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Discrete Group Actions and Spectral Geometry of Crossed Products

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Abstract

The (twisted) crossed product construction is fundamental in the theory of C^*-algebras and in noncommutative topology since it represents the operation of forming a quotient when this is a singular, badly-behaved space. For instance, the study of noncommutative coverings, in the special case of finite abelian structure groups, shows that twisted crossed products are the noncommutative analogue of topological regular coverings. Since spectral triples are a central notion in noncommutative geometry, this makes the task of constructing spectral triples on crossed products a natural subject of interest.

In this thesis, we construct and study spectral triples on reduced twisted crossed products $A \rtimes^{\sigma}_{\alpha,r} G$, where A is a unital C^*-algebra, G a discrete group and (α, σ) a twisted action in the sense of Busby and Smith [19]. For this construction we follow, as in [47], the guiding principle of the Kasparov external product, combining the given Dirac operator on A with a matrix valued length-type function on the group. In particular, we provide sufficient conditions so that this triple on $A \rtimes^{\sigma}_{\alpha,r} G$ satisfies some of the *axioms* of noncommutative manifolds [23]: summability, regularity, compatibility with real structures, first and second order conditions and orientation cycles.

Our guide example is the spectral triple on the noncommutative 2-torus [44, 95], regarded as the crossed product $C(S^1) \rtimes \mathbb{Z}$. We show that our constructions generalize the usual ones on its triple.

Declaration of Authorship

The work in this thesis is my own except where otherwise stated. Some of the original material has appeared previously in the following articles:

(1) A. Rubin and L. Dąbrowski. “Real Spectral Triples on Crossed Products”. Preprint arXiv:2012.15698 (2020).

(2) P. Antonini, D. Guido, T. Isola and A. Rubin. “A note on twisted crossed products and spectral triples”. Preprint arXiv:2110.05345 (2021).

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Chapter 1

Introduction

1.1 Background and History

The notion of a "spectral triple" (or unbounded Fredholm module) was introduced by Alain Connes in the course of studying a generalization of the Atiyah-Singer index theorem to noncommutative spaces. Its prototype is given by the commutative $*$-algebra $C^\infty(M)$ of smooth functions on a compact spin manifold M and the Dirac operator on the Hilbert space of square-integrable spinors. Under precise additional assumptions, any commutative spectral triple must be of this form and so it is possible to recover the original manifold and its geometry from these data [23, 26, 94]. The range of applicability of this paradigm is vast, going from the foundational example of spin manifolds to foliated manifolds, group C^*-algebras, quantum groups, quantum deformations and fractals. This list is not exhaustive and we direct the reader to [20, 24, 25, 28, 44] and the references therein for more information.

So far, most research has focused on investigating the properties of particular known spectral triples. However, despite their importance and extensive study, it is not yet fully understood under what conditions it is possible to define a spectral triple on a given C^*-algebra and examples have been constructed only for some specific classes of them, as for instance in [35, 41, 42, 63, 70, 73, 76, 77, 80].

An example of a noncommutative space which does not yet have a fully satisfactory description of its spectral geometry is the (twisted) crossed product of a C^*-algebra A with a locally compact group G. This construction is of particular interest in noncommutative topology since it is recognized as the right noncommutative generalization of a quotient space by a group action, even when the quotient is a singular, badly-behaved space. Indeed, it is known that given a group G acting on a locally compact and Hausdorff topological space X, the quotient X/G can easily fail to be also locally compact and Hausdorff. However, when the action is free and proper, the quotient X/G has these properties and the algebra $C_0(X/G)$ of continuous functions vanishing at infinity is Morita equivalent to the maximal crossed product $C_0(X) \rtimes G$, meaning that they have the same topological information (e.g. isomorphic K-theory). The advantage of the crossed product is that it may distinguish two quotients even when these are not locally compact and Hausdorff, a lack that prevents the use of the standard Gelfand machinery. Furthermore, it naturally retains the information of the isotropy of the action. A more detailed discussion can be found in [61, Chapter 2].

Twisting the crossed product construction with a 2-cocycle $\sigma\colon G \rtimes G \to A$ is also a rather natural geometric operation since studying noncommutative coverings with finite abelian structure group one discovers that any twisted crossed product is a regular covering and any regular covering is a twisted crossed product, c.f. [5, 107]. This makes the task of the construction of spectral triples on crossed products a natural subject of interest.

The first example of a spectral triple on a crossed product was introduced in [22] using a length function ℓ on a discrete group G to define a multiplication operator by ℓ on $\ell^2(G)$. The reduced group algebra $C^*_r(G)$, which is in fact a crossed product with the trivial C^*-algebra $\mathbb{C}$, is taken acting on $\ell^2(G)$ via the left regular representation. There Connes also develops a notion of *metric* on a compact noncommutative space and shows that there are some necessary conditions for a C^*-algebra to admit a summable spectral triple on itself.

The first generalization of this construction for any C^*-algebra A is in [13], where the authors construct a canonical spectral triple on the crossed product $A \rtimes \mathbb{Z}$ to characterize the metric properties of a dynamical system $(A, \mathbb{Z}, \alpha)$. The great achievement therein is the understanding that these properties are strictly related to the equicontinuity of the action in a Lipschitz sense (namely, as a compact quantum metric space [22, 97]).

Their work has been subsequently generalized in [47] for any discrete group G endowed with a length-type function, using as building blocks the spectral triples on the reduced group C^*-algebra $C^*_r(G)$ as defined by Connes. There the action is assumed to act smoothly and equicontinuously on the spectral triple $(\mathcal{A}, H, D)$, in the sense that the action α has to preserve the dense subalgebra $\mathcal{A}$ and

$$\sup_{g\in G} \|[D, \pi(\alpha_g(a))]\| < \infty$$

for all $a \in \mathcal{A}$. As pointed out by the authors, *the key idea is to use (a representative of the) external unbounded Kasparov product to produce a spectral triple on the tensor product $A \otimes C^*_r(G)$ and then check under which conditions the same formula still defines a triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ on the reduced crossed product.* Despite the use of the Kasparov product, the reason why this construction should bear any relation to the (external) Kasparov product is so far not completely clear.

Besides the structure of a quantum metric space, in [47] the authors begin the study of the spectral properties of their triple on $A \rtimes_\alpha G$: it is proved, for instance, that the summability of spectral triples is preserved (under some additional assumptions) under the passage from $(\mathcal{A}, H, D)$ to $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$, and that the *dimension* is additive. Further properties of this triple were found in [83], where the author shows that the triple is equivariant with respect to the coaction of a group and that this coaction satisfies a condition known as *contractivity*.

A complete and satisfactory discussion when G is not discrete but just locally compact and Hausdorff is far from being understood. A great step in this direction was given by [52], where the authors are able to relax the hypothesis on the existence of a length-type function on the group at the price of producing a twisted spectral triple. Despite the extremely general range of applicability, it is not clear whether this construction preserves all the spectral properties of the starting triple or bears any relation with Kasparov's bivariant K-theory (the same authors note that their resulting triple does not define in general an equivariant Kasparov module [52, Remark 2.8]).

1.2 Purpose of the Thesis and Main Results

Following the stream pursued by the aformentioned authors [13, 22, 47, 52, 83], in this thesis we construct spectral triples on reduced twisted crossed products $A \rtimes^{\sigma}_{\alpha,r} G$, where A is a unital C^*-algebra, G a discrete group and (α, σ) a twisted action in the sense of Busby and Smith [19], and then we study under which conditions we have some of the Connes axioms for a spectral manifold. Note that, rather than the more advanced discussion developed in [52], we build upon the construction given in [47].

More precisely, in Chapter 6 we construct spectral triples on the reduced crossed product of a twisted group algebra when the group G is endowed with a proper Dirac weight with matrix values. We characterize the summability property in terms of a generalized notion of growth of the group (Proposition 6.8) and provide sufficient conditions such that the regularity condition holds true (Propositions 6.9 and 6.11). Eventually, when the weight is scalar valued, we define an involutive antilinear map, characterize when this map is a real structure and show under which conditions this map satisfies the first and the second order conditions (Propositions 6.13, 6.14 and 6.16).

In Chapter 7 we generalize the constructive result of [47] when the length-type function l on the group takes values in a matrix algebra and the equicontinuity condition is suitably adapted to take into consideration the presence of the twisting cocycle σ (see Theorem 7.1 and condition (7.1)). Then, following the idea in [47, Remark 2.9], we assume that the triple $(\mathcal{A}, H, D)$ is G-equivariant with respect to a map $u\colon G \to \mathcal{U}(H)$ and we show that there exists another construction of a triple on the twisted crossed product $A \rtimes^{\sigma}_{\alpha,r} G$ which turns out to be K-homologically equivalent to the previous construction when a uniform bound condition of the form

$$\sup_{g\in G} \|[D, u_g]\| < \infty \tag{1.1}$$

holds true. We further show that the resulting triple of this construction:

- is equivariant with respect to the dual coaction of G (Proposition 7.6)
- naturally generalizes the spectral triple on the noncommutative 2-torus [44, 95], regarded as the crossed product $C(S^1) \rtimes \mathbb{Z}$ (Example 7.5).
- represents a Kasparov class which is the result of natural operation in KK-theory applied to the starting data (see Section 7.2, in particular Theorem 7.13 and Proposition 7.11).
- is regular when the spectral triple $(\mathcal{A}, H, D, u)$ is regular and the map u is smooth in a natural sense (see Theorem 7.17).

To prove the existence of a real structure and an orientation cycle on this triple, we assume that the cocycle is trivial, the length function takes scalar values and G acts by isometries, i.e., the operators u_g commute with the Dirac operator D. Then:

- We construct a real structure $\widehat{J}$ on $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ provided $(\mathcal{A}, H, D)$ admits a real structure J such that $Ju_g = u_g J$ for any $g \in G$ (see Theorem 7.21).
- If G is abelian, we construct a second inequivalent real structure $\widetilde{J}$ provided $Ju_g = u_g^* J$ for any $g \in G$ (see Theorem 7.27).

- In both cases the relationship between J and u can be interpreted as the equivariance of J with respect to the action of $\mathbb{C}G$ endowed with a suitable $*$-structure, and find that both $\widehat{J}$ and $\widetilde{J}$ are equivariant for the dual coaction of G.
- We compute the KO-dimension of $\widehat{J}$ and $\widetilde{J}$ in terms of the KO-dimension of J, and show that, under suitable assumptions, the first and the second order conditions are preserved.

Lastly, in Section 7.6, using a suitably twisted shuffle product between Hochschild cycles, we induce an equivariant orientation cycle on $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ from an equivariant orientation cycle on $(\mathcal{A}, H, D)$ (Theorem 7.41).

1.3 Structure of the Thesis

The thesis is divided in three parts: preliminary material, the body of the thesis and the appendix. In the first part we review the main ingredients that we need to present the research project and our novel contributions. In particular

- in Chapter 3 we fix the main definitions and properties about twisted crossed products and their covariant representations. Furthermore, we present the example of the rotation algebra, which will guide our exposition, and the theory of noncommutative coverings with finite abelian structure group.
- in Chapter 4 we give a brief introduction to Kasparov's bivariant K-theory, with particular emphasis to its unbounded version. We also present the equivariant theory and how is related to the KK-theory of crossed products.
- in Chapter 5 we review the notion of spectral triple and its relations with spectral geometry, from the Connes axioms for the reconstruction theorem to the definition of a compact quantum metric space. We remind the well known construction of the spectral triple on the rotation algebra as a guiding example.

In Part II we expose our main results, namely the construction of a spectral triple on the reduced twisted crossed product $A \rtimes^{\sigma}_{\alpha,r} G$ and the study of the Connes axioms of a spectral manifold, discussing at first the case in which $A = \mathbb{C}$. We conclude this part with some remarks and possible future research directions.

Part III is the appendix and we collect as a reference some basic facts and conventions about Hilbert modules and operator $*$-modules that are needed in order to properly discuss KK-theory and to compute the internal Kasparov product of unbounded modules. We further review the basic facts about Hopf algebras and compact quantum groups aimed at discussing their actions and coactions on spectral triples.

Chapter 2

Conventions and Notations

All Hilbert spaces and C^*-algebras in this thesis are assumed to be separable and algebras are assumed to be unital unless specified otherwise. Furthermore, every group G is assumed to be topological with a locally compact second-countable Hausdorff topology. We denote its neutral element by e and its left Haar measure by μ. We adopt the multiplicative notation.

2.1 List of Symbols

Symbol	Meaning
A^+	Minimal unitization of a C^*-algebra A
$\mathcal{U}(A)$	Group of unitaries of the C^*-algebra A (or its unitization A^+)
$M_{m,n}(A)$	Space of $m \times n$ matrices with values in the C^*-algebra A.
S^1	The unit circle $\mathbb{R}/\mathbb{Z}$
T^2	The 2-dimensional torus $S^1 \times S^1$
$U(1)$	The unitary group in $\mathbb{C}$
E_A	Right Hilbert C^*-module over A
$\mathcal{L}_A(E)$	Adjointable operators over the Hilbert C^*-module E_A
$\mathcal{K}_A(E)$	Compact operators over the Hilbert C^*-module E_A
$\mathcal{L}(H)$	Space of bounded linear operators on a Hilbert space H
$\mathcal{U}(H)$	Space of unitary operators on a Hilbert space H
(A, G, α, σ)	Twisted C^*-dynamical system with action α and cocycle σ
$A \rtimes_\alpha^\sigma G$	Twisted crossed product of the system (A, G, α, σ)
$A \rtimes_{\alpha,r}^\sigma G$	Reduced twisted crossed product of the system (A, G, α, σ)
$C^*_\sigma(G)$	Twisted group algebra of G
$\pi \rtimes u$	Integrated form of a covariant representation (π, u)
A_θ^2	The 2-dimensional rotation algebra
τ	The canonical trace on A_θ^2
H_τ	The GNS representation of A_θ^2
$\widehat{G}$	Pontryagin dual group of an abelian group G

λ	Left regular representation of G on $\ell^2(G)$
ρ	Right regular representation of G on $\ell^2(G)$
$\mathbb{C}\ell_n$	n-th complex Clifford Algebra
$C_c(X)$	Algebra of continuous functions on X with compact support
$C_0(X)$	Algebra of continuous functions on X which vanish at infinity
$\mathcal{H}, \varepsilon, S$	A Hopf algebra $\mathcal{H}$ with counit ε and antipode S
$\triangleleft$, $\triangleright$	Right, left action
$KK_\bullet(A, B)$	KK-theory group
$\otimes$	Either maximal tensor product or Kasparov product of modules
$\mathfrak{b}(D)$	bounded transform of a Dirac operator D
∇	Connection over an operator $*$-module
$C^{Lip}(A)$	Lipschitz subalgebra of A for a spectral triple $(\mathcal{A}, H, D)$
V	Complex and finite dimensional vector space
Sp	The spectrum of an element in a C^*-algebra
d_G	Growth of the group G

Part I

Preliminaries

Chapter 3

Twisted Crossed Products

Given a C^*-algebra A and a group G acting on it, a crossed product $A \rtimes G$ is the smallest C^*-algebra containing A in which G acts only by inner automorphisms. This is not unique, depending on in which sense one means "the smallest". The crossed product contains both geometric and analytic information: indeed, on the one hand it can be regarded as the noncommutative version of a classical quotient space (see e.g. [61, Chapter 2]); on the other hand (when $A = \mathbb{C}$ and G is abelian) it is the natural environment to do Fourier Analysis on G. In this chapter we briefly recall some basic facts, definitions and examples about twisted crossed products following mainly [12, 78, 103].

Assumptions 3.1. *In this chapter we denote by A a separable unital C^*-algebra and by G a (countable) discrete group with neutral element e.*

Note that most of the contents of this chapter can be adapted to be true also when A is not unital (by passing to the multiplier algebra) and when G is just a locally compact topological group (by using its Haar measure).

3.1 Basic Definitions, Examples and Properties

Definition 3.2 (cf. [19, 78]). *Let A and G as in Assumptions 3.1. A **twisted C^*-dynamical system** is a pair of maps $\alpha\colon G \to \mathrm{Aut}(A)$ and $\sigma\colon G \times G \to \mathcal{U}(A)$ satisfying*

$$\alpha_x \circ \alpha_y = \mathrm{Ad}(\sigma(x,y)) \circ \alpha_{xy} \tag{3.1}$$

$$\sigma(x,y)\sigma(xy,z) = \alpha_x(\sigma(y,z))\sigma(x,yz) \tag{3.2}$$

$$\sigma(x,e) = \sigma(e,x) = 1 \tag{3.3}$$

$$\alpha_e = \mathrm{id}_A \tag{3.4}$$

*for all $x, y, z \in G$. The map σ is called a **2-cocycle** with values in $\mathcal{U}(A)$.*

Note that if σ takes values in the centre of A, then α is a homomorphism from G to $\mathrm{Aut}(A)$, i.e. is an ordinary action. This happens for example when A is commutative or when σ takes value in $U(1)$. In particular, if $\sigma \equiv 1$ then the system is an ordinary C^*-dynamical system (see e.g. [109]) and, for sake of clearness, we call this case as *untwisted.*

From the properties of the couple (α, σ) we can readily deduce also that for $x = z = g$ and $y = g^{-1}$ in (3.2) we have

$$\sigma(g, g^{-1}) = \alpha_g \left(\sigma(g^{-1}, g)) \right). \tag{3.5}$$

Further, using equation (3.5) one can easily check that the inverse map of α is

$$\alpha_g^{-1} = \alpha_{g^{-1}} \circ \mathrm{Ad}\left(\sigma(g, g^{-1})^* \right) = \mathrm{Ad}\left(\sigma(g^{-1}, g)^* \right) \circ \alpha_{g^{-1}}. \tag{3.6}$$

Example 3.3. Let X be a locally compact Hausdorff topological space and G a discrete group. For any continuous left action $G \times X \to X$, the map $\alpha\colon G \to \mathrm{Aut}\, \mathcal{C}_0(X)$ defined by

$$\alpha_s(f)(x) := f(s^{-1} \cdot x)$$

for $s \in G$ and $x \in X$ is a group homomorphism and the triple $(\mathcal{C}_0(X), G, \alpha)$ is an untwisted C^*-dynamical system. Vice versa, a standard application of the Gelfand-Najmark duality between topological spaces and C^*-algebras (see e.g. [75, 109]) shows that given an untwisted C^*-dynamical system $(\mathcal{C}_0(X), G, \alpha)$, there exists a continuous left action $G \times X \to X$ such that

$$\alpha_s(f)(x) = f(s^{-1} \cdot x)$$

for $s \in G$ and $x \in X$. This happens for instance when $X = G$ and G acts on itself by left or right multiplication. □

Given a twisted dynamical system (A, G, α, σ), we define a twisted convolution $*$-algebra structure on the space $C_c(G, A)$ of the finitely supported function from G to A by

$$\begin{aligned} (f \star g)(x) &:= \sum_{y \in G} f(y) \alpha_y(g(y^{-1}x)) \sigma(y, y^{-1}x), \\ f^*(x) &:= \sigma(x, x^{-1})^* \alpha_x(f(x^{-1})^*), \end{aligned} \tag{3.7}$$

for $f, g \in C_c(G, A)$. Hereafter we denote by $a\delta_x$ the A-valued function on G which is 0 everywhere except at the point $x \in G$ where it takes the value $a \in A$. As G is discrete, every function $f \in C_c(G, A)$ will be then identified with the (finite) sum $\sum_{x \in G} f(x) \delta_x$ and operations (3.7) are translated into the following relations:

$$\delta_x \star \delta_y = \sigma(x, y) \delta_{xy}, \qquad \delta_x^* = \sigma(x^{-1}, x)^* \delta_{x^{-1}}, \qquad \delta_x a \delta_x^* = \alpha_x(a) \tag{3.8}$$

for $x \in G$ and $a \in A$. Such rules extend to $L^1(G, A)$ making it the Banach $*$-algebra $L^1(A, G, \alpha, \sigma)$. Obviously $C_c(G, A)$ is unital since A is unital and the unit is $1_A \delta_e$. Using the previous axioms it is easy to check that

$$(a\delta_g)^* = \sigma(g^{-1}, g)^* \alpha_{g^{-1}}(a^*) \delta_{g^{-1}}.$$

3.1.1 Twisted Covariant Representations

Definition 3.4. *A **representation** of a unital C^*-algebra A is a pair (H, π) given by a Hilbert space H and a $*$-homomorphism $\pi\colon A \to \mathcal{L}(H)$. We say that a representation (H, π) is **non degenerate** if $\overline{\pi(a)H} = H$.*

According to our convention, any representation of a unital C^*-algebra need not be a unital homomorphism. Upon restricting to the subspace $\overline{\pi(A)H}$, it is always possible to assume that a representation is non degenerate. Note that any unital representation is non degenerate.

Definition 3.5. *A* ***twisted covariant representation*** *of* (A, G, α, σ) *is a couple* (π, u) *where* $\pi\colon A \to \mathcal{L}(H)$ *is a non degenerate representation and* $u\colon G \to \mathcal{U}(H)$ *satisfies*

$$u_x u_y = \pi(\sigma(x, y))u_{xy} \tag{3.9}$$
$$u_x \pi(a)u_x^* = \pi(\alpha_x(a)) \tag{3.10}$$

for all $x, y \in G$ *and* $a \in A$.

Note that from these relations we get that $u_e = \pi(1_A)$ and that

$$u_x^* = u_{x^{-1}}\pi\left(\sigma(x, x^{-1})^*\right) = \pi\left(\sigma(x^{-1}, x)^*\right)u_{x^{-1}}. \tag{3.11}$$

Example 3.6. Let G act on itself by left translation and let $\mathrm{lt}\colon G \to \mathrm{Aut}(C_0(G))$ be the associated dynamical system as in Example 3.3. Let $M\colon C_0(G) \to \mathcal{L}(L^2(G))$ be given by point-wise multiplication:

$$M(f)h(s) := f(s)h(s)$$

and let $\lambda\colon G \to \mathcal{U}(L^2(G))$ be the left-regular representation $\lambda_x f(y) = f(x^{-1}y)$. Then (M, λ) is a covariant representation of $(C_0(G), G, \mathrm{lt})$. □

There exists a $1-1$ correspondence between covariant representations and non degenerate representations of $L^1(A, G, \alpha, \sigma)$. On the one hand, to the twisted covariant representation (π, u) one associates its **integrated form**, which is the $*$-homomorphism $\pi \rtimes u\colon L^1(A, G, \alpha, \sigma) \to \mathcal{L}(H)$ given on the dense subspace $C_c(G, A)$ by

$$\pi \rtimes u\Big(\sum_{x\in G} a_x\delta_x\Big) := \sum_{x\in G} \pi(a_x)u_x. \tag{3.12}$$

Note that the integrated form $\pi \rtimes u$ is non degenerate since π is non degenerate. Further, it is unital when π is unital. On the other hand, given a $*$-representation ϕ of $C_c(G, A)$ it is possible to check that the couple of representations

$$\pi(a) := \phi(a\delta_e) \qquad u_g := \phi(1_A\delta_g)$$

is covariant and $\phi = \pi \rtimes u$.

Remark 3.7. When $\sigma \equiv 1$, covariant representations of the system $(A, \{ e \}, \mathrm{id})$ correspond exactly to representations of A and covariant representations of the system $(\mathbb{C}, G, \mathrm{id})$, where id is the trivial action that sends all the elements of G to the identity automorphism of A, correspond to unitary representations of G.

3.1.2 Twisted Crossed Products

Given a twisted dynamical system (A, G, α, σ), the quantity

$$\|f\| := \sup\{ \|\pi \rtimes u(f)\| \,:\, (\pi, u) \text{ is a covariant repr. of } (A, G, \alpha, \sigma) \} \tag{3.13}$$

on $C_c(G, A)$ is well defined and finite as it can be controlled by the L^1 norm of f. In particular, it is possible to prove that (3.13) is a norm which is called the **universal norm**.

Definition 3.8. *The* ***twisted crossed product*** *C^*-algebra $A \rtimes_\alpha^\sigma G$ for (A, G, α, σ) is the C^*-completion of $C_c(G, A)$ with respect to the universal norm* (3.13).

Since A is unital, the crossed product $A \rtimes_\alpha^\sigma G$ contains a copy of both A and G. More precisely, there is a non degenerate homomorphism $i_A \colon A \to A \rtimes_\alpha^\sigma G$ and a map $i_G \colon G \to \mathcal{U}(A \rtimes_\alpha^\sigma G)$ such that:

(1) the couple (i_A, i_G) is covariant in the sense that

$$i_G(x) i_G(y) = i_A(\sigma(x,y)) i_G(xy) \quad \text{and} \quad i_A \circ \alpha_x = \mathrm{Ad}(i_G(x)) \circ i_A$$

for every $x, y \in G$.

(2) for every covariant representation (π, U) of (A, G, α, σ) on the Hilbert space H there is a non degenerate representation $\pi \rtimes U \colon A \rtimes_\alpha^\sigma G \to \mathcal{L}(H)$ making the diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[dr, bend right, "U"'] & A \rtimes_\alpha^\sigma G \arrow[d, "\pi \rtimes U"] & A \arrow[l, "i_A"'] \arrow[dl, bend left, "\pi"] \\ & \mathcal{L}(H) & \end{tikzcd}$$

commute.

These maps are given by $i_A(a) \coloneqq a\delta_e$ and $i_G(\delta_g) \coloneqq 1_A \delta_g$. Note that the image of the map $i_A \rtimes i_G$ defined on $L^1(G, A)$ is dense in $A \rtimes_\alpha^\sigma G$ as G is discrete. The existence of these maps constitutes the **universal property** of the crossed product $A \rtimes_\alpha^\sigma G$ in the sense that if there exists a (unital) C^*-algebra B endowed with maps (j_A, j_G) with the two previous properties, then B is isomorphic to $A \rtimes_\alpha^\sigma G$ and, under this isomorphism, $j_A = i_A$ and $j_G = i_G$ (see e.g. [78, Proposition 2.7]).

Definition 3.9. *The* ***twisted group algebra*** *of G is the twisted crossed product $\mathbb{C} \rtimes_\alpha^\sigma G$ with respect to the trivial action of G on $\mathbb{C}$ and is denoted by $C^*_\sigma(G)$.*

Example 3.10 (Clifford Algebras)**.** As proved in [3], we can regard (complex) Clifford algebras as twisted group algebras of a suitable finite group. Consider indeed $\mathbb{C}\ell_n$ as the complex algebra generated by the skew-adjoint anti-commuting elements $e_1, \ldots, e_n$ such that $e_i^2 = -1$ and graded in the standard way. Consider further the cyclic multiplicative group $\mathbb{Z}_2$ and let $g = -1$ be its generator; any element in $G = \mathbb{Z}_2^n$ is of the form $x = (g^{x_1}, \ldots, g^{x_n})$ where $x_i = 0, 1 \in \mathbb{Z}$ for $i = 1, \ldots, n$. Define

$$\sigma_n(x,y) \coloneqq (-1)^{\sum_{j<i} x_i y_j}$$

for $x, y \in G$ and consider the twisted group algebra $C^*_{\sigma_n}(G)$ for the trivial action of G on $\mathbb{C}$. Then the map $C^*_{\sigma_n}(G) \to \mathbb{C}\ell_n$, given on a basis by

$$(g^{x_1}, \ldots, g^{x_n}) \longmapsto (ie_1)^{x_1} \cdots (ie_n)^{x_n},$$

is an isomorphism of C^*-algebras. Note that the cocycle σ_n is trivial when $n = 1$ and so $\mathbb{C}\ell_1$ is an untwisted crossed product. □

Remark 3.11. Note that the twist by a cocycle is a genuine noncommutative phenomenon in the sense that the C^*-algebra $C^*_\sigma(G)$ is commutative if and only if G is abelian and the cocycle is trivial (see e.g. [39]).

Remark 3.12. It can be shown that the vector spaces $A \rtimes^{\sigma}_{\alpha} G$ and $A \otimes C^*_{\sigma}(G)$ are isomorphic for any twisted action of G on A but they are in general not isomorphic *as algebras*. When $\alpha\colon G \to \mathrm{Aut}(A)$ is (exterior equivalent to) the trivial action, they are isomorphic and this motivates the assertion that *a crossed product can be though of as a "twisted" maximal tensor product of A and $C^*_{\sigma}(G)$*. For more details we refer to [109, Section 2.5].

3.1.3 Reduced Twisted Crossed Products

Let us now consider back the definition of the twisted crossed product. In general, it is not obvious that there are any covariant representations of a given dynamical system. However, starting from a suitable representation of the algebra A on a Hilbert space H, one can induce a representation of $A \rtimes^{\sigma}_{\alpha} G$ on a larger space. Consider indeed a twisted dynamical system (A, G, α, σ) and let $\pi\colon A \to \mathcal{L}(H)$ be a faithful $*$-representation (provided e.g. by standard GNS theory [75]). Define a couple of maps from A and G onto $\mathcal{L}(H \otimes \ell^2(G))$ by

$$\begin{cases} \tilde{\pi}(a)(\xi \otimes \delta_x) \coloneqq \pi\left(\alpha_{x^{-1}}(a)\right) \xi \otimes \delta_x \\ \tilde{\lambda}_y(\xi \otimes \delta_x) \coloneqq \pi\left(\sigma(x^{-1}y^{-1}, y)\right) \xi \otimes \delta_{yx}. \end{cases} \tag{3.14}$$

It is easy to check that the couple $(\tilde{\pi}, \tilde{\lambda})$ is a twisted covariant representation and its integrated form is a representation of $A \rtimes^{\sigma}_{\alpha} G$ on $H \otimes \ell^2(G)$. Note in particular that

$$\tilde{\lambda}^*_y(\xi \otimes \delta_x) = \pi\left(\sigma(x^{-1}, y)^*\right) \xi \otimes \delta_{y^{-1}x}.$$

The homomorphism $\tilde{\pi} \rtimes \tilde{\lambda}\colon C_c(G, A) \to \mathcal{L}(H \otimes \ell^2(G))$ is non degenerate whenever π is non degenerate. Furthermore, if π is faithful then $\tilde{\pi} \rtimes \tilde{\lambda}$ is also faithful. This allows to define a norm on $C_c(G, A)$ by setting

$$\|f\| \coloneqq \left\|\tilde{\pi} \rtimes \tilde{\lambda}(f)\right\| \qquad f \in C_c(G, A) \tag{3.15}$$

where the left hand side is the operatorial norm of $\mathcal{L}(H \otimes \ell^2(G))$.

Definition 3.13. *The **reduced twisted crossed product** $A \rtimes^{\sigma}_{\alpha,r} G$ (or just $A \rtimes_r G$) is the C^*-algebra completion of $C_c(G, A)$ with respect to the norm* (3.15).

It can be shown that the reduced crossed product is independent of the choice of the faithful representation π (see e.g. [78, Remark 3.12]). The reduced twisted group algebra $\mathbb{C} \rtimes^{\sigma}_{r} G$ is denoted by $C^*_{r,\sigma}(G)$.

Remark 3.14. We have defined the induced representation using a *twisted left* regular representation λ of G on $\ell^2(G)$. Analogous formulas can be defined for a suitable *twisted right* regular representation but they are unitarily equivalent under the involution $V_G\colon H \otimes \ell^2(G) \to H \otimes \ell^2(G)$ given by $\xi \otimes \delta_x \mapsto \xi \otimes \delta_{x^{-1}}$. More details can be found in [12, 19].

Obviously, there exists a map $\Lambda\colon A \rtimes^{\sigma}_{\alpha} G \to A \rtimes^{\sigma}_{\alpha,r} G$ induced by the identity map which is always surjective but in general fails to be injective. It is known that this map is an isomorphism when the group G is *amenable*, a topological property which is satisfied by abelian groups and by compact groups (see e.g. [79, 110]). Other conditions which imply the injectivity of that map are provided also in [4] and [12].

3.1.4 A Fell Absorption Principle

It is a well known fact in representation theory that the left regular representation λ of a group G on $\ell^2(G)$ is able to absorb any other unitary representation u of G on a Hilbert space H in the sense that the tensor product action $u\otimes\lambda$ on $H\otimes\ell^2(G)$ is unitarily equivalent to the action $1\otimes\lambda$ (see e.g. [33]). This principle due to Fell can be easily generalized to twisted actions as follows; for sake of convenience (and with a slight abuse of notation) we denote the map $\tilde{\lambda}$ in (3.14) by $\sigma\otimes\lambda$.

Lemma 3.15 (cf. [12])**.** *Let (π, u) be a twisted covariant representation of (A, G, α, σ) on H and $W\colon H\otimes\ell^2(G)\to H\otimes\ell^2(G)$ be given by*

$$W(\xi\otimes\delta_g) := \pi(\sigma(g,g^{-1})^*)u_g\xi\otimes\delta_g.$$

Then $W(\sigma\otimes\lambda)W^* = u\otimes\lambda$.

Proof. Given $\xi\otimes\delta_g\in H\otimes\ell^2(G)$ we have

$$W(\sigma\otimes\lambda)_hW^*(\xi\otimes\delta_g) = \pi(\sigma(hg,g^{-1}h^{-1})^*)u_{hg}\pi(\sigma(g^{-1}h^{-1},h))u_g^*\pi(\sigma(g,g^{-1}))\xi\otimes\delta_{hg}.$$

for any $h\in G$. Using equation (3.11) this becomes

$$\begin{aligned} W(\sigma\otimes\lambda)_hW^*(\xi\otimes\delta_g) &= \pi(\underbrace{\sigma(hg,g^{-1}h^{-1})^*\alpha_{hg}(\sigma(g^{-1}h^{-1},h))\sigma(gh,g^{-1})}_{\sigma(e,h)\text{ by equation (3.2)}})u_h\xi\otimes\delta_{hg}\\ &= u_h\xi\otimes\delta_{hg}\end{aligned}$$

which is just $(u\otimes\lambda)_h$ applied to $\xi\otimes\delta_g$. ☐

As noticed in [12], a Fell absorption principle holds true also for twisted representations of dynamical systems in the following way. Let (π, u) a twisted covariant representation of (A, G, α, σ) on a Hilbert space H and define the maps $\hat{\pi}\colon A\to\mathcal{L}(H\otimes\ell^2(G))$ and $\hat{\lambda}\colon G\to\mathcal{L}(H\otimes\ell^2(G))$ by

$$\begin{cases}\hat{\pi}(a)(\xi\otimes\delta_g) := \pi(a)\xi\otimes\delta_g\\ \hat{\lambda}_h(\xi\otimes\delta_g) := u_h\xi\otimes\delta_{hg}\end{cases}\tag{3.16}$$

for $a\in A$, $\xi\in H$ and $g,h\in G$. These clearly forms a twisted covariant representation of (A, G, α, σ) on $H\otimes\ell^2(G)$ as

$$\hat{\lambda}_h\hat{\lambda}_g(\xi\otimes\delta_x) = u_hu_g\xi\otimes\delta_{hgx} = \hat{\pi}(\sigma(h,g))\hat{\lambda}_{hg}(\xi\otimes\delta_g)$$

and

$$\hat{\lambda}_h\hat{\pi}(a)\hat{\lambda}_h^*(\xi\otimes\delta_x) = u_h\pi(a)u_h^*\xi\otimes\delta_g = \hat{\pi}\left(\alpha_h(a)\right)(\xi\otimes\delta_g).$$

We then have the following result.

Proposition 3.16 (cf. [12])**.** *The integrated form $\hat{\pi}\rtimes\hat{\lambda}$ of the covariant system* (3.16) *is unitarily equivalent to the integrated form $\tilde{\pi}\rtimes\tilde{\lambda}$ of the covariant system* (3.14)*.*

Proof. We have already seen in Lemma 3.15 that $W\tilde{\lambda}_h W^* = \hat{\lambda}_h$ for any $h \in G$. An easy calculation shows that

$$\begin{aligned} W\tilde{\pi}(a)W^*(\xi \otimes \delta_g) &= \pi(\sigma(g,g^{-1})^*)\left[u_g \pi(\alpha_{g^{-1}}(a))u_g^*\right]\pi(\sigma(g,g^{-1}))\xi \otimes \delta_g \\ &= \pi(\alpha_{gg^{-1}}(a))\xi \otimes \delta_g = \pi(a)\xi \otimes \delta_g \\ &= \hat{\pi}(a)(\xi \otimes \delta_g). \end{aligned}$$

for any $a \in A$. In particular, $W\tilde{\pi} \rtimes \tilde{\lambda}(a\delta_h)W^* = W\tilde{\pi}(a)W^*W\tilde{\lambda}_h W^* = \hat{\pi} \rtimes \hat{\lambda}(a\delta_h)$ for any $a\delta_h \in A \rtimes_\alpha^\sigma G$.

□

3.2 An Example: The Rotation Algebra

In this section we collect some of the well known facts about the rotation algebra [95], the simplest and richest example of a noncommutative topological space. We analyze its differential structure, from the point of view of Connes spectral geometry [25], in Chapter 5. For more references and details we refer to [32, 75, 84, 89] and in particular to [44, Ch. 12].

Consider the unit circle S^1 as the additive group $\mathbb{R}/\mathbb{Z}$ endowed with the quotient structure. We identify functions in $C(S^1)$ to continuous periodic functions on $\mathbb{R}$ with period 1 thanks to the isomorphism $S^1 \to U(1)$, $[t] \mapsto e^{2\pi i t}$.

Definition 3.17. *Let θ be a real number and $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(C(S^1))$ be the action induced by the rigid rotation of the circle by the angle $2\pi\theta$ given by $\alpha_n(f)(t) := f(t+n\theta)$ for $t \in \mathbb{R}$. The* ***rotation algebra*** *A_θ^2 is defined as the (maximal) crossed product $C(S^1) \rtimes_\alpha \mathbb{Z}$.*

It is often useful to realize the rotation algebra as an algebra of bounded operators acting on a Hilbert space: to this aim, consider the representation M of $C(S^1)$ on $L^2(S^1)$ given by point-wise multiplication and the unitary shift operator $V^n f(t) = f(t + n\theta)$ for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. An easy computation shows that (M, V) is a faithful covariant representation of $(C(S^1), \mathbb{Z}, \alpha)$ on $L^2(S^1)$. Further, if we expand elements g of $C(S^1)$ as Fourier series

$$g(t) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n t}, \tag{3.17}$$

we notice that the multiplication operator by g is generated by the unitary multiplication operator $Uf(t) = e^{2\pi i t} f(t)$ and thus any element in A_θ^2 is of the form $\sum_{n,m\in\mathbb{Z}} a_{n,m} U^n V^m$. A straightforward computation shows that

$$VU = e^{2\pi i\theta} UV \tag{3.18}$$

and it is possible to interpret A_θ^2 as the universal C^*-algebra generated by two unitaries U, V satisfying (3.18). This picture immediately gives clear isomorphisms between the various A_θ^2: firstly, as (3.18) is unchanged for the transformation $\theta \mapsto \theta + n$ with $n \in \mathbb{Z}$, we deduce that there is an isomorphism $A_\theta^2 \simeq A_{\theta+n}^2$. Therefore we can, whenever convenient, restrict the range of parameter θ to the interval $[0, 1)$. Furthermore, as $UV = e^{-2\pi i\theta} VU = e^{2\pi i(1-\theta)} VU$, the swap of U with V in (3.18) extends to an isomorphism $A_\theta^2 \simeq A_{1-\theta}^2$ for any $\theta \in [0, 1)$. This means that we can further restrict the interval down to $[0, \frac{1}{2}]$. It is known, however, that it is not possible to further reduce this interval and, more precisely, that for $\theta \in [0, \frac{1}{2}]$ the algebras A_θ^2 are mutually not isomorphic (see [51]), even tough some may be Morita equivalent.

Remark 3.18. From (3.18) it is clear that A^2_θ is abelian if and only if θ is an integer. This means that the only commutative rotation algebra with θ in the range $[0, \frac{1}{2}]$ is A^2_0 and this is isomorphic to the C^*-algebra $C(T^2)$ of continuous functions on the 2-torus $T^2 = S^1 \times S^1$ with angular coordinates (φ_1, φ_2), by taking $U = e^{2\pi i \varphi_1}$ and $V = e^{2\pi i \varphi_2}$.

It is known that the rotation algebra A^2_θ comes equipped with a distinguished (unique if θ is irrational) tracial state

$$\tau\colon A^2_\theta \longrightarrow \mathbb{C}, \qquad \tau\left(\sum_{n,m\in\mathbb{Z}} a_{nm} U^n V^m\right) := a_{00}$$

which is clearly normalized in the sense that $\tau(1) = 1$. Furthermore, one can check that

$$\tau(a^* a) = \sum_{n,m\in\mathbb{Z}} |a_{nm}|^2 ,$$

so τ is also faithful. The GNS representation theory for τ is defined as follows: the tracial state τ defines a sesquilinear form $\langle a, b\rangle = \tau(a^* b)$ on A^2_θ. The completion of A^2_θ with respect to this scalar product yields a Hilbert space $H_\tau = L^2(A^2_\theta, \tau)$ which carries a $*$-representation of A^2_θ by left multiplication operators. This representation has $1 \in A^2_\theta \subseteq H_\tau$ as cyclic and separating vector.

Theorem 3.19. *Consider the covariant representation (M, V) of $C(S^1) \rtimes_\alpha \mathbb{Z}$ on $L^2(S^1)$ where M is the multiplication operator and V the canonical unitary shift. There exists an isomorphism of Hilbert spaces $H_\tau \to L^2(S^1) \otimes \ell^2(\mathbb{Z})$ which maps the GNS representation of A^2_θ into the integrated form of the covariant system $(M \otimes 1, V \otimes \lambda)$ as defined in* (3.16).

