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Analogue gravity: between analogy and emergence

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Basic conventions

Throughout the thesis the signature of the metric is diag $(-1, 1, 1, 1)$. We work in natural units $c = \hbar = 1$ unless stated otherwise. For simplicity we omit the hat notation for the operators.

Publications

- Stefano Liberati, Giovanni Tricella, and Matt Visser. Towards a Gordon form of the Kerr spacetime. *Class. Quant. Grav.*, 35(15):155004, 2018.
- Stefano Liberati, Sebastian Schuster, Giovanni Tricella, and Matt Visser. Vorticity in analogue spacetimes. *Phys. Rev. D*, 99(4):044025, 2019.
- Stefano Liberati, Giovanni Tricella, and Andrea Trombettoni. The information loss problem: an analogue gravity perspective. *Entropy*, 21(10):940, 2019.
- Stefano Liberati, Giovanni Tricella, and Andrea Trombettoni. Back-reaction in canonical analogue black holes. 10 2020. arXiv: 2010.09966

Contents

1 Introduction

This thesis presents the research I carried on during my PhD on the topics of analogue gravity, which is the set of techniques that allow to simulate in table-top experiments phenomena that are typically due to the presence of gravitational fields. Before entering the discussion, let us clarify some premises.

First, it is worth underlining that analogue gravity is not intended to simulate the dynamics of spacetime, *i.e.* it is not intended to describe geometrodynamic objects that follow the same equations as of general relativity or of some other theory of gravity; analogue gravity is intended to simulate the dynamics of objects living on a spacetime, as described by classical or quantum field theory in curved spacetime.

Second, analogue gravity should not be considered a physical theory, in the sense that it does not encode the scientific paradigms of a specific branch of physics into a set of quantitative mathematical expressions. By scientific paradigm we mean a set of techniques and methods for interrogating nature and interpreting the relations between observed phenomena, which a theory then formalizes [1]. Analogue gravity is therefore not a theory but rather a paradigm: looking at known and understood physical phenomena of the various branches of physics, each already described by consolidated theories, analogue gravity consists in finding in these phenomena, and in their theoretical descriptions, (conceptual) relations of analogy.

In particular, analogue gravity focuses on the analogy between the fields that propagate through gravitational fields and the perturbations that propagate in continuous media that can be tested experimentally, media with well-studied and well-known behaviors. In this relation between the specific gravitational case and all the others, emphasis is placed on the first because it is the one which is particularly difficult to examine directly with experiments, while the systems with which the analogy is established are those in which experiments and measures can be made.

In this work we investigate the nature of the analogy between systems made with Bose–Einstein condensates and gravitational systems, with the aim of studying how the analogy can be extended, what deductions can be drawn and what perspectives open towards a broader understanding of gravity.

In particular, analogue models in which gravitational phenomena can be simulated consist of (classical or quantum) perturbations propagating in a medium, an analogue spacetime. The description of the medium is made in terms of the classical and continuum limit of a microscopic underlying structure, and it is therefore emergent from a more fundamental theory.

We therefore make a brief review of the principle of analogy and of the concept of emergence, to which we want to refer in the discussion that will follow, and which guided our research.

1.1 Analogy

From the point of view of logic, the relation of analogy is a symmetrical relation (or an equivalence) between objects: if there are A and B such that A is analogous to B, then B is analogous to A:

$$A \sim B \iff B \sim A\,, \tag{1.1}$$

indicating, in this section, with the symbol "$\sim$" the analogy relation, and with the arrow "$\Rightarrow$" the logical implication of necessity.

We are interested in the analogies between physical phenomena, in which a causal structure is typically present for which, in the study of physical theories, effects follow from causes: from the cause A follows the effect C,

$$\begin{array}{c} A \\ \Downarrow \\ C\,. \end{array} \tag{1.2}$$

The principle of analogy establishes that the relation of analogy is inherited in the causal succession of phenomena: if there are A, B and C such that A is analogous to B, and such that from A necessarily follows C, then there must be D such that C is analogous to D, and such that from B necessarily follows D:

$$\begin{array}{ccc} A & \sim & B \\ \Downarrow & & \Downarrow \\ C & \sim & D\,. \end{array} \tag{1.3}$$

This principle is applied in research fields where it is not possible to collect empirical evidence corroborating the theses. In quantitative science, for an analogy to be established between two systems, we require them to be described with the same formalism: if two physical systems allow predictive mathematical descriptions related to each other by a morphism, we can say that they are analogous.

While this definition can be applied in very simple cases (*e.g.* for algebraic proportions), analogies can be established in more complex theories. Typically, in physics we can establish analogies between theories that involve fields of the same nature (*e.g.*

discrete or differentiable, classical or quantum, scalar or tensorial). A few examples of applications of the principle of analogy are between systems like the spring, the uniform circular motion and the harmonic pendulum, or between quantum harmonic oscillators and quantum field theory.

Analogue gravity is a powerful example in which the principle of analogy is applied after observing that the equations describing the dynamics of perturbations in a continuous medium are the same as those describing fields in curved spacetime. Therefore they are physical systems analogous to each other by virtue of the mathematical analogy between the equations that describe them.

According to the principle of analogy, analogous descriptions lead to analogous predictions and realizations: the theoretical predictions made in one system must also apply to the other, the phenomena observed in one system must also be (analogously) observed in the other. This is true as long as we stay within regimes where the analogy remains valid.

To corroborate the analogy, one typically has to be in a certain regime of approximation in which two descriptions are analogous. When the conditions that validate the approximations are not verified, then the analogies between the consequent predictions must be considered unverified as well. But by loosening the analogy it is possible to extend it, going beyond the rigorous analytical and quantitative analogy, looking for a weaker (*e.g.* a qualitative) analogy.

In fact, it must be remarked that the principle of analogy follows the cause-effect connections only in the direction of the necessary implications. Even if C follows necessarily from A as in equation (1.2), it is not always true that C is sufficient for A to exist. If C were sufficient for A, then any D analogous to C would be sufficient for a B analogous to A (we would have simply applied the principle of analogy, just reversing causes and effects). But if C is not sufficient for A, then no D analogous to C can be sufficient for a B analogous to A, or there would be a contradiction:

$$\begin{array}{ccc} A & \sim & B \\ \not\Uparrow & & \not\Uparrow \\ C & \sim & D\,. \end{array} \tag{1.4}$$

For completeness, we add that if C is not sufficient for A, and there is a D analogous to C and sufficient for B, we can exclude that B is analogous to A, or there would be a contradiction:

$$\begin{array}{ccc} A & \nsim & B \\ \not\Uparrow & & \Uparrow \\ C & \sim & D\,. \end{array} \tag{1.5}$$

To apply the principle of analogy starting from the effects it is therefore necessary to find what sufficient implications can be defined, characterizing the causes in terms that are general enough.

It is with this approach that one can think of obtaining, from analogue models, hints to deepen the understanding of gravity and for which it is particularly worth considering the concept of emergence. Table-top experiments that provide models of analogue gravity — where the description which is analogous to field theory in curved spacetime emerges from an underlying microscopic theory — must be studied to understand whether the emergence of their description is a sufficient condition for making claims that can be applied, by analogy, to the gravitational case.

1.2 Emergence

In physics there is a constant tension between the attempt to analyze a system in terms of its components and the study of collective behaviors. The first is a reductionist approach, based on reducing a problem to the characteristics of the constituent elements of the system. The second is an approach based on emergence, that is the idea that even if a system is composed of constituent elements, on sufficiently large scales — with respect to which these constituents are microscopic — new collective properties emerge.

These different approaches to the problem of connecting different scales are not necessarily in contrast with each other, what ultimately changes is what they are focused on. In one case one is interested in phenomena occurring on shorter scales, in the other one is interested in collective phenomena that emerge at larger scales. In both cases, maintaining that the components have collective behaviors.

These emerging collective behaviors generally require a separate study, which leads to the formulation of a coherent theory, that only later can be explained in microscopic terms. An emblematic example is thermodynamics, a branch of physics that began about a century before the kinetic theory of gases was formulated, which allowed to explain the thermodynamics of gases in atomic terms. With the development of statistical physics it has become clear that thermodynamic relations are ultimately expressions that connect the thermodynamic variables of a system to ensemble averages evaluated in terms of its constituent elements.

In the case of analogue gravity realized with classical and quantum hydrodynamic systems — in this thesis we will privilege Bose–Einstein condensates — the models are described by theories of clear emergent nature. In the hydrodynamic formalism a mesoscopic description is employed, in which a fluid is treated on sufficiently large scales such that the individual description of its atomic or molecular elements is not necessary. But scales small enough with respect to the extension of the system, that it is possible to observe and study local phenomena such as the propagation of perturbations.

Therefore analogue gravity realized in hydrodynamic systems, as we will discuss in detail, studies perturbations in a continuous medium in which the notion of continuity is emergent, an approximation of an underlying structure composed of discrete objects. The definition of continuous differentiable fields to describe the theories used in the realization of such analogue models is an abstraction which however provides a very reliable approximate description of the physics of these systems.

The question that naturally arises is therefore whether spacetime itself can have an underlying discrete nature, *i.e.* whether general relativity — analogously to hydrodynamics — is a continuous (and differentiable) theory that emerges from a discrete theory of gravity, in particular a theory of quantum gravity. Speculations of this kind are not novel, both for what regards the emergence and as well as for the description of discretized spacetime.

However, in general, the existence of a hydrodynamic description (*i.e.* in the continuum and classical limit) is not a sufficient condition to state that there is an underlying description from which it emerges. Consequently, analogue gravity cannot prove that general relativity must emerge from a more fundamental theory of quantum gravity. However, there are various open physical issues pointing in the direction of quantum gravity. And so if a theory of quantum gravity really proved to be a valid description of the nature of spacetime, then it would be necessary that general relativity (and hence, in hierarchical order, also quantum field theory in curved spacetime and then semiclassical gravity) is a theory that emerges in a limit of collective and large-scale phenomena. And at the same time it would be necessary for the analogies between gravitational systems and analogue models made with quantum fluids to become more stringent.

1.3 Plan of the thesis

This thesis is structured in such a way as to initially introduce the descriptive tools used subsequently.

In chapter 2 we present the theory of Bose–Einstein condensates. In particular, we focus on discussing the number conserving formalism, which is the tool we use for a formulation of analogue gravity that keeps track of all the quantum degrees of freedom of the system. In chapter 3 we discuss gravity and its open issues, and how they can be addressed with analogue gravity. We discuss the formulation of analogue models made in classical and quantum hydrodynamic systems. We are particularly interested in condensates, where the condensate wavefunction is a classical background for the propagation of quantum fluctuations.

In chapter 4 we present a model of analogue cosmological particle creation in a condensate, which we studied by applying the number conserving formalism. By tracking how the system evolves, we discuss the differences that emerge from the standard mean-field description and how to get insights into the information loss problem. In chapter 5 we present the study of canonical analogue black holes, and we present how from purely quantum effects it is possible, in a condensate, to study the problem of the back-reaction of quantum fields on geometry. Furthermore, we show how the evaporation problem, which is typically a semi-classical gravity phenomenon, can be addressed with analogue gravity. Finally, in chapter 6 we address the mathematical problem of the realization of an analogue Kerr black hole and the physical problem of its realization in a Bose–Einstein condensate, a system in which it is necessary to include vorticity in the description to obtain an analogue model.

We conclude our study by discussing the scenarios that can open up to analogue gravity, assuming the quantum gravity paradigm.

2

Bose–Einstein condensates

2.1 Introduction to Bose–Einstein condensation

2.1.1 Bosonic systems and second quantization

In physics there are two kinds of quantum systems of identical particles: bosonic and fermionic. According with the spin-statistic theorem, bosons have integer spins and the wavefunction of a many-body bosonic system is totally symmetric, while fermions have half-integer spins and the wavefunction of a many-body fermionic system is totally antisymmetric [2–4]. The relation between spin and symmetry is ultimately due to how states transform under Lorentz transformations [5].

These many-body systems can be described with the second quantization formalism: the particles are assumed pointlike, and every state can be obtained from the vacuum through the action of the ladder operators, forming the Fock space to which the many-body state belong. Each state in the 1-particle Hilbert space corresponds to a creation operator acting in the Fock space, and its conjugate is a destruction operator. The action of a creation operator is such that it adds a new particle in the corresponding 1-particle state to the many-body state, while a destruction operator removes a particle from the many-body state if there was (at least) one occupying that 1-particle state, otherwise it annihilates the state. A complete basis for the Fock space is given in terms of the occupation numbers of the 1-particle states occupied by the particles: for any complete basis of the 1-particle Hilbert space, the set of many-body states with all the possible combinations of occupation numbers relative to the states of the 1-particle basis is a basis of the Fock space.

The symmetry or antisymmetry of the wavefunctions is included in the construction of the Fock space by setting the appropriate commutation relations between the ladder operators: bosonic operators have commutative relations, while fermionic operators have anticommutative relations. These commutation relations have immediate consequence in the fact that the number of bosons that can occupy the same 1-particle state is unlimited, while any 1-particle fermionic state can be only empty or occupied (it is the Pauli exclusion principle).

