#\Sigma(1).$$

If also the rank of M is $d+r+1$, then the inclusion $M \to \mathbb{Z}^{\Sigma(1)}$ induces a finite cover

$$h^\vee \colon T^*(\mathbb{R}^{d+r+1}/M) \to T^*(\mathbb{R}^{d+r+1}/\mathbb{Z}^{d+r+1}).$$

Consider the tailored pants $\tilde{P}_{d+r}$ in the target induced by the pants

$$\{1 + \Sigma_{i=1}^{d+r} x_i x_{d+r+1} = 0\} = \{x_{d+r+1}^{-1} + \Sigma_{i=1}^{d+r} x_i = 0\} \subset (\mathbb{C}^*)^{d+r+1}.$$

Let $\tilde{P}_{d+r-1} \subset \tilde{P}_{d+r}$ be its closed subset induced by setting $x_{d+r+1} = 1$. From $G^\vee \cong \mathbb{Z}^{d+r+1}/M$ it follows

$$\tilde{P}_{d+r-1} \cong (h^\vee)^{-1}(\tilde{P}_{d+r-1})/G^\vee.$$

Hence we obtain

$$\mathrm{Fuk}(\tilde{P}_{d+r-1}) \simeq \mathrm{Fuk}((h^\vee)^{-1}(\tilde{P}_{d+r-1}))^{G^\vee} \simeq \mathrm{Fuk}((h^\vee)^{-1}(\tilde{P}_{d+r-1}))_{BG}.$$

Taking G-invariants, we obtain

$$\mathrm{Fuk}(\tilde{P}_{d+r-1})^G \simeq (\mathrm{Fuk}((h^\vee)^{-1}(\tilde{P}_{d+r-1}))_{BG})^G \simeq \mathrm{Fuk}((h^\vee)^{-1}(\tilde{P}_{d+r-1})).$$

Via HMS for pairs of pants the left most term is equivalent to

$$\mathrm{MF}(\mathbb{A}^{d+r+1}, y_1 \cdots y_{d+r+1})^G \simeq \mathrm{MF}(\mathbb{A}^{d+r+1}/G, y_1 \cdots y_{d+r+1}).$$

Note that by Lemma 7.2 the product $y_1 \cdots y_{d+r+1}$ is invariant under the G-action. Thus we obtain

$$\mathrm{MF}(\mathbb{A}^{d+r+1}/G, y_1 \cdots y_{d+r+1}) \simeq \mathrm{Fuk}((h^\vee)^{-1}(\tilde{P}_{d+r-1})).$$

8. Intersections and categories

As explained in Remark 6.5, 6.13, the critical loci $\mathrm{Crit}(W_{\mathbf{X}}), \mathrm{Crit}(W_{\mathbf{Y}})$ dually project onto $\bigcap_{i=1}^r \Pi_{\Sigma_i}$ under $\mathbf{f}, \mathbf{g}$. We introduce a topology on $\bigcap_{i=1}^r \Pi_{\Sigma_i}$ induced from that on Π_{Σ_i} defined as in Definition 4.1. Then $\bigcap_{i=1}^r \Pi_{\Sigma_i}$ admits an open cover by the intersections $\bigcap_{i=1}^r S_i^{(k_i)}$ of k_i-strata $S_i^{(k_i)}$ of Π_{Σ_i}. In this section, we establish equivalences of corresponding categories over $\bigcap_{i=1}^r S_i^{(k_i)}$ and glue them to yield HMS for complete intersections of very affine hypersurfaces.

8.1. Covering complete intersections.

By Lemma 6.4 we have

$$\mathrm{Crit}(W_{\mathbf{X}}) = \bigcap_{i=1}^r \{u_i = 0\} \cap \bigcap_{i=1}^r \{W_i = 0\} \subset (\mathbb{C}^*)^{d+1} \times \mathbb{C}^r,$$

where $\{W_i = 0\}$ are given by the union $\bigcup_{v^i \in \mathrm{Vert}(\Pi_{\Sigma_i})} \tilde{P}_{v^i} \times \mathbb{C}^r$. We denote by $L_{v^i}(S_i^{(k_i)})$ the intersection of k_i legs of $\tilde{P}_{v^i}$ corresponding to $S_i^{(k_i)}$. Since by assumption $H_1, \ldots, H_r$ are in general position, we may assume that $\mathrm{Crit}(W_{\mathbf{X}})$ is the union

$$\bigcup_{v^1, \ldots, v^r} \bigcup_{S_1^{(k_1)}, \ldots, S_r^{(k_r)}} \left(\bigcap_{i=1}^r \{u_i = 0\} \cap \bigcap_{i=1}^r \left(L_{v^i}(S_i^{(k_i)}) \times \mathbb{C}^r \right) \right).$$

Under $\mathbf{f}$ it projects onto $\bigcup_{S_1^{(k_1)}, \ldots, S_r^{(k_r)}} \bigcap_{i=1}^r S_i^{(k_i)}$.

We denote by $\sigma(S_i^{(k_i)})$ the cones in the subfans $\Sigma_i \subset \Sigma_{\mathbf{Y}} \subset \mathbb{R}^{d+r+1}$ corresponding to $S_i^{(k_i)}$ and by $\xi_1(S_i^{(k_i)}), \ldots, \xi_{d+2-k_i}(S_i^{(k_i)}) \in \Sigma_i(1)$ the rays spanning $\sigma(S_i^{(k_i)})$. Since by assumption $\Pi_{\Sigma_1}, \ldots, \Pi_{\Sigma_r}$ intersect transversely, the rays

$$\xi_1(S_1^{(k_1)}), \ldots, \xi_{d+2-k_1}(S_1^{(k_1)}), \ldots, \xi_1(S_r^{(k_r)}), \ldots, \xi_{d+2-k_r}(S_r^{(k_r)}) \in \Sigma_{\mathbf{Y}}(1) \tag{8.1}$$

span a $(\Sigma_{i=1}^r d + 2 - k_i)$-dimensional cone $\sigma(S_1^{(k_1)}, \ldots, S_r^{(k_r)}) \in \Sigma_{\mathbf{Y}}$. Note that we have

$$\sum_{i=1}^r d + 2 - k_i \leq d + r + 1.$$

Under $\mathbf{g}$ the union of intersections of

$$U(S_1^{(k_1)}, \ldots, S_r^{(k_r)}) = \operatorname{Spec} \mathbb{C}[\sigma^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}) \cap (M \times \mathbb{Z}^r)]$$

and $\operatorname{Crit}(W_{\mathbf{Y}})$ projects onto $\bigcup_{S_1^{(k_1)}, \ldots, S_r^{(k_r)}} \bigcap_{i=1}^r S_i^{(k_i)}$.

8.2. **Local A-side categories for complete intersections.** Let $\vec{\beta}$ be a $\Sigma_{i=1}^r l_i$-tuple of vertices

$$\beta^{1,1}, \ldots, \beta^{1,l_1} \in A_1, \ldots, \beta^{r,1}, \ldots, \beta^{r,l_r} \in A_r$$

which together with the origin define rays

$$\xi_{d+3-k_1}(S_1^{(k_1)}), \ldots, \xi_{d+2-k_1+l_1}(S_1^{(k_1)}) \in \Sigma_1(1), \ldots, \xi_{d+3-k_r}(S_r^{(k_r)}), \ldots, \xi_{d+2-k_r+l_r}(S_r^{(k_r)}) \in \Sigma_r(1)$$

spanning a top dimensional cone $\sigma(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta}) \in \Sigma_{\mathbf{Y},\max}$ together with the rays (8.1). We denote by $\Delta^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$ the $(d + r + 1)$-simplex

$$\operatorname{Conv}(0, (-\alpha^{1,1}, e_1), \ldots, (-\alpha^{1,d+2-k_1+l_1}, e_1), \ldots, (-\alpha^{r,1}, e_r), \ldots, (-\alpha^{r,d+2-k_r+l_r}, e_r))$$

with $\xi_j(S_i^{(k_i)}) = \operatorname{Cone}(-\alpha^{i,j}, e_i)$ and $\alpha^{i,d+2-k_i+j_i} = \beta^{i,j_i}$.

Recall that the tailored $\Delta^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$-pants $\tilde{P}(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$ is the inverse image of $\tilde{P}_{d+r}$ under the map

$$h^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta}) \colon (\mathbb{C}^*)^{d+r+1} \to (\mathbb{C}^*)^{d+r+1},$$

whose restriction gives a finite cover of $\tilde{P}_{d+r}$. Here, $h^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$ is induced by a homomorphism

$$h(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta}) \colon \mathbb{Z}^{d+r+1} \to \mathbb{Z}^{d+r+1}$$

of lattices which sends $\Delta_{d+r+1}^\vee$ to $\Delta^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$. Now, assume that $\Delta^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$ is unimodular. Then $h^\vee(S_1^{(k_1)}, \ldots, S_r^{(k_r)}, \vec{\beta})$ becomes an isomorphism and the monomials

$$x^{-\alpha^{1,1}, e_1}, \ldots, x^{-\alpha^{1,d+2-k_1+l_1}, e_1}, \ldots, x^{-\alpha^{r,1}, e_r}, \ldots, x^{-\alpha^{r,d+2-k_r+l_r}, e_r}$$

give coordinates on the target $(\mathbb{C}^*)^{d+r+1}$.

Consider the product

$$\tilde{H}(S_1^{(k_1)}, \vec{\beta}) \times \cdots \times \tilde{H}(S_r^{(k_r)}, \vec{\beta}) \subset (\mathbb{C}^*)^{d+2-k_1+l_1} \times \cdots \times (\mathbb{C}^*)^{d+2-k_r+l_r} = (\mathbb{C}^*)^{d+r+1}$$

of very affine hypersurfaces

$$\tilde{H}(S_i^{(k_i)}, \vec{\beta}) = \{ \sum_{j_i=1}^{d+2-k_i+l_i} x^{-\alpha^{i,j_i}, e_i} = 0 \} \subset (\mathbb{C}^*)^{d+2-k_i+l_i} \subset (\mathbb{C}^*)^{d+r+1}.$$

We denote by $H(S_i^{(k_i)}, \vec{\beta})$ the quotients of $\tilde{H}(S_i^{(k_i)}, \vec{\beta})$ by the $\mathbb{C}_{u_i}^*$-action

$$(u_i, x^{-\alpha^{i,1}, e_i}, \ldots, x^{-\alpha^{i,d+2-k_i+l_i}, e_i}) \mapsto (u_i x^{-\alpha^{i,1}, e_i}, \ldots, u_i x^{-\alpha^{i,d+2-k_i+l_i}, e_i}),$$

which are isomorphic to $(d-k_i+l_i)$-dimensional tailored pants up to deformation. Locally, $\mathbf{H}$ is given by a product of r lower dimensional tailored pants.

Lemma 8.1. *Assume that $\Delta^\vee(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta})$ is unimodular. Then the product*

$$H(S_1^{(k_1)},\vec{\beta})\times\cdots\times H(S_r^{(k_r)},\vec{\beta})\subset(\mathbb{C}^*)^{d+1-k_1+l_1}\times\cdots\times(\mathbb{C}^*)^{d+1-k_r+l_r}=(\mathbb{C}^*)^{d+1}$$

is isomorphic to the intersection

$$\bigcap_{i=1}^r\{u_i=0\}\cap\bigcap_{i=1}^r\{\sum_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i}}=0\}\subset(\mathbb{C}^*)^{d+1}\times\mathbb{C}^r.$$

Proof. The quotient of $\times_{i=1}^r\tilde{H}(S_i^{(k_i)},\vec{\beta})$ by the $(\mathbb{C}^*)^r_{u_1,\dots,u_r}$-action

$$\begin{aligned}&(u_1,\dots,u_r,x^{-\alpha^{1,1},e_1},\dots,x^{-\alpha^{1,d+2-k_1+l_1},e_1},\dots,x^{-\alpha^{r,1},e_r},\dots,x^{-\alpha^{r,d+2-k_r+l_r},e_r})\\ \mapsto&(u_1x^{-\alpha^{1,1},e_1},\dots,u_1x^{-\alpha^{1,d+2-k_1+l_1},e_1},\dots,u_rx^{-\alpha^{r,1},e_r},\dots,u_rx^{-\alpha^{r,d+2-k_r+l_r},e_r})\end{aligned}$$

is $\times_{i=1}^rH(S_i^{(k_i)},\vec{\beta})$. Since we have

$$\times_{i=1}^r\tilde{H}(S_i^{(k_i)},\vec{\beta})=\bigcap_{i=1}^r\left(\{\sum_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i},e_i}=0\}\times(\mathbb{C}^*)^{r-1+k_i-l_i}_{\hat{x}^{-\alpha^{i,1},e_i},\dots,\hat{x}^{-\alpha^{i,d+2-k_i+l_i},e_i}}\right),$$

the quotient is isomorphic to

$$\bigcap_{i=1}^r\{u_i=\epsilon\}\cap\bigcap_{i=1}^r\{\sum_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i},e_i}=0\}\subset(\mathbb{C}^*)^{d+r+1},$$

which in turn is isomorphic to

$$\bigcap_{i=1}^r\{u_i=0\}\cap\bigcap_{i=1}^r\{\sum_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i}}=0\}\subset(\mathbb{C}^*)^{d+1}\times\mathbb{C}^r.$$

□

Corollary 8.2. *Assume that $\Delta^\vee(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta})$ is unimodular. Then there is an equivalence*

$$\mathrm{Fuk}(\bigcap_{i=1}^r\{u_i=0\}\cap\bigcap_{i=1}^r\{\Sigma_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i}}=0\})\simeq\bigotimes_{i=1}^r\mathrm{Fuk}(H(S_i^{(k_i)},\vec{\beta})).$$

Proof. As an open submanifold of a closed submanifold $\mathbf{H}$ of $(\mathbb{C}^*)^{d+r+1}$, the intersection $\bigcap_{i=1}^r\{u_i=0\}\cap\bigcap_{i=1}^r\{\Sigma_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i}}=0\}$ carries a Stein manfold structure, which in turn defines a Weinstein structure. Then the claim follows from [GPS2, Theorem 1.5, Corollary 1.18]. □

8.3. **Local B-side categories for complete intersections.** Also in this subsection, we assume $\Delta^\vee(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta})$ to be unimodular. Let

$$y_1(S_1^{(k_1)}),\dots,y_{d+2-k_1+l_1}(S_1^{(k_1)}),\dots,y_1(S_r^{(k_r)}),\dots,y_{d+2-k_r+l_r}(S_r^{(k_r)})$$

be local coordinates for

$$\begin{aligned}\mathbb{A}^{d+2-k_1+l_1}\times\cdots\times\mathbb{A}^{d+2-k_r+l_r}&\cong U(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta})\\&=\operatorname{Spec}\mathbb{C}[\sigma^\vee(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta})\cap(M\times\mathbb{Z}^r)]\subset\mathbf{Y}.\end{aligned}$$

Lemma 8.3. *Assume that $\Delta^\vee(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta})$ is unimodular. Then there is an equivalence*

$$\mathrm{MF}^\infty(U(S_1^{(k_1)},\dots,S_r^{(k_r)},\vec{\beta}),W_{\mathbf{Y}})\simeq\bigotimes_{i=1}^r\mathrm{MF}^\infty(\mathbb{A}^{d+2-k_i+l_i},y_1(S_i^{(k_i)})\cdots y_{d+2-k_i+l_i}(S_i^{(k_i)})).$$

Proof. Since we have

$$W_{\mathbf{Y}}|_{U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})} = y_1(S_1^{(k_1)})\cdots y_{d+2-k_1+l_1}(S_1^{(k_1)}) + \cdots + y_1(S_r^{(k_r)})\cdots y_{d+2-k_r+l_r}(S_r^{(k_r)}),$$

the claim follows from Lemma 3.16, Lemma 6.12 and [Pre, Theorem 4.1.3]. □

8.4. **Gluing equivalences.**

Lemma 8.4. *Assume that $\Delta^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ is unimodular. Then there is an equivalence*

$$\mathrm{Fuk}(\bigcap_{i=1}^{r}\{u_i=0\}\cap\bigcap_{i=1}^{r}\{\Sigma_{j_i=1}^{d+2-k_i+l_i}x^{-\alpha^{i,j_i}}=0\})_{\mathbb{Z}_2}\simeq \mathrm{MF}^\infty(U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}),W_{\mathbf{Y}}).$$

Proof. Due to Corollary 8.2 and Lemma 8.3 it suffices to show an equivalence

$$\mathrm{Fuk}(H(S_i^{(k_i)},\vec{\beta}))_{\mathbb{Z}_2}\simeq \mathrm{MF}^\infty(\mathbb{A}^{d+2-k_i+l_i},y_1^i(S_i^{(k_i)})\cdots y^i_{d+2-k_i+l_i}(S_i^{(k_i)})). \tag{8.2}$$

This follows from Theorem 5.4, since by construction $H(S_i^{(k_i)},\vec{\beta})$ is isomorphic to $(d-k_i+l_i)$-dimensional tailored pants up to deformation. □

Theorem 8.5. *Assume that $\Delta^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ are unimodular for all $S_1^{(k_1)},\ldots,S_r^{(k_r)}$ whose intersection is nonempty. Then there is an equivalence*

$$\mathrm{Fuk}(\mathbf{H})_{\mathbb{Z}_2}\simeq \lim_{S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}} \mathrm{MF}^\infty(U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}),W_{\mathbf{Y}})=\mathrm{MF}^\infty(\mathbf{Y},W_{\mathbf{Y}})$$

Proof. We glue the local equivalences from Lemma 8.4. Again, due to Corollary 8.2 and Lemma 8.3 it suffices to show the compatibility of (8.2) with gluing, which follows from the same argument as the proof of Theorem 5.5. □

8.5. **Nonunimodular case.** Finally, we drop the assumption on $\Delta^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ to be unimodular. The simplicial toric variety $U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ becomes isomorphic to the geometric quotient

$$\mathbb{A}^{d+r+1}/G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}).$$

by the finite abelian subgroup $G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ of $(\mathbb{C}^*)^{d+r+1}$ with canonically induced action. The inclusion extends to a short exact sequence

$$0\to G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})\to(\mathbb{C}^*)^{d+r+1}\to M^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})_{\mathbb{C}}\to 0, \tag{8.3}$$

where $M^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ is the cocharacter lattice associated with $U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$. Then we have

$$\mathrm{MF}^\infty(\mathbb{A}^{d+r+1}/G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}),W_{\mathbf{Y}})\simeq \mathrm{MF}^\infty(\mathbb{A}^{d+r+1},W_{\mathbf{Y}})^{G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})}.$$

On the A-side, $\tilde{P}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ becomes a finite cover of $\tilde{P}_{d+r}$. Since $U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ has no torus factors, (8.3) is obtained from the short exact sequence

$$0\to M(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})\to\mathbb{Z}^{d+r+1}\to G^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})\to 0$$

by taking $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$. Here, we use the symbol $G^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ to denote the divisor class group of $U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$, which acts on

$$(\mathbb{C}^*)^{d+r+1}\cong T^*(\mathbb{R}^{d+r+1}/M(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})).$$

Then the finite cover is given by

$$\tilde{P}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})\to\tilde{P}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})/G^\vee(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})=\tilde{P}_{d+r}.$$

Similarly replace

$$\times_{i=1}^{r}\tilde{H}(S_i^{(k_i)},\vec{\beta}) = \bigcap_{i=1}^{r}\left(\{\sum_{j_i=1}^{d+2-k_i+l_i} x^{-\alpha^{i,j_i,e_i}} = 0\} \times (\mathbb{C}^*)^{r-1+k_i-l_i}_{\hat{x}^{-\alpha^{i,1,e_i}},\ldots,\hat{x}^{-\alpha^{i,d+2-k_i+l_i,e_i}}}\right) = \bigcap_{i=1}^{r}\{\sum_{j_i=1}^{d+2-k_i+l_i} x^{-\alpha^{i,j_i,e_i}} = 0\}$$

and $\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta})$ with their inverse images. Then we have

$$\text{(8.4)} \qquad \mathrm{Fuk}((\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))/G^{\vee}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})) \simeq \mathrm{Fuk}(\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))^{G^{\vee}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})}.$$

As explained above, there is an equivalence to the de-equivariantization

$$\text{(8.5)} \qquad \mathrm{Fuk}(\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))^{G^{\vee}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})} \simeq \mathrm{Fuk}(\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))_{BG(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})}$$

with respect to $G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$. Combining (8.4), (8.5) and the equivalence for unimodular case, we obtain

$$\mathrm{MF}^{\infty}(\mathbb{A}^{d+r+1}, W_{\mathbf{Y}}) \simeq (\mathrm{Fuk}(\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))_{\mathbb{Z}_2})_{BG(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})}.$$

Passing to the equivariantization, we obtain

$$\mathrm{MF}^{\infty}(\mathbb{A}^{d+r+1}/G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}), W_{\mathbf{Y}}) \simeq ((\mathrm{Fuk}(\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))_{\mathbb{Z}_2})_{BG(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})})^{G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})}$$

which by Lemma 7.1 implies

$$\text{(8.6)} \qquad \mathrm{MF}^{\infty}(U(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta}), W_{\mathbf{Y}}) \simeq \mathrm{Fuk}((\times_{i=1}^{r}H(S_i^{(k_i)},\vec{\beta}))).$$

We glue the above local equivalences. It suffices to show the compatibility of (8.6) with gluing, which follows from the same argument as the proof of Theorem 5.5 extended in a straight forward way. Indeed, the compatibility of the actions of $G(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ and $G^{\vee}(S_1^{(k_1)},\ldots,S_r^{(k_r)},\vec{\beta})$ with gluing follows from Lemma 7.2 and the combinatorial duality.

References

[Abo06] M. Abouzaid, *Homogeneous coordinate rings and mirror symmetry for toric varieties*, Geom. Topol. 10, 1097-1157 (2006).

[AA] M. Abouzaid and D. Auroux, *Homological mirror symmetry for hypersurfaces in* $(\mathbb{C}^*)^N$, arXiv:2111.06543

[AAK16] M. Abouzaid, D. Auroux, and L. Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. IHÉS. 123-199 (2016).

[BB96] V. Batyrev and L. A. Borisov, *On Calabi-Yau complete intersections in toric varieties*, Higher-dimensional complex varieties (Trento, 1994), 39-65 (1996).

[CE12] K. Cieliebak and Y. Eliashberg, *From Stein and Weinstein and back*, Symplectic Geometry of Affine Complex Manifolds, Colloquium Publications Vol. 59, AMS, (2012).

[CLS11] D. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics Vol. 124, AMS, (2011).

[DK18] T. Dyckerhoff and M. Kapranov, *Triangulated surfaces in triangulated categories*, J. Eur. Math. Soc. 20(6), 1473-1524 (2018).

[Eli] Y. Eliashberg, *Weinstein manifolds revisited*, arXiv:1707.03442

[FOOO1] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction, Part I*, Studies in Advanced Mathematics Vol. 46.1, AMS/IP, (2010).

[FOOO2] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction, Part II*, Studies in Advanced Mathematics Vol. 46.2, AMS/IP, (2010)

[Fuk93] K. Fukaya, *Morse homotopy, A_∞-category, and Floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93, Lecture Notes Ser. Vol.18, 1–102, Seoul Nat. Univ., Seoul, (1993).

[GKR17] M. Gross, L. Katzarkov, and H. Ruddat. *Towards mirror symmetry for varieties of general type*, Adv. Math. 308, 208-275 (2017).

[GPS1] S. Ganatra, J. Pardon, and V. Shende, *Covariantly functorial wrapped Floer theory on Liouville sectors*, Publ. math. IHES. 131, 73–200 (2020).

[GPS2] S. Ganatra, J. Pardon, and V. Shende, *Sectorial descent for wrapped Fukaya categories*, arXiv:1809.03427v3

[GPS3] S. Ganatra, J. Pardon, and V. Shende, *Microlocal Morse theory of wrapped Fukaya categories*, arXiv:1809.08807v2

[GS1] B. Gammage and V. Shende, *Mirror symmetry for very affine hypersurfaces*, arXiv:1707.02959v3

[GS2] B. Gammage and V. Shende, *Homological mirror symmetry at large volume*, arXiv:2104.11129v2

[Jef22] M. Jeffs, *Mirror symmetry and Fukaya categories of singular hypersurfaces*, Adv. Math. 397, 108116 (2022).

[KW07] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. 1(1), 1-236 (2007).