Proof. Let us first define the isomorphism. The map $H_\tau \to L^2(T^2)$ given on the generators by $U \mapsto e^{2\pi i \varphi_1}$, $V \mapsto e^{2\pi i \varphi_2}$ extends to an isomorphism of Hilbert spaces: in particular, an element $h = \sum_{n,m\in\mathbb{Z}} h_{nm} U^n V^m$ goes to the function

$$\sum_{m\in\mathbb{Z}} \underbrace{\left(\sum_{n\in\mathbb{Z}} h_{nm} e^{2\pi i n \varphi_1}\right)}_{f_m(\varphi_1)\in L^2(S^1)} e^{2\pi i m \varphi_2}.$$

Under Fourier transform on the second entry, we identify $L^2(S^1 \times S^1)$ with $L^2(S^1) \otimes \ell^2(\mathbb{Z})$ and the function $e^{2\pi i m \varphi_2} \in L^2(S^1)$ goes to the delta function δ_m in $\ell^2(\mathbb{Z})$ (note that, according to our convention, the character $n \in \mathbb{Z} \simeq \widehat{S^1}$ maps $z \mapsto z^{-n}$). The isomorphism $\Phi\colon H_\tau \to L^2(S^1) \otimes \ell^2(\mathbb{Z})$ then just takes an element $f \otimes \delta_m \in C(S^1) \rtimes_\alpha \mathbb{Z} \subseteq H_\tau$ and maps it to $f \otimes \delta_m$. In this way it is easy to understand how $C(S^1) \rtimes_\alpha \mathbb{Z}$ acts on $L^2(S^1) \otimes \ell^2(\mathbb{Z})$: a pure element $g \in C(S^1)$ just acts by multiplication on the first factor; on the other hand, for $\delta_m \in C(S^1) \rtimes \mathbb{Z}$ we have

$$\delta_m \triangleright (f(\varphi_n)\delta_n) = \delta_m f(\varphi_1) \delta_m^* \delta_m \delta_n = V^m(f)(\varphi_1)\delta_{n+m}.$$

These are precisely the representations defining the system (3.16). $\square$

Another way to look at the rotation algebra which is often useful is as the twisted group algebra of $\mathbb{Z}^2$ in the following way. Given a real number θ, we define $\sigma\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to U(1)$ as

$$\sigma(x, y) := e^{-\pi i \theta (x_1 y_2 - x_2 y_1)}. \tag{3.19}$$

An easy computation shows that σ is a 2-cocycle for the (additive) group $\mathbb{Z}^2$, such that $\overline{\sigma(x,y)} = \sigma(y,x)$ and $\sigma(x,-x) = 1$ for any $x, y \in \mathbb{Z}^2$.

Proposition 3.20. *There exists an isomorphism $A_\theta^2 \simeq C^*(\mathbb{Z}^2, \sigma)$ of C^*-algebras.*

Proof. It is enough to show that in $C^*(\mathbb{Z}^2, \sigma)$ there are two unitaries satisfying (3.18). Consider the canonical basis $e_1 = (1,0), e_2 = (0,1)$ of $\mathbb{Z}^2$ and set $u = \delta_{e_1}, v = \delta_{e_2}$. Then $uv = e^{-\pi i\theta}\delta_{(1,1)}$ and $vu = e^{\pi i\theta}\delta_{(1,1)}$. In particular, $vu = e^{2\pi i\theta}uv$. □

3.3 Noncommutative Finite Abelian Coverings

In this section we introduce regular noncommutative finite coverings with abelian finite structure group and we show that they are isomorphic to twisted crossed products with respect to the dual group, a result that apparently appeared in [107] for the first time. We also discuss in detail the meaning of the notion that was called regularity in [2] by comparing it with the familiar notion of free action. We refer to [86] and [99] for further properties of group actions on C^*-algebras. This section is taken from [5].

Given a topological abelian group G, the **Pontryagin dual** $\widehat{G}$ is the group of continuous homomorphisms $\chi \colon G \to S^1$, where S^1 is the circle group $\mathbb{R}/\mathbb{Z}$ with quotient structure. The group $\widehat{G}$ is topological once endowed with the compact-open topology and clearly abelian so that we can consider its dual: the theorem of Pontryagin (known as *Pontryagin duality*) states that

$$G \simeq \widehat{\widehat{G}} \tag{3.20}$$

where every point $x \in G$ goes in the map that evaluates every character of G in x^{-1}. Note that if G is discrete then $\widehat{G}$ is compact.

Convention 3.21. *In the following we denote elements of G with Latin letters and those of $\widehat{G}$ with Greek letters.*

Note that, according to this convention, the pairing between $\widehat{G}$ and G is given by $\langle \gamma, x\rangle = \gamma(x)$ while the pairing between $G \simeq \widehat{\widehat{G}}$ and $\widehat{G}$ is given by $\langle x, \gamma\rangle = \gamma(x^{-1}) = \overline{\gamma(x)}$. For a reference about this convention we direct the reader to [109, pp. 194–195]. Till the end of this section we consider G finite and abelian.

Definition 3.22 (cf. [2])**.** *A **finite abelian (noncommutative) covering** of a unital C^*-algebra A is an inclusion of C^*-algebras $A \subseteq B$ together with an action β of a finite abelian group G on B such that A is the fixed point algebra B^G of B. We say that G is the **deck transformation group** and denote this structure by $G \curvearrowright B \supseteq A$.*

Given a noncommutative covering, we have that action of G on B decomposes B in its closed **spectral subspaces**

$$B_\gamma := \{b \in B \mid \beta_g(b) = \gamma(g)b \quad \forall g \in G\}$$

for $\gamma \in \widehat{G}$. Every B_γ is a Hilbert bimodule over A with scalar right product $\langle b_1, b_2\rangle := b_1^* b_2$. and left product $\langle b_1, b_2\rangle := b_1 b_2^*$. Notice that the induced norm on every B_γ coincides with the norm of B so that every spectral subspace is complete from the beginning.

Proposition 3.23 (cf. [2]). *With the above notation we have that:*

(1) $B_\gamma B_\mu \subseteq B_{\gamma\mu}$.

(2) Given $b \in B_\gamma$, *then* $b^* \in B_{\overline{\gamma}}$.

(3) If $b \in B_\gamma$ *is invertible, then* $b^{-1} \in B_{\overline{\gamma}}$.

(4) Each $b \in B$ *may be written as* $\sum_{\gamma\in\widehat{G}} b_\gamma$ *with* $b_\gamma \in B_\gamma$ *given by*

$$b_\gamma := \frac{1}{|G|}\sum_{g\in G}\overline{\gamma(g)}\beta_g(b).$$

Let us now consider the ordinary crossed product $B \rtimes G$. We can construct a canonical $B \rtimes G - A$ - Hilbert bimodule ${}_{B\rtimes G}\mathcal{E}_A$ in the following way: consider B endowed with

- left action $b\delta_g \triangleright x := b\beta_g(x)$ and left inner product

$$_{B\rtimes G}\langle b_1, b_2\rangle := \sum_{g\in G} b_1\beta_g(b_2^*)\delta_g,$$

- right action $x \triangleleft a := xa$ and right inner product

$$\langle b_1, b_2\rangle_A := \frac{1}{|G|}\sum_{g\in G}\beta_g(b_1^* b_2).$$

Then ${}_{B\rtimes G}\mathcal{E}_A$ is the bimodule obtained by completing B in the usual way. It turns out that ${}_{B\rtimes G}\mathcal{E}_A$ is almost a Morita equivalence bimodule for it may lack the fullness of the left product ${}_{B\rtimes G}\langle\cdot,\cdot\rangle$.

Definition 3.24. *The group action is* ***free*** *when* ${}_{B\rtimes G}\mathcal{E}_A$ *is a Morita equivalence bimodule.*

Note that a group action on a C^*-algebra which is free in the sense of Definition 3.24 is not free in the set theoretic sense: for example, $a = 0 \in B$ is a fixed-point with respect to the action β for any $g \in G$. To avoid possible confusion, in literature a group action for which ${}_{B\rtimes G}\mathcal{E}_A$ is a Morita equivalence bimodule is often called *saturated*. The two meanings are linked in the following way: when $B = C(X)$ with X Hausdorff and compact, Rieffel has proved that the action $\beta\colon G \to \mathrm{Aut}(C(X))$ is free (saturated) if and only if the corresponding action $X \curvearrowleft G$ is free in the familiar sense (see e.g. Proposition 7.1.12 and Theorem 7.2.6 in [85]).

Definition 3.25. *The* ***canonical map*** can: $B \odot B \to B \otimes C(G)$ *defined on the algebraic tensor product is the* $B - A$*-module map given by*

$$\mathrm{can}(x \odot y) := \sum_{g\in G} x\beta_g(y) \otimes \delta_g. \tag{3.21}$$

We say that the G*-action satisfies the* ***Elwood condition*** *if the map* can *has dense range in* $B \otimes C(G)$ *with respect to the crossed product* C^**-norm.*

Theorem 3.26 (cf. [86, 99]). *Let $G \curvearrowright B \supseteq A$ be a finite abelian covering. The following conditions are equivalent:*

(1) The action β is free.

(2) The action β satisfies the Elwood condition.

(3) Every spectral subspace B_γ for $\gamma \in \widehat{G}$ is a $A - A$ Morita equivalence bimodule.

(4) The multiplication map induces an isomorphism of Hilbert $A - A$-modules between the balanced tensor product $B_{\overline{\gamma}} \otimes_A B_\gamma$ and A for every $\gamma \in \widehat{G}$.

Moreover, if the action is free any B_γ is finitely generated and projective as a right A-module.

We say that an element x in a right A-module X is a generator if $X = \{xa, a \in A\}$. Analogously, an element x in a left A-module X is a generator if $X = \{ax, a \in A\}$.

Proposition 3.27. *For any $\gamma \in \widehat{G}$, the following facts are equivalent:*

(1) B_γ contains an element which is unitary in B.

(2) B_γ contains an element which is invertible in B.

(3) B_γ is a free, rank-1, right A-module and the action is free.

(4) B_γ has a generator as a right A-module and the action is free.

(5) B_γ is a free, rank-1, left A-module and the action is free.

(6) B_γ has a generator as a left A-module and the action is free.

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1): if $\mu_\gamma \in B_\gamma$ is invertible in B, $a_\gamma = \sqrt{\mu_\gamma^* \mu_\gamma}$ is invertible in A, hence $u_\gamma = \mu_\gamma a_\gamma^{-1} \in B_\gamma$ is invertible in B. Since $\mu_\gamma = u_\gamma a_\gamma$ is the polar decomposition, u_γ is unitary and is clearly a generator for B_γ both as a right and as a left A-module.

(2) $\Rightarrow$ (3): the map $a \in A \mapsto \mu_\gamma a \in B_\gamma$ is a right A-module isomorphism. As for the freeness, we have to show that, for any $\gamma \in \widehat{G}$, the A-bimodule map

$$\begin{array}{rcccc} p: & B_{\overline{\gamma}} & \otimes_A & B_\gamma & \to & A \\ & b_{\overline{\gamma}} & \otimes & b_\gamma & \mapsto & b_{\overline{\gamma}} b_\gamma. \end{array}$$

is a bijection. Since the μ_γ are invertible, for any element $x \in B_{\overline{\gamma}} \otimes_A B_\gamma$ there is a unique element $a \in A$ such that $\mu_\gamma \otimes \mu_\gamma a \in B_{\overline{\gamma}} \otimes B_\gamma$ is a representative of x, and $p(\mu_{\overline{\gamma}} \otimes \mu_\gamma a) = \mu_{\overline{\gamma}} \mu_\gamma a$ is clearly injective. Setting $a = \mu_\gamma^{-1} \mu_{\overline{\gamma}}^{-1}$, we have $a \in A$ and $p(\mu_{\overline{\gamma}} \otimes \mu_\gamma a) = 1$, which implies that p is surjective.

(3) $\Rightarrow$ (4): if $\Phi\colon A \to B_\gamma$ is a right A-module isomorphism, namely $\Phi(a_1)a_2 = \Phi(a_1 a_2)$, $\mu_\gamma := \Phi(1)$ is clearly a generator.

(4) $\Rightarrow$ (2): reasoning as above, since the μ_γ are left-generators, any element in $B_{\overline{\gamma}} \otimes_A B_{\overline{\gamma}}$ as a representative in $B_{\overline{\gamma}} \otimes B_\gamma$ of the form $\mu_{\overline{\gamma}} \otimes \mu_\gamma a$ with $p(\mu_{\overline{\gamma}} \otimes \mu_\gamma a) = \mu_{\overline{\gamma}} \mu_\gamma a$, even though a is not unique in principle. Exploiting the surjectivity of p, $\exists a\colon \mu_{\overline{\gamma}} \mu_\gamma a = 1$, namely $\mu_\gamma a \in B_\gamma$ is a right inverse of $\mu_{\overline{\gamma}}$. This shows that each μ_γ has a right-inverse $\lambda_\gamma \in B_{\overline{\gamma}}$.

Now the injectivity of p implies that if $\mu_{\overline{\gamma}}\mu_\gamma a = 0$ then $\mu_\gamma a = 0$. Setting $a = \lambda_\gamma(1 - \lambda_{\overline{\gamma}}\mu_{\overline{\gamma}})\mu_\gamma$ we get

$$\mu_{\overline{\gamma}}\mu_\gamma a = \mu_{\overline{\gamma}}\mu_\gamma\lambda_\gamma(1 - \lambda_{\overline{\gamma}}\mu_{\overline{\gamma}})\mu_\gamma = \mu_{\overline{\gamma}}(1 - \lambda_{\overline{\gamma}}\mu_{\overline{\gamma}})\mu_\gamma = 0,$$

therefore $0 = \mu_\gamma a = \mu_\gamma\lambda_\gamma(1 - \lambda_{\overline{\gamma}}\mu_{\overline{\gamma}})\mu_\gamma = (1 - \lambda_{\overline{\gamma}}\mu_{\overline{\gamma}})\mu_\gamma$. Multiplying by λ_γ to the right we get $1 - \lambda_{\overline{\gamma}}\mu_{\overline{\gamma}} = 0$, namely $\mu_{\overline{\gamma}} \in B_{\overline{\gamma}}$ is invertible in B.

The equivalences (3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) are proved using the adjoint map from B_γ to $B_{\overline{\gamma}}$. $\square$

Definition 3.28. *We say that the finite abelian covering $G \curvearrowright B \supseteq A$ is **rank-1 regular** if one of the equivalent conditions of Proposition 3.27 is satisfied. In this case, the action is said to be **rank-1 free**. Any map $s\colon \widehat{G} \longmapsto U(B)$ with $s_\gamma \in B_\gamma$ and $s_e = I$ is called a **frame**.*

Condition (1) in Proposition 3.27 has been called regularity in [2] and it has been proved to imply freeness by making use of the Elwood condition.

Remark 3.29. Rank-1 freeness implies that the action is faithful. Indeed, if there exists a nontrivial $g \in G$ acting trivially, we may find $\gamma \in \widehat{G}$ such that $\gamma(g) \neq 1$. Therefore, the equation $b = \beta_g(b) = \gamma(g)b$ is satisfied only for $b = 0$ and B_γ does not contain any invertible element.

Remark 3.30. As $\mu_\gamma \in B_\gamma$, we have that $\mu_\gamma A \mu_\gamma^* = A$ for any $\gamma \in \widehat{G}$.

Remark 3.31. A finite abelian noncommutative covering $G \curvearrowright B \supseteq A$ is rank-1 regular if and only if B is strongly $\widehat{G}$-graded (that is, the inclusion in Proposition 3.23 point (1) is an equality for any $\gamma, \mu \in \widehat{G}$). Indeed, assume that the covering is regular. Take $a \in B_{\gamma\mu}$ and let $s\colon \widehat{G} \to U(B)$ be a frame. A straightforward computation shows that as_μ^* is in B_γ and so $a = as_\mu^* s_\mu \in B_\gamma B_\mu$. On the other hand, assume that B is strongly $\widehat{G}$-graded. For any $\mu \in \widehat{G}$ we have that $1 \in B_e = B_{\mu\overline{\mu}} = B_\mu B_{\overline{\mu}}$. In particular, there are elements $c \in B_\mu$ and $d \in B_{\overline{\mu}}$ such that $1 = cd$. We have found in any spectral subspace B_μ an element c which is invertible in B and so the covering is rank-1 regular.

It is known that the condition that B is strongly graded with G finite and abelian is equivalent to the Hopf-Galois condition.

In general, the rank-1 regularity assumption is not always satisfied as shown by [2, Example 1.7]. Later we shall give a commutative example too. Let us now consider when this is instead satisfied.

Example 3.32. Let G be a finite abelian group and A a unital C^*-algebra. Assume that $\widehat{G}$ has a twisted action (α, σ) on A. Consider the twisted crossed product $B \coloneqq A \rtimes_\alpha^\sigma \widehat{G}$ and the action β of G on it given by the dual action

$$\beta_g(a\delta_\gamma) = \overline{\langle g, \gamma\rangle} a\delta_\gamma = \gamma(g) a\delta_\gamma.$$

The C^*-algebra B contains A in the obvious way and clearly the fixed point algebra of B under β is precisely A. Moreover, the spectral subspaces are $B_\gamma = A\delta_{\overline{\gamma}} = \delta_\gamma A$ for every $\gamma \in \widehat{G}$ and of course every δ_γ is unitary. This means that $G \curvearrowright B \supseteq A$ is a rank-1 free noncommutative covering with deck transformation group G. $\square$

In Example 3.32 we have shown that $A \subseteq A \rtimes_\alpha^\sigma \widehat{G}$ is a rank-1 regular finite noncommutative covering. The converse holds true: any finite rank-1 regular covering is a twisted crossed product. This was proved by Wagner in [107] with a slightly different terminology.

Theorem 3.33. *Let $G \curvearrowright B \supseteq A$ be a finite rank-1 regular noncommutative covering with G abelian. Any frame $s\colon \widehat{G} \longmapsto U(B)$ determines a twisting pair (α, σ) by*

$$\alpha_\gamma(a) := s_\gamma a s_\gamma^*, \quad \textit{and} \quad \sigma(\gamma, \mu) := s_\gamma s_\mu\, s_{\gamma\mu}^*, \quad \gamma, \mu \in \widehat{G}$$

together with an isomorphism $\Phi\colon A \rtimes_\alpha^\sigma \widehat{G} \to B$ which on the generators is given by

$$\Phi(a\delta_\gamma) := a s_\gamma.$$

This isomorphism is equivariant with respect to the dual action of G on $A \rtimes_\alpha^\sigma \widehat{G}$.

Proof. It is straightforward to check that (α, σ) is a twisting pair. Moreover since the group is finite we don't have to worry about completions and the maximal reduced crossed product coincides with the reduced one. Let's check that Φ is a morphism of C^*-algebras. Then it will be clearly invertible. We have

$$\Phi(a\delta_\gamma \star b\delta_\mu) = \Phi(a\alpha_\gamma(b)\sigma(\gamma,\mu)\delta_{\gamma\mu}) = a(s_\gamma b s_\gamma^*)(s_\gamma s_\mu s_{\gamma\mu}^*)s_{\gamma\mu} = \Phi(a\delta_\gamma)\Phi(b\delta_\mu)$$

and

$$\Phi((a\delta_\gamma)^*) = \Phi(\sigma_{\overline{\gamma},\gamma}^* \alpha_{\overline{\gamma}}(a^*)\delta_{\overline{\gamma}}) = s_e s_\gamma^* s_{\overline{\gamma}}^* (s_{\overline{\gamma}} a^* s_{\overline{\gamma}}^*) s_{\overline{\gamma}} = s_\gamma^* a^* = \Phi(a\delta_\gamma)^*.$$

The statement about the equivariance is trivial. □

Remark 3.34. Following [19], let $B(\widehat{G}, A)$ be the group of all the maps $p\colon \widehat{G} \to U(A)$ satisfying $p(e) = 1$. There is a natural action of $B(\widehat{G}, A)$ on the set of twisting pairs, where $p \in B(\widehat{G}, A)$ acts on (α, σ) by $p \cdot (\alpha, \sigma) := (\alpha^p, \sigma^p)$, with

$$\alpha_\gamma^p = \mathrm{Ad}_{p(\gamma)} \circ \alpha_\gamma, \quad \text{and} \quad \sigma^p(\gamma, \mu) = p(\gamma)\alpha_x(p(\mu))\sigma(\gamma,\mu)p(\gamma\mu)^*,$$

for $\gamma, \mu \in G$. The isomorphism class of $A \rtimes^{\alpha,\sigma} \widehat{G}$ only depends on the orbit of the twisting pair under the above $B(\widehat{G}, A)$-action. We see that any two frames give rise to twisting pairs *in the same cohomology class* i.e. in the same orbit. Indeed, if $s, s'\colon \widehat{G} \to B$ are frames with corresponding twisting pairs (α, σ) and (α', σ'), then putting $p(\gamma) := s_\gamma'^* s_\gamma$ for $\gamma \in \widehat{G}$, we find $(\alpha', \sigma') = p \cdot (\alpha, \sigma)$.

Let us examine now the commutative case. Consider a compact space X with right action $X \curvearrowleft G$ and quotient $Y = X/G$. The induced action on the functions corresponds to

$$(\beta_g f)(x) = f(x \cdot g),$$

for $g \in G$ and $f \in C(X)$, and the spectral subspaces read as

$$C(X)_\gamma = \{f \in C(X)\colon \beta_g(f) = \langle \gamma, g\rangle f\}, \quad \gamma \in \widehat{G}.$$

As is well known, in the case when the projection $\pi\colon X \to X/G$ is a regular covering, we can identify $C(X)_\gamma$ with the $C(Y)$-module of sections of a unitary complex line bundle $V_\gamma \to Y$ associated to the character $\gamma\colon G \to U(1)$. The total space of V_γ is the quotient space $(X \times \mathbb{C})/G$ with respect the the right diagonal action $(x, z) \cdot g := (x \cdot g, \langle \gamma, g\rangle\, z)$ and the canonical identification

$$C(X)_\gamma \cong \Gamma(Y, V_\gamma),$$

associates to the equivariant function $f\colon X \to \mathbb{C}$ the section $F\colon Y \to V_\gamma$ defined by $F([x]) = [x, f(x)]$ for every $[x] \in Y$. The following facts are well known but it is interesting to specialize them to the rank-1 case.

Proposition 3.35. *If the covering $C(Y) \subseteq C(X) \curvearrowleft G$ is rank-1 regular, the projection $X \to Y$ is a regular covering in the topological sense. Moreover the spectral subspaces for every $\gamma \in \widehat{G}$ satisfy all the following equivalent properties:*

(1) there is an invertible $\omega_\gamma \in C(X)_\gamma$.

(2) The bundle V_γ is topologically trivial $V_\gamma \simeq Y \times \mathbb{C}$ (the isomorphism is not required to preserve the flat structure).

(3) The module $C(X)_\gamma$ has rank one over $C(Y)$.

Vice versa if $X \to Y$ is a regular covering and one of the equivalent properties (1), (2), (3) *is satisfied for every $\gamma \in \widehat{G}$ then the covering is rank-1 free.*

In this case, if all the spaces are reasonably good (say CW-complexes) then the above properties are equivalent to the vanishing of the first Chern class $[c_1(V_\gamma)] \in H^2(Y;\mathbb{Z})$.

Proof. At this point the proof is immediate. We give some details and some remarks. First of all, since the rank-1 property is quite strong, the implication: rank-1 regular $\Longrightarrow$ topologically regular can be proven directly since $x \cdot g = x$ implies

$$\langle \gamma, g\rangle \omega_\gamma(x) = (\beta_g \omega_\gamma)(x) = \omega_\gamma(x \cdot g) = \omega_\gamma(x)$$

for every γ. This is impossible unless $g = e$. Concerning the equivalence of the properties, we comment on the (3) $\Longrightarrow$ (2). Assume that $C(X)_\gamma$ has an algebraic generator ω_γ. This corresponds to a section of V_γ that has to be different to zero on every point on Y, and so $\omega_\gamma(x) \neq 0$ for every $x \in X$. The vice versa is clear once we observe that the assumption of topological regularity makes the construction of the vector bundles V_γ possible. In other words the corresponding modules are finitely generated projective. For Chern classes we refer to [8]. $\square$

Notice that if X and Y are manifolds, by Chern-Weil theory since our bundles are flat, the real Chern classes $[c_1(V_\gamma)] \in H^2(Y;\mathbb{R})$ vanish for every $\gamma \in \widehat{G}$. In particular the integer classes $[c_1(V_\gamma)] \in H^2(Y;\mathbb{Z})$ are torsion classes. It follows that if $H^2(Y;\mathbb{Z})$ is torsion free then the regularity assumption (3.28) is satisfied. In general this is not the case as shown by the following example.

Example 3.36 (cf. [57])**.** Let $\mathbb{Z}_2$ act on the 2-sphere as the universal covering of the real projective space $\mathbb{P}^2(\mathbb{R})$. Then the character $\gamma\colon \mathbb{Z}_2 \to SO(2)$ mapping the generator to the antipode produces a flat $SO(2)$-bundle which is non trivial because its Euler class generates $H^*(\mathbb{P}^2(\mathbb{R});\mathbb{Z})$. Under the isomorphism $SO(2) \cong U(1)$, V_γ has a natural complex structure and the first Chern class corresponds to the Euler class. $\square$

Chapter 4

Kasparov's Bivariant K-Theory

The main object of Kasparov's bivariant K-theory [59] consists of a bifunctor which associates a $\mathbb{Z}_2$-graded group $KK_\bullet(A, B)$ to any pair of (suitable) C^*-algebras A and B. This bifunctor unifies both K-theory and K-homology in the sense that

$$KK_\bullet(\mathbb{C}, B) \simeq K_\bullet(B) \qquad \text{and} \quad KK_\bullet(A, \mathbb{C}) \simeq K^\bullet(A).$$

The strength of this construction relies on the existence of a product which generalizes the index pairing between K-theory and K-homology. In this chapter we recall some definitions and facts about KK-theory and its unbounded version, the Kasparov product and the relations of these with group actions. For more details we refer to [10, 16, 53]. For a brief introduction to Hilbert modules, we refer to Appendix A.

4.1 Kasparov Modules and KK-Theory

The main ingredient of Kasparov's KK-theory is the Kasparov module, an object that is modeled on a family of abstract elliptic operators on Hilbert modules.

Definition 4.1 (cf. [59])**.** *Let A and B be graded C^*-algebras. A **Kasparov** $A-B$**-module** is a triple (E, ϕ, F) where E is a countably generated graded right Hilbert B-module, $\phi\colon A \to \mathcal{L}_B(E)$ is a graded $*$-homomorphism and F is an odd operator such that*

$$[F, \phi(a)],\ (F^2 - 1)\phi(a) \quad \text{and} \quad (F - F^*)\phi(a) \in \mathcal{K}_B(E) \tag{4.1}$$

for all $a \in A$. In the following we let $\mathbb{E}(A, B)$ denote the set of Kasparov $A-B$-modules.

Note that the commutator in (4.1) is a graded commutator. Furthermore, there is a binary operation on $\mathbb{E}(A, B)$ given by direct sum.

Example 4.2. Let A and B be graded C^*-algebras. Any graded $*$-homomorphism $\varphi\colon A \to B$ defines a Kasparov $A-B$-module $(B, \phi, 0)$. □

Two Kasparov modules $(E_0, \phi_0, F_0), (E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ are **unitarily equivalent** if there is an even unitary $u \in \mathcal{L}_B(E_0, E_1)$ such that $u\phi_0(a)u^* = \phi_1(a)$ for any $a \in A$ and $uF_0u^* = F_1$. In this case we write $(E_0, \phi_0, F_0) \simeq (E_1, \phi_1, F_1)$.

Two Kasparov $A-B$-modules $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{E}(A, B)$ are said to be **operator homotopic** when there are a graded Hilbert module E, a graded homomorphism $\phi\colon A \to \mathcal{L}_B(E)$ and a norm continuous path $(F_t)_{t\in[0,1]}$ in $\mathcal{L}_B(E)$ such that:

(1) $\mathcal{F}_t := (E, \phi, F_t) \in \mathbb{E}(A, B)$ for all $t \in [0, 1]$

(2) $\mathcal{F}_0 \simeq \mathcal{E}_0$ and $\mathcal{F}_1 \simeq \mathcal{E}_1$.

A Kasparov module $(E, \phi, F) \in \mathbb{E}(A, B)$ is said to be **degenerate** if $[F, \phi(a)] = 0$, $(F^2 - 1)\phi(a) = 0$ and $(F - F^*)\phi(a) = 0$ for any $a \in A$. We write $\mathcal{E}_0 \sim_{oh} \mathcal{E}_1$ if there are degenerate Kasparov modules $\mathcal{D}_0, \mathcal{D}_1 \in \mathbb{E}(A, B)$ such that $\mathcal{E}_0 \oplus \mathcal{D}_0$ is operator homotopic to $\mathcal{E}_1 \oplus \mathcal{D}_1$. This is an equivalence relation on $\mathbb{E}(A, B)$ [53, Lemma 2.1.17].

Definition 4.3. *We define $KK(A, B)$ as the set of equivalence classes of Kasparov $A-B$-modules in $\mathbb{E}(A, B)$ under the relation $\sim_{oh}$.*

The operator homotopy equivalence relation respects direct sums and so $KK(A, B)$ becomes an abelian semigroup under the operation:

$$[(E_1, \phi_1, F_1)] \oplus [(E_1, \phi_1, F_1)] := [(E_0 \oplus E_1, \phi_0 \oplus \phi_1, F_0 \oplus F_1)]$$

If $(E, \phi, F) \in \mathbb{E}(A, B)$ is degenerate, then it is homotopic to the zero module (see [16, Proposition 17.2.3]). A remarkable result of the theory is that this semigroup is actually a group (see e.g. [53, Theorem 2.1.23] or [16, Proposition 17.3.3]).

Remark 4.4. The standard definition of the Kasparov group relies on a slightly different notion of homotopy [16, Section 17.2 and 17.3]. However, since our algebras are assumed separable, these definitions are all equivalent, cf. [53, Theorem 2.2.17].

Remark 4.5. A Kasparov module $(E, \phi, F) \in \mathbb{E}(A, B)$ is said to be a **compact perturbation** of $(E, \phi, F') \in \mathbb{E}(A, B)$ if $(F - F')\phi(a) \in \mathcal{K}_B(E)$ for all $a \in A$. In this case, the straight line homotopy from F to F' provides an operator homotopy between the two modules (see e.g. [16, Prop. 17.2.5]) and thus they represent the same KK-class.

The construction of the KK-group is functorial in both entries:

(1) If $f: A_1 \to A_2$ is a graded homomorphism, then for any B there is an induced map $\mathbb{E}(A_2, B) \to \mathbb{E}(A_1, B)$ given by $(E, \phi, F) \to (E, \phi \circ f, F)$. This map respects direct sum and homotopy and so defines a homomorphism $f^*: KK(A_2, B) \to KK(A_1, B)$ (see [53, Lemma 2.1.25]). Analogous for $KK_{oh}(A, B)$.

(2) If $g: B_1 \to B_2$ is a graded homomorphism, then for any A there is an induced map $\mathbb{E}(A, B_1) \to \mathbb{E}(A, B_2)$ given by $(E, \phi, F) \to (E \widehat{\otimes}_g B, \phi \widehat{\otimes} 1, F \widehat{\otimes} 1)$. This respects direct sum and homotopy and so defines a homomorphism $g_*: KK(A, B_1) \to KK(A, B_2)$ (see [53, Lemma 2.1.26]). Analogous for $KK_{oh}(A, B)$.

Combining these constructions together, we deduce that KK is a bifunctor from pairs of C^*-algebras to abelian groups which is contravariant in the first variable and covariant in the second. Functoriality is particularly meaningful in view of the Example 4.2.

Example 4.6. Let $f: A \to D$ and $g: D \to B$ be graded homomorphism and denote by $[f]$ and $[g]$ the classes of the morphisms respectively in $KK(A, D)$ and $KK(D, B)$. Then $f^*([g])$ and $[g \circ f] \in KK(A, B)$ are both represented by $(B, g \circ f, 0)$ so $f^*([g]) = [g \circ f]$.

On the other hand, the class $g_*([f])$ is represented by $(\overline{g(D)B}, g \circ f, 0)$ which is homotopic to $(B, g \circ f, 0)$. We deduce in particular that $g_*([f]) = [g \circ f]$. □

There is a functorial construction known as **amplification** which is worth to mention. If A, B, D are C^*-algebras, there is a map from $\mathbb{E}(A,B)$ to $\mathbb{E}(A\widehat{\otimes}D, B\widehat{\otimes}D)$ given by $(E,\phi,F) \mapsto (E\widehat{\otimes}D, \phi\widehat{\otimes}1, F\widehat{\otimes}1)$. This map preserves direct sums and the equivalence relation (see e.g. [53, Lemma 2.1.27]) and so induces homorphisms $\tau_D\colon KK(A,B) \to KK(A\widehat{\otimes}D, B\widehat{\otimes}D)$. Note that this homomorphism is natural in both variable.

Proposition 4.7. *For any C^*-algebras A and B, the map*

$$\tau_{\mathcal{K}}\colon KK(A,B) \to KK(A\widehat{\otimes}\mathcal{K}, B\widehat{\otimes}\mathcal{K})$$

is an isomorphism. In particular, KK is a stable bifunctor.

Complex Clifford Algebras are used to define the higher-order KK-groups. In our convention, $\mathbb{C}\ell_n$ is the complex algebra generated by the skew-adjoint anti-commuting elements $e_1, \dots, e_n$ such that $e_i^2 = -1$ and graded in the standard way.

Definition 4.8. *We define $KK_n(A,B) \coloneqq KK(A, B\widehat{\otimes}\mathbb{C}\ell_n)$.*

Using the fact that $\mathbb{C}\ell_n \simeq \mathrm{End}(\mathbb{C}^{2^{\frac{n}{2}}})$ for n even and that KK is a stable bifunctor, we deduce that we have only two groups to consider and that we have the so called **formal Bott periodicity**:

$$KK(A\hat{\otimes}\mathbb{C}\ell_1, B) \simeq KK(A, B\hat{\otimes}\mathbb{C}\ell_1). \tag{4.2}$$

4.2 The Kasparov Product

The Kasparov product [59, §4, Theorem 4] is a pairing

$$\widehat{\otimes}_D\colon KK_i(A_1, B_1\widehat{\otimes}D) \times KK_j(D\widehat{\otimes}A_2, B_2) \longrightarrow KK_{i+j}(A_1\widehat{\otimes}A_2, B_1\widehat{\otimes}B_2)$$

for suitable C^*-algebras which has many functorial properties and generalizes composition and tensor product of $*$-homomorphism. It is the greatest achievement of the theory, both for the great generality of the assumptions and the technicalities of the non-constructive proof. A more constructive description of it was introduced and developed in [21, 102] thanks to the notion of a *connection* which we now recall.

Let E_1 be a graded Hilbert A-module, E_2 a graded Hilbert B-module and $\phi\colon A \to \mathcal{L}_B(E_2)$ be a graded homomorphism. We can form the internal tensor product $E_{12} = E_1\widehat{\otimes}_\phi E_2$ as in Subsection A.1.1 which is a graded Hilbert B-module under the map S_{12} given by $S_{12}(e_1 \otimes e_2) = S_{E_1}(e_1) \otimes S_{E_2}(e_2)$. For each $x \in E_1$ we can define the creation operator $T_x \in \mathcal{L}_B(E_2, E_{12})$ by $T_x(e_2) = x \otimes e_2$ for $e_2 \in E_2$. An easy computation shows that T_x^* is given by $T_x^*(e_1 \otimes e_2) = \phi(\langle x, e_1\rangle)e_2$ for $e_1 \in E_1$ and $e_2 \in E_2$. Note that T_x depends linearly on x and that $\|T_x\| = \|T_x^*\| \leq \|x\|$ for any $x \in E_1$.

Definition 4.9. *Let $F_2 \in \mathcal{L}_B(E_2)$. An element $F \in \mathcal{L}_B(E_{12})$ is called an F_2-**connection** for E_1 when*

$$\begin{aligned} T_x \circ F_2 - (-1)^{\deg(x)\deg(F_2)} F \circ T_x \in \mathcal{K}_B(E_2, E_{12}) \\ F_2 \circ T_x^* - (-1)^{\deg(x)\deg(F_2)} T_x^* \circ F \in \mathcal{K}_B(E_{12}, E_2) \end{aligned} \tag{4.3}$$

*for any $x \in E_1$. A Kasparov $A-C$-module (E_{12}, ϕ, F) is called a **Kasparov product** for (E_1, ϕ_1, F_1) in $\mathbb{E}(A,B)$ and $(E_2, \phi_2, F_2) \in \mathbb{E}(B,C)$ when F is an F_2-connection for E_1 and $\phi(a)[F_1\widehat{\otimes}1, F]\phi(a)^* \geq 0 \mod \mathcal{K}_C(E_{12})$ for all $a \in A$.*

It is possible to prove that for any $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$ there exists a Kasparov product $(E_{12}, \phi, F) \in \mathbb{E}(A, C)$ and that this is unique up to operator homotopy [53, Theorem 2.2.8].

Theorem 4.10. *There exists a bilinear map*

$$\widehat{\otimes}_D\colon KK(A, D) \times KK(D, B) \longrightarrow KK(A, B)$$

which takes two classes $[(E_1, \phi_1, F_1)]$ and $[(E_2, \phi_2, F_2)]$ and gives back the class of the Kasparov product of the representatives. Furthermore, this map is associative in the sense that

$$\mathbf{x}\widehat{\otimes}_{D_1}(\mathbf{y}\widehat{\otimes}_{D_2}\mathbf{z}) = (\mathbf{x}\widehat{\otimes}_{D_1}\mathbf{y})\widehat{\otimes}_{D_2}\mathbf{z}$$

for any $\mathbf{x} \in KK(A, D_1)$, $\mathbf{y} \in KK(D_1, D_2)$ and $\mathbf{z} \in KK(D_2, B)$.

Example 4.11. Let $f\colon A \to D$ be a morphism and $(D, f, 0)$ its Kasparov module in $\mathbb{E}(A, D)$ as in Example 4.2. Given any $(E, \phi, F) \in \mathbb{E}(D, B)$ with ϕ essential, we have that $D\widehat{\otimes}\phi E \simeq E$ and, under this identification, $(E, \phi \circ f, F)$ is a Kasparov product for $(D, f, 0)$ and (E, ϕ, F). In particular, the product represents the pullback (see Example 4.6):

$$f^*([(E, \phi, F)]) = [f]\widehat{\otimes}_D[(E, \phi, F)] \tag{4.4}$$

Analogously, given $g\colon B \to A$, we have that $g_*([(E, \phi, F)]) = [(E, \phi, F)]\widehat{\otimes}_B[g]$. □

Having Example 4.11 in mind, one may regard the elements in the KK groups as morphisms over separable algebras where the composition rule is given by the Kasparov product. This category has the remarkable property of being triangulated and is universal in the sense that split exact stable functor from the category of C^*-algebras to abelian groups factors through the category KK (see for instance [30, 48]). The next proposition summarizes the important functoriality properties of the product.

Proposition 4.12. *Let $A, A', B, B'D, D'$ be graded C^*-algebras. Let $f\colon A' \to A$, $g\colon B \to B'$, $h\colon D \to D'$ be graded $*$-homomorphisms. Let $\mathbf{x} \in KK(A, D)$, $\mathbf{y} \in KK(D, B)$ and $\mathbf{z} \in KK(D', B)$. Then:*

(1) $f^*(\mathbf{x}\widehat{\otimes}_D\mathbf{y}) = f^*(\mathbf{x})\widehat{\otimes}_D\mathbf{y}$

(2) $h_*(\mathbf{x}\widehat{\otimes}_{D'}\mathbf{z}) = \mathbf{x}\widehat{\otimes}_{D'}h^*(\mathbf{z})$

(3) $g_*(\mathbf{x}\widehat{\otimes}_D\mathbf{y}) = \mathbf{x}\widehat{\otimes}_D g_*(\mathbf{y})$

Proof. See [16, Proposition 18.7.1]. □

4.3 Unbounded KK-Theory

It is sometimes convenient to define Kasparov modules and classes using an unbounded operator D instead of a bounded Hilbert module map. In doing so, the issues in the theory of unbounded operators are even more delicate as, differently from Hilbert spaces, closed submodule of a C^*-module need not be orthogonally complemented.

Definition 4.13 (cf. [10])**.** *Let B be a C^*-algebra and E a (right) Hilbert B-module. A densely defined closed operator $D\colon \operatorname{Dom} D \to E$ is called **regular** if*

(1) D^* is densely defined in E

(2) $1 + D^*D$ has dense range

*A regular operator D is **symmetric** if $\operatorname{Dom} D \subseteq \operatorname{Dom} D^*$ and $D = D^*$ on $\operatorname{Dom} D$. It is **selfadjoint** if it is symmetric and $\operatorname{Dom} D = \operatorname{Dom} D^*$.*

It is possible to show that if $D\colon \operatorname{Dom} D \to E$ is regular, then D is automatically B-linear and $\operatorname{Dom} D$ is a B-submodule of E. Furthermore, in this case, D^*D is self-adjoint and regular (see [71, Proposition 1.3.2]).

Definition 4.14 (cf. [10]). *Let A and B be $\mathbb{Z}_2$-graded C^*-algebras. An **unbounded Kasparov (A, B)-module** is a triple (E, ϕ, D) where E is a graded right Hilbert B-module, $\phi\colon A \to \mathcal{L}_B(E)$ a graded C^*-algebra homomorphism and $D\colon \operatorname{Dom}(D) \to E$ a self-adjoint regular odd operator such that:*

(1) there exists a dense subalgebra $\mathcal{A} \subseteq A$ such that $\phi(a)\operatorname{Dom}(D) \subseteq \operatorname{Dom}(D)$ and the commutators $[D, \phi(a)]$ are bounded for every $a \in \mathcal{A}$

(2) the operator $(1 + D^2)^{-1/2}\phi(a)$ extends to a compact operator on E for any $a \in A$

We denote the space of such modules by $\Psi(\mathcal{A}, B)$.

As in the bounded case we set $\Psi_j(A, B) \coloneqq \Psi(A, B\widehat{\otimes}\mathbb{C}\ell_j)$ where $\mathbb{C}\ell_j$ is the complex Clifford algebra with j generators. Bounded and unbounded Kasparov modules are related by the following result.

Theorem 4.15 (cf. [10]). *Let A and B be $\mathbb{Z}_2$-graded C^*-algebras and (E, ϕ, D) an unbounded Kasparov (A, B)-module. The **bounded transform** of D, defined by*

$$\mathfrak{b}(D) \coloneqq D(1 + D^2)^{-\frac{1}{2}}, \tag{4.5}$$

is a bounded operator on E_B. Furthermore, the triple $(E, \phi, \mathfrak{b}(D))$ is a (bounded) Kasparov module.