We shall focus here only on bosons, where the possibility to occupy a single state with an arbitrary number of particles leads to the concept of condensation. Bose and Einstein first described condensation considering the statistical distribution of non-interacting identical particles at equilibrium at finite temperature: the particles of the system occupy the excited states following a distribution that for high energy approaches asymptotically the Boltzmann distribution, and diverges for energy equal to that of the ground state.

In systems at thermodynamic equilibrium and with fixed number of particles, the occupation number distribution predicts which fraction of the particles occupy the excited states, and the remaining must all occupy the 1-particle ground state. When the ground state is macroscopically occupied the system is in the phase known as Bose–Einstein condensation.

With this formalism it is possible not only to describe fundamental particles — originally Bose was interested in a fully quantum approach to describe the black-body radiation of photons — but it applies in good approximation to cold atoms: even if atoms are composite and extended objects themselves, when they are in dilute systems, weakly interacting atoms are most likely to be all in their spin-0 ground state and approximately behave as point-like spin-0 bosons.

2.1.2 Condensates in statistical physics

In statistical physics the description of many-particle systems requires the concept of ensemble: the system is in a statistical superposition of independent microstates — corresponding to a basis of many-particle states of the Fock space — each with a different statistical weight. The ensemble therefore defines a density matrix, containing all the information. The density matrix is a Hermitian operator that is diagonalized by the basis of its orthonormal eigenstates

$$\rho = \sum_\sigma \rho_\sigma \Big|\sigma\Big\rangle\Big\langle\sigma\Big| \,, \tag{2.1}$$

where each $|\sigma\rangle$ is a many-body eigenstate and ρ_σ is its statistical weight; the weights are normalized $\sum_\sigma \rho_\sigma = 1$. Statistical predictions of the physical observables are made by taking the trace over the microstates of the product of the density matrix and the second-quantized Hermitian operators associated with the observables of interest. For a generic operator O the statistical average is

$$\Big\langle O \Big\rangle = \mathrm{Tr}\rho O = \sum_\sigma \Big\langle \sigma \Big| \rho O \Big| \sigma \Big\rangle = \sum_\sigma \rho_\sigma \Big\langle \sigma \Big| O \Big| \sigma \Big\rangle \,. \tag{2.2}$$

At equilibrium — the condition at which the appropriate free energy, depending on what the system can exchange with the environment, is minimized — the ensemble has a specific statistical distribution: the normalized weight of each microstate is the exponential of the product of the inverse temperature ($\beta = 1/T$) and the difference between the free energy $\langle G\rangle - TS$ and the corresponding generator G evaluated on that microstate

$$\langle G \rangle - TS = -\frac{1}{\beta} \ln \sum_\sigma e^{-\beta \langle \sigma | G | \sigma \rangle} \,, \tag{2.3}$$

$$\rho_\sigma = e^{-\beta(\langle \sigma | G | \sigma \rangle - \langle G \rangle + TS)} \,, \tag{2.4}$$

where S is the entropy of the ensemble. Therefore the weight of each microstate depends only on the values of the global observables, being them intensive (the temperature T, the pressure P, the chemical potential μ) or extensive (the internal energy $U = \langle H \rangle$, the volume V, the total number of particles N): microstates associated with the same global quantities have equal weight, as prescribed by the ergodic hypothesis. As an example, the canonical ensemble describes systems at equilibrium that have a fixed number of particles; its generator is simply the Hamiltonian operator H and its free energy is the Helmholtz free energy, *i.e.* the difference between the internal energy and the product of entropy and temperature $\langle H \rangle - TS$.

In terms of statistical physics, a bosonic system at equilibrium displays condensation when the temperature is extremely low and reaches the critical temperature defined in terms of the particle number density, at which the particles are forced to occupy the 1-particle ground state. A well known calculation that shows this is for the occupation numbers in the grand canonical ensemble of a non-interacting bosonic gas. In the grand canonical ensemble the free energy is given by the grand potential Φ, which

$$\Phi = \langle H \rangle - \mu \langle N \rangle - TS \,, \tag{2.5}$$

and therefore it is simply $\Phi = -PV$. Considering non-interacting particles, each microstate σ has an energy which is the sum of the energies ϵ_i of the 1-particle state labeled with index i, and each 1-particle state is occupied by a number n_i of particles. The grand canonical ensemble differs from the canonical ensemble by not having constraints on the total number of particles, but in the thermodynamical limit, at equilibrium, they must give the same thermodynamical predictions.

$$e^{\beta \langle P \rangle V} = \sum_\sigma e^{-\beta \sum_i (\epsilon_i - \mu) n_i} \,, \tag{2.6}$$

$$\langle N \rangle = \frac{1}{\beta} \partial_\mu \sum_\sigma e^{-\beta \left(\sum_i (\epsilon_i - \mu) n_i + \langle P \rangle V \right)} = \tag{2.7}$$

$$= \frac{1}{\beta} \partial_\mu e^{-\beta \langle P \rangle V} \prod_i \sum_{n_i = 0}^{\infty} e^{-\beta(\epsilon_i - \mu) n_i} = \tag{2.8}$$

$$= \langle N_0 \rangle - \frac{1}{\beta} \sum_i \partial_\mu \ln \left(1 - e^{-\beta(\epsilon_i - \mu)} \right) = \tag{2.9}$$

$$= \langle N_0 \rangle + \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \,, \tag{2.10}$$

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \,. \tag{2.11}$$

Observe that the step between (2.8) and (2.9) can be performed only when considering states with energy $\epsilon_i > \mu$. The 1-particle ground state must be treated separately, and it contributes to the total number of particles $\langle N \rangle$ separately, with the total number of particles in the 1-particle ground state being $\langle N_0 \rangle$. The state with minimum energy $\epsilon_0 = \mu$ must be macroscopically occupied if the sum of the other occupation numbers gives a total which is less than the number of particles in the system, therefore resulting in condensation.

The occupation number of each excited 1-particle state is given by the bosonic distribution (2.11), and depends on the difference between its energy ϵ_i and the chemical potential μ, *i.e.* the energy of the ground state, and the inverse temperature β.

The expected occupation numbers $\langle n_i \rangle$ are strictly decreasing with decreasing temperature, as it is clear considering the dependence on the inverse temperature β. Therefore the lower the temperature, the smaller will be the number of excited particles. This is a corroborating element to the assumption that this description can be applied to cold atoms, approximated to pointlike bosons: at low temperatures the atoms are most likely in their spin-0 ground state, with respect to their internal degrees of freedom, and at the same time they are most likely to be in the state of zero total momentum, with respect to their kinetic degrees of freedom.

The total number of particles predicted to occupy the excited states, which we call $\langle N_{\text{exc}} \rangle$, in the case of a non-relativistic and non-interacting massive boson is

$$\langle N_{\text{exc}} \rangle = \langle N \rangle - \langle N_0 \rangle \ , \tag{2.12}$$

$$\langle N_{\text{exc}} \rangle = V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon(k)-\mu)} - 1} = \tag{2.13}$$

$$= V \int \frac{\mathrm{d}k}{2\pi^2} \frac{k^2}{e^{\beta \frac{k^2}{2m}} - 1} = \tag{2.14}$$

$$= V \left(\frac{m}{2\pi} \frac{1}{\beta} \right)^{\frac{3}{2}} \zeta \left(\frac{3}{2} \right) \ , \tag{2.15}$$

where ζ is the Riemann zeta function. This implies that for a system with a fixed particle number density $\langle N \rangle / V$, calculated with respect to the total number of particles in the system, the critical temperature is set at

$$T_{\text{crit}} = \frac{2\pi}{m k_B} \left(\frac{1}{\zeta\left(\frac{3}{2}\right)} \frac{\langle N \rangle}{V} \right)^{\frac{2}{3}} \ . \tag{2.16}$$

Keeping the density fixed, for temperatures lower than T_{crit}, there must be condensation, and the lower the temperature, the smaller the fraction of particles in the excited part.

In the context of statistical physics the phenomenon of condensation therefore depends on the temperature and the dispersion relation for the energy ϵ_i of the 1-particle state.

Photons, such as ultra-relativistic massive particles, show an appreciably different behavior given by the different dispersion relation. For this particles, the energy of the

1-particle state with momentum k is $\epsilon_k = c\,|k|$, where c is the speed of light. The number occupation distribution is still given by equation (2.11), with $\mu = 0$ and with this linear dispersion relation. This distribution is known as, in the case of photons, the black body distribution, and describes the radiation emitted at thermal equilibrium with a black body, the ideal emitter. We will see that this distribution is the same predicted in the case of Hawking radiation.

2.2 Mean-field approximation

2.2.1 The mean-field approximation and the Gross–Pitaevskii equation

A many-body system is a condensate when a macroscopic number of particles is in the same 1-particle state. When this condition is verified, we can study in detail the system through the condensate wavefunction. It should effectively be considered as the macroscopically occupied 1-particle state, rescaled to match the total number of particles in the system.

We consider a system of cold-atoms and we study it as a non-relativistic massive bosonic field, with an ultralocal two-body interaction and an external potential. This is the system we have considered in our works, and it is most commonly employed in analogue gravity.

The Hamiltonian operator of this system can be written as

$$H = \int \mathrm{d}x \left[\phi^\dagger(x)\left(-\frac{\nabla^2}{2m}\phi(x)\right) + \frac{\lambda}{2}\phi^\dagger(x)\,\phi^\dagger(x)\,\phi(x)\,\phi(x) + V_{\text{ext}}(x)\,\phi^\dagger(x)\,\phi(x)\right], \tag{2.17}$$

where we dropped for notational convenience the time dependence of the bosonic field operator ϕ. Considering the usual commutation relation for the field operators — which in general in non-relativistic second quantization are effectively the ladder operator creating or destroying a particle in a point — we can derive the equation of motion of the field from the Hamiltonian

$$\left[\phi(x)\,,\phi^\dagger(y)\right] = \delta(x,y)\ , \tag{2.18}$$

$$\left[\phi(x)\,,\phi(y)\right] = 0\,, \tag{2.19}$$

$$i\partial_t\phi(x) = \left[\phi(x)\,,H\right] = -\frac{\nabla^2}{2m}\phi(x) + \lambda\phi^\dagger(x)\,\phi(x)\,\phi(x) + V_{\text{ext}}\phi(x)\ . \tag{2.20}$$

In equation (2.20) the parameter m is the mass of the atom and λ is the interaction strength, proportional to the scattering length, and can generally be taken as time-dependent.

In the mean-field approximation we assume that the field operator can be replaced with a classical function $\phi(x) \to \langle\phi_0(x)\rangle$. Applying this substitution in the equation of motion of the field, we get that $\langle\phi_0(x)\rangle$ is therefore the solution of the equation

$$i\partial_t\langle\phi_0(x)\rangle = -\frac{\nabla^2}{2m}\langle\phi_0(x)\rangle + \lambda\overline{\langle\phi_0(x)\rangle}\,\langle\phi_0(x)\rangle\,\langle\phi_0(x)\rangle + V_{\text{ext}}(x)\,\langle\phi_0(x)\rangle\ , \tag{2.21}$$

which is known as the Gross–Pitaevskii equation [4], and it is the most commonly adopted expression to approximate the dynamics of the condensate wavefunction. The initial conditions are set to match the physical requirements and constraints; *e.g.* in the physics of Bose–Einstein condensates typically one searches for the solution with a fixed normalization which minimizes the energy density.

When the solution of equation (2.21) is an L^2 square-integrable function, its norm is constant and it is identified with the total number of atoms in the system

$$\int \mathrm{d}x \, |\langle \phi_0 (x) \rangle|^2 = \langle N \rangle \ . \tag{2.22}$$

The integral over a generic domain $\mathcal{D}$ of the squared modulus of the wavefunction gives the total number of atoms contained in that domain, and the variation in time of that number is given by the flux of atoms across the boundary of that domain. When the domain considered is the entire space, *i.e.* the boundary is a surface where the flux in equation (2.24) must vanish, we obtain that the number $\langle N \rangle$ must be constant

$$\partial_t \int_{\mathcal{D}} \mathrm{d}x \, |\langle \phi_0 (x) \rangle|^2 = i \int_{\mathcal{D}} \mathrm{d}x \left[\left(-\frac{\nabla^2}{2m} \overline{\langle \phi_0 (x) \rangle} \right) \langle \phi_0 (x) \rangle - \overline{\langle \phi_0 (x) \rangle} \left(-\frac{\nabla^2}{2m} \langle \phi_0 (x) \rangle \right) \right] = \tag{2.23}$$

$$= -\frac{i}{2m} \int_{\partial \mathcal{D}} \mathrm{d}\Sigma \cdot \left[\left(\nabla \overline{\langle \phi_0 (x) \rangle} \right) \langle \phi_0 (x) \rangle - \overline{\langle \phi_0 (x) \rangle} \left(\nabla \langle \phi_0 (x) \rangle \right) \right] , \tag{2.24}$$

$$\Downarrow$$

$$\partial_t \int \mathrm{d}x \, |\langle \phi_0 (x) \rangle|^2 = 0 \, . \tag{2.25}$$

Here we have considered the solution of the Gross–Pitaevskii equation to completely describe the condensate, therefore the conservation of the total number; taking into account the perturbations will introduce depletion.