[Kon09] M. Kontsevich, *Symplectic geometry of homological algebra*, https://www.ihes.fr/ maxim/TEXTS/Symplectic_AT2009.pdf

[Lee16] H. Lee, *Homological mirror symmetry for open riemann surfaces from pair-of-pants decompositions*, arXiv:1608.04473

[Mik04] G. Mikhalkin, *Decomposition into pairs of pants for complex algebraic hypersurfaces*, Topology. 43(5), 1035-1065 (2004).

[Nad] D. Nadler, *Wrapped microlocal sheaves on pairs of pants*, arXiv:1604.00114

[NZ09] D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. 22(1), 233-286 (2009).

[Orl04] D. Orlov, *Triangulated categories of singularities and D-branes in Landau–Ginzburg models*, Proc. Steklov Inst. Math. 246, 227-249 (2004).

[Pre] A. Preygel, *Thom–Sebastiani & duality for matrix factrizations*, arXiv:1101.5834

[PS1] J. Pascaleff and N. Sibilla, *Topological Fukaya category and mirror symmetry for punctured surfaces*, Compos. Math. 155(3), 599-644 (2019).

[PS2] J. Pascaleff and N. Sibilla, *Fukaya categories of higher-genus surfaces and pants decompositions*, arXiv:2103.03366v3

[PS3] J. Pascaleff and N. Sibilla, *Singularity categories of normal crossings surfaces, descent, and mirror symmetry*, arXiv:2208.03896

[PZ98] A. Polishchuk and E. Zaslow *Categorical mirror symmetry: The elliptic curve*, Adv. Theor. Math. Phys. 2(2), 443-470 (1998).

[Sei03] P. Seidel, *Homological mirror symmetry for the quartic surface*, Mem. Am. Math. Soc. 236(1116), 1-129, AMS, (2015).

[Sei10] P. Seidel, *Some speculations on pairs-of-pants decompositions and Fukaya categories*, arXiv:1004.0906v5

[She15] N. Sheridan, *Homological mirror symmetry for Calabi–Yau hypersurfaces in projective space*, Invent. Math. 199(1), 1-186 (2015).

[She22] V. Shende, *Toric mirror symmetry revisited*, Comptes Rendus Math. 360, 751-759 (2022).

[STZ14] N. Sibilla, D. Treumann, and E. Zaslow. *Ribbon graphs and mirror symmetry*, Selecta Math. 20(4), 979-1002 (2014).

SISSA, via Bonomea 265, 34136 Trieste, Italy
Email address: hmorimur@sissa.it

SISSA, via Bonomea 265, 34136 Trieste, Italy
Email address: nsibilla@sissa.it

University of California, Berkeley, CA 94720-3840, Berkeley, USA
Email address: pzhou.math@berkeley.edu

LARGE VOLUME LIMIT FIBRATIONS OVER FANIFOLDS

HAYATO MORIMURA

Abstract. We lift the stratified torus fibration over a fanifold constructed by Gamage–Shende to the associated Weinstein manifold, which is homotopic to a filtered stratified integrable system with noncompact fibers. When the fanifold admits a dual stratified space in a suitable sense, we give a stratified fibration over it completing SYZ picture. For the fanifold associated with a very affine hypersurface, we realize the latter fibration as a restriction of SYZ fibrations over the tropical hypersurface proposed by Abouzaid–Auroux–Katzarkov.

1. Introduction

Given a Calabi–Yau manifold X, Homological Mirror Symmetry(HMS) conjecture [Kon00] claims the existence of another Calabi–Yau manifold $\check{X}$ called its mirror partner whose Fukaya category $\mathrm{Fuk}(\check{X})$ and dg category of coherent sheaves $\mathrm{Coh}(\check{X})$ are respectively equivalent to $\mathrm{Coh}(X)$ and $\mathrm{Fuk}(X)$. The equivalences are believed to connect the symplectic and complex geometry of the mirror pair $(X, \check{X})$. Nowadays, HMS concerns more general spaces but neither precise definition nor systematic construction of mirror pairs are available.

One way to go is indicated by Strominger–Yau–Zaslow(SYZ) conjecture [SYZ96], which has been elaborated independently by Kontsevich–Soibelman [KS00, KS06] and Gross–Siebert [GS11]. Roughly speaking, $(X, \check{X})$ should admit so called SYZ fibrations, i.e., dual special Lagrangian torus fibrations $X \to B \leftarrow \check{X}$ over a common base. Then $\check{X}$ should be obtained by dualizing $X \to B$ over the smooth locus and compactifying the result in a suitable way. SYZ conjecture is hard to correctly formulate and still largely open.

Recently, Gamage–Shende introduced fanifolds [GS1, Definition 2.4] by gluing rational polyhedral fans of cones, which provide the organizing topological and discrete data for HMS at large volume [GS1, Theorem 5.4]. To a fanifold Φ they associated an algebraic space $\mathbf{T}(\Phi)$ [GS1, Proposition 3.10] obtained as the gluing of the toric varieties T_Σ associated with the fans Σ along their toric boundaries. Based on an idea from SYZ fibrations, they constructed its mirror partner $\widetilde{\mathbf{W}}(\Phi)$ by inductive Weinstein handle attachements.

Theorem 1.1 ([GS1, Theorem 4.1]). *Let $\Phi \subset \mathcal{M}$ be a fanifold. Then there exists a triple $(\widetilde{\mathbf{W}}(\Phi), \widetilde{\mathbb{L}}(\Phi), \pi)$ of a subanalytic Weinstein manifold $\widetilde{\mathbf{W}}(\Phi)$, a conic subanalytic Lagrangian $\widetilde{\mathbb{L}}(\Phi) \subset \widetilde{\mathbf{W}}(\Phi)$, and a map $\pi \colon \widetilde{\mathbb{L}}(\Phi) \to \Phi$ satisfying the following conditions.*

(1) *Let $S \subset \Phi$ be a stratum of codimension d. Then:*
- *$\pi^{-1}(S) \cong T^d \times S$ where T^d is a real d-dimensional torus.*
- *$\pi^{-1}(\mathrm{Nbd}(S)) \cong \mathbb{L}(\Sigma_S) \times S$ where $\mathrm{Nbd}(S)$ is an appropriate neighborhood and $\mathbb{L}(\Sigma_S)$ is the FLTZ Lagrangian associated with the normal fan Σ_S of S.*
- *In a neighborhood of $\pi^{-1}(S) \cong T^d \times S$, there is a symplectomorphism of pairs*

$$(T^*T^d \times T^*S, \mathbb{L}(\Sigma_S) \times S) \hookrightarrow (\widetilde{\mathbf{W}}(\Phi), \widetilde{\mathbb{L}}(\Phi)). \tag{1.1}$$

(2) *If Φ is closed, then we have $\widetilde{\mathbb{L}}(\Phi) = \mathrm{Core}(\widetilde{\mathbf{W}}(\Phi))$ for the skeleton $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi))$ of $\widetilde{\mathbf{W}}(\Phi)$.*

(3) *A subfanifold $\Phi' \subset \Phi$ determines a Weinstein sector $\mathbf{W}(\Phi') \subset \widetilde{\mathbf{W}}(\Phi)$ with skeleton $\mathbb{L}(\Phi') = \mathbf{W}(\Phi') \cap \widetilde{\mathbb{L}}(\Phi)$*

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(4) *The Weinstein manifold $\widetilde{\mathbf{W}}(\Phi)$ carries a Lagrangian polarization given in the local charts* (1.1) *by taking the fiber direction in T^*T^d and the base direction in T^*S.*

The guiding principle behind their construction of $\widetilde{\mathbf{W}}(\Phi)$ was that the FLTZ Lagrangians $\mathbb{L}(\Sigma)$ and the projections $\mathbb{L}(\Sigma) \to \Sigma$ should glue to yield $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi))$ and π. Theorem 1.1(4) gives rise to canonical grading/orientation data [GPS3, Section 5.3], which one needs to define the partially wrapped Fukaya category $\mathrm{Fuk}(\widetilde{\mathbf{W}}(\Phi), \partial_\infty \widetilde{\mathbb{L}}(\Phi))$. By [GPS3, Theorem 1.4] we have

$$\mathrm{Fuk}(\widetilde{\mathbf{W}}(\Phi), \partial_\infty \widetilde{\mathbb{L}}(\Phi)) \cong \Gamma(\widetilde{\mathbb{L}}(\Phi), \mu\mathrm{sh}_{\widetilde{\mathbb{L}}(\Phi)})^{op} \cong \Gamma(\Phi, \pi_* \mu\mathrm{sh}_{\widetilde{\mathbb{L}}(\Phi)})^{op}$$

where $\mu\mathrm{sh}_{\widetilde{\mathbb{L}}(\Phi)}$ is a certain constructible sheaf of dg categories. Over neighborhoods of strata, local sections of $\pi_* \mu\mathrm{sh}_{\widetilde{\mathbb{L}}(\Phi)}$ are computed in [GS2, Section 7.3]. The third bullet in Theorem 1.1(1) gives rise to gluing data [GS1, Proposition 4.34], which one needs to determine $\pi_* \mu\mathrm{sh}_{\widetilde{\mathbb{L}}(\Phi)}$.

SYZ fibrations often give candidates for mirror pairs and [GS1, Theorem 5.4] is just one of such examples. Moreover, [GS1, Corollary 5.8] implies that HMS at large volume limit is obtained by the local-to-global principle. Hence it is natural to expect that the canonical lift of the projections $\mathbb{L}(\Sigma) \to \Sigma$ should glue to yield a version of A-side SYZ fibration $\widetilde{\mathbf{W}}(\Phi) \to \Phi$, as predicted by Gamage–Shende [GS1, Remark 4.5]. Indeed, taking wrapped Fukaya categories and gluing of local pieces of $\widetilde{\mathbf{W}}(\Phi)$, which respects the gluing of $\mathbb{L}(\Sigma)$, intertwine in the sense of [GS1, Corollary 5.9]. In this paper, we prove the following.

Theorem 1.2. *There is a filtered stratified fibration $\bar{\pi}\colon \widetilde{\mathbf{W}}(\Phi) \to \Phi$ restricting to π, which is homotopic to a filtered stratified integrable system with noncompact fibers. If the fan Σ_S associated to any stratum $S \subset \Phi$ is proper, then the homotopy becomes trivial.*

Note that by construction of $\widetilde{\mathbf{W}}(\Phi)$ one can expect isotrivial fibers only on each stratum of the induced stratification by Φ. Moreover, Φ admits a filtration [GS1, Remark 2.12] and fibers would vary when passing through the induced filters. Since in general the canonical lift of π does not land in Φ, we must compose the map induced by a retraction, which might not be smooth, even C^1. After that, as suggested in [GS1, Remark 4.5], it remains to show compatibility with the handle attachment process. Then we need to fully understand the proofs of Theorem 1.1, especially (4) and the third bullet in (1). As their details are skipped in [GS1], we include them for completeness.

If Φ admits a well defined dual stratified space Ψ, then one expects the gluing $\mathbf{T}(\Phi) \to \Psi$ of algebraic moment maps to be the B-side SYZ fibration as mentioned in [GS1, Remark 4.5]. We obtain a stratified fibration which should be its SYZ dual, modifying the canonical lift of π.

Theorem 1.3. *Assume that Φ has the dual stratified space Ψ in the sense of Definition 6.1. Then there is a stratified fibration $\underline{\pi}\colon \widetilde{\mathbf{W}}(\Phi) \to \Psi$. Let $S \subset \Phi$ be a stratum of codimension d. Over a point of its dual stratum $S^\perp \subset \Psi$, the fiber of $\underline{\pi}$ is isomorphic to $T^d \times T^*S$.*

Finally, we provide an evidence to convince the readers that $\underline{\pi}$ should be the A-side SYZ fibration for $(\widetilde{\mathbf{W}}(\Phi), \mathbf{T}(\Phi))$. Consider the fanifold $\Phi = \Sigma \cap S^{n+1}$ from [GS1, Example 4.22] generalized as in [GS1, Section 6]. It is associated with the mirror pair $(H, \partial \mathbf{T}_\Sigma)$ [GS2, Theorem 1.0.1] in the sense that $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi)) = \mathrm{Core}(\mathbf{W}(\Phi)) \cong \mathrm{Core}(H)$ and $\mathbf{T}(\Phi) = \partial \mathbf{T}_\Sigma$ for a very affine hypersuface $H \subset (\mathbb{C}^*)^{n+1}$ and the toric boundary divisor $\partial \mathbf{T}_\Sigma$ of the toric stack $\mathbf{T}_\Sigma$. The fanifold Φ has the dual stratified space Ψ. We show that the SYZ dual fibrations for $(H, \partial \mathbf{T}_\Sigma)$ over the tropical hypersurface Π_Σ of H [AAK16, Section 3] restricts to that for $(\mathbf{W}(\Phi), \mathbf{T}(\Phi))$ over Ψ in the following sense.

Theorem 1.4. *Let Φ be the fanifold from* [GS1, Example 4.22] *generalized as in* [GS1, Section 6]. *Then there are a stratified homeomorphism $\Psi \hookrightarrow \Pi_\Sigma$, a symplectomorphism of pairs*

$(\mathbf{W}(\Phi), \mathbb{L}(\Phi)) \hookrightarrow (H, \mathrm{Core}(H))$ *and a map* $\mathbb{T}(\Phi) \to \Psi$ *which makes the diagram*

$$\begin{tikzcd}
\mathbb{L}(\Phi) \arrow[r, hook] \arrow[d, equal] & \mathbf{W}(\Phi) \arrow[r, "\pi"] \arrow[d, hook] & \Psi \arrow[d, hook] & \mathbf{T}(\Phi) \arrow[l] \arrow[d, hook] \\
\mathrm{Core}(H) \arrow[r, hook] & H \arrow[r] & \Pi_\Sigma & -K_{\mathbf{T}_\Sigma} \arrow[l]
\end{tikzcd}$$

commute, where $H \to \Pi_\Sigma \leftarrow -K_{\mathbf{T}_\Sigma}$ *are canonical extensions to toric stacks of SYZ fibrations from* [AAK16, Section 3].

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2. Review on fanifolds

Fanifolds are introduced by Gamage–Shende in [GS1, Section 2] as a formulation of stratified manifolds for which the geometry normal to each stratum is equipped with the structure of a fan. Let Φ be a stratified space. As in [GS1], throughout the paper, we assume Φ to satisfy the following conditions.

(i) Φ has finitely many strata.
(ii) Φ is conical in the complement of a compact subset.
(iii) Φ is given as a germ of a closed subset in an ambient manifold $\mathcal{M}$.
(iv) The strata of Φ are smooth submanifolds of $\mathcal{M}$.
(v) The strata of Φ are contractible.

We will express properties of Φ in terms of the chosen ambient manifold $\mathcal{M}$ as long as they only depend on the germ of Φ. Taking the normal cone $C_S\Phi \subset T_S\mathcal{M}$ for each stratum $S \subset \Phi$, one obtains a stratification on $C_S\Phi$ induced by that of a sufficiently small tubular neighborhood $T_S\mathcal{M} \to \mathcal{M}$.

Definition 2.1. The stratified space Φ is *smoothly normally conical* if for each stratum $S \subset \Phi$ some choice of tubular neighborhood $T_S\mathcal{M} \to \mathcal{M}$ induces locally near S a stratified diffeomorphism $C_S\Phi \to \Phi$, which in turn induces the identity $C_S\Phi \to C_S\Phi$.

Example 2.2. Associating to each cone its interior as a stratum, one may regard a fan Σ of cones as a stratified space satisfying the conditions (i), ... , (v). Clearly, Σ is smoothly normally conical. By abuse of notation, we use the same symbol $\sigma \in \Sigma$ to denote a stratum of Σ corresponding to σ. We introduce a partial order in Σ canonically descending to the stratified space. Namely, for two strata $\sigma, \tau \in \Sigma$ we define $\sigma < \tau$ if and only if $\sigma \subset \bar{\tau}$ for the closure $\bar{\tau}$ of τ in Σ.

Definition 2.3. We write $\mathrm{Exit}(\Phi)$ for the exit path category. For each stratum $S \subset \Phi$ we write $\mathrm{Exit}_S(\Phi)$ for the full subcategory of exit paths starting at S contained inside a sufficiently small neighborhood of S.

Definition 2.4. We write $\mathrm{Fan}^{\twoheadrightarrow}$ for the category whose objects are pairs (M, Σ) of a laticce M and a stratified space Σ by finitely many rational polyhedral cones in $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. For any $(M, \Sigma), (M', \Sigma') \in \mathrm{Fan}^{\twoheadrightarrow}$ a morphism $(M, \Sigma) \to (M', \Sigma')$ is given by the data of a cone $\sigma \in \Sigma$ and an isomorphism $M/\langle\sigma\rangle \cong M'$ such that $\Sigma' = \Sigma/\sigma = \{\tau/\langle\sigma\rangle \subset M_\mathbb{R}/\langle\sigma\rangle \mid \tau \in \Sigma, \sigma \subset \bar{\tau}\}$. We denote by $\mathrm{Fan}^{\twoheadrightarrow}_{\Sigma/}$ for an object $(M, \Sigma) \in \mathrm{Fan}^{\twoheadrightarrow}$ the full subcategory of objects (M', Σ') with $\Sigma' = \Sigma/\sigma$ for some cone $\sigma \in \Sigma$.

We have a natural identification

$$\mathrm{Exit}(\Sigma) \cong \mathrm{Fan}^{\twoheadrightarrow}_{\Sigma/}, \ \sigma \mapsto [\Sigma \mapsto \Sigma/\sigma]$$

of posets. In addition, the normal geometry to σ is the geometry of Σ/σ. This is the local model of fanifolds introduced by Gamege–Shende.

Definition 2.5 ([GS1, Definition 2.4])**.** A *fanifold* is a smoothly normally conical stratified space $\Phi \subset \mathcal{M}$ satisfying the conditions (i), ..., (v) and equipped with the following data:

- A functor $\mathrm{Exit}(\Phi) \to \mathrm{Fan}^{\twoheadrightarrow}$ whose value on each stratum S is a pair (M_S, Σ_S) of a lattice M_S and a rational polyhedral fan $\Sigma_S \subset M_{S,\mathbb{R}}$ called the *associated normal fan*.
- For each stratum $S \subset \Phi$ a trivialization $\phi_S : T_S\mathcal{M} \cong M_{S,\mathbb{R}}$ of the normal bundle carrying the induced stratification on $C_S\Phi$ to the standard stratification induced by Σ_S.

These data must make the diagram

$$\begin{tikzcd} T_S\mathcal{M} \arrow[r, "\phi_S"] \arrow[d] & M_{S,\mathbb{R}} \arrow[d] \\ T_{S'}\mathcal{M}|_S \arrow[r, "\phi_{S'}"'] & M_{S',\mathbb{R}}, \end{tikzcd}$$

commute for any stratum S' of the induced stratification on $\mathrm{Nbd}(S)$, where the left vertical arrow is the quotient by the span of S'. The right vertical arrow corresponds to the map $M_S \to M_{S'}$ on lattices.

Remark 2.6. In the original definition, the conditions (i), ..., (iv) were assumed in advance and the condition (v) was added later [GS1, Assumption 2.5]. Due to (v), $\mathrm{Exit}_S(\Phi)$ is equivalent to the poset $\mathrm{Exit}(\Sigma_S)$. Although we have already assumed Φ to satisfy (i), ..., (v) above, here we mention it again to emphasize the difference from the original form.

Example 2.7. A manifold $\mathcal{M}$ regarded as a trivially stratified space is obviously a smoothly normally conical satisfying (i), ..., (v). Associating to the unique stratum $\mathcal{M}$ a pair $(M_{\mathcal{M}} = \{0\}, \Sigma_{\mathcal{M}} = \{0\})$ defines a trivial fanifold structure on $\mathcal{M}$. It follows that the product of a manifold and a fanifold is canonically a fanifold.

Example 2.8 ([GS1, Example 2.7])**.** As explained in Example 2.2, a fan $\Sigma \subset \mathbb{R}^n$ of cones regarded as a stratified space is a smoothly normally conical satisfying (i), ..., (v). Associating to each stratum $\sigma \in \Sigma$ a pair $(M/\langle\sigma\rangle, \Sigma/\sigma)$ of the quotient lattice $M/\langle\sigma\rangle$ and the normal fan Σ/σ defines a fanifold structure on Σ.

Example 2.9 ([GS1, Example 2.10])**.** Given a fanifold $\Phi \subset \mathcal{M}$, if a submanifold $\mathcal{M}' \subset \mathcal{M}$ intersects transversely all strata of Φ, then $\Phi \cap \mathcal{M}' \subset \mathcal{M}'$ canonically inherits the fanifold structure. In particular, the ideal boundary $\partial_\infty\Phi$ of the fanifold carries a canonical fanifold structure.

Lemma 2.10 ([GS1, Remark 2.12])**.** *Let Φ be a fanifold of dimension n. Then it admits a filtration*

$$\Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_n = \Phi \tag{2.1}$$

where Φ_k are subfanifolds defined as sufficiently small neiborhoods of k-skeleta $\mathrm{Sk}_k(\Phi)$*, the closure of the subset of k-strata.*

Proof. The normal geometry to a 0-stratum $P \subset \Phi$ is the geometry of the normal fan $\Sigma_P \subset M_{P,\mathbb{R}}$ by definition. Let

$$\Phi_0 = \bigsqcup_P \Sigma_P$$

be the disjoint union of Σ_P equipped with the canonically induced fanifold structure, where P runs through all 0-strata of Φ. Then $\Phi_0 \subset \Phi$ is clearly a subfanifold containing $\mathrm{Sk}_0(\Phi)$.

Suppose that Φ_{k-1} has the desired filtration $\Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_{k-1}$. The normal geometry to a k-stratum $S \subset \Phi$ is the geometry of the normal fan $\Sigma_S \subset M_{S,\mathbb{R}}$ by definition. The ideal boundary

$\partial_\infty S$ might have some subset $\partial_{in}S$ which is in the direction to the interior of Φ. Perform the gluing $\Phi_{k-1}\#_{(\Sigma_S\times\partial_{in}S)}(\Sigma_S\times S)$ which is equal to $\Sigma_S\times S$ when Φ_{k-1} is empty and equal to Φ_{k-1} unless S is an *interior* k-stratum, i.e., $\partial_{in}S = \partial S_\circ$ where $S_\circ = S\setminus\Phi_{k-1}$ is a manifold-with-boundary. Note that by Example 2.9 the induced fanifold structure on $\partial_{in}S$ is compatible with that of Φ_{k-1}. Let

$$\Phi_k = \Phi_{k-1}\#_{\bigsqcup_S(\Sigma_S\times\partial_{in}S)}\bigsqcup_S(\Sigma_S\times S)$$

be the result of such gluings for all k-strata S in Φ. Then $\Phi_k\subset\Phi$ is a subfanifold containing $\mathrm{Sk}_k(\Phi)$ since the products $\Sigma_S\times S$ are canonically fanifolds by Example 2.7, 2.8. □

Definition 2.11. A fanifold $\Phi\subset\mathcal{M}$ is *closed* if its all strata are interior. Namely, any k-stratum S of Φ satisfies $\partial_{in}S = \partial S_\circ$ where $S_\circ = S\setminus\Phi_{k-1}$.

Example 2.12 ([GS1, Example 4.21])**.** Consider a 2-dimensional fanifold $\Phi = [0,1]\times[0,1]$ stratified by vertices $P_1 = (0,1), P_2 = (0,0), P_3 = (1,0), P_4 = (1,1)$, edges $I_{12} = \{0\}\times[0,1], I_{23} = [0,1]\times\{0\}, I_{34} = \{1\}\times[0,1], I_{23} = [0,1]\times\{1\}$ and a face $F = (0,1)\times(0,1)$. As all strata are interior, Φ is closed. The filtration from Lemma 2.10 is given by

$$\Phi_0 = \bigsqcup_{i=1}^{4}\Sigma_{P_i},\ \Phi_1 = \Phi_0\#_{\bigsqcup_{1\le i<j\le4}\Sigma_{I_{ij}}\times\partial_{in}I_{ij}}\bigsqcup_{1\le i<j\le4}(\Sigma_{I_{ij}}\times I_{ij}),\ \Phi_2 = \Phi_1\#_{\partial_{in}F}F \tag{2.2}$$

where $\Sigma_{P_i},\Sigma_{I_{ij}}$ are the fans $\Sigma_{\mathbb{A}^2}\subset\mathbb{R}^2, \Sigma_{\mathbb{A}^1}\subset\mathbb{R}$. We place the origins of Σ_{P_i} at P_i and that of $\Sigma_{I_{ij}}$ along I_{ij} so that $\Sigma_{P_i}\cap F\neq\emptyset$ and $\Sigma_{I_{ij}}\cap F\neq\emptyset$.