The bounded transform induces a map $\mathfrak{b}\colon \Psi(A, B) \to KK(A, B)$ which send any unbounded module to the KK-class of its bounded transform. As pointed out in [10], if $(E, \phi, D) \in \Psi(A, B)$ and $T \in \mathcal{L}_B(E)$ is odd self-adjoint, then the class induced by (E, ϕ, D) and by $(E, \phi, D + T)$ in $KK(A, B)$ is the same. In this case we say that $(E, \phi, D + T)$ is a **bounded perturbation** of (E, ϕ, D).

Theorem 4.16 (cf. [10]). *The bounded transform $\mathfrak{b}\colon \Psi(A, B) \to KK(A, B)$ is surjective.*

The bounded transform map $\mathfrak{b}\colon \Psi(A, B) \to KK(A, B)$ is clearly not injective in general as we need to suitably identify unbounded modules up to homotopy and "quotienting" the kernel of the transform; this kernel is identified by a degeneracy condition known as spectral decomposability.

Definition 4.17 (cf. [55]). *An unbounded Kasparov module (E, ϕ, D) is **spectrally decomposable** when there exists an orthogonal projection $P\colon E \to E$ such that:*

(1) P preserves the domain of D and $[D, P] = 0$ on $\operatorname{Dom} D$

(2) *DP and $D(P-1)$ are unbounded positive and regular operators*

(3) *$\phi(a)P = P\phi(a)$ for all even elements $a \in A$ and $\phi(a)P = (1-P)\phi(a)$ for all odd elements $a \in A$*

(4) *$\chi P = (1-P)\chi$, where $\chi \colon E \to E$ is the $\mathbb{Z}_2$-grading.*

Proposition 4.18 (cf. [55])**.** *Let A and B be $\mathbb{Z}_2$-graded C^*-algebras and (E, ϕ, D) an unbounded Kasparov $A-B$-module. If (E, ϕ, D) is spectrally decomposable, then the bounded Kasparov class $[(E, \phi, \mathfrak{b}(D))] \in KK(A, B)$ is the null class.*

Similarly to the bounded case, it is possible to define an homotopy between unbounded Kasparov modules up to spectrally decomposable modules (which in [55] is called "stable homotopy") and it turns out that this is an equivalence relation. The quotient of $\Psi_1(A, B)$ under this relation is denoted by $UK(A, B)$ and the bounded transform $\mathfrak{b}$ induces a map

$$\mathfrak{b} \colon UK(A, B) \longrightarrow KK(A, B) \tag{4.6}$$

which is an isomorphism (see [55, Theorem 7.1]).

One of the greatest challenges of the bounded picture of Kasparov's bivariant K-theory is the fact there is not a closed formula to describe the interior product of its modules: one has to make a guess and then show that it has the properties of a product. As shown in [72] and its subsequent developments [18, 56], in the unbounded picture it is possible to define directly (by an algebraic formula) the interior product of Kasparov modules by introducing the notion of a (suitably) smooth connection on an operator space, notion which we now briefly recall in a form which will be of interest for us. We take for granted the definitions in Appendix B.

Let A_1 be an operator $*$-algebra and assume that it is dense inside a C^*-algebra A. The inclusion $A_1 \to A$ is completely bounded. Let $P \colon \mathbb{H}_{A_1} \to \mathbb{H}_{A_1}$ be a completely bounded self-adjoint idempotent and $X_1 = P\mathbb{H}_{A_1}$ an operator $*$-module; P extends to an orthogonal projection $P \colon \mathbb{H}_A \to \mathbb{H}_A$ and $X \coloneqq P\mathbb{H}_A$ is a Hilbert A-module. The inclusion $X_1 \to X$ is completely bounded and compatible with both the inner products and the module actions. Suppose now we are given a right Hilbert B-module Y together with a $*$-representation $\pi \colon A \to \mathcal{L}_B(Y)$ and a self-adjoint densely defined unbounded operator D on Y such that each $a \in A_1$ maps the domain of D into itself and the commutator with D yields a completely bounded map $[D, \cdot] \colon A_1 \to \mathcal{L}_B(Y)$. We denote by $X \otimes_A \mathcal{L}_B(Y)$ the interior tensor product over A; this is a right Hilbert A-module via the action $(x \otimes T) \cdot a = x \otimes T\pi(a)$.

Definition 4.19. *A* ***connection*** *for the triple (A, Y_B, D) is a completely bounded linear map $\nabla \colon X_1 \to X \otimes_A \mathcal{L}_B(Y)$ such that*

$$\nabla(x \cdot a) = \nabla(x) \cdot a + x \otimes [D, a]$$

for all $x \in X_1$ and $a \in A_1$.

Given a connection $\nabla \colon X_1 \to X \otimes_A \mathcal{L}_B(Y)$, we write $c(\nabla) \colon X_1 \otimes_A Y \to X \otimes_A Y$ for the composition of maps

$$\begin{aligned} c(\nabla) \colon X_1 \otimes_A Y \xrightarrow{\nabla \otimes 1} X \otimes_A \mathcal{L}_B(Y) \otimes Y &\xrightarrow{\ c\ } X \otimes_A Y \\ x \otimes T \otimes y &\longrightarrow x \otimes Ty \end{aligned} \tag{4.7}$$

where c is the contraction map.

The inner product on X induces the pairing $X \times X \otimes_A \mathcal{L}_B(Y) \to \mathcal{L}_B(Y)$ "on the first entry" by $(x, y \otimes T) = \langle x, y \rangle \cdot T$.

Definition 4.20. *A connection for the triple (A, Y_B, D) is said to be **hermitian** if*

$$[D, \langle x, y \rangle] = (x, \nabla(y)) - (y, \nabla(x))^*$$

for all $x, y \in X_1$.

As shown in the following example, under mild assumptions any operator $*$-module X_1 carries a hermitian connection and this is essentially the commutator $[D, \cdot]$ under the identification $X_1 = P\mathbb{H}_{A_1}$.

Example 4.21 (Grassmann connection)**.** We define Ω^1_D to be the smallest C^*-subalgebra of $\mathcal{L}_B(Y)$ generated by the elements of the form $[D, a_1]$ and $\pi(a)$ for $a_1 \in A_1$ and $a \in A$. We endow Ω^1_D with its natural structure of an $A - \Omega^1_D$ Hilbert C^*-module. We further suppose that the action of A on Ω^1_D is **essential**, namely such that $A \cdot \Omega^1_D$ is dense in Ω^1_D. This condition provides an isomorphism $\mathbb{H}_A \otimes_A \Omega^1_D \simeq \mathbb{H}_{\Omega^1_D}$ of Hilbert C^*-modules. The composition of maps

$$X_1 \longrightarrow \mathbb{H}_{A_1} \xrightarrow{[D,\cdot]} \mathbb{H}_{\Omega^1_D} \simeq \mathbb{H}_A \otimes_A \Omega^1_D \xrightarrow{P \otimes 1} X \otimes_A \Omega^1_D \longrightarrow X \otimes_A \mathcal{L}_B(Y)$$

is a hermitian connection (cf. [56]) on X_1 called the **Grassmann connection**. □

In general, the operator $1 \otimes D$ on $X \otimes_A Y$ is not well defined as D is not A-linear. This can be fixed as follows thanks to the existence of a connection. Suppose now that A_1 is σ-unital, that the action of A on $\mathcal{L}_B(Y)$ is essential and that there exists a connection ∇ on X_1. We define the densely operator $1 \otimes_\nabla D$ on $X_1 \otimes_A Y$ by

$$1 \otimes_\nabla D \coloneqq 1 \otimes D + c(\nabla), \tag{4.8}$$

with $c(\nabla)$ defined in (4.7). Regarding X_1 as $P\mathbb{H}_{A_1}$, the operator (4.8) is nothing but the diagonal action of D on every entry of the sequence [56, Lemma 5.1]. It is well known that (4.8) is self-adjoint regular and well defined on the interior tensor product. We are now ready to compute the internal Kasparov product of two modules.

Theorem 4.22 (cf. [56])**.** *Let (X, D_1) and (Y, D_2) be two odd unbounded Kasparov modules for (A, B) and (B, C) respectively and suppose that $B \cdot Y$ is dense in Y and that $B \cdot \Omega^1_{D_2}$ is dense in $\Omega^1_{D_2}$. Suppose that there is a correspondence (X_1, ∇^0) from (X, D_1) to (Y, D_2), namely an operator $*$-module X_1 over a σ-unital operator $*$-algebra B_1 and a completely bounded hermitian D_2-connection $\nabla^0 \colon X_1 \to X \otimes_B \mathcal{L}_C(Y)$ such that*

(1) the operator $*$-module $X_1 \subseteq X$ is a dense subspace of X and the operator $*$-algebra $B_1 \subseteq B$ is a dense $*$-subalgebra of B. The inclusions are completely bounded and compatible with the module structures and inner products.

(2) each $b \in B_1$ maps the domain of D_2 into itself and the derivation $[D_2, \cdot] \colon B_1 \to \mathcal{L}_C(Y)$ is completely bounded on B_1.

(3) the commutator $[1 \otimes_{\nabla^0} D_2, a]$ is well defined and extends to a bounded operator on $X \otimes_B Y$ for all $a \in A$.

(4) for any $\mu \in \mathbb{R} \setminus \{0\}$, the unbounded operator

$$[D_1 \otimes 1, 1 \otimes_{\nabla^0} D_2]\,(D_1 \otimes 1 - i\mu)^{-1}$$

is well-defined and extends to a bounded operator on $X \otimes_B Y$.

Then, for any hermitian D_2-connection $\nabla_{D_2} \colon X_1 \to X \otimes_B \mathcal{L}_C(Y)$, the operator

$$D_1 \times_\nabla D_2 := \begin{pmatrix} 0 & D_1 \otimes 1 - i1 \otimes_\nabla D_2 \\ D_1 \otimes 1 + i1 \otimes_\nabla D_2 & 0 \end{pmatrix}$$

on $(X \otimes_B Y) \oplus (X \otimes_B Y)$ is an even unbounded Kasparov $A-C$-module which represents the interior Kasparov product of (X, D_1) and (Y, D_2).

The proof of this theorem relies on the well known Kucerovsky criterion [64], which provides sufficient conditions to establish when a candidate Kasparov module is the interior product of others two given. We mention that his work has been recently improved and generalized in [36], with the advantage that the positivity condition in Kucerovsky's original result (which was a global condition) is there replaced by a "local" positivity condition, in the full spirit of the Connes-Skandalis approach to the bounded Kasparov product [21, 102]. Let us now show how the exterior Kasparov product looks in the unbounded picture.

Theorem 4.23 (cf. [10])**.** *Let (E_i, D_i) be unbounded bimodules for (A_i, B_i), $i = 1, 2$. The operator*

$$D_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} D_2 \colon \operatorname{Dom}(D_1) \widehat{\otimes} \operatorname{Dom}(D_2) \longrightarrow E_1 \widehat{\otimes} E_2$$

extends to a self-adjoint regular operator with compact resolvent. Moreover, the diagram

$$\begin{array}{ccc} \Psi_i(A_1, B_1) \times \Psi_j(A_2, B_2) & \longrightarrow & \Psi_{i+j}(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2) \\ \Big\downarrow \mathfrak{b} & & \Big\downarrow \mathfrak{b} \\ KK_i(A_1, B_1) \times KK_j(A_2, B_2) & \xrightarrow{\widehat{\otimes}} & KK_{i+j}(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2) \end{array}$$

commutes.

4.4 Equivariant KK-Theory

In this section we describe the KK-theory of algebras which carry an action of a locally compact topological group G and give a survey of the most important properties. The original theory is contained in [58, 59]; we refer also to [16, §20] and [37] for more details.

Definition 4.24. *A C^*-algebra B with a strongly continuous action $\beta \colon G \to \operatorname{Aut}(B)$ of a locally compact Hausdorff topological group G is called a G-**algebra**.*

Definition 4.25. *Let B be a G-algebra and E a (right) Hilbert B-module. A continuous action of G on E is a homomorphism from G into the space of bounded linear transformations on E (not necessarily the space of module homomorphisms $\mathcal{L}_B(E)$) such that it is continuous in the strong operator topology and*

$$g \cdot (xb) = (g \cdot x)\beta_g(b)$$

*for any $g \in G$, $x \in E$ and $b \in B$. A Hilbert B-module with a continuous action of G is called a G-**equivariant Hilbert** B-**module**.*

If E_1 and E_2 are G-equivariant Hilbert B-modules, there is a natural induced action of G on $\mathcal{L}(E_1, E_2)$ given by conjugation. In general, this action $g \mapsto g \cdot T$ for $T \in \mathcal{L}(E_1, E_2)$ is not norm-continuous: we say that T is G-continuous if it is. Obviously every G-equivariant map is G-continuous. Furthermore, if G is compact, any continuous map can be averaged over G to give a canonically associated G-equivariant map. In the case of a graded C^*-algebras and graded Hilbert modules, we require that the action of the group G preserves the subspaces of homogeneous elements.

Definition 4.26. *Let A and B be graded G-algebras. A G-equivariant Kasparov $A - B$-module is a triple (E, ϕ, F) where E is a G-equivariant countably generated Hilbert B-module, $\phi\colon A \to \mathcal{L}_B(E)$ is an equivariant graded $*$-homomorphism and F is an even G-continuous operator in $\mathcal{L}(E)$ such that*

$$[F, \phi(a)],\ (F^2 - 1)\phi(a),\ (F - F^*)\phi(a) \text{ and } (g \cdot F - F)\phi(a) \tag{4.9}$$

are compact operators on E for all $a \in A$ and $g \in G$. The set of G-equivariant Kasparov $A - B$-modules is denoted by $\mathbb{E}_G(A, B)$. The set $\mathbb{D}_G(A, B)$ of degenerate equivariant Kasparov modules is defined correspondingly.

The equivalence relation $\sim_{oh}$ is defined as in Section 4.1 and $KK^G(A, B)$ is the quotient of $\mathbb{E}_G(A, B)$ by $\sim_{oh}$. As in the non-equivariant situation, we define

$$KK_n^G(A, B) \coloneqq KK^G(A, B \hat{\otimes} \mathbb{C}\ell_n)$$

where $\mathbb{C}\ell_n$ is the complex Clifford algebra and $KK_\bullet^G(A, B)$ is an abelian group for the direct sum. Furthermore, $KK^G(\cdot, \cdot)$ is a bifunctor from pairs of G-algebras to abelian groups, contravariant in the first variable and covariant in the second.

Remark 4.27. When G is compact, it is always possible to assume that the representative (E, ϕ, F) of any equivariant Kasparov class is such that $g \cdot F = F$. Indeed, averaging F over the group will yield a compact perturbation with the desired property.

Example 4.28. Let A and B be graded G-algebras and $f\colon A \to B$ a $*$-homomorphism which is G-equivariant, namely such that

$$f(\alpha_g(a)) = \beta_g(f(a)).$$

The triple $(B, f, 0)$ defines canonically an element $[f] \in KK^G(A, B)$. □

Also in this case there exists an intersection product and the proofs are similar. More precisely, given the G-algebras A_1, A_2, B_1, B_2 and D, there exists a bilinear pairing

$$\hat{\otimes}_D\colon KK_m^G(A_1, B_1 \hat{\otimes} D) \times KK_n^G(D \hat{\otimes} A_2, B_2) \longrightarrow KK_{n+m}^G(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$$

which is associative and functorial in all possible senses. As a consequence of this pairing, one has Bott periodicity.

In the following subsections we present two fundamental relations between equivariant KK-theory and KK-theory of crossed products.

4.4.1 The Green-Julg Isomorphism

The Green-Julg isomorphisms [54] allow to describe the equivariant K-theory or K-homology of a certain algebra in terms of the K-theory or K-homology respectively of a crossed product. More precisely:

(1) If G is compact, then $KK^G_\bullet(\mathbb{C}, B) \simeq KK_\bullet(\mathbb{C}, B \rtimes G)$.

(2) If G is discrete, then $KK^G_\bullet(A, \mathbb{C}) \simeq KK_\bullet(A \rtimes G, \mathbb{C})$.

Note that the crossed products are maximal and that this isomorphisms can be easily generalized to the case where $\mathbb{C}$ is replaced by any C^*-algebra which carries a trivial action of G. Furthermore, the two isomorphisms can be given explicitely on the representatives of the equivalence classes. Let us begin by describing the case of G compact.

Let E be a G-equivariant Hilbert B-module for G compact and B a G-C^*-algebra. Then E becomes a pre-Hilbert $B \rtimes_\beta G$-module if we define the right action of $B \rtimes_\beta G$ on E and the $B \rtimes_\beta G$-valued inner products by the formulas:

$$x \cdot f := \int_G u_s(x \cdot f(s^{-1}))ds \qquad \langle x_1, x_2\rangle_{B\rtimes_\beta G}(s) := \langle x_1, u_s(x_2)\rangle_B$$

for $x, x_1, x_2 \in E$ and $f \in C(G, B) \subseteq B \rtimes_\beta G$. We denote by $E_{B\rtimes_\beta G}$ its completion. It is easy to see that if (E, ϕ, T) represents an element of $KK^G(\mathbb{C}, B)$ with T being G-invariant (and note that this can always been done for G compact according to Remark 4.27), then T extends to an operator on $E_{B\rtimes_\beta G}$ such that $(E_{B\rtimes_\beta G}, \phi, T)$ represents an element of $KK(\mathbb{C}, B \rtimes_\beta G)$ [38, Lemma 0.1]. We have the following result.

Theorem 4.29 (Green-Julg Isomorphism [10, 38])**.** *Let G be a compact group and B a G-C^*-algebra. The map $\Pi\colon KK^G(\mathbb{C}, B) \to KK(\mathbb{C}, B \rtimes_\beta G)$ given by*

$$\Pi([(E_B, \phi, T)]) := [(E_{B\rtimes_\beta G}, \phi, T)] \tag{4.10}$$

(for T the G-invariant representative) is an isomorphism.

Sketch of the proof. We define a map $\nu\colon KK(\mathbb{C}, B \rtimes_\beta G) \to KK^G(\mathbb{C}, B)$ and show that it is inverse to Π. Let $L^2(G, B)$ denote the Hilbert B-module defined as the completion of $C(G, B)$ with respect to the B-valued inner product

$$\langle f, g\rangle_B := \int_G \beta_s\left(f(s^{-1})^* g(s^{-1})\right) ds$$

and the action of B on $L^2(G, B)$ given by $(f \cdot b)(t) := f(t)\beta_t(b)$ for $f \in C(G, B)$ and $b \in B$. There is a well defined left action of $B \rtimes_\beta G$ on $L^2(G, B)$ given by convolution (when restricted to $C(G, B) \subseteq B \rtimes_\beta G$ and to $C(G, B) \subseteq L^2(G, B)$). Assume now that (E, ϕ, T) represents an element of $KK(\mathbb{C}, B \rtimes_\beta G)$. We can therefore consider the tensor product $E \otimes_{B\rtimes_\beta G} L^2(G, B)$ which is a G-equivariant Hilbert B-module under the action $\mathrm{id} \otimes \rho$ for $\rho\colon G \to \mathrm{Aut}L^2(G, B)$ defined by $\rho_s(f)(t) = f(ts)$ for $f \in C(G, B)$. Then $(E \otimes L^2(G, B), \phi \otimes 1, T \otimes 1)$ represents an element of $KK^G(\mathbb{C}, B)$; the map ν is then defined by

$$\nu[(E, \phi, T)] := [(E \otimes L^2(G, B), \phi \otimes 1, T \otimes 1)] \tag{4.11}$$

$\square$

Let us now discuss the isomorphism for G discrete (see [37, Example 4.9]). Given a G-equivariant Kasparov $A - \mathbb{C}$-module (E, ϕ, T, u), we have that the couple (ϕ, u) is a covariant representation for the action α of G on A. The integrated form $\phi \rtimes u$ represents the maximal crossed product $A \rtimes G$ on H and so this defines a Kasparov $A \rtimes_\alpha G - \mathbb{C}$-module $(E, \phi \rtimes u, T)$. Note that the discreteness of G is needed to prove that the quantities in (4.9) are compacts.

Theorem 4.30 (Green-Julg Isomorphism, Dual Version [10, 37, 38]). *Let G be a discrete group and (A, G, α) a dynamical system. The map $I_G\colon KK^G(A, \mathbb{C}) \to KK(A \rtimes_\alpha G, \mathbb{C})$ given by*

$$I_G([(E, \phi, T, U)]) \coloneqq [(E, \phi \rtimes u, T)] \tag{4.12}$$

is an isomorphism.

The construction of the inverse map relies on the universal property of the maximal crossed product for which any non degenerate homomorphism $\Phi\colon A \rtimes_\alpha G \to \mathcal{L}(H)$ must be of the form $\Phi = \phi \rtimes u$ for a covariant couple (ϕ, u).

4.4.2 The Kasparov Descent

Let (B, G, β) a C^*-dynamical system and E a right Hilbert B-module. The algebra $C_c(G, B)$ acts on $C_c(G, E)$ by

$$(x \cdot f)(t) \coloneqq \int_G x(s)\beta_s\left(f(s^{-1}t)\right) ds \tag{4.13}$$

for $f \in C_c(G, B)$ and $x \in C_c(G, E)$. We define the crossed product $E \rtimes_\beta G$ as the right Hilbert $B \rtimes_\beta G$-module obtained by completing the right $C_c(G, B)$-module $C_c(G, E)$ with respect to the $C_c(G, B)$-valued scalar product

$$\langle x, y\rangle(t) \coloneqq \int_G \beta_{s^{-1}}\left(\langle x(s), y(st)\rangle_B\right) ds, \qquad x, y \in C_c(G, E). \tag{4.14}$$

Remark 4.31. Note that the module $E \rtimes_\beta G$ as just defined is different from the module $E \rtimes_\beta G$ defined for the Green-Julg isomorphism and that, in this case, for its construction a G-action on E is not needed.

Suppose now to have a covariant Kasparov module $(E, \phi, F) \in \mathbb{E}_G(A, B)$. The action $\phi\colon A \to \mathcal{L}_B(E)$ induces a left action $\psi\colon A \rtimes_\alpha G \longrightarrow \mathcal{L}(E \rtimes_\beta G)$ by

$$(\psi(a)x)(t) \coloneqq \int_G \phi(a(s)) \cdot [s \cdot x(s^{-1}t)] ds \tag{4.15}$$

for $a \in C_c(G, A)$ and $x \in C_c(G, E)$. Endowed with this action and the operator $\widetilde{F} \in \mathcal{L}(E \rtimes_\beta G)$ defined by

$$(\widetilde{F}x)(t) \coloneqq F(x(t)) \qquad x \in C_c(G, E),$$

one can show that $(E \rtimes_\beta, \psi, \widetilde{F})$ becomes a $A \rtimes_\alpha G - B \rtimes_\beta G$ Kasparov module.

Theorem 4.32 (Kasparov Descent [59]). *Let (B, G, β) a C^*-dynamical system and $(E, \phi, F) \in \mathbb{E}_G(A, B)$. The map J_G sending the equivariant Kasparov module (E, ϕ, F) to $(E \rtimes_\beta G, \psi, \widetilde{F})$ induces a homomorphism of groups*

$$J_G\colon KK^G(A, B) \longrightarrow KK(A \rtimes_\alpha G, B \rtimes_\beta G)$$

which is functorial in A and B and is compatible with the intersection product. Furthermore, when $A = B$, the map J_G is unital in the sense that $J_G(\mathbf{1}_A) = \mathbf{1}_{A \rtimes_\alpha G}$.

An analogous statement holds true also for reduced crossed products. Note that this construction works also for unbounded modules as leaves the operator essentially untouched: an easy computation shows indeed that the Kasparov descent construction commutes with the bounded transform.

Remark 4.33. When G is discrete, it is known (cf. [6, Remark 1.15]) that there is a commutative diagram

$$\begin{tikzcd} KK^G(A, \mathbb{C}) \arrow[rr, "I_G"] \arrow[d, "J_G"'] & & KK(A \rtimes_\alpha G, \mathbb{C}) \\ KK(A \rtimes_\alpha G, C^*(G)), \arrow[urr, "\epsilon"'] & & \end{tikzcd}$$

where the map ϵ is induced by the trivial representation of G.

Chapter 5

Spectral Triples and Geometry

Spectral triples on C^*-algebras are a central notion in noncommutative geometry, being modeled on the geometric structure codified by the properties of the commutative $*$-algebra $C^\infty(M)$ of smooth functions on a compact spin manifold M and the Dirac operator on the Hilbert space of square-integrable spinors. In this chapter we recall how a spectral triple can determine a geometric space and its interactions with (co)actions of a group.

5.1 Connes' Spectral Manifolds

In this section we recall some basic well known definitions, facts and examples about spectral triples, with particular emphasis on some of the Connes axioms for the spectral reconstruction theorem. For more details we refer to [23–26, 28, 44, 68].

Definition 5.1. *An **odd spectral triple** $(\mathcal{A}, H, D)$ on a unital C^*-algebra A consists of a unital dense $*$-subalgebra $\mathcal{A} \subseteq A$, a representation $\pi\colon A \to \mathcal{L}(H)$ on a Hilbert space H and a self-adjoint operator D (called a **Dirac operator**) densely defined on H such that $(1+D^2)^{-\frac{1}{2}}$ is compact, $\pi(a)(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ and the commutator $[D, \pi(a)]$ extends to a bounded operator on H for every $a \in \mathcal{A}$.*

If the representation π is not clear from the context, we will use the notation (A, H, D, π). In general, the algebra $\mathcal{A}$ is not necessarily coincident with the **Lipschitz algebra** $C^{Lip}(A)$, i.e. the algebra of all those $a \in A$ leaving invariant the domain of D and for which the operator $[D, \pi(a)]$ is bounded. In this sense, the operator $a \mapsto [D, \pi(a)]$ can be seen as a closable derivation from A to $\mathcal{L}(H)$, where the domain of its closure is precisely $C^{Lip}(A)$. It is known that the Lipschitz algebra is a Banach $*$-algebra with respect to the graph norm

$$\|a\|_1 \coloneqq \|a\| + \|[D, \pi(a)]\| \,.$$

For a proof, we direct the reader to [13, Lemma 1].

Example 5.2 (cf. [20, 44, 68])**.** Let M be a compact spin manifold [66]. Denote by $\pi\colon S \to M$ the spin bundle and let $H = L^2(S)$ be the Hilbert space completion of the space of smooth sections $\Gamma(M, S)$, ∇^S the lift of the Levi-Civita connection to S and γ the Clifford action of the dual tangent space on S. Define the **Dirac operator** as $D = \gamma \circ \nabla^S$. Then $(C^\infty(M), L^2(S), D)$ is a spectral triple. □

Definition 5.3. *A spectral triple $(\mathcal{A}, H, D)$ on a unital C^*-algebra A is called* ***non degenerate*** *when it is the case that the representation π is faithful and $[D, \pi(a)] = 0$ for $a \in \mathcal{A}$, if and only if $a \in \mathbb{C}1_A$. It is called* ***irreducible*** *if there is no closed subspace of H invariant under the action of A and D.*

An **even spectral triple** on A is given by the same data with the addition of a $\mathbb{Z}_2$-grading, namely a self-adjoint operator $\chi\colon H \to H$ called a **grading operator** such that $\chi^2 = 1_H$, $\pi(a)\chi = \chi\pi(a)$ for all $a \in A$, $\chi(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ and $D\chi = -\chi D$.

In the case of an even spectral triple, it is always possible to fix a basis of H in such a way that $H = H_0 \oplus H_1$ and

$$\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \pi = \begin{pmatrix} \pi_0 & 0 \\ 0 & \pi_1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix}.$$

Sometimes it is useful to think of odd spectral triples as even triples under the grading $\chi = \mathrm{id}_H$. In this way, it is possible to consider the two situations at the same time.

As already mentioned, in [26, 94] it is shown that any spectral triple over a commutative C^*-algebra satisfying a suitable set of additional properties must be of the form of Example 5.2. We will now introduce some of them, properly stated in a noncommutative setup.

5.1.1 Summability

Definition 5.4. *Given a spectral triple $(\mathcal{A}, H, D)$ on a unital C^*-algebra A, we consider the function*

$$\zeta_D(z) := \operatorname{tr}\left((1 + D^2)^{-z/2}\right)$$

for $z \in \mathbb{C}$, with the convention that it may take infinite values. The numbers $z \in \mathbb{C}$ for which $\zeta_D(z)$ is finite are called ***summability exponents*** *and the triple is called* ***finitely summable*** *if there exists at least one summability exponent. In this case, we define the* ***abscissa of convergence*** *of ζ_D as*

$$\operatorname{abs}(\zeta_D) := \inf\{\, t > 0 \,:\, \zeta_D(t) < \infty \,\}.$$

We recall that $\operatorname{abs}(\zeta_D)$ is the unique number d, if any, for which the Dixmier trace $\operatorname{tr}_\omega((I + D^2)^{-d/2})$ is finite non zero (cf. e.g. [45] Theorem 2.7).

There exists a direct way to compute the abscissa of convergence of the zeta function associated to a Dirac operator D and goes as follows.

Definition 5.5. *Let T be a positive compact operator on a Hilbert space H, denote by $\{\mu_n(T)\}_{n\geq 0}$ the non-increasing sequence (with multiplicity) of its eigenvalues and set*

$$\lambda_t(T) := \#\{\, n \geq 0 : \mu_n(T) > t \,\}$$

for $t > 0$, cf. e.g. [40]. We call ***infinitesimal order*** *of T the number*

$$o(T) := \inf\{\, s > 0 : \operatorname{tr} T^s < \infty \,\}.$$

Theorem 5.6. *Let T be a positive compact operator on a Hilbert space H. Then*

$$o(T)=\left(\liminf_{n\to\infty}\frac{\log(\mu_n(T))}{\log(1/n)}\right)^{-1}=\limsup_{t\to 0}\frac{\log(\lambda_t(T))}{\log(1/t)}=\limsup_{n\to\infty}\frac{\log(\lambda_{1/n}(T))}{\log n}.$$

As a consequence, note that if T is finite rank then $o(T)=0$. Furthermore, the abscissa of convergence is just

$$\mathrm{abs}(\zeta_D)=o\left((1+D^2)^{-1/2}\right). \tag{5.1}$$

The first equality in the statement of Theorem 5.6 has been proved in [45, Theorem 1.4] and the second equality has been proved in [46, Proposition 1.13]. For the sake of completeness, we prove them explicitely here. We first need a Lemma.

Lemma 5.7. *Let $f:(0,\infty)\to(0,\infty)$ be a right-continuous, non-increasing, piecewise-constant function.*

(1) If $\lim_{t\to\infty} f(t)=0$ and the set of discontinuity points consists of an unbounded increasing sequence x_n, then

$$\limsup_{t\to\infty}\frac{\log(1/f(t))}{\log t}=\limsup_{n}\frac{\log(1/f(x_n))}{\log x_n}$$
$$\liminf_{t\to\infty}\frac{\log(1/f(t))}{\log t}=\liminf_{n}\frac{\log(1/f(x_n))}{\log x_{n+1}}.$$

(2) If $\lim_{t\to 0} f(t)=+\infty$ and the set of discontinuity points consists of an infinitesimal decreasing sequence x_n, then

$$\limsup_{t\to 0}\frac{\log f(t)}{\log 1/t}=\limsup_{n}\frac{\log f(x_n)}{\log 1/x_{n-1}}$$
$$\liminf_{t\to 0}\frac{\log f(t)}{\log 1/t}=\liminf_{n}\frac{\log f(x_n)}{\log 1/x_n}.$$

Proof. (1) Let t_k be an increasing sequence such that $\lim_k \frac{\log(1/f(t_k))}{\log t_k}$ exists. Possibly passing to a subsequence, we may assume that, for any $n\in\mathbb{N}$, there is at most one k such that $x_n\le t_k<x_{n+1}$, denote by n_k the indices for which $x_{n_k}\le t_k<x_{n_k+1}$. Since $f(t)$ is constant in $[x_{n_k},x_{n_k+1})$ and $1/\log t$ is decreasing, we have, for any $y_n\in[x_n,x_{n+1})$ such that $y_{n_k}\ge t_k$, the inequalities

$$\frac{\log(1/f(y_{n_k}))}{\log y_{n_k}}\le\frac{\log(1/f(t_k))}{\log t_k}\le\frac{\log(1/f(x_{n_k}))}{\log x_{n_k}}.$$

On the one hand we get

$$\lim_k\frac{\log(1/f(t_k))}{\log t_k}\le\limsup_n\frac{\log(1/f(x_{n_k}))}{\log x_{n_k}}\le\limsup_n\frac{\log(1/f(x_n))}{\log x_n}\le\limsup_{t\to\infty}\frac{\log(1/f(t))}{\log t}.$$

On the other hand,

$$\liminf_{t\to\infty}\frac{\log(1/f(t))}{\log t}\le\liminf_n\frac{\log(1/f(y_n))}{\log y_n}\le\liminf_k\frac{\log(1/f(y_{n_k}))}{\log y_{n_k}}\le\lim_k\frac{\log(1/f(t_k))}{\log t_k}.$$

Finally, by choosing y_n close enough to x_{n+1} so that $\frac{\log x_{n+1}}{\log y_n} \to 1$, we get

$$\liminf_n \frac{\log(1/f(y_n))}{\log y_n} = \liminf_n \frac{\log(1/f(x_n))}{\log x_{n+1}}.$$

Since t_k may be chosen as to reach the lim inf or the lim sup, part (1) follows.

As for part (2), let t_k be a decreasing sequence such that $\lim_k \frac{\log(f(t_k))}{\log 1/t_k}$ exists. As above, we may assume that, for a suitable subsequence x_{n_k}, $x_{n_k} \leq t_k < x_{n_k - 1}$, and consider a sequence $y_n \in [x_n, x_{n-1})$ such that $y_{n_k} > t_k$. Again,

$$\frac{\log f(x_{n_k})}{\log 1/x_{n_k}} \leq \frac{\log f(t_k)}{\log 1/t_k} \leq \frac{\log f(y_{n_k})}{\log 1/y_{n_k}}.$$

The rest of the proof follows as in part (1). □

Proof of Theorem 5.6. During this proof, we suppress the dependence on T. We first extend the sequence $(\mu_n)_{n\in\mathbb{N}}$ to a right-continuous non increasing function $[0,\infty) \to \mathbb{R}$ by posing $\mu(t) = \mu_n$ for $n \leq t < n+1$. We also write $\lambda(t)$ instead of λ_t.

Let us prove the first equality. We observe first that

$$\operatorname{tr} T^\alpha = \sum_{n\geq 0} \mu_n(T)^\alpha = \int_0^\infty \mu(t)^\alpha dt,$$

then set

$$d = \left(\liminf_{n\to\infty} \frac{\log(\mu_n(T))}{\log(1/n)}\right)^{-1} = \left(\liminf_{t\to\infty} \frac{\log(\mu(t))}{\log(1/t)}\right)^{-1}$$

and $\Omega = \{\alpha > 0 : \int_0^\infty \mu(t)^\alpha dt < \infty\}$. We prove that $o(T) \leq d$ and that $d \leq o(T)$.

- To prove that $o(T) \leq d$, set $a(t) = \frac{\log 1/\mu(t)}{\log(t)}$. In particular $\mu(t) = t^{-a(t)}$ and $\liminf_{t\to\infty} a(t) = 1/d$. If $\alpha > d$, then $\liminf_{t\to\infty} \alpha a(t) = \alpha/d > 1$, hence there exists $\beta > 1$ such that $\alpha a(t) \geq \beta$ for t sufficiently large. Therefore

$$\int_0^\infty \mu(t)^\alpha dt = \int_0^\infty t^{-\alpha a(t)} dt \leq const + \int_0^\infty t^{-\beta} dt < \infty,$$

  which implies $\alpha \in \Omega$, namely $(d,\infty) \subseteq \Omega$.

- On the other hand, to prove that $d \leq o(T)$ we may assume $d > 0$, namely $1/d < \infty$. Now let $t_k \to \infty$ be such that

$$\ell_k := \frac{\log 1/\mu(t_k)}{\log t_k} \longrightarrow 1/d.$$

  We have $\mu(t_k) = t_k^{-\ell_k}$. Let now $\alpha < d$, namely $\alpha\ell_k \to \alpha/d < 1$, and choose $\varepsilon > 0$ such that $\alpha\ell_k \leq 1 - \varepsilon$ eventually. Then

$$\int_0^{t_k} \mu(t)^\alpha dt \geq t_k \mu(t_k)^\alpha = t_k \cdot t_k^{-\alpha\ell_k} \geq t_k^\varepsilon \to \infty, \quad \text{for } k \to \infty.$$

  Therefore $\alpha \leq o(T)$, i.e. $d \leq o(T)$.

We now prove the second equality. Let us note that $\mu(t)$ satisfies the hypotheses of part (1) of the Lemma above, where the set of discontinuity points consists of a sequence $\{p_n, n \in \mathbb{N}\} \subseteq \mathbb{N}$, while $\lambda(t)$ satisfies the hypotheses of part (2), where the set of discontinuity points consists of the sequence $\{\mu(p_n), n \in \mathbb{N}\}$. We also note that $\lambda \circ \mu(p_n) = p_n$. By Lemma above,

$$\liminf_{t\to\infty} \frac{\log(1/\mu(t))}{\log t} = \liminf_{n} \frac{\log(1/\mu(p_n))}{\log p_{n+1}},$$
$$\limsup_{t\to 0} \frac{\log(\lambda(t))}{\log 1/t} = \limsup_{n} \frac{\log(\lambda(\mu_{p_n}))}{\log 1/\mu_{p_n - 1}}.$$

Finally,

$$\limsup_{n} \frac{\log(\lambda(\mu_{p_n}))}{\log 1/\mu_{p_n-1}} = \limsup_{n} \frac{\log p_n}{\log 1/\mu_{p_n-1}} = \left(\liminf_{n} \frac{\log 1/\mu_{p_n-1}}{\log p_n}\right)^{-1}.$$

The third equality comes from Lemma 5.7. □

5.1.2 Reality

The following notion generalizes the existence of the charge conjugation operator on Dirac spinors on spin$_c$ manifolds [44, 90].

Definition 5.8 (cf. [23, 28])**.** *A **real structure** for an (even or odd) spectral triple $(\mathcal{A}, H, D, \pi, \chi)$ on a unital C^*-algebra A is an anti-linear isometry $J\colon H \to H$ such that*

*(1) $[\pi(a), J\pi(b)J^{-1}] = 0$ for all $a, b \in A$ (**zeroth order condition**)*

(2) there are signs $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ for which

$$J^2 = \varepsilon \qquad DJ = \varepsilon' JD \qquad \chi J = \varepsilon'' J\chi.$$

*In this case $(\mathcal{A}, H, D, J)$ is called a **real spectral triple**.*

There are eight possible triples of signs $(\varepsilon, \varepsilon', \varepsilon'')$ and they determine the so-called **KO-dimension** $n \in \mathbb{Z}_8$ of the real spectral triple according to the following table[1]:

n	0	1	2	3	4	5	6	7
ε	$+1$	$+1$	-1	-1	-1	-1	$+1$	$+1$
ε'	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$
ε''	$+1$		-1		$+1$		-1	

Accordingly, even triples have even KO-dimension and odd triples have odd KO-dimension. We mention that a real structure on a spectral triple determines a KKR-cycle in Real K-homology and the periodicity modulo 8 of the previous table is related to the periodicity of real Clifford algebras [44, Section 9.5].

Remark 5.9. In the commutative case (see Example 5.2), the KO-dimension of the real structure coincides (modulo 8) with the dimension of the underlying manifold.

[1] If n is even, $J' = J\chi$ can be used as another real structure with the new signs $(\varepsilon\varepsilon'', -\varepsilon', \varepsilon'')$ [34].

The zeroth order condition makes the Hilbert space H into a bimodule over A thanks to the right action

$$\psi \triangleleft b = J\pi(b^*)J^{-1}\psi, \qquad \psi \in H, b \in A.$$

Equilavently, we can regard H as a $A \otimes A^{\mathrm{op}}$-module under the action

$$a \otimes b^{\mathrm{op}} \triangleright \psi = \pi(a)J\pi(b^*)J^{-1}\psi$$

for $a, b \in A$ and $\psi \in H$. The following property is the noncommutative analogue of requiring D to be a first order differential operator.

Definition 5.10 (cf. [25])**.** *A real spectral triple $(\mathcal{A}, H, D, J)$ on a unital C^*-algebra A satisfies the* ***first order condition*** *if*

$$\left[[D, \pi(a)]\,, J\pi(b)J^{-1}\right] = 0. \tag{5.2}$$

for every $a, b \in \mathcal{A}$.

Remark 5.11*.* Using Jacobi identity, one can show that (5.2) is equivalent to

$$\left[\left[D, J\pi(b)J^{-1}\right], \pi(a)\right] = 0$$

for any $a, b \in \mathcal{A}$. In particular, this means that the first order condition is "symmetric" in A and A^{op}.

Motivated by the properties of the real structure operator on the spectral triple of the noncommutative Standard Model of particle physics (see e.g. [28, Chapter 1] and the references therein), we consider also the following property.

Definition 5.12 (cf. [17])**.** *A real spectral triple $(\mathcal{A}, H, D, J)$ on a unital C^*-algebra A satisfies the* ***second order condition*** *if*

$$\left[[D, \pi(a)]\,, J[D, \pi(b)]J^{-1}\right] = 0. \tag{5.3}$$

for every $a, b \in A$.

5.1.3 Orientability

The orientability condition involves the generalization of a a differential top form in terms of Hochschild homology, which we now briefly recall [67]. Let A be a complex unital algebra and M an A-bimodule. For every positive $n \in \mathbb{N}$ define the A-module of **Hochschild n-chains** (with coefficients in M) to be $C_n(M, A) = M \otimes A^{\otimes n}$ and $C_0(M, A) = M$. The **Hochschild boundary** is the family of maps $b_n \colon C_n(M, A) \to C_{n-1}(M, A)$ given on pure elements by

$$\begin{aligned} b_n\left(m \otimes a_1 \otimes \cdots \otimes a_n\right) \coloneqq\; & m a_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned} \tag{5.4}$$

if $n \geq 1$ and $b_0(m) = 0$, and extended by linearity. It turns out that $(C_\bullet(M, A), b)$ is a chain complex and its homology is the Hochschild homology with coefficients in M.