2.2.2 Mean-field and coherent states

The Gross–Pitaevskii equation (2.21) can be directly related to the full equation of motion (2.20) by taking the expectation value of both sides. Let us then discuss the conditions under which the field operator ϕ can be replaced with its expectation value $\langle \phi \rangle$, and when the expectation value itself can be approximated with the solution of the Gross–Pitaevskii equation (2.21), *i.e.* with the classical function $\langle \phi_0 \rangle$.

The mean-field approximation is an intuitive way to restrict the description only to the condensate wavefunction, but not all many-body states allow to replace consistently the field operator with a classical function. Consider the expectation value of the equation of motion (2.20). It always holds exactly that

$$i \partial_t \langle \phi (x) \rangle = -\frac{\nabla^2}{2m} \langle \phi (x) \rangle + \lambda \left\langle \phi^\dagger (x) \, \phi (x) \, \phi (x) \right\rangle + V_{\mathrm{ext}} (x) \langle \phi (x) \rangle \ . \tag{2.26}$$

However this equation is not a closed equation for the expectation value of the field operator, *i.e.* for the mean-field, since it includes the expectation value $\langle \phi^\dagger(x)\,\phi(x)\,\phi(x)\rangle$. One can obtain a closed equation only if it is possible to approximate the expectation value of every normal ordered composition of field operators with the corresponding composition of mean-field functions, *e.g.* having

$$\left\langle \phi^\dagger(x)\,\phi(x)\,\phi(x)\right\rangle \approx \overline{\langle\phi(x)\rangle}\,\langle\phi(x)\rangle\,\langle\phi(x)\rangle\ . \tag{2.27}$$

This approximation holds exactly for a wide class of states in the Fock space: the coherent states. A coherent state is such that it is the eigenstate of a destruction operator. For any normalized 1-particle state ψ and any total number of particles N, it is possible to obtain a coherent state such that it is an eigenstate of a destruction operator with a specific eigenvalue, and it is annihilated by any other destruction operator commuting with the former

$$a_\psi\left|coh_{\sqrt{N}\psi}\right\rangle = \sqrt{N}\left|coh_{\sqrt{N}\psi}\right\rangle, \tag{2.28}$$

$$\Downarrow$$

$$\phi(x)\left|coh_{\sqrt{N}\psi}\right\rangle = \sqrt{N}\psi(x)\left|coh_{\sqrt{N}\psi}\right\rangle. \tag{2.29}$$

Knowing the commutation relations between creation and destruction operators, it is easy to show that a coherent state can be defined as the action of the exponential of the creation operator times the eigenvalue $\sqrt{N}$ applied to the vacuum state $|\emptyset\rangle$, the state in the Fock space in which no 1-particle state is occupied

$$\left[a_\psi, a_\psi^\dagger\right] = 1\,, \tag{2.30}$$

$$\left|coh_{\sqrt{N}\psi}\right\rangle = e^{\sqrt{N}a_\psi^\dagger}\left|\emptyset\right\rangle, \tag{2.31}$$

$$a_\psi\left|coh_{\sqrt{N}\psi}\right\rangle = a_\psi\sum_{i=0}^{\infty}\frac{\sqrt{N}^i a_\psi^{\dagger i}}{i!}\left|\emptyset\right\rangle = \sum_{i=1}^{\infty}\frac{\sqrt{N}^i a_{\psi N}^{\dagger(i-1)}}{(i-1)!}\left|\emptyset\right\rangle = \sqrt{N}\left|coh_{\sqrt{N}\psi}\right\rangle. \tag{2.32}$$

It follows that a coherent state is the quantum superposition of infinite many-body states: all the states with the only 1-particle state ψ occupied, but with every possible occupation number. All these states are orthogonal to each other. So we remark that coherent states do not represent states with a fixed number of particles, but require an elaborate quantum superposition. This is a related but different condition from the previously mentioned grand canonical ensemble, where we have considered systems that can exchange particles with the environment: in such thermodynamical systems the superposition is statistical, not necessarily quantum. However, the two formalisms could be related to each other more strictly, as the coherent states do provide a basis for the Fock space and are used in statistical physics in the context of Bose–Einstein condensation.

The mean-field approximation is therefore best applicable when the system is in a coherent state, and from equation (2.20) it immediately follows that

$$\phi\left(x\right)\left|coh_{\langle\phi\rangle}\right\rangle = \left\langle\phi\left(x\right)\right\rangle\left|coh_{\langle\phi\rangle}\right\rangle , \tag{2.33}$$

$$\left\langle coh_{\langle\phi\rangle}\right|\phi^{\dagger}\left(x\right) = \overline{\left\langle\phi\left(x\right)\right\rangle}\left\langle coh_{\langle\phi\rangle}\right| , \tag{2.34}$$

$$\Downarrow$$

$$\left\langle coh_{\langle\phi\rangle}\right|\phi^{\dagger}\left(x\right)\phi\left(x\right)\phi\left(x\right)\left|coh_{\langle\phi\rangle}\right\rangle = \overline{\left\langle\phi\left(x\right)\right\rangle}\left\langle\phi\left(x\right)\right\rangle\left\langle\phi\left(x\right)\right\rangle . \tag{2.35}$$

And we verify that, in this case, equation (2.27) would be satisfied exactly.

It is important to remark that even assuming that the field operator can be replaced by a classical function, the equivalence between the expectation value of the field and the condensate wavefunction should not be taken for granted. As we have seen, it holds when the state is coherent, which is a quantum superposition of states with every possible occupation number; when a system is in a state with an exact number of particles, *i.e.* an eigenstate of the total number operator, the expectation value of the field — a ladder operator that changes by one the total number of particles in a state — would vanish.

Moreover, the approximation to a coherent state can be exact at a single moment of time, but the evolution of the system will inevitably spoil this approximation because of the many-body interaction term, which contains a creation operator: there are no eigenstates of a creation operator in the Fock space (only in its dual space), therefore the operator on the RHS of equation (2.20) does not admit many-body states of the kind in equation (2.31) as eigenstates. The dynamical evolution of the system results in not being verified that every normal ordered composite operators can be reduced to the corresponding composition of expectation values of the field operators, and the many-point correlation functions of the system increasingly become less trivial.

Nevertheless, we will show that, following the evolution of a generic system, the mean-field approximation can still be a good approximation of the condensate wavefunction in the regime of small perturbations.

2.3 Bogoliubov description and quasi-particles

2.3.1 The Bogoliubov–de Gennes equation

The mean-field approximation is an effective description of the condensate wavefunction, but this description cannot include the quantum excitations of the system. To describe the excitation of the condensate, one uses the Bogoliubov description.

Previously we have considered the replacement of the field operator ϕ with the mean-field $\langle\phi\rangle$, which focuses only on the dynamics of condensate wavefunction. An approach that would enable to study more completely the system, while being consistent with the mean-field description, would be to consider the field as the sum of the mean-field and of its quantum perturbation. The field operator and its conjugate therefore take the form

$$\phi(x) = \langle\phi(x)\rangle + \delta\phi(x) \ , \tag{2.36}$$

$$\phi^\dagger(x) = \overline{\langle\phi(x)\rangle} + \delta\phi^\dagger(x) \ . \tag{2.37}$$

The field operator and its conjugate are therefore redefined by translation. However this redefinition could also be interpreted as splitting each operator in two parts, the condensate and the quantum fluctuation, where the condensate is described by the classical mean-field function. One should therefore recall the previous discussion reminding that this formalism would be best suited for the description of the perturbation of a coherent state. In general, it would be better not to lose track of the quantum nature of the bosonic particles in the condensate.

In this description all the quantum features of the system are assumed to be retained by the quantum perturbation; the operators $\delta\phi$ and $\delta\phi^\dagger$ are effectively the ladder operators of the quantum perturbations over the condensate wavefunction, describing a new Fock space that is a deformation of the initial one. These operators inherit the commutation relation from the initial bosonic field

$$\left[\delta\phi(x), \delta\phi^\dagger(y)\right] = \delta(x,y) \ , \tag{2.38}$$

$$[\delta\phi(x), \delta\phi(y)] = 0 \, . \tag{2.39}$$

While one might want to consider these operators as describing the excited atoms, it would not be a completely accurate description: these new ladder operators describe another Fock space, the space of the quantum perturbations of the condensate wavefunction. It is a complete space where the subspace associated with perturbations proportional to the condensate wavefunction are not projected away.

But the order of magnitude of the perturbation operators $\delta\phi$ and $\delta\phi^\dagger$ in any correlation function is of order of unity, inheriting this property from the excited atom states, with occupation numbers n_i of the order of unity, as one would argue from equation (2.11): under the hypothesis of condensation, only the condensed 1-particle state described by $\langle\phi_0\rangle$ is macroscopically occupied. Therefore, while the square modulus of the condensate wavefunction $|\langle\phi_0(x)\rangle|^2$ must be approximately equal to the expectation value of the local number density operator $\left\langle\phi^\dagger(x)\,\phi(x)\right\rangle$, the corrections due to the quantum perturbations must be of the order of unity.

Moreover, for consistency with the definition of mean-field as the expectation value of the field operator, the expectation value of the quantum perturbation (and also the expectation value of its conjugate) must vanish, giving:

$$\langle\delta\phi(x)\rangle = 0 \, , \tag{2.40}$$

$$\left\langle\phi^\dagger(x)\,\phi(y)\right\rangle = \overline{\langle\phi_0(x)\rangle}\,\langle\phi_0(y)\rangle + \left\langle\delta\phi^\dagger(x)\,\delta\phi(y)\right\rangle = \tag{2.41}$$

$$= \overline{\langle\phi_0(x)\rangle}\,\langle\phi_0(y)\rangle\left(1 + \mathcal{O}\left(\frac{1}{\langle N\rangle}\right)\right) \ . \tag{2.42}$$

Under the redefinition of the field operators, the equation of motion can be expressed in a new form, by substitution of the expressions (2.36)–(2.37) into equation (2.20). We can then take the expectation value of the equation of motion, obtaining two different equations: one classical equation for the mean-field and, by subtraction, one operator equation for the quantum fluctuation

$$\begin{aligned} i\partial_t \langle\phi\rangle &= -\frac{\nabla^2}{2m}\langle\phi\rangle + \lambda\overline{\langle\phi\rangle}\langle\phi\rangle\langle\phi\rangle + V_{\text{ext}}\langle\phi\rangle + \\ &+ \lambda\overline{\langle\phi\rangle}\langle\delta\phi\delta\phi\rangle + 2\lambda\langle\phi\rangle\left\langle\delta\phi^\dagger\delta\phi\right\rangle + \lambda\left\langle\delta\phi^\dagger\delta\phi\delta\phi\right\rangle , \qquad (2.43) \\ i\partial_t\delta\phi &= -\frac{\nabla^2}{2m}\delta\phi(x) + \lambda\langle\phi\rangle\langle\phi\rangle\delta\phi^\dagger + 2\lambda\overline{\langle\phi\rangle}\langle\phi\rangle\delta\phi + V_{\text{ext}}\delta\phi + \\ &+ 2\lambda\langle\phi\rangle\left(\delta\phi^\dagger\delta\phi - \left\langle\delta\phi^\dagger\delta\phi\right\rangle\right) + \lambda\overline{\langle\phi\rangle}\left(\delta\phi\delta\phi - \langle\delta\phi\delta\phi\rangle\right) + \\ &+ \lambda\left(\delta\phi^\dagger\delta\phi\delta\phi - \left\langle\delta\phi^\dagger\delta\phi\delta\phi\right\rangle\right) . \qquad (2.44) \end{aligned}$$

We have dropped the dependence from the coordinates, for convenience, but all the terms in the mean-field and the quantum perturbation should be assumed as evaluated at the same point x.

We can recognize the first line in equation (2.43) as the Gross–Pitaevskii equation, while the second line includes new terms: we will refer to them as the back-reaction. We will focus in particular on the first two terms, which are quadratic in the quantum perturbation $\delta\phi$.

The equation (2.44) for the dynamics of the quantum perturbation is more articulate, and its various lines include many-body interaction terms inherited by the initial equation (2.20).

The approach developed by Bogoliubov consists in considering, for the dynamics of the quantum perturbation, only the terms linear in the operators $\delta\phi$ and $\delta\phi^\dagger$, *i.e.* consists in retaining only the first line in equation (2.44) and neglecting the others. The non-linear terms are small contributions safely assumed to be negligible, since every operator $\delta\phi$ and $\delta\phi^\dagger$ keeps the correlation functions in which they appear in place of mean-field term smaller by an $\sqrt{\langle N\rangle}$, as discussed previously.