As explained in [GS1, Section 6], one can generalize fanifolds in terms of stacky fans. First, we recall the definition of stacky fans.

Definition 2.13 ([GS15, Definition 2.4])**.** A *stacky fan* is the data of a map of lattices $\beta\colon\tilde{M}\to M$ with finite cokernel, together with fans $\tilde{\Sigma}\subset\tilde{M}_{\mathbb{R}}$ and $\Sigma\subset M_{\mathbb{R}}$ such that β induces a combinatorial equivalence on the fans.

Stacky fans form a category $\mathrm{stFan}^{\twoheadrightarrow}$. Morphisms

$$(M,\tilde{M},\Sigma,\tilde{\Sigma}).\to(M',\tilde{M}',\Sigma',\tilde{\Sigma}')$$

are given by the choices of cones $\tilde{\sigma}$ with $\beta(\tilde{\sigma}) = \sigma$ and compatible isomorphisms

$$(M/\langle\sigma\rangle,\tilde{M}/\langle\tilde{\sigma}\rangle,\Sigma/\sigma,\tilde{\Sigma}/\tilde{\sigma})\cong(M',\tilde{M}',\Sigma',\tilde{\Sigma}').$$

Now, replace the functor $\mathrm{Exit}(\Phi)\to\mathrm{Fan}^{\twoheadrightarrow}$ in Definition 2.5 with $\mathrm{Exit}(\Phi)\to\mathrm{stFan}^{\twoheadrightarrow}$. Defining the *associated normal fan* as Σ_S for each value $(M_S,\tilde{M}_S,\Sigma_S,\tilde{\Sigma}_S)$, one obtains the generalization.

3. Review on Weinstein handle attachments

The Weinstein manifold $\widetilde{\mathbf{W}}(\Phi)$ is obtained by inductively attaching products of cotangent bundles of real tori and strata of Φ. Each step requires us to modify Weinstein structures near gluing regions. In order to show the compatibility of such modifications with candidate maps, we need to fully understand the attachment process.

3.1. **Weinstein manifolds.**

Definition 3.1 ([CE12, Section 11.1], [Eli, Section 1])**.** A *Liouville domain* (W,λ) is a compact symplectic manifold $(W,\omega = d\lambda)$ with smooth boundary ∂W whose Liouville vector field $Z = \omega^\#\lambda$ points outwardly along ∂W. We call the positive half $(\partial W\times\mathbb{R}_{\ge0}, e^t(\lambda|_{\partial W}))$ of the symplectization of ∂W with Liouville form $e^t(\lambda|_{\partial W})$ the *cylindrical end*. Any Liouville domain (W,λ) can be completed to a *Liouville manifold* $(\widehat{W},\hat{\lambda})$ by attaching the cylindrical end, i.e., $\widehat{W} = W\cup(\partial W\times\mathbb{R}_{\ge0})$, $\hat{\lambda}|_W = \lambda$ and $\hat{\lambda}|_{\partial W\times\mathbb{R}_{\ge0}} = e^t(\lambda|_{\partial W})$.

Remark 3.2. Strictly speaking, the completion $(\widehat{W}, \hat{\lambda})$ is a *Liouville manifold of finite type*. Every Liouville manifold of finite type is the completion of some Liouville domain. Throughout the paper, following [GS1, Section 4], by a *Liouville manifold* we will mean the completion of some Liouville domain. Note that the skeleton, which will be defined below, of a Liouville manifold of finite type is compact.

Definition 3.3 ([CE12, Section 11.1], [GS1, Definition 4.8])**.** Let (W, λ) be a Liouville manifold. Its *skeleton* is the attractor

$$\mathrm{Core}(W) = \bigcap_{t>0} Z^{-t}(W)$$

of the negative Liouville flow Z^{-t}. Equivalently, $\mathrm{Core}(W)$ is the union of all stable manifolds, i.e., the maximal compact subset invariant under the Liouville flow. We denote by $([W], [\lambda])$ a Liouville domain which completes to (W, λ) and contains $\mathrm{Core}(W)$. The *ideal boundary* $\partial_\infty W$ of W is the intersection $W \cap \partial[W]$. For a subset $\mathcal{L}$ of $\partial_\infty W$ the *relative skeleton* $\mathrm{Core}(W, \mathcal{L})$ of (W, λ) associated with $\mathcal{L}$ is the disjoint union

$$\mathrm{Core}(W) \sqcup \mathbb{R}\mathcal{L} \subset W$$

of $\mathrm{Core}(W)$ and the saturation of $\mathcal{L}$ by the Liouville flow.

Definition 3.4 ([CE12, Definition 11.10], [Eli, Section 1])**.** A *Weinsten domain* (W, λ, ϕ) is a Liouville domain (W, λ) whose Liouville vector field is gradient-like for a Morse–Bott function $\phi \colon W \to \mathbb{R}$ which is constant on ∂W. We call the positive half $(\partial W \times \mathbb{R}_{\geq 0}, e^t(\lambda|_{\partial W}), \phi)$ of the symplectization of ∂W with Liouville form $e^t(\lambda|_{\partial W})$ and canonically extended ϕ the *cylindrical end*. Any Weinstein domain (W, λ, ϕ) can be completed to a *Weinstein manifold* $(\widehat{W}, \hat{\lambda}, \hat{\phi})$ by attaching the cylindrical end.

Remark 3.5. The completion $(\widehat{W}, \hat{\lambda}, \hat{\phi})$ is a *Weinstein manifold of finite type*, i.e., ϕ has only finitely many critical points. Every Weinstein manifold of finite type is the completion of some Weinstein domain. Throughout the paper, following [Nad, Definition 5.5], by a *Weinstein manifold* we will mean the completion of some Weinstein domain. Note that in view of [GS1, Example 4.23] one may regard mirror symmetry established in [GS1] as a generalization of that in [GS2], which relies on [Nad, Theorem 5.13].

Remark 3.6. For a Weinstein manifold (W, λ, ϕ) the skeleton $\mathrm{Core}(W)$ is isotropic by [CE12, Lemma 11.13(a)]. Moreover, the stable manifold of the critical locus of ϕ contains the zero locus of Z.

Example 3.7 ([CE12, Definition 11.12(2)(3)])**.** Consider the cotangent bundle T^*Y of a closed manifold Y with the standard symplectic form $\omega_{st} = d\lambda_{st}$ where $\lambda_{st} = pdq$ is the standard Liouville form. The associated Liouville vector field $Z_{st} = p\partial_p$ is gradient-like for a Morse–Bott function $\phi_{st}(q, p) = \frac{1}{2}|p|^2$. The product $T^*Y \times T^*Y'$ of two such Weinstein manifolds with symplectic form $\omega_{st} \oplus \omega'_{st}$, Liouvlle form $\lambda_{st} \oplus \lambda'_{st}$ and Morse–Bott function $\phi_{st} \oplus \phi'_{st}$ is a Weinstein manifold.

3.2. **Weinstein pairs.**

Definition 3.8 ([Eli, Section 2])**.** Let (Y, ξ) be a contact manifold. We call a codimension 1 submanifold $H \subset Y$ with smooth boundary a *Weinstein hypersurface* if there exists a contact form λ for ξ such that $(H, \lambda|_H)$ is compatible with a Weinstein structure on H, i.e., $\omega_H = d\lambda|_H$ is a symplectic form, $Z_H = \omega_H^{\#}\lambda|_H$ points outwardly along ∂H and is gradient-like for some Morse–Bott function $\phi_H \colon H \to \mathbb{R}$. Its *contact surrounding* $U_\epsilon(H)$ is the neighborhood of H in Y defined as follows. Let $\tilde{H}$ be a slightly extended Weinstein hypersurface satisfying

$\lambda|_{\tilde{H}\setminus H} = t\lambda|_{\partial H}$ for $t \in [1, 1+\epsilon]$. There is a neighborhood $\tilde{U}$ of $\tilde{H}$ diffeomorphic to $\tilde{H}\times(-\epsilon, \epsilon)$ with $\lambda|_{\tilde{U}} = \mathrm{pr}_1^*(\lambda|_{\tilde{H}}) + du$ where u is the coordinate of the second factor. For a nonnegative function $h\colon \tilde{H} \to \mathbb{R}$ which is equal to 0 on H and to $t-1$ near $\partial\tilde{H}$, set $U_\epsilon(H) = \{h^2 + u^2 \leq \epsilon^2\} \subset \tilde{U}$.

Remark 3.9. Although the induced Weinstein structure on H depends on the choice of λ, the skeleton $\mathrm{Core}(H)$ is independent of the choice [CE12, Lemma 12.1].

Example 3.10 ([Eli, Example 2.1(i)])**.** Let $\mathcal{L}$ be a Legendrian submanifold of a contact manifold (Y, ξ). It admits a neighborhood $U(\mathcal{L})$ isomorphic to

$$(J^1(\mathcal{L}), du - pdq),\ q \in \mathcal{L},\ \|p\|^2 + u^2 \leq \epsilon.$$

Then $H(\mathcal{L}) = U(\mathcal{L}) \cap \{u = 0\}$ is a Weinstein hypersuface called a *Weinstein thickening* of $\mathcal{L}$. It is symplectomorphic to the cotangent ball bundle of $\mathcal{L}$. Up to Weinstein isotopy, $H(\mathcal{L})$ is independent of all the choices.

Definition 3.11 ([Eli, Section 2])**.** A *Weinstein pair* (W, H) consists of a Weinstein domain (W, λ, ϕ) together with a Weinstein hypersurface $(H, \lambda|_H)$ in ∂W. The *skeleton* $\mathrm{Core}(W, H)$ of (W, H) is the relative skeleton of the Liouville manifold (W, λ) associated with H.

Definition 3.12 ([Eli, Section 2])**.** Let (W, H) be a Weinstein pair, $\phi_H\colon H \to \mathbb{R}$ a Morse–Bott function for which $Z_H = (\omega|_H)^\#\lambda|_H$ is gradient-like and $U_\epsilon(H) \subset \partial W$ the contact surrounding of H. We call a pair (λ_0, ϕ_0) of a Liouville form λ_0 for ω and a smooth function $\phi_0\colon W \to \mathbb{R}$ *adjusted* to (W, H) if

- $Z_0 = \omega^\#\lambda_0$ is tangent to ∂W on $U_\epsilon(H)$ and transverse to ∂W elsewhere;
- $Z_0|_{U_\epsilon(H)} = Z_H + u\partial_u$;
- the attractor $\bigcap_{t>0} Z_0^{-t}(W)$ coincides with $\mathrm{Core}(W, H)$;
- the function ϕ_0 is Morse–Bott for which $Z_0 = (\omega)^\#\lambda_0$ is gradient-like satisfying $\phi_0|_{U_\epsilon(H)} = \phi_H + \frac{1}{2}u^2$ and whose critical values are not more than $\phi_0|_{\partial U_\epsilon(H)}$.

Given a Weinstein pair (W, H), one can always modify the Liouville form λ for ω and the Morse–Bott function ϕ_H to be adjusted.

Lemma 3.13 ([Eli, Proposition 2.9])**.** *Let (W, H) be a Weinstein pair. Then there exist a Liouville form λ_0 for ω and a smooth function $\phi_0\colon W \to \mathbb{R}$ such that (λ_0, ϕ_0) are adjusted to (W, H) and $\lambda_0|_{W\setminus U_\epsilon(H)} = \lambda|_{W\setminus U_\epsilon(H)}$.*

3.3. **Gluing of Weinstein pairs.**

Definition 3.14 ([Eli, Section 3.1])**.** Let (W, λ, ϕ) be a Weinstein domain. A *splitting* for (W, λ, ϕ) is a hypersurface $(P, \partial P) \subset (W, \partial W)$ satisfying the following conditions.

- ∂P and P respectively split ∂W and W into two parts $\partial W = Y_- \cup Y_+$ with $\partial Y_- = \partial Y_+ = Y_- \cap Y_+ = \partial P$ and $W = W_- \cup W_+$ with $\partial W_- = P \cup Y_-, \partial W_+ = P \cup Y_+, W_- \cap W_+ = P$.
- The Liouville vector field $Z = \omega^\#\lambda$ is tangent to P.
- There exists a hypersurface $(H, \partial H) \subset (P, \partial P)$ which is Weinstein for $\lambda|_H$, tangent to Z and intersects transversely all leaves of the characteristic foliation of P.

The above hypersurface $(H, \partial H)$ is called the *Weinstein soul* for the splitting hypersurface P. Due to Lemma 3.16 below, P is contactomorphic to the contact surrounding $U_\epsilon(H)$ of its Weinstein soul.

Definition 3.15 ([Eli, Section 2])**.** Let H be a closed hypersurface in a $(2n-1)$-dimensional manifold and ξ a germ of a contact structure along H which admits a transverse contact vector field Z. The *invariant extension* of the germ ξ is the canonical extension $\hat{\xi}$ on $H \times \mathbb{R}$, which is invariant with respect to translations along the second factor and whose germ along any slice $H \times \{t\}, t \in \mathbb{R}$ is isomorphic to ξ.

Lemma 3.16 ([Eli, Lemma 2.6]). *Let H be a closed $(2n-2)$-dimensional manifold and ξ a contact structure on $P = H \times [0,\infty)$ which admits a contact vector field Z inward transverse to $H \times \{0\}$ such that its trajectories intersecting $H \times \{0\}$ fill the whole manifold P. Then (P, ξ) is contactomorphic to $(H \times [0,\infty), \hat{\xi})$, where $\hat{\xi}$ is the invariant extension of the germ of ξ along $H \times \{0\}$. Moreover, for any compact set $C \subset P$ with $H \times \{0\} \subset \operatorname{Int} C$ there exists a contactomorphism $(Y, \xi) \to (H \times [0,\infty), \hat{\xi})$, which is equal to the identity on $H \times \{0\}$ and sends $Z|_C$ to the vector field ∂_t.*

Definition 3.17 ([Eli, Section 3.1]). Let $(W, H), (W', H')$ be Weinstein pairs with adjusted Liouville forms and Morse–Bott functions $(\lambda, \phi), (\lambda', \phi')$. Suppose that there is an isomorphism $(H, \lambda|_H, \phi|_H) \cong (H', \lambda'|_H, \phi'|_H)$ of Weinstein manifolds. Extend it to their contact surroundings morphism $U_\epsilon(H) \cong U_\epsilon(H')$. Then the *gluing* of $(W, H), (W', H')$ is given by $W \#_{H \cong H'} W' = W \cup_{U_\epsilon(H) \cong U_\epsilon(H')} W'$ with glued Liouville form and Morse–Bott function.

By definition we have

$$\operatorname{Core}(W \#_{H \cong H'} W') = \operatorname{Core}(W, H) \cup_{\operatorname{Core}(H) \cong \operatorname{Core}(H')} \operatorname{Core}(W', H').$$

3.4. **Weinstein handle attachments.** Let W be a Weinstein domain with a smooth Legendrian $\mathcal{L} \subset \partial_\infty W$. Fix a standard neighborhood of $\mathcal{L}$ in $\partial_\infty W$

$$\eta\colon \operatorname{Nbd}_{\partial_\infty W}(\mathcal{L}) \hookrightarrow J^1\mathcal{L} = T^*\mathcal{L} \times \mathbb{R}$$

which extends to a neighborhood in the Weinstein domain W

$$\xi\colon \operatorname{Nbd}_W(\mathcal{L}) \hookrightarrow J^1\mathcal{L} \times \mathbb{R}_{\leq 0} \cong T^*(\mathcal{L} \times \mathbb{R}_{\leq 0}),$$

where the Liouville flow on W gets identified with the translation action on $\mathbb{R}_{\leq 0}$. Note that $\eta^{-1}(T^*\mathcal{L} \times \{0\})$ gives a Weinstein thichkening of $\mathcal{L}$.

Due to Lemma 3.13, one can modify the Weinstein structure near $\mathcal{L}$ so that the Liouville flow gets identified with the cotangent scaling on $T^*(\mathcal{L} \times \mathbb{R}_{\leq 0})$. In other words, the Liouville form and the Morse–Bott function become adjusted to the Weinstein pair $(W, \eta^{-1}(T^*\mathcal{L} \times \{0\}))$. We denote by $\widetilde{W}$ its conic completion. Then η yields a neiborhood of $\mathcal{L}$ in $\partial\widetilde{W}$

$$\widetilde{\eta}\colon \operatorname{Nbd}_{\partial\widetilde{W}}(\mathcal{L}) \hookrightarrow J^1\mathcal{L}$$

which extends to a neighborhood in $\widetilde{W}$

$$\widetilde{\xi}\colon \operatorname{Nbd}_{\widetilde{W}}(\mathcal{L}) \hookrightarrow J^1\mathcal{L} \times \mathbb{R}_{\leq 0} \cong T^*(\mathcal{L} \times \mathbb{R}_{\leq 0}).$$

Remark 3.18. As a Liouville domain, the modification is canonical up to contractible choice, since any two Liouville structures on a compact symplectic manifold are canonically homotopic [Eli, page 2].

Remark 3.19. The result $\widetilde{W}$ is a Liouville sector in the sense of [GPS1, Definition 2.4]. Indeed, $\widetilde{W}$ is a Liouville manifold-with-boundary, whose boundary consists of $\operatorname{Nbd}_{\partial\widetilde{W}}(\mathcal{L})$ and the conic completion of $\partial \operatorname{Nbd}_{\partial_\infty W}(\mathcal{L})$. The characteristic foliations of the former and the latter are respectively given by the Reeb vector field on ∂W and the restricted Liouville vector field on $\widetilde{W}$. Now, it is clear that $\partial\partial_\infty\widetilde{W}$ is convex and there is a diffeomorphism $\partial\widetilde{W} \cong \mathbb{R} \times F$ sending the characteristic foliation of $\partial\widetilde{W}$ to the foliation of $\mathbb{R} \times F$ by leaves $\mathbb{R} \times \{p\}$.

Given Weinstein domains W, W' with smooth Legendrian embeddings $\partial_\infty W \hookleftarrow \mathcal{L} \hookrightarrow \partial_\infty W'$, we write $W \#_{\mathcal{L}} W'$ for the gluing

$$W \#_{\eta^{-1}(T^*\mathcal{L} \times \{0\}) \cong \eta'^{-1}(T^*\mathcal{L} \times \{0\})} W' = \widetilde{W} \cup_{J^1\mathcal{L}} \widetilde{W'}$$

which yields another Weinstein domain with skeleton

$$\operatorname{Core}(W \#_{\mathcal{L}} W') = \operatorname{Core}(W, \mathcal{L}) \cup_{\mathcal{L}} \operatorname{Core}(W', \mathcal{L}).$$

Definition 3.20 ([GS1, Definition 4.9])**.** Let W be a Weinstein domain with a smooth Legendrian $\mathcal{L} \subset \partial_\infty W$. A conic subset $\mathbb{L} \subset W$ is *biconic* along $\mathcal{L}$ if the image of $\mathbb{L} \cap \mathrm{Nbd}_W(\mathcal{L})$ under some $\xi\colon \mathrm{Nbd}_W(\mathcal{L}) \hookrightarrow T^*(\mathcal{L} \times \mathbb{R}_{\leq 0})$, where the Liouville flow on W gets identified with the translation action on $\mathbb{R}_{\leq 0}$, is invariant also under the cotangent scaling of $T^*(\mathcal{L} \times \mathbb{R}_{\leq 0})$.

By construction any biconic subset $\mathbb{L} \subset W$ remains conic in $\widetilde{W}$. We write $\widetilde{\mathbb{L}} \subset \widetilde{W}$ for its saturation under the Liouville flow.

Lemma 3.21 ([GS1, Lemma 4.10])**.** *Let W be a Weinstein domain with a smooth Legendrian $\mathcal{L} \subset \partial_\infty W$. Then a Lagrangian $\mathbb{L} \subset W$ is biconic along $\mathcal{L}$ if and only if it is conic and*

$$\xi(\mathbb{L} \cap \mathrm{Nbd}_W(\mathcal{L})) \subset T^*\mathcal{L} \times \{0\} \times \mathbb{R}_{\leq 0} \subset J^1\mathcal{L} \times \mathbb{R}_{\leq 0} \cong T^*(\mathcal{L} \times \mathbb{R}_{\leq 0})$$

for some $\xi\colon \mathrm{Nbd}_W(\mathcal{L}) \hookrightarrow T^*(\mathcal{L} \times \mathbb{R}_{\leq 0})$, where the Liouville flow on W gets identified with the translation action on $\mathbb{R}_{\leq 0}$.

Proof. The second factor $\mathbb{R}$ in the product $T^*\mathcal{L} \times \mathbb{R} = J^1\mathcal{L}$ is responsible for the additional cotangent scaling of $T^*(\mathcal{L} \times \mathbb{R}_{\leq 0})$ with respect to $T^*\mathcal{L}$. □

Given biconic Lagrangians $\mathbb{L} \subset W, \mathbb{L}' \subset W'$ with matching ends in the sense that

$$\widetilde{\eta}(\mathbb{L} \cap \mathrm{Nbd}_{\partial_\infty W}(\mathcal{L})) = \widetilde{\eta}'(\mathbb{L}' \cap \mathrm{Nbd}_{\partial_\infty W'}(\mathcal{L}))$$

in $W\#_{\mathcal{L}}W'$, we write $\mathbb{L}\#_{\mathcal{L}}\mathbb{L}'$ for the gluing $\widetilde{\mathbb{L}} \cup_{\widetilde{\eta}(\mathbb{L}\cap\mathrm{Nbd}_{\partial_\infty W}(\mathcal{L}))} \widetilde{\mathbb{L}'}$ in $W\#_{\mathcal{L}}W'$. Since any biconic subsets in W, W' remain conic in $W\#_{\mathcal{L}}W'$, the gluing $\mathbb{L}\#_{\mathcal{L}}\mathbb{L}'$ is a conic Lagrangian.

For a closed manifold $\widehat{M}$ and a manifold-with-boundary S, consider the Weinstein domain $W' = [T^*\widehat{M} \times T^*S]$ with a smooth Legendrian $\mathcal{L} = \widehat{M} \times \partial S$ taken to be a subset of the zero section. The Liouville flow on W' near $\mathcal{L}$ is the cotangent scaling. We write $\widetilde{W'}$ for the completion $T^*\widehat{M} \times T^*S$, which is an exact symplectic manifold without boundary. One can check that the above gluing procedure carries over, although $\mathcal{L}$ does not belong to $\partial_\infty W'$.

Definition 3.22 ([GS1, Definition 4.11])**.** Let W be a Weinstein domain with a smooth Legendrian embedding $\widehat{M} \times \partial S \hookrightarrow \partial_\infty W$.

(1) A *handle attachement* is the gluing

$$W\#_{\widehat{M}\times\partial S}[T^*\widehat{M} \times T^*S]$$

respecting the product structure.

(2) Let further $\mathbb{L} \subset W$ be a biconic subset along $\widehat{M} \times \partial S$ which locally factors as

$$\eta(\mathbb{L} \cap \mathrm{Nbd}_{\partial_\infty W}(\widehat{M} \times \partial S)) = \mathbb{L}_S \times \partial S \subset T^*\widehat{M} \times T^*\partial S$$

for some fixed conic Lagrangian $\mathbb{L}_S$. The *extension* of $\mathbb{L}$ through the handle is the gluing

$$\mathbb{L}\#_{\widehat{M}\times\partial S}(\mathbb{L}_S \times S)$$

respecting the product structure.

(3) Let further $\Lambda \subset \partial_\infty W$ be a subset satisfying

$$\eta(\Lambda \cap \mathrm{Nbd}_{\partial_\infty W}(\widehat{M} \times \partial S)) \cong \mathbb{L}_S \times \partial S \subset T^*\widehat{M} \times T^*\partial S \times \{0\} \subset J^1(\widehat{M} \times \partial S)$$

for $\mathbb{L}_S$. The *extension* of Λ through the handle is the ideal boundary

$$\partial_\infty(\mathrm{Core}(W, \Lambda)\#_{\widehat{M}\times S}(\mathbb{L}_S \times S))$$

of the extension of the biconic Lagrangian $\mathrm{Core}(W, \Lambda)$ thorough the handle.