Choosing $M = A$ as an A-bimodule with the usual left and right multiplication, we get the Hochschild chain complex of A.

Let now $(\mathcal{A}, H, D, \chi)$ be an even or (with the convention that $\chi = 1$) odd spectral triple on A, and let J be a real structure of KO-dimension $n \in \mathbb{Z}_8$. For a Hochschild n-chain $c = \sum a_0 \otimes a_1 \otimes \cdots \otimes a_n \in C_n(\mathcal{A}, \mathcal{A})$ set

$$\pi_D(c) \coloneqq \sum \pi(a_0)[D, \pi(a_1)] \cdots [D, \pi(a_n)].$$

Definition 5.13. *A spectral triple $(\mathcal{A}, H, D, \chi)$ on A is **strongly orientable** if there exists a Hochschild n-cycle $c \in C_n(\mathcal{A}, \mathcal{A})$ such that $\pi_D(c) = \chi$.*

It is useful to consider also a weaker notion of orientability. Consider the case in which the A-module is $M = A \otimes A^{\mathrm{op}}$, where A^{op} denotes the opposite algebra, with the left and right actions of A given on $m \otimes n \in A \otimes A^{\mathrm{op}}$ by:

$$a(m \otimes n)b \coloneqq amb \otimes n \qquad a, b \in A.$$

For a Hochschild n-chain $c = \sum (a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_n$ in $C_n(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathcal{A})$ we define the map

$$\pi_D(c) \coloneqq \sum \pi(a_0) J \pi(b_0^*) J^{-1} [D, \pi(a_1)] \cdots [D, \pi(a_n)]. \tag{5.5}$$

Definition 5.14. *A real spectral triple $(\mathcal{A}, H, D, J, \chi)$ on A is **orientable** if there exists a Hochschild n-cycle $c \in C_n(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathcal{A})$ such that $\pi_D(c) = \chi$.*

Note that if $\mathcal{A}$ is unital and $\pi(1_{\mathcal{A}}) = \mathrm{id}_H$, then every strong orientation cycle $c = \sum a_0 \otimes a_1 \otimes \cdots \otimes a_n$ in $C_n(\mathcal{A}, \mathcal{A})$ induces a (weak) orientation cycle $c' = \sum (a_0 \otimes 1_A) \otimes a_1 \otimes \cdots \otimes a_n$ in $C_n(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathcal{A})$ as

$$\begin{aligned} \pi_D(c') &= \sum \pi(a_0) J \pi(1_A) J^{-1} [D, \pi(a_1)] \cdots [D, \pi(a_n)] \\ &= \sum \pi(a_0) [D, \pi(a_1)] \cdots [D, \pi(a_n)] \\ &= \pi_D(c) = \chi. \end{aligned}$$

5.1.4 Regularity

The regularity condition originates from the development a well-defined noncommutative differential calculus on spectral triples and encodes the information of what elements can be regarded as *smooth* inside the algebra A. For more details we refer to [44, Chapter 10.3].

Definition 5.15. *Let $(\mathcal{A}, H, D)$ be a spectral triple over a unital C^*-algebra A and*

$$\delta(T) \coloneqq [|D|, T] \tag{5.6}$$

be the unbounded derivation on the domain $\mathrm{Dom}(\delta)$ of elements $T \in \mathcal{L}(H)$ which preserve $\mathrm{Dom}(|D|)$ and for which $[|D|, T]$ extends to a bounded operator on H. For $k \geq 2$ we define δ^k inductively on the domain $\mathrm{Dom}(\delta^k) \coloneqq \left\{ T \in \mathrm{Dom}(\delta) \;\middle|\; \delta(T) \in \mathrm{Dom}(\delta^{k-1}) \right\}$ and set

$$\mathrm{Dom}^\infty(\delta) \coloneqq \bigcap_{k=1}^{\infty} \mathrm{Dom}(\delta^k).$$

*We say that $(\mathcal{A}, H, D)$ is **regular** if both $\pi(\mathcal{A})$ and $[D, \pi(\mathcal{A})]$ belong to $\mathrm{Dom}^\infty(\delta)$.*

In the case the operator $|D|$ is not invertible, we can replace $|D|$ with its invertible bounded perturbation $\Delta^{\frac{1}{2}} \coloneqq (1 + D^2)^{\frac{1}{2}}$ thanks to the following result.

Theorem 5.16 (cf. [49, 105])**.** *A spectral triple* $(\mathcal{A}, H, D)$ *over a unital* C^**-algebra* A *is regular if and only if* $\pi(\mathcal{A})$ *and* $[D, \pi(\mathcal{A})])$ *belong to* $\mathrm{Dom}^\infty([\Delta^{\frac{1}{2}}, \cdot\,])$*.*

The regularity of a spectral triple is closely related to the existence of a so-called algebra of generalized differential operators [29, 49, 50]. We recall some basic facts (mainly following the exposition of [105]). In the next following we denote by Δ a self-adjoint positive and invertible operator on a Hilbert space H (and so strictly positive). When dealing with a spectral triple, we take $\Delta = 1 + D^2$ so that Δ is thought of order two.

Definition 5.17. *The* ***Δ-Sobolev space*** *of order* $s \in \mathbb{R}$*, denoted* $W^s = W^s(\Delta)$*, is the Hilbert space completion of* $\mathrm{Dom}(\Delta^{\frac{s}{2}})$ *with respect to the inner product*

$$\langle \xi, \eta \rangle_{W^s} \coloneqq \langle \Delta^{\frac{s}{2}} \xi, \Delta^{\frac{s}{2}} \eta \rangle_H \tag{5.7}$$

for every $\xi, \eta \in H$*.*

For any $t \leq s$ there exists a constant C such that

$$\|\xi\|_{W^t} \leq C \, \|\xi\|_{W^s} \tag{5.8}$$

for $\xi \in W^t(\Delta)$, and so there is a continuous inclusion $W^s \subseteq W^t$. Moreover, $\mathrm{Dom}(\Delta^{\frac{s}{2}})$ is complete in the norm induced by (5.7) for any $s \geq 0$. (see [105] for a proof).

Definition 5.18. *The space of* ***Δ-smooth vectors*** *is*

$$W^\infty \coloneqq \bigcap_{s \in \mathbb{R}} W^s = \bigcap_{n=0}^{\infty} W^{2n} = \bigcap_{n=0}^{\infty} \mathrm{Dom}(\Delta^n).$$

As the notation suggests, the space W^∞ is dense in W^s for any $s \in \mathbb{R}$ (in particular, also in $W^0 = H$). We denote the space of linear maps $P \colon W^\infty \to W^\infty$ by $\mathrm{End}(W^\infty)$.

Definition 5.19. *We say that a linear map* $P \colon W^\infty \to W^\infty$ *has* ***analytic order*** *at most* $t \in \mathbb{R}$ *if it extends by continuity to a bounded linear operator* $P \colon W^{s+t} \to W^s$ *for any* $s \in \mathbb{R}$*. The space of such operators is denoted with* $\mathrm{Op}^t = \mathrm{Op}^t(\Delta)$*. We further define*

$$\mathrm{Op} = \mathrm{Op}(\Delta) \coloneqq \bigcup_{t \in \mathbb{R}} \mathrm{Op}^t(\Delta).$$

Notice in particular that operators with analytic order at most 0 extend to bounded linear operators on $H = W^0$, allowing us to identify Op^0 with a subspace of $\mathcal{L}(H)$.

Lemma 5.20 (cf. [105])**.** *Operators with finite analytic order form a filtered algebra:*

(1) $\mathrm{Op}^s \subseteq \mathrm{Op}^t$ *for any* $s \leq t$

(2) $\mathrm{Op}^s \cdot \mathrm{Op}^t \subseteq \mathrm{Op}^{s+t}$

In particular, Op^0 *is a subalgebra of* $\mathcal{L}(H)$*.*

The algebra Op^0 plays a central role in the regularity of a spectral triple.

Proposition 5.21 (cf. [44], Lemma 10.22)**.** *Let* $(\mathcal{A}, H, D)$ *be a regular spectral triple over a unital* C^**-algebra* A *and* $\Delta \coloneqq 1 + D^2$*. We have that both* $\pi(\mathcal{A})$ *and* $[D, \pi(\mathcal{A})]$ *are in* $\mathrm{Op}^0(\Delta)$*.*

Sketch of the proof. The idea is to show by induction that

$$\Delta^{\frac{n}{2}}\pi(a)\Delta^{-\frac{n}{2}} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\delta^j(\pi(a))\mathfrak{b}(D)^j$$

where δ is the derivation (5.6) and $\mathfrak{b}(D) = D(1+D^2)^{-\frac{1}{2}}$ is the bounded transform of D. In particular $\Delta^{\frac{n}{2}}\pi(a)\Delta^{-\frac{n}{2}}$ is bounded for any $n \in \mathbb{N}$ (0-included) and so

$$\|\pi(a)\xi\|_{W^n} = \left\|\Delta^{\frac{n}{2}}\pi(a)\xi\right\| = \left\|\Delta^{\frac{n}{2}}\pi(a)\Delta^{-\frac{n}{2}}\Delta^{\frac{n}{2}}\xi\right\| \leq \left\|\Delta^{\frac{n}{2}}\pi(a)\Delta^{-\frac{n}{2}}\right\| \|\xi\|_{W^n}.$$

By interpolation, one gets the thesis for general exponents. □

A theorem of Higson relates the notion of regularity with the existence of an algebra of generalized differential operators.

Definition 5.22 (cf. [49])**.** *An $\mathbb{N}$-filtered subalgebra $\mathcal{D} \subseteq \mathrm{Op}(\Delta)$ is called an algebra of **generalized differential operators** (GDO) if it is closed under the derivation $[\Delta, \cdot]$ and satisfies*

$$[\Delta, \mathcal{D}^k] \subseteq \mathcal{D}^{k+1} \tag{5.9}$$

for any $k \in \mathbb{N}$.

Theorem 5.23 (cf. [49])**.** *A spectral triple $(\mathcal{A}, H, D)$ over a unital C^*-algebra A is regular if and only if there exists an algebra of generalized differential operators (with respect to $\Delta = 1 + D^2$) containing $\pi(\mathcal{A})$ and $[D, \pi(\mathcal{A})]$ in degree zero.*

We maintain the notation $\Delta = 1 + D^2$; then observe that $D \in \mathrm{Op}^1(\Delta)$. Indeed for any $\xi \in W^\infty$ and $s \in \mathbb{R}$ we have

$$\begin{aligned}\|D\xi\|^2_{W^s} &= \langle D^2(1+D^2)^s\xi, \xi\rangle = \langle \mathfrak{b}(D)^2(1+D^2)^{s+1}\xi, \xi\rangle \\ &= \left\|\mathfrak{b}(D)^2\Delta^{\frac{s+1}{2}}\xi\right\|^2 \leq \|\mathfrak{b}(D)\|^2\,\|\xi\|^2_{W^{s+1}}\end{aligned}$$

where $\mathfrak{b}(D) = D(1+D^2)^{-\frac{1}{2}}$ is the bounded transform of D. Now define the $\mathbb{N}$-filtered algebra $\mathcal{E} \subseteq \mathrm{End}(W^\infty)$ inductively by:

(1) $\mathcal{E}^0$ is the subalgebra generated by $\pi(\mathcal{A})$ and $[D, \pi(\mathcal{A})]$

(2) $\mathcal{E}^1 = \mathcal{E}^0 + [\Delta, \mathcal{E}^0] + \mathcal{E}^0[\Delta, \mathcal{E}^0]$.

(3) $\mathcal{E}^k = \mathcal{E}^{k-1} + \sum_{j=1}^{k-1}\mathcal{E}^j\mathcal{E}^{k-j} + [\Delta, \mathcal{E}^{k-1}] + \mathcal{E}^0[\Delta, \mathcal{E}^{k-1}]$ for $k \geq 2$

We have the following result.

Theorem 5.24 (cf. [29, 49, 105])**.** *Let $(\mathcal{A}, H, D)$ be a spectral triple over A*

(1) If $(\mathcal{A}, H, D)$ is regular then $\mathcal{E}^k \subseteq \mathrm{Op}^k$ for any $k \geq 0$.

(2) If $\pi(\mathcal{A})W^\infty \subseteq W^\infty$ and $\mathcal{E}^k \subseteq \mathrm{Op}^k$ for any $k \geq 0$, then $(\mathcal{A}, H, D)$ is regular.

Proof. If the triple is regular there exists an algebra $\mathcal{D}$ of generalized differential operators containing $\pi(\mathcal{A})$ and $[D, \pi(\mathcal{A})]$ in degree zero by Theorem 5.23. By induction, $\mathcal{E}^k \subseteq \mathcal{D}^k$ for any $k \in \mathbb{N}$. In particular, $\mathcal{E}^k \subseteq \mathcal{D}^k \subseteq \mathrm{Op}^k$ for any $k \geq 0$.

On the other hand, if $\mathcal{E}^k \subseteq \mathrm{Op}^k$ for any $k \geq 0$ then $\mathcal{E}$ is an algebra of generalized differential operators containing $\pi(\mathcal{A})$ and $[D, \pi(\mathcal{A})]$ in degree zero; by Theorem 5.23 the triple is regular. □

5.2 Equivariant Spectral Triples

In this section we recall the standard definitions and some basic facts about actions and coactions of Hopf algebras and compact quantum groups on spectral triples, with particular emphasis on the case where the Hopf algebra is $\mathbb{C}G$ and the CQG is $C_r^*(G)$.

Definition 5.25. *Let (A, G, α, σ) be a twisted dynamical system. A spectral triple $(\mathcal{A}, H, D)$ on A is G-**equivariant** if there exists a map $u\colon G \to \mathcal{U}(H)$ such that:*

(1) (π, u) is a twisted covariant representation of (A, G, α, σ) on H.

(2) The operators u_g leave the domain of D invariant for all $g \in G$ and the commutator $[D, u_g]$ extends to a bounded operator on H for every $g \in G$.

When $[D, u_g] = 0$ for every $g \in G$, we just say that the triple $(\mathcal{A}, H, D)$ is G-invariant.

Remark 5.26. Differently from our references [82, 100, 101], we admit a non-trivial commutator $[D, u_g]$. What in their papers is called a "G-equivariant spectral triple", it's a G-invariant spectral triple for us. Note that the invariance is not referred to the action on the algebra A.

When the cocycle is trivial, it is known that the bounded transform of an equivariant spectral triple defines an equivariant Kasparov module as introduced in Section 4.4 (see [87]). The idea of the proof is to check that the commutator $[\mathfrak{b}(D), u_g]$ is compact for any $g \in G$ in the same way the commutator $[\mathfrak{b}(D), \pi(a)]$ is proved to be compact in [10, Proposition 2.2] using the boundedness of the commutators between D and a.

Example 5.27 (cf. [44, 95])**.** Consider the unit circle S^1 as in Section 3.2. Fix now $\theta \in \mathbb{R}$ and consider the action of $\mathbb{Z}$ on $C(S^1)$ given by

$$\alpha_n(f)(t) \coloneqq f(t + n\theta)$$

for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Then $(C(S^1), \mathbb{Z}, \alpha)$ is a C^*-dynamical system. The spectral triple $(C(S^1), L^2(S^1), D)$, where D is the self-adjoint extension of the operator $-i\frac{\partial}{\partial x}$ on $L^2(S^1)$, is $\mathbb{Z}$-equivariant with respect to the unitary representation u of $\mathbb{Z}$ on $L^2(S^1)$ given by $u_n(f)(t) \coloneqq f(t + n\theta)$. More precisely, note that $[D, u] = 0$ and so the triple is actually $\mathbb{Z}$-invariant. □

Remark 5.28. When the group G is compact and the 2-cocycle trivial, it is always possible to find a G-invariant spectral triple in the same K-homology class just by averaging the Dirac operator over the group.

Remark 5.29. If $(\mathcal{A}, H, D)$ is G-invariant, then $[D, \pi(\sigma(g, h))] = 0$ for any $g, h \in G$.

We recall now some constructions involved with equivariant spectral triples regarded as unbounded Kasparov modules.

Example 5.30 (Green-Julg Isomorphism)**.** Let $\sigma \equiv 1$. Any G-equivariant spectral triple (A, H, D, π, u) defines a spectral triple $(C_c(G, A), H, D, \pi \rtimes u)$ on the *maximal* crossed product $A \rtimes_\alpha G$ as the commutators

$$[D, \pi(a)u_g] = [D, \pi(a)]u_g + \pi(a)[D, u_g] \tag{5.10}$$

are bounded by hypothesis. Vice versa, using the universal property of the maximal crossed product, any spectral triple on $A \rtimes_\alpha G$ comes from a G-equivariant spectral triple. We can think of this association as the unbounded version of the well known Green-Julg isomorphism $KK^G(A, \mathbb{C}) \simeq KK(A \rtimes_\alpha G, \mathbb{C})$ for discrete groups (see Section 4.4.1). □

Example 5.31 (Kasparov Descent)**.** Let $\sigma \equiv 1$. Let us now discuss the Kasparov descent map of a G-equivariant spectral triple. From formula (4.13), consider $B = \mathbb{C}$ and $C_c(G)$ acting on the right on $C_c(G, H)$ by right multiplication:

$$(\xi \otimes \delta_g) \triangleleft \delta_h \coloneqq \xi \otimes \delta_{gh}$$

for $\xi \in H$ and $\delta_g, \delta_h \in C_c(G)$. The completion of $C_c(G, H)$ with the $C^*(G)$-valued scalar product

$$\langle \xi \otimes \delta_g, \mu \otimes \delta_h \rangle \coloneqq \langle \xi, \mu \rangle \delta_{g^{-1}h},$$

as introduced in formula (4.14), defines the right Hilbert $C^*(G)$-Hilbert module $H \rtimes G \simeq H \otimes C^*(G)$. Using formula (4.15), we see that the representation $\pi \colon A \to \mathcal{L}(H)$ induces a representation ψ of $A \rtimes G$ on $H \otimes C^*(G)$ by

$$\psi(a\delta_g)(\xi \otimes \delta_h) = \pi(a) u_g \xi \otimes \delta_{gh}.$$

The image of $(\mathcal{A}, H, D, u)$ under the descent is then $(C_c(G, A), H \otimes C^*(G), D \otimes 1, \psi)$.

□

The notion of equivariance has been widely studied in literature in many contexts. In particular, having in mind the case in which $\mathcal{H} = \mathbb{C}G$, it is possible to define the notion of equivariance even for any Hopf algebra [82, 100, 101].

Definition 5.32. *Let $\mathcal{H}$ be a Hopf algebra and $(\mathcal{A}, H, D)$ be a spectral triple over an $\mathcal{H}$-module algebra A. We say that the triple is $\mathcal{H}$-**equivariant** if there exists a dense subspace $W \subseteq H$ for which:*

(1) W is an $\mathcal{H}$-equivariant $\mathcal{A}$-module, that is:

$$h \triangleright (\pi(a)v) = \pi(h_{(1)} \triangleright a)(h_{(2)} \triangleright v)$$

for any $h \in \mathcal{H}$, $v \in W$ and $a \in \mathcal{A}$.

(2) the commutator $[D, h \triangleright]$ is bounded on its domain for any $h \in \mathcal{H}$.

*If the commutators $[D, h \triangleright]$ vanish for every $h \in \mathcal{H}$, we say that the triple is **invariant**.*

Analogously, having in mind the case $Q = C^*_r(G)$, we formulate a notion of equivariance for compact quantum groups.

Definition 5.33 (cf. [15])**.** *Let $(\mathcal{A}, H, D, \chi)$ be an (even or odd) spectral triple on a Q-comodule unital C^*-algebra B. We say that the triple is **equivariant for coactions** of Q if there exists a dense subspace $W \subseteq H$ and a unitary corepresentation $\Theta \colon H \to H \otimes Q$ for which:*

(1) W is a Q-equivariant B-module in the sense that for every $b \in B$ and $x \in W$,

$$\Theta(b \triangleright x) = b_{(-1)} \triangleright x_{(-1)} \otimes b_{(0)} x_{(0)},$$

(2) the operatorial form X of Θ as in (C.8) *commutes with $D \otimes 1_Q$ and with $\chi \otimes 1_Q$*

(3) $(\mathrm{id} \otimes \varphi)\mathrm{Ad}_X(b) \in B''$ for every $b \in B$ and every state φ on Q.

5.3 An Example: The Noncommutative Torus

In Section 3.2 we have introduced the rotation algebra A_θ^2 as a deformation of the C^*-algebra of continuous functions on the 2-torus $T^2 = S^1 \times S^1$ and studied its properties as a noncommutative topological space. In this section we focus our attention on its differential and metric structure, namely in the existence of a spectral triple over A_θ^2 and its properties. For more details, we refer to [44, Section 12.3] and the references therein.

Definition 5.34. *The* ***noncommutative torus*** $\mathcal{A}_\theta^2$ *is the unital dense subalgebra in* A_θ^2 *given by*

$$\mathcal{A}_\theta^2 := \left\{ \sum_{m,n\in\mathbb{Z}} a_{mn} U^m V^n \;\middle|\; (a_{mn})_{mn} \in \mathcal{S}(\mathbb{Z}^2) \right\},$$

where $\mathcal{S}(\mathbb{Z}^2)$ *is the space of Schwarz functions on* $\mathbb{Z}^2$*, i.e. those sequences that satisfy*

$$\sup_{m,n\in\mathbb{Z}} (1+m^2+n^2)^k \left|a_{mn}\right|^2 < \infty$$

for all $k \in \mathbb{N}$*.*

The Lie group T^2 acts on A_θ^2 by $(z,w) \triangleright U^m V^n = z^m w^n U^m V^n$ for $z, w \in U(1)$ and it is easy to see that $\mathcal{A}_\theta^2$ is just the subalgebra of smooth elements for this action. In particular, this proves that $\mathcal{A}_\theta^2$ is a unital pre-C^*-algebra (see [44, Proposition 3.45]).

Remember now from Section 3.2 that A_θ^2 comes equipped with a faithful normalized tracial state τ which defines the GNS representation $H_\tau = L^2(A_\theta^2, \tau)$. Note that the trace τ is invariant under the aformentioned action of the torus T^2.

Definition 5.35. *We define on the dense subspace* $\mathcal{A}_\theta^2 \subseteq H_\tau$ *the* ***basic derivations*** ∂_i *for* $i = 1, 2$ *as the operators*

$$\begin{aligned} \partial_1 U &= 2\pi i U \qquad & \partial_2 U &= 0 \\ \partial_1 V &= 0 \qquad & \partial_2 V &= 2\pi i V. \end{aligned} \tag{5.11}$$

We have the following facts:

(1) The operators ∂_1 and ∂_2 satisfy the *Leibniz derivation rule*:

$$\partial_i(ab) = (\partial_i a)b + a(\partial_i b)$$

for any $a, b \in \mathcal{A}_\theta^2$. This condition implies that $\partial_i(1_{\mathcal{A}_\theta^2}) = 0$ for any $i = 1, 2$.

(2) $\partial_i(a^*) = (\partial_i a)^*$ for any $a \in \mathcal{A}_\theta^2$ and $i = 1, 2$. In particular, the map ∂_i^* defined by

$$\partial_i^* a := (\partial_i a^*)^*$$

is a derivation and $\partial_i^* = \partial_i$. Thus, the basic derivations are symmetric operators.

(3) The basic derivations ∂_1 and ∂_2 commute. Moreover, they span the abelian Lie algebra of infinitesimal generators of the action of T^2 on A_θ^2. In this sense, the pre-C^*-algebra $\mathcal{A}_\theta^2$ is precisely their common smooth domain.

(4) Using the previous points, one can check that $\tau(\partial_i a) = 0$ for any $a \in \mathcal{A}^2_\theta$ and any $i = 1, 2$. As a consequence, we have that

$$\tau((\partial_i a)^* b) = -\tau(a^* \partial_i b)$$

for any $a, b \in \mathcal{A}^2_\theta$ and so any basic derivation extends to a closed (unbounded) skew-adjoint operator on H_τ.

Using the basic derivations, we can define the Dirac operator

$$D = -i \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \end{pmatrix} = -i(\sigma_1 \partial_1 + \sigma_2 \partial_2) \tag{5.12}$$

on $H_\tau \oplus H_\tau$ and this turns out to be self-adjoint, to have compact resolvent and to have the commutator with $a \in \mathcal{A}^2_\theta$ (acting diagonally on $H_\tau \oplus H_\tau$) bounded. We deduce that

$$\left(\mathcal{A}^2_\theta, H_\tau \oplus H_\tau, D\right) \tag{5.13}$$

is a spectral triple on A^2_θ. We now discuss the Connes axioms for this triple.

Definition 5.36. *The **Tomita conjugation** associated to the noncommutative torus spectral triple* $\left(\mathcal{A}^2_\theta, H_\tau \oplus H_\tau, D\right)$ *is the anti-unitary map* $J_0 \colon H_\tau \to H_\tau$ *given by*

$$J_0 a \coloneqq a^*.$$

for $a \in \mathcal{A}^2_\theta \subseteq H_\tau$.

Clearly $J_0^2 = 1$ and the right multiplication map $\pi^{\mathrm{op}} \colon A^2_\theta \to \mathcal{L}(H_\tau)$ given by

$$\pi^{\mathrm{op}}(b)a \coloneqq J_0 \pi(b^*) J_0^{-1} a = ab$$

is a representation of the opposite algebra $(A^2_\theta)^{\mathrm{op}}$. Since left and right multiplication commute, we have that $[\pi(a), J_0\pi(b)J_0^{-1}] = 0$ for any $a, b \in A^2_\theta$. In particular, the operator

$$J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$$

defines on $H_\tau \oplus H_\tau$ a real structure of KO-dimension 2. A straightforward computation shows that we also have the first order condition.

To show that the triple is orientable, consider the Hochschild 2-chain

$$c \coloneqq -\frac{i}{2(2\pi)^2} \left(V^* U^* \otimes U \otimes V - U^* V^* \otimes V \otimes U\right). \tag{5.14}$$

A straightforward computation shows that $bc = 0$ so this is a Hochschild 2-cycle. Furthermore, using equations (5.11), we deduce that

$$\pi_D(c) = -\frac{i(2\pi)^2}{2(2\pi)^2}(\sigma_1\sigma_2 - \sigma_2\sigma_1) = \sigma_3 = \chi$$

and so the orientation axiom is satisfied.

Remark 5.37. This example served then as a building block for large class of theta-deformed spaces, with similar properties, see e.g. [27].

5.4 Spectral Triples From Equicontinuous Actions

Given a spectral triple on a unital C^*-algebra A and an equicontinuous action of a discrete group G on A, a spectral triple on the reduced crossed product C^*-algebra $A \rtimes_r G$ was constructed by Hawkins, Skalski, White and Zacharias in [47], extending the construction by Belissard, Marcolli and Reihani in [13]. The main idea was to use the Kasparov product to make an ansatz for the Dirac operator. In this section we recall the relevant results from their work, as a preparation for our new contributions that are discussed in Part II.

5.4.1 Spectral Triples on Group Algebras

The main ingredient in their construction of a triple on $A \rtimes_{\alpha,r} G$ is a spectral triple on the reduced group algebra $C^*_r(G)$. It is known that unbounded Kasparov modules on groups and groupoids exist when they are endowed with a weight type function, see e.g. [9, 14, 22, 70]. We recall some basic facts and definitions specializing the discussion for our purposes.

Definition 5.38. *A* ***weight*** *on a non-necessarily topological group G is a function $l\colon G \to \mathbb{R}$. A weight is* ***proper*** *if the level sets $\{ g \in G \mid -n \leq l(g) \leq n \}$ are finite for each $n \in \mathbb{N}$. We say that a weight is* ***non degenerate*** *when $l(g) = 0$ if and only if $g = e$.*

Differently from [47], we consider weights on groups taking values in $\mathbb{R}$ and we admit the possibility for them to be degenerate. We will discover that this flexibility will not influence the nature of their results.

Example 5.39. Any group homomorphism $l\colon G \to \mathbb{R}$ is clearly a weight on G. In this case, being $\mathbb{R}$ abelian, l must be cyclic in the sense that

$$l(gh) = l(hg)$$

for any $g, h \in G$. Note that if the homomorphism l is non degenerate as a weight, then G must be abelian as well. $\square$

Definition 5.40. *A weight $l\colon G \to \mathbb{R}$ is said to be a* ***Dirac weight*** *if for every $g \in G$ the (left) translation function $l_g\colon G \to \mathbb{R}$ given by*

$$l_g(x) \coloneqq l(x) - l(g^{-1}x) \qquad x \in G$$

is bounded. A Dirac weight l is said to be of ***first order type*** *if for every $g \in G$ the (left) translation function l_g is constant.*

Example 5.41 (cf. [22, 97])**.** A **pseudo-length function** on a group G is a weight $\ell\colon G \to \mathbb{R}$ such that:

(1) $\ell(e) = 0$,

(2) $\ell(xy) \leq \ell(x) + \ell(y)$ for every $x, y \in G$,

(3) $\ell(x) = \ell(x^{-1})$ for every $x \in G$.

A length function is a pseudo-length function which is non-degenerate as a weight. The prototypical length function is the *word metric* on a finitely generated group G which associates to any element $g \in G$ the minimum number of generators needed to write g (for an *a priori* fixed generating set). Note that any length function is a non-negative Dirac weight as

$$0 = \ell(e) = \ell(xx^{-1}) \leq \ell(x) + \ell(x^{-1}) = 2\ell(x) \qquad \forall x \in G$$

and $\ell_g(x) = \ell(x) - \ell(g^{-1}x) \leq \ell(g)$ for any $x, g \in G$. Given a pseudo-length function $\ell\colon G \to \mathbb{R}$, the quantity

$$d_\ell(x, y) := \ell(xy^{-1}) \tag{5.15}$$

clearly defines a right invariant pseudo-metric on the group G. If ℓ is a length function, then d_ℓ is a metric.

□

Remark 5.42. Let G be a topological group. If there is a G-equivariant spectral triple $(\mathcal{A}, H, D, u)$ on a certain unital C^*-algebra A, then $\ell(g) = \|[D, u_g]\|$ is a length function on G.

The following lemma provides a complete characterization of first order Dirac weights on groups.

Lemma 5.43. *Let $l\colon G \to \mathbb{R}$ be a weight on a group G. The following facts are equivalent:*

(1) l is a first-order type Dirac weight

(2) $l = \alpha + \varphi$ where $\alpha = l(e)$ is a constant and $\varphi\colon G \to \mathbb{R}$ is a group homomorphism

(3) $l(xzy^{-1}) - l(zy^{-1}) = l(xz) - l(z)$ for every $x, y, z \in G$.

Proof. See [14, Lemma 3.13] □

Remark 5.44. Differently from Example 5.41, given a Dirac weight $l\colon G \to \mathbb{R}$ there is no reason for $d(x, y) := l(xy^{-1})$ to be a metric or a pseudo-metric on the group. However, if l is of first order then the quantity

$$\ell(x) := |l(x) - l(e)|$$

is a pseudo-length function on G since the function $l(x) - l(e)$ is a group homomorphism by Lemma 5.43. In particular, $d_\ell(x, y) := \ell(xy^{-1})$ is a right invariant pseudo-metric on G. Note that if the first order Dirac weight l is non-degenerate, then ℓ is non-degenerate and d_ℓ a metric on G. As an example, the Dirac weight $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ gives back precisely the standard euclidean distance on $G = \mathbb{R}$. Note that this condition is however rather restrictive: if l is non-degenerate, then it is an injective homomorphism and G is a subgroup of $\mathbb{R}$. Standard results show then that G must be either dense in $\mathbb{R}$ or of the form $a\mathbb{Z}$ for $a \in \mathbb{R}$ (namely, cyclic).

Remark 5.45. The properness condition for a Dirac weight on a group is also extremely restrictive as it forces the weight to grow. For example, there is not any constant proper weight on a group unless the group is finite. Furthermore, the space of proper weights on a group may be a priori empty. To avoid this condition, one is somewhat forced to fall into Kasparov's KK-theory (e.g. [70, Section 3.7]). However, avoiding this condition is not always convenient: for example, any positive weight (such as length functions) will yield a spectral triple whose K-homology class is trivial.

Till the end of this section we shall consider a discrete group G endowed with a proper Dirac weight $l\colon G \to \mathbb{R}$. Let M_l be the multiplication operator by l on the domain of finitely supported elements of $\ell^2(G)$ and let us denote also by M_l its self adjoint extension to $\ell^2(G)$. The group algebra $\mathbb{C}G$ acts on $\ell^2(G)$ via the left regular representation $\lambda_g \delta_h := \delta_{gh}$ for $g, h \in G$, and the data

$$(\mathbb{C}G, \ell^2(G), M_l, \lambda) \tag{5.16}$$

form an odd spectral triple on the reduced group C^*-algebra $C_r^*(G)$. Indeed, M_l is self-adjoint as l takes values in $\mathbb{R}$, the properness of l implies that the resolvent of M_l is compact and the fact that l is a Dirac weight guarantees that the commutators $[M_l, \lambda_g] = M_{l_g}\lambda_g$ are bounded for every $g \in G$. Note further that if the weight l is non degenerate, then the triple (5.16) is also non degenerate.

Example 5.46. Consider the discrete group $G = \mathbb{Z}$ endowed with the non degenerate proper Dirac weight $\imath\colon \mathbb{Z} \to \mathbb{R}$ given by the inclusion. It is well known (see e.g. [70, pag. 240]) that the spectral triple as defined in (5.16) agrees with the usual spectral triple on $C(S^1)$ arising from the Dirac operator $-i\frac{\partial}{\partial x}$ under the Fourier transform $C_r^*(\mathbb{Z}) \simeq C(S^1)$. Note in particular that, at the level of K-homology, it is a generator of the cyclic group $K^1(C_r^*(\mathbb{Z})) \simeq \mathbb{Z}$. □

5.4.2 Spectral Triples on Crossed Products

Consider now an odd spectral triple $(\mathcal{A}, H, D, \pi)$ on a unital C^*-algebra A and assume that there is a C^*-dynamical system (A, G, α). Let us assume further that G is equipped with a proper Dirac weight $l\colon G \to \mathbb{R}$ so that this determines a spectral triple $(\mathbb{C}G, \ell^2(G), M_l, \lambda)$ over $C_r^*(G)$. Following the notation in Chapter 3, the pair of maps

$$\begin{cases} \tilde{\pi}_1(a)(\xi \otimes \delta_g) := \pi\left(\alpha_{g^{-1}}(a)\right)\xi \otimes \delta_g \\ \hat{\lambda}_h(\xi \otimes \delta_g) := \xi \otimes \delta_{hg}, \end{cases} \tag{5.17}$$

where $a \in A$, $\xi \in H$ and $g, h \in G$, provide representations of A and G on $H \otimes \ell^2(G)$ which are covariant and the integrated form $\tilde{\pi}_1 \rtimes \hat{\lambda}$ is a $*$-representation of $C_c(G, \mathcal{A})$ on $H \otimes \ell^2(G)$. Following the prescription for the external Kasparov product, e.g. [24, Chapter 4.A], we define a Dirac operator $\widehat{D}$ on $\widehat{H} = H \otimes \ell^2(G) \otimes \mathbb{C}^2 = (H \otimes \ell^2(G)) \oplus (H \otimes \ell^2(G))$ by

$$\begin{aligned} \widehat{D} &:= D \otimes 1 \otimes \sigma_1 + 1 \otimes M_l \otimes \sigma_2 \\ &= \begin{pmatrix} 0 & D \otimes 1 - i \otimes M_l \\ D \otimes 1 + i \otimes M_l & 0 \end{pmatrix}, \end{aligned} \tag{5.18}$$

where σ_1 and σ_2 are Pauli matrices and we consider $A \rtimes_{r,\alpha} G$ acting diagonally on $\widehat{H}$. The operator (5.18) is clearly densely defined and self-adjoint with compact resolvent. The only non-trivial fact is to check that the commutator of $\widehat{D}$ with the representation of $A \rtimes_{\alpha,r} G$ is bounded: on the one hand, we have that

$$\left[1 \otimes M_l, \tilde{\pi}_1(a)\hat{\lambda}_h\right] = (1 \otimes M_{l_h})\tilde{\pi}_1(a)\hat{\lambda}_h \tag{5.19}$$

for any $h \in G$. As l is a Dirac weight, the operator M_{l_h} of multiplication by the function $l_h(x) = l(x) - l(h^{-1}x)$ on $\ell^2(G)$ is bounded. On the other hand, for $a \in \mathcal{A}$ and $g, h \in G$, we have

$$\left[D \otimes 1, \tilde{\pi}_1(a)\hat{\lambda}_h\right](\xi \otimes \delta_g) = \left[D, \pi(\alpha_{g^{-1}h^{-1}}(a))\right]\xi \otimes \delta_{hg}. \tag{5.20}$$

A priori this quantity is not well defined as $\pi(\alpha_g(a))$ might not belong to the domain of D; and even in that case the right hand side is not necessarily bounded. These difficulties are overcame by some ad hoc hypothesis.

Theorem 5.47 (cf. [47])**.** *Let (A, G, α) be a C^*-dynamical system with G discrete and $(\mathcal{A}, H, D)$ be an odd spectral triple on A. Assume G is equipped with a proper Dirac weight $l\colon G \to \mathbb{R}$ and that the action on A is also*

*(1) **smooth**, in the sense that $\alpha_g(\mathcal{A}) \subseteq \mathcal{A}$ for all $g \in G$;*

*(2) **equicontinuous**, namely such that*

$$\sup_{g\in G} \|[D, \pi(\alpha_g(a))]\| < \infty \tag{5.21}$$

for all $a \in \mathcal{A}$.

The triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi}_1 \rtimes \hat{\lambda})$ is an even spectral triple on $A \rtimes_{\alpha,r} G$. Furthermore, if the weight l is non degenerate and the triple $(\mathcal{A}, H, D)$ is non degenerate, then the triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi}_1 \rtimes \hat{\lambda})$ is also non degenerate.

Remark 5.48. Even though the operator $\widehat{D}$ resembles (a representative for) the external Kasparov product of D and M_l, the triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ is *not* the external Kasparov product of the triple (A, H, D) with the triple $(C^*_r(G), \ell^2(G), M_l)$ as the isomorphic vector spaces $A \rtimes_{\alpha,r} G$ and $A \otimes C^*_r(G)$ are in general not isomorphic as algebras.

Remark 5.49. Of course, not every action is equicontinuous. Various examples are given for instance in [13, Section 4.1].

It is well known that the construction in Theorem 5.47 can be thought as a generalization of the boundary map in the K-homology Pimsner-Voiculescu sequence (cf. [88]) in the following sense. Applying the six-term exact sequence in K-homology to the generalized Toeplitz extension

$$0 \longrightarrow A \otimes \mathcal{K}(\ell^2(\mathbb{Z})) \longrightarrow C_0(X, A) \rtimes \mathbb{Z} \longrightarrow A \rtimes_\alpha \mathbb{Z} \longrightarrow 0, \tag{5.22}$$

where $X = \mathbb{Z} \cup \{ +\infty \}$ is the one point compactification of $\mathbb{Z}$, and identifying the K-theory of $C_0(X, A) \rtimes \mathbb{Z}$ with the K-theory of A, one obtains the Pimsner-Voiculescu exact sequence

$$\begin{tikzcd}
K^0(A) \arrow[d, "\partial^0"'] & K^0(A) \arrow[l, "1-\alpha^*"'] & K^0(A \rtimes_\alpha \mathbb{Z}) \arrow[l, "\varepsilon"'] \\
K^1(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\varepsilon"] & K^1(A) \arrow[r, "1-\alpha^*"] & K^1(A). \arrow[u, "\partial^1"']
\end{tikzcd}$$

Here ε is the pull-back map. Note that the boundary maps ∂^0 and ∂^1 are just the left Kasparov multiplication by the class in $KK_1(A \rtimes_\alpha \mathbb{Z}, A \otimes \mathcal{K}) \simeq KK_1(A \rtimes_\alpha \mathbb{Z}, A)$ defined by the extension (5.22) (see [16, Sections 19.5-6]).

Consider now the discrete group $G = \mathbb{Z}$ and the Dirac weight $\imath \colon \mathbb{Z} \to \mathbb{R}$ given by the inclusion. It is clear that the triple $(A \rtimes_\alpha \mathbb{Z}, \widehat{H}, \widehat{D})$ is the Kasparov product of (A, H, D) with the class of the generalized Toeplitz extension (5.22). This proves that

$$\partial^1 \left[(A, H, D)\right] = \left[(A \rtimes_\alpha \mathbb{Z}, \widehat{H}, \widehat{D})\right]. \tag{5.23}$$

A more refined method to see this fact can be found for instance in [93].

Remark 5.50. By passing to higher order spectral triples and applying a logarithmic dampening of the Dirac operator, it is possible to handle also non-equicontinuous actions and get similar results. As an example of this strategy, consider a non-isometric diffeomorphism on the circle which give rise to a twisted spectral triple on S^1 [43, Example 1.6] of which the logarithmic dampening is an ordinary spectral triple [43, Example 1.9]. The pullback action of the diffeomorphism generates an action of $\mathbb{Z}$ on the circle which is not equicontinuous. Deforming the standard Dirac operator on the circle and following a prescription similar to that of Theorem 5.47, one can define a higher order spectral triple on $C(S^1) \rtimes \mathbb{Z}$ whose K-homology class coincides with the class of its dampening [43, Corollary 1.41] and represents the Pimsner-Voiculescu boundary map image of the aforementioned dampened class on $C(S^1)$ [43, Prop. 1.30 and the preceeding discussion].