The Bogoliubov description is therefore formalized in terms of the two equations obtained from equations (2.43)–(2.44) by not including the back-reaction of the quantum perturbation on the mean-field, and keeping only the linear terms in the dynamics of the perturbation itself:

$$i\partial_t \langle\phi_0\rangle = -\frac{\nabla^2}{2m}\langle\phi_0\rangle + \lambda\overline{\langle\phi_0\rangle}\langle\phi_0\rangle\langle\phi_0\rangle + V_{\text{ext}}\langle\phi_0\rangle , \qquad (2.45)$$

$$i\partial_t\delta\phi = -\frac{\nabla^2}{2m}\delta\phi + 2\lambda\overline{\langle\phi_0\rangle}\langle\phi_0\rangle\delta\phi + \lambda\langle\phi_0\rangle\langle\phi_0\rangle\delta\phi^\dagger + V_{\text{ext}}\delta\phi . \qquad (2.46)$$

The first equation is again the Gross–Pitaevskii equation which, as we have described before, is a closed differential equation for a complex classical functions that approximates the condensate wavefunction within the mean-field description. We use a notation with which we distinguish the exact mean-field function $\langle\phi\rangle$ from the solution of the Gross–Pitaevskii equation $\langle\phi_0\rangle$ that approximates it.

The equation (2.46) for the quantum perturbation is known as the Bogoliubov–de Gennes equation. It is a linear differential operator equation for $\delta\phi$ and $\delta\phi^\dagger$, and in principle it is possible to diagonalize it combining it with its conjugate equation, finding the propagating modes and their dispersion relations. This means that it is possible to find a transformation of the operators $\delta\phi$ and $\delta\phi^\dagger$ — which are sufficient to generate a basis of the Fock space — to a different set of operators, with two properties: this new set of operators is itself sufficient to generate a basis of the Fock; the dynamics of each of these operators is described by a separate closed linear differential equation.

These new propagating modes are the quasi-particles, which, by previous definition, would propagate as non-interacting particles in the new effective theory defined by the Bogoliubov–de Gennes equation over the background defined by the solution of the Gross–Pitaevskii equation.

In the case of a homogeneous condensate, it is possible to study in detail the Bogoliubov–de Gennes equation, and the analysis of the quasi-particles is well established. The condensate wavefunction is assumed to change in time through a linear phase, and its number density is a constant and homogeneous real number

$$i\partial_t \langle\phi_0\rangle = \varpi \langle\phi_0\rangle \ , \tag{2.47}$$

$$|\langle\phi_0\rangle|^2 = \langle\rho_0\rangle \ , \tag{2.48}$$

where the frequency ϖ must be equal to $\varpi = \lambda\langle\rho_0\rangle + V_{\text{ext}}$, as can be easily derived from the Gross–Pitaevskii equation under the assumption of homogeneity and the dependence of its solution on the time coordinate only.

The Bogoliubov–de Gennes equation (2.46) becomes

$$(i\partial_t - \varpi)\,\delta\phi = -\frac{\nabla^2}{2m}\delta\phi + \lambda\langle\rho_0\rangle\,\delta\phi + \lambda\langle\rho_0\rangle\,\delta\phi^\dagger \,. \tag{2.49}$$

which can be Fourier transformed in space and time, to give

$$\omega\delta\phi_{\omega k} = \frac{k^2}{2m}\delta\phi_{\omega k} + \lambda\langle\rho_0\rangle\,\delta\phi_{\omega k} + \lambda\langle\rho_0\rangle\,\delta\phi^\dagger_{-\omega -k}\,, \tag{2.50}$$

where we have considered the eigenvalue of the time derivative to be $-i(\omega+\varpi)$, translating the energy axis, as the constant ϖ is, in fact, of arbitrary definition.

The equation (2.50) can be studied to find which are the propagating modes that diagonalize the dynamics, the Bogoliubov quasi-particles.

Before diagonalizing the equation, it is already possible to derive the dispersion relation of the quasi-particles. By considering equation (2.50) and its conjugate (appropriately changing the signs of the Fourier variables ω and k), we obtain an equation for the operator $\delta\phi_{\omega k}$ only, of second degree with respect to the energy ω. The resulting dispersion relation (considering the positive energies) is therefore

$$\omega(k) = \sqrt{\frac{\lambda\langle\rho_0\rangle}{m}k^2}\sqrt{1+\frac{k^2}{4\lambda\langle\rho_0\rangle m}} = \tag{2.51}$$

$$= c_s|k|\sqrt{1+\frac{1}{4}\xi^2k^2} = \tag{2.52}$$

$$= c_s|k|\left(1+\mathcal{O}\left(k^2\right)\right) . \tag{2.53}$$

The dispersion relation is approximately linear for small momenta, and resembles the dispersion relation of photons in vacuum. For higher momenta the linearity is broken, as the terms of higher order in the momentum k cannot be neglected anymore. We have introduced two expressions, c_s and ξ, which are parameters that allow to express the physical scales of the problem

$$c_s = \sqrt{\frac{\lambda\langle\rho_0\rangle}{m}} , \tag{2.54}$$

$$\xi = \frac{1}{\sqrt{\lambda\langle\rho_0\rangle m}} . \tag{2.55}$$

Here c_s is the speed of sound in the condensate, which provides the relation between lengths and time scales in the low-momentum linear dispersion relation in equation(2.53); ξ is the healing length, and represents a minimum length below which the linearity of the dispersion relation would not be a good approximation, implying that at sufficiently high momenta the quasi-particles are affected by the underlying atomic structure. Recalling the comparison with the linear dispersion relation of photons, in that case a modified dispersion relation would imply a Lorentz symmetry breaking at a short length scale, a phenomenon that one might expect at the Planck length scale should the spacetime have an underlying microscopic structure.

The equation (2.50) is diagonalized looking for the operators obtained as linear combinations of the ladder operators that preserve the canonical commutation relations

$$\gamma_{\omega(k)k} = u_k\delta\phi_{\omega(k)k} + v_k\delta\phi^\dagger_{-\omega(k)-k} , \tag{2.56}$$

$$\left[\gamma_k, \gamma^\dagger_{k'}\right] = \delta_{k,k'} , \tag{2.57}$$

$$\left[\gamma_k, \gamma_{k'}\right] = 0 , \tag{2.58}$$

where the commutation relations imply that the parameters u_k and v_k must satisfy a normalization constraint

$$1 = u_k^2 - v_k^2 , \tag{2.59}$$

$$\Downarrow$$

$$u_k = \cosh\Theta_k , \tag{2.60}$$

$$v_k = \sinh\Theta_k , \tag{2.61}$$

where here the use of hyperbolic functions with respect to some transformation parameter Θ_k is mentioned just for its practical convenience in the step by step calculations made to obtain the final result. This class of transformations is known as the Bogoliubov transformations, were the initial ladder operators of both types, creation and destruction operators, are combined into a new set of ladder operators.

Substituting the expression in equation (2.56) into equation (2.50) and imposing the diagonalization one obtains the coefficients of the Bogoliubov transformations

$$u_k^2 = \frac{1}{2}\left(\frac{\frac{k^2}{2m} + \lambda\left\langle\rho_0\right\rangle}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda\left\langle\rho_0\right\rangle\right)}} + 1\right) , \tag{2.62}$$

$$v_k^2 = \frac{1}{2}\left(\frac{\frac{k^2}{2m} + \lambda\left\langle\rho_0\right\rangle}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda\left\langle\rho_0\right\rangle\right)}} - 1\right) . \tag{2.63}$$

The case of the homogeneous condensate is particularly simple, but explicative: the dispersion relation we have found, linear for small momenta, is at the basis of the idea of analogue gravity in Bose–Einstein condensates. In this linear regime the speed of sound for the propagation of the Bogoliubov quasi-particles in the condensate is analogous to the speed of light in the propagation of photons in empty space. This behavior is the first suggestion that with Bose–Einstein condensates it is possible to simulate the phenomenology of massless fields in spacetime.

This case is peculiar for having both speed of sound and healing length which are homogeneous and constant, but from their definitions in equations (2.54)–(2.55) we see that, in the case of a less trivial coupling λ and number density $\langle\rho_0\rangle$, they could depend on both space and time.

2.3.2 The back-reaction of the Bogoliubov quasi-particles on the condensate wavefunction

In the Bogoliubov description, as seen previously, the standard approach is to consider the mean-field as evolving separately from the quantum perturbation. The two equations that describe their dynamics are hierarchically ordered and are not studied at the same time. At first, one must find a solution of the Gross–Pitaevskii equation (2.21), providing an approximate expression for the mean-field, and only afterwards it is possible to take the solution thus obtained and plug it into the Bogoliubov–de Gennes equation (2.46) as a background for the dynamics of the Bogoliubov quasi-particles.

But the description can be extended to include the back-reaction of the perturbation on the condensate wavefunction. We have seen how the Gross–Pitaevskii equation is obtained from the full dynamical equation after the field translation, as described in equation (2.43), by considering only the terms of leading order.

But we can include more terms in the description. If we keep the terms which are quadratic in the perturbation, usually called the anomalous terms [6], we can account for their back-reaction on the condensate wavefunction as they appear in equation (2.43)

$$n = \frac{\left\langle \delta\phi^\dagger \delta\phi \right\rangle}{\overline{\langle \phi \rangle} \langle \phi \rangle}\,, \tag{2.64}$$

$$m = m_R + i m_I = \frac{\langle \delta\phi \delta\phi \rangle}{\langle \phi \rangle \langle \phi \rangle}\,. \tag{2.65}$$

The anomalous density n is strictly real, while the anomalous mass m has both real and imaginary part. We have chosen these instead of other expressions because their adimensionality makes clearer the perturbative analysis we will present later, as the order of magnitude of each of these terms is of order $\mathcal{O}(\langle N \rangle^{-1})$.

The other back-reaction term which appears in equation (2.43) is a cubic composition of perturbation operators, which we can neglect with very small loss of generality. It is reasonable to assume that the cubic term is identically vanishing because, when the dynamics of the quantum perturbation is approximately linear, the many-points correlation functions such as $\left\langle \delta\phi^\dagger \delta\phi \delta\phi \right\rangle$ are just the multilinear propagation of the initial conditions. By definition no many-point interactions contribute to a linear dynamics and if the many point correlation functions can be set to zero initially, they remain zero in time.

The two equations resulting from considering the linear dynamics of the perturbation and the quadratic back-reaction described by the anomalous mass and density are therefore a modified Gross–Pitaevskii equation coupled to a Bogoliubov–de Gennes equation where the role of the mean-field is not played by the solution of the original Gross–Pitaevskii equation, calculated separately beforehand, but by the solution of the modified equation

$$i\partial_t \langle \phi \rangle = -\frac{\nabla^2}{2m} \langle \phi \rangle + \lambda \overline{\langle \phi \rangle} \langle \phi \rangle \langle \phi \rangle (1 + 2n + m) + V_{\text{ext}} \langle \phi \rangle\,, \tag{2.66}$$

$$i\partial_t \delta\phi = -\frac{\nabla^2}{2m} \delta\phi + 2\lambda \overline{\langle \phi \rangle} \langle \phi \rangle \delta\phi + \lambda \langle \phi \rangle \langle \phi \rangle \delta\phi^\dagger + V_{\text{ext}} \delta\phi\,. \tag{2.67}$$

Here we go back to using the expression $\langle \phi \rangle$ for the condensate wavefunction, instead of the previous $\langle \phi_0 \rangle$. This choice is motivated by the need to distinguish the two expressions and by the fact that, under the reasonable hypothesis that the cubic term vanishes identically, $\left\langle \delta\phi^\dagger \delta\phi \delta\phi \right\rangle = 0$, the equation (2.66) is exact. The difference between $\langle \phi \rangle$ and the exact mean-field is only due to the approximation made in the dynamics of the perturbation.

Operatively one would either solve the two equations together, or iteratively, calculating from one equation increasingly better expressions from to put in the other.

At the first step of this iteration, one would find $\langle \phi_0 \rangle$ from the standard Gross–Pitaevskii equation, and then use it to find the dynamics of the quantum perturbation $\delta\phi$; this first expression would be sufficient to evaluate the first approximation of the anomalous terms. In particular one could find the depletion:

$$\partial_t \int_{\mathcal{D}} \mathrm{d}x \, |\langle\phi\rangle|^2 = i \int_{\partial\mathcal{D}} \mathrm{d}\Sigma \cdot \left(- \langle\phi\rangle \frac{\nabla}{2m} \overline{\langle\phi\rangle} + \overline{\langle\phi\rangle} \frac{\nabla}{2m} \langle\phi\rangle \right) + \\ - 2\lambda \int_{\mathcal{D}} \mathrm{d}x \overline{\langle\phi\rangle\langle\phi\rangle} \langle\phi\rangle \langle\phi\rangle \, m_I \,, \tag{2.68}$$

and we see that the back-reaction directly impact the number of atoms in the condensate. When we include the back-reaction, not only there will be a change in the number of atoms in the condensate in any region $\mathcal{D}$ locally due to the flow, but even globally the excitation of quantum perturbation leads, through the imaginary part of the anomalous mass, to the depletion of atoms from the condensate to the excited part.