Lemma 3.23 ([GS1, Lemma 4.12])**.** *Let $E \to Y$ be a vector bundle. Then near the Legendrian $\mathcal{L} = \partial_\infty T^*_Y E \subset T^*E$ there are local coordinates such that if $\Lambda \subset \bigcup_\alpha T^*_{S_\alpha}E$ for any collection $\{S_\alpha\}_\alpha$ of conic subsets $S_\alpha \subset E$ with respect to the scaling of E, then Λ is biconic along $\mathcal{L}$.*

Proof. As a bundle over Y, we have $T^*E = E \oplus E^\vee \oplus T^*Y$. Let $P \subset T^*E$ be the polar hypersurface defined as the kernel of the pairing between E and $E^\vee$. Then P contains $T^*_{S_\alpha}E$ for any conic subset $S_\alpha \subset E$. Indeed, a point $(y, u, v, w) \in T^*_y E$ belongs to the fiber $(T^*_{S_\alpha}E)_y \subset (T^*E|_{S_\alpha})_y$ only if $(y, u) \in S_\alpha$ and $v(u) = 0$. Locally, the ideal boundary $\partial_\infty P \subset \partial_\infty T^*E$ can be identified with $T^*\mathcal{L}$ compatibly with their standard Liouville structures. Indeed, any point of $(\partial_\infty P)_y$ is expressed as $(y, u, [v], w)$ with $(y, u, v, w) \in P_y$, while any point of $T^*_y\mathcal{L}$ is expressed as $(y, [v], w, u)$ with $(y, [v]) \in (\partial_\infty T^*_Y E)_y$, $w \in T^*_Y$, and $u \in E_y$. Here, we identify E_y with the hyperplane orthogonal to v passing through the origin. Hence locally $\partial_\infty P$ is a Weinstein thickening of $\mathcal{L}$ from Example 3.10 Transporting to $\mathrm{Nbd}_P(\mathcal{L})$ the Reeb vector flow which intersects $T^*\mathcal{L}$ transversely, we obtain local coordinates $\eta\colon \mathrm{Nbd}_P(\mathcal{L}) \hookrightarrow J^1\mathcal{L}$ whose biconicity follows from Lemma 3.21. □

Corollary 3.24 ([GS1, Corollary 4.13])**.** *Let $Y \subset Y'$ be a submanifold. Then near the Legendrian $\partial_\infty T^*_Y Y'$ there are local coordinates such that if $\Lambda \subset \bigcup_\alpha T^*_{S_\alpha} Y'$ for any collection $\{S_\alpha\}_\alpha$ of subsets $S_\alpha \subset Y$, then Λ is biconic.*

Proof. Locally identify a tubular neighborhood $N_Y Y'$ of Y in Y' with Y'. Since any subset $S_\alpha \subset Y$ trivially becomes conic with respect to the scaling of $N_Y Y'$, one can apply Lemma 3.23 for $E = N_Y Y'$ and $\mathcal{L} = \partial_\infty T^*_Y N_Y Y'$ to obtain the desired local coordinates. □

Definition 3.25 ([FLTZ12, Section 3.1])**.** Let $\Sigma \subset M_\mathbb{R}$ be a rational polyhedral fan and $\widehat{M}$ the real n-torus $\mathrm{Hom}(M, \mathbb{R}/\mathbb{Z}) = M^\vee_\mathbb{R}/M^\vee$. The *FLTZ Lagrangian* is the union

$$\mathbb{L}(\Sigma) = \bigcup_{\sigma\in\Sigma} \mathbb{L}_\sigma = \bigcup_{\sigma\in\Sigma} \sigma^\perp \times \sigma$$

of conic Lagrangians $\sigma^\perp \times \sigma \subset \widehat{M} \times M_\mathbb{R} \cong T^*\widehat{M}$, where $\sigma^\perp$ is the real $(n - \dim\sigma)$-subtorus

$$\{x \in \widehat{M} \mid \langle x, v\rangle = 0 \text{ for all } v \in \sigma\}.$$

Remark 3.26 ([FLTZ14, Definition 6.3])**.** When Σ is a stacky fan, the FLTZ Lagrangian $\mathbb{L}(\Sigma)$ becomes the union $\bigcup_{\sigma\in\Sigma} G_\sigma \times \sigma$, where $G_\sigma = \mathrm{Hom}(M_\sigma, \mathbb{R}/\mathbb{Z})$ are possibly disconnected subgroup of $\widehat{M}$. Here, we denote by M_σ the quotients of the lattice M by the stacky primitives for σ.

Lemma 3.27 ([GS1, Lemma 4.16])**.** *Let $\Sigma \subset M_\mathbb{R}$ be a rational polyhedral fan. Then for each cone $\sigma \in \Sigma$ near the Legendrian $\partial_\infty\mathbb{L}_\sigma \subset \partial_\infty T^*\widehat{M}$ there are local coordinates*

$$\eta_\sigma\colon \mathrm{Nbd}(\partial_\infty\mathbb{L}_\sigma) \hookrightarrow J^1\partial_\infty\mathbb{L}_\sigma = T^*\partial_\infty\mathbb{L}_\sigma \times \mathbb{R}$$

with the following properties.

(1) *The Lagranzian $\mathbb{L}_\sigma$ is biconic along $\partial_\infty\mathbb{L}_\sigma$.*

(2) *For any nonzero cones $\sigma, \tau \in \Sigma$ with $\sigma \subset \bar{\tau}$, we have*

$$\eta_\sigma(\partial_\infty\mathbb{L}_\tau \cap \mathrm{Nbd}(\partial_\infty\mathbb{L}_\sigma)) = \mathbb{L}_{\tau/\sigma} \times \partial_\infty\sigma \times \{0\} \subset T^*\sigma^\perp \times T^*\partial_\infty\sigma \times \{0\} = T^*\partial_\infty\mathbb{L}_\sigma \times \{0\}. \tag{3.1}$$

(3) *The FLTZ Lagranzian $\mathbb{L}(\Sigma)$ is biconic along each $\partial_\infty\mathbb{L}_\sigma$.*

(4) *For each $\sigma \in \Sigma$ the local coordinates η_σ define a Weinstein hypersuface*

$$R_\sigma = \eta_\sigma^{-1}(T^*\partial_\infty\mathbb{L}_\sigma \times \{0\}) \subset \partial_\infty T^*\widehat{M}$$

containing $\partial_\infty\mathbb{L}_\sigma$ as its skeleton and we have $R_\tau \cap \mathrm{Nbd}(\partial_\infty\mathbb{L}_\sigma) \subset R_\sigma$ for any $\tau \in \Sigma$ with $\sigma \subset \bar{\tau}$.

Proof. Applying Corollary 3.24 to each submanifold $\sigma^\perp \subset \widehat{M}$, we obtain local coordinates $\eta_\sigma\colon \mathrm{Nbd}(\partial_\infty\mathbb{L}_\sigma) \hookrightarrow J^1\partial_\infty\mathbb{L}_\sigma$ near $\partial_\infty\mathbb{L}_\sigma$ such that $\mathbb{L}_\sigma$ is biconic along $\partial_\infty\mathbb{L}_\sigma$. Provided (1), the property (3) follows from (2), which is a consequence of the identification

$$\partial_\infty\mathbb{L}_\tau \cong \tau^\perp \times \partial_\infty(\tau/\langle\sigma\rangle \times \sigma) \cong (\tau/\langle\sigma\rangle)^\perp \times \tau/\langle\sigma\rangle \times \partial_\infty\sigma \subset \mathbb{L}(\Sigma/\sigma) \times \partial_\infty\sigma$$

near σ of subsets of the stratified spaces Σ and Σ/σ for any $\tau \in \Sigma$ with $\{0\} \neq \sigma \subset \bar{\tau}$. Note that subtori $(\tau/\langle\sigma\rangle)^\perp, \tau^\perp$ of respective tori $\widehat{M/\langle\sigma\rangle} \subset \widehat{M}$ are the same. When $\sigma = \{0\}$ and taking the ideal boundary $\partial_\infty\sigma$ makes no sense, we have instead

$$\eta_\sigma(\partial_\infty\mathbb{L}_\tau \cap \mathrm{Nbd}(\partial_\infty\mathbb{L}_\sigma)) = \partial_\infty\mathbb{L}_{\tau/\sigma} \times \{0\} \subset T^*\sigma^\perp \times \{0\}.$$

It remains to show (4). Let P_σ be the polar hypersurface defined as the kernel of the pairing

$$f_\sigma \colon T^*U_\sigma \cong T^*N_{\sigma^\perp}\widehat{M} = N_{\sigma^\perp}\widehat{M} \oplus (N_{\sigma^\perp}\widehat{M})^\vee \oplus T^*\sigma^\perp \to \mathbb{R},$$

which gives rise to the above local coordinates $\mathrm{Nbd}_{P_\sigma}(\partial_\infty T^*_{\sigma^\perp}\widehat{M}) \hookrightarrow J^1\partial_\infty T^*_{\sigma^\perp}\widehat{M}$ as in the proof of Lemma 3.23. However, there is no inclusion between $\partial_\infty P_\sigma, \partial_\infty P_\tau$ for $\sigma \subset \bar{\tau}$. To remedy this issue, one needs to modify the above local coordinates as follows. For a 1-dimensional cone σ, replace f_σ with $g_\sigma = f_\sigma|_{V_\sigma}$ where $V_\sigma \cong N_{\mathbb{L}_\sigma}T^*\widehat{M}$ is a conic tubular neighborhood projecting to U_σ and $W_\sigma \cong N_\sigma M_\mathbb{R}$ via the identification $T^*\widehat{M} \cong \widehat{M} \times M_\mathbb{R}$, chosen so that $V_\sigma \cap V_{\sigma'}$ when $\bar{\sigma} \cap \bar{\sigma}' = \{0\}$. Locally, we have

$$R_\sigma = g_\sigma^{-1}(0) \cap T^*\partial_\infty T^*_{\sigma^\perp}\widehat{M} \times \{0\} \cong T^*\partial_\infty T^*_{\sigma^\perp}\widehat{M}$$

and the modified local coordinates

$$\mathrm{Nbd}_{g_\sigma^{-1}(0)}(\partial_\infty T^*_{\sigma^\perp}\widehat{M}) \hookrightarrow J^1\partial_\infty T^*_{\sigma^\perp}\widehat{M}$$

satisfies the properties (1), (2), (3).

For a higher dimensional cone τ, one extends g_σ as follows. When $\dim\tau = 2$ and τ is spanned by σ_1, σ_2, replace f_τ with

$$g_\tau = a_1 f_{\sigma_1}|_{V_\tau} + a_2 f_{\sigma_2}|_{V_\tau} \colon V_\tau \to \mathbb{R}$$

where $(a_1, a_2) \colon V_\tau \to \tau = \langle\sigma_1, \sigma_2\rangle$ is the canonical projection. The functions a_i take values in $\mathbb{R}_{\geq 0}$ and satisfy $a_1|_{V_{\sigma_2}\cap V_\tau} = a_2|_{V_{\sigma_1}\cap V_\tau} = 0$. Since we have $(N_{\tau^\perp}\widehat{M})^\vee \supset N_\tau M_\mathbb{R} \subset N_{\sigma_i}M_\mathbb{R} \subset (N_{\sigma_i^\perp}\widehat{M})^\vee$ as a bundle over $\tau^\perp$, the restrictions $(f_{\sigma_i}|_{V_\tau})|_{V_{\sigma_i}\cap V_\tau}$ coincide with $(g_{\sigma_i})|_{V_{\sigma_i}\cap V_\tau}$. Hence g_τ is an extension of $(g_{\sigma_1}, g_{\sigma_2}) \colon V_{\sigma_1} \sqcup V_{\sigma_2} \to \mathbb{R}$. Locally, we have

$$R_\tau = g_\tau^{-1}(0) \cap T^*\partial_\infty T^*_{\tau^\perp}\widehat{M} \times \{0\} \cong T^*\partial_\infty T^*_{\tau^\perp}\widehat{M}$$

and the modified local coordinates

$$\mathrm{Nbd}_{g_\tau^{-1}(0)}(\partial_\infty T^*_{\tau^\perp}\widehat{M}) \hookrightarrow J^1\partial_\infty T^*_{\tau^\perp}\widehat{M}$$

satisfies the properties (1), (2), (3) and

$$R_\tau \cap \mathrm{Nbd}_{g_{\sigma_i}^{-1}(0)}(\partial_\infty T^*_{\sigma_i^\perp}\widehat{M}) \subset R_{\sigma_i}.$$

Inductively, one obtains the desired extensions for the other cases. □

Remark 3.28. Some readers of [GS1] might wonder why the authors stated Lemma 3.27(4). As explained in Section 3.3, we glue two Weinstein domains along their isomorphic Weinstein hypersurfaces. Here, each Weinstein hypersurface R_σ is defined in the modified Polar hypersurface $g_\sigma^{-1}(0)$ and coincides with a Weinstein thickening of $\partial_\infty\mathbb{L}_\sigma$. Lemma 3.27(4) guarantees compatibility of such local Weinstein pair structures with the union $\mathbb{L}(\Sigma) = \bigcup_{\sigma\in\Sigma}\mathbb{L}_\sigma$.

4. The proof of Theorem 1.1 explained

In this section, we review the proof of Theorem 1.1, filling in the details. By Lemma 2.10 the fanifold Φ of dimension n admits a filtration (2.1) We proceed by induction on k.

4.1. **Base case.** When $k = 0$, let

$$\widetilde{\mathbf{W}}(\Phi_0) = \mathbf{W}(\Phi_0) = \bigsqcup_P T^*\widehat{M}_P, \ \widetilde{\mathbb{L}}(\Phi_0) = \mathbb{L}(\Phi_0) = \bigsqcup_P \mathbb{L}(\Sigma_P)$$

where each $T^*\widehat{M}_P$ is equipped with the canonical Liouville structure. We fix an identification $T^*\widehat{M}_P \cong \widehat{M}_P \times M_{P,\mathbb{R}}$ to regard each $\mathbb{L}(\Sigma_P)$ as a conic Lagrangian submanifold of $T^*\widehat{M}_P$.

Lemma 4.1. *The manifold* $\widetilde{\mathbf{W}}(\Phi_0)$ *is subanalytic Weinstein.*

Proof. Equipped with the canonical Liouville structure, $T^*\widehat{M}_P$ become Weinstein as explained in Example 3.7. In general, real analytic subsets of real analytic manifolds are subanalytic and the product of subanalytic subsets of real analytic manifolds is subanalytic. □

Lemma 4.2. *The subset* $\widetilde{\mathbb{L}}(\Phi_0) \subset \widetilde{\mathbf{W}}(\Phi_0)$ *is a subanalytic conic Lagrangian and contains the skeleton* $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi_0))$ *of the Weinstein manifold* $\widetilde{\mathbf{W}}(\Phi_0)$.

Proof. The skeleton $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi_0))$ is the disjoint union $\bigsqcup_P \widehat{M}_P$ of the zero sections $\widehat{M}_P \subset T^*\widehat{M}_P$ for all 0-strata P of Φ, each of which is contained in the conic Lagrangian $\mathbb{L}(\Sigma_P)$ via the fixed identification $T^*\widehat{M}_P \cong \widehat{M}_P \times M_{P,\mathbb{R}}$. In general, the collection of subanalytic subsets of a real analytic manifold forms a Boolean algebra. Hence $\mathbb{L}(\Sigma_P)$ is subanalytic, as it is the union of the products of a real torus and the union of an Euclidean space and its algebraic subsets, which are subanalytic subsets of $\widehat{M}_P \times M_{P,\mathbb{R}}$. □

Let $\pi_0 \colon \widetilde{\mathbb{L}}(\Phi_0) \to \Phi_0$ be the map induced by the projection to cotangent fibers.

Lemma 4.3. *The triple* $(\widetilde{\mathbf{W}}(\Phi_0), \widetilde{\mathbb{L}}(\Phi_0), \pi_0)$ *satisfies the conditions (1), ..., (4).*

Proof. (1) It suffices to show the claim for a neighborhood $\mathrm{Nbd}(P)$ of a single 0-stratum $P \subset \Phi$. Let $S_P \subset \Phi_0$ be the stratum of codimension d corresponding to a cone $\sigma_P \in \Sigma_P$ via ϕ_P. Then by definition of π_0 we have

$$\pi_0^{-1}(S_P) = \mathbb{L}_{\sigma_P} = \sigma_P^{\perp} \times \sigma_P \cong T^d \times S_P.$$

From the isomorphism

$$\mathbb{L}_{\tau_P} = \tau_P^{\perp} \times \tau_P \cong (\tau_P/\langle\sigma_P\rangle)^{\perp} \times \tau_P/\langle\sigma_P\rangle \times \sigma_P = \mathbb{L}_{\tau_P/\sigma_P} \times \sigma_P \tag{4.1}$$

for any stratum $\tau_P \in \Sigma_P$ with $\sigma_P \subset \bar{\tau}_P$ it follows

$$\pi_0^{-1}(\mathrm{Nbd}(S_P)) = \bigcup_{\tau_P \in \Sigma_P, \ \sigma_P \subset \bar{\tau}_P} \mathbb{L}_{\tau_P} \cong \mathbb{L}(\Sigma_P/\sigma_P) \times S_P.$$

Consider the cotangent bundle $T^*T^d \times T^*S_P$ with the canonical symplectic form. Since we have $T^*S_P \cong S_P \times \mathbb{R}^{\dim \widehat{M}_P - d} \cong S_P \times S_P$, there is a symplectomorphism

$$T^*T^d \times T^*S_P \cong T^*T^d \times T^*S_P, \ ((\theta,\eta),(x,y)) \mapsto ((\theta,\eta),(-y,x)). \tag{4.2}$$

Then an open embedding

$$T^d \times T^*_x S_P \hookrightarrow T^d \times T^{\dim \widehat{M}_P - d}, \ (\theta, y) \mapsto (\theta, -y)$$

canonically extends to a symplectomorphism

$$T^*T^d \times T^*S_P \hookrightarrow T^*\widehat{M}_P, \ ((\theta,\eta),(x,y)) \mapsto (\theta,-y,\eta,x). \tag{4.3}$$

Since it restricts to isomorphisms

$$(\tau_P/\langle\sigma_P\rangle)^{\perp} \times \tau_P/\langle\sigma_P\rangle \times \sigma_P \cong \tau_P^{\perp} \times \tau_P, \ ((\theta,\eta),(x,0)) \mapsto (\theta,0,\eta,x), \tag{4.4}$$

the symplectomorphism (4.3) embeds $\mathbb{L}(\Sigma_{S_P}) \times S_P$ into $\mathbb{L}(\Sigma_P)$.

(2) The fanifold Φ_0 is closed only if $\dim M_{P,\mathbb{R}} = 0$ for all 0-strata P. Then the claim is trivial.

(3) Since any subfanifold $\Phi_0' \subset \Phi_0$ is the disjoint union of fans Σ_P for some 0-strata of Φ, by definition we have $\widetilde{\mathbf{W}}(\Phi_0') = \mathbf{W}(\Phi_0') \subset \widetilde{\mathbf{W}}(\Phi_0)$ and $\widetilde{\mathbb{L}}(\Phi_0') = \mathbb{L}(\Phi_0') = \widetilde{\mathbf{W}}(\Phi_0') \cap \widetilde{\mathbb{L}}(\Phi_0)$.

(4) Recall that a Lagrangian polarization of a symplectic manifold W is a global section of the Lagrangian Grassmannian bundle $\mathrm{LGr}(W)$ over W. Up to homotopy, a Lagrangian polarization of W is equivalent to a real vector bundle B over W with an isomorphim $B \otimes_{\mathbb{R}} \mathbb{C}$. When W is a cotangent bundle $T^*\widehat{M}$, the tautological foliation by cotangent fibers yields an isomorphism $TT^*\widehat{M} = T^*\widehat{M} \otimes_{\mathbb{R}} \mathbb{C}$. Consider the Lagrangian foliation of $T^*T^d \times T^*S_P$ with leaves

$$(T^*_\theta T^d \times \{y\}) \times (\{\theta\} \times S_P),\ \theta \in T^d,\ y \in \mathbb{R}^{\dim \widehat{M}_P - d} \cong S_P.$$

Via the symplectomorphism (4.2) the leaf space get identified with the zero section, which in turn is isomorphic to $\pi_0^{-1}(S_P)$. Along the base direction in T^*S_P, it is compatible with the inclusion $S_P \hookrightarrow \overline{S'_P}$ to any stratum S'_P of codimension d' of the induced stratification on $\mathrm{Nbd}(S_P)$. Along the fiber direction in T^*T^d, it is compatible with the inclusions $T^*_\theta T^{d'} \hookrightarrow T^*_\theta T^d$ induced by the quotient map $T_{S_P} M_{P,\mathbb{R}} \to T_{S'_P} M_{P,\mathbb{R}}|_{S_P}$ from the definition of fanifolds for $\theta \in T^{d'} \cap T^d \subset \widehat{M}_P$. Hence one obtains the desired polarization. □

4.2. **Special case.** Before moving to general cases, we explicitly write down the relevant results and their proofs for Example 2.12, where the filtration of Φ from Lemma 2.10 is given by (2.2). Let $\mathcal{L}_1$ be the disjoint union of

$$\mathcal{L}_{I_{ij}} = \pi_0^{-1}(I_{ij}) \cap \partial_\infty \mathbb{L}(\Phi_0),\ 1 \le i < j \le 4.$$

for all interior 1-strata.