Proposition 5.51 (cf. [47]). *Let $(\mathcal{A}, H, D)$ be an odd spectral triple on a unital C^*-algebra A and G a discrete group equipped with a proper Dirac weight l acting smoothly and equicontinuously on A. If the triple $(\mathcal{A}, H, D)$ is p-summable and the triple $(\mathbb{C}G, \ell^2(G), M_l)$ is q-summable, then the triple $(A \rtimes_{\alpha,r} G, \widehat{H}, \widehat{D}, \hat{\pi}_1 \rtimes \hat{\lambda})$ is $(p+q)$-summable.*

Sketch of the Proof. This follows as in the case of the external product of two spectral triples: if λ_n and μ_m are respectively the sequences of the eigenvalues of D and M_l, by assumption the sequences $((1+\lambda_n^2)^{-p/2})$ and $((1+\mu_m^2)^{-q/2})$ are convergent. Then, using the inequality

$$(x+y-1)^{\alpha+\beta} \geq x^\alpha y^\beta, \qquad x, y > 1,\ \alpha, \beta > 0,$$

the double sequence $(1+\lambda_n^2+\mu_m^2)^{-(p+q)/2}$ also is proved convergent. □

Proposition 5.52. *Let $(\mathcal{A}, H, D)$ be an odd spectral triple on a unital C^*-algebra A and G a discrete group equipped with a proper Dirac weight l acting smoothly and equicontinuously on A. If (A, H, D) is irreducible then $(A \rtimes_{\alpha,r} G, \widehat{H}, \widehat{D}, \hat{\pi}_1 \rtimes \hat{\lambda})$ is also irreducible.*

Proof. It is a straightforward check. □

5.4.3 The Iteration Procedure

Similarly to what described so far, we can construct odd spectral triples on crossed products starting from even spectral triples and an equicontinuous action (in a suitable sense). Let

$$\left(\mathcal{A}, H = H_0 \oplus H_1, D = \begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix}\right)$$

be an even spectral triple on a unital C^*-algebra A with the $\mathbb{Z}^2$-grading $H_0 \oplus H_1$ and $\pi = \pi_0 \oplus \pi_1$. Let α be an action of a discrete group G on A and let $l \colon G \to \mathbb{R}$ be a proper Dirac weight. We have a diagonal representation of the reduced crossed product $A \rtimes_{\alpha,r} G$

on $(H_0 \otimes \ell^2(G)) \oplus (H_1 \otimes \ell^2(G))$. Provided $\alpha_g(\mathcal{A}) \subseteq \mathcal{A}$ for all $g \in G$ and the equicontinuity condition

$$\sup_{g\in G} \|\pi_0(\alpha_g(x))D_0 - D_0\pi_1(\alpha_g(x))\| < +\infty, \qquad \forall x \in \mathcal{A}$$

then the Dirac operator

$$\widehat{D} = \begin{pmatrix} 1 \otimes M_l & D_0 \otimes 1 \\ D_0^* \otimes 1 & -1 \otimes M_l \end{pmatrix} \tag{5.24}$$

can be used to define an odd spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ on $A \rtimes_{\alpha,r} G$ for $\widehat{H} = H \otimes \ell^2(G)$. In the same way as for odd spectral triples, we have that this triple:

(1) is non degenerate whenever the Dirac weight l and the triple (A, H, D) are non degenerate

(2) represents the image of the even spectral triple (A, H, D, χ) under the Pimsner-Voiculescu boundary map when one considers the group $G = \mathbb{Z}$ and the Dirac weight given by the inclusion $\imath\colon \mathbb{Z} \to \mathbb{R}$.

As noticed in [47], the combination of the construction in Theorem 5.47 and its even version discussed above leads to the construction of satisfactory spectral triples on crossed products by (smooth and equicontinuous) actions of $\mathbb{Z}^d$ for any $d \in \mathbb{N}$ under the standard identification

$$A \rtimes_\alpha \mathbb{Z}^d \simeq ((A \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z} \cdots) \rtimes_{\alpha_d} \mathbb{Z}, \tag{5.25}$$

where $\alpha_1, \ldots, \alpha_d$ denote the coordinate $\mathbb{Z}$-actions of α. Note that this triple is odd if d is even and even if d is odd. The only non trivial thing to check is that the equicontinuity of α implies the equicontinuity of all its coordinate $\mathbb{Z}$-actions: a precise formulation of this fact is Proposition 2.8 in [47]. Note that the spectral triple on $A \rtimes_\alpha \mathbb{Z}^d$ is not in general a spectral triple constructed from a Dirac weight on $\mathbb{Z}^d$ as introduced in Subsection 5.4.1. However, they can be viewed as spectral triples from matrix valued Dirac weight as suitably defined in the next chapter.

Part II

Spectral Geometry of Crossed Products

Chapter 6

Spectral Triples on $C^*_{r,\sigma}(G)$

Following the method recalled in Section 5.4 to create a spectral triple on a crossed product, we now study the preliminary case of a spectral triple on a group algebra. However, differently from [47] and [98], we will consider a more general situation of twisted group algebras and matrix valued Dirac weights on discrete groups, as in [5].

6.1 Matrix Valued Dirac Weights

Let V be a finite dimensional complex vector space and denote by $\mathcal{L}(V)$ the space of linear operators on V. We denote by $\mathrm{Sp}(T)$ the spectrum of the operator T.

Definition 6.1. *A (matrix valued)* ***weight*** *on a group G is a function $l\colon G \to \mathcal{L}(V)$ such that $l(g)$ is self-adjoint for any $g \in G$. We say that a weight is* ***non degenerate*** *when $l(g) = 0$ if and only if $g = e$. A weight $l\colon G \to \mathcal{L}(V)$ is* ***proper*** *if*

(1) the union of all the spectra $\mathcal{S} = \bigcup_{g\in G} \mathrm{Sp}(l(g))$ is a discrete set in $\mathbb{R}$

(2) for any $t \in \mathcal{S}$ there are only finite elements $g_1, \dots, g_n \in G$ such that $t \in \bigcap_{i=1}^n \mathrm{Sp}(l(g_i))$.

A weight $l\colon G \to \mathcal{L}(V)$ is said to be a ***Dirac weight*** *if for every $g \in G$ the (left) translation function $l_g\colon G \to \mathbb{R}$ given by*

$$l_g(x) := l(x) - l(g^{-1}x) \qquad x \in G$$

is bounded.

Example 6.2 (Clifford Length Functions)**.** Let $\mathbb{C}\ell_n$ be the complex Clifford algebra where $vw + wv = -2\langle v, w\rangle$ for $v, w \in \mathbb{R}^n$. A Clifford representation $\varepsilon : \mathbb{C}\ell_n \to \mathrm{End}_{\mathbb{C}}(V)$ is unitary if $\varepsilon(v)$ is unitary for any $v \in \mathbb{R}^n$ with $\|v\| = 1$, and in this case it follows that $\varepsilon(v)^* = -\varepsilon(v)$. Given a norm preserving group embedding $\imath : \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$, we can define

$$l(m) := i\varepsilon(\imath(m))$$

for $m \in \mathbb{Z}^n$. In other words $l(m) = \sum_{j=1}^n m_j f_j$ for $f_j := i\varepsilon(\imath(e_j))$. The elements $l(m)$ are clearly self-adjoint and l is a Dirac weight since ε is a representation. Further, since

$$l(m)^2 = \|m\|^2 \cdot \mathrm{Id}_n,$$

we have that l is a proper weight. We call these particular weights **Clifford length functions**. Notice that such weights can be pulled back: indeed, let $s : G_1 \to G_2$ be a map of sets such that

(1) $s(g) = 0$ if and only if $g = 0$

(2) has finite fibers.

In particular, if s is a group morphism then it has to be injective. Then if l is a Clifford length function also $l \circ s$ is a proper Dirac weight; however it is not Clifford, in general. □

Example 6.3 (cf. [47]). We provide an example of a Clifford length function on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$. Let M_n^{sa} denote the space of $n \times n$ hermitian matrices over $\mathbb{C}$. Define $l^{(1)} \colon \mathbb{Z} \to M_2^{sa}$ be given by

$$l^{(1)}(n) := \begin{pmatrix} 0 & -in \\ in & 0 \end{pmatrix}.$$

For $d > 1$ define inductively $l^{(d)} \colon \mathbb{Z}^d \to M^{sa}_{\lceil (d+1)/2 \rceil}$ by

$$l^{(d)}(n_1, \dots, n_d) := l^{(d-1)}(n_1, \dots, n_{d-1}) + \begin{pmatrix} n_d 1_{2^{d/2-1}} & 0 \\ 0 & -n_d 1_{2^{d/2-1}} \end{pmatrix}$$

if d is even, and by

$$l^{(d)}(n_1, \dots, n_d) := \begin{pmatrix} 0 & l^{(d-1)}(n_1, \dots, n_{d-1}) - in_d 1_{2^{(d-1)/2}} \\ l^{(d-1)}(n_1, \dots, n_{d-1})^* + in_d 1_{2^{(d-1)/2}} & 0 \end{pmatrix}$$

if d is odd. A straightforward computation shows that they are proper Dirac weights on $\mathbb{Z}^d$ satisfying the smoothness condition, the equicontinuity condition and the Clifford condition $(l^{(d)}(n))^2 = \|n\|^2 \cdot 1$.

□

Remark 6.4 (cf. [47]). The Clifford length function introduced in Example 6.3 allow to give a concise description of the iterated spectral triple on $A \rtimes_\alpha \mathbb{Z}^d$ as introduced at the end of Section 5.4. Fixed $d \in \mathbb{N}$, set $r = \lceil d/2 \rceil$ and write

$$\widehat{H} = H \otimes \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2^r} \simeq \left(H \otimes \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2^{r-1}} \right) \oplus \left(H \otimes \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2^{-1}} \right).$$

Then, the Dirac operator on H inducing a spectral triple on $A \rtimes \mathbb{Z}^d$ can be written in the form

$$\widehat{D} = \begin{pmatrix} 0 & D \otimes 1_{\ell^2(\mathbb{Z}^d, V)} \\ D \otimes 1_{\ell^2(\mathbb{Z}^d, V)} & 0 \end{pmatrix} + 1_H \otimes M_l$$

with $V = \mathbb{C}^{2^{r-1}}$. □

6.2 Construction and Basic Properties

Till the end of this section we shall consider the following setup.

Assumptions 6.5. *Let G be a discrete group and $\sigma \colon G \times G \to U(1)$ a 2-cocycle such that $(\mathbb{C}, G, \mathrm{id}, \sigma)$ is a twisted C^*-dynamical system as in Definition 3.2. Assume further that G is endowed with a proper Dirac weight $l : G \to \mathcal{L}(V)$.*

Let M_l be the multiplication operator by l on the domain of finitely supported elements of $\ell^2(G, V) \simeq \ell^2(G) \otimes V$, explicitely

$$M_l(\delta_g \otimes v) \coloneqq \delta_g \otimes l(g)v$$

for $g \in G$ and $v \in V$, and let us denote also by M_l its self adjoint extension to $\ell^2(G, V)$. The twisted group algebra $\mathbb{C}G$ acts on $\ell^2(G, V)$ via the amplification $\lambda \otimes 1$ of the left regular representation λ of G on $\ell^2(G)$ given explicitely by

$$\lambda_g \delta_h \coloneqq \sigma(g, h)\delta_{gh} \qquad \forall g, h \in G.$$

Sometimes, we write λ instead of $\lambda \otimes 1$ for sake of simplicity. The data

$$(\mathbb{C}G, \ell^2(G, V), M_l, \lambda) \tag{6.1}$$

form then an odd spectral triple on the reduced twisted group C^*-algebra $C^*_{r,\sigma}(G)$. Indeed, M_l is self-adjoint as l takes values in self-adjoint matrices, the properness of l implies that the resolvent of M_l is compact and the fact that l is a Dirac weight guarantees that the commutators $[M_l, \lambda_g] = M_{l_g}\lambda_g$ are bounded for every $g \in G$. Note further that if the weight is non degenerate, then the triple $(\mathbb{C}G, \ell^2(G) \otimes V, M_l)$ is non degenerate.

Example 6.6 (cf. [81])**.** Consider $G = \mathbb{Z}^2$ and define a cocycle σ on it as in (3.19). We have seen with Proposition 3.20 that $C^*_\sigma(\mathbb{Z}^2)$ is isomorphic to the algebra of the noncommutative 2-torus. We want to show that one can recover the standard spectral triple on the torus using matrix valued Dirac weights. On the one hand, using Theorem 3.19 and the details contained in its proof, it is easy to see that the GNS representation H_τ is isomorphic to the space $L^2(S^1 \times S^1) \simeq \ell^2(\mathbb{Z}^2)$ via Fourier transform and that the multiplication operator of $C(S^1) \rtimes \mathbb{Z}$ goes to the twisted left regular representation of the twisted group algebra. Fix now $V = \mathbb{C}^2$ and consider the matrix valued Dirac weight $l^{(2)}$ as given in Example 6.3. Using the fact that the pointwise multiplication by the length function $i\colon \mathbb{Z} \to \mathbb{R}$ is the Fourier transform of the standard Dirac operator $D = -i\frac{\partial}{\partial x}$ on the circle S^1 (see Example 5.46), a straightforward computation shows that the matrix multiplication by $l^{(2)}$ is the Fourier transform of the Dirac operator (5.12). □

Let us now discuss when the spectral triple $(\mathbb{C}G, \ell^2(G, V), M_l)$ is finitely summable. We will adapt a well known argument from [22, 74], generalizing their results for matrix valued weights and twisted crossed products.

Definition 6.7. *Let G and $l : G \to \mathcal{L}(V)$ be as in Assumptions 6.5. Define*

$$B_n \coloneqq \{\, g \in G \mid \min \operatorname{Sp}(|l(g)|) \leq n \,\}.$$

*Note that Dirac weight l being proper, the cardinality $\#B_n$ of B_n is finite. We shall say that G has polynomial growth with respect to the weight l when $\#B_n$ grows at most polynomially for $n \to \infty$. In this case, we call **growth** of G the number*

$$d_G \coloneqq \limsup_n \frac{\log(\#B_n)}{\log n}. \tag{6.2}$$

Proposition 6.8. *Let G and $l : G \to \mathcal{L}(V)$ be as in Assumptions 6.5. The spectral triple $(\mathbb{C}G, \ell^2(G) \otimes V, M_l)$ defined in* (6.1) *is finitely summable if and only if G has polynomial growth with respect to the weight l. In this case,* $\operatorname{abs}(\zeta_{M_l})$ *coincides with the growth of G.*

Proof. Let's first estimate the size of the sets B_n. The eigenvectors of the Dirac operator for the spectral triple $(\mathbb{C}G, \ell^2(G)\otimes V, M_l)$ are the vectors $\delta_g \otimes v_j(g) \in \ell^2(G)\otimes V$, where the $v_j(g)$'s are the eigenvectors of the matrix $l(g)$. For any $g \in G$, we denote by $s_j(g)$, $j = 1, \dots, \dim V$, the singular values (with multiplicity) of $l(g)$ and set

$$\Sigma_n \coloneqq \{(g,k) \in G \times \{1,\dots,\dim V\} : s_k(g) < n\}.$$

The family of singular values (with multiplicity) for M_l is then given by

$$\{\ s_j(g) \ \mid\ (g,j) \in G \times \{\ 1,\dots,\dim V\ \}\ \}.$$

On the one hand $\Sigma_n \subseteq B_n \times \{1,\dots,\dim V\}$, therefore $\#\Sigma_n \le \#B_n \cdot \dim V$. On the other hand, for any $g \in B_n$ there exists $k \in \{1,\dots,\dim V\}$ such that $(g,k) \in \Sigma_{n+1}$, therefore $\#B_n \le \#\Sigma_{n+1}$. This implies that

$$d_G = \limsup_n \frac{\log(\#\Sigma_n)}{\log n}.$$

Let us now show that this quantity is equal to the abscissa of convergence. Using equation (5.1) we get that

$$\operatorname{abs}(\zeta_{M_l}) = \limsup_n \frac{\log(\lambda_{1/n}((1+M_l^2)^{-1/2}))}{\log n}.$$

We then observe that

$$\begin{aligned}\lambda_{1/n}((1+M_l^2)^{-1/2}) &= \#\left\{\ k\in\mathbb{N} : \mu_k((1+M_l^2)^{-1/2}) > 1/n\ \right\}\\ &= \#\left\{\ (g,k)\in G\times\{1,\dots,\dim V\} : \sqrt{1+s_k(g)^2} < n\ \right\}\end{aligned}$$

and that

$$\#\Sigma_{n-1} \le \#\{(g,k)\in G\times\{1,\dots,\dim V\} : \sqrt{1+s_k(g)^2} \le n\} \le \#\Sigma_n$$

since $|M_l| \le (1+M_l^2)^{1/2} \le 1+|M_l|$. Finally,

$$d_G = \limsup_n \frac{\log(\#\Sigma_n)}{\log n} = \limsup_n \frac{\log(\lambda_{1/n}((1+M_l^2)^{-1/2}))}{\log n} = \operatorname{abs}(\zeta_{M_l}).$$

□

Let us now move to discuss the regularity of the triple. A first general result, independent of the proper Dirac weight $l\colon G \to \mathcal{L}(V)$, is the following.

Proposition 6.9. *Let G and $l : G \to \mathcal{L}(V)$ be as in Assumptions 6.5. The triple $(\mathbb{C}G, \ell^2(G)\otimes V, M_l)$ is **Lipschitz regular**, that is the derivation δ defined in* (5.6) *is bounded.*

Proof. It is clear that the operator $|M_l|$ is the (self-adjoint extension of the) multiplication operator by $h \mapsto |l(h)|$. Given $\delta_h \otimes v \in \ell^2(G)\otimes V$ we have

$$\begin{aligned}[|M_l|, \lambda_g](\delta_h\otimes v) &= |M_l|\,(\delta_{gh}\otimes v) - \lambda_g\delta_h \otimes |l(h)|\,v\\ &= \delta_{gh}\otimes(|l(gh)| - |l(h)|)\,v.\end{aligned} \tag{6.3}$$

It is known that for any couple of square matrices S, T we have

$$\||S| - |T|\| \leq C \|S - T\| \tag{6.4}$$

for a suitable constant C which does not depend on S, T but depends on the dimension of the vector space V (see [62, Corollary 14]) and this is $C = 1$ for $\dim V = 1$ and proportional to $\log(\dim V)$ for $\dim V \geq 2$. This proves that the commutator $[|M_l|, \lambda_g]$ is bounded for any $g \in G$. $\square$

Remark 6.10. Estimate (6.4) is true only for operators on finite dimensional vector spaces (see [60]). This constitutes a serious obstruction to the generalization of the previous result to weights with values in $\mathcal{L}(H)$ for H infinite-dimensional Hilbert space.

In general, the left regular representation λ_g does not lie in the domain of the iterated derivation δ^k for $k \geq 2$ as the elements $|l(h)|$ do not commute each other. If this is the case, then

$$\delta^k(\lambda_g)(\delta_h \otimes v) = \delta_{gh} \otimes (|l(gh)| - |l(h)|)^k \, v$$

and this is bounded by (6.4). Understanding when the commutator $[M_l, \lambda_g]$ lies in the domain of δ^k for any $k \geq 1$ leads to a similar issue: for $\delta_h \otimes v \in \ell^2(G) \otimes V$ we have

$$\begin{aligned}
\delta([M_l, \lambda_g])(\delta_h \otimes v) &= |M_l| \, \delta_{gh} \otimes (l(gh) - l(h)) \, v - [M_l, \lambda_g] \, |M_l| \, (\delta_h \otimes v) \\
&= \delta_{gh} \otimes (|l(gh)| \, (l(gh) - l(h)) - (l(gh) - l(h)) \, |l(h)|) \, v
\end{aligned}$$

which is in general not bounded. However, as before, if we make the technical assumption that

$$[l(g), |l(h)|] = 0, \quad \forall g, h \in G \tag{6.5}$$

then

$$\delta^k([M_l, \lambda_g]) = \delta_{gh} \otimes (|l(gh)| - |l(h)|)^k \, (l(gh) - l(h)) v$$

which is bounded by (6.4). This discussion can be summarized in the following result.

Proposition 6.11. *Let G and $l : G \to \mathcal{L}(V)$ be as in Assumptions 6.5. If the Dirac weight l satisfies* (6.5)*, the spectral triple* $(\mathbb{C}G, \ell^2(G) \otimes V, M_l)$ *is regular.*

Remark 6.12. Note that the condition expressed in Equation (6.5) is satisfied whenever l is a Clifford length function on $\mathbb{Z}^n$. Indeed, in this situation $|l(m)| = \|m\| \cdot \mathrm{Id}_n$.

Let us now discuss the existence of a real structure on $(\mathbb{C}G, \ell^2(G, V), M_\ell)$. For sake of simplicity, we just consider the case $V = \mathbb{C}$. To this aim, we study the standard real structure given by the flip map $g \mapsto g^{-1}$ as in [14, 96] and adapt the argument in [98] to take in consideration the presence of the twisting cocycle.

Proposition 6.13. *Let G be a discrete group and $l \colon G \to \mathbb{R}$ a proper Dirac weight. The anti-unitary involutive map $J_G \colon \ell^2(G) \to \ell^2(G)$ given by the anti-linear extension of*

$$J_G \delta_g \coloneqq \sigma(g^{-1}, g)^* \delta_{g^{-1}} \qquad \forall g \in G \tag{6.6}$$

is a real structure on the odd spectral triple (5.16) *if and only if for every $g \in G$, $l(g^{-1}) = \varepsilon' l(g)$, where $\varepsilon' = \pm 1$. In this case the KO-dimension of the real structure is given by the pair $(+1, \varepsilon')$ and can be either* 1 *or* 7*.*

Proof. Clearly $J_G^2 = 1$. The zeroth order condition is a consequence of the cocycle properties: to see this, let us first note that

$$J_G\lambda_h J_G^{-1}\delta_x = \overline{\sigma(h,x^{-1})}\sigma(x^{-1},x)\overline{\sigma(xh^{-1},hx^{-1})}\delta_{xh^{-1}} = \overline{\sigma(xh^{-1},h)}\delta_{xh^{-1}}, \tag{6.7}$$

where to prove the second passage it is enough to substitute $x \mapsto xh^{-1}$, $y \mapsto h$ and $z \mapsto x^{-1}$ into Equation (3.2). Then

$$[\lambda_g, J_G\lambda_h J_G^{-1}]\delta_x = \left(\sigma(g,xh^{-1})\overline{\sigma(xh^{-1},h)} - \overline{\sigma(gxh^{-1},h)}\sigma(g,x)\right)\delta_{gxh^{-1}}.$$

To prove that this quantity is zero, just note that

$$\sigma(g,xh^{-1})\sigma(gxh^{-1},h) = \sigma(g,x)\sigma(xh^{-1},h) \tag{6.8}$$

by substituting $x \mapsto g$, $y \mapsto xh^{-1}$ and $z \mapsto h$ into Equation (3.2).

To conclude the proof, just note that the equation $M_l J_G = \pm J_G M_l$ is fulfilled if and only if $l(g^{-1}) = \pm l(g)$ for every $g \in G$. ☐

Proposition 6.14. *Let G be a discrete group and $l\colon G \to \mathbb{R}$ a proper Dirac weight. Suppose that the map J_G given in* (6.6) *is a real structure for the spectral triple* (5.16)*. Then $(\mathbb{C}G, \ell^2(G), M_l, J_G)$ satisfies the first order condition if and only if l is either a constant or a homomorphism.*

Proof. A straightforward computation, involving the use of (6.7) and (6.8), shows that the first order condition holds true if and only if we have that

$$l(xzy^{-1}) - l(zy^{-1}) = l(xz) - l(z)$$

for every $x, y, z \in G$. By Lemma 5.43, this means that l must be the sum of a constant and a homomorphism. But from Proposition 6.13 we know that $l(g^{-1}) = \varepsilon' l(g)$ for every $g \in G$ and it is easy to see that if $\varepsilon' = 1$ then l must be constant and that if $\varepsilon' = -1$ then l must be a homomorphism. ☐

Remark 6.15. Proposition 6.14 imposes a serious restriction to the possibility that the involution (6.6) is a real structure on the aformentioned triple. Indeed, if we require that

(1) G is discrete and not finite

(2) J_G is a real structure which satisfies the first order condition

(3) the triple $(\mathbb{C}G, \ell^2(G), M_l, J_G)$ is non degenerate

then the group must be cyclic and of the form $a\mathbb{Z}$ with $a \in \mathbb{R}$. This is a straightforward consequence of Remark 5.44. We deduce that if G is not of this form, then the real structure (6.6) is compatible only with some "degenerate" situations: either G is finite and M_l bounded or G is infinite but M_l has a non-trivial kernel.

Proposition 6.16. *Let G be a discrete group and $l\colon G \to \mathbb{R}$ a proper Dirac weight. Suppose that the map J_G given in* (6.6) *is a real structure for the spectral triple* (5.16)*. If $(\mathbb{C}G, \ell^2(G), M_l, J_G)$ satisfies the first order condition, then it satisfies the second order condition.*

Proof. A straightforward computation shows that $\Big[[M_l, \lambda_g], J_G[M_l, \lambda_h]J_G^{-1}\Big]$ applied to δ_x is equal to

$$\begin{aligned}
&\sigma(g, xh^{-1})\overline{\sigma(xh^{-1}, h)}\left(l(hx^{-1}) - l(x^{-1})\right)\left(l(gxh^{-1}) - l(xh^{-1})\right)\delta_{gxh^{-1}} \\
&\qquad - \sigma(g, x)\overline{\sigma(gxh^{-1}, h)}\,(l(gx) - l(x))\left(l(hx^{-1}g^{-1}) - l(x^{-1}g^{-1})\right)\delta_{gxh^{-1}}.
\end{aligned}$$

for any $g, h, x \in G$. The thesis comes from Proposition 6.14 and equation (6.8). $\square$

Chapter 7

Spectral Triples on $A \rtimes^{\sigma}_{\alpha,r} G$

In this chapter we construct a spectral triple on a twisted crossed product $A \rtimes^{\sigma}_{\alpha,r} G$ starting from a spectral triple on A satisfying some assumptions. More precisely, we provide two different constructions that under some additional properties of uniform boundedness of the action of G is a Lipschitz sense are equivalent. Moreover, we provide a description in the framework of KK-theory and then study the Connes axioms as spectral manifolds.

7.1 Constructions and Basic Properties

We first generalize the constructive result of [47] to the case of twisted actions and possibly degenerate matrix valued Dirac weights.

Theorem 7.1 (First Construction, cf. [5]). *Let (A, G, α, σ) be a twisted C^*-dynamical system with G discrete and $(\mathcal{A}, H, D)$ be an odd spectral triple on a unital C^*-algebra A with π faithful. Assume G is equipped with a proper matrix valued Dirac weight $l\colon G \to \mathcal{L}(V)$ and that the twisted action on A is also*

(1) ***smooth****, in the sense that it restricts to a twisted action on $\mathcal{A}$ (namely, $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ and $\sigma : G \times G \to U(\mathcal{A})$).*

(2) ***equicontinuous****, in the sense that*

$$\sup_{x \in G} \| [D, \pi(\alpha_x(a)\sigma(x, y))] \| < \infty \tag{7.1}$$

for every $a \in \mathcal{A}$ and $y \in G$.

Then we have an induced even spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ on $A \rtimes^{\sigma}_{\alpha,r} G$ with

$$\widehat{H} = (H \otimes \ell^2(G, V)) \oplus (H \otimes \ell^2(G, V)),$$

the representation is the direct sum $\Pi \oplus \Pi$ of the integrated form of the induced covariant representation (3.14) which is explicitly given by

$$\Pi(a\delta_x)(\xi \otimes \delta_y \otimes v) = \pi\big(\alpha_{y^{-1}x^{-1}}(a)\sigma(y^{-1}x^{-1}, x)\big)\xi \otimes \delta_{xy} \otimes v$$

for the basic elements $a\delta_x \in C_c(G,\mathcal{A})$, $v \in V$ and $\xi \in H$, and the Dirac operator has the form

$$\widehat{D} = \begin{pmatrix} 0 & D \otimes 1_{\ell^2(G,V)} - i1_H \otimes M_l \\ D \otimes 1_{\ell^2(G,V)} + i1_H \otimes M_l & 0 \end{pmatrix}. \tag{7.2}$$

Furthermore, if the weight l is non degenerate and the triple $(\mathcal{A}, H, D)$ is non degenerate, then the triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D})$ is also non degenerate.

Proof. As in [47], most of the properties will follow because we are following the prescription for the exterior Kasparov product. In particular the Dirac operator is a sum $\widehat{D} = D_1 + iD_2$ with the two blocks

$$D_1 := \begin{pmatrix} 0 & D \otimes 1_{\ell^2(G,V)} \\ D \otimes 1_{\ell^2(G,V)} & 0 \end{pmatrix}, \quad D_2 := \begin{pmatrix} 0 & -1_H \otimes M_l \\ 1_H \otimes M_l & 0 \end{pmatrix}.$$

The smooth algebra $C_c(G,\mathcal{A})$ preserves the domain of $\widehat{D}$ and it allows us to estimate the commutators $[D_1, \Pi \oplus \Pi(a)]$ and $[D_2, \Pi \oplus \Pi(a)]$, for $a \in C_c(G,\mathcal{A})$ separately. Of course we just have to compute these commutators for elements of the form $a\delta_x$. It follows by a straightforward computation that

$$[D_1, \Pi \oplus \Pi(a\delta_x)] \begin{pmatrix} \xi \otimes \delta_y \otimes v \\ \eta \otimes \delta_z \otimes w \end{pmatrix} = \begin{pmatrix} [D, \pi(\alpha_{z^{-1}x^{-1}}(a)\sigma(z^{-1}x^{-1}, x))]\eta \otimes \delta_{xz} \otimes w \\ [D, \pi(\alpha_{y^{-1}x^{-1}}(a)\sigma(y^{-1}x^{-1}, x)]\xi \otimes \delta_{xy} \otimes v \end{pmatrix}$$

which is bounded by condition (7.1). Concerning the second commutator we have

$$[D_2, \Pi \oplus \Pi(a\delta_x)] = \begin{pmatrix} 0 & -\big[1_H \otimes M_l, \Pi(a\delta_x) \otimes 1_V\big] \\ \big[1_H \otimes M_l, \Pi(a\delta_x) \otimes 1_V\big] & 0 \end{pmatrix}.$$

Now

$$\big[1_H \otimes M_l, \Pi(a\delta_x) \otimes 1_V\big](\xi \otimes \delta_y \otimes v) = \Pi(a\delta_x)(\xi \otimes \delta_y) \otimes (l(xy) - l(y))v$$

which is bounded by the property of the Dirac weight. Compactness of the resolvent and non degeneracy follow from the corresponding statements for the Kasparov exterior product [10].

$\square$

Remark 7.2. If the spectral triple on A is even with $H = H^+ \oplus H^-$, representation $\pi = \pi^+ \oplus \pi^-$ and $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$, then we can construct an odd spectral triple $(C_c(G,\mathcal{A})\, \widehat{H}, \widehat{D})$ on $A \rtimes^{\sigma}_{\alpha,r} G$ with Hilbert space $\widehat{H} = (H^+ \otimes \ell^2(G) \otimes V) \oplus (H^- \otimes \ell^2(G) \otimes V)$ and the representation Π is induced by π. It is diagonal whose components are induced by $\pi^{\pm}$. The Dirac operator is:

$$\widehat{D} = \begin{pmatrix} 1_{H^+} \otimes M_l & D^- \otimes 1_{\ell^2(G,V)} \\ D^+ \otimes 1_{\ell^2(G,V)} & -1_{H^-} \otimes M_l \end{pmatrix}. \tag{7.3}$$

$\square$

We want now to discuss another similar construction. The most important difference is the fact that in this second case the spectral triple is assumed being equivariant.

Theorem 7.3 (Second Construction, cf. [98])**.** *Let (A, G, α, σ) be a twisted C^*-dynamical system with G discrete and assume that it is equipped with a proper matrix valued Dirac weight $l\colon G \to \mathcal{L}(V)$. Let $(\mathcal{A}, H, D, u)$ be an odd spectral triple with π faithful and suppose that it G-equivariant, namely that there exists a map $u\colon G \to \mathcal{U}(H)$ as in Definition 5.25. We have an induced spectral triple*

$$(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda}) \tag{7.4}$$

on $A \rtimes^\sigma_{\alpha,r} G$ with $\widehat{H} = H \otimes \ell^2(G, V) \otimes \mathbb{C}^2 = (H \otimes \ell^2(G, V)) \oplus (H \otimes \ell^2(G, V))$, the representation is the direct sum of the integrated form of the covariant representation

$$\begin{cases} \hat{\pi}(a)(\xi \otimes \delta_g \otimes v) := \pi(a)\xi \otimes \delta_g \otimes v \\ \hat{\lambda}_h(\xi \otimes \delta_g \otimes v) := u_h\xi \otimes \delta_{hg} \otimes v, \end{cases} \tag{7.5}$$

for $a \in A$, $\xi \in H$, $v \in V$ and $g, h \in G$ defined following the prescription as in Subsection 3.1.4, the Dirac operator $\widehat{D}$ on $\widehat{H}$ by

$$\begin{aligned} \widehat{D} &:= D \otimes 1 \otimes \sigma_1 + 1 \otimes M_l \otimes \sigma_2 \\ &= \begin{pmatrix} 0 & D \otimes 1 - i1_H \otimes M_l \\ D \otimes 1 + i1_H \otimes M_l & 0 \end{pmatrix}, \end{aligned} \tag{7.6}$$

where σ_1 and σ_2 are Pauli matrices.

Proof. The only non trivial thing to check is that the operator (7.6) has bounded commutators with $C_c(G, \mathcal{A})$. First of all we note that

$$\alpha_g(C^{Lip}(A)) \subseteq C^{Lip}(A) \tag{7.7}$$

for any $g \in G$. Indeed, as $\pi(\alpha_g(a)) = u_g\pi(a)u_g^*$, any element $\alpha_g(a)$ for $a \in C^{Lip}(A)$ preserves the domain of D. Then, using the covariance condition, a straightforward computation shows that

$$[D, \pi(\alpha_g(a))] = -\Big[u_g[D, u_g^*], \pi(\alpha_g(a))\Big] + u_g[D, \pi(a)]u_g^*. \tag{7.8}$$

and so the commutator $[D, \pi(\alpha_g(a))]$ is bounded when $a \in C^{Lip}(A)$. Since $\mathcal{A} \subseteq C^{Lip}(A)$ by construction, using equation (5.10) it is easy to see that the operator (7.6) has bounded commutators with $C_c(G, \mathcal{A})$ and so (7.4) is a spectral triple. □

Remark 7.4. Note that, differently from what was discussed in Theorem 7.1, in Theorem 7.3 action α need neither be smooth nor equicontinuous in order to ensure a well defined and bounded commutation relation between $\widehat{D}$ and $A \rtimes^\sigma_{\alpha,r} G$.

Example 7.5. If we apply this procedure to the equivariant spectral triple of Example 5.27 where $\mathbb{Z}$ is endowed with the proper Dirac weight $i\colon \mathbb{Z} \to \mathbb{R}$ given by the inclusion, we just get the canonical spectral triple on the noncommutative 2-torus $C(S^1) \rtimes_\alpha \mathbb{Z}$. Indeed, we have seen in Theorem 3.19 that the GNS representation H_τ is isomorphic to $L^2(S^1) \otimes \ell^2(\mathbb{Z})$ and the corresponding multiplication representation of $C(S^1) \rtimes \mathbb{Z}$ goes precisely to integrated form of the covariant couple $(\hat{\pi}, \hat{\lambda})$, where $\hat{\pi}$ is the multiplication operator of $C(S^1)$ and $\hat{\lambda}$ is defined as in (7.5) for u as in Example 5.27. We also known that the Fourier transform of the Dirac weight $i\colon \mathbb{Z} \to \mathbb{R}$ gives precisely the derivation operator on the circle (see Example 5.46), then the Dirac operator $\widehat{D}$ as defined in (7.6) goes precisely to the standard Dirac operator (5.12) on the noncommutative torus.

□

The spectral triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda})$ on $A \rtimes^{\sigma}_{\alpha,r} G$ as defined in Theorem (7.3) has been constructed starting from an equivariant spectral triple: as we are going to show, this triple is also equivariant with respect to the dual coaction of G.

Proposition 7.6. *Assume $\sigma \equiv 1$. The spectral triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda})$ is equivariant with respect to the dual coaction $\widehat{\alpha}$ of Example C.9.*

Remark 7.7. If the group G is abelian and $\sigma \equiv 1$, Proposition 7.6 means that the triple on $A \rtimes^{\sigma}_{\alpha,r} G$ is $\widehat{G}$-invariant under the unitary representation $V \coloneqq v \oplus v\colon \widehat{G} \to \mathcal{L}(\widehat{H})$ where $v\colon \widehat{G} \to \mathcal{L}(H \otimes \ell^2(G))$ is given by

$$v_\chi(\xi \otimes \delta_g) \coloneqq \overline{\chi(g)}\xi \otimes \delta_g$$

for $\xi \in H$, $g \in G$ and $\chi \in \widehat{G}$.

Proof. Consider $C^*_r(G)$ coacting on itself via the map $\Delta(\delta_g) = \delta_g \otimes \delta_g$ and the unitary corepresentation map $\Theta\colon H \otimes \ell^2(G) \to H \otimes \ell^2(G) \otimes C^*_r(G)$ given by

$$\Theta(\xi \otimes \delta_g) = \xi \otimes \delta_g \otimes \delta_g, \qquad \xi \in H,\ g \in G.$$

According to (C.8), Θ is equivalent to the unitary operator $X \in \mathcal{L}(H \otimes \ell^2(G) \otimes C^*_r(G))$ given by

$$X(\xi \otimes \delta_x \otimes \delta_g) \coloneqq \xi \otimes \delta_x \otimes \delta_{xg}, \qquad x, g \in G$$

In this way $H \otimes \ell^2(G)$ becomes a $C^*_r(G)$-equivariant $C^*_r(G)$-module. Consider in fact $b = a\delta_g \in C_c(G,\mathcal{A})$ and $x = \xi \otimes \delta_h \in H \otimes C^*_r(G)$, then by definition $b_{(-1)} = a\delta_g$, $b_{(0)} = \delta_g$, $x_{(-1)} = \xi \otimes \delta_h$ and $x_{(0)} = \delta_h$. Next,

$$\begin{aligned}\Theta(b \triangleright x) &= \Theta(\pi(a)u_g\xi \otimes \delta_{gx}) = \pi(a)u_g\xi \otimes \delta_{gx} \otimes \delta_{gx} \\ &= b_{(-1)} \triangleright x_{(-1)} \otimes b_{(0)}x_{(0)}.\end{aligned}$$

Moreover, it is easy to check that $[D \otimes 1 \otimes 1, X] = 0$ and $[1 \otimes M_l \otimes 1, X] = 0$ so that $[\widehat{D}, X \oplus X] = 0$. Finally, we have that $X\hat{\pi}(a)\hat{\lambda}_h X^*(\xi \otimes \delta_x \otimes \delta_g) = \pi(a)u_h\xi \otimes \delta_{hx} \otimes \delta_{hg}$ for any $x, g, h \in G$ and so

$$(\mathrm{id} \otimes \varphi)\mathrm{Ad}_X(a\delta_h) = \varphi(\delta_{hg})a\delta_{hx} \in C_c(G,\mathcal{A}) \subseteq (A \rtimes_{\alpha,r} G)''$$

for any state φ on $C^*_r(G)$. □

Remark 7.8. Note that what we have described so far extends to a construction of an equivariant odd spectral triple on the crossed product starting from an equivariant *even* spectral triple.

Let us now discuss the relationships between the two constructions. It turns out that, under some natural assumptions, they are K-homologically equivalent. Consider the setup given in Theorem 7.3. It's easy to adapt the contents of Proposition 3.16 to the case in which V is not trivial and then prove that the unitary operator $W\colon H\otimes\ell^2(G,V)\to H\otimes\ell^2(G,V)$ given by

$$W(\xi\otimes\delta_g\otimes v):=\pi(\sigma(g,g^{-1})^*)u_g\xi\otimes\delta_g\otimes v \tag{7.9}$$

intertwines the representations of the algebra $C_c(G,\mathcal{A})$ on $\widehat{H}$ in the two constructions, in the sense that

$$W\tilde{\pi}\rtimes\tilde{\lambda}(a\delta_h)W^*=\hat{\pi}\rtimes\hat{\lambda}(a\delta_h)$$

for any $a\delta_h\in A\rtimes^{\sigma}_{\alpha,r}G$. At this point it is clear that the unitary $\mathcal{W}=W\oplus W$ conjugates the operator $\widehat{D}$ to $\mathcal{W}\widehat{D}\mathcal{W}^*=\widehat{D}+\mathcal{W}[\widehat{D},\mathcal{W}^*]$. In general this is not a bounded perturbation of $\widehat{D}$ because the commutators $[D,u_g]$ are just pointwise bounded. However, we have the following result.