2.4 Original approach to the number conserving formalism with natural orbitals for Bose–Einstein condensates

In this section we present part of the work we have published in [7].

The mean-field approach is based on strict requirements on the many-body state of the system and on its properties; as we have discussed previously, the mean-field description is most accurate for coherent states.

Following the Penrose–Onsager criterion, we present a formulation of the theory of condensation which allows a generalization, defining the condensate wavefunction as an eigenfunction of the 2-point correlation function.

We connect it to the number conserving formalism of second quantization, well known in the theoretical description of many-body systems, in order to describe the production of quasi-particles in a generic Bose–Einstein condensate without losing information on the quantum nature of the atoms in the 1-particle condensed state.

2.4.1 The Penrose–Onsager criterion and the natural orbitals

We have seen how the conditions for condensation are addressed in statistical physics, but the description of Bose–Einstein condensation can be extended beyond the equilibrium states selected in that theory: we have discussed at length the mean-field approximation for the condensate wavefunction in the case of coherent states and their perturbations, which are pure states and not statistical ensembles.

In statistical physics the whole set of statistical states describing a system at thermodynamical equilibrium is defined by mixed density matrices associated with the equilibrium thermodynamical ensembles; at equilibrium every density matrix is fully characterized by few thermodynamical variables.

In the case of coherent states, the state is pure and is completely described in terms of a 1-particle state and the total occupation number. Then we have assumed that it is possible to consider perturbations to states of this kind and include the study of perturbations. By statistical superposition one could pass from coherent states to the ensembles of statistical physics.

The only fundamental characteristic that we must preserve in generalizing the description of condensation is that the expected value of the occupation number of a specific 1-particle state is macroscopically occupied.

As it is stated in the Penrose–Onsager criterion for off-diagonal long-range order [4,8], the condensation phenomenon is best defined considering the properties of the 2-point correlation function.

The 2-point correlation function is the expectation value on the quantum state of an operator composed of the creation of a particle in a position x after the destruction of a particle in a different position y: $\left\langle \phi^\dagger(x)\,\phi(y)\right\rangle$. Since by definition the 2-point correlation function is Hermitian, $\overline{\langle \phi^\dagger(y)\,\phi(x)\rangle} = \left\langle \phi^\dagger(x)\,\phi(y)\right\rangle$, it can always be diagonalized as

$$\left\langle \phi^\dagger(x)\,\phi(y)\right\rangle = \sum_I \langle N_I\rangle\, \overline{f_I}(x)\, f_I(y)\,, \tag{2.69}$$

where the eigenfunctions are orthogonal and normalized

$$\int \mathrm{d}x \overline{f_I}(x)\, f_J(x) = \delta_{IJ}\,. \tag{2.70}$$

The orthonormal functions f_I, eigenfunctions of the 2-point correlation function, are known as the natural orbitals, and define a complete basis for the 1-particle Hilbert space. In the case of a time-dependent Hamiltonian (or during the dynamics) they are in turn time-dependent. As for the field operator, to simplify the notation we are not going to always write explicitly the time dependence of the f_I.

The eigenvalues $\langle N_I\rangle$ are the occupation numbers of these wavefunctions. The sum of these eigenvalues gives the total number of particles in the state (or the mean value, in the case of superposition of quantum states with different number of particles):

$$\langle N\rangle = \sum_I \langle N_I\rangle\ . \tag{2.71}$$

The time-dependent orbitals define creation and destruction operators, and consequently the relative number operators (having as expectation values the eigenvalues of the 2-point correlation function):

$$a_I = \int \mathrm{d}x \overline{f_I}(x)\, \phi(x)\ , \tag{2.72}$$

$$\left[a_I, a_J^\dagger\right] = \delta_{IJ}\,, \tag{2.73}$$

$$\left[a_I, a_J\right] = 0\,, \tag{2.74}$$

$$N_I = a_I^\dagger a_I\,. \tag{2.75}$$

The state is called "condensate" [8] when one of these occupation numbers is macroscopic (comparable with the total number of particles) and the others are small when compared to it.

In the weakly interacting limit, the condensed fraction $\langle N_0 \rangle / \langle N \rangle$ is approximately equal to 1, and the depletion factor $\sum_{I \neq 0} \langle N_I \rangle / \langle N \rangle$ is negligible. This requirement is satisfied by coherent states which define perfect condensates, as the 2-point correlation functions are a product of the mean-field and its conjugate:

$$\langle coh| \phi^\dagger(x) \phi(y) |coh\rangle = \overline{\langle \phi(x) \rangle} \langle \phi(y) \rangle \,, \tag{2.76}$$

with

$$f_0(x) = \langle N_0 \rangle^{-1/2} \langle \phi(x) \rangle \,, \tag{2.77}$$

$$\langle N_0 \rangle = \int \mathrm{d}y \overline{\langle \phi(y) \rangle} \langle \phi(y) \rangle \,, \tag{2.78}$$

$$\langle N_{I \neq 0} \rangle = 0 \,. \tag{2.79}$$

Therefore, in this case, the set of time-dependent orbitals is given by the proper normalization of the mean-field function with a completion that is the basis for the subspace of the Hilbert space orthogonal to the mean-field. The latter set can be redefined arbitrarily, as the only non-vanishing eigenvalue of the 2-point correlation function is the one relative to mean-field function. The fact that there is a non-vanishing macroscopic eigenvalue implies that there is total condensation, *i.e.* $\langle N_0 \rangle / \langle N \rangle = 1$.

2.4.2 Time-dependent orbitals formalism

It is important to understand how we can study the condensate state even if we are not considering coherent states and how the description is related to the mean-field approximation. In this framework, we shall see that the mean-field approximation is not a strictly necessary theoretical requirement for condensation (and secondly for analogue gravity).

With respect to the basis of time-dependent orbitals and their creation and destruction operators, we can introduce a new expression for the atomic field operator, projecting it on the sectors of the Hilbert space as

$$\phi(x) = \phi_0(x) + \phi_1(x) = \tag{2.80}$$

$$= f_0(x)\, a_0 + \sum_{I \neq 0} f_I(x)\, a_I = \tag{2.81}$$

$$= f_0(x) \left(\int \mathrm{d}y \overline{f_0}(y) \phi(y) \right) + \sum_{I \neq 0} f_I(x) \left(\int \mathrm{d}y \overline{f_I}(y) \phi(y) \right) \,. \tag{2.82}$$

The two parts of the atomic field operator so defined are related to the previous mean-field $\langle \phi \rangle$ and quantum fluctuation $\delta\phi$ expressions given in equation (2.36). The standard canonical commutation relation of the background field is of order V^{-1}, where V denotes the volume of the system

$$\left[\phi_0(x) , \phi_0^\dagger(y) \right] = f_0(x) \overline{f_0}(y) = \mathcal{O}\left(V^{-1}\right) \,, \tag{2.83}$$

Note that although the commutator (2.83) does not vanish identically, it is negligible in the limit of large V.

In the formalism (2.82) the condensed part of the field is described by the operator ϕ_0 and by the orbital producing it through projection, the 1-particle wavefunction f_0. The dynamics of the function f_0, the 1-particle wavefunction that best describes the collective behavior of the condensate, can be extracted by using the relations

$$\left\langle \phi^\dagger (x) \phi (y) \right\rangle = \sum_I \langle N_I \rangle \overline{f_I} (x) f_I (y) \ , \tag{2.84}$$

$$\left\langle \left[a_K^\dagger a_J , H \right] \right\rangle = i\partial_t \langle N_J \rangle \delta_{JK} + i \left(\langle N_K \rangle - \langle N_J \rangle \right) \left(\int \mathrm{d}x \overline{f_J} (x) \dot{f}_K (x) \right) \ , \tag{2.85}$$

and the evolution of the condensate 1-particle wavefunction

$$i\partial_t f_0 (x) = \left(\int \mathrm{d}y \overline{f_0} (y) \left(i\partial_t f_0 (y) \right) \right) f_0 (x) + \sum_{I\neq 0} \left(\int \mathrm{d}y \overline{f_0} (y) \left(i\partial_t f_I (y) \right) \right) f_I (x) = \tag{2.86}$$

$$= \left(\int \mathrm{d}y \overline{f_0} (y) \left(i\partial_t f_0 (y) \right) \right) f_0 (x) + \sum_{I\neq 0} \frac{1}{\langle N_0 \rangle - \langle N_I \rangle} \left\langle \left[a_0^\dagger a_I , H \right] \right\rangle f_I (x) = \tag{2.87}$$

$$= -\frac{i}{2} \frac{\partial_t \langle N_0 \rangle}{\langle N_0 \rangle} f_0 (x) + \left(-\frac{\nabla^2}{2m} f_0 (x) \right) + \frac{1}{\langle N_0 \rangle} \left\langle a_0^\dagger \left[\phi (x) , V \right] \right\rangle + \\ + \sum_{I\neq 0} \frac{\langle N_I \rangle \left\langle a_0^\dagger \left[a_I , V \right] \right\rangle + \langle N_0 \rangle \left\langle \left[a_0^\dagger , V \right] a_I \right\rangle}{\langle N_0 \rangle \left(\langle N_0 \rangle - \langle N_I \rangle \right)} f_I (x) \ , \tag{2.88}$$

(we assumed at any time $\langle N_0 \rangle \neq \langle N_{I\neq 0} \rangle$). The above equation is valid for a condensate when the dynamics is driven by a Hamiltonian operator composed of a kinetic term and a generic potential V_{ext}, but we are interested in the specific case of equation (2.17). Furthermore, $f_0 (x)$ can be redefined through an overall phase transformation, $f_0 (x) \to e^{i\Theta} f_0 (x)$ with any arbitrary time-dependent real function Θ. We have chosen the overall phase to satisfy the final expression equation (2.88), as it is the easiest to compare with the Gross–Pitaevskii equation (2.21).

2.4.3 Connection with the Gross–Pitaevskii equation

We have already seen how the Gross–Pitaevskii equation is the starting point for the description of the propagation of quasi-particles in the condensate. So let us here discuss the relation between the function f_0 — the eigenfunction of the 2-point correlation function with macroscopic eigenvalue — and the solution of the Gross–Pitaevskii equation, approximating the mean-field function for quasi-coherent states. In particular, we aim at comparing the equations describing their dynamics, detailing under which approximations they show the same behavior. This discussion provides a preliminary technical basis for the study of the effect of the quantum correlations between the background condensate and the quasi-particles, which are present when the quantum nature of the condensate field operator is retained and it is not just approximated by a number, as done when performing the standard Bogoliubov approximation. We refer to [9] for a review on the Bogoliubov approximation in weakly imperfect Bose gases and to [10] for a presentation of rigorous results on the condensation properties of dilute bosons.

The Gross–Pitaevskii equation (2.21) is an approximated equation for the mean-field dynamics. It holds when the back-reaction of the fluctuations $\delta\phi$ on the condensate — described by a coherent state — is negligible, and includes a notion of number conservation.

We can compare the Gross–Pitaevskii equation for the mean-field with the equation for $\langle N_0\rangle^{1/2} f_0(x)$ approximated to leading order, *i.e.* $\langle \phi_0(x)\rangle$ should be compared to the function $f_0(x)$ under the approximation that there is no depletion from the condensate. If we consider the approximations

$$\left\langle a_0^\dagger \left[\phi(x), V\right]\right\rangle \approx \lambda \langle N_0\rangle^2 \overline{f_0}(x) f_0(x) f_0(x) \,, \tag{2.89}$$

$$i\partial_t \langle N_0\rangle = \left\langle \left[a_0^\dagger a_0, V\right]\right\rangle \approx 0 \,, \tag{2.90}$$

$$\sum_{I\neq 0} \frac{\langle N_I\rangle \left\langle a_0^\dagger \left[a_I, V\right]\right\rangle + \langle N_0\rangle \left\langle \left[a_0^\dagger, V\right] a_I\right\rangle}{\langle N_0\rangle - \langle N_I\rangle} f_I(x) \approx 0 \,, \tag{2.91}$$

we obtain that $\langle N_0\rangle^{1/2} f_0(x)$ satisfies the Gross–Pitaevskii equation.