Lemma 4.4. *There are smooth Legendrian embeddings*

$$\partial_\infty \mathbf{W}(\Phi_0) \hookleftarrow \mathcal{L}_1 \hookrightarrow \partial \bigsqcup_{1 \le i < j \le 4} (T^*\widehat{M}_{I_{ij}} \times T^* I_{ij,\circ}).$$

Proof. It suffices to show the claim for $\mathcal{L}_{I_{12}}$. Since $\partial_{in} I_{12}$ consists of two 0-strata P_1, P_2 of Φ, there is a smooth Legendrian embedding

$$\mathcal{L}_{I_{12}} \cong \partial_\infty \mathbb{L}_{\sigma^{P_1}_{I_{12}}} \sqcup \partial_\infty \mathbb{L}_{\sigma^{P_2}_{I_{12}}} \hookrightarrow \partial_\infty \mathbf{W}(\Phi_0) \tag{4.5}$$

for the cones $\sigma^{P_i}_{I_{12}} \in \Sigma_{P_i}$ corresponding to I_{12}. The quotient maps $M_{P_i,\mathbb{R}} \to M_{P_i,\mathbb{R}}/\langle \sigma^{P_i}_{I_{12}} \rangle = M_{I_{12},\mathbb{R}}$ from the definition of fanifolds identify the images of $\sigma^{P_i}_{I_{12}}$ with the origin of $\Sigma_{I_{12}}$. Hence $\partial_\infty \mathbb{L}_{\sigma^{P_i}_{I_{12}}}$ are isomorphic to $\widehat{M}_{I_{12}}$. Regarding $\widehat{M}_{I_{12}} \times \partial I_{12,\circ}$ as a subset of $T^*\widehat{M}_{I_{12}} \times \partial T^* I_{12,\circ}$, one obtains another smooth Legendrian embedding

$$\mathcal{L}_{I_{12}} \cong \widehat{M}_{I_{12}} \times \partial I_{12,\circ} \hookrightarrow \partial(T^*\widehat{M}_{I_{12}} \times T^* I_{12,\circ}).$$

□

We define $\widetilde{\mathbf{W}}(\Phi_1)$ as the handle attachment

$$\mathbf{W}(\Phi_0) \#_{\mathcal{L}_1} \bigsqcup_{1 \le i < j \le 4} [T^*\widehat{M}_{I_{ij}} \times T^* I_{ij,\circ}].$$

Lemma 4.5. *The manifold $\widetilde{\mathbf{W}}(\Phi_1)$ is subanalytic Weinstein.*

Proof. As it is the result of handle attachments, $\widetilde{\mathbf{W}}(\Phi_1)$ is Weinstein. Each handle $T^*\widehat{M}_{I_{ij}} \times T^* I_{ij,\circ}$ is subanalytic. Since the union of subanalytic subsets is subanalytic, $\widetilde{\mathbf{W}}(\Phi_1)$ is subanalytic by Lemma 4.1. □

Lemma 4.6. *There is a standard neighborhood*

$$\eta_1\colon \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_0)}(\mathcal{L}_1) \hookrightarrow J^1\mathcal{L}_1 \tag{4.6}$$

near $\mathcal{L}_1$ for which $\mathbb{L}(\Phi_0)$ is biconic along $\mathcal{L}_1$ and $\partial_\infty\mathbb{L}(\Phi_0)$ locally factors as

$$\eta_1(\mathbb{L}(\Phi_0) \cap \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_0)}(\mathcal{L}_1)) = \bigsqcup_{1\le i<j\le 4} (\mathbb{L}(\Sigma_{I_{ij}}) \times \partial I_{ij,\circ}).$$

Proof. It suffices to show the claim for $\mathcal{L}_{I_{12}}$. Since we have (4.5), the disjoint union $\mathbb{L}(\Sigma_{P_1}) \sqcup \mathbb{L}(\Sigma_{P_2})$ is biconic along $\mathcal{L}_{I_{12}}$ for the standard coordinates

$$\eta_{\sigma^{P_1}_{I_{12}}} \sqcup \eta_{\sigma^{P_2}_{I_{12}}}\colon \operatorname{Nbd}_{\partial_\infty T^*\widehat{M}_{P_1} \sqcup \partial_\infty T^*\widehat{M}_{P_2}}(\mathcal{L}_{I_{12}}) \hookrightarrow J^1\mathcal{L}_{I_{12}}$$

from Lemma 3.27. Then the local factorization of $\partial_\infty\mathbb{L}(\Sigma_{P_1}) \sqcup \partial_\infty\mathbb{L}(\Sigma_{P_2})$ follows from (3.1). □

We define $\widetilde{\mathbb{L}}(\Phi_1)$ as the extension through the disjoint union of the handles $T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}$

$$\mathbb{L}(\Phi_0)\#_{\mathcal{L}_1} \bigsqcup_{1\le i<j\le 4} [\mathbb{L}(\Sigma_{I_{ij}}) \times I_{ij,\circ}].$$

Lemma 4.7. *The subset $\widetilde{\mathbb{L}}(\Phi_1) \subset \widetilde{\mathbf{W}}(\Phi_1)$ is a subanalytic conic Lagrangian and contains the skeleton* $\operatorname{Core}(\widetilde{\mathbf{W}}(\Phi_1))$ *of the Weinstein manifold $\widetilde{\mathbf{W}}(\Phi_1)$.*

Proof. Since by Lemma 4.6 and Lemma 3.23 the Lagrangians

$$\mathbb{L}(\Phi_0) \subset \mathbf{W}(\Phi_0), \quad \bigsqcup_{1\le i<j\le 4} [\mathbb{L}(\Sigma_{I_{ij}}) \times I_{ij,\circ}] \subset \bigsqcup_{1\le i<j\le 4} [T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}]$$

are biconic along $\mathcal{L}_1$, the gluing $\widetilde{\mathbb{L}}(\Phi_1)$ remains conic. The part of $\operatorname{Core}(\widetilde{\mathbf{W}}(\Phi_1))$ newly formed by the handle attachment is the cone $\mathbb{R}(\mathbb{L}(\Phi_0) \cap \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_0)}(\mathcal{L}_1))$. It is the saturation of the zero set of the Liouville vector field on the gluing

$$\eta_1(\operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_0)}(\mathcal{L}_1))\#_{\mathcal{L}_1} \bigsqcup_{1\le i<j\le 4} \operatorname{Nbd}_{\partial_\infty[T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}]}(\mathcal{L}_{I_{ij}}) \subset \mathbf{W}(\Phi_0)\#_{\mathcal{L}_1} \bigsqcup_{1\le i<j\le 4} [T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}],$$

which implies $\operatorname{Cone}(\widetilde{\mathbf{W}}(\Phi_1)) \subset \widetilde{\mathbb{L}}(\Phi_1)$. The extension $\widetilde{\mathbb{L}}(\Phi_1)$ is subanalytiic, as it is the union of the products of a real torus and the union of an Euclidean space and its algebraic subsets. □

Let $\pi_1\colon \widetilde{\mathbb{L}}(\Phi_1) \to \Phi_1$ be the map induced by π_0 and the projections from $T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}$ to the cotangent fiber direction in $T^*\widehat{M}_{I_{ij}}$ and the base direction in $T^*I_{ij,\circ}$.

Lemma 4.8. *The triple $(\widetilde{\mathbf{W}}(\Phi_1), \widetilde{\mathbb{L}}(\Phi_1), \pi_1)$ satisfies the conditions (1), …, (4).*

Proof. (1) It suffices to show the claim when attaching the handle $T^*\widehat{M}_{I_{12}} \times T^*I_{12,\circ}$ to $\widetilde{\mathbf{W}}(\Phi_0)$. Then we may assume that

$$\Phi_0 = \Sigma_{P_1} \sqcup \Sigma_{P_2},\ \Phi_1 = \Phi_0\#_{\Sigma_{I_{12}} \times \partial_{in} I_{12}}(\Sigma_{I_{12}} \times I_{12}). \tag{4.7}$$

Let $S \subset \Phi_1$ be the stratum

$$(S_{P_1} \sqcup S_{P_2})\#_{\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle \times \partial I_{12,\circ}}(\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle \times I_{12,\circ})$$

where S_{P_i} are the strata of codimension d corresponding to cones $\sigma_{P_i} \in \Sigma_{P_i}$ via ϕ_{P_i}. Then by definition of π_1 and Lemma 4.6 the inverse image $\pi_1^{-1}(S)$ is equal to

$$\left(\bigsqcup_{i=1}^{2}[\mathbb{L}_{\sigma_{P_i}}]\right)\#_{\mathbb{L}_{\sigma_{P_1}/\sigma^{P_1}_{I_{12}}} \times \partial I_{12,\circ}}[\mathbb{L}_{\sigma_{P_1}/\sigma^{P_1}_{I_{12}}} \times I_{12,\circ}] \cong T^d \times \left(\left(\bigsqcup_{i=1}^{2} S_{P_i}\right)\#_{\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle \times \partial I_{12,\circ}}(\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle \times I_{12,\circ})\right).$$

From the isomorphisms (4.1) and

$$\tau_{P_i}^{\perp} \times \tau_{P_i} \cong (\tau_{P_i}/\langle \sigma_{I_{12}}^{P_i} \rangle)^{\perp} \times \tau_{P_i}/\langle \sigma_{P_i} \rangle \times \sigma_{P_i}/\langle \sigma_{I_{12}}^{P_i} \rangle \times I_{12} \tag{4.8}$$

for any stratum $\tau_{P_i} \in \Sigma_{P_i}$ with $\sigma_{P_i} \subset \bar{\tau}_{P_i}$ it follows

$$\pi_1^{-1}(\mathrm{Nbd}(S)) = \bigcup_{\tau_{P_1}, \tau_{P_2}} \left([\mathbb{L}_{\tau_{P_1}} \sqcup \mathbb{L}_{\tau_{P_2}}] \#_{\mathbb{L}_{\tau_{P_1}/\sigma_{I_{12}}^{P_1}} \times \partial I_{12,\circ}} [\mathbb{L}_{\tau_{P_1}/\sigma_{I_{12}}^{P_1}} \times I_{12,\circ}] \right) \cong \mathbb{L}(\Sigma_{P_1}/\sigma_{P_1}) \times S,$$

where τ_{P_i} run through cones in Σ_{P_i} with $\sigma_{P_i} \subset \bar{\tau}_{P_i}$ mapping to the same cone under the quotient maps $M_{P_i,\mathbb{R}} \to M_{P_i,\mathbb{R}}/\langle \sigma_{P_i} \rangle = M_{S,\mathbb{R}}$ from the definition of fanifolds.

Consider the symplectomorphism

$$T^*T^d \times T^*S \hookrightarrow [T^*\widehat{M}_{P_1} \sqcup T^*\widehat{M}_{P_2}] \#_{\mathcal{L}_{I_{12}}} [T^*\widehat{M}_{I_{12}} \times T^*I_{12,\circ}] \tag{4.9}$$

induced by the symplectomorphisms

$$T^*T^d \times T^*S_{P_i} \hookrightarrow T^*\widehat{M}_{P_i}, \ T^*T^d \times T^*(\sigma_{P_1}/\langle \sigma_{I_{12}}^{P_1} \rangle) \hookrightarrow T^*\widehat{M}_{I_{12}} \tag{4.10}$$

defined as (4.3) and the identity on $T^*I_{12,\circ}$. Since it restricts to isomorphisms

$$\mathbb{L}(\Sigma_{S_{P_i}}) \times S_{P_i} \hookrightarrow \mathbb{L}(\Sigma_{P_i}), \ \mathbb{L}(\Sigma_{I_{12}}/(\sigma_{P_1}/\langle \sigma_{I_{12}}^{P_1} \rangle)) \times \sigma_{P_1}/\langle \sigma_{I_{12}}^{P_1} \rangle \hookrightarrow \mathbb{L}(\Sigma_{I_{12}}) \tag{4.11}$$

defined as (4.4), the symplectomorphism (4.9) embeds $\mathbb{L}(\Sigma_S) \times S$ into $\mathbb{L}(\Phi_1)$.

(2) By assumption Φ_1 is closed. The saturation of the zero set of the Liouville vector field on the gluing

$$\eta_1(\mathrm{Nbd}_{\partial_\infty \mathbf{W}(\Phi_0)}(\mathcal{L}_{I_{12}})) \#_{\mathcal{L}_{I_{12}}} \mathrm{Nbd}_{\partial_\infty [T^*\widehat{M}_{I_{12}} \times T^*I_{12,\circ}]}(\mathcal{L}_{I_{12}}) \subset \mathbf{W}(\Phi_0) \#_{\mathcal{L}_{I_{12}}} [T^*\widehat{M}_{I_{12}} \times T^*I_{12,\circ}]$$

gives the newly formed part of $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi_1))$ by the handle attachment. Due to the absence of higher dimensional strata, it projects onto I_{12} under π_1 and connects 1-dimensional components of $\mathbf{W}(\Phi_0)$. Since I_{12} is interior, by Lemma 4.3(2) the union of the saturation and $\widetilde{\mathbb{L}}(\Phi_0)$ coincide with $\widetilde{\mathbb{L}}(\Phi_1)$.

(3) It suffices to show the claim for $\Phi_1' = \Phi_0$. Then by definition and Remark 3.19

$$\widetilde{\mathbf{W}}(\Phi_1') = T^*\widehat{M}_{P_1} \sqcup T^*\widehat{M}_{P_2}$$

determines a Weinstein sector $\mathbf{W}(\Phi_1') \subset \widetilde{\mathbf{W}}(\Phi_1)$ with skeleton $\mathbb{L}(\Phi_1') = \mathbf{W}(\Phi_1') \cap \widetilde{\mathbb{L}}(\Phi_1)$. Here, one obtains $\mathbf{W}(\Phi_1')$ by completing $[T^*\widehat{M}_{P_1}] \sqcup [T^*\widehat{M}_{P_2}]$ along the modified Liouville flow.

(4) Consider the Lagrangian foliation of $T^*T^d \times T^*S$ with leaves

$$(T_\theta^*T^d \times \{y\}) \times (\{\theta\} \times T_y^*S), \ \theta \in T^d, \ y \in S.$$

Via the symplectomorphism $T^*T^d \times T^*S \cong T^*T^d \times T^*S$ induced by (4.9), the leaf space get identified with the zero section, which in turn is isomorphic to $\pi_1^{-1}(S)$. For the rest, the proof of Lemma 4.3(4) carries over. □

In our current setting, there is only one 2-dimensional stratum F of Φ, which is interior. Let

$$\mathcal{L}_2 = \mathcal{L}_F = \pi_1^{-1}(F) \cap \partial_\infty \mathbb{L}(\Phi_1).$$

Lemma 4.9. *There are smooth Legendrian embeddings*

$$\partial_\infty \mathbf{W}(\Phi_1) \hookleftarrow \mathcal{L}_2 \hookrightarrow \partial(T^*\widehat{M}_F \times T^*F_\circ).$$

Proof. Since $\pi_1^{-1}(F)$ is the gluing of

$$\bigsqcup_{i=1}^{4} \mathbb{L}_{\sigma_F^{P_i}}, \ \bigsqcup_{1 \le i < j \le 4} \mathbb{L}_{\sigma_F^{P_i}/\sigma_{I_{ij}}^{P_i}} \times I_{ij,\circ},$$

by the second bullet of Lemma 4.8(1), there is a smooth Legendrian embedding

$$\mathcal{L}_F \cong \widehat{M}_F \times \partial F_\circ \hookrightarrow \partial_\infty \mathbf{W}(\Phi_1).$$

On the other hand, $\widehat{M}_F \times \partial F_\circ$ can be regarded as a smooth Legendrian in

$$T^*\widehat{M}_F \times T^*\partial F_\circ \subset \partial(T^*\widehat{M}_F \times T^*F_\circ).$$

□

We define $\widetilde{\mathbf{W}}(\Phi_2)$ as the handle attachment

$$\mathbf{W}(\Phi_1)\#_{\mathcal{L}_2}[T^*\widehat{M}_F \times T^*F_\circ].$$

Lemma 4.10. *The manifold $\widetilde{\mathbf{W}}(\Phi_2)$ is subanalytic Weinstein.*

Proof. The proof of Lemma 4.5 carries over. □

Lemma 4.11. *There is a standard neighborhood*

$$\eta_2\colon \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_1)}(\mathcal{L}_2) \hookrightarrow J^1\mathcal{L}_2 \tag{4.12}$$

near $\mathcal{L}_2$ for which $\mathbb{L}(\Phi_1)$ is biconic along $\mathcal{L}_2$ and $\partial_\infty\mathbb{L}(\Phi_1)$ locally factors as

$$\eta_2(\mathbb{L}(\Phi_1) \cap \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_1)}(\mathcal{L}_2)) = \mathbb{L}(\Sigma_F) \times \partial F_\circ.$$

Proof. Along the smooth Legendrian

$$\bigsqcup_{i=1}^{4} \partial_\infty \mathbb{L}_{\sigma_F^{P_i}} \subset \mathcal{L}_F \cap \partial_\infty \mathbb{L}(\Phi_0),$$

the disjoint union $\bigsqcup_{i=1}^4 \mathbb{L}(\Sigma_{P_i})$ is biconic for the standard coordinates

$$\bigsqcup_{i=1}^{4} \eta_{\sigma_F^{P_i}}\colon \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_0)}(\mathcal{L}_F \cap \partial_\infty \mathbb{L}(\Phi_0)) \hookrightarrow J^1\mathcal{L}_F$$

from Lemma 3.27. Along the smooth Legendrian

$$\bigsqcup_{1\le i<j\le 4} (\partial_\infty \mathbb{L}_{\sigma_F^{P_i}/\sigma_{I_{ij}}} \times I_{ij,\circ}) \subset \bigsqcup_{1\le i<j\le 4} \partial_\infty T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ} \subset \bigsqcup_{1\le i<j\le 4} \partial_\infty (T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}),$$

the Lagrangian $\bigsqcup_{1\le i<j\le 4}(\mathbb{L}(\Sigma_{I_{ij}}) \times I_{ij,\circ})$ is biconic for the disjoint union

$$\bigsqcup_{1\le i<j\le 4} (\eta_{\sigma_F^{P_i}/\sigma_{I_{ij}}} \times \operatorname{can}_{I_{ij}})\colon \operatorname{Nbd}_{\partial_\infty(T^*\widehat{M}_{I_{ij}}\times T^*I_{ij,\circ})}(\mathcal{L}_F \cap \bigsqcup_{1\le i<j\le 4} \partial_\infty(\mathbb{L}(\Sigma_{I_{ij}}) \times I_{ij,\circ})) \hookrightarrow J^1\mathcal{L}_F$$

of the product of the standard coordinates from Lemma 3.27 and the canonical coordinates $\operatorname{can}_{I_{ij}}$ on $\bigsqcup_{1\le i<j\le 4} T^*I_{ij,\circ}$. By construction of $\mathbb{L}(\Phi_1)$ these coordinates glue to define a standard neighborhood

$$\eta_F\colon \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_1)}(\mathcal{L}_F) \hookrightarrow J^1\mathcal{L}_F.$$

Then the factorization of $\left(\bigsqcup_{i=1}^4 \partial_\infty \mathbb{L}(\Sigma_{P_i})\right) \# \left(\bigsqcup_{1\le i<j\le 4} \partial_\infty(\mathbb{L}(\Sigma_{I_{ij}}) \times I_{ij,\circ})\right)$ as

$$\left(\bigsqcup_{i=1}^{4} (\mathbb{L}(\Sigma_{P_i}/\sigma_F^{P_i}) \times \partial_\infty \sigma_F^{P_i})\right) \# \left(\bigsqcup_{1\le i<j\le 4} (\mathbb{L}(\Sigma_{I_{ij}}/(\sigma_F^{P_i}/\langle\sigma_{I_{ij}}^{P_i}\rangle)) \times \partial_\infty \sigma_F^{P_i}/\langle\sigma_{I_{ij}}^{P_i}\rangle \times I_{ij,\circ})\right) \cong \mathbb{L}(\Sigma_F) \times F_\circ$$

follows from (3.1) for any strata $\tau_{P_i} \in \Sigma_{P_i}$ with $\sigma_F^{P_i} \subset \bar{\tau}_{P_i}$ mapping to the same cone in Σ_F under the quotient maps $\Sigma_{P_i} \to \Sigma_{P_i}/\sigma_F^{P_i}$. □

We define $\widetilde{\mathbb{L}}(\Phi_2)$ as the extension through the handle $T^*\widehat{M}_F \times T^*F_\circ$

$$\mathbb{L}(\Phi_1)\#_{\mathcal{L}_2}[\mathbb{L}_{\Sigma_F} \times F_\circ].$$

Lemma 4.12. *The subset* $\widetilde{\mathbb{L}}(\Phi_2) \subset \widetilde{\mathbf{W}}(\Phi_2)$ *is a subanalytic conic Lagrangian and contains the skeleton* $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi_2))$ *of the Weinstein manifold* $\widetilde{\mathbf{W}}(\Phi_2)$.

Proof. Since by Lemma 4.11 and Lemma 3.23 the Lagrangians $\mathbb{L}(\Phi_1) \subset \mathbf{W}(\Phi_1)$ and $[\mathbb{L}(\Sigma_F) \times F_\circ] \subset [T^*\widehat{M}_F \times T^*F_\circ]$ are biconic along $\mathcal{L}_2$, the gluing $\widetilde{\mathbb{L}}(\Phi_2)$ remains conic. For the rest of the claim, the proof of Lemma 4.7 carries over. □

Let $\pi_2 \colon \mathbb{L}(\Phi_2) \to \Phi_2$ be the map induced by π_1 and the projections from $T^*\widehat{M}_F \times T^*F_\circ$ to the cotangent fiber direction in $T^*\widehat{M}_F$ and the base direction in $T^*F_\circ$.

Lemma 4.13. *The triple* $(\widetilde{\mathbf{W}}(\Phi_2), \widetilde{\mathbb{L}}(\Phi_2), \pi_2)$ *satisfies the conditions (1), ..., (4).*

Proof. (1) Let $S \subset \Phi_2$ be the stratum

$$\left(\left(\bigsqcup_{i=1}^{4} S_{P_i}\right)\#_{\bigsqcup_{1\le i<j\le 4}(\sigma_{P_i}/\langle\sigma^{P_i}_{I_{ij}}\rangle\times\partial I_{ij,\circ})}\left(\bigsqcup_{1\le i<j\le 4}\sigma_{P_i}/\langle\sigma^{P_i}_{I_{ij}}\rangle\times I_{ij,\circ}\right)\right)\#_{\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle\times\partial F_\circ}(\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle\times F_\circ)$$

where $S_{P_i} \subset \Phi_0, 1 \le i \le 4$ are the strata of codimension d corresponding to cones $\sigma_{P_i} \in \Sigma_{P_i}$ via ϕ_{P_i}. Then by definition of π_2 and Lemma 4.11 the inverse image $\pi_2^{-1}(S)$ is equal to

$$\begin{aligned}
&\left([\bigsqcup_{i=1}^{4}\mathbb{L}_{\sigma_{P_i}}]\#_{\bigsqcup_{1\le i<j\le 4}(\mathbb{L}_{\sigma_{P_i}/\sigma^{P_i}_{I_{ij}}}\times\partial I_{ij,\circ})}[\bigsqcup_{1\le i<j\le 4}\mathbb{L}_{\sigma_{P_i}/\sigma^{P_i}_{I_{ij}}}\times I_{ij,\circ}]\right)\#_{\mathbb{L}_{\sigma_{P_1}/\sigma^{P_1}_F}\times\partial F_\circ}[\mathbb{L}_{\sigma_{P_1}/\sigma^{P_1}_F}\times F_\circ]\\
\cong& T^d \times\left(\left(\left(\bigsqcup_{i=1}^{4} S_{P_i}\right)\#_{\bigsqcup_{1\le i<j\le 4}(\sigma_{P_i}/\langle\sigma^{P_i}_{I_{ij}}\rangle\times\partial I_{ij,\circ})}\left(\bigsqcup_{1\le i<j\le 4}(\sigma_{P_i}/\langle\sigma^{P_i}_{I_{ij}}\rangle\times I_{ij,\circ})\right)\right)\#_{\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle\times\partial F_\circ}(\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle\times F_\circ)\right).
\end{aligned}$$

From the isomorphisms (4.1), (4.8) and

$$\tau_{P_i}^{\perp}\times\tau_{P_i} \cong (\tau_{P_i}/\langle\sigma_{P_i}\rangle)^{\perp}\times\tau_{P_i}/\langle\sigma_{P_i}\rangle\times\sigma_{P_i}/\langle\sigma^{P_i}_F\rangle\times F \tag{4.13}$$

for any stratum $\tau_{P_i} \in \Sigma_{P_i}$ with $\sigma_{P_i} \subset \bar{\tau}_{P_i}$ it follows that $\pi_2^{-1}(\mathrm{Nbd}(S))$ is equal to

$$\begin{aligned}
&\bigcup_{\tau_{P_1},\dots,\tau_{P_4}}\left(\left([\bigsqcup_{i=1}^{4}\mathbb{L}_{\tau_{P_i}}]\#_{\bigsqcup_{1\le i<j\le 4}\mathbb{L}_{\tau_{P_i}/\sigma^{P_i}_{I_{ij}}}\times\partial I_{ij,\circ}}[\bigsqcup_{1\le i<j\le 4}\mathbb{L}_{\tau_{P_i}/\sigma^{P_i}_{I_{ij}}}\times I_{ij,\circ}]\right)\#_{\mathbb{L}_{\tau_{P_1}/\sigma^{P_1}_F}\times\partial F_\circ}[\mathbb{L}_{\tau_{P_1}/\sigma^{P_1}_F}\times F_\circ]\right)\\
\cong&\mathbb{L}(\Sigma_{P_1}/\sigma_{P_1})\times S,
\end{aligned}$$

where $\tau_{P_1},\dots,\tau_{P_4}$ run through cones in $\Sigma_{P_1},\dots,\Sigma_{P_4}$ with $\sigma_{P_i} \subset \bar{\tau}_{P_i}$ mapping to the same cone under the quotient maps $M_{P_i,\mathbb{R}} \to M_{P_i,\mathbb{R}}/\langle\sigma_{P_i}\rangle = M_{S,\mathbb{R}}$ from the definition of fanifolds.

Consider the symplectomorphism

$$T^*T^d\times T^*S \hookrightarrow \left([\bigsqcup_{i=1}^{4}T^*\widehat{M}_{P_i}]\#_{\bigsqcup_{1\le i<j\le 4}\mathcal{L}_{I_{ij}}}[\bigsqcup_{1\le i<j\le 4}T^*\widehat{M}_{I_{ij}}\times T^*I_{ij,\circ}]\right)\#_{\mathcal{L}_F}[T^*\widehat{M}_F\times T^*F_\circ] \tag{4.14}$$

induced by the symplectomorphisms (4.10) and

$$T^*T^d\times T^*(\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle) \hookrightarrow T^*\widehat{M}_F$$

defined as (4.3), and the identity on $T^*I_\circ, T^*F_\circ$. Since it restricts to such isomorphisms as (4.11) and an isomorphism

$$\mathbb{L}(\Sigma_F/(\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle))\times\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle \hookrightarrow \mathbb{L}(\Sigma_F) \tag{4.15}$$

defined as (4.4), the symplectomorphism (4.14) embeds $\mathbb{L}(\Sigma_S)\times S$ into $\mathbb{L}(\Phi_2)$.

(2) By assumption Φ_2 is closed. The saturation of the zero set of the Liouville vector field on the gluing

$$\eta_2(\mathrm{Nbd}_{\partial_\infty\mathbf{W}(\Phi_1)}(\mathcal{L}_2))\#_{\mathcal{L}_2}\mathrm{Nbd}_{\partial_\infty[T^*\widehat{M}_F\times T^*F_\circ]}(\mathcal{L}_F) \subset \mathbf{W}(\Phi_1)\#_{\mathcal{L}_2}[T^*\widehat{M}_F\times T^*F_\circ]$$

gives the newly formed part of Core($\mathbf{W}(\Phi_2)$) by the handle attachment. Due to the absence of higher dimensional strata, it projects onto F under π_2 and connects 2-dimensional components of $\mathbf{W}(\Phi_1)$. Since F is interior, by Lemma 4.8(2) the union of the saturation and $\widetilde{\mathbb{L}}(\Phi_1)$ coincide with $\widetilde{\mathbb{L}}(\Phi_2)$.