Lemma 7.9. *Let (A,G,α,σ) be a twisted C^*-dynamical system and $(\mathcal{A},H,D,u)$ a G-equivariant spectral triple on A. We have the following facts:*

(1) If the commutator $[u_g,D]$ is uniformly bounded in norm for all $g\in G$, then the twisted action α of G on A is equicontinuous in the sense of (7.1)*.*

(2) If $(\mathcal{A},H,D,u)$ is G-invariant, then the action is Lip-isometric in the sense that

$$\|[D,\pi(\alpha_g(a))]\|=\|[D,\pi(a)]\|\,.$$

Proof. The second point is a consequence of equation (7.8). To prove point (1), note that

$$[D,\pi\left(\alpha_x(a)\sigma(x,y)\right)]=[D,\pi\left(\alpha_x(a)\right)]\pi(\sigma(x,y))+\pi(\alpha_x(a))[D,\pi\left(\sigma(x,y)\right)]$$

and so it is enough to prove that $[D,\pi\left(\alpha_x(a)\right)]$ and $[D,\pi\left(\sigma(x,y)\right)]$ are uniformly bounded in norm in the x variable. On the one hand, from equation (7.8) we have

$$\|[D,\pi(\alpha_x(a))]\|\le 2\,\|[D,u_x]\|\,\|\pi(a)\|+\|[D,\pi(a)]\|$$

for any $a\in A$ and $g\in G$. On the other hand, since $\|[D,u_g]\|=\left\|[D,u_g^*]\right\|$, we have that

$$\|[D,\pi(\sigma(x,y))]\|=\left\|[D,\pi\left(u_xu_yu_{xy}^*\right)]\right\|\le 3\sup_x\|[D,u_x]\|$$

and this concludes. □

The relation between $(C_c(G,\mathcal{A}),\widehat{H},\widehat{D},\hat{\pi}\rtimes\hat{\lambda})$ and $(C_c(G,\mathcal{A}),\widehat{H},\widehat{D},\hat{\pi}_1\rtimes\hat{\lambda})$ as triples on $A\rtimes^{\sigma}_{\alpha,r}G$ is then easy to understand: *if the twisted action is smooth and the commutator $[u_g,D]$ is uniformly bounded in norm for all $g\in G$, then the triple $(C_c(G,\mathcal{A}),\widehat{H},\widehat{D},\tilde{\pi}\rtimes\tilde{\lambda})$ is well defined by Lemma 7.9 and defines the same K-homology class of* $(C_c(G,\mathcal{A}),\widehat{H},\widehat{D},\hat{\pi}\rtimes\hat{\lambda})$ since it is just the bounded perturbation of a triple which is unitarily equivalent to it. Note in particular that, when $[D,u_g]=0$ for any $g\in G$, then the two triples are well defined and unitarily equivalent.

Example 7.10. We provide an example of a G-equivariant spectral triple $(\mathcal{A},H,D,u)$ for which the action is equicontinuous but the commutators $[D,u_g]$ are not uniformly bounded, so that the vice versa of Lemma 7.9 is in general not true. Consider the spectral triple $(C^\infty(S^1), L^2(S^1), D = -i\frac{\partial}{\partial x})$, denote by α the trivial automorphism of $C(S^1)$ and define a unitary operator X on $L^2(S^1)$ by

$$(Xf)(x) \coloneqq e^{2\pi i x} f(x).$$

Clearly, the representation M and the operator X commute so (M, X) is a covariant representation of $(C(S^1), \mathbb{Z}, \alpha)$ and $(C^\infty(S^1), L^2(S^1), D = -i\frac{\partial}{\partial x}, X)$ is an equivariant spectral triple. The trivial action is clearly equicontinuous (even Lipschitz isometric). However, a straightforward computation shows that

$$[D, X^n]f(x) = -i\frac{\partial}{\partial x}\left(e^{2\pi i n x} f(x)\right) + ie^{2\pi i n x} f'(x) = 2\pi n e^{2\pi i n x} f(x),$$

and so $\|[D, X^n]\| = 2\pi n$. This is clearly not uniformly bounded in $n \in \mathbb{Z}$. □

At the end of this section we show that in the equivariant case, even when neither σ nor V are trivial, the spectral triple (7.4) can be understood as the restriction of an external product of spectral triples. The key point of the construction is the existence of a morphism

$$\phi^r_A : A \rtimes^{\sigma}_{\alpha,r} G \longrightarrow A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G)$$

such that $\phi^r_A(a\delta_g) = a\delta_g \otimes \delta_g$. This map is the twisted version of the one used in [7, 31]. Integrating (π, u) defines a non degenerate representation $\pi \rtimes u : A \rtimes^{\sigma}_{\alpha} G \to \mathcal{L}(H)$. Let $B \subseteq \mathcal{L}(H)$ be the (separable) C^*-algebra image of $\pi \rtimes u$. We get a non degenerate morphism

$$\psi_A : A \rtimes^{\sigma}_{\alpha,r} G \longrightarrow B \otimes C^*_r(G).$$

The analytical details will be included in the proof of the next proposition. The algebra B carries a spectral triple $(\mathcal{B}, H, D)$ with the same D as the triple on A and smooth algebra $\mathcal{B}$ given by finite sums $\sum_{g\in G} a_g U_g$ with $a_g \in \mathcal{A}$; on the other hand, we consider on $C^*_r(G)$ the spectral triple $(C_c(G), \ell^2(G) \otimes V, M_l, \lambda)$ constructed with the length function and regular representation.

Proposition 7.11. *The equivariant spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda})$ is the restriction via ψ_A of the product of the triples $(\mathcal{B}, H, D)$ and $(C_c(G), \ell^2(G) \otimes V, M_l, \lambda)$.*

Proof. We begin by constructing a morphism $\phi^{\max}_A : A \rtimes^{\sigma}_{\alpha} G \to A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G)$. This one exists because is the integrated form of a covariant couple with values in $A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G)$; precisely the couple (j_A, j_G) with

$$\begin{aligned} j_A : A &\longrightarrow A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G), \qquad & j_G : G &\longrightarrow U(A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G)),\\ a &\longmapsto a \otimes 1 & g &\longmapsto \delta_g \otimes \delta_g. \end{aligned}$$

Indeed we have presented the universal property of the twisted crossed product with respect to covariant representations on Hilbert spaces but we may also use covariant couples with values in the multipliers of any C^*-algebra using Hilbert modules. In practice we can just represent faithfully $A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G)$ on a Hilbert space to make the construction rest on the Hilbert space based definition. To see that $\phi^{\max}_A$ descends to the reduced crossed product, we adapt the argument in [7]. Fix a faithful representation $\mu : A \rtimes^{\sigma}_{\alpha} G \to \mathcal{L}(H_\mu)$ and combine it with the left regular representation λ to obtain a faithful representation $\mu \otimes \lambda$ of $A \rtimes^{\sigma}_{\alpha} G \otimes C^*_r(G)$ on $H_\mu \otimes \ell^2(G)$. Then μ corresponds to a covariant representation (π^μ, u^μ) on H_μ and $(\mu\otimes\lambda)\circ\phi_A^{\max}$ corresponds to the covariant representation $(\pi^\mu\otimes\mathrm{Id}_{\ell^2(G)}, u^\mu\otimes\lambda_g)$. On the other hand, we consider the reduced crossed product as being defined by the covariant representation induced by π^μ and we call this couple $(\underline{\pi}^\mu, L^\mu)$, namely

$$\underline{\pi}^\mu(a)(\xi\otimes\delta_x) := \pi^\mu(\alpha_{x^{-1}}(a))\xi\otimes\delta_x, \quad L_x^\mu(\xi\otimes\delta_y) := \pi^\mu(\sigma(y^{-1}x^{-1},x))\xi\otimes\delta_{xy}.$$

The unitary

$$\begin{aligned}\Theta : H_\mu\otimes\ell^2(G) &\longrightarrow H_\mu\otimes\ell^2(G)\\ \xi\otimes\delta_g &\longmapsto \pi^\mu(\sigma(g,g^{-1})^*)u_g^\mu\xi\otimes\delta_g\end{aligned}$$

satisfies

$$\Theta\,\underline{\pi}^\mu\,\Theta^* = \pi^\mu\otimes\mathrm{Id}_{\ell^2(G)} \quad\text{and}\quad \Theta\, L_g^\mu\,\Theta^* = u_g^\mu\otimes\lambda_g,$$

so that $\Theta\,(\underline{\pi}^\mu\rtimes L^\mu)\,\Theta^* = (\mu\otimes\lambda)\circ\phi_A^{\max}$. This means that the representation $(\mu\otimes\lambda)\circ\phi_A^{\max}$ is unitarily equivalent to the left regular one. It follows that $\phi_A^{\max}$ descends to an injection $\phi_A^r : A\rtimes_{\alpha,r}^\sigma G\to A\rtimes_\alpha^\sigma G\otimes C_r^*(G)$ defined on the reduced crossed product.

Now it is clear that D on H is a Dirac operator for a spectral triple $(\mathcal{B},H,D)$ because the starting triple is equivariant. The rest of the proof is straightforward because the operator $\widehat{D}$ in (7.2) is exactly the product of the amplification of D and the amplification of M_l. □

7.2 A KK-Theory Description

In this section we give a more precise description of the constructions presented in previous section in terms of natural maps in equivariant KK-theory (namely, the Green-Julg map and the Kasparov descent).

Assumptions 7.12. *In this section we fix the assumptions as in Theorem 7.3 together with $\sigma\equiv 1$ and $V=\mathbb{C}$.*

First of all we want to show that the spectral triple (7.4) can be factorized with the Kasparov product under the Green-Julg isomorphism for discrete groups. Indeed, the weight l defines a G equivariant odd spectral triple $(\mathbb{C},\ell^2(G),M_l)$ where the group action on $\ell^2(G)$ is given by the left regular representation. The Kasparov product of $[D]\in KK_1^G(A,\mathbb{C})$ and $[M_l]\in KK_1^G(\mathbb{C},\mathbb{C})$ is represented by the even G-equivariant triple

$$(\mathcal{A},\widehat{H},\widehat{D}), \tag{7.10}$$

on A where $\widehat{H} = H\otimes\ell^2(G)\otimes\mathbb{C}^2$, the Dirac operator $\widehat{D}$ is defined as in (5.18), the representation of A on $\widehat{H}$ is given by (two copies of)

$$\hat{\pi}(a)(\xi\otimes\delta_g) := \pi(a)\xi\otimes\delta_g, \qquad a\in A, \xi\in H, g\in G$$

and the equivariance is implemented by (two copies of) the representation $\hat{\lambda}\colon G\to\mathcal{L}(\widehat{H})$ given by

$$\hat{\lambda}_h(\xi\otimes\delta_g) := u_h\xi\otimes\delta_{hg} \tag{7.11}$$

for $g,h\in G$. Under the Green-Julg isomorphism, the class of the triple (7.10) is represented in $KK(A\rtimes_\alpha G,\mathbb{C})$ by the triple

$$(C_c(G,\mathcal{A}),\widehat{H},\widehat{D}) \tag{7.12}$$

where the action of the algebra is now given by (two copies of) the integrated form of the covariant representation $(\hat{\pi}, \hat{\lambda})$. We have seen that the unitary map W conjugates the representation of the triple (7.4) to the representation defining the reduced crossed product and so the spectral triple (7.12), which is a priori just defined on the maximal crossed product, descends to a triple on the *reduced* crossed product $A \rtimes_{\alpha,r} G$.

The spectral triple (7.12) can be seen also as a representative of an *interior* Kasparov product as follows. It is known that in bounded KK-theory there is a commutative diagram

$$\begin{tikzcd}
KK_1^G(A,\mathbb{C}) \times KK_1^G(\mathbb{C},\mathbb{C}) \arrow[r, "\otimes_{\text{ext}}"] \arrow[d, "J_G \times I_G"] & KK_0^G(A,\mathbb{C}) \arrow[d, "I_G"] \\
KK_1(A \rtimes_\alpha G, C^*(G)) \times KK_1(C^*(G),\mathbb{C}) \arrow[r, "\otimes_{C^*(G)}"] & KK_0(A \rtimes_\alpha G, \mathbb{C}).
\end{tikzcd}$$

Here J_G is the Kasparov descent map (see Subsection 4.4.2), I_G is the Green-Julg isomorphism, $\otimes_{C^*(G)}$ represents the interior Kasparov product and $\otimes_{\text{ext}}$ is the exterior Kasparov product. This is a consequence of the fact that the Kasparov descent is compatible with the product and that it factorizes the Green-Julg isomorphism (see Remark 4.33). We will now show the following fact:

Theorem 7.13. *Let* $[D] \in KK_1^G(A,\mathbb{C})$ *and* $[M_l] \in KK_1^G(\mathbb{C},\mathbb{C})$ *be classes defined by two spectral triples as before and let* $[\widehat{D}] \in KK(A \rtimes_\alpha G, \mathbb{C})$ *be the class defined by the spectral triple* (7.12). *Then*

$$[\widehat{D}] = J_G[D] \otimes_{C^*(G)} I_G[M_l].$$

Proof. First of all, let us describe explicitely the image of the equivariant spectral triple $(\mathcal{A}, H, D, u)$ under the Kasparov descent (we take for granted the definitions and conventions in Subsection 4.4.2). From formula (4.13), consider $B = \mathbb{C}$ and $C_c(G)$ acting on the right on $C_c(G, H)$ by right multiplication:

$$(\xi \otimes \delta_g) \triangleleft \delta_h \coloneqq \xi \otimes \delta_{gh}$$

for $\xi \in H$ and $\delta_g, \delta_h \in C_c(G)$. The completion of $C_c(G, H)$ with the $C^*(G)$-valued scalar product

$$\langle \xi \otimes \delta_g, \mu \otimes \delta_h \rangle \coloneqq \langle \xi, \mu \rangle \delta_{g^{-1}h},$$

as introduced in formula (4.14), defines the right Hilbert $C^*(G)$-Hilbert module $H \rtimes G \simeq H \otimes C^*(G)$. Using formula (4.15), we see that the representation $\pi \colon A \to \mathcal{L}(H)$ induces a representation ψ of $A \rtimes G$ on $H \otimes C^*(G)$ by

$$\psi(a\delta_g)(\xi \otimes \delta_h) \coloneqq \pi(a) u_g \xi \otimes \delta_{gh}.$$

The image of $(\mathcal{A}, H, D, u)$ under J_G is then just $(C_c(G, A), H \otimes C^*(G), D \otimes 1, \psi)$. To compute the product of this element together with $(C_c(G), \ell^2(G), M_l, \lambda)$, we now want to show that the hypothesis of Theorem 4.22 hold true (as before, we take for granted the definitions and notations from Subsection 4.4.2).

Consider $B_1 = C_c(G) \subseteq B = C^*(G)$ and $X_1 = H \otimes C_c(G) \subseteq H \otimes C^*(G) = X$. The map $\nabla\colon X_1 \to X \otimes_{C^*(G)} \mathcal{L}(\ell^2(G))$ given by

$$\nabla(\xi \otimes \delta_g) \coloneqq (\xi \otimes \delta_e) \otimes_{C^*(G)} [M_l, \lambda_g].$$

is a connection on X_1 as

$$\begin{aligned}\nabla(\xi \otimes \delta_g \triangleleft \delta_h) &= \nabla(\xi \otimes \delta_{gh}) = (\xi \otimes \delta_e) \otimes_{C^*(G)} [M_l, \lambda_{gh}] \\ &= (\xi \otimes \delta_e) \otimes_{C^*(G)} \lambda_g [M_l, \lambda_h] + (\xi \otimes \delta_e) \otimes_{C^*(G)} [M_l, \lambda_g]\lambda_h \\ &= (\xi \otimes \delta_g) \otimes_{C^*(G)} [M_l, \lambda_h] + \nabla(\xi \otimes \delta_g) \triangleleft \delta_h.\end{aligned}$$

Actually, an easy computation shows that ∇ is the Grassmann connection on X_1 and so it is also hermitian. Let us now show that (X_1, ∇) is a correspondence from $(C_c(G, A), H \otimes C^*(G), D \otimes 1, \psi)$ to $(C_c(G), \ell^2(G), M_l, \lambda)$:

(1) the operator $*$-module $X_1 \subseteq X$ is clearly a dense subspace of X and the operator $*$-algebra $B_1 \subseteq B$ is a dense $*$-subalgebra of B. The inclusions are completely bounded and compatible with the module structures and inner products.

(2) each $b \in B_1$ maps the domain of M_l into itself and $[M_l, \cdot]\colon B_1 \to \mathcal{L}(\ell^2(G))$ is completely bounded on B_1.

(3) Let us prove that the commutator $[1 \otimes_\nabla M_l, a\delta_g]$ is bounded on $X \otimes_B Y$ for all $a \in A$ and $g \in G$. A straightforward computation shows that

$$\begin{aligned}[1 \otimes_\nabla M_l, \psi(a\delta_g) \otimes 1](e \otimes_B \delta_h) &= \psi(a\delta_g)e \otimes_B M_l\delta_h + c(\nabla)(\psi(a\delta_g)e \otimes_B \delta_h) \\ &\quad - \psi(a\delta_g)e \otimes_B M_l\delta_h - (\psi(a\delta_g) \otimes 1)c(\nabla)(e \otimes_B \delta_h) \\ &= [c(\nabla), \psi(a\delta_g) \otimes 1](e \otimes_B \delta_h).\end{aligned}$$

In particular, we have that

$$\begin{aligned}[c(\nabla), \psi(a\delta_g) \otimes 1](\xi \otimes \delta_x \otimes_B \delta_h) &= \pi(a)u_g\xi \otimes \delta_e \otimes_B [M_l, \lambda_{gx}]\delta_h \\ &\quad - \pi(a)u_g\xi \otimes \delta_g \otimes_B [M_l, \lambda_x]\delta_h \\ &= \pi(a)u_g\xi \otimes \delta_e \otimes_B [M_l, \lambda_g]\lambda_x\delta_h.\end{aligned}$$

This is clearly bounded.

(4) Instead of proving in full generality that, for any $\mu \in \mathbb{R} \setminus \{ 0 \}$, the unbounded operator

$$[D \otimes 1, 1 \otimes_\nabla M_l]\,(D \otimes 1 - i\mu)^{-1}$$

is well-defined and extends to a bounded operator on $X \otimes_B Y$, we will actually prove that the commutator $[D \otimes 1, 1 \otimes_\nabla M_l]$ is zero. Indeed, $[D \otimes 1, 1 \otimes M_l] = 0$ for obvious reasons and

$$\begin{aligned}[D \otimes 1, c(\nabla)](\xi \otimes \delta_x \otimes_B \delta_h) = D\xi \otimes \delta_e \otimes_B [M_l\lambda_x]\delta_h &\qquad = 0. \\ - D\xi \otimes \delta_e \otimes_B [M_l\lambda_x]\delta_h &\end{aligned}$$

Following the prescription of Theorem 4.22, we deduce that the operator

$$D \otimes 1 \times_\nabla M_l \coloneqq \begin{pmatrix} 0 & D \otimes 1 \otimes 1 - i1 \otimes_\nabla M_l \\ D \otimes 1 \otimes 1 + i1 \otimes_\nabla M_l & 0 \end{pmatrix} \tag{7.13}$$

on $(H \otimes C^*(G) \otimes_{C^*(G)} \ell^2(G)) \oplus (H \otimes C^*(G) \otimes_{C^*(G)} \ell^2(G))$ is an even unbounded Kasparov $A-\mathbb{C}$-module which represents the interior Kasparov product of $(C_c(G, A), H \otimes C^*(G), D \otimes 1, \psi)$ together with $(C_c(G), \ell^2(G), M_l, \lambda)$. To relate this spectral triple with the spectral triple (7.12), consider the unitary transformation

$$\Phi \colon H \otimes C^*(G) \otimes_{C^*(G)} \ell^2(G) \longrightarrow H \otimes \ell^2(G), \quad \Phi(\xi \otimes \delta_g \otimes_{C^*(G)} \delta_h) = \xi \otimes \delta_{gh}$$

which is clearly well defined and its inverse is given by $\xi \otimes \delta_g \mapsto \xi \otimes \delta_e \otimes_{C^*(G)} \delta_g$. We also denote by Φ the map $\Phi \oplus \Phi$ and claim that

$$\widehat{D} = \Phi(D \otimes 1 \times_\nabla M_l)\Phi^{-1}. \tag{7.14}$$

Indeed,

$$\begin{aligned}\Phi(D \otimes 1 \pm i \otimes_\nabla M_l)\Phi^{-1}(\xi \otimes \delta_g) &= \Phi(D \otimes 1 \pm i \otimes_\nabla M_l)(\xi \otimes \delta_e \otimes_{C^*(G)} \delta_g) \\ &= \Phi(D\xi \otimes \delta_e \otimes_{C^*(G)} \delta_g \pm i\xi \otimes \delta_e \otimes_{C^*(G)} M_l\delta_g \\ &\qquad \pm i\, \underbrace{c(\nabla)(\xi \otimes \delta_e \otimes_{C^*(G)} \delta_g)}_{0}) \\ &= D\xi \otimes \delta_g \pm i\xi \otimes M_l\delta_g \\ &= (D \otimes 1 \pm i \otimes M_l)(\xi \otimes \delta_g).\end{aligned}$$

Let us now show that also the representations are unitarily equivalent:

$$\begin{aligned}\Phi(\psi(a\delta_g) \otimes 1)\Phi^{-1}(\xi \otimes \delta_h) &= \Phi(\psi(a\delta_g) \otimes 1)(\xi \otimes \delta_e \otimes_{C^*(G)} \delta_h \\ &= \Phi(\pi(a)u_g\xi \otimes \delta_g \otimes_{C^*(G)} \delta_h) = \pi(a)u_g\xi \otimes \delta_{gh} \\ &= \hat{\pi} \rtimes \hat{\lambda}_h(\xi \otimes \delta_h).\end{aligned}$$

We conclude then that the Kasparov product (7.13) is unitary equivalent to the spectral triple (7.12). ☐

7.3 Finite Summability

Consider the spectral triple $(\mathcal{A}, \widehat{H}, \widehat{D})$ as constructed in Theorem 7.1 and recall Definition 6.7. We have the following fact.

Theorem 7.14. *If the triple $(\mathcal{A}, H, D)$ is finitely summable and G has polynomial growth w.r.t. the proper Dirac weight $l \colon G \to \mathcal{L}(V)$, then also the spectral triple $A \rtimes^{\sigma}_{\alpha,r} G$ is finitely summable and*

$$\operatorname{abs}(\zeta_{\widehat{D}}) \leq \operatorname{abs}(\zeta_D) + d_G, \tag{7.15}$$

where d_G is the growth of G. If $\lim_n \frac{\log(\#B_n)}{\log n}$ exists, then the equality in (7.15) holds true.

We notice that if the group G is finitely generated and $l(g)$ is given by the word length, then d_G does not depend on the choice of the generators and the limit $\lim_n \frac{\log(\#B_n)}{\log n}$ exists.

Proof. Let us note that, if T is a positive invertible operator with compact inverse and we denote by $N_t(T)$ the number (with multiplicity) of the eigenvalues of T lower than t, we have $N_t(T) = \#\{n \geq 0 : \mu_n(T^{-1}) > t\}$ and $o(T^{-1}) = \limsup_{t\to\infty} \frac{\log(N_t(T))}{\log t}$. Therefore

$$\operatorname{abs}(\zeta_{\widehat{D}}) = \limsup_{t\to\infty} \frac{\log(N_t((1 + \widehat{D}^2)^{1/2}))}{\log t}.$$

Since $\widehat{D}^2 = (1 \otimes M_l^2 + D^2 \otimes 1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the eigenvalues of $1 \otimes M_l^2 + D^2 \otimes 1$ are $\mu^2 + \nu^2$ where μ, resp. ν is a singular value of M_l, resp. D, we have

$$N_t\left((1+\widehat{D}^2)^{1/2}\right) = 2N_t\left(\sqrt{1 \otimes (1+M_l^2) + D^2 \otimes 1}\right)$$

and

$$\begin{aligned} N_{t/\sqrt{2}}\left(\sqrt{1+M_l^2}\right) N_{t/\sqrt{2}}\left(\sqrt{1+D^2}\right) &\leq N_t\left(\sqrt{1 \otimes (1+M_l^2) + D^2 \otimes 1}\right) \\ &\leq N_t\left(\sqrt{1+M_l^2}\right) N_t\left(\sqrt{1+D^2}\right), \end{aligned}$$

hence

$$\begin{aligned} \operatorname{abs}(\zeta_{\widehat{D}}) &= \limsup_{t\to\infty} \frac{\log(N_t((1+\widehat{D}^2)^{1/2}))}{\log t} \\ &= \limsup_{t\to\infty} \frac{\log\left(N_t(\sqrt{1+M_l^2})\right)}{\log t} + \frac{\log\left(N_t(\sqrt{1+D^2})\right)}{\log t} \\ &\leq d_G + \operatorname{abs}(\zeta_D), \end{aligned}$$

where the equality holds if any of the limits exists. □

Remark 7.15. The hypothesis about the polynomial growth of the Dirac weight guarantees the finitely summability of the spectral triple on the twisted group algebra (remember Proposition 6.8). Theorem 7.14 relies on the fact that the exterior product of finitely summable spectral triples is finitely summable.

Remark 7.16. The proof of the previous result can be easily rephrased for the triple (7.4) defined in Theorem 7.3 since the representation of the algebra $A \rtimes_{\alpha,r}^{\sigma} G$ is not involved in the proof, which just uses the peculiar form of the Dirac operator.

7.4 The Regularity Condition

In this section we discuss the regularity of the triple (7.4) on $A \rtimes_{\alpha,r}^{\sigma} G$. Recall in particular the spectral triples

$$(\mathcal{B}, H, D) \quad \text{and} \quad (C_c(G), \ell^2(G) \otimes V, M_l, \lambda) \tag{7.16}$$

where $\mathcal{B}$ is the algebra of finite sums $\sum_{g\in G} a_g u_g$ with $a_g \in \mathcal{A}$. The completion of $\mathcal{B}$ is the algebra $B \subseteq \mathcal{L}(H)$ image of the full crossed product by the integrated form of the covariant representation (π, u). By Proposition 7.11 the equivariant triple is the restriction of the product of the triples (7.16). Since the notion of regularity is manifestly well behaved with respect to the operation of restriction, identifying sufficient conditions for the regularity of the two factor triples will give sufficient conditions for the regularity of our triple.

Theorem 7.17. *Let $(\mathcal{A}, H, D, \pi, u)$ be a G-equivariant odd spectral triple over a unital C^*-algebra A and $l : G \to \mathcal{L}(V)$ a proper Dirac weight.*

(1) If the spectral triples $(\mathcal{B}, H, D)$ and $(C_c(G), \ell^2(G) \otimes V, M_l)$ are regular, then the triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda})$ on $A \rtimes_{\alpha,r}^{\sigma} G$ of the equivariant construction is regular.

(2) If $(\mathcal{A}, H, D)$ is regular and the group action is by order zero i.e. $u_g \in \mathrm{Op}^0(\Delta)$ (the Laplacian being relative to D on H) assume further that

- $[D, u_g] \in \mathrm{Op}^0(\Delta)$
- *for every integer k:*

$$\underbrace{[\Delta,[\Delta,\dots[\Delta}_{k\ \text{times}},u_g]]] \quad \text{and} \quad \underbrace{[\Delta,[\Delta,\dots[\Delta}_{k\ \text{times}},[D,u_g]]]] \in \mathrm{Op}^k(\Delta) \tag{7.17}$$

Then the triple $(\mathcal{B}, H, D)$ is regular.

(3) the converse of point (2) holds. If $(\mathcal{B}, H, D)$ is regular then $(\mathcal{A}, H, D)$ is regular and the group action satisfies: $u_g \in \mathrm{Op}^0(\Delta)$, $[D, u_g] \in \mathrm{Op}^0(\Delta)$ and (7.17) for every integer k.

Proof. The first statement is clear. Concerning the second one, assume that the triple on A is regular and denote with $\mathcal{E}_A$ its canonical algebra of GDO. Assuming u_g of order zero implies that $\mathcal{B}W^\infty \subseteq W^\infty$ and we can construct the canonical algebra of operators relative to B as explained right before Theorem 5.24. We call it $\mathcal{E}_B$. Now the remaining assumptions imply that $\mathcal{E}_B \supset \mathcal{E}_A$ is the canonical algebra of GDO that, according to Higson's theorem 5.24 witnesses that the triple over $\mathcal{B}$ is regular.
Let's discuss point (3). For every covariant representation $U_e = \mathrm{Id}$ so that we have an inclusion $\mathcal{A} \subseteq \mathcal{B}$ and the triple on $\mathcal{A}$ is the restriction of the one on $\mathcal{B}$. It follows that $(\mathcal{A}, H, D)$ is regular. Moreover since $\mathcal{A}$ is unital we get all the stated conditions on the u_g's just by direct examination of its canonical GDO algebra. □

Of course the hypothesis (7.17) are satisfied when the commutator $[D, u_g]$ is zero for any $g \in G$. We present an example in which these hypothesis are fulfilled and the commutator is non trivial.

Example 7.18 (cf. [29], Section 1.2)**.** Let M be a a smooth compact manifold M with an integrable subbundle $V \subseteq TM$. Let $N = TM/V$ be the transverse bundle and assume that both N and V are oriented euclidean even dimensional bundles. Combining a longitudinal signature operator of order 2 with the usual signature operator in the transverse direction, Connes constructs a hypoelliptic differential operator Q corresponding to the signature of M, which modulo lower order only depends upon the Euclidean structures of both V and N but not upon a choice of Riemannian metric on M. He then defines a first-order operator D by the equation $Q = D\,|D|$ and shows in Theorem I.1 that D gives rise to a spectral triple on the crossed product $A = C(M) \rtimes \Gamma$, where Γ is any group of diffeomorphisms preserving the triangular structure of V and N. In particular, he also proves that given a $\varphi \in \Gamma$ which preserves the foliation V and is isometric on both V and N, the corresponding unitary u_φ and and the commutator $[D, u_\varphi]$ belong to $\mathrm{Dom}^\infty(\delta)$.
□

Let us now briefly discuss the nature of the condition $[D, u_g] \in \mathrm{Op}^0(\Delta)$ of point (2) in Theorem 7.17 in the manifold case. Intuitively, it says that the group is acting in an isometric fashion. Indeed, if we consider the case of an odd Spin manifold, we can show the following fact, which is certainly well known, but it is proved here for completeness.

Proposition 7.19. *Let M be a spin (Riemannian) compact odd dimensional manifold with spinor bundle S and let $U : M \to M$ be a diffeomorphism that is covered by a linear* *map $\widetilde{U} : S \to S$ which is unitary on the fibers. This means that we have a commutative diagram*

$$\begin{tikzcd} S \arrow[r, "\widetilde{U}"] \arrow[d] & S \arrow[d] \\ M \arrow[r, "U"] & M. \end{tikzcd}$$

Let $\underline{U} : \Gamma(M,S) \longrightarrow \Gamma(M,S)$ be the induced operator on sections:

$$(\underline{U}s)(x) := (\widetilde{U}_x)^* s(U(x))$$

for $s \in \Gamma(M,S)$ and $x \in M$. Denote with D the Dirac operator associated to M. We have $\underline{U} \in \mathrm{Op}^0(\Delta)$ and it follows that $[D,\underline{U}] \in \mathrm{Op}^0(\Delta)$ if and only if U is an isometry.

Proof. Since M is compact, $\underline{U}$ is order zero (it does not involve derivatives) and invertible. It follows that $[D,\underline{U}]$ is order zero if and only if the operator $A := \underline{U}^*[D,\underline{U}] = \underline{U}^* D\underline{U} - D$ is order zero. Though $\underline{U}$ is not a differential operator (it is non local), the operator A is a differential operator of order no more than one so it will be order zero if and only if its principal symbol (of order one) vanishes. More precisely let's check that $\underline{U}^* D\underline{U}$ is differential of order one. Following [11], we show that for every fixed section $s \in \Gamma(M,S)$ and $\varphi \in \Gamma(M,S^*)$ the operator

$$C^\infty(M) \ni f \longmapsto \varphi(\underline{U}^* D\underline{U}(fs)) \in C^\infty(M)$$

is a differential operator. Since $\underline{U}(fs) = (f \circ U)\underline{U}s$, we can compute

$$\varphi(\underline{U}^* D\underline{U}(fs)) = \varphi(\underline{U}^* c(d(f \circ U))\underline{U}s) + \varphi(f\underline{U}^* D\underline{U}s). \tag{7.18}$$

Here for a function f we denote with $c(df)$ the corresponding Clifford multiplication; indeed we have used the property of the Dirac operator: $D(fs) = c(df)s + fDs$.

From Equation (7.18) we have that $[A,f]$ is C^∞-linear and so A is a differential operator. The same equation can be used to compute the principal symbol of A: let s be a section of the spinor bundle and fix a point $x = U(y) \in M$. Then

$$\begin{aligned} \sigma^1(A)(d_x f)s_x = i[A,f]s\,\big|_x &= i\underline{U}^* c(d(f \circ U))\underline{U}s - ic(df)s\,\big|_x \\ &= i\widetilde{U}_y\, c(d_y U^t(d_x f))\,(\widetilde{U}_y)^* s_x - ic(d_x f)s_x. \end{aligned}$$

The thesis follows quickly because this is zero if and only if we have a commutative diagram

$$\begin{tikzcd} S_y \arrow[r, "\widetilde{U}_y"] \arrow[d, "c_y(d_y U^t\,(d_x f))"'] & S_{U(y)} \arrow[d, "c(d_x f)"] \\ S_y \arrow[r, "\widetilde{U}_y"'] & S_{U(y)} \end{tikzcd} \tag{7.19}$$

for $d_y U^t$ the transpose of dU. Let us now show that this means that dU^t is unitary: given a function f such that $\|df\| = 1$, we have that the linear map $c(df)$ is unitary. Since $\widetilde{U}$ is an isometry, we deduce that $c_y(d_y U^t\,(d_x f))$ is an isometry so that $\|d_y U^t\,(d_x f)\| = 1$. Then dU^t preserves the norms.

□

7.5 The Existence of a Real Structure

In this section we construct a real structure on the equivariant spectral triple (7.4), and present sufficient conditions for the first and second order conditions. The idea is to employ the tensor product $J_1 \otimes J_2$ of two real structures on two spectral triples $(\mathcal{A}_1, H_1, D_1)$ and $(\mathcal{A}_2, H_2, D_2)$ which defines a real structure on the tensor product spectral triple such that the resultant KO-dimension is the sum of the two initial KO-dimensions (with some minor modifications in the case of a grading), cf. [34]. We will check that this construction remains valid also in the case of a crossed product extension. We discuss two cases depending on how the real structure J on the triple $(\mathcal{A}, H, D, u)$ interacts with the map $u\colon G \to \mathcal{L}(H)$.

Assumptions 7.20. *In this section we assume $\sigma \equiv 1$ and $V = \mathbb{C}$.*

7.5.1 First Case (J unitarily invariant)

Let G be a discrete group endowed with a proper Dirac weight $l\colon G \to \mathbb{R}$ and $(\mathcal{A}, H, D, u)$ a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a real structure J of KO-dimension $n \in \mathbb{Z}_8$ which is **unitarily invariant**, i.e.,

$$u_g J u_g^* = J \tag{7.20}$$

for every $g \in G$. This is for example the case when the triple discussed in Example 5.27 is endowed with the antilinear operator J_1 on $L^2(S^1)$ given by the complex conjugation (which gives a real structure on the triple of KO-dimension 1). We state now our first main result:

Theorem 7.21. *Suppose $(\mathcal{A}, H, D, J)$ has KO-dimension $n \in \mathbb{Z}_8$. If $l\colon G \to \mathbb{R}$ satisfies $l(g^{-1}) = -l(g)$ for all $g \in G$, then the equivariant spectral triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D}, \tilde{\pi}_2 \rtimes \hat{\Gamma})$ on $A \rtimes_{\alpha,r} G$ admits a real structure $\widehat{J}$ of KO-dimension $n+1$.*

In view of Proposition 6.13, a similar result which provides a real structure of KO-dimension $n-1$ holds for weights such that $l(g^{-1}) = l(g)$ for any $g \in G$. However, if we want also the first order condition, Remark 5.45 and Proposition 6.14 force G to be finite. As this situation is K-homologically trivial, we will not discuss this case.

Remark 7.22. Applying Theorem 7.21 to the triple in Example 5.27 with real structure J_1, one recovers precisely the real structure on the noncommutative 2-torus as described for instance in [44, Chapter 12.3].

The proof of Theorem 7.21 is constructive and relies on the following auxiliary map.

Lemma 7.23. *Let $j\colon H \otimes \ell^2(G) \to H \otimes \ell^2(G)$ be the antilinear map defined by*

$$j(\xi \otimes \delta_g) \coloneqq u_g^* J\xi \otimes J_G \delta_g = u_g^* J\xi \otimes \delta_{g^{-1}}, \tag{7.21}$$

where J_G is given by (6.6). Then:

(1) j is isometric.

(2) If $J^2 = \varepsilon$ then $j^2 = \varepsilon$.

(3) j maps $A \rtimes_\alpha G$ into its commutant.

(4) If $DJ = \varepsilon' JD$, then $(D \otimes 1)j = \varepsilon' j(D \otimes 1)$.

(5) If l satisfies $l(g^{-1}) = -l(g)$ for any $g \in G$, then $(1 \otimes M_l)j = -j(1 \otimes M_l)$.

Proof. Point (1) is clear as both u and J are isometric. To prove point (2) note that

$$j^2(\xi \otimes \delta_g) = u^*_{g^{-1}} J u^*_g J\xi \otimes \delta_g = u_g u^*_g J^2 \xi \otimes \delta_g = \varepsilon(\xi \otimes \delta_g).$$

Point (3) comes by a straightforward computation: for any $a, b \in A$ and $g, h \in G$ we have

$$\begin{aligned}\left[\hat\pi(a)\hat\lambda_h, j\hat\pi(b)\hat\lambda_g j^{-1}\right](\xi \otimes \delta_x) &= \pi(a)u_h u^*_{gx^{-1}} J\pi(b)u_g J^{-1} u^*_x \xi \otimes \delta_{hxg^{-1}} \\ &\quad - u^*_{gx^{-1}h^{-1}} J\pi(b)u_g J^{-1} u^*_{hx}\pi(a)u_h \xi \otimes \delta_{hxg^{-1}} \\ &= \left[\pi(a), J\pi(\alpha_{hxg^{-1}}(b))J^{-1}\right] u_h \xi \otimes \delta_{hxg^{-1}} \\ &= 0\end{aligned}$$

since J satisfies the zeroth order condition. To prove point (4) note that

$$(D \otimes 1)j(\xi \otimes \delta_g) = Du^*_g J\xi \otimes \delta_{g^{-1}} = \varepsilon' u^*_g JD\xi \otimes \delta_{g^{-1}} = \varepsilon' j(D \otimes 1)(\xi \otimes \delta_g)$$

by the invariance of D. Point (5) is clear. □

We then claim that the equivariant real structure $\widehat{J}$ for $\widehat{D}$ when n is odd is given by

- $\widehat{J} = j \otimes cc$ for $n = 3, 7$
- $\widehat{J} = j \otimes cc \circ \sigma_2$ for $n = 1, 5$

on the Hilbert space $\widehat{H} = H \otimes \ell^2 G \otimes \mathbb{C}^2$ (here cc denotes the complex conjugation operator). When n is even, the equivariant real structure $\widehat{J}$ for $\widehat{D}$ is instead given by

- $\widehat{J} = \chi J \otimes J_G$ for $n = 0, 4$
- $\widehat{J} = J \otimes J_G$ for $n = 2, 6$

on the Hilbert space $\widehat{H} = H \otimes \ell^2(G)$, where J_G is the flip morphism defined in (6.6).

Proof of Theorem 7.21. Suppose as a first case that the triple $(\mathcal{A}, H, D)$ is odd. The zeroth-order condition directly comes from the zeroth order condition in Lemma 7.23 and the peculiar diagonal/anti-diagonal form of $\widehat{J}$. Let us now discuss the triple of signs $(\varepsilon, \varepsilon', \varepsilon')$: since l is a homomorphism, j anti-commutes with M_l on $\ell^2(G)$ and so the real structure $\widehat{J}$ has the same ε sign as J when $n = 3, 7$ and the opposite when $n = 1, 5$. Analogously, the ε' sign remains the same for $n = 3, 7$ and changes for $n = 1, 5$ (that is, is always $+1$). Further $\widehat{J}$ is always even with respect to the grading $\chi = \sigma_3$ when $n = 3, 7$ and odd when $n = 1, 5$. We have therefore checked the theorem for all the possible cases of $(\mathcal{A}, H, D)$ odd.

Suppose now that $(\mathcal{A}, H, D, J)$ is an even real triple with respect to the grading $\chi = \sigma_3$, $H = H_0 \oplus H_1$ and

$$D = \begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix}.$$

Recall that $\widehat{D}$ is given by (5.24). By the compatibility conditions of J with the grading χ we deduce that J must be of the following form:

- $J = \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix}$ for $n = 0, 4$,
- $J = \begin{pmatrix} 0 & j_1 \\ j_2 & 0 \end{pmatrix}$ for $n = 2, 6$.

So now we have just to check case by case: if $n = 0, 4$ then by assumption

$$\begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix} \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix} = \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix} \begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix}.$$

Using this equation it is easy to check that

$$\begin{pmatrix} 1 \otimes M_l & D_0 \otimes 1 \\ D_0^* \otimes 1 & -1 \otimes M_l \end{pmatrix} \begin{pmatrix} j_1 \otimes J_G & 0 \\ 0 & -j_2 \otimes J_G \end{pmatrix} = - \begin{pmatrix} j_1 \otimes J_G & 0 \\ 0 & -j_2 \otimes J_G \end{pmatrix} \begin{pmatrix} 1 \otimes M_l & D_0 \otimes 1 \\ D_0^* \otimes 1 & -1 \otimes M_l \end{pmatrix}$$

so that the two triples have different ε' signs. The ε sign remains the same and this proves that we get dimensions 1 and 5 respectively. In an analogous way one shows that the dimensions 2 and 6 go to the dimensions 3 and 7 respectively. □

Remark 7.24. The assumption that $[D, u_g] = 0$ for any $g \in G$ is essentially necessary in order to prove Theorem 7.21. Indeed, suppose there is a real structure J on $(\mathcal{A}, H, D)$, so that $DJ = \varepsilon' JD$ by definition. An essential step to prove Theorem 7.21 is Lemma 7.23 point (4), namely the fact that

$$Du_g J = \varepsilon' u_g JD, \qquad \forall g \in G. \tag{7.22}$$

These two conditions together imply that D must be G-invariant: indeed, using (7.22) we see that $Du_g = \varepsilon' u_g JDJ^{-1}$ and so

$$[D, u_g] = \varepsilon' u_g JDJ^{-1} - u_g DJJ^{-1} = u_g \left(\varepsilon' JD - DJ\right) J^{-1} = 0.$$

Note that this computation is independent of the fact that J is unitarily equivalent, which is instead an assumption needed to show that $\widehat{J}$ satisfies the zeroth order condition. □

The following results show that the crossed product spectral triple construction is compatible with the first and second order conditions.