The approximation in equation (2.89) is easily justified, since we are retaining only the leading order of the expectation value $\left\langle a_0^\dagger [\phi, V]\right\rangle$ and neglecting the others, which depend on the operators ϕ_1 and $\phi_1^\dagger$ and are of order smaller than $\langle N_0\rangle^2$. The second equation (2.90) is derived from the previous one as a direct consequence, since the depletion of N_0 comes from the subleading terms $\left\langle a_0^\dagger \phi_1^\dagger \phi_1 a_0\right\rangle$ and $\left\langle a_0^\dagger a_0^\dagger \phi_1 \phi_1\right\rangle$. The first of these two terms is of order $\langle N_0\rangle$, having its main contributions from separable expectation values — $\left\langle a_0^\dagger \phi_1^\dagger \phi_1 a_0\right\rangle \approx \langle N_0\rangle \left\langle \phi_1^\dagger \phi_1\right\rangle$ — and the second is of the same order due to the dynamics. The other terms are even more suppressed, as can be argued considering that they contain an odd number of operators ϕ_1. Taking their time derivatives, we observe that they arise from the second order in the interaction, making these terms negligible in the regime of weak interaction. The terms $\left\langle a_0^\dagger a_0^\dagger a_0 \phi_1\right\rangle$ are also subleading with respect to those producing the depletion, since the separable contributions — $\left\langle a_0^\dagger a_0\right\rangle \left\langle a_0^\dagger \phi_1\right\rangle$ — vanish by definition, and the remaining describe the correlation between small operators, acquiring relevance only while the many-body quantum state is mixed by the depletion of the condensate:

$$\left\langle a_0^\dagger a_0^\dagger a_0 \phi_1\right\rangle = \left\langle a_0^\dagger \phi_1 \left(N_0 - \langle N_0\rangle\right)\right\rangle \,. \tag{2.92}$$

Using the same arguments we can assume the approximation in equation (2.91) to hold, as the denominator of order $\langle N_0\rangle$ is sufficient to suppress the terms in the numerator, which are negligible with respect to the leading term in equation (2.89).

The leading terms in equation (2.91) do not affect the depletion of N_0, but they may be of the same order. They depend on the expectation value

$$\left\langle a_0^\dagger \left[a_I, V\right]\right\rangle \approx \lambda \left(\int \mathrm{d}x \overline{f_I}\left(x\right) \overline{f_0}\left(x\right) f_0\left(x\right) f_0\left(x\right)\right) \left\langle N_0\right\rangle^2 . \tag{2.93}$$

Therefore, these terms with the mixed action of ladder operators relative to the excited part and the condensate are completely negligible when the integral $\int \overline{f_I f_0} f_0 f_0$ is sufficiently small. This happens when the condensed 1-particle state is approximately $f_0 \approx V^{-1/2} e^{i\theta_0}$, *i.e.* when the atom number density of the condensate is approximately homogeneous.

Moreover, in many cases of interest, it often holds that the terms in the LHS of equation (2.91) vanish identically: if the quantum state is an eigenstate of a conserved charge — *e.g.* total momentum or total angular momentum — the orbitals must be labeled with a specific value of charge. The relative ladder operators act by adding or removing from the state such charge, and for any expectation value not to vanish the charges must cancel out. In the case of homogeneity of the condensate and translational invariance of the Hamiltonian, this statement regards the conservation of momentum. In particular, if the state is invariant under translations, we have

$$\left\langle a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}\right\rangle = \delta_{k_1+k_2,k_3+k_4} \left\langle a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}\right\rangle , \tag{2.94}$$

$$\left\langle a_0^\dagger a_0^\dagger a_0 a_k\right\rangle = 0 , \tag{2.95}$$

(for $k \neq 0$).

In conclusion, we obtain that for a condensate with a quasi-homogeneous density a good approximation for the dynamics of the function $\langle N_0\rangle^{1/2} f_0$, the rescaled 1-particle wavefunction macroscopically occupied by the condensate, is provided by

$$\begin{aligned} & i\partial_t \left(\langle N_0\rangle^{1/2} f_0\left(x\right)\right) \\ & = -\frac{\nabla^2}{2m}\left(\langle N_0\rangle^{1/2} f_0\left(x\right)\right) + \lambda \langle N_0\rangle^{3/2} \overline{f_0}\left(x\right) f_0\left(x\right) f_0\left(x\right) + \\ & \quad + \lambda \langle N_0\rangle^{-1} \left(\left\langle a_0^\dagger \phi_1\left(x\right) \phi_1^\dagger\left(x\right) a_0\right\rangle + \left\langle a_0^\dagger \phi_1^\dagger\left(x\right) \phi_1\left(x\right) a_0\right\rangle\right) \left(\langle N_0\rangle^{1/2} f_0\left(x\right)\right) + \\ & \quad + \lambda \langle N_0\rangle^{-1} \left\langle a_0^\dagger \phi_1\left(x\right) a_0^\dagger \phi_1\left(x\right)\right\rangle \left(\langle N_0\rangle^{1/2} \overline{f_0}\left(x\right)\right) + \mathcal{O}\left(\lambda \langle N_0\rangle^{1/2} V^{-3/2}\right) . \end{aligned} \tag{2.96}$$

This equation is equivalent to the Gross–Pitaevskii equation (2.21) when we consider only the leading terms, *i.e.* the first line of equation (2.96). If we also consider the remaining lines of the equation (2.96), *i.e.* if we include the effect of the depletion, we obtain an equation which should be compared to the equation for the mean-field function up to the terms quadratic in the operators $\delta\phi$. The two equations are analogous when making the identification

$$\langle N_0 \rangle^{1/2} f_0 \approx \langle \phi_0 \rangle \ , \tag{2.97}$$

$$\left\langle \phi_0^\dagger \phi_1^\dagger \phi_1 \phi_0 \right\rangle \approx \overline{\langle \phi_0 \rangle} \langle \phi_0 \rangle \left\langle \delta\phi^\dagger \delta\phi \right\rangle \ , \tag{2.98}$$

$$\left\langle \phi_0^\dagger \phi_0^\dagger \phi_1 \phi_1 \right\rangle \approx \overline{\langle \phi_0 \rangle \langle \phi_0 \rangle} \langle \delta\phi\delta\phi \rangle \ . \tag{2.99}$$

The possible ambiguities in comparing the two equations come from the arbitrariness in fixing the overall time-dependent phases of the functions f_0 and $\langle \phi_0 \rangle$, and from the fact that the commutation relations for the operators ϕ_1 and the operators $\delta\phi$ differ from each other by a term going as $\overline{f_0} f_0$, as seen in equation (2.83). This causes an apparent difference when comparing the two terms

$$\lambda \langle N_0 \rangle^{-1} \left(\left\langle a_0^\dagger \phi_1 (x) \phi_1^\dagger (x) a_0 \right\rangle + \left\langle a_0^\dagger \phi_1^\dagger (x) \phi_1 (x) a_0 \right\rangle \right) \left(\langle N_0 \rangle^{1/2} f_0 (x) \right)$$
$$\approx 2\lambda \left\langle \delta\phi^\dagger (x) \delta\phi (x) \right\rangle \langle \phi_0 (x) \rangle \ . \tag{2.100}$$

However the difference can be reabsorbed — manipulating the RHS — in a term which only affects the overall phase of the mean-field, not the superfluid velocity.

The equation for the depletion can be easily derived for the number-conserving approach and compared to the result in the Bogoliubov approach. As we have seen, the dynamical equation for f_0 contains the information for the time derivative of its occupation number. Projecting the derivative along the function itself and taking the imaginary part, one gets

$$\frac{1}{2} i\partial_t \langle N_0 \rangle = \frac{1}{2} \left\langle \left[a_0^\dagger a_0, V \right] \right\rangle = \tag{2.101}$$

$$= i\mathrm{Im} \left(\int \mathrm{d}x \, \langle N_0 \rangle^{1/2} \overline{f_0} (x) \, i\partial_t \left(\langle N_0 \rangle^{1/2} f_0 (x) \right) \right) = \tag{2.102}$$

$$\approx \frac{\lambda}{2} \left(\int \mathrm{d}x \left\langle \phi_0^\dagger (x) \phi_0^\dagger (x) \phi_1 (x) \phi_1 (x) \right\rangle - \left\langle \phi_1^\dagger (x) \phi_1^\dagger (x) \phi_0 (x) \phi_0 (x) \right\rangle \right) . \tag{2.103}$$

We can compare this to the equation (2.68) describing the depletion in the mean-field description

$$\frac{1}{2} i\partial_t N = \frac{1}{2} i\partial_t \left(\int \mathrm{d}x \overline{\langle \phi (x) \rangle} \langle \phi (x) \rangle \right) = \tag{2.104}$$

$$= i\mathrm{Im} \left(\int \mathrm{d}x \overline{\langle \phi (x) \rangle} i\partial_t \langle \phi (x) \rangle \right) = \tag{2.105}$$

$$\approx \frac{\lambda}{2} \left(\int \mathrm{d}x \overline{\langle \phi (x) \rangle \langle \phi (x) \rangle} \langle \delta\phi (x) \delta\phi (x) \rangle - \langle \phi (x) \rangle \langle \phi (x) \rangle \left\langle \delta\phi^\dagger (x) \delta\phi^\dagger (x) \right\rangle \right) . \tag{2.106}$$

The two expressions are consistent with each other under the relation

$$\left\langle \phi_0^\dagger(x)\,\phi_0^\dagger(x)\,\phi_1(x)\,\phi_1(x)\right\rangle \approx \overline{\langle\phi_0(x)\rangle\langle\phi_0(x)\rangle}\,\langle\delta\phi(x)\,\delta\phi(x)\rangle\,. \tag{2.107}$$

For coherent states, one expects to find equivalence between $\delta\phi$ and ϕ_1. To do so, we need to review the number-conserving formalism that can provide the same description used for analogue gravity in the general case, *e.g.* when there is a condensed state with features different from those of coherent states.

2.4.4 Number-conserving formalism

Within the mean-field framework, the splitting of the field obtained by translating the field operator ϕ by the mean-field function produces the new field $\delta\phi$. This redefinition of the field also induces a corresponding one of the Fock space which, to a certain extent, hides the physical atom degrees of freedom, as the field $\delta\phi$ describes the quantum fluctuations over the mean-field.

The fact that $\delta\phi$ is obtained by translation provides this field with canonical commutation relation. The mean-field description for condensates holds for coherent states and is a good approximation for quasi-coherent states.

When we consider states with fixed number of atoms, therefore not coherent states, it is better to consider different operators to study the fluctuations. One can do it following the intuition that the fluctuation represents a shift of a single atom from the condensate to the excited fraction and vice versa. Our main reason here to proceed this way is that we are interested in retaining the quantum nature of the condensate. We therefore want to adopt the established formalism of number-conserving ladder operators (see, *e.g.*, the work by the authors of [3]) to obtain a different expression for the Bogoliubov–de Gennes equation, studying the excitations of the condensate in these terms. We can adapt this discussion to the time-dependent orbitals.

An important remark is that the qualitative point of introducing the number-conserving approach is conceptually separated from the fact that higher-order terms are neglected by Bogoliubov approximations anyway [3, 6]. Indeed, neglecting the commutation relations for a_0 would always imply the impossibility to describe the correlations between quasi-particles and condensate, even when going beyond the Bogoliubov approximation (*e.g.* by adding terms with three quasi-particle operators). Including such terms, in a growing level of accuracy (and complexity), the main difference would be that the true quasi-particles of the systems do no longer coincide with the Bogoliubov ones. From a practical point of view, this makes clearly a (possibly) huge quantitative difference for the energy spectrum, correlations between quasi-particles, transport properties and observables. Nevertheless this does not touch at heart that the quantum nature of the condensate is not retained. A discussion of the terms one can include beyond Bogoliubov approximation, and the resulting hierarchy of approximations, is presented in [6]. Here our point is rather of principle, *i.e.* the investigation of the consequences of retaining the operator nature of a_0. Therefore, we used the standard Bogoliubov approximation, improved via the introduction of number-conserving operators.

If we consider the ladder operators a_I, satisfying by definition the relations in equations (2.73)–(2.74), and keep as reference the state a_0 for the condensate, it is a straightforward procedure to define the number-conserving operators $\alpha_{I\neq 0}$, one for each excited wavefunction, according to the relations

$$\alpha_I = N_0^{-1/2} a_0^\dagger a_I \,, \tag{2.108}$$

$$\left[\alpha_I, \alpha_J^\dagger\right] = \delta_{IJ} \qquad \forall I, J \neq 0 \,, \tag{2.109}$$

$$\left[\alpha_I, \alpha_J\right] = 0 \qquad \forall I, J \neq 0 \,. \tag{2.110}$$

The degree of freedom relative to the condensate is absorbed into the definition, from the hypothesis of number conservation. These relations hold for $I, J \neq 0$, and obviously there is no number-conserving ladder operator relative to the condensed state. The operators α_I are not a complete set of operators to describe the whole Fock space, but they span any subspace of given number of total atoms. To move from one another it would be necessary to include the operator a_0.

This restriction to a subspace of the Fock space is analogous to what is implicitly done in the mean-field approximation, where one considers the subspace of states which are coherent with respect to the action of the destruction operator associated to the mean-field function.