(3) It suffices to show the claim for $\Phi_2' = \Phi_1$. Then by definition and Remark 3.19

$$\widetilde{\mathbf{W}}(\Phi_2') = [\bigsqcup_{i=1}^{4} T^*\widehat{M}_{P_i}]\#_{\bigsqcup_{1\le i<j\le 4}\mathcal{L}_{I_{ij}}}[\bigsqcup_{1\le i<j\le 4} T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}]$$

determines a Weinstein sector $\mathbf{W}(\Phi_2') \subset \widetilde{\mathbf{W}}(\Phi_2)$ with skeleton $\mathbb{L}(\Phi_2') = \mathbf{W}(\Phi_2') \cap \widetilde{\mathbb{L}}(\Phi_2)$. Here, one obtains $\mathbf{W}(\Phi_2')$ by completing the gluing of $[\bigsqcup_{i=1}^4 T^*\widehat{M}_{P_i}]$ with $[\bigsqcup_{1\le i<j\le 4} T^*\widehat{M}_{I_{ij}} \times T^*I_{ij,\circ}]$ along the modified Liouville flow.

(4) Consider the Lagrangian foliation of $T^*T^d \times T^*S$ with leaves

$$(T^*_\theta T^d \times \{y\}) \times (\{\theta\} \times T^*_y S), \ \theta \in T^d, \ y \in S.$$

Via the symplectomorphism $T^*T^d \times T^*S \cong T^*T^d \times T^*S$ induced by (4.14), the leaf space get identified with the zero section, which in turn is isomorphic to $\pi_2^{-1}(S)$. For the rest, the proof of Lemma 4.3(4) carries over. □

4.3. **General case.** Suppose that Theorem 1.1 and the relevant results hold for the subfanifold Φ_{k-1}. Let $\mathcal{L}_k$ be the disjoint union of

$$\mathcal{L}_{S^{(k)}} = \pi_{k-1}^{-1}(S^{(k)}) \cap \partial_\infty \mathbb{L}(\Phi_{k-1})$$

for all interior k-strata $S^{(k)}$ of Φ.

Lemma 4.14. *There are smooth Legendrian embeddings*

$$\partial_\infty \mathbf{W}(\Phi_{k-1}) \hookleftarrow \mathcal{L}_k \hookrightarrow \partial \bigsqcup_{S^{(k)}} (T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ).$$

Proof. It suffices to show the claim for $\mathcal{L}_{S^{(k)}}$. Since $\pi_{k-1}^{-1}(S^{(k)})$ is the gluing of

$$\mathbb{L}_{\sigma^P_{S^{(k)}}}, \ \mathbb{L}_{\sigma^P_{S^{(k)}}/\sigma^P_I} \times I_\circ, \ \mathbb{L}_{\sigma^P_{S^{(k)}}/\sigma^P_F} \times F_\circ, \ \dots$$

by the second bullet of Theorem 1.1(1) for $n = k-1$, there is a smooth Legendrian embedding

$$\mathcal{L}_{S^{(k)}} \cong \widehat{M}_{S^{(k)}} \times \partial S^{(k)}_\circ \hookrightarrow \partial_\infty \mathbf{W}(\Phi_{k-1}).$$

Here, $P, I, F, \dots$ run through strata of dimension $0, 1, 2, \dots$ in $\partial S^{(k)}_\circ$ with $P \subset \partial_{in} I, P, I \subset \partial_{in} F, \dots$. Note that by definition of fanifolds $\sigma^P_{S^{(k)}}/\langle\sigma^P_I\rangle, \sigma^P_{S^{(k)}}/\langle\sigma^P_F\rangle, \dots$ regarded as cones in $\Sigma_I, \Sigma_F, \dots$ do not depend on the choice of P and we fix some P when such $I, F, \dots$ run. On the other hand, $\widehat{M}_{S^{(k)}} \times \partial S^{(k)}_\circ$ can be regarded as a smooth Legendrian in

$$T^*\widehat{M}_{S^{(k)}} \times T^*\partial S^{(k)}_\circ \subset \partial \bigsqcup_{S^{(k)}} (T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ).$$

□

We define $\widetilde{\mathbf{W}}(\Phi_k)$ as the handle attachment

$$\mathbf{W}(\Phi_{k-1})\#_{\mathcal{L}_k} \bigsqcup_{S^{(k)}} [T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ].$$

Lemma 4.15. *The manifold $\widetilde{\mathbf{W}}(\Phi_k)$ is subanalytic Weinstein.*

Proof. The proof of Lemma 4.5 carries over. □

Lemma 4.16. *There is a standard neighborhood*

$$\eta_k\colon \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_{k-1})}(\mathcal{L}_k) \hookrightarrow J^1\mathcal{L}_k \tag{4.16}$$

near $\mathcal{L}_k$ for which $\mathbb{L}(\Phi_{k-1})$ is biconic along $\mathcal{L}_k$ and $\partial_\infty\mathbb{L}(\Phi_{k-1})$ locally factors as

$$\eta_k(\mathbb{L}(\Phi_{k-1}) \cap \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_{k-1})}(\mathcal{L}_k)) = \bigsqcup_{S^{(k)}} (\mathbb{L}(\Sigma_{S^{(k)}}) \times \partial S^{(k)}_\circ).$$

Proof. It suffices to show the claim for $\mathcal{L}_{S^{(k)}}$. Along

$$\partial_\infty \mathbb{L}_{\sigma^P_{S^{(k)}}},\ \partial_\infty \mathbb{L}_{\sigma^P_{S^{(k)}}/\sigma^P_I} \times I_\circ,\ \partial_\infty \mathbb{L}_{\sigma^P_{S^{(k)}}/\sigma^P_F} \times F_\circ,\ \ldots$$

where $P, I, F, \ldots$ run through strata of dimension $0, 1, 2, \ldots$ in $\partial S_\circ$ with $P \subset \partial_{in} I$, $P, I \subset \partial_{in} F, \ldots$ as above,

$$\mathbb{L}(\Sigma_P),\ \mathbb{L}(\Sigma_I) \times I_\circ,\ \mathbb{L}(\Sigma_F) \times F_\circ,\ \ldots \tag{4.17}$$

are biconic for the products

$$\eta_{\sigma^P_{S^{(k)}}},\ \eta_{\sigma^P_{S^{(k)}}/\sigma^P_I} \times \operatorname{can}_I,\ \eta_{\sigma^P_{S^{(k)}}/\sigma^P_F} \times \operatorname{can}_F,\ \ldots$$

of the standard coordinates from Lemma 3.27 and the canonical coordinates $\operatorname{can}_I, \operatorname{can}_F, \ldots$ on $T^*I_\circ, T^*F_\circ, \ldots$. By construction of $\mathbb{L}(\Phi_{k-1})$ these coordinates glue to define a standard neighborhood

$$\eta_{S^{(k)}}\colon \operatorname{Nbd}_{\partial_\infty \mathbf{W}(\Phi_{k-1})}(\mathcal{L}_{S^{(k)}}) \hookrightarrow J^1\mathcal{L}_{S^{(k)}}.$$

Then the factorization of the gluing of (4.17) as the gluing of

$$\mathbb{L}(\Sigma_P/\sigma^P_{S^{(k)}}),\ \mathbb{L}(\Sigma_I/(\sigma^P_{S^{(k)}}/\langle\sigma^P_I\rangle)) \times \sigma^P_{S^{(k)}}/\langle\sigma^P_I\rangle \times I_\circ,\ \mathbb{L}(\Sigma_F/(\sigma^P_{S^{(k)}}/\langle\sigma^P_F\rangle)) \times \sigma^P_{S^{(k)}}/\langle\sigma^P_F\rangle \times F_\circ,\ \ldots$$

follows from (3.1) for any strata $\tau_P \in \Sigma_P, \tau_P/\langle\sigma^P_I\rangle \in \Sigma_I, \tau_P/\langle\sigma^P_F\rangle \in \Sigma_F, \ldots$ with $\sigma^P_{S^{(k)}} \subset \bar\tau_P$ mapping to the same cone in $\Sigma_{S^{(k)}}$ under the quotient maps

$$\Sigma_P \to \Sigma_P/\sigma^P_{S^{(k)}},\ \Sigma_I \to \Sigma_I/(\sigma^P_{S^{(k)}}/\langle\sigma^P_I\rangle),\ \Sigma_F \to \Sigma_F/(\sigma^P_{S^{(k)}}/\langle\sigma^P_F\rangle),\ \ldots.$$

□

We define $\widetilde{\mathbb{L}}(\Phi_k)$ as the extension through the disjoint union of the handles $T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ$

$$\mathbb{L}(\Phi_{k-1})\#_{\mathcal{L}_k} \bigsqcup_{S^{(k)}} [\mathbb{L}(\Sigma_{S^{(k)}}) \times S^{(k)}_\circ].$$

Lemma 4.17. *The subset $\widetilde{\mathbb{L}}(\Phi_k) \subset \widetilde{\mathbf{W}}(\Phi_k)$ is a subanalytic conic Lagrangian and contains the skeleton* $\operatorname{Core}(\widetilde{\mathbf{W}}(\Phi_k))$ *of the Weinstein manifold $\widetilde{\mathbf{W}}(\Phi_k)$.*

Proof. Since by Lemma 4.16 and Lemma 3.23 the Lagrangians $\mathbb{L}(\Phi_{k-1}) \subset \mathbf{W}(\Phi_{k-1})$ and $\bigsqcup_{S^{(k)}}[\mathbb{L}(\Sigma_{S^{(k)}}) \times S^{(k)}_\circ] \subset \bigsqcup_{S^{(k)}}[T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ]$ are biconic along $\mathcal{L}_k$, the gluing $\widetilde{\mathbb{L}}(\Phi_k)$ remains conic. For the rest of the claim, the proof of Lemma 4.7 carries over. □

Let $\pi_k\colon \mathbb{L}(\Phi_k) \to \Phi_k$ be the map induced by π_{k-1} and the projections from $T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ$ to the cotangent fiber direction in $T^*\widehat{M}_{S^{(k)}}$ and the base direction in $T^*S^{(k)}_\circ$.

Lemma 4.18. *The triple $(\widetilde{\mathbf{W}}(\Phi_k), \widetilde{\mathbb{L}}(\Phi_k), \pi_k)$ satisfies the conditions (1), …, (4).*

Proof. (1) It suffices to show the claim when attaching the handle $T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ$ to $\widetilde{\mathbf{W}}(\Phi_{k-1})$ for a single interior k-stratum $S^{(k)} \subset \Phi$ with $k \le \dim S$. Then by definition of π_k and Lemma 4.16 the inverse image $\pi_k^{-1}(S)$ is the gluing of

$$\mathbb{L}_{\sigma_P} \cong T^d \times \sigma_P,\ \mathbb{L}_{\sigma_P/\sigma^P_I} \times I_\circ \cong T^d \times \sigma_P/\langle\sigma^P_I\rangle \times I_\circ,\ \mathbb{L}_{\sigma_P/\sigma^P_F} \times F_\circ \cong T^d \times \sigma_P/\langle\sigma^P_F\rangle \times F_\circ,\ \ldots$$

and $\mathbb{L}_{\sigma^P_{S^{(k)}}} \times S^{(k)}_\circ \cong T^d \times \sigma_P/\langle\sigma^P_{S^{(k)}}\rangle \times S^{(k)}_\circ$ where $P, I, F, \dots$ run through strata of dimension $0, 1, 2, \dots$ in $\partial S^{(k)}_\circ$ with $P \subset \partial_{in} I, P, I \subset \partial_{in} F, \dots$ as above. Hence by Lemma 2.10 we obtain $\pi_k^{-1}(S) \cong T^d \times S$.

From the isomorphisms (4.1), (4.8), (4.13), ... and

$$\tau_P^\perp \times \tau_P \cong (\tau_P/\langle\sigma_P\rangle)^\perp \times \tau_P/\langle\sigma_P\rangle \times \sigma_P/\langle\sigma^P_{S^{(k)}}\rangle \times S^{(k)} \tag{4.18}$$

for any stratum $\tau_P \in \Sigma_P$ with $\sigma_P \subset \bar{\tau}_P$, it follows that $\pi_k^{-1}(S')$ for a stratum S' of the induced stratification on $\mathrm{Nbd}(S)$ is the gluing of

$$\mathbb{L}_{\tau_P} \cong \mathbb{L}_{\tau_P/\sigma_P} \times \sigma_P,\ \mathbb{L}_{\tau_P/\sigma^P_I} \times I_\circ \cong \mathbb{L}_{\tau_P/\sigma_P} \times \sigma_P/\langle\sigma^P_I\rangle \times I_\circ,\ \mathbb{L}_{\tau_P/\sigma^P_F} \times F_\circ \cong \mathbb{L}_{\tau_P/\sigma_P} \times \sigma_P/\langle\sigma^P_F\rangle \times F_\circ,\ \dots$$

and $\mathbb{L}_{\tau_P/\sigma_P} \times \sigma_P/\langle\sigma^P_{S^{(k)}}\rangle \times S^{(k)}_\circ$ where $P, I, F, \dots$ run through strata of dimension $0, 1, 2, \dots$ in $\partial S^{(k)}_\circ$ with $P \subset \partial_{in} I, P, I \subset \partial_{in} F, \dots$ as above and where τ_P are the cones in Σ_P corresponding to S'. Since τ_P map to the same cone under the quotient maps $M_{P,\mathbb{R}} \to M_{P,\mathbb{R}}/\langle\sigma_P\rangle = M_{S,\mathbb{R}}$ from the definition of fanifolds, by Lemma 2.10 we obtain

$$\pi_k^{-1}(S') \cong T^{d'} \times \tau_P/\sigma_P \times S,\ \pi_k^{-1}(\mathrm{Nbd}(S)) \cong \mathbb{L}(\Sigma_P/\sigma_P) \times S.$$

Consider the symplectomorphism

$$T^*T^d \times T^*S \hookrightarrow \mathbf{W}(\Phi_{k-1})\#_{\mathcal{L}_{S^{(k)}}}[T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ] \tag{4.19}$$

induced by the symplectomorphisms

$$T^*T^d \times T^*S_P \hookrightarrow T^*\widehat{M}_P,\ T^*T^d \times T^*(\sigma_P/\langle\sigma^P_I\rangle) \hookrightarrow T^*\widehat{M}_I,\ T^*T^d \times T^*(\sigma_P/\langle\sigma^P_F\rangle) \hookrightarrow T^*\widehat{M}_F,\ \dots$$

and the symplectomorphism

$$T^*T^d \times T^*(\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle) \hookrightarrow T^*\widehat{M}^{(k)}_S$$

defined as (4.3), and the identity on $T^*I_\circ, T^*F_\circ, \dots$ and $T^*S^{(k)}_\circ$. Since it restricts to isomorphisms

$$\mathbb{L}(\Sigma_S) \times S \hookrightarrow \mathbb{L}(\Sigma_P),\ \mathbb{L}(\Sigma_S) \times \sigma_P/\langle\sigma^P_I\rangle \hookrightarrow \mathbb{L}(\Sigma_I),\ \mathbb{L}(\Sigma_S) \times \sigma_P/\langle\sigma^P_F\rangle \hookrightarrow \mathbb{L}(\Sigma_F),\ \dots$$

and an isomorphism

$$\mathbb{L}(\Sigma_{S^{(k)}}/(\sigma_{P_1}/\langle\sigma^{P_1}_{S^{(k)}}\rangle)) \times \sigma_{P_1}/\langle\sigma^{P_1}_{S^{(k)}}\rangle = \mathbb{L}(\Sigma_S) \times \sigma_{P_1}/\langle\sigma^{P_1}_{S^{(k)}}\rangle \hookrightarrow \mathbb{L}(\Sigma_{S^{(k)}}) \tag{4.20}$$

defined as (4.4), the symplectomorphism (4.19) embeds $\mathbb{L}(\Sigma_S) \times S$ into $\mathbb{L}(\Phi_k)$.

(2) The fanifold Φ_k is closed only if:

- There are no strata of dimension more than k.
- All strata of dimension $1, 2, \dots, k-1$ are interior.

Then the saturation of the zero set of the Liouville vector field on the gluing

$$\eta_k(\mathrm{Nbd}_{\partial_\infty \mathbf{W}(\Phi_{k-1})}(\mathcal{L}_k))\#_{\mathcal{L}_k} \bigsqcup_{S^{(k)}} \mathrm{Nbd}_{\partial_\infty[T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ]}(\mathcal{L}_{S^{(k)}}) \subset \mathbf{W}(\Phi_{k-1})\#_{\mathcal{L}_{S^{(k)}}} \bigsqcup_{S^{(k)}} [T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ]$$

gives the newly formed part of $\mathrm{Core}(\widetilde{\mathbf{W}}(\Phi_k))$ by the handle attachment. Due to the absence of higher dimensional strata, it projects onto $\bigsqcup_{S^{(k)}} S^{(k)} \subset \Phi_k$ under π_k and connects k-dimensional components of $\mathbf{W}(\Phi_{k-1})$. Since all k-strata are interior, by Theorem 1.1(2) for $n = k-1$ the union of the saturation and $\widetilde{\mathbb{L}}(\Phi_{k-1})$ coincide with $\widetilde{\mathbb{L}}(\Phi_k)$.

(3) It suffices to show the claim for

$$\Phi_k = \Phi_{k-1}\#_{\Sigma_{S^{(k)}} \times \partial_{in} S^{(k)}}(\Sigma_{S^{(k)}} \times S^{(k)}),\ \Phi'_k = \Phi_{k-1}.$$

Then by definition and Remark 3.19 $\widetilde{\mathbf{W}}(\Phi'_k)$ determines a Weinstein sector $\mathbf{W}(\Phi'_k) \subset \widetilde{\mathbf{W}}(\Phi_k)$ with skeleton $\mathbb{L}(\Phi'_k) = \mathbf{W}(\Phi'_k) \cap \widetilde{\mathbb{L}}(\Phi_k)$. Here, one obtains $\mathbf{W}(\Phi'_k)$ by completing the gluing of the domains along the modified Liouville flow.

(4) Consider the Lagrangian foliation of $T^*T^d \times T^*S$ with leaves

$$(T^*_\theta T^d \times \{y\}) \times (\{\theta\} \times T^*_y S),\ \theta \in T^d,\ y \in S.$$

Via the symplectomorphism $T^*T^d \times T^*S \cong T^*T^d \times T^*S$ induced by (4.19), the leaf space get identified with the zero section, which in turn is isomorphic to $\pi_k^{-1}(S)$. Along the base direction in T^*S, it is compatible with the inclusion $S \hookrightarrow \overline{S'}$ to any stratum S' of codimension d' of the induced stratification on $\mathrm{Nbd}(S)$. Along the fiber direction in T^*T^d, it is compatible with the inclusions $T^*_\theta T^{d'} \hookrightarrow T^*_\theta T^d$ induced by the quotient maps

$$T_{S_P} M_{P,\mathbb{R}} \to T_{S'_P} M_{P,\mathbb{R}}|_{S_P},\ T_{\sigma_P/\langle\sigma_I^P\rangle} M_{I,\mathbb{R}} \to T_{\sigma'_P/\langle\sigma_I^P\rangle} M_{I,\mathbb{R}}|_{\sigma_P},\ T_{\sigma_P/\langle\sigma_F^P\rangle} M_{F,\mathbb{R}} \to T_{\sigma'_P/\langle\sigma_F^P\rangle} M_{F,\mathbb{R}}|_{\sigma_P},\ \ldots$$

and $T_{\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle} M_{S^{(k)},\mathbb{R}} \to T_{\sigma'_P/\langle\sigma^P_{S^{(k)}}\rangle} M_{S^{(k)},\mathbb{R}}|_{\sigma_P}$ from the definition of fanifolds for $\theta \in T^{d'} \cap T^d \subset \widehat{M}_P$. Hence one obtains the desired polarization. □

Remark 4.19. The symplectomorphism (4.19) sends the cotangent fibers of

$$T^*S_P,\ T^*(\sigma_P/\langle\sigma_I^P\rangle),\ T^*(\sigma_P/\langle\sigma_F^P\rangle),\ \ldots$$

and $T^*(\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle)$ to the bases of $T^*\widehat{M}_P,\ T^*\widehat{M}_I,\ T^*\widehat{M}_F,\ \ldots$ and $T^*\widehat{M}_{S^{(k)}}$ with negation.

Remark 4.20. If $\Sigma_P, \Sigma_I, \Sigma_F, \ldots$ and $\Sigma_{S^{(k)}}$ are stacky fans, then we consider stacky FLTZ skeleta $\mathbb{L}(\Sigma_{P_i}), \mathbb{L}(\Sigma_{I_{ij}}), \mathbb{L}(\Sigma_F), \ldots$ and $\mathbb{L}(\Sigma_{S^{(k)}})$. According to how many of copies of tori there, duplicate the corresponding handles $T^*\widehat{M}_I \times T^*I_\circ,\ T^*\widehat{M}_F \times T^*F_\circ,\ \ldots$ and $T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ$. Then our proof generalizes in a straightforward way.

5. The proof of Theorem 1.2

Recall that a *fibration* is a map which satisfies the homotopy lifting property for all topological spaces. Any fiber bundle over a paracompact Hausdorff base gives an example. In this section, we first construct an intermediate filtered stratified fibration $\tilde{\pi}$ from $\widetilde{\mathbf{W}}(\Phi)$ restricting to π, which defines a filtered stratified integrable system with noncompact fibers. The composition with a certain map induced by retractions yields $\bar{\pi}$. When Σ_S is proper for any $S \subset \Phi$, the map is trivial and $\tilde{\pi}$ defines the integrable system. As in the previous section, we proceed by induction on k.

5.1. Base case.

Lemma 5.1. *There is a stratified fibration $\bar{\pi}_0 \colon \widetilde{\mathbf{W}}(\Phi_0) \to \Phi_0$ restricting to π_0.*

Proof. Define $\tilde{\pi}_0$ as the disjoint union of the projections to the cotangent fibers $T^*\widehat{M}_P \to M_{P,\mathbb{R}}$. Clearly, its restriction to $\widetilde{\mathbb{L}}(\Phi_0)$ coincides with π_0. Let $\mathrm{ret}_0 \colon \bigsqcup_P M_{P,\mathbb{R}} \to \bigsqcup_P \Sigma_P$ be the disjoint union of maps induced by retractions which are the canonical extensions of piecewise projections onto facets in $\partial\Phi$ from outwards along their normal directions. Then the composition $\bar{\pi}_0 = \mathrm{ret}_0 \circ \tilde{\pi}_0$ gives the desired fibration. □

5.2. Special case.

Again, before moving to general cases, we explicitly write down the proof for Example 2.12.

Lemma 5.2. *There is a filtered stratified fibration $\bar{\pi}_1 \colon \widetilde{\mathbf{W}}(\Phi_1) \to \Phi_1$ restricting to π_1.*

Proof. It suffices to show the claim when attaching the handle $T^*\widehat{M}_{I_{12}} \times T^*I_{12,\circ}$ to $\widetilde{\mathbf{W}}(\Phi_0)$. Then we may assume (4.7). Define $\tilde{\pi}_1$ as the map canonically induced by $\tilde{\pi}_0, \tilde{\pi}_{0,I_{12}}$ and the projection $T^*I_{12,\circ} \to I_{12,\circ}$ to the base, where $\tilde{\pi}_{0,I_{12}} \colon T^*\widehat{M}_{I_{12}} \to M_{I_{12},\mathbb{R}}$ is the projection to the cotangent fibers. Here, we precompose the contraction

$$\mathrm{cont}_1 \colon \widetilde{\mathbf{W}}(\Phi_1) \to \widetilde{\mathbf{W}}(\Phi_1)$$

of the cylindrical ends along the negative Liouville flow to the union of $\partial_\infty \mathbf{W}(\Phi_0)$ and $\partial T^* I_{12,\circ}$. Let $\mathrm{ret}_1 \colon M_{I_{12},\mathbb{R}} \to \Sigma_{I_{12}}$ be the map induced by a retraction which is the canonical extension of piecewise projections onto facets in $\partial\Phi$ from outwards along their normal directions. Define $\bar{\pi}_1$ as the map canonically induced by $\bar{\pi}_0, \bar{\pi}_{0,I_{12}} = \mathrm{ret}_1 \circ \tilde{\pi}_{0,I_{12}}$ and the projection $T^* I_{12,\circ} \to I_{12,\circ}$ to the base.