Proposition 7.25. *Let G be a discrete group endowed with a proper group homomorphism $l\colon G \to \mathbb{R}$ and let $(\mathcal{A}, H, D, u)$ be a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a unitarily invariant real structure J. If $(\mathcal{A}, H, D, J)$ satisfies the first order condition, then the spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \widehat{J})$ on $A \rtimes_{\alpha,r} G$ also satisfies the first order condition.*

Proof. We prove the first order condition for $(\mathcal{A}, H, D, u, J)$ odd; the even case is similar. For any $a, b \in A$ and $g, h \in G$, the desired commutator

$$\left[\left[D \otimes 1 \pm i \otimes M_l, \hat{\pi} \rtimes \hat{\lambda}(a\delta_g)\right], j\hat{\pi} \rtimes \hat{\lambda}(b\delta_h)j^{-1}\right]$$

is equal to the sum of the following two pieces:

$$C_1 = \left[\left[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_g\right], j\hat{\pi}(b)\hat{\lambda}_h j^{-1}\right]$$

$$C_2 = \pm i \left[\left[1 \otimes M_l, \hat{\pi}(a)\hat{\lambda}_g \right], j\hat{\pi}(b)\hat{\lambda}_h j^{-1} \right].$$

We will separately prove that they are vanishing. On the one hand, using the invariance of D with respect to the action of G, we have that

$$\begin{aligned}
C_1(\xi \otimes \delta_x) &= \left[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_g \right] \left(u^*_{hx^{-1}} J\pi(b) u_h J^{-1} u^*_x \xi \otimes \delta_{xh^{-1}} \right) \\
&\quad - j\hat{\pi}(b)\hat{\lambda}_h j^{-1} \left(D\pi(a) u_g \xi \otimes \delta_{gx} - \pi(a) u_g D\xi \otimes \delta_{gx} \right) \\
&= [D, \pi(a)] u_{gxh^{-1}} J\pi(b) u_h J^{-1} u^*_x u^*_g u_g \xi \otimes \delta_{gxh^{-1}} \\
&\quad - u_{gxh^{-1}} J\pi(b) u_h J^{-1} u^*_{gx} [D, \pi(a)] u_g \xi \otimes \delta_{gxh^{-1}} \\
&= \left[[D, \pi(a)], J u_{gxh^{-1}} \pi(b) u^*_{gxh^{-1}} J^{-1} \right] u_g \xi \otimes \delta_{gxh^{-1}} \\
&= 0
\end{aligned}$$

by the first order condition for J. On the other hand

$$\begin{aligned}
\pm i C_2(\xi \otimes \delta_x) &= \left[1 \otimes M_l, \hat{\pi}(a)\hat{\lambda}_g \right] \left(u^*_{hx^{-1}} J\pi(b) u_h J^{-1} u^*_x \xi \otimes \delta_{xh^{-1}} \right) \\
&\quad - j\hat{\pi}(b)\hat{\lambda}_h j^{-1} \left(\pi(a) u_g \xi \otimes l(gx)\delta_{gx} - \pi(a) u_g \xi \otimes l(x)\delta_{gx} \right) \\
&= \left(l(gxh^{-1}) - l(xh^{-1}) \right) \pi(a) u_{gxh^{-1}} J\pi(b) u^*_{gxh^{-1}} J^{-1} u_g \xi \otimes \delta_{gxh^{-1}} \\
&\quad - \left(l(gx) - l(x) \right) u_{gxh^{-1}} J\pi(b) u^*_{gxh^{-1}} J^{-1} \pi(a) u_g \xi \otimes \delta_{gxh^{-1}}.
\end{aligned}$$

As $l\colon G \to \mathbb{R}$ is a homomorphism, we have that $l(gxh^{-1}) - l(xh^{-1}) = l(gx) - l(x)$. Then

$$\pm i C_2(\xi \otimes \delta_x) = \left(l(gx) - l(x) \right) \left[\pi(a), J\pi(\alpha_{gxh^{-1}}(b)) J^{-1} \right] u_g \xi \otimes \delta_{gxh^{-1}}$$

and this is zeroth as J implements the zeroth order condition. □

Proposition 7.26. *Let G be a discrete group endowed with a proper group homomorphism $l\colon G \to \mathbb{R}$ and let $(\mathcal{A}, H, D, u)$ be a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a unitarily invariant real structure J which satisfies the first order condition. If $(\mathcal{A}, H, D, J)$ satisfies the second order condition, then the spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \widehat{J})$ on $A \rtimes_{\alpha,r} G$ also satisfies the second order condition.*

Proof. Let us focus on $(\mathcal{A}, H, D, u, J)$ odd as the even case is similar. With a slight abuse of notation, let us denote $\widehat{D} = D \otimes 1 \pm i \otimes M_l$. To prove the required commutation relation

$$[\widehat{D}, \hat{\pi}(a)\hat{\lambda}_h] j [\widehat{D}, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} = j [\widehat{D}, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} [\widehat{D}, \hat{\pi}(a)\hat{\lambda}_h]$$

we will show that the following four commutators are vanishing:

$$\begin{aligned}
C_1 &= \left[[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_h], j[D \otimes 1, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right] \\
C_2 &= \left[[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_h], j[\pm i \otimes M_l, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right] \\
C_3 &= \left[[\pm i \otimes M_l, \hat{\pi}(a)\hat{\lambda}_h], j[D \otimes 1, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right] \\
C_4 &= \left[[\pm i \otimes M_l, \hat{\pi}(a)\hat{\lambda}_h], j[\pm i \otimes M_l, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right]
\end{aligned}$$

for any $a, b \in A$ and $g, h \in G$. First, note that:

$$\begin{aligned}
j[D \otimes 1, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} (\xi \otimes \delta_x) &= u^*_{gx^{-1}} J [D, \pi(b)] u_g J^{-1} u^*_x \xi \otimes \delta_{xg^{-1}} \\
&= J [D, \pi(\alpha_{xg^{-1}}(b))] J^{-1} \xi \otimes \delta_{xg^{-1}}
\end{aligned}$$

and that

$$\begin{aligned} j[\pm i \otimes M_l, \hat{\pi}(b)\hat{\lambda}_g]j^{-1}(\xi \otimes \delta_x) &= \mp i u^*_{gx^{-1}} J\pi(b)u_g J^{-1}u^*_x\xi \otimes l(g)\delta_{xg^{-1}} \\ &= \mp i J\pi(\alpha_{xg^{-1}}(b))J^{-1}\xi \otimes l(g)\delta_{xg^{-1}} \end{aligned}$$

as l is a homomorphism. It is then easy to see that

$$C_1 = \Big[[D, \pi(a)], J[D, \pi(\alpha_{hxg^{-1}}(b))]J^{-1}\Big]\, u_h\xi \otimes \delta_{hxg^{-1}}$$

which vanishes since J implements the second order condition. Next, since $l(h)l(g) = l(g)l(h)$ for any $g, h \in G$, we have that

$$C_4 = \Big[\pi(a), J\pi(\alpha_{hxg^{-1}}(b))J^{-1}\Big]\, u_h\xi \otimes l(g)l(h)\delta_{hxg^{-1}}$$

vanishes by the zeroth order condition for J. Furthermore, the two mixed terms

$$\begin{aligned} C_2 &= i\Big[[D, \pi(a)], J\pi(\alpha_{hxg^{-1}}(b))J^{-1}\Big]\, u_h\xi \otimes l(g)\delta_{hxg^{-1}} \\ C_3 &= \pm i\Big[\pi(a), J[D, \pi(\alpha_{hxg^{-1}}(b))]J^{-1}\Big]\, u_h\xi \otimes l(h)\delta_{hxg^{-1}} \end{aligned}$$

vanish for the first order condition for J. The peculiar diagonal/anti-diagonal form of $\widehat{J}$ then brings the thesis.

□

7.5.2 Second Case (J twisted invariant)

Let G be discrete group endowed with a proper Dirac weight $l\colon G \to \mathbb{R}$ and $(\mathcal{A}, H, D, u)$ a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a real structure J of KO-dimension $n \in \mathbb{Z}_8$ which is **twisted invariant**, namely such that

$$u_g J u_g = J \tag{7.23}$$

for every $g \in G$. This is for example the case when the triple discussed in Example 5.27 is endowed with the antilinear operator J_2 on $L^2(S^1)$ given by $J_2 f(x) = \overline{f(-x)}$ (which gives a real structure on the triple of KO-dimension 7). We will prove the following fact.

Theorem 7.27. *Suppose $(\mathcal{A}, H, D, J)$ has KO-dimension $n \in \mathbb{Z}_8$. If G is abelian, the $\widehat{G}$-invariant spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi}_2 \rtimes \hat{\Gamma})$ on $A \rtimes_{\alpha,r} G$ admits a real structure $\widetilde{J}$ of KO-dimension $n - 1$.*

Remark 7.28. The assumption that G is abelian might seem unnecessarily restrictive (and thus of low interest) compared to the discussion in Subsection 7.5.1, where G need not be commutative. However, this flexibility is just apparent. Indeed, when we focus our attention to spectral triples which are non degenerate (a condition that is necessary for example to have a compact quantum metric space structure) we discover that the hypothesis that $l\colon G \to \mathbb{R}$ is a homomorphism as in Theorem 7.21 forces the commutativity of G (cf. Example 5.39).

The proof of Theorem 7.27 is constructive and relies on the properties of the following auxiliary map.

Lemma 7.29. *Let G be an abelian discrete group and $j\colon H \otimes \ell^2(G) \to H \otimes \ell^2(G)$ an antilinear map defined by*

$$j(\xi \otimes \delta_g) := u_g J\xi \otimes cc\, \delta_g. \tag{7.24}$$

Then:

(1) j is isometric

(2) If $J^2 = \varepsilon$ then $j^2 = \varepsilon$.

(3) j maps $A \rtimes_\alpha G$ into its commutant.

(4) If $DJ = \varepsilon' JD$, then $(D \otimes 1)j = \varepsilon' j(D \otimes 1)$

(5) $(\pm i \otimes M_l)j = -j(\pm i \otimes M_l)$

Proof. Point (1) is clear as both u and J are isometric. To prove point (2) note that

$$j^2(\xi \otimes \delta_g) = u_g J u_g J\xi \otimes \delta_g = u_g u_g^* J^2 \xi \otimes \delta_g = \varepsilon(\xi \otimes \delta_g).$$

Point (3) is a straightforward computation: for any $a, b \in A$ and $g, h \in G$ we have

$$\begin{aligned}\left[\hat{\pi}(a)\hat{\lambda}_h, j\hat{\pi}(b)\hat{\lambda}_g j^{-1}\right](\xi \otimes \delta_x) &= \pi(a)u_h u_{xg} J\pi(b)u_g J^{-1}u_x^* \xi \otimes \delta_{xgh} \\ &\quad - u_{xhg} J\pi(b)u_g J^{-1} u_{xh}^* \pi(a) u_h \xi \otimes \delta_{xhg} \\ &= \left[\pi(a), J\pi(\alpha_{xgh}^{-1}(b))J^{-1}\right] u_h \xi \otimes \delta_{xgh} \\ &= 0\end{aligned}$$

as J satisfies the zero order condition and G is abelian. To prove point (4) note that

$$(D \otimes 1)j(\xi \otimes \delta_g) = Du_g J\xi \otimes \delta_g = \varepsilon' u_g JD\xi \otimes \delta_g = \varepsilon' j(D \otimes 1)(\xi \otimes \delta_g)$$

by the invariance of D. Point (5) is clear as J is anti-linear. □

We then claim that the equivariant real structure $\widetilde{J}$ for $\widehat{D}$ when n is odd is given by

- $\widetilde{J} = j \otimes cc \circ \sigma_1$ for $n = 3, 7$
- $\widetilde{J} = j \otimes cc \circ \sigma_3$ for $n = 1, 5$

on the Hilbert space $\widehat{H} = H \otimes \ell^2 G \otimes \mathbb{C}^2$. When n is even, the equivariant real structure $\widetilde{J}$ for $\widehat{D}$ is instead given by

- $\widetilde{J} = J \otimes \mathrm{cc}$ for $n = 0, 4$
- $\widetilde{J} = \chi J \otimes \mathrm{cc}$ for $n = 2, 6$

on the Hilbert space $\widehat{H} = H \otimes \ell^2(G)$.

Proof of Theorem 5.24. Suppose as a first case that the triple $(\mathcal{A}, H, D)$ is odd. The zeroth-order condition comes directly from the zeroth order condition in Lemma 7.29 and the diagonal/anti-diagonal form of $\widetilde{J}$. Let us now discuss the triple of signs $(\varepsilon, \varepsilon', \varepsilon'')$: from the previous lemma we easily deduce that the real structure $\widetilde{J}$ has the same ε sign as J. An easy computation shows that the ε' sign remains the same for $n = 3, 7$ and changes for $n = 1, 5$ (that is, it is always $+1$). Further $\widetilde{J}$ is always odd with respect to the grading $\chi = \sigma_3$ when $n = 3, 7$ and even when $n = 1, 5$. We have therefore checked the theorem for all the possible cases of $(\mathcal{A}, H, D)$ odd.

Suppose now that the real triple $(\mathcal{A}, H, D, J)$ is even with respect to the grading χ and suppose that $\chi = \sigma_3$, $H = H_0 \oplus H_1$ and

$$D = \begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix}.$$

By the compatibility conditions of J with the grading χ we deduce that J must be of the following form:

- $J = \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix}$ for $n = 0, 4$,
- $J = \begin{pmatrix} 0 & j_1 \\ j_2 & 0 \end{pmatrix}$ for $n = 2, 6$.

Now we check case by case. If $n = 0, 4$ then by assumption

$$\begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix} \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix} = \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix} \begin{pmatrix} 0 & D_0 \\ D_0^* & 0 \end{pmatrix}$$

and, recalling that $\widehat{D}$ is given by (5.24), we get

$$\begin{pmatrix} 1 \otimes M_l & D_0 \otimes 1 \\ D_0^* \otimes 1 & -1 \otimes M_l \end{pmatrix} \begin{pmatrix} j_1 \otimes 1 & 0 \\ 0 & j_2 \otimes 1 \end{pmatrix} = \begin{pmatrix} j_1 \otimes 1 & 0 \\ 0 & j_2 \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes M_l & D_0 \otimes 1 \\ D_0^* \otimes 1 & -1 \otimes M_l \end{pmatrix}.$$

Thus the triples on A and on $A \rtimes_{\alpha,r} G$ have the same sign ε'. Also the sign ε remains the same, but since the grading disappears, the dimension 0 (or $8 \mod 8$) goes to 7 and the dimension 4 goes to 3. In an analogous way one shows that dimensions 2 and 6 go to dimensions 1 and 5 respectively. □

The following results show that the crossed product spectral triple construction is compatible with the first and second order conditions.

Proposition 7.30. *Let G be an abelian discrete group endowed with a proper first-order Dirac weight $l \colon G \to \mathbb{R}$ and let $(\mathcal{A}, H, D, u)$ be a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a twisted invariant real structure J. If $(\mathcal{A}, H, D, J)$ satisfies the first order condition, then $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \widetilde{J})$ also satisfies the first order condition.*

Proof. We prove the first order condition only for $(\mathcal{A}, H, D, u, J)$ odd; the even case is similar. For any $a, b \in \mathcal{A}$ and $g, h \in G$, the desired commutator

$$\left[\left[D \otimes 1 \pm i \otimes M_l, \hat{\pi} \rtimes \hat{\lambda}(a\delta_g)\right], j\hat{\pi} \rtimes \hat{\lambda}(b\delta_h)j^{-1}\right]$$

is equal to the sum of the following two pieces:

$$C_1 = \left[\left[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_g\right], j\hat{\pi}(b)\hat{\lambda}_h j^{-1}\right]$$

$$C_2 = \pm i \left[\left[1 \otimes M_l, \hat{\pi}(a)\hat{\lambda}_g \right], j\hat{\pi}(b)\hat{\lambda}_h j^{-1} \right].$$

We prove that they are separately vanishing. On the one hand, using the invariance of D with respect to the action of G, we have that

$$\begin{aligned}
C_1(\xi \otimes \delta_x) &= \left[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_g \right] \left(u_{hx} J\pi(b) u_h J^{-1} u_x^* \xi \otimes \delta_{hx} \right) \\
&\qquad - j\hat{\pi}(b)\hat{\lambda}_h j^{-1} \left(D\pi(a) u_g \xi \otimes \delta_{gx} - \pi(a) u_g D\xi \otimes \delta_{gx} \right) \\
&= [D, \pi(a)] u_{xgh} J\pi(b) u_h J^{-1} u_x^* \xi \otimes \delta_{xgh} \\
&\qquad - u_{xgh} J\pi(b) u_h J^{-1} u_{xg}^* [D, \pi(a)] u_g \xi \otimes \delta_{xgh} \\
&= \left[[D, \pi(a)], J u_{xgh}^* \pi(b) u_{xgh} J^{-1} \right] u_g \xi \otimes \delta_{xgh} \\
&= 0
\end{aligned}$$

since J implements the first order condition. On the other hand

$$\begin{aligned}
\pm i C_2(\xi \otimes \delta_x) &= \left[1 \otimes M_l, \hat{\pi}(a)\hat{\lambda}_g \right] u_{xh} J\pi(b) u_h J^{-1} u_x^* \xi \otimes \delta_{xgh} \\
&\qquad - j\hat{\pi}(b)\hat{\lambda}_h j^{-1} \left(\pi(a) u_g \xi \otimes l(xg)\delta_{xg} - \pi(a) u_g \xi \otimes l(x)\delta_{xg} \right) \\
&= \left(l(xgh) - l(xh) \right) \pi(a) u_{xgh} J\pi(b) u_h J^{-1} u_x^* \xi \otimes \delta_{xgh} \\
&\qquad - \left(l(xg) - l(x) \right) u_{xgh} J\pi(b) u_h J^{-1} u_{xg}^* \pi(a) u_g \xi \otimes \delta_{xgh}.
\end{aligned}$$

Since $l \colon G \to \mathbb{R}$ is of first order, we have that $l(xgh) - l(xh) = l(xg) - l(x)$. Then

$$\pm i C_2(\xi \otimes \delta_x) = \left(l(xg) - l(x) \right) \left[\pi(a), J\pi(\alpha_{xgh}^{-1}(b)) J^{-1} \right] u_g \xi \otimes \delta_{xgh}$$

which is zero since J implements the zeroth order condition. □

Proposition 7.31. *Let G be an abelian discrete group endowed with a proper first-order Dirac weight $l \colon G \to \mathbb{R}$ and let $(\mathcal{A}, H, D, u)$ be a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a twisted invariant real structure J which satisfies the first order condition. If $(\mathcal{A}, H, D, J)$ satisfies the second order condition, then $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \widetilde{J})$ also satisfies the second order condition.*

Proof. Let us focus on $(\mathcal{A}, H, D, u, J)$ odd cause the even case is similar. With a slight abuse of notation, let us denote $\widehat{D} = D \otimes 1 \pm i \otimes M_l$. To prove the required commutation relation

$$[\widehat{D}, \hat{\pi}(a)\hat{\lambda}_h] j [\widehat{D}, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} = j[\widehat{D}, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} [\widehat{D}, \hat{\pi}(a)\hat{\lambda}_h]$$

we will prove that the following four commutators are vanishing:

$$\begin{aligned}
C_1 &= \left[[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_h], j[D \otimes 1, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right] \\
C_2 &= \left[[D \otimes 1, \hat{\pi}(a)\hat{\lambda}_h], j[\pm i \otimes M_l, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right] \\
C_3 &= \left[[\pm i \otimes M_l, \hat{\pi}(a)\hat{\lambda}_h], j[D \otimes 1, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right] \\
C_4 &= \left[[\pm i \otimes M_l, \hat{\pi}(a)\hat{\lambda}_h], j[\pm i \otimes M_l, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} \right]
\end{aligned}$$

for any $a, b \in \mathcal{A}$ and $g, h \in G$. The diagonal/anti-diagonal form of $\widetilde{J}$ then brings the thesis. First of all note that:

$$\begin{aligned}
j[D \otimes 1, \hat{\pi}(b)\hat{\lambda}_g] j^{-1} (\xi \otimes \delta_x) &= u_{gx} J [D, \pi(b)] u_g J^{-1} u_x^* \xi \otimes \delta_{gx} \\
&= J[D, \pi(\alpha_{gx}^{-1}(b))] J^{-1} \xi \otimes \delta_{gx}
\end{aligned}$$

and that

$$\begin{aligned} j[\pm i \otimes M_l, \hat{\pi}(b)\hat{\lambda}_g]j^{-1}(\xi \otimes \delta_x) &= \mp i u_{gx} J\pi(b) u_g J^{-1} u_x^* \xi \otimes l(g)\delta_{gx} \\ &= \mp i J \alpha_{gx}^{-1}(b) J^{-1} \xi \otimes l(g)\delta_{gx} \end{aligned}$$

as l is of first order. It is relatively easy then to compute

$$C_1 = \Big[[D, \pi(a)], J[D, \pi(\alpha_{hgx}^{-1}(b))]J^{-1}\Big] u_h \xi \otimes \delta_{hgx}$$

that vanishes since J implements the second order condition. Furthermore, as $l(h)l(g) = l(g)l(h)$ for any $g, h \in G$, we have that

$$C_4 = \Big[\pi(a), J\pi(\alpha_{hgx}^{-1}(b))J^{-1}\Big] u_h \xi \otimes l(g)l(h)\delta_{hgx}$$

vanishes since J implements the zeroth order condition. Finally, the two mixed terms

$$\begin{aligned} C_2 &= \mp i \Big[[D, \pi(a)], J\pi(\alpha_{hgx}^{-1}(b))J^{-1}\Big] u_h \xi \otimes l(g)\delta_{hgx} \\ C_3 &= \pm i \Big[\pi(a), J[D, \pi(\alpha_{hgx}^{-1}(b))]J^{-1}\Big] u_h \xi \otimes l(h)\delta_{hgx} \end{aligned}$$

vanish by the first order condition for J.

□

7.5.3 Equivariant Real Structures

In the previous two subsections we constructed real structures $\widehat{J}$ and (for abelian G) $\widetilde{J}$ on $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ starting from a real structure J on $(\mathcal{A}, H, D, u)$ which is suitably G-invariant. In this subsection we give a unifying picture interpreting the relations between J with u in terms of the (unitary) action of the Hopf $*$-algebra $\mathbb{C}G$ endowed with a suitable $*$-structure. This will explain the reason why in the case of $\widetilde{J}$ we must assume the group is abelian. Furthermore, we will show that in both cases $\widehat{J}$ and $\widetilde{J}$ are equivariant under the dual coaction of $\mathbb{C}G$ consistently with J.

Definition 7.32 (cf.[100])**.** *Let $\mathcal{H}$ be a Hopf $*$-algebra and $(\mathcal{A}, H, D)$ an $\mathcal{H}$-equivariant spectral triple over the $\mathcal{H}$-module C^*-algebra A. A real structure J is said to be **equivariant** if there exists a dense subspace $V \subseteq H$ such that for any $h \in \mathcal{H}$*

$$J h \triangleright J^{-1} = (Sh)^* \triangleright \tag{7.25}$$

as operators on V.

We can now explain the commutation relations (7.20) and (7.23) between J and u_g as the $\mathcal{H}$-equivariance in the sense of Definition 5.32 corresponding to the two different $*$-structures (C.1) and (C.2) respectively on the Hopf algebra $\mathcal{H} = \mathbb{C}G$ as in Example C.2. In the first case we use the obvious actions of $\mathbb{C}G$ on H and A

$$h \triangleright \xi = u_h \xi, \qquad g \triangleright a = \alpha_g(a)$$

to make A a $\mathbb{C}G$-module algebra and H a $\mathbb{C}G$-equivariant $\mathcal{A}$-module. Then equation (7.25) for the $*$-structure (C.1) on $\mathbb{C}G$ becomes

$$J u_g J^{-1} = u_{(Sg)^*} = u_g$$

which means precisely that J is unitarily invariant (7.20). In the second case (when G is abelian) we use the (less) obvious actions of $\mathbb{C}G$ on H and A

$$h \triangleright \xi = u_h^* \xi, \qquad g \triangleright a = \alpha_{g^{-1}}(a)$$

to make A a $\mathbb{C}G$-module algebra and H a $\mathbb{C}G$-equivariant $\mathcal{A}$-module. Then equation (7.25) for the $*$-structure (C.2) on $\mathbb{C}G$ becomes

$$J u_g J^{-1} = u_{(Sg)^*} = u_{g^{-1}} = u_g^*$$

which means precisely that J is twisted invariant (7.23). Note that in both cases the compatibility condition (C.3) holds true.

With this unifying picture, we summarize the two constructions of this section in the following table:

Group G	Discrete	Discrete and abelian	
$*$-structure on $\mathbb{C}G$	$*\delta_g = \delta_{g^{-1}}$	$\star\delta_g = \delta_g$	
Equivariance of J	$u_g J u_g^* = J$	$u_g J u_g = J$	
weight $l\colon G \to \mathbb{R}$	homomorphism	constant + homom.	
auxiliary map j	$j(\xi \otimes \delta_g) = u_g^* J\xi \otimes \delta_{g^{-1}}$	$j(\xi \otimes \delta_g) = u_g J\xi \otimes \delta_g$	
real structure on $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D})$	$\widehat{J} = \begin{cases} j \otimes \mathrm{cc} \\ j \otimes cc \circ \sigma_2 \\ \chi J \otimes J_G \\ J \otimes J_G \end{cases}$	$\widetilde{J} = \begin{cases} j \otimes cc \circ \sigma_1 \\ j \otimes cc \circ \sigma_3 \\ J \otimes \mathrm{cc} \\ \chi J \otimes \mathrm{cc} \end{cases}$	for $n = \begin{cases} 3,7 \\ 1,5 \\ 0,4 \\ 2,6 \end{cases}$
KO-dim	$n+1$	$n-1$	

Let us now prove that the real structures $\widehat{J}$ and $\widetilde{J}$ are equivariant for coactions of G. First, we need a definition.

Definition 7.33 (cf. [15])**.** *Let $(\mathcal{B}, H, D, \chi)$ an (even or odd) spectral triple equivariant for coaction of G as in Definition 5.33 and let $X \in \mathcal{L}(H \otimes C_r^*(G))$ be the unitary corepresentation of G on H. A real structure J on $(\mathcal{B}, H, D, \chi)$ is said to be* ***equivariant for coactions of*** *G if*

$$(J \otimes *)X = X(J \otimes 1) \tag{7.26}$$

on $H \otimes 1_{C_r^*(G)}$.

Proposition 7.34. *Let G be a discrete group endowed with a proper group homomorphism $l\colon G \to \mathbb{R}$ and $(\mathcal{A}, H, D, u)$ a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a unitarily invariant real structure J. The real structure $\widehat{J}$ on the equivariant spectral triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D}, \tilde{\pi} \rtimes \hat{\lambda})$ on $A \rtimes_{\alpha,r} G$ defined in Theorem 7.21 is equivariant for the dual coaction of G.*

Remark 7.35. If G is abelian and the dual coaction $\widehat{\alpha}$ is Fourier-transformed into the dual action of $\widehat{G}$, one can show that if J is unitarily invariant then $\widehat{J}$ is also unitarily invariant under the action V given in Remark 7.7.

Proof. For any $\xi \in H$ and $g \in G$ we have

$$\begin{aligned}(j \otimes *)X(\xi \otimes \delta_g \otimes \delta_e) &= (j \otimes *)(\xi \otimes \delta_g \otimes \delta_g) = u_g^* J\xi \otimes \delta_{g^{-1}} \otimes \delta_{g^{-1}} \\ &= X(u_g^* J\xi \otimes \delta_{g^{-1}} \otimes \delta_e) \\ &= X(j(\xi \otimes \delta_g) \otimes \delta_e).\end{aligned}$$

The diagonal/anti-diagonal form of $\widehat{J}$ leads to the thesis. □

Proposition 7.36. *Let G be a discrete abelian group endowed with a proper Dirac weight $l\colon G \to \mathbb{R}$ and $(\mathcal{A}, H, D, u)$ a G-invariant (even or odd) spectral triple on a unital C^*-algebra A endowed with a twisted invariant real structure J. The real structure $\widetilde{J}$ on the equivariant spectral triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda})$ on $A \rtimes_{\alpha,r} G$ defined in Theorem 7.21 is twisted invariant.*

Proof. Noting that $jv_\chi = v_\chi^* j$ since characters $\chi \in \widehat{G}$ are complex-valued, the diagonal/anti-diagonal form of $\widetilde{J}$ leads to the thesis. □

7.6 The Existence of an Orientation Cycle

In this section we induce an Hochschild orientation cycle on the triple (7.4) from an Hochschild orientation cycle on the triple on A. The first step is to properly define a group action on Hochschild chains.

Assumptions 7.37. *In this section we assume $\sigma \equiv 1$ and $V = \mathbb{C}$.*

Definition 7.38. *Let (A, G, α) be a C^*-dynamical system. For every Hochschild n-chain $c = \sum (a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_n \in C_n(A \otimes A^{\mathrm{op}}, A)$, we define*

$$\alpha_g(c) := \sum (\alpha_g(a_0) \otimes b_0) \otimes \alpha_g(a_1) \otimes \cdots \otimes \alpha_g(a_n) \tag{7.27}$$

*and say that c is **G-invariant** if $\alpha_g(c) = c$ for every $g \in G$.*

Remark 7.39. For any Hochschild n-chain c we have that $b\alpha_g(c) = \alpha_g(bc)$ by the definition of the Hochschild boundary. In particular, if c is a cycle then $\alpha_g(c)$ is also a cycle.

In equation (7.27) the elements b_0 play no essential role. When dealing with the orientation property, this is reflected in the following fact.

Lemma 7.40. *Let $(\mathcal{A}, H, D, \chi, u)$ be a G-invariant real spectral triple on A with a unitarily invariant real structure J and let $c = \sum (a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_n$ be a Hochschild cycle in $C_n(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathcal{A})$. Define*

$$c_g := \sum (a_0 \otimes \alpha_g(b_0)) \otimes a_1 \otimes \cdots \otimes a_n \tag{7.28}$$

for any $g \in G$. Then $\pi_D(c_g) = \mathrm{Ad}u_g \circ \pi_D(\alpha_g(c))$.

Proof. Using the fact that $[D, u_g] = 0$ for any $g \in G$ and that $Ju_g = u_g J$ for any $g \in G$, we have that

$$\begin{aligned}\pi_D(c_g) &= \sum \pi(a_0) J\pi(\alpha_g(b_0)^*) J^{-1} [D, \pi(a_1)] \cdots [D, \pi(a_n)] \\ &= u_g \sum \pi(\alpha_g(a_0)) J\pi(b_0^*) J^{-1} [D, \pi(\alpha_g(a_1))] \cdots [D, \pi(\alpha_g(a_n))] u_g^* \\ &= \mathrm{Ad}u_g \circ \pi_D(\alpha_g(c)).\end{aligned}$$

□

Theorem 7.41. *Let G be a discrete group and $l\colon G \to \mathbb{R}$ a proper homomorphism. Let $(\mathcal{A}, H, D, u)$ be an (even or odd) G-invariant spectral triple on a unital C^*-algebra A and J a unitarily invariant real structure. Then:*

(1) If the triple $(\mathcal{A}, H, D)$ is orientable and the orientation cycle c is G-invariant, then the real spectral triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \tilde{\pi} \rtimes \hat{\lambda}, \widehat{J})$ on $A \rtimes_{\alpha,r} G$ admits an orientation cycle $\hat{c}$.

(2) If c is a strong orientation cycle, then $\hat{c}$ is also a strong orientation cycle.

As suggested in [111, Chapter 6], the idea of the proof is to twist the prescription described in [34], where the shuffle product is used to create a cycle on a tensor product spectral triple.

Definition 7.42. *For any Hochschild n-chain*

$$c = \sum (a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_n \in C_n(A \otimes A^{\mathrm{op}}, A)$$

*and any 1-chain $\delta = \sum (\delta_g \otimes \delta_h) \otimes \delta_f \in C_1(Q \otimes Q^{\mathrm{op}}, Q)$ for $Q = C_r^*(G)$, we define their **twisted shuffle product** as the Hochschild $(n+1)$-chain in $C_{n+1}(B \otimes B^{\mathrm{op}}, B)$ for $B = A \otimes C_r^*(G)$:*

$$\begin{aligned} c \rtimes_\alpha \delta := {} & \sum (a_0\delta_g \otimes b_0\delta_h) \otimes \delta_f \otimes a_1 \otimes \cdots \otimes a_n \\ & + \sum_{j=2}^{n} (-1)^{j-1} \sum (a_0\delta_g \otimes b_0\delta_h) \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_{j-1}) \otimes \delta_f \otimes a_j \otimes \cdots \otimes a_n \\ & + (-1)^n \sum (a_0\delta_g \otimes b_0\delta_h) \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_n) \otimes \delta_f. \end{aligned}$$

Note that for simplicity we denote by a the element $a\delta_e$ and by δ_f the element $1_A\delta_f$. Under the assumption of covariance, namely that $\delta_g a = \alpha_g(a)\delta_g$ for any $a \in A$ and $g \in G$, the twisted shuffle product also defines a chain over the crossed product $A \rtimes_{\alpha,r} G$. Note also that if $\alpha = \mathrm{id}$ then the twisted shuffle product is really the shuffle product of the two chains as defined in [67, Chapter 4.2] (up to a sign depending on the length of the chain).

Proposition 7.43. *For any G-invariant $c \in C_n(A \otimes A^{\mathrm{op}}, A)$ and any $\delta \in C_1(Q \otimes Q^{\mathrm{op}}, Q)$ we have that*

$$b(c \rtimes_\alpha \delta) = bc \rtimes_\alpha \delta + c \rtimes b\delta \tag{7.29}$$

as chains over $A \rtimes_{\alpha,r} G$.

Proof. By bilinearity, we can suppose that c and δ are pure tensors:

$$c = (a_0 \otimes b_0) \otimes a_1 \otimes \cdots \otimes a_n, \qquad \delta = (\delta_g \otimes \delta_h) \otimes \delta_f.$$

Since $b\delta = \delta_{gf} \otimes \delta_h - \delta_{fg} \otimes \delta_h \in Q \otimes Q^{\mathrm{op}}$, the untwisted shuffle product with c is equal to

$$\begin{aligned} c \rtimes b\delta = {} & (a_0\delta_{gf} \otimes b_0\delta_h) \otimes a_1 \otimes \cdots \otimes a_n \\ & - (a_0\delta_{fg} \otimes b_0\delta_h) \otimes a_1 \otimes \cdots \otimes a_n. \end{aligned} \tag{7.30}$$

For the sake of simplicity, we set $m = a_0\delta_g \otimes b_0\delta_h$ and write $c \rtimes_\alpha \delta$ in Definition 7.42 as $c \rtimes_\alpha \delta = \sum_{j=1}^{n+1} c_j$. Let us compute bc_j for every $j = 1, \dots, n+1$. First,

$$\begin{aligned} bc_1 = (a_0\delta_{gf} \otimes b_0\delta_h) \otimes a_1 \otimes \cdots \otimes a_n \\ - m \otimes \delta_f a_1 \otimes a_2 \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^{i+1} m \otimes \delta_f \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^{n+1} a_n m \otimes \delta_f \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned} \tag{7.31}$$

Next, for $j = 2, \dots, n$ we have:

$$\begin{aligned} bc_j = (-1)^{j-1} m \otimes \alpha_f(a_1) \otimes \cdots \otimes \delta_f \otimes a_j \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{j-2} (-1)^{i+j-1} m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_i a_{i+1}) \otimes \cdots \\ \cdots \otimes \alpha_f(a_{j-1}) \otimes \delta_f \otimes a_j \otimes \cdots \otimes a_n \\ + (-1)^{j-1}(-1)^{j-1} m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_{j-1})\delta_f \otimes a_j \otimes \cdots \otimes a_n \\ + (-1)^{j-1}(-1)^{j} m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_{j-1}) \otimes \delta_f a_j \otimes \cdots \otimes a_n \\ + \sum_{i=j}^{n-1} (-1)^{i+j} m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_{j-1}) \otimes \delta_f \otimes a_j \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^{n+1}(-1)^{j-1} a_n m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_{j-1}) \otimes \delta_f \otimes a_j \otimes \cdots \otimes a_{n-1} \end{aligned}$$

with the convention that for $j = 2$ the first summation is neglected. Finally:

$$\begin{aligned} bc_{n+1} = (-1)^n m \otimes \alpha_f(a_1) \otimes \alpha_f(a_2) \otimes \cdots \otimes \alpha_f(a_n) \otimes \delta_f \\ + (-1)^n \sum_{i=1}^{n-1} (-1)^i m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_i a_{i+1}) \otimes \cdots \otimes \alpha_f(a_n) \otimes \delta_f \\ + (-1)^n(-1)^n m \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_{n-1}) \otimes \alpha_f(a_n)\delta_f \\ + (-1)^n(-1)^{n+1} (\delta_f a_0 \delta_g \otimes b_0\delta_h) \otimes \alpha_f(a_1) \otimes \cdots \otimes \alpha_f(a_n). \end{aligned} \tag{7.32}$$

Since $\delta_f a_0 \delta_g = \alpha_f(a_0)\delta_{fg}$ and the cycle c is G-invariant, the last line of (7.32) can be rewritten as

$$-(a\delta_{fg} \otimes b_0\delta_h) \otimes a_1 \otimes \cdots \otimes a_n$$

Together with the first line of (7.31), these summands are precisely $c \rtimes b\delta$ (see (7.30)). The two central lines of every bc_j for $j = 2, \dots, n$ form a telescopic summation that, together with the second line in (7.31) and the third line in (7.32), sum up to zero. What remains is precisely $bc \rtimes_\alpha \delta$. ☐

We can now prove the main theorem of this section.

Proof of Theorem 7.41. Since the Dirac weight l is proper, there exists $g \in G$ such that $l(g) \neq 0$. Consider then the Hochschild 1-cycle

$$\Delta_g \coloneqq (\delta_{g^{-1}} \otimes \delta_e) \otimes \delta_g \in C_1(Q \otimes Q^{\mathrm{op}}, Q).$$

If c is the G-invariant orientation cycle of the triple (A, H, D, J, u), then the twisted shuffle product $c \rtimes_\alpha \Delta_g$ is also a cycle by Proposition 7.43. We will show that the normalized shuffle product

$$\hat{c} = \frac{1}{M} c \rtimes_\alpha \Delta_g \tag{7.33}$$

is an orientation cycle for the triple $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda}, \widehat{J})$ on $A \rtimes_{\alpha,r} G$, where the normalization factor M is given by

$$M = \begin{cases} -il(g)(n+1) & \text{if } (A, H, D) \text{ is odd} \\ l(g)(n+1) & \text{if } (A, H, D) \text{ is even.} \end{cases}$$

Indeed, since

$$\begin{aligned} c \rtimes_\alpha \Delta_g = &\sum (a_0 \delta_{g^{-1}} \otimes b_0) \otimes \delta_g \otimes a_1 \otimes \cdots \otimes a_n \\ &+ \sum_{j=2}^{n} (-1)^{j-1} \sum (a_0 \delta_{g^{-1}} \otimes b_0) \otimes \alpha_g(a_1) \otimes \cdots \otimes \alpha_g(a_{j-1}) \otimes \delta_g \otimes a_j \otimes \cdots \otimes a_n \\ &+ (-1)^n \sum (a_0 \delta_{g^{-1}} \otimes b_0) \otimes \alpha_g(a_1) \otimes \cdots \otimes \alpha_g(a_n) \otimes \delta_g \end{aligned}$$

we have that

$$\begin{aligned} \pi_{\widehat{D}}(c \rtimes_\alpha \Delta_g) = &\sum \hat{\pi}(a_0) \hat{\lambda}_g^* \widehat{J} \hat{\pi}(b_0^*) \widehat{J}^{-1} [\widehat{D}, \hat{\lambda}_g][\widehat{D}, \hat{\pi}(a_1)] \cdots [\widehat{D}, \hat{\pi}(a_n)] \\ &+ \sum_{j=2}^{n} (-1)^{j-1} \sum \hat{\pi}(a_0) \hat{\lambda}_g^* \widehat{J} \hat{\pi}(b_0^*) \widehat{J}^{-1} [\widehat{D}, \hat{\pi}(\alpha_g(a_1))] \cdots \\ &\qquad \cdots [\widehat{D}, \hat{\pi}(\alpha_g(a_{j-1}))][\widehat{D}, \hat{\lambda}_g][\widehat{D}, \hat{\pi}(a_j)] \cdots [\widehat{D}, \hat{\pi}(a_n)] \\ &+ (-1)^n \sum \hat{\pi}(a_0) \hat{\lambda}_g^* \widehat{J} \hat{\pi}(b_0^*) \widehat{J}^{-1} [\widehat{D}, \hat{\pi}(\alpha_g(a_1))] \cdots [\widehat{D}, \hat{\pi}(\alpha_g(a_n))][\widehat{D}, \hat{\lambda}_g] \end{aligned}$$

with a slight abuse of notation ($\hat{\pi}$ denotes two copies of the representation $\hat{\pi}$ and $\hat{\lambda}$ two copies of $\hat{\lambda}$).