In this setup, we need to relate the excited part described by ϕ_1 to the usual translated field $\delta\phi$, and obtain an equation for its dynamics related to the Bogoliubov–de Gennes equation. To do so, we need to study the linearization of the dynamics of the operator ϕ_1, combined with the proper operator providing the number conservation

$$N_0^{-1/2} a_0^\dagger \phi_1 = N_0^{-1/2} a_0^\dagger \left(\phi - \phi_0\right) = \tag{2.111}$$

$$= \sum_{I\neq 0} f_I \alpha_I \,. \tag{2.112}$$

As long as the approximations needed to write a closed dynamical equation for ϕ_1 are compatible with those approximations under which the equation for the dynamics of f_0 resembles the Gross–Pitaevskii equation, *i.e.* as long as the time derivative of the operators α_I can be written as a combination of the α_I themselves, we can expect to have a setup for analogue gravity.

Therefore we consider the order of magnitude of the various contributions to the time derivative of $\left(N_0^{-1/2} a_0^\dagger \phi_1\right)$. We have already discussed the time evolution of the function f_0: from the latter it depends the evolution of the operator a_0, since it is the projection along f_0 of the full field operator ϕ. At first we observe that the variation in time of N_0 must be of smaller order, both for the definition of condensation and because of the approximations considered in the previous section

$$i\partial_t N_0 = i\partial_t \left(\int \mathrm{d}x\mathrm{d}y f_0\left(x\right) \overline{f_0}\left(y\right) \phi^\dagger\left(x\right) \phi\left(y\right) \right) = \tag{2.113}$$

$$= \left[a_0^\dagger a_0, V\right] + \sum_{I\neq 0} \frac{1}{\langle N_0\rangle - \langle N_I\rangle} \left(\left\langle \left[a_0^\dagger a_I, V\right] \right\rangle a_I^\dagger a_0 + \left\langle \left[a_I^\dagger a_0, V\right] \right\rangle a_0^\dagger a_I \right) . \tag{2.114}$$

Of these terms, the summations are negligible due to the prefactors $\left\langle \left[a_I^\dagger a_0, V\right] \right\rangle$ and their conjugates. Moreover, as $\left\langle a_I^\dagger a_0 \right\rangle$ is vanishing, the dominant term is the first, which has contributions of at most the order of the depletion factor.

The same can be argued for both the operators a_0 and $N_0^{-1/2}$: their derivatives do not provide leading terms when we consider the derivative of the composite operator $\left(N_0^{-1/2} a_0^\dagger \phi_1\right)$, only the derivative of the last operator ϕ_1 being relevant. At leading order, we have

$$i\partial_t \left(N_0^{-1/2} a_0^\dagger \phi_1\right) \approx i \left(N_0^{-1/2} a_0^\dagger\right) \left(\partial_t \phi_1\right) . \tag{2.115}$$

We, therefore, have to analyze the properties of $\partial_t \phi$, considering the expectation values between the orthogonal components ϕ_0 and ϕ_1 and their time derivatives:

$$\left\langle \phi_0^\dagger\left(y\right) \left(i\partial_t \phi_0\left(x\right)\right) \right\rangle = \left(\langle N_0\rangle^{1/2} \overline{f_0}\left(y\right) \right) i\partial_t \left(\langle N_0\rangle^{1/2} f_0\left(x\right) \right) , \tag{2.116}$$

$$\left\langle \phi_0^\dagger\left(y\right) \left(i\partial_t \phi_1\left(x\right)\right) \right\rangle = - \sum_{I\neq 0} \overline{f_0}\left(y\right) f_I\left(x\right) \frac{\langle N_I\rangle \left\langle a_0^\dagger \left[a_I, V\right] \right\rangle + \langle N_0\rangle \left\langle \left[a_0^\dagger, V\right] a_I \right\rangle}{\langle N_0\rangle - \langle N_I\rangle} = \tag{2.117}$$

$$= - \left\langle \left(i\partial_t \phi_0^\dagger\left(y\right) \right) \phi_1\left(x\right) \right\rangle , \tag{2.118}$$

$$\left\langle \phi_1^\dagger\left(y\right) \left(i\partial_t \phi_1\left(x\right)\right) \right\rangle = \left\langle \phi_1^\dagger\left(y\right) \left(-\frac{\nabla^2}{2m} \phi_1\left(x\right) \right) \right\rangle + \left\langle \phi_1^\dagger\left(y\right) \left[\phi_1\left(x\right), V\right] \right\rangle + \\ - \sum_{I\neq 0} \overline{f_I}\left(y\right) f_0\left(x\right) \frac{\langle N_I\rangle \left\langle \left[a_I^\dagger a_0, V\right] \right\rangle}{\langle N_0\rangle - \langle N_I\rangle} . \tag{2.119}$$

The first equation shows that the function $\langle N_0\rangle^{1/2} f_0\left(x\right)$ assumes the same role of the solution of the Gross–Pitaevskii equation in the mean-field description. As long as the expectation value $\left\langle \left[a_0^\dagger a_I, V\right] \right\rangle$ is negligible, we have that the mixed term described by the second equation is also negligible — as it can be said for the last term in the third equation — so that the excited part ϕ_1 can be considered to evolve separately from ϕ_0 in first approximation. Leading contributions from $\left\langle \phi_1^\dagger \left[\phi_1, V\right] \right\rangle$ must be those quadratic in the operators ϕ_1 and $\phi_1^\dagger$, and therefore the third equation can be approximated as

$$\left\langle \phi_1^\dagger i\partial_t \phi_1 \right\rangle \approx \left\langle \phi_1^\dagger \left(-\frac{\nabla^2}{2m}\phi_1 + 2\lambda\phi_0^\dagger\phi_0\phi_1 + \lambda\phi_1^\dagger\phi_0\phi_0 \right) \right\rangle . \tag{2.120}$$

This equation can be compared to the Bogoliubov–de Gennes equation. If we rewrite it in terms of the number-conserving operators, and we consider the fact that the terms mixing the derivative of ϕ_1 with ϕ_0 are negligible, we can write an effective linearized equation for $N_0^{-1/2} a_0^\dagger \phi_1$:

$$\begin{aligned} i\partial_t \left(N_0^{-1/2} a_0^\dagger \phi_1(x) \right) \approx -\frac{\nabla^2}{2m} \left(N_0^{-1/2} a_0^\dagger \phi_1(x) \right) + 2\lambda\rho_0(x) \left(N_0^{-1/2} a_0^\dagger \phi_1(x) \right) + \\ + \lambda\rho_0(x)\, e^{2i\theta_0(x)} \left(\phi_1^\dagger(x)\, a_0 N_0^{-1/2} \right) . \end{aligned} \tag{2.121}$$

In this equation we use the functions ρ_0 and θ_0 which are obtained from the condensed wavefunction, by writing it as $\langle N_0\rangle^{1/2} f_0 = \rho_0^{1/2} e^{i\theta_0}$. One can effectively assume the condensed function to be the solution of the Gross–Pitaevskii equation, as the first corrections will be of a lower power of $\langle N_0\rangle$ (and include a back-reaction from this equation itself).

Assuming that ρ_0 is, at first approximation, homogeneous, implies that the term $\left\langle \left[a_0^\dagger a_I, V \right] \right\rangle$ is negligible. If ρ_0 and θ_0 are ultimately the same as those obtained from the Gross–Pitaevskii equation, the same equation that holds for the operator $\delta\phi$ can be assumed to hold for the operator $N_0^{-1/2} a_0^\dagger \phi_1$. The solution for the mean-field description of the condensate is therefore a general feature of the system in studying the quantum perturbation of the condensate, not strictly reserved to coherent states.

While having strongly related dynamical equations, the substantial difference between the operators $\delta\phi$ of equation (2.36) and $N_0^{-1/2} a_0^\dagger \phi_1$ is that the number-conserving operator does not satisfy the canonical commutation relations with its Hermitian conjugate, as we have extracted the degree of freedom relative to the condensed state

$$\left[\phi_1(x), \phi_1^\dagger(y) \right] = \delta(x,y) - f_0(x)\,\overline{f_0}(y) . \tag{2.122}$$

Although this does not imply a significant obstruction, one must remind that the field ϕ_1 should never be treated as a canonical quantum field. What has to be done, instead, is considering its components with respect to the basis of time-dependent orbitals. Each mode of the projection ϕ_1 behaves as if it is a mode of the quantum perturbation field $\delta\phi$ used in the mean-field description. We will be able to use either one or the other consistently.

2.5 On the consistency between the mean-field and the natural orbitals approach

We present here an unpublished calculation by which we reinforce the argument that the solution of the Gross–Pitaevskii equation can be safely used as an approximation of the condensate wavefunction, instead of exact natural orbital corresponding to the condensed state.

We assume to know the two point correlation function of the field ϕ on a state which we assume to be a condensate with respect to the Penrose–Onsager criterion, *i.e* in the spectral decomposition of the 2-point correlation function there must be exactly one eigenvalue large with respect to the sum of all the others. With the same expressions used before

$$\left\langle \phi^\dagger(x_1)\,\phi(x_0)\right\rangle = \langle N_0\rangle\,\overline{f_0}(x_1)\,f_0(x_0) + \sum_{I\neq 0}\langle N_I\rangle\,\overline{f_I}(x_1)\,f_I(x_0) = \tag{2.123}$$

$$= \overline{\langle\phi(x_1)\rangle}\,\langle\phi(x_0)\rangle + \left\langle \delta\phi^\dagger(x_1)\,\delta\phi(x_0)\right\rangle\,. \tag{2.124}$$

Our aim is understanding how well the mean-field $\langle\phi\rangle$ can approximate the condensed 1-particle state f_0. Given that f_0 is an eigenfunction, we can apply it to the 2-point correlation function, whether it is expressed in terms of the field ϕ or in its translated mean-field expression

$$\langle N_0\rangle\,f_0(x_0) = \left(\int \mathrm{d}x_1 f_0(x_1)\,\overline{\langle\phi(x_1)\rangle}\right)\langle\phi(x_0)\rangle + \int \mathrm{d}x_1 f_0(x_1)\left\langle \delta\phi^\dagger(x_1)\,\delta\phi(x_0)\right\rangle\,, \tag{2.125}$$

but the first term on the RHS is a rescaling of the mean-field by an integral between the mean-field and the natural orbital f_0.

Projecting $\langle\phi\rangle$ onto f_0 we define an auxiliary parameter ϵ that represents how well we could approximate $\langle N_0\rangle^{1/2} f_0 \approx \langle\phi\rangle$:

$$\int \mathrm{d}y_0\overline{f_0}(y_0)\,\langle\phi(y_0)\rangle = \sqrt{(1-\epsilon)\,\langle N_0\rangle}\,. \tag{2.126}$$

We can write f_0 as a series, applying a process of recursion from equation (2.125)

$$f_0(x_0) = \sqrt{\frac{1-\epsilon}{\langle N_0\rangle}}\,\langle\phi(x_0)\rangle + \int \mathrm{d}x_1 f_0(x_1)\,\frac{\left\langle \delta\phi^\dagger(x_1)\,\delta\phi(x_0)\right\rangle}{\langle N_0\rangle}\,, \tag{2.127}$$

$$\Downarrow$$

$$\frac{f_0(x_0)}{\sqrt{1-\epsilon}} = \frac{\langle\phi(x_0)\rangle}{\sqrt{\langle N_0\rangle}} + \sum_{i=1}^{\infty}\int \mathrm{d}y_1\ldots\mathrm{d}y_i\frac{\langle\phi(y_i)\rangle}{\sqrt{\langle N_0\rangle}}\,\frac{\left\langle \delta\phi^\dagger(y_i)\,\delta\phi(y_{i-1})\right\rangle}{\langle N_0\rangle}\cdots\frac{\left\langle \delta\phi^\dagger(y_1)\,\delta\phi(x_0)\right\rangle}{\langle N_0\rangle}\,. \tag{2.128}$$

The transformation from the Bogoliubov description, of mean-field and quantum perturbations, to the condensate wavefunction can therefore be defined without carrying out the full process of diagonalization, but one need to find appropriate expressions for ϵ and for $\langle N_0\rangle$. To do so we can make use of the implicit definition of ϵ in equation (2.126), substituting the series expression of f_0, and we can also make use of the normalization of the function.

For convenience in the notation, we define the integrals

$$F_0 = \int \mathrm{d}y_0 \left\langle \phi\left(y_0\right)\right\rangle \overline{\left\langle \phi\left(y_0\right)\right\rangle}\,, \tag{2.129}$$

$$F_i = \int \mathrm{d}y_0 \ldots \mathrm{d}y_i \left\langle \phi\left(y_i\right)\right\rangle \left\langle \delta\phi^\dagger\left(y_i\right)\delta\phi\left(y_{i-1}\right)\right\rangle \ldots \left\langle \delta\phi^\dagger\left(y_1\right)\delta\phi\left(y_0\right)\right\rangle \overline{\left\langle \phi\left(y_0\right)\right\rangle}\,. \tag{2.130}$$

We consider the normalization of f_0, *i.e.* we take the equation $\int \mathrm{d}x \overline{f_0} f_0 = 1$, and the definition of ϵ in equation (2.126), and then we apply the series expression for f_0 of equation (2.128). We get

$$\frac{1}{1-\epsilon} = \sum_{i=0}^{\infty} \frac{i+1}{\left\langle N_0\right\rangle^i} \frac{F_i}{\left\langle N_0\right\rangle}\,, \tag{2.131}$$

$$1 = \sum_{i=0}^{\infty} \frac{1}{\left\langle N_0\right\rangle^i} \frac{F_i}{\left\langle N_0\right\rangle}\,. \tag{2.132}$$

In principle, knowing the expressions of the mean-field $\left\langle \phi\left(x\right)\right\rangle$ and of the correlation function $\left\langle \delta\phi^\dagger\left(x\right)\delta\phi\left(y\right)\right\rangle$, one should be able to extract from equations (2.131)–(2.131) the expressions for ϵ and $\left\langle N_0\right\rangle$.