Recall that the symplectomorphism (4.9) sends the bases of $T^* S_{P_i}$ and $T^*(\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle)$ to the cotangent fibers of $T^* \widehat{M}_{P_i}$ and $T^* \widehat{M}_{I_{12}}$ preserving $T^* I_{12,\circ}$. In the gluing

$$\widetilde{\mathbf{W}}(\Phi_1) = [T^* \widehat{M}_{P_1} \sqcup T^* \widehat{M}_{P_2}] \#_{\mathcal{L}_{I_{12}}} [T^* \widehat{M}_{I_{12}} \times T^* I_{12,\circ}] \tag{5.1}$$

the cotangent fibers of the images of $T^* S_{P_1}, T^* S_{P_2}$ get connected through $J^1 \mathcal{L}_{I_{12}}$ with the product of the cotangent fibers of the image of $T^*(\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle)$ and the base of $T^* I_{12,\circ}$. In other words, (4.9) respects the gluing (5.1) and

$$T^* T^d \times ((T^* S_{P_1} \sqcup T^* S_{P_2}) \# (T^*(\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle) \times T^* I_{12,\circ}))$$

induced by the gluing of the zero sections.

By definition $\bar{\pi}_1$ projects the former and the latter parts of (5.1) respectively onto

$$S_{P_1} \sqcup S_{P_2}, \ \sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle \times I_{12,\circ}$$

in the gluing

$$S = (S_{P_1} \sqcup S_{P_2}) \#_{\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle \times \partial I_{12,\circ}} (\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle \times I_{12,\circ}).$$

Hence $\bar{\pi}_1$ is compatible with the relevant gluing procedure. As explained in the proof of Lemma 4.8(4), the source $T^* T^d \times T^* S$ contains $\pi_1^{-1}(S)$ as the zero section. Then by Lemma 4.8(1) the restriction of $\bar{\pi}_1$ to $\widetilde{\mathbb{L}}(\Phi_1)$ coincides with π_1. $\square$

Lemma 5.3. *There is a filtered stratified fibration $\bar{\pi}_2 \colon \widetilde{\mathbf{W}}(\Phi_2) \to \Phi_2$ restricting to π_2.*

Proof. Define $\bar{\pi}_2$ as the map canonically induced by $\bar{\pi}_1, \tilde{\pi}_{0,F}$ and the projection $T^* F_\circ \to F_\circ$ to the base, where $\tilde{\pi}_{0,F} \colon T^* \widehat{M}_F \to M_{F,\mathbb{R}}$ is the projection to the cotangent fibers. Here, we precompose the contraction

$$\mathrm{cont}_2 \colon \widetilde{\mathbf{W}}(\Phi_2) \to \widetilde{\mathbf{W}}(\Phi_2)$$

of the cylindrical ends along the negative Liouville flow to the union of $\partial_\infty \mathbf{W}(\Phi_1)$ and $\partial T^* F_\circ$. Let $\mathrm{ret}_2 \colon M_{F,\mathbb{R}} \to \Sigma_F$ be the map induced by a retraction which is the canonical extension of piecewise projections onto facets in $\partial\Phi$ from outwards along their normal directions. Define $\bar{\pi}_2$ as the map canonically induced by $\bar{\pi}_1, \bar{\pi}_{0,F} = \mathrm{ret}_2 \circ \tilde{\pi}_{0,F}$ and the projection $T^* F_\circ \to F_\circ$ to the base.

Recall that the symplectomorphism (4.14) sends the bases of $T^* S_{P_i}, T^*(\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle)$ and $T^*(\sigma_{P_1}/\langle \sigma^{P_1}_F \rangle)$ to the cotangent fibers of $T^* \widehat{M}_{P_i}, T^* \widehat{M}_{I_{12}}$ and $T^* \widehat{M}_F$ preserving $T^* I_{12,\circ}, T^* F_\circ$. In the gluing

$$\widetilde{\mathbf{W}}(\Phi_2) = \left([T^* \widehat{M}_{P_1} \sqcup T^* \widehat{M}_{P_2}] \#_{\mathcal{L}_{I_{12}}} [T^* \widehat{M}_{I_{12}} \times T^* I_{12,\circ}] \right) \#_{\mathcal{L}_F} [T^* \widehat{M}_F \times T^* F_\circ] \tag{5.2}$$

the cotangent fibers of the images of $T^* S_{P_i}$ get connected through $J^1 \mathcal{L}_{I_{12}}$ with the product of the cotangent fibers of the image of $T^*(\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle)$ and the base of $T^* I_{12,\circ}$. The result gets connected through $J^1 \mathcal{L}_F$ with the product of the cotangent fibers of the image of $T^*(\sigma_{P_1}/\langle \sigma^{P_1}_F \rangle)$ and the base of $T^* F_\circ$. In other words, (4.14) respects the gluing (5.2) and

$$T^* T^d \times \left([T^* S_{P_1} \sqcup T^* S_{P_2}] \# [T^*(\sigma_{P_1}/\langle \sigma^{P_1}_{I_{12}} \rangle) \times T^* I_{12,\circ}] \right) \# [T^*(\sigma_{P_1}/\langle \sigma^{P_1}_F \rangle) \times T^* F_\circ]$$

induced by the gluing of the zero sections.

By definition $\bar{\pi}_2$ projects the former and the latter parts of (5.2) respectively onto

$$(S_{P_1} \sqcup S_{P_2})\#_{\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle\times\partial I_{12,\circ}}(\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle \times I_{12,\circ}),\ \sigma_{P_1}/\langle\sigma^{P_1}_F\rangle \times F_\circ$$

in the gluing

$$S = \Big((S_{P_1} \sqcup S_{P_2})\#_{\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle\times\partial I_{12,\circ}}(\sigma_{P_1}/\langle\sigma^{P_1}_{I_{12}}\rangle \times I_{12,\circ})\Big)\#_{\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle\times\partial F_\circ}(\sigma_{P_1}/\langle\sigma^{P_1}_F\rangle \times F_\circ).$$

Hence $\bar{\pi}_2$ is compatible with the relevant gluing procedure. As explained in the proof of Lemma 4.13(4), the source $T^*T^d \times T^*S$ contains $\pi_2^{-1}(S)$ as the zero section. Then by Lemma 4.13(1) the restriction of $\bar{\pi}_2$ to $\widetilde{\mathbb{L}}(\Phi_2)$ coincides with π_2. □

5.3. **General case.** Suppose that Theorem 1.2 holds for the subfanifold Φ_{k-1}.

Lemma 5.4. *There is a filtered stratified fibration $\bar{\pi}_k\colon \widetilde{\mathbf{W}}(\Phi_k) \to \Phi_k$ restricting to π_k.*

Proof. It suffices to show the claim when attaching the handle $T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ$ to $\widetilde{\mathbf{W}}(\Phi_{k-1})$ for a single interior k-stratum $S^{(k)} \subset \Phi$ with $k \le \dim S$. Define $\tilde{\pi}_k$ as the map canonically induced by $\tilde{\pi}_{k-1}, \tilde{\pi}_{0,S^{(k)}}$ and the projection $T^*S^{(k)}_\circ \to S^{(k)}_\circ$ to the base, where $\tilde{\pi}_{0,S^{(k)}}\colon T^*\widehat{M}_{S^{(k)}} \to M_{S^{(k)},\mathbb{R}}$ is the projection to the cotangent fibers. Here, we precompose the contraction

$$\mathrm{cont}_k\colon \widetilde{\mathbf{W}}(\Phi_k) \to \widetilde{\mathbf{W}}(\Phi_k)$$

of the cylindrical ends along the negative Liouville flow to the union of $\partial_\infty\mathbf{W}(\Phi_{k-1})$ and $\partial T^*S^{(k)}_\circ$. Let $\mathrm{ret}_k\colon M_{S^{(k)},\mathbb{R}} \to \Sigma_{S^{(k)}}$ be the map induced by a retraction which is the canonical extension of piecewise projections onto facets in $\partial\Phi$ from outwards along their normal directions. Define $\bar{\pi}_k$ as the map canonically induced by $\bar{\pi}_{k-1}, \bar{\pi}_{0,S^{(k)}} = \mathrm{ret}_k \circ \tilde{\pi}_{0,S^{(k)}}$ and the projection $T^*S^{(k)}_\circ \to S^{(k)}_\circ$ to the base.

Recall that the symplectomorphism (4.19) sends the bases of

$$T^*S_P,\ T^*(\sigma_P/\langle\sigma^P_I\rangle),\ T^*(\sigma_P/\langle\sigma^P_F\rangle),\ \ldots$$

and $T^*(\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle)$ to the cotangent fibers of

$$T^*\widehat{M}_P,\ T^*\widehat{M}_I,\ T^*\widehat{M}_F,\ \ldots$$

and $T^*\widehat{M}_{S^{(k)}}$ preserving $T^*I_\circ, T^*F_\circ, \ldots$, where $P, I, F, \ldots$ run through strata of dimension $0, 1, 2, \ldots$ in $\partial S^{(k)}_\circ$ with $P \subset \partial_{in}I$, $P, I \subset \partial_{in}F, \ldots$ as above. In the gluing

$$\widetilde{\mathbf{W}}(\Phi_k) = \mathbf{W}(\Phi_{k-1})\#_{\mathcal{L}_{S^{(k)}}}[T^*\widehat{M}_{S^{(k)}} \times T^*S^{(k)}_\circ] \tag{5.3}$$

the cotangent fibers of the images of T^*S_P get connected through $J^1\mathcal{L}_I$ with the products of the cotangent fibers of the images of $T^*(\sigma_P/\langle\sigma^P_I\rangle)$ and the bases of $T^*I_\circ$. The results get connected through $J^1\mathcal{L}_F$ with the product of the cotangent fibers of the images of $T^*(\sigma_P/\langle\sigma^P_F\rangle)$ and the bases of $T^*F_\circ$. Inductively, the results for $n = k-1$ get connected through $J^1\mathcal{L}_{S^{(k)}}$ with the product of the cotangent fibers of the image of $T^*(\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle)$ and the base of $T^*S^{(k)}_\circ$. In other words, (4.19) respects the gluing (5.3) and the gluing of

$$T^*T^d \times T^*S_P,\ T^*T^d \times T^*(\sigma_P/\langle\sigma^P_I\rangle) \times T^*I_\circ,\ T^*T^d \times T^*(\sigma_P/\langle\sigma^P_F\rangle) \times T^*F_\circ,\ \ldots$$

and $T^*T^d \times T^*(\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle) \times T^*S^{(k)}_\circ$ induced by the gluing of the zero sections.

By definition $\bar{\pi}_k$ projects the former part of (5.3) onto the gluing of

$$S_P,\ \sigma_P/\langle\sigma^P_I\rangle \times I_\circ,\ \sigma_P/\langle\sigma^P_F\rangle \times F_\circ,\ \ldots$$

and the latter part onto $\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle \times S^{(k)}_\circ$ in the gluing S. Hence $\bar{\pi}_k$ is compatible with the relevant gluing procedure. As explained in the proof of Lemma 4.18(4), the source $T^*T^d \times T^*S$ contains

$\pi_k^{-1}(S)$ as the zero section. Then by Lemma 4.18(1) the restriction of $\bar{\pi}_k$ to $\widetilde{\mathbb{L}}(\Phi_k)$ coincides with π_k. □

Remark 5.5. If Σ_S is proper for any $S \subset \Phi$, then $\mathrm{ret}_i, i = 0, 1, \ldots, k$ are trivial and $\bar{\pi}_k = \tilde{\pi}_k$.

Remark 5.6. From the above proof and the definition of fanifolds it follows that $\tilde{\pi}_k$ projects the former part of (5.3) onto the gluing of

$$M_{P,\mathbb{R}},\ M_{I,\mathbb{R}} \times I_\circ,\ M_{F,\mathbb{R}} \times F_\circ,\ \ldots$$

and the latter part onto $M_{S^{(k)},\mathbb{R}} \times S^{(k)}_\circ$ in the gluing

$$\bigsqcup_P M_{P,\mathbb{R}} \# \bigsqcup_I (M_{I,\mathbb{R}} \times I_\circ) \# \bigsqcup_F (M_{F,\mathbb{R}} \times F_\circ) \cdots \# (M_{S^{(k)},\mathbb{R}} \times S^{(k)}_\circ).$$

Hence $\tilde{\pi}_k$ is also compatible with the relevant gluing procedure.

5.4. **The associated integrable system.**

Definition 5.7 ([KS06, Section 3.1])**.** Let (W, ω) be a $2n$-dimensional symplectic manifold, B an n-dimensional manifold and $\varpi\colon W \to B$ a smooth surjective map. A triple (W, ϖ, B) is an *integrable system* if ϖ satisfies

$$\{\varpi^{-1}(f), \varpi^{-1}(g)\} = 0,\ f, g \in C^\infty(B)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on W.

Remark 5.8. Here, we do not require the fibers of ϖ to be compact.

Example 5.9. A collection $(H_1, \ldots, H_n)$ of Hamiltonian functions on W defines a typical example of integrable systems. In particular, on local coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ the projection to the base $(q, p) \mapsto q$ defines an integrable system, as we have

$$\{q_i, q_j\} = \sum_{k=1}^n dq_k \wedge dp_k(Z_{q_i}, Z_{q_j}) = \sum_{k=1}^n dq_k \wedge dp_k(\partial_{p_i}, \partial_{p_j}) = 0$$

where Z_{q_i} are Hamiltonian vector fields for q_i. Similarly, the projection to the cotangent fibers $(q, p) \mapsto p$ defines another integrable system.

Consider the map

$$\tilde{\pi} = \tilde{\pi}_n\colon \widetilde{\mathbf{W}}(\Phi) \to \bigsqcup_P M_{P,\mathbb{R}} \# \bigsqcup_I (M_{I,\mathbb{R}} \times I_\circ) \# \bigsqcup_F (M_{F,\mathbb{R}} \times F_\circ) \cdots \# \bigsqcup_{S^{(n)}} (M_{S^{(n)},\mathbb{R}} \times S^{(n)}_\circ)$$

where $P, I, F, \ldots, S^{(n)}$ run through strata of dimension $0, 1, 2, \ldots, n$ of Φ. If

$$P \subset \partial_{in} I,\ P, I \subset \partial_{in} F,\ \ldots P, I, F, \ldots, S^{(n-1)} \subset \partial_{in} S^{(n)},$$

then by definition of fanifolds

$$\sigma^P_{S^{(n)}}/\langle \sigma^P_I \rangle,\ \sigma^P_{S^{(n)}}/\langle \sigma^P_F \rangle,\ \ldots \sigma^P_{S^{(n)}}/\langle \sigma^P_{S^{(n)}} \rangle$$

regarded as cones in $\Sigma_I, \Sigma_F, \ldots, \Sigma_{S^{(n)}}$ do not depend on the choice of P and we fix some P when such $I, F, \ldots, S^{(n)}$ run. By construction $\tilde{\pi}$ respects the gluing.

Lemma 5.10. *The map $\tilde{\pi}$ defines a filtered stratified integrable system with noncompact fibers.*

Proof. When restricted to each stratum in the filter over $\bar{\Phi}_k \setminus \Phi_{k-1}$, clearly $\widetilde{\mathbf{W}}(\Phi)$ becomes a smooth surjective submersion. Moreover, $\tilde{\pi}$ is the gluing of products of the projection to the cotangent fibers from $T^*\widehat{M}_{S^{(k)}}$ and the projection to the base from $T^*S^{(k)}_\circ$. Hence the restriction is an integrable system defined by a collection of Hamiltonian functions. □

Consider the map

$$\bar{\pi} = \bar{\pi}_n \colon \widetilde{\mathbf{W}}(\Phi) \to \Phi.$$

In general, $\bar{\pi}$ does not define an integrable system. For instance, on 0-strata $P_1, \dots, P_4$ of the fanifold from Example 2.12 it is not even C^1. Nevertheless, from construction and Remark 5.5 for $k = n$ it follows

Lemma 5.11. *The map $\bar{\pi}$ is homotopic to $\tilde{\pi}$. If Σ_S is proper for any $S \subset \Phi$, then the homotopy becomes trivial and $\bar{\pi}$ defines a filtered stratified integrable system with noncompact fibers.*

6. SYZ picture

6.1. **The associated dual stratified spaces.** First, we recall a fundamental piece of B-side SYZ fibrations over stratified spaces dual in a certin sense to fanifolds. Let X_Σ be the n-dimensional toric variety associated with a fan $\Sigma \subset M_\mathbb{R}$ for a lattice $M \cong \mathbb{Z}^n$. Consider the map $X_\Sigma \to (X_\Sigma)_{\geq 0}$ induced by the retraction to the nonnegative real points. By [CLS11, Proposition 12.2.3] the fiber over a point of $O(\sigma)_{\geq 0}$ for each cone $\sigma \in \Sigma$ is isomorphic to $T^{n - \dim \sigma}$. Here, $O(\sigma)$ is the orrbit corresponding to σ via [CLS11, Theorem 3.2.6]. Assume that Σ is the normal fan of a very ample full dimensional lattice polytope Q. Then we have the algebraic moment map $X_\Sigma \to M^\vee_\mathbb{R}$. By [CLS11, Theorem 12.2.5] the image of its restriction $(X_\Sigma)_{\geq 0} \to M^\vee_\mathbb{R}$ is homeomorphic to Q. Hence the composition

$$\mathrm{mom}_Q \colon X_\Sigma \to (X_\Sigma)_{\geq 0} \to Q \tag{6.1}$$

gives a stratified torus fibration. Note that Q gives the dual cell complex to Σ. Since mom_Q is compatible with taking subfans, one can also define it when Q is noncompact.

Now, in order to define the associated dual stratified space, we assume Φ to satisfy the following additional condition.

(vi) There is a collection of full dimensional lattice polytopes $Q_P \subset M^\vee_{P,\mathbb{R}}$ such that Σ_P are subfans of the normal fans of Q_P and for some collection $\{l_P\}_{P \in \Phi_0}$ of integers $l_P Q_P$ are very ample and $\mathrm{mom}_{l_P Q_P}(X_{\Sigma_P})$ glue along the inclusions obtained by the definition of fanifolds.

Here, we explain more about the condition (vi). Given a fanifold Φ, we have the disjoint union of toric varieties X_{Σ_P} associated with the fans $\Sigma_P \subset M_{P,\mathbb{R}}$ for all 0-strata. Consider an exit path

$$P \to I \to F \to \cdots \to S^{(k)}.$$

By definition of fanifolds, we have the sequence of quotients

$$T_P\mathcal{M} \cong M_{P,\mathbb{R}} \to T_I\mathcal{M}|_P \cong M_{I,\mathbb{R}} \to T_F\mathcal{M}|_P \cong M_{F,\mathbb{R}} \to \cdots \to T_{S^{(k)}}\mathcal{M}|_P \cong M_{S^{(k)},\mathbb{R}}.$$

Fixing inner products, we obtain the sequence of inclusions

$$M^\vee_{S^{(k)},\mathbb{R}} \hookrightarrow \cdots \hookrightarrow M^\vee_{F,\mathbb{R}} \hookrightarrow M^\vee_{I,\mathbb{R}} \hookrightarrow M^\vee_{P,\mathbb{R}},$$

which induces a sequence of inclusions of cones

$$(\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle)^\perp \hookrightarrow \cdots \hookrightarrow (\sigma_P/\langle\sigma^P_F\rangle)^\perp \hookrightarrow (\sigma_P/\langle\sigma^P_I\rangle)^\perp \hookrightarrow (\sigma_P)^\perp$$

for the cone $\sigma_P \in \Sigma_P$ corresponding to $S^{(k)}$. Note that

$$(\sigma_P)^\perp, (\sigma_P/\langle\sigma^P_I\rangle)^\perp, (\sigma_P/\langle\sigma^P_F\rangle)^\perp, \dots, (\sigma_P/\langle\sigma^P_{S^{(k)}}\rangle)^\perp$$

can be regarded as fibers of $T_P\mathcal{M}, T_I\mathcal{M}, , T_F\mathcal{M}, \cdots, T_{S^{(k)}}\mathcal{M}$.

Another exit path

$$P' \to I \to F \to \cdots \to S^{(k)}$$

induces a sequence of inclusions of cones

$$(\sigma_{P'}/\langle\sigma^{P'}_{S^{(k)}}\rangle)^\perp \hookrightarrow \cdots \hookrightarrow (\sigma_{P'}/\langle\sigma^{P'}_F\rangle)^\perp \hookrightarrow (\sigma_{P'}/\langle\sigma^{P'}_I\rangle)^\perp \hookrightarrow (\sigma_{P'})^\perp$$

for the cone $\sigma_{P'} \in \Sigma_{P'}$ corresponding to $S^{(k)}$. Then the identifications

$$(\sigma_P/\langle\sigma^{P}_{S^{(k)}}\rangle)^\perp = (\sigma_{P'}/\langle\sigma^{P'}_{S^{(k)}}\rangle)^\perp,\ \ldots, (\sigma_P/\langle\sigma^{P}_F\rangle)^\perp = (\sigma_{P'}/\langle\sigma^{P'}_F\rangle)^\perp,\ (\sigma_P/\langle\sigma^{P}_I\rangle)^\perp = (\sigma_{P'}/\langle\sigma^{P'}_I\rangle)^\perp$$

from the definition of fanifolds give a gluing datum for $(\sigma_P)^\perp$ and $(\sigma_{P'})^\perp$. Identifying $(\sigma_P)^\perp$ with $\mathrm{mom}_{l_P Q_P}(\sigma_P)$ for each $P \in \Phi_0$, the condition (vi) requires such gluing data to be compatible with all exit paths.

Definition 6.1. Let Φ be a fanifold satisfying the additional condition (vi). We define its *associated dual stratified space* Ψ as the gluing of $\mathrm{mom}_{l_P Q_P}(X_{\Sigma_P})$ with the canonical stratification in a sufficiently large ambient space $\mathcal{N}$. For a k-stratum $S^{(k)}$ of Φ, its *dual stratum* $S^{(k),\perp}$ of Ψ is the stratum defined by the cone $\sigma_P \in \Sigma_P$ corresponding to $S^{(k)}$.

Example 6.2. Let Φ be the fanifold from Example 2.12. Adding rays to P_1, P_2, P_3, P_4 parallel to vectors $(-1,1), (-1,-1), (1,-1), (1,1)$, we obtain the normal fans $\Sigma_{Q_{P_i}}$ of full dimensional lattice polytopes Q_{P_i}. Since we have $\dim M_{P_i,\mathbb{R}} = 2$, by [CLS11, Corollary 2.2.19] any full dimensional lattice polytope is very ample. Then $\mathrm{mom}_{Q_{P_i}}(X_{\Sigma_{P_i}})$ glue to yield a stratified space $\Psi \subset M^\vee_{\mathbb{R}}$. Its 0-stratum $F^\perp$ is a point $(1/2, 1/2)$, 1-strata $I^\perp_{12}, I^\perp_{23}, I^\perp_{34}, I^\perp_{14}$ are defined by rays from $F^\perp$ parallel to vectors $(-1,0), (0,-1), (1,0), (0,1)$, and 2-strata $P^\perp_1, P^\perp_2, P^\perp_3, P^\perp_4$ are defined by quadrants with the origin placed at $F^\perp$ bounded by the pairs $(I^\perp_{12}, I^\perp_{14}), (I^\perp_{12}, I^\perp_{23}), (I^\perp_{23}, I^\perp_{34}), (I^\perp_{14}, I^\perp_{34})$.

Lemma 6.3. *Let Φ be a fanifold of dimension n satisfying the additional condition (vi). Then it admits a filtration*

$$\Psi_0 \subset \Psi_1 \subset \cdots \subset \Psi_n = \Psi \tag{6.2}$$

where Ψ_k is a stratified subspace defined as the complement in Ψ of k-skeleta $\mathrm{Sk}_{n-k-1}(\Psi)$, the closure of the subset of $n-k-1$-strata.

Proof. It follows immediately from the construction of Ψ. □

In [GS1, Section 3], to a fanifold Φ Gamage–Shende associated the colimit

$$\mathbf{T}(\Phi) = \varinjlim_S X_{\Sigma_S}$$

along closed embeddings induced by quotient maps between fans, where S runs through all strata of Φ. By [GS1, Proposition 3.10] the colimit $\mathbf{T}(\Phi)$ always exists as an algebraic space. We will recall later that $\mathbf{T}(\Phi)$ is a mirror partner of $\widetilde{\mathbf{W}}(\Phi)$. When Φ has the associated dual stratified space, B-side SYZ fibration over Ψ for the pair $(\widetilde{\mathbf{W}}(\Phi), \mathbf{T}(\Phi))$ should be the following.