Consider the case when $(\mathcal{A}, H, D)$ is odd and thus $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda}, \widehat{J})$ is even. Then

$$[\widehat{D}, \hat{\pi}(a)] = [D, \pi(a)] \otimes 1 \otimes \sigma_1, \qquad \hat{\lambda}_g^* [\widehat{D}, \hat{\lambda}_g] = l(g) \otimes 1 \otimes \sigma_2,$$

for any $a \in \mathcal{A}$ and $g \in G$. In particular $\hat{\lambda}_g^* [\widehat{D}, \hat{\pi}(\alpha_g(a))] \hat{\lambda}_g = [D, \pi(a)] \otimes 1 \otimes \sigma_1$ and so

$$\begin{aligned} \pi_{\widehat{D}}(c \rtimes_\alpha \Delta_g)(\xi \otimes \delta_x \otimes v) = l(g) &\sum \pi(a_0) J \pi(\alpha_x^{-1}(b_0^*)) J^{-1} [D, \pi(a_1)] \cdots [D, \pi(a_n)] \xi \otimes \delta_x \\ &\otimes \left(\sigma_2 \sigma_1^n + \sum_{j=2}^{n} (-1)^{j-1} \sigma_1^{j-1} \sigma_2 \sigma_1^{n-j+1} + (-1)^n \sigma_1^n \sigma_2 \right) v \end{aligned}$$

by the zeroth order condition of $\widehat{J}$. The summation in the brackets is just $(n+1)$ times $\sigma_2 \sigma_1^n$ which is $-i(n+1)\sigma_3$ as n is odd, by the properties of the algebra of Pauli matrices. The factor

$$\sum \pi(a_0) J \pi(\alpha_x^{-1}(b_0^*)) J^{-1} [D, \pi(a_1)] \cdots [D, \pi(a_n)] \tag{7.34}$$

is just $\pi_D(c_{x^{-1}})$ (according to the notation of (7.28)). By Lemma 7.40 this is $\mathrm{Ad} u_x^* \circ \pi_D(\alpha_x^{-1}(c))$. Since c is G-invariant and $\pi_D(c) = \mathrm{id}_H$ we deduce that (7.34) is trivial. Then since $\sigma_3 = \chi$, the normalization factor brings the thesis.

Consider now the case when $(\mathcal{A}, H, D)$ is even and thus $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D}, \hat{\pi} \rtimes \hat{\lambda}, \widehat{J})$ is odd. Then

$$[\widehat{D}, \hat{\pi}(a)] = [D, \pi(a)] \otimes 1, \qquad \hat{\lambda}^*_g[\widehat{D}, \hat{\lambda}_g] = l(g)(\chi \otimes 1),$$

for any $a \in \mathcal{A}$ and $g \in G$. In particular, $\hat{\lambda}^*_g[\widehat{D}, \hat{\pi}(\alpha_g(a))]\hat{\lambda}_g = [D, \pi(a)] \otimes 1$. Since

$$\chi[D, \pi(a_j)] = -[D, \pi(a_j)]\chi$$

we get

$$\begin{aligned}
\pi_{\widehat{D}}(c \rtimes_\alpha \Delta_g) &= l(g) \sum \pi(a_0)\chi J \pi(b_0^*)J^{-1}[D, \pi(a_1)] \cdots [D, \pi(a_n)] \otimes \\
&\qquad \otimes \left(1 + \sum_{j=2}^{n} (-1)^{j-1}(-1)^{j-1} + (-1)^n(-1)^n\right) \\
&= l(g)(n+1)\chi \underbrace{\left(\sum \pi(a_0) J\pi(b_0^*)J^{-1}[D, \pi(a_1)] \cdots [D, \pi(a_n)]\right)}_{\chi} \otimes 1
\end{aligned}$$

by the zeroth order condition for $\widehat{J}$. Since $\chi^2 = \mathrm{id}_H$ by assumption, the normalization factor M completes the proof. □

Remark 7.44. The method of the twisted shuffle product is not suitable for the case of the $\star$-equivariance: indeed, the shuffle product sums the degree of the Hochschild chains that are multiplied and it is not possible to pass from dimension n to dimension $n-1$ (apart from multiplying a hypothetical 7-cycle whose existence is not guaranteed).

Example 7.45. Applying Theorem 7.41 and formula (7.33) to the triple in Example 5.27, one recovers (up to a multiplicative constant) the standard orientation cycle on the noncommutative 2-torus as described in [44, Chapter 12.3]. Indeed, if we regard $C(S^1)$ as the C^*-algebra generated by $U = e^{2\pi i \varphi_1}$ with $\varphi_1 \in S^1$, then $c = U^* \otimes U$ is a Hochschild orientation 1-cycle for the spectral triple over $C(S^1)$ as defined in Example 5.27. Doing the shuffle product with the 1-cocycle $\delta = V^* \otimes V$ (where V is the generator of the action of $\mathbb{Z}$) we have by definition

$$\begin{aligned}
c \rtimes \delta &= U^*V^* \otimes V \otimes U - U^*V^* \otimes \alpha(U) \otimes V \\
&= U^*V^* \otimes V \otimes U - e^{2\pi i\theta} U^*V^* \otimes U \otimes V \\
&= U^*V^* \otimes V \otimes U - V^*U^* \otimes U \otimes V
\end{aligned}$$

which is equation (5.14) up to a constant. □

After having constructed an orientation cycle $\hat{c}$ on the equivariant triple $(C_c(G,\mathcal{A}), \widehat{H}, \widehat{D})$, we now examine its equivariance for the coaction of G in a suitable sense.

Definition 7.46. *Let $\delta \colon B \to B \otimes C^*_r(G)$ be a coaction of G on a unital C^*-algebra B. For any Hochschild chain $c = \sum (b_0 \otimes p) \otimes b_1 \otimes \cdots \otimes b_n \in C_n(B \otimes B^{\mathrm{op}}, B)$, we define*

$$\delta(c) := \sum (b_{0(-1)} \otimes p) \otimes b_{1(-1)} \otimes \cdots \otimes b_{n(-1)} \otimes \left(b_{0(0)} \cdots b_{n(0)}\right) \in C_n(B \otimes B^{\mathrm{op}}, B) \otimes C^*_r(G) \tag{7.35}$$

using the Sweedler notation (C.5)*. We say that c is **δ-invariant** if $\delta(c) = c \otimes 1$.*

Proposition 7.47. *Let G be a discrete group and $l\colon G \to \mathbb{R}$ a proper homomorphism. Let $(\mathcal{A}, H, D, u)$ be an (even or odd) G-invariant spectral triple on a unital C^*-algebra A, J a unitarily invariant real structure and c a G-invariant orientation cycle. The orientation cycle $\hat{c}$ of $(C_c(G, \mathcal{A}), \widehat{H}, \widehat{D})$ given in (7.33) is invariant for the dual coaction $\widehat{\alpha}$.*

Proof. Let us write $c \rtimes_\alpha \Delta_g = \sum_{j=1}^{n+1} c_j$ as a shorthand notation and recall that by definition $\widehat{\alpha}(a\delta_g) = a\delta_g \otimes \delta_g$ for any $a\delta_g \in C_c(G, \mathcal{A})$. Concerning $c_1 = \sum (a_0\delta_{g^{-1}} \otimes b_0) \otimes \delta_g \otimes a_1 \otimes \cdots \otimes a_n$ we have by definition

$$\widehat{\alpha}(c_1) = \sum \Big((a_0\delta_{g^{-1}} \otimes b_0) \otimes \delta_g \otimes a_1 \otimes \cdots \otimes a_n \Big) \otimes (\delta_{g^{-1}}\delta_g\delta_e \cdots \delta_e) = c_1 \otimes \delta_e.$$

The other cases are similar; any factor $a\delta_e \in C_c(G, \mathcal{A})$ of c_j brings a trivial contribution to the piece in $C_r^*(G)$. Since any c_j contains precisely one term δ_g and one term $\delta_{g^{-1}}$, the total contribution is trivial. $\square$

Remark 7.48. If G is abelian, then the orientation cycle $\hat{c}$ given in (7.33) is indeed invariant under the dual action $\widehat{\alpha}$ of the Pontryagin group $\widehat{G}$ as in Definition 7.38.

Chapter 8

Conclusions

In this thesis we have proposed two constructions of a spectral triple on a twisted crossed product $A \rtimes^{\sigma}_{\alpha,r} G$ and we have investigated some of the Connes axioms for a spectral manifold. In both cases we have used as a building block a spectral triple on the twisted group algebra $C^*_{\sigma,r}(G)$, defined via a matrix-valued Dirac weight function $l\colon G \to \mathcal{L}(V)$, for which we have given full results about the regularity condition, its summability and the existence of a real structure. These results have been successfully reproduced when dealing with the triple on $A \rtimes^{\sigma}_{\alpha,r} G$, even though some of them require some more restrictive assumptions (e.g. on the cocycle or the dimension of the vector space V).

Between the two aformentioned constructions, the one developed in Theorem 7.3 has been proved to be more directly connected to KK-theory, giving in particular a preliminary insight (Proposition 7.11) into the reason why the original ansatz made in [47] for the exterior Kasparov product makes their construction work. Note that it is precisely thanks to the external Kasparov product that we are able to extend to the triple on $A \rtimes^{\sigma}_{\alpha,r} G$ some of the Connes axioms originally assumed on the triples on A and on $C^*_{r,\sigma}(G)$, such as the real structure and the regularity condition.

Although passing through the exterior Kasparov product is a rather direct strategy, some of the results obtained depend heavily on the exact commutation of the operator D with the representation u of the group G on H, a hypothesis which forces the action of the group to be Lipschitz isometric (and in particular equicontinuous). This hypothesis seems to be very difficult to work without: as noted in Remark 7.24, the triviality of the commutator may be unavoidable. Furthermore, apart from the summability condition (Section 7.3), which depends on the Dirac operator but not on the representation of the algebra, the problem of obtaining these structures and properties when the starting triple is not equivariant is still unsolved.

When constructing the building block spectral triple on $C^*_{\sigma,r}(G)$, we have assumed the existence of a length-type function on a discrete group, inspired by the (Fourier transform of the) canonical spectral triple on the circle. As noted in Remarks 5.44 and 5.45, these elements are not free from difficulties. The generalization of our results from a Dirac weight with scalar values to a Dirac weight with matrix values has led to serious technical problems, mostly related to the fulfillment of the zeroth order condition of the candidate real structures. Furthermore, it is not completely clear how far the construction of the triple on the twisted crossed product can be generalized when the building block spectral triple is given by an arbitrary element in $KK^G(\mathbb{C},\mathbb{C})$ with G not necessarily discrete.

It would be desirable to produce a construction encompassing the locally compact case as well. However, our approach relies on the Green-Julg map in KK-theory for discrete groups and it is not clear how to overcome this obstruction.

In summation, we have constructed spectral triples on a crossed product $A \rtimes^{\sigma}_{\alpha,r} G$ starting from a spectral triple on A which is equivariant with respect to a (twisted) action of G. We have found that, whenever this action is sufficiently regular, the spectral geometry on A can be naturally promoted to the triple on $A \rtimes^{\sigma}_{\alpha,r} G$. How to control the situations when the interaction of the group with the spectral triple is badly-behaved remains a challenging and interesting open question for future research.

Part III

Appendix

Appendix A

Hilbert C^*-Modules

In this appendix (which is mainly based on [16, 53, 65, 92]) we shall give a brief account of definitions and properties of Hilbert C^*-modules.

Definition A.1. *Let A be a C^*-algebra and X a complex vector space which is also a right A-module. We say that X is a right **pre-Hilbert A-module** if it is equipped with a pairing $\langle \cdot, \cdot \rangle_A \colon X \times X \to A$ which is $\mathbb{C}$-linear in the second variable and such that:*

(1) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A \cdot a$

(2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$

(3) $\langle x, x \rangle_A \geq 0 \quad \text{(as an element of } A)$

(4) $\langle x, x \rangle_A = 0$ *implies* $x = 0$

for all $x, y \in X$ and $a \in A$.

We say that X is a left A-module if satisfies similar properties with respect to an action $A \times X \to X$ and we write X_A or ${}_AX$ to emphasize whether we are regarding X as a right or a left A-module. We put the label A on the right on the symbol $\langle \cdot, \cdot \rangle$ both to recall that the scalar product is A-valued and to denote in which entry it is A-linear.

Definition A.2. *We say that a (right) pre-Hilbert A-module X is a **Hilbert module** if it is complete in the norm $\|x\|_A := \|\langle x, x \rangle_A\|^{\frac{1}{2}}$. The Hilbert module is **full** if the ideal $I := \operatorname{span}\{ \langle x, y \rangle_A \mid x, y \in X \}$ is dense in A.*

Example A.3. Let A be a C^*-algebra. We can regard A as a module over itself by right multiplication. Furthermore, A is a full Hilbert A-module with respect to the inner product

$$\langle a, b \rangle_A := a^* b.$$

Note that fullness comes from the existence of approximate units. □

Example A.4 (Direct Sums)**.** Let A be a C^*-algebra. Suppose that X and Y are Hilbert A-modules. Then $Z = X \oplus Y$ is a right A-module in the obvious way. The quantity defined by

$$\langle (x, y), (x', y') \rangle_A := \langle x, x' \rangle_A + \langle y, y \rangle_A$$

is an A-valued inner product on Z which makes it a Hilbert A-module. □

Example A.5 (Standard A-Module)**.** As a special case of the sum construction, we allow also infinite sums in the following way. Let E be the space of sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ with values in A that are eventually 0 and define the pairing

$$\langle \mathbf{a}, \mathbf{b} \rangle_A \coloneqq \sum_{n=1}^{+\infty} a_n^* b_n.$$

By construction this is a finite sum and so it is well defined. Then E is a pre-Hilbert A-module under the right action $\mathbf{a} \cdot a = (a_n a)_{n\in\mathbb{N}}$ and the Hilbert A-module obtained by norm completion is denoted by $\mathbb{H}_A$. Note that $\mathbb{H}_A$ with $A = \mathbb{C}$ is just the Hilbert space $\ell^2(\mathbb{N})$.

□

While in a Hilbert space every bounded operator admits an adjoint, this is no longer true for Hilbert modules (see [92, Example 2.19]): it turns out that the the right notion of morphisms for Hilbert modules is the following.

Definition A.6. *Let X, Y be Hilbert A-modules. A map $T\colon X \to Y$ is* ***adjointable*** *if there exists a map $T^*\colon Y \to X$ such that*

$$\langle Tx, y \rangle_A = \langle x, T^* y \rangle_A$$

for every $x \in X$ and $y \in Y$. The map T^* is called the ***adjoint*** *of T. We denote by $\mathcal{L}_A(X, Y)$ the set of all adjointable operators from X to Y and write $\mathcal{L}_A(X)$ for $\mathcal{L}_A(X, X)$.*

Proposition A.7 (cf. [92])**.** *Let X, Y be Hilbert A-modules. Every adjointable map $T\colon X \to Y$ is bounded and A-linear.*

It is easy to prove that the space $\mathcal{L}_A(X)$ has all the expected properties, for instance:

(1) If $T \in \mathcal{L}_A(X)$ then the adjoint T^* is unique and still belongs to $\mathcal{L}_A(X)$ with $T^{**} = T$.

(2) If $T, S \in \mathcal{L}_A(X)$, then $TS \in \mathcal{L}_A(X)$ and $(TS)^* = S^* T^*$

(3) The quantity $\|T\| = \sup_{\|x\|_A = 1} \|Tx\|_A$ is a norm that makes $\mathcal{L}_A(X)$ a Banach space.

(4) With the $*$-structure given by adjointion, the space $\mathcal{L}_A(X)$ is a C^*-algebra.

We want now to find the analogues of compact operators on a Hilbert space for Hilbert modules. It turns out that the right way passes through the well known fact that the set of finite rank operators on a Hilbert space H is dense in $\mathcal{K}(H)$ and is linearly spanned by the rank-one projections (see for instance [75, pag. 55 – 56]).

Definition A.8. *Let X and Y be Hilbert A-modules. For every $x \in X$ and $y \in Y$ we define $\Theta_{y,x}\colon X \to Y$ by*

$$\Theta_{y,x}(x') \coloneqq y \cdot \langle x, x' \rangle_A$$

This map is adjointable with $\Theta_{y,x} = \Theta_{x,y}^*$ and we denote by $\mathcal{K}_A(X, Y)$ the closed linear subspace of $\mathcal{L}_A(X, Y)$ spanned by the elements $\Theta_{y,x}$ for $x \in X, y \in Y$.

Proposition A.9. *$\mathcal{K}_A(X)$ is a closed two-sided ideal in $\mathcal{L}_A(X)$.*

Proof. See [92, Lemma 2.25]. □

A.1 Tensor Products of Hilbert Modules

Let us now discuss the tensor product constructions of Hilbert modules.

A.1.1 Internal Tensor Product

Let X_1 and X_2 be Hilbert modules over C^*-algebras B_1 and B_2, respectively. Let $\varphi\colon B_1 \to \mathcal{L}_{B_2}(X_2)$ be a $*$-homomorphism and regard in this way X_2 as a left B_1-module. The algebraic inner tensor product $X_1 \odot_\varphi X_2$ is defined as the free vector space generated by elements of the form $x_1 \otimes x_2$ such that

$$(x_1 \cdot b) \otimes x_2 = x_1 \otimes (b \cdot x_2)$$

for $b \in B_1$. This is a right B_2-module by setting $(x_1 \otimes x_2) \cdot b = x_1 \otimes x_2 b$. We define a B_2-valued inner product on the tensor product by

$$\langle\!\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle\!\rangle_{B_2} \coloneqq \big\langle x_2, \varphi(\langle x_1, y_1\rangle_{B_1}) y_2 \big\rangle_{B_2} \tag{A.1}$$

We consider the submodule N of vectors in $X_1 \odot_\varphi X_2$ of length 0 with respect to the inner product (A.1); the **internal tensor product** $X_1 \otimes_\varphi X_2$ is the completion of the algebraic tensor product $X_1 \odot_\varphi X_2$ quotiented by N with respect to the inner product (A.1). This tensor product is sometimes denoted also by $X_1 \otimes_{B_1} X_2$.

Note that if $\varphi\colon B_1 \to B_2$ is a $*$-homomorphism, then $B_1 \otimes_\varphi B_2$ is isomorphic to the closed right ideal $\overline{\varphi(B_1)B_2}$ of B_2 generated by B_1. In particular, if φ is a unital homomorphism of unital C^*-algebras (or, more generally, if φ is **essential** in the sense that $\varphi(B_1)$ contains an approximate units for B_2), then $B_1 \otimes_\varphi B_2$ is isomorphic to B_2.

A.1.2 External Tensor Product

Let X_1 and X_2 be Hilbert modules over C^*-algebras B_1 and B_2, respectively. The algebraic external tensor product $X_1 \odot X_2$ is defined as the free vector space generated by elements of the form $x_1 \otimes x_2$ such that $\lambda x_1 \otimes x_2 = x_1 \otimes \lambda x_2$ for any $\lambda \in \mathbb{C}$ endowed with a right module over the algebraic tensor product $B_1 \odot B_2$ by

$$(x_1 \otimes x_2)(b_1 \otimes b_2) = x_1 b_1 \otimes x_2 b_2 \tag{A.2}$$

for $x_i \in X_i$ and $b_i \in B_i$. We define a $B_1 \otimes B_2$-inner product on $X_1 \odot X_2$ by

$$\langle\!\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle\!\rangle_{B_1 \otimes B_2} \coloneqq \big\langle x_1, y_1 \big\rangle_{B_1} \otimes \big\langle x_2, y_2 \big\rangle_{B_2} \tag{A.3}$$

We consider the submodule N of vectors in $X_1 \odot X_2$ of length 0 with respect to the inner product (A.3); the **external tensor product** $X_1 \otimes X_2$ is the completion of the algebraic tensor product $X_1 \odot X_2$ quotiented by N with respect to the inner product (A.3).

A.2 Graded Algebras and Modules

Definition A.10. *Let A be a C^*-algebra. A grading on A is a $*$-automorphism β such that $\beta^2 = \mathrm{id}_A$; we say that A is $\mathbb{Z}_2$-graded if it admits a grading.*

When A is $\mathbb{Z}_2$-graded, it decomposes into the eigenspaces of β, i.e. $A = A_0 \oplus A_1$ where $A_0 \coloneqq \{\, a \in A \mid \beta(a) = a \,\}$ and $A_1 \coloneqq \{\, a \in A \mid \beta(a) = -a \,\}$. Note that this is just a Banach space decomposition; in particular A_0 is a C^*-algebra but A_1 not. An element

a which is in A_j for $j = 0$ or 1 is called **homogeneous**; in this case the index j is called its **degree** and we write $\deg(a) = j$. Note that if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$ where the sum $i + j$ is taken in $\mathbb{Z}_2$. Given $a \in A = A_0 \oplus A_1$, we denote by $a = a^{(0)} + a^{(1)}$ the decomposition of a; in this case, the action of β is given by $\beta(a^{(0)} + a^{(1)}) = a^{(0)} - a^{(1)}$. The **graded commutator** of a $\mathbb{Z}_2$-graded C^*-algebra A is defined to be the unique bilinear map $[\cdot,\cdot]\colon A \times A \to A$ satisfying $[a, b] = ab + (-1)^{ij} ba$ for $a \in A_i$ and $b \in A_j$. A homomorphism $\varphi\colon A \to B$ of graded C^*-algebras is a **graded homomorphism** if $\varphi \circ \beta_A = \beta_B \circ \varphi$, namely if $\varphi(A_j) \subseteq B_j$ for $j = 0, 1$.

Example A.11. Let A an ungraded C^*-algebra. The grading on $A \oplus A$ given by $(a, b) \mapsto (a, -b)$ is called the **standard even grading**; the even and odd elements are given respectively by $(A \oplus A)_0 = \{ (a, 0) \mid a \in A \}$ and $(A \oplus A)_1 = \{ (0, a) \mid a \in A \}$.

Example A.12. Let A an ungraded C^*-algebra. The grading on $A \oplus A$ given by $(a, b) \mapsto (b, a)$ is called the **standard odd grading**; the even and odd elements are given respectively by $(A \oplus A)_0 = \{ (a, a) \mid a \in A \}$ and $(A \oplus A)_1 = \{ (a, -a) \mid a \in A \}$.

Definition A.13. *Let A be a $\mathbb{Z}_2$-graded C^*-algebra. A* ***graded Hilbert A-module*** *is a Hilbert A-module X equipped with a linear bijection $S\colon X \to X$ (called the grading operator) such that $S^2 = \mathrm{Id}_X$ and*

(1) $S(xa) = S(x)\beta(a)$ for any $x \in X$ and $b \in A$

(2) $\langle Sx, Sy\rangle = \beta(\langle x, y\rangle_A)$ for any $x, y \in X$

As in the case of C^*-algebras, we have that $X = X_0 \oplus X_1$ (where X_0 and X_1 are the eigenspaces of S). Note that $X_i B_j \subseteq X_{i+j}$ and that $\langle X_i, X_j\rangle \subseteq A_{i+j}$ for any $i, j \in \mathbb{Z}_2$. If X is a graded Hilbert A-module, we denote by X^{op} the graded Hilbert A-module obtained from X by interchanging X_0 and X_1; analogously, we write $\widehat{\mathbb{H}}_A$ for $\mathbb{H}_A \oplus \mathbb{H}_A^{\mathrm{op}}$ endowed with the standard even grading. Note that any grading S on a Hilbert A-module X induces naturally a grading on the space $\mathcal{L}_A(X)$ by $T \mapsto STS^{-1}$.

Let us now discuss graded tensor products. Let A and B be graded C^*-algebras and $A \odot B$ their algebraic tensor product. We define a product and involution on $A \odot B$ by

(1) $(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) := (-1)^{\deg(b_1)\cdot\deg(a_2)}(a_1 a_2 \hat{\otimes} b_1 b_2)$

(2) $(a_1 \hat{\otimes} b_1)^* := (-1)^{\deg(b)\cdot\deg(a)}(a^* \hat{\otimes} b^*)$

for homogeneous elementary tensors. The algebraic tensor product with this operations is a $*$-algebra denoted by $A \widehat{\odot} B$; the maximal graded tensor product $A \widehat{\otimes} B$ of A and B is then the universal enveloping algebra of $A \widehat{\odot} B$.

If X_1 and X_2 are graded Hilbert modules over A and B respectively, and φ is a graded $*$-homomorphism from A to B, we define the internal tensor product $X_1 \widehat{\otimes} X_2$ as their ordinary tensor product with grading $\deg(x \hat{\otimes} y) = \deg(x) + \deg(y)$. The formula for the external product must be modified as well: (A.2) and (A.3) become respectively

$$(x_1 \otimes x_2)(b_1 \otimes b_2) = (-1)^{\deg(x_2)\cdot\deg(b_1)} x_1 b_1 \otimes x_2 b_2$$

$$\langle\!\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle\!\rangle_{B_1 \otimes B_2} = (-1)^{\deg(x_2)(\deg(x_1)+\deg(y_1))} \langle x_1, y_1\rangle_{B_1} \otimes \langle x_2, y_2\rangle_{B_2}$$

Appendix B

Operator $*$-Modules

Let X be a Banach space over the complex numbers and denote by $\|\cdot\|_X : X \to [0, +\infty)$ its norm. We denote by $M(\mathbb{C})$ the $*$-algebra of infinite matrices over $\mathbb{C}$ with only finitely many entries different from 0 and $M(X) \coloneqq M(\mathbb{C}) \otimes X$ (where $\otimes$ is the algebraic tensor product).

Definition B.1. *A Banach space $(X, \|\cdot\|)$ is called an **operator space** if there exists a norm $\|\cdot\|_X : M(X) \to [0, +\infty)$ such that:*

(1) $\|vxw\| \leq \|v\|_{\mathbb{C}} \|x\|_X \|w\|_{\mathbb{C}}$ for any $v, w \in M(\mathbb{C})$ and $x \in M(X)$

(2) $\|pxp + qyq\|_X = \max\{\|pxp\|_X, \|qyq\|_X\}$ for any $x, y \in M(X)$ and any pair of projections $p, q \in M(\mathbb{C})$ with $pq = 0$

(3) $\|p \otimes x\|_X = \|x\|$ for any rank-1 projection $p \in M(\mathbb{C})$ and $x \in X$.

Note that the last condition implies that the norm $\|\cdot\|_X$ is compatible with the given norm on X and so we will always write $\|\cdot\|_X$. Let $\overline{M}(X)$ denote the completion of $M(X)$ in the operator norm; this can be given the structure of an operator space by using the identification $M_n(\overline{M}(X)) \simeq \overline{M}(M_n(X))$.

Definition B.2. *We say that a continuous linear map $T \colon X \to Y$ between operator spaces is **completely bounded** if the quantity*

$$\|T\|_{cb} \coloneqq \sup_{n \in \mathbb{N}} \|\mathrm{id} \otimes T \colon M_n(\mathbb{C}) \otimes X \to M_n(\mathbb{C}) \otimes Y\|$$

is finite. We denote by $CB(X, Y)$ the space of completely bounded linear maps from X to Y.

We remark that a map $T \colon X \to Y$ is completely bounded if an only if it induces a bounded map $\overline{T} \colon \overline{M}(X) \to \overline{M}(Y)$. We say that the operator spaces X and Y are **completely isomorphic** if there exists a completely bounded vector space isomorphism $U \colon X \to Y$ with completely bounded inverse.

Definition B.3. *Let X be an operator space which is at the same time an algebra over $\mathbb{C}$. We say that X is an **operator algebra** if the multiplication map $m \colon X \times X \to X$ is completely bounded, namely if there exists a constant $K > 0$ such that*

$$\|xy\|_X \leq K \|x\|_X \|y\|_X$$

for any $x, y \in M(X)$. We say that X is an ***operator $*$-algebra*** *if X has a completely bounded involution $*\colon X \to X$.*

Example B.4 (cf. [56])**.** Suppose we are given:

(1) A C^*-algebra B and a right Hilbert B-module E.

(2) A $*$-algebra $\mathcal{A}$ and a $*$-homomorphism $\pi\colon \mathcal{A} \to \mathcal{L}_B(E)$

(3) A self-adjoint unbounded operator D on E such that for each $\mathcal{A}$ the operator $\pi(a)$ maps the domain of D into itself and the commutator $[D, \pi(a)]$ is in $\mathcal{L}_B(E)$.

Let $A_1 \subseteq \mathcal{L}_B(E)$ denote the completion of $\pi(\mathcal{A})$ in the norm

$$\|\pi(a)\|_1 \coloneqq \|\pi(a)\| + \|[D, \pi(a)]\|\,.$$

Then the $*$-subalgebra $A_1 \subseteq \mathcal{L}_B(E)$ can be given the structure on an operator $*$-algebra by embedding it into $\mathcal{L}(E \oplus E)$ as follows:

$$a \longmapsto \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix}.$$

Furthermore, the inclusion $A_1 \to A$ into its C^*-completion and $[D, \cdot]\colon A_1 \to \mathcal{L}_B(E)$ are completely bounded maps. $\square$

Definition B.5. *let A be an operator algebra and let X be a right-module over A. We say that X is a* ***right operator module*** *over A if X is equipped with the structure of an operator space such that the right action $X \times A \to X$ is completely bounded, namely if there exists a constant $K > 0$ such that*

$$\|\xi a\|_X \leq K \|\xi\|_X \|a\|_A$$

for all $\xi \in M(X)$ and $a \in M(A)$.

Definition B.6. *Let A be an operator $*$-algebra and X a right operator module over A. We say that X is a* ***hermitian operator module*** *if there exists a completely bounded pairing $\langle\cdot,\cdot\rangle_X\colon X \times X \to A$ satisfying:*

(1) $\langle x, y\lambda + z\mu\rangle = \langle x, y\rangle\lambda + \langle x, z\rangle\mu$

(2) $\langle x, ya\rangle = \langle x, y\rangle a$

(3) $\langle x, y\rangle^* = \langle y, x\rangle$

for any $x, y, z \in X$, $a \in A$ and $\lambda, \mu \in \mathbb{C}$.

Example B.7 (The standard module)**.** Let A be an operator $*$-algebra. The standard operator module $\mathbb{H}_A$ over A is the completion of the space of finite sequences in $M(A)$ in its norm. Under the pairing $\langle\cdot,\cdot\rangle\colon \mathbb{H}_A \times \mathbb{H}_A \to A$ given by

$$\langle (a_n)_n, (b_n)_n\rangle \coloneqq \sum_n a_n^* b_n,$$

$\mathbb{H}_A$ is a hermitian operator module. $\square$

Definition B.8. *Let X be a hermitian operator module over the operator $*$-algebra A. We say that X is an* ***operator $*$-module*** *if there exist a completely bounded self-adjoint idempotent $P\colon \mathbb{H}_A \to \mathbb{H}_A$ and a completely bounded isomorphism of hermitian operator modules $X \simeq P\mathbb{H}_A$.*

Appendix **C**

Hopf Algebras and CQG

This Appendix gives a brief introduction to Hopf algebras and compact quantum groups and their action and coactions. For a more detailed treatment we refer to [1, 69, 104].

C.1 Hopf Algebras

Given an unital algebra $\mathcal{H}$, we denote by $m\colon \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ the multiplication operator $m(e\otimes f) \coloneqq ef$ and by $\eta\colon \mathbb{C}\to\mathcal{H}$ the unit map $\eta(\lambda)\coloneqq \lambda\cdot 1_{\mathcal{H}}$.

Definition C.1. *A **Hopf algebra** is a unital associative algebra $\mathcal{H}$ equipped with:*

*(1) a unital algebra homomorphism $\Delta\colon \mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$, called the **coproduct**, which is coassociative in the sense that*

$$(\Delta\otimes \mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$$

*(2) a homomorphism $\varepsilon\colon \mathcal{H}\to\mathbb{C}$, called the **counit**, such that*

$$(\varepsilon\otimes \mathrm{id})\circ\Delta = (\mathrm{id}\otimes\varepsilon)\circ\Delta = \mathrm{id}$$

*(3) a linear map $S\colon \mathcal{H}\to\mathcal{H}$, called the **antipode**, such that*

$$m\circ(S\otimes \mathrm{id})\circ\Delta = m\circ(\mathrm{id}\otimes S)\circ\Delta = \eta\circ\varepsilon$$

We shall adopt Sweedler's notation for which $\Delta h = \sum h_{(1)}\otimes h_{(2)}$ for any $h\in\mathcal{H}$. Further, the summation symbol is often omitted for sake of clearness. Using the axioms, it is easy to see that the antipode map S must be an anti-linear homomorphism of algebras [1, Theorem 2.1.4] and that it must be unique [69, Proposition 1.3.1].

Example C.2. Let G be a discrete group and $\mathcal{H}=\mathbb{C}G$ its group algebra. One can make $\mathcal{H}$ into a Hopf algebra by defining maps Δ, ε and S as follows:

$$\Delta(\delta_g)\coloneqq\delta_g\otimes\delta_g \qquad \varepsilon(\delta_g)\coloneqq 1 \qquad S(\delta_g)\coloneqq\delta_{g^{-1}}.$$

Then it is easy to check that $\mathbb{C}G$ is **cocommutative** (in the sense that $F\circ\Delta=\Delta$ for $F\colon \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$ the flip map $F(g\otimes f)=f\otimes g$) and that the antipode S is invertible with $S^{-1}=S$. □

Definition C.3. *A **Hopf $*$-algebra** is a Hopf algebra $\mathcal{H}$ equipped with an anti-linear involution $*\colon \mathcal{H} \to \mathcal{H}$ which makes $\mathcal{H}$ into an associative $*$-algebra and such that*

(1) $\Delta(h^*) = h^*_{(1)} \otimes h^*_{(2)}$ *for every* $h \in \mathcal{H}$

(2) $\varepsilon(h^*) = \overline{\varepsilon(h)}$ *for every* $h \in \mathcal{H}$

(3) $(S \circ *)^2 = \mathrm{id}$

Note that point (3) in Definition C.3 means that the antipode S is invertible and that $(Sh)^* = S^{-1}(h^*)$ for any $h \in E$.

Example C.4. The Hopf algebra $\mathbb{C}G$ of a discrete group G admits a canonical star structure given by the anti-linear extension of the map

$$* \, \delta_g := \delta_{g^{-1}}. \tag{C.1}$$

However, differently from the antipode map which is unique, the star structure might not be unique: for instance, if G is abelian, also the anti-linear extension of the map

$$\star \, \delta_g := \delta_g \tag{C.2}$$

is a star structure. Note that we have used two different symbols for the two stars. □

Given a Hopf algebra $\mathcal{H}$, we say that an algebra A is a **left $\mathcal{H}$-module algebra** if A is a left $\mathcal{H}$-module and the representation is compatible with the algebra structure in A, namely if

$$h \triangleright (a_1 a_2) = (h_{(1)} \triangleright a_1)(h_{(2)} \triangleright a_2)$$

for any $h \in \mathcal{H}$ and $a_1, a_2 \in A$. If A is unital, we further require that

$$h \triangleright 1 = \varepsilon(h)$$

for any $h \in \mathcal{H}$. We say that a left A-module M over a left $\mathcal{H}$-module algebra A is a **left $\mathcal{H}$-equivariant A-module** if M is a left $\mathcal{H}$-module and

$$h \triangleright (am) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright m)$$

for any $h \in \mathcal{H}$, $a \in A$ and $m \in M$. In this thesis, when dealing with a Hopf $*$-algebra $\mathcal{H}$ and a $\mathcal{H}$-module algebra A endowed with a $*$-involution, we will always assume that the action of $\mathcal{H}$ is compatible with the star structure of A in the sense that

$$(h \triangleright a)^* = (Sh)^* \triangleright a^* \tag{C.3}$$

for all $a \in A$ and $h \in \mathcal{H}$.

C.2 Compact Quantum Groups

It is well known that the C^*-completion $C^*_r(G)$ of the group algebra $\mathbb{C}G$ is in general not a Hopf algebra as the counit map and the antipode map may not be bounded (see for instance [106, Remark 3.3]). It turns out that the right notion to deal with this situation is the following.

Definition C.5 (cf. [104])**.** *A C^*-algebraic* ***compact quantum group*** *(or CQG) is a unital C^*-algebra Q equipped with a unital $*$-homomorphism $\Delta\colon Q \to Q \otimes Q$ such that:*

(1) Δ is coassociative in the sense that $(\Delta \otimes \mathrm{id}_Q) \circ \Delta = (\mathrm{id}_Q \otimes \Delta) \circ \Delta$

(2) The subspaces

$$\operatorname{span}\{\, (p \otimes 1_Q)\Delta(q) \mid p, q \in Q \,\} \qquad \operatorname{span}\{\, (1_Q \otimes p)\Delta(q) \mid p, q \in Q \,\} \tag{C.4}$$

are norm dense in $Q \otimes Q$.

Example C.6. Let G be a compact topological group and $Q = C(G)$ the unital C^*-algebra of continuous functions on G with values in $\mathbb{C}$ and pointwise multiplication. Then Q is a CQG with respect to the comultiplication $\Delta\colon C(G) \to C(G) \otimes C(G) \simeq C(G \times G)$ given by

$$(\Delta f)(g, h) \coloneqq f(gh)$$

for $g, h \in G$. In this case, the coassociativity condition is the associativity of the product of G and conditions (C.4) represent the cancellation property of the product in G. □

Example C.7. Under the comultiplication introduced in Example C.2, the C^*-algebra $C_r^*(G)$ for G discrete is a CQG. □

Differently from algebras, the natural environment to study "actions" of compact quantum groups on spectral triples is the one of coactions, corepresentations and comodules which we now briefly recall.

Definition C.8. *We say that a CQG (Q, Δ) coacts on a unital C^*-algebra A if there exists a unital C^*-homomorphism (called* ***coaction****) $\theta\colon A \to A \otimes Q$ such that:*

(1) $(\theta \otimes \mathrm{id}_Q)\theta = (\mathrm{id}_A \otimes \Delta)\theta$

(2) $\operatorname{span}\{\, \theta(a)(1_A \otimes b) \mid a \in A, b \in Q \,\}$ is norm dense in $A \otimes Q$.

In this case we say that A is a right Q-comodule. We adopt an analogue of the Sweedler notation for the coproduct and we denote an element $\theta(b) = \sum_{i=1}^n b_i \otimes c_i$ with $b_i \in B$ and $c_i \in Q$ just by

$$\theta(b) = \sum b_{(-1)} \otimes b_{(0)}, \tag{C.5}$$

omitting the summation index.

It is well known (cf. [108], [91]) that condition (2) in Definition C.8 is equivalent to the existence of a norm-dense unital $*$-subalgebra A_0 of A such that $\theta(A_0) \subseteq A_0 \odot Q_0$ and $(\mathrm{id} \otimes \varepsilon)\theta = \mathrm{id}_{A_0}$.

Example C.9 (Dual Coaction)**.** Consider a C^*-dynamical system (A, G, α) with A unital and (for simplicity) assume that G is discrete, and set $B = A \rtimes_{\alpha,r} G$. The maps $i_A\colon A \to B \otimes C_r^*(G)$ and $i_G\colon G \to B \otimes C_r^*(G)$ given by

$$\begin{cases} i_A(a) \coloneqq a\delta_e \otimes \delta_e \\ i_G(g) \coloneqq 1_A \delta_g \otimes \delta_g, \end{cases} \tag{C.6}$$

form a covariant representation of (A, G, α) on $B \otimes C_r^*(G)$, i.e.,

$$i_G(g) i_A(a) i_G(g)^* = \delta_g a \delta_g^* \otimes \delta_g \delta_e \delta_g^* = \alpha_g(a)\delta_e \otimes \delta_e = i_A(\alpha_g(a)).$$

The integrated form $\widehat{\alpha} := i_A \rtimes i_G \colon B \to B \otimes C_r^*(G)$, $a\delta_g \mapsto a\delta_g \otimes \delta_g$, is a coaction (known as the **dual coaction**) of G on $A \rtimes_{\alpha,r} G$ since

$$(\widehat{\alpha} \otimes \mathrm{id}) \circ \widehat{\alpha}(a\delta_g) = a\delta_g \otimes \delta_g \otimes \delta_g = (\mathrm{id} \otimes \Delta) \circ \widehat{\alpha}(a\delta_g)$$

and the density condition is trivially satisfied. When G is abelian, it is known that any coaction $\delta \colon B \to B \otimes C_r^*(G)$ is equivalent via Fourier transform to an action of the Pontryagin dual group $\widehat{G}$. In particular, the dual coaction $\widehat{\alpha} = i_A \rtimes i_G$ corresponds to the *dual action* $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_{\alpha,r} G$ given by

$$\widehat{\alpha}_\gamma(a\delta_g) = \overline{\gamma(g)} a\delta_g, \qquad \gamma \in \widehat{G}.$$

We use $\widehat{\alpha}$ to denote both the action and the coaction with a slight abuse of notation. $\square$

Definition C.10 (cf. [104]). *A (unitary)* ***corepresentation*** *of a CQG (Q, Δ) on a Hilbert space H is a linear map $\Theta \colon H \to H \otimes Q$ such that:*

(1) $(\Theta \otimes \mathrm{id}_Q)\Theta = (\mathrm{id}_H \otimes \Delta)\Theta$

(2) $\Theta(H)Q$ *is linearly dense in* $H \otimes Q$

(3) $\langle \Theta(x) \mid \Theta(y) \rangle = \langle x, y \rangle \cdot 1_Q$ *for all $x, y \in H$, where $\langle \cdot \mid \cdot \rangle$ is the usual scalar product on the external tensor product of the Hilbert modules $H_{\mathbb{C}}$ and Q_Q.*

It turns out that when dealing with spectral triples, it is more convenient to see the corepresentations of a (C^*-algebraic) quantum group (Q, Δ) on a Hilbert space H as unitary operators $X \in \mathcal{L}_Q(H \otimes Q) \simeq M(\mathcal{K}(H) \otimes Q)$ such that

$$(\mathrm{id} \otimes \Delta)(X) = X_{(12)} X_{(13)}. \tag{C.7}$$

As shown in [104, Proposition 5.2.2], these two notions coincide in the following sense:

(1) If $\Theta \colon H \to H \otimes Q$ is a unitary corepresentation, then the map

$$X \colon H \odot Q \to H \otimes Q \qquad x \otimes q \mapsto \Theta(x)q \tag{C.8}$$

extends to a unitary operator $X \in \mathcal{L}_Q(H \otimes Q)$ satisfying (C.7).

(2) If a unitary $X \in \mathcal{L}_Q(H \otimes Q)$ satisfies (C.7), then the map

$$\Theta \colon H \to H \otimes Q \qquad x \mapsto X(x \otimes 1_Q)$$

is a unitary corepresentation as in Definition C.10.

Definition C.11. *Let A be a unital C^*-algebra represented on a Hilbert space H and (Q, Δ) be a CQG coacting on A by $\theta \colon A \to A \otimes Q$. We say that H is a* ***Q-equivariant A-module*** *if there is a unitary corepresentation $\Theta \colon H \to H \otimes Q$ such that*

$$\Theta(ax) = a_{(-1)} x_{(-1)} \otimes a_{(0)} x_{(0)}$$

for every $a \in A$ and $x \in H$.

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