We want to argue that one can always choose a field translation, as defined in equation (2.36), such that the mean-field function together with the quantum perturbation are a good approximation of the full 2-point correlation function, and that the mean-field is a good approximation of its eigenfunction f_0, *i.e.* such that the mean-field function is a good approximation of the condensate wavefunction for a generic state that satisfies the Penrose–Onsager criterion.

We have already seen that f_0 and $\left\langle \phi\right\rangle$ satisfy, at leading order, the Gross–Pitaevskii equation. Now we argue that the mean-field can be selected such that it differs from $\left\langle N_0\right\rangle^{1/2} f_0$ by negligible corrections, even including the back-reaction in the description. To do so we focus on the integrals F_i that characterize the magnitude of the corrections in orders of $\delta\phi$.

We assume $\left\langle N_0\right\rangle = F_0$, which means that we translate the field by a solution of the Gross–Pitaevskii equation such that it is normalized appropriately. We consider the first terms of the equations (2.131)–(2.132).

$$\frac{\epsilon}{1-\epsilon} = \frac{1}{\left\langle N_0\right\rangle^2} \frac{F_2}{\left\langle N_0\right\rangle}\,, \tag{2.133}$$

$$\frac{1}{\left\langle N_0\right\rangle} \frac{F_1}{\left\langle N_0\right\rangle} = -\frac{1}{\left\langle N_0\right\rangle^2} \frac{F_2}{\left\langle N_0\right\rangle}\,. \tag{2.134}$$

We see that F_1, while having a quadratic dependence on the quantum perturbation, must be of the same order as $F_2/\left\langle N_0\right\rangle$, by geometrical properties. Then it follows that ϵ, which is a tracer for the compatibility between $\left\langle N_0\right\rangle^{1/2} f_0$ and $\left\langle \phi\right\rangle$, has a quartic dependence from the quantum perturbation field, and it is suppressed as $\left\langle N_0\right\rangle^{-2}$, proving that the difference between the two functions is negligible, even in the generic condensate state.

2.6 Final remarks

We have made an overview of the theory of Bose–Einstein condensation, showing some key points for what is relevant in analogue gravity: the statistical physics approach and the occupation number distribution, the consistency of the theory for cold atoms, the properties of the condensate states in the Fock space.

We have provided a short overview of the mean-field theory formulated in terms of the Gross–Pitaevskii equation and the Bogoliubov–de Gennes equation, focusing on the definition of quasi-particles in Bose–Einstein condensates and their dispersion relation which is modified at short distances; these are fundamental aspects of the theory for a consistent formulation of analogue gravity.

We have also described the problem of the back-reaction: the effect that the quantum perturbations, the quasi-particles, have on the condensate are of paramount interest in studying the dynamics of the condensate.

In the second part of this chapter we have presented at length our original work, partly published in [7], were we have presented a study of the number conserving formalism applied in the specific case of the time dependent basis of the 1-particle Hilbert space given by the natural orbitals. The natural orbitals can be directly associated with the dynamical quasi-particle modes, and they are even more interesting in the number conserving formalism because they allow to keep track of the creation of quasi-particles in terms of excitation of the atoms previously in the condensed 1-particle state. In the usual mean-field formalism it is impossible to preserve the full quantum nature of the atoms in the condensate, because the condensate is treated separately as a classical object or as a separate coherent state.

The number conserving formalism allows to describe in any many-body condensate state, as defined in accordance with the Penrose–Onsager criterion, how the correlations between excited atoms and condensate atoms are created and depend on the dynamics.

Finally we have presented a calculation that corroborates the consistency between the mean-field description and the number conserving formalism, showing how the mean-field function calculated as a solution of the Gross–Pitaevskii equation, is generally a valid approximation of the condensate wavefunction.

3

Gravity and analogue models

3.1 General relativity and its open issues

General relativity is the strongly predictive theory of gravitation formulated by Einstein that describes the dynamics of objects in space and time and of spacetime itself [11–13]. Its soundness has been proven with numerous tests since its early days, with the accurate description of the perihelion precession of Mercury and the deflection of light by the Sun. Later, other experimental confirmations followed, like the measure of gravitational redshift of light and the increasingly improving evidence of the identity between inertial and gravitational mass, up to the recent experimental measures of gravitational waves produced by the merging of extended objects [14].

The most remarkable difference between the theories of gravity that preceded it (Newtonian and Nordstrom gravity) and general relativity is the fact that Einstein's theory describes spacetime itself as a fully dynamic object. Let us discuss the founding principles of this theory, its general features and the problems encountered at the limits of its validity.

3.1.1 The principles underlying general relativity

A physical theory is a system that: is descriptive of physics; organizes knowledge into an ordered science; is predictive of physical phenomena.

The formulation of a complete and self-consistent physical theory requires two sets of assumptions. At first it is necessary to identify the physical principles of the theory and the nature of the objects described; then it is necessary to define a mathematical framework, *i.e.* the tools which are compatible with the physical assumptions and through which it is possible to translate them in an operative language.

By completeness we mean that the outcome of every conceivable experiment within the framework of the theory should be predictable.

By self-consistency we mean that there should be unambiguousness and uniqueness in the procedure to be followed to provide such predictions.

The objects of interest for the theory of gravity are moving bodies, whether massless or massive, whether point-like or extended. The aim is to describe their trajectories in space and time, and how extended bodies remain cohesive — *i.e.* how they self-gravitate — or disperse, as well as how spacetime reacts to these bodies.

Although there is no widely accepted axiomatic formulation of gravitation, and specifically of general relativity, we can identify the physical principles underlying the current understanding of the theory: what here we define as the "reality principle" and the causality principle, which are of primary importance in all the current formulation of physical theories, and the strong equivalence principle, which is well-defined and well-established within the framework of gravitation [15].

Under the name of reality principle we indicate the statement that, in a broad sense, a physical theory describes predictable events, where a describable occurrence to the object of the theory constitutes an event.

The causality principle states that events are caused by preceding conditions, and therefore that there must be an ordered relation of cause and effect between events. The final results therefore depend on the initial conditions. Causality and determinism must not be confused: this principle can be formulated in a strong or weak connotation, remaining compliant with the request for completeness and self-consistency.

The strong causality principle states that it is always possible to set initial conditions in such a way as to cause unique predictable effects; this principle is compatible with deterministic, unitary, theories. The weak causality principle states that it is possible to set initial conditions in such a way as to cause a set of predictable effects; this principle is compatible with statistical theories.

Whether we are describing deterministic or statistical theories, we would argue that the causality principle is a logical requirement to formulate any physical theory: a theory admitting violations of the causality principle is equivalent to it admitting the occurrence of intrinsically unpredictable events, falsifying the predictivity of the theory. [1]

From the existence of the set of the events, causally related to each other, it is possible to identify an ordered pre-geometric structure on which to start building — at least locally — the notions we associate with spacetime. The global set of events defines the manifold, *i.e.* the spacetime or the universe, to which the appropriate mathematical description is then applied.

But before we get to the mathematical description of gravity, the principle of strong equivalence is introduced as a physical principle. The strong equivalence principle is composed of three parts: the universality of free fall, the local Lorentz invariance and the local position invariance.

By universality of free fall we here mean that all test, self-gravitating bodies, *i.e.* bodies whose compactness is sufficiently small such as to neither be altered by the

[1] The local causality of the theory must not be confused with the possibility that particular realizations may present violations of causality, in which it is not topologically possible to establish an absolute past and future. In general relativity this problem is encountered with Cauchy horizons, null hypersurfaces, which however are generally considered unstable. Realizations of this kind are problematic and it is customary to look for physical mechanisms to avoid them, either within the given theory or in a modified one.

surrounding environment nor deform it, follow the same trajectories as point-like bodies regardless of their mass or composition. This can generally be quantified by saying that a self-gravitating body has an extension not much larger than the Schwarzschild radius associated with its mass [11]. While by local Lorentz invariance and local position invariance we mean that in any point of the spacetime it would be possible to define a local inertial frame where to verify respectively the local validity of special relativity and to make physical experiments (including gravitational ones) with the same results.

Einstein codified these principles into a mathematical formulation based on differential geometry. General relativity describes the spacetime as a differentiable pseudo-Riemannian manifold $\mathcal{M}$ (in particular, a $(1,3)$ Lorentzian manifold), of which each point corresponds to an event. The description is made through a metric tensor $g_{\mu\nu}$, which defines the geometry, and a connection $\Gamma^{\sigma}_{\mu\nu}$ (assumed metric, or affine, and symmetric), which defines the kinematics. It is from these notions that it is possible to formulate a theory of gravity that satisfies the set objectives, *i.e.* that describes the motion of free-falling bodies and of distributions of matter, respecting the principles of reality, causality and a strong equivalence.

The metric tensor defines the causal structure of spacetime, and the causality principle is enforced through the definition of the light cones, which describe the past and future associated with each event. The metric tensor allows to quantify the distance between events $\int \mathrm{d}s$, defining the shortest and the straightest trajectories (being the connection the Christoffel symbol of the metric) between two events in spacetime: these trajectories are the geodesics $\gamma(\tau)$, which are assumed to be the natural paths followed by free-falling, self-gravitating bodies.

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \,, \tag{3.1}$$

$$g^{\sigma\rho}g_{\rho\nu} = \delta^{\sigma}_{\nu} \,, \tag{3.2}$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}\right) \,, \tag{3.3}$$

$$\frac{\mathrm{d}\gamma^{\sigma}}{\mathrm{d}\tau^2} = -\Gamma^{\sigma}_{\mu\nu}\frac{\mathrm{d}\gamma^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}\gamma^{\nu}}{\mathrm{d}\tau} \,. \tag{3.4}$$

The geometry of the spacetime is defined by the metric, the affine connection and the Riemann curvature tensor $R^{\rho}_{\mu\sigma\nu}$ and its contractions (the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the Einstein tensor $G_{\mu\nu}$):

$$R^{\rho}_{\mu\sigma\nu} = \partial_{\sigma}\Gamma^{\rho}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \,, \tag{3.5}$$

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} \,, \tag{3.6}$$

$$R = g^{\mu\nu}R_{\mu\nu} \,, \tag{3.7}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \,. \tag{3.8}$$

The evolution of extended distributions of matter is described by the Einstein field equations, the Lagrange equation of an action including a purely gravitational term — the Einstein–Hilbert action — and a term associated with the matter fields, relating the

Einstein tensor to the stress-energy tensor $T_{\mu\nu}$ of the distribution:

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}_{\text{matter}}\,, \tag{3.9}$$

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}\,, \tag{3.10}$$

The resulting second order differential tensor equation describes the evolution of the six components of the symmetric metric tensor, whose dynamics is determined by the stress-energy tensor; vice versa, the dynamics of a matter distribution is determined by the Einstein curvature tensor: matter tells spacetime how to curve, and curved spacetime tells matter how to move.

In this descriptions the mathematical principles applied are most notably the principle of covariance of the theory and the background independence. Although we have made use of coordinates to write the previous equations, this theory does not require to choose them in a specific way, the spacetime manifold can be described with a generic set of coordinates such that an atlas of charts covering the manifold exists and is well-defined. General relativity therefore admits a covariant formulation which does not require a specific set of coordinates. Moreover, the theory is background independent: not only it does not privilege a coordinate system over another, but it does not even require to fix structures (other than topology and signature of the manifold), since the theory appears to be completely formulated in terms of the dynamic objects described through the Einstein equations themselves — *e.g.* the theory does not require to set an expression for the flat metric, it naturally results as the solution of the equation itself in absence of a source.

The resulting theory has the peculiarity of not only being invariant under change of coordinates, or passive diffeomorphisms, but also invariant under active diffeomorphisms: the infinitesimal transformation of the metric tensor along a vector field:

$$g_{\mu\nu} \to g_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}\,, \tag{3.11}$$

This infinitesimal transformation leaves invariant the action, its variation being

$$\delta S = \int d^4x\sqrt{-g}\xi_\nu\nabla_\mu\left(\frac{1}{8\pi G_N}G^{\mu\nu} - T^{\mu\nu}\right) + \text{boundary terms}\,. \tag{3.12}$$

This variation vanishes on shell, for solutions of the Einstein equations: the divergence of the Einstein tensor is zero by virtue of the Bianchi identities, and the divergence of the stress-energy tensor vanishes for a solution of the equations of motion. It follows that four of the six degrees of freedom of the theory are not physical and can be transformed