Definition 6.4. Let Φ be a fanifold satisfying the additional condition (vi). We define

$$\mathrm{mom}_\Phi \colon \mathbf{T}(\Phi) \to \Psi$$

as the gluing of $\mathrm{mom}_{l_P Q_P}$ where P runs through all 0-strata of Φ.

6.2. **The proof of Theorem 1.3.** Suppose that Φ satisfies the additional condition (vi). By Lemma 6.3 the dual space Ψ admits a filtration (6.2). Now, we construct a fibration $\underline{\pi} \colon \widetilde{\mathbf{W}}(\Phi) \to \Psi$ as an integrable system with noncompact fibers, which should be SYZ dual to mom_Φ. We proceed by induction on k.

Definition 6.5. We define

$$\underline{\pi}_0 \colon \widetilde{\mathbf{W}}(\Phi_0) \to \Psi_0$$

as the composition of $\tilde{\pi}_0$ with the disjoint union of diffeomorphisms $T^*_\theta \widehat{M}_P \cong M_{P,\mathbb{R}} \cong P^\perp$. Here, the first and the second diffeomorphisms respectively follow from the fixed identifications and the definition of Ψ.

Remark 6.6. The map $\underline{\pi}$ is nothing but the disjoint union of moment maps

$$\mathrm{Log}_P\colon T^*\widehat{M}_P \cong (\mathbb{C}^*)^n \to \mathbb{R}^n,\ (\theta, \xi) \mapsto \xi$$

for the lifts to $T^*\widehat{M}_P$ of the self $\widehat{M}_P$-actions.

Suppose that we construct $\underline{\pi}_{k-1}$ for the subfanifold Φ_{k-1}.

Definition 6.7. We define

$$\underline{\pi}_k\colon \widetilde{\mathbf{W}}(\Phi_k) \to \Psi_k$$

as follows. First, consider the map canonically induced by $\underline{\pi}_{k-1}, \underline{\pi}_{0,S^{(k)}}$ and the projections $T^*S^{(k)}_\circ \to S^{(k)}_\circ$ to the base, where $\underline{\pi}_{0,S^{(k)}}$ are the compositions of $\tilde{\pi}_{0,S^{(k)}}$ with the diffeomorphisms $T^*_\theta \widehat{M}^{(k)}_S \cong M_{S^{(k)},\mathbb{R}} \cong S^{(k),\perp}$. Here, we regard the images $S^{(k)}_\circ$ as a fiber of the normal bundles $T_{S^{(k)},\perp}\mathcal{N}$. Next, contract the images $S^{(k)}_\circ$ to obtain $\underline{\pi}_k$.

Remark 6.8. Since we contract $T^*S^{(k)}_\circ$ to single points when defining $\underline{\pi}_k$, by Remark 5.6 one can regard $\underline{\pi}_k$ as the gluing of $\underline{\pi}_{k-1}$ with $\underline{\pi}_{0,S^{(k)}}$. As the contraction of the images $S^{(k)}_\circ$ corresponds to shrinking the Lagrangians $\widehat{M}_{S^{(k)}} \times S^{(k)}_\circ$ in the gluing $\widetilde{\mathbb{L}}(\Phi_k)$, the image of $\underline{\pi}_k$ is homeomorphic to that of $\tilde{\pi}_k$.

We denote $\underline{\pi}_n$ by $\underline{\pi}$. From Remark 6.8 for $k = n$ one sees that $\underline{\pi}$ is the gluing of the disjoint unions of the projections to the cotangent fibers from $T^*\widehat{M}_P$ along the canonical inclusions induced by

$$\widehat{M}_{S^{(n)}} \hookrightarrow \ldots \hookrightarrow \widehat{M}_F \hookrightarrow \widehat{M}_I \hookrightarrow \widehat{M}_P,$$

which are obtained by the definition of fanifolds. Here, $P, I, F, \ldots, S^{(n)}$ run through strata of dimension $0, 1, 2, \ldots, n$ of Φ. with $P \subset \partial_{in} I, P, I \subset \partial_{in} F, \ldots, P, I, F, \ldots, S^{(n-1)} \subset \partial_{in} S^{(n)}$.

Lemma 6.9. *The map $\underline{\pi}$ defines a stratified fibration, whose fiber over a point of any k-stratum $S^{(k),\perp}$ is given by $T^k \times T^*S^{(k)}$.*

Proof. It immediately follows from definition and Remark 6.8 □

6.3. **Review on very affine hypersurfaces.** First, we recall HMS for very affine hypersurfaces established by Gamage–Shende. Let $\Sigma \subset M_\mathbb{R} \cong \mathbb{R}^{n+1}$ be a smooth quasiprojective stacky fan [GS15, Definition 2.4] whose primitive ray generators span a convex lattice polytope $\Delta^\vee$ containing the origin. Then Σ defines a smooth Deligne–Mumford stack $\mathbf{T}_\Sigma$ [GS15, Definition 2.5] with toric boundary divisor $\partial\mathbf{T}_\Sigma$ and an adapted star-shaped triangulation $\mathcal{T}$ of $\Delta^\vee$ [GS2, Definition 3.3.1]. Recall that a triangulation $\mathcal{T}$ is *adapted* if there is a convex piecewise function $\mu\colon \Delta^\vee \to \mathbb{R}$ whose corner locus is $\mathcal{T}$. We denote by $\mathbb{T}_\mathbb{C}$ the complex torus $M_\mathbb{R}/M \otimes_\mathbb{R} \mathbb{C}$ acting on $\mathbf{T}_\Sigma$, where $\mathbb{T}$ is a real $(n+1)$-dimensional torus with character lattice $M^\vee$ and cocharacter lattice M.

Consider a Laurent polynomial

$$W_t\colon \mathbb{T}^\vee_\mathbb{C} \to \mathbb{C},\ z \mapsto \sum_{\alpha \in \mathrm{Vert}(\mathcal{T})} c_\alpha t^{-\mu(\alpha)} z^\alpha$$

in coordinates $z = (z_1, \ldots, z_{n+1})$ on $\mathbb{T}^\vee_\mathbb{C}$, where $c_\alpha \in \mathbb{C}^*$ are arbitrary constants and $t \gg 0$ is a tropicalization parameter. For sufficiently general t, the hypersurface $H_t = W_t^{-1}(0)$ is smooth and called *very affine*. Restricting the canonical Liouville structure on $\mathbb{T}^\vee_\mathbb{C} \cong T\mathbb{T}^\vee_\mathbb{C} \cong T^*\mathbb{T}^\vee$ for a fixed inner product, we may regard H_t as a Liouville submanifold.

Theorem 6.10 ([GS2, Theorem 1.0.1])**.** *There is an equivalence*

$$\mathrm{Fuk}(H_t) \simeq \mathrm{Coh}(\partial \mathbf{T}_\Sigma)$$

between the wrapped Fukaya category on H_t and the dg category of coherent sheaves on $\partial \mathbf{T}_\Sigma$.

Next, following [AAK16, Section 3], we explain SYZ fibrations associated with mirror symmetry for very affine hypersurfaces. For simplicity, we temporarily assume that Σ is an ordinary simplicial fan. Consider the moment map

$$\mathrm{Log}\colon \mathbb{T}^\vee_\mathbb{C} \to \mathbb{R}^{n+1},\ z \mapsto (\log|z_1|, \dots, \log|z_{n+1}|)$$

for the lift to $T^*\mathbb{T}^\vee$ of the self $\mathbb{T}^\vee$-action. The image $\Pi_t = \mathrm{Log}(H_t)$ is called the *amoeba* of H_t.

Definition 6.11. The *tropical hypersurface* Π_Σ associated with H_t is the hypersurface defined by the *tropical polynomial*

$$\varphi\colon M^\vee_\mathbb{R} \to \mathbb{R},\ \varphi(m) = \max\{\langle m, n\rangle - \mu(n) \mid n \in \Delta^\vee\}.$$

Namely, Π_Σ is the set of points where the maximum is achieved more than once.

It is known that Π_Σ is a deformation retract of Π_t. According to [Mik04, Corollary 6.4], when $t \to \infty$ the rescaled amoeba $\Pi_t/\log t$ converges to Π_Σ. Combinatorially, Π_Σ is the dual cell complex of $\mathcal{T}$. In particular, the set of connected components of $\mathbb{R}^{n+1} \setminus \Pi_\Sigma$ bijectively corresponds to the set $\mathrm{Vert}(\mathcal{T})$ of vertices of $\mathcal{T}$ according to which $\alpha \in \mathrm{Vert}(\mathcal{T})$ achieves the maximum of $\langle m, \alpha\rangle - \mu(\alpha)$ for $m \in \mathbb{R}^{n+1} \setminus \Pi_\Sigma$. Note that $\mathbb{R}^{n+1} \setminus \Pi_t$ for $t \gg 0$ has the same combinatrics as $\mathbb{R}^{n+1} \setminus \Pi_\Sigma$. Since we assume $\Delta^\vee$ to contain the origin, each connected component of $\mathbb{R}^{n+1} \setminus \Pi_\Sigma$ is the locus where the monomial $c_\alpha t^{-\mu(\alpha)} z^\alpha$ becomes dominant. In the sequel, we will fix a general $t \gg 0$ and drop t from the notation.

Let $\mathrm{ret}\colon \Pi \to \Pi_\Sigma$ be the continuous map induced by the retraction. Then the composition

$$H \hookrightarrow \mathbb{T}^\vee_\mathbb{C} \xrightarrow{\mathrm{Log}} \Pi \xrightarrow{\mathrm{ret}} \Pi_\Sigma \tag{6.3}$$

gives the A-side SYZ fibration. Recall that H admits a pants decomposition [Mik04, Theorem 1']. By [Mik04, Proposition 4.6] the k-th intersection of $i_1, \cdots, i_k$-th legs [GS2, Definition 5.2.2] of an n-dimensional tailored pants $\tilde{P}_n$ is isomorphic to a product

$$\mathbb{C}^*_{z_{i_1}} \times \cdots \times \mathbb{C}^*_{z_{i_k}} \times \tilde{P}_{n-k}.$$

Under (6.3) a k-th intersection of legs maps to a subset of k-stratum away from lower dimensional strata. In particular, the fiber over a general point of a k-stratum contains T^k. On the other hand, the B-side SYZ fibration is induced by the map from the total space of the anticanonical sheaf on $\mathbf{T}_\Sigma = X_\Sigma$ defined as (6.1).

Now, we return to the case where Σ is a smooth quasiprojective stacky fan. Since $\mathcal{T}$ is star-shaped, each $(n+1)$-simplex is a polytope

$$\Delta^\vee_\sigma = \mathrm{Conv}(0, \alpha^1, \dots, \alpha^n) \subset M_\mathbb{R}$$

spanned by the origin and primitive ray generators $\alpha^1, \dots, \alpha^n$ of some maximal dimensional cone σ. Consider the map

$$\Delta^\vee_{n+1} \to \Delta^\vee_\sigma,\ e_i \mapsto \alpha^i$$

from the standard $(n+1)$-simplex with the standard basis $e_1, \dots, e_n$, whose dual induces a map $f_{\Delta^\vee_\sigma}\colon \mathbb{T}^\vee_\mathbb{C} \to \mathbb{T}^\vee_\mathbb{C}$.

Definition 6.12 ([GS2, Definition 5.1.4]). The $\Delta_\sigma^\vee$-*pants* is the inverse image $\tilde{P}_{\Delta_\sigma^\vee} = f_{\Delta_\sigma^\vee}^{-1}(\tilde{P}_n)$ of the tailored pants. We write $\tilde{A}_{\Delta_\sigma^\vee}$ for its amoeba $\mathrm{Log}(\tilde{P}_{\Delta_\sigma^\vee})$. For $b = (b_1, \ldots, b_{n+1}), b_i \gg 0$ we denote by $\tilde{P}^b_{\Delta_\sigma^\vee}$ and $\tilde{A}^b_{\Delta_\sigma^\vee}$ respectively the *translated tailored* $\Delta_\sigma^\vee$-*pants* obtained by scaling the coefficients of the defining polynomial by e^{-b_i} and its amoeba obtained by translation to the first orthant.

Lemma 6.13 ([GS2, Lemma 5.3.6]). *Let $\partial^0 \tilde{A}^b_{\Delta_\sigma^\vee}$ be the component of $\partial \tilde{A}^b_{\Delta_\sigma^\vee}$ which bounds the region of $\mathbb{R}^{n+1}$ containing the inverse image of the all-negative orthant. Restrict the canonical Liouille structure on $T^*\mathbb{T}^\vee$ to $\tilde{P}^b_{\Delta_\sigma^\vee}$. Let $\Sigma_\sigma \subset \Sigma$ be the stacky subfan whose primitive ray generators are $\alpha_1, \ldots, \alpha_n$. Then we have*

$$\mathrm{Core}(\tilde{P}^b_{\Delta_\sigma^\vee}) = \mathrm{Log}^{-1}(\partial^0 \tilde{A}^b_{\Delta_\sigma^\vee}) \cap \mathbb{L}(-\Sigma_\sigma).$$

From [GS2, Theorem 6.2.4] it follows that $\mathrm{Core}(H)$ is the gluing of $\mathrm{Core}(\tilde{P}^b_{\Delta_\sigma^\vee})$ for all $\sigma \in \Sigma_{\max}$. Hence in this case the fiber of (6.3) over a point of k-stratum away from lower dimensional strata becomes isomorphic to a product

$$\mathbb{C}^*_{z_{i_1}} \times \cdots \times \mathbb{C}^*_{z_{i_k}} \times \tilde{P}^b_{\Delta^\vee_{\sigma/\langle i_1,\ldots,i_k\rangle}}.$$

Here, $\tilde{P}^b_{\Delta^\vee_{\sigma/\langle i_1,\ldots,i_k\rangle}}$ is the k-th intersection of $i_1, \cdots, i_k$-th legs [GS2, Definition 5.2.5] of $\tilde{P}^b_{\Delta_\sigma^\vee}$ with

$$\Delta^\vee_{\sigma/\langle i_1,\ldots,i_k\rangle} = \mathrm{Conv}(0, \alpha^1, \ldots, \hat{\alpha}^{i_1}, \ldots, \hat{\alpha}^{i_k}, \ldots, \alpha^n).$$

On the other hand, the B-side SYZ fibration becomes the composition of the structure morphism $\mathbf{T}_\Sigma \to X_\Sigma$ to the coarse moduli space with the map defined as (6.1). The fiber over any point of a k-stratum is the real part of the corresponding subgroup of the Deligne–Mumford torus [FMN10, Definition 2.4, Proposition 2.6] acting on $\mathbf{T}_\Sigma$.

6.4. **The proof of Theorem 1.4.** For a smooth quasiprojective stacky fan $\Sigma \subset M_\mathbb{R}$, consider the fanifold $\Phi = \Sigma \cap S^{n+1}$ from [GS1, Example 4.22] generalized as in [GS1, Section 6]. In particular, to each 0-stratum P we associate the stacky fan $\Sigma_P = \Sigma/\rho_P$ where $\rho_P \in \Sigma$ is the ray passing through P. Hence $\mathbf{T}(\Phi)$ coincide with the toric boundary divisor $\partial\mathbf{T}_\Sigma$ of $\mathbf{T}_\Sigma$. Fix a stratified homeomorphism $h_\Phi \colon \partial\Delta^\vee \to \Phi$. Then strata of $\partial\Delta^\vee$ inherit the labels from Φ which induces the labels on strata of Π_Σ.

Definition 6.14. Let Π^0_Σ be the connected component of $\mathbb{R}^{n+1} \setminus \Pi_\Sigma$ corresponding to the origin of $\Delta^\vee$. We define Ψ as the image of a fixed stratified homeomorphism $h_\Psi \colon \partial\Pi^0_\Sigma \hookrightarrow S^{n+1} \subset M^\vee_\mathbb{R}$, which makes the inherited labels on strata of Ψ compatible with that on strata of Φ transported to $M^\vee_\mathbb{R}$ via a fixed inner product.

Note that $\partial\Pi^0_\Sigma$ is the image of the composition

$$\partial\mathbf{T}_\Sigma \hookrightarrow \mathbf{T}_\Sigma \to X_\Sigma \to (X_\Sigma)_{\geq 0} \to M^\vee_\mathbb{R}.$$

Hence Φ satisfies (vi) and has the dual stratified space Ψ. In particular, the B-side fibration $\mathbf{T}(\Phi) \to \Psi$ is the gluing of the compositions $\mathbf{T}_{\Sigma_P} \to X_{\Sigma_P} \to Q_P$ and compatible with that $\partial\mathbf{T}_\Sigma \to \partial\Pi^0_\Sigma$ for very affine hypersurface. Namely, we have the commutative diagram

$$\begin{tikzcd} \mathbf{T}(\Phi) \arrow[r, equal] \arrow[d] & \partial\mathbf{T}_\Sigma \arrow[d] \\ \Psi \arrow[r, "h_\Psi^{-1}"] & \partial\Pi^0_\Sigma. \end{tikzcd}$$

Lemma 6.15. *There is a stratified homeomorphism of the image of* $\mathrm{Core}(H)$ *under the composition* $\mathrm{ret} \circ \mathrm{Log}$ *and Φ transported to $M^\vee_\mathbb{R}$ via the fixed inner product.*

Proof. As explained above, $\mathrm{Core}(H)$ is the gluing of $\mathrm{Core}(\tilde{P}^b_{\Delta^\vee_\sigma})$ for all $\sigma \in \Sigma_{\max}$. By Lemma 6.13 each piece is the Legendrian boundary $\partial_\infty \mathbb{L}(-\Sigma_\sigma)$ of the FLTZ Lagrangian $\mathbb{L}(-\Sigma_\sigma)$ associated with the stacky subfan $\Sigma_\sigma \subset \Sigma$. Unwinding the proof of [Nad, Theorem 5.13], one sees that $\mathrm{ret} \circ \mathrm{Log}$ projects each k-dimensional torus in $\partial_\infty \mathbb{L}(-\Sigma_\sigma)$ to a point of the corresponding cone in $\Sigma_\sigma(n+1-k)$ transported to $M^\vee_{\mathbb{R}}$. Such k-dimensional tori are connected with lower dimensional tori by the Liouville flow. Hence the composition

$$h_\Psi \circ \mathrm{ret} \circ \mathrm{Log} \colon \mathrm{Core}(H) \to \Psi$$

projects each k-dimensional torus in $\mathrm{Core}(H)$ to a point of the corresponding $(n-k)$-stratum of Φ transported to $M^\vee_{\mathbb{R}}$. Note that tori in different $\partial_\infty \mathbb{L}(-\Sigma_\sigma)$ mapping to the same point get identified in $\mathrm{Core}(H)$. The images of k-dimensional tori are connected with that of lower dimensional tori by the images of the Liouville flow, which become parallel to the corresponding strata of Φ. Thus one can find a homeomorphism of the image of each stratum of $\mathrm{Core}(H)$ under $h_\Psi \circ \mathrm{ret} \circ \mathrm{Log}$ to the corresponding stratum of Φ transported to $M^\vee_{\mathbb{R}}$. By construction such homomorphisms glue to yield the desired map. □

Corollary 6.16. *There is a diffeomorphism of* $\mathbb{L}(\Phi)$ *and* $\mathrm{Core}(H)$ *over* Φ *transported to* $M^\vee_{\mathbb{R}}$.

Proof. For any k-stratum $S^{(k)}_P$ of Φ adjacent to a 0-stratum P, its inverse image under the restriction $h_\Psi \circ \mathrm{ret} \circ \mathrm{Log}|_{\mathrm{Core}(H)}$ is diffeomorphic to the zero section of $T^*\widehat{M}_{S^{(k)}_P} \times T^* S^{(k)}_P$. By construction

$$(h_\Psi \circ \mathrm{ret} \circ \mathrm{Log}|_{\mathrm{Core}(H)})^{-1}(S^{(k)}_{P_\circ}),\ \widehat{M}_{S^{(k)}_P} \times S^{(k)}_{P,\circ}$$

respectively glue along the boundaries to yield $\mathrm{Core}(H), \mathbb{L}(\Phi)$. □

Corollary 6.17. *There is a symplectomorphism of pairs*

$$(\mathbf{W}(\Phi), \mathbb{L}(\Phi)) \hookrightarrow (H, \mathrm{Core}(H)) \tag{6.4}$$

over Ψ *with respect to* $\underline{\pi}$ *and a certain modification of* $h_\Psi \circ \mathrm{ret} \circ \mathrm{Log}$.

Proof. Contracting cotangent fibers along the negative Liouville flow if necessary, one can make $(T^*\widehat{M}_{S^{(k)}_P} \times T^* S^{(k)}_P, \widehat{M}_{S^{(k)}_P} \times S^{(k)}_P)$ symplectomorphic to

$$(\mathrm{Nbd}((h_\Psi \circ \mathrm{ret} \circ \mathrm{Log}|_{\mathrm{Core}(H)})^{-1}(S^{(k)}_P)), (h_\Psi \circ \mathrm{ret} \circ \mathrm{Log}|_{\mathrm{Core}(H)})^{-1}(S^{(k)}_P))$$

for any k-stratum $S^{(k)}_P$ of Φ adjacent to a 0-stratum P. By construction they glue to yield a symplectomorphism of pairs

$$(\mathbf{W}(\Phi), \mathbb{L}(\Phi)) \hookrightarrow (H, \mathrm{Core}(H))$$

over the pair $(\tilde{\pi}(\mathbf{W}(\Phi)), \Phi)$. Since it is an embedding, the same modification as in the proof of Theorem 1.3 yields compatible fibrations over Ψ. □

References

[AAK16] M. Abouzaid, D. Auroux, and L. Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. IHÉS. 123-199 (2016).

[CE12] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and back*, Simplectic Geometry of Affine Complex Manifolds, Colloquim publication Vol. 59, AMS (2012).

[CLS11] D. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics 124, AMS (2011).

[Eli] Y. Eliashberg, *Weinstein manifolds revisited*, arXiv:1707.03442

[FLTZ12] B. Fang, C. Melissa Liu, D. Treumann and E. Zaslow. *T-duality and homological mirror symmetry for toric varieties*, Adv. Math. 229(3), 1873 – 1911 (2012).

[FLTZ14] B. Fang, C. Melissa Liu, D. Treumann and E. Zaslow. *The coherent-constructible correspondence for toric Deligne-Mumford Stacks*, Int. Math. Res. 2014(4), 914–954 (2014).

[FMN10] B. Fantechi, E. Mann, and F. Nironi, *Smooth toric Deligne–Mumford stacks*, J. Reine Angew. Math. 648, 201-244 (2010).

[GPS1] S. Ganatra, J. Pardon, and V. Shende, *Covariantly functorial wrapped Floer theory on Liouville sectors*, Publ. math. IHES. 131, 73–200 (2020).

[GPS3] S. Ganatra, J. Pardon and V. Shende, *Microlocal Morse theory of wrapped Fukaya categories*, arXiiv:1809.08807v2

[GS11] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, Ann. of Math. 174(3) 1301-1428 (2011).

[GS15] A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans*, Trans. Amer. Math. Soc. 367(2), 1033–1071 (2015).

[GS1] B. Gammage and V. Shende, *Homological mirror symmetry at large volume*, arXiv:2104.11129

[GS2] B. Gammage and V. Shende, *Mirror symmetry for very affine hypersurfaces*, arXiv:1707.02959v3

[Kon00] M. Kontsevich, *Homological algebra of mirror symmetry*, In: Vol. 1 of Proceedings of ICM. Birkhäuser Boston, Inc., Boston, MA (1995).

[KS00] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, In: Symplectic geometry and mirror symmetry (Seoul, 2000). World Sci. Publ., River Edge, NJ (2001).

[KS06] M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, In: Vol. 244 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA (2006).

[Mik04] G. Mikhalkin, *Decomposition into pairs of pants for complex algebraic hypersurfaces*, Topology. 43(5), 1035-1065 (2004).

[Nad] D. Nadler *Wrapped microlocal sheaves on pairs of pants*, arXiv:1604.00114

[SYZ96] A. Strominger, S.T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. B. 479(1-2), 243–259 (1996).

SISSA, via Bonomea 265, 34136 Trieste, Italy
Email address: hmorimur@sissa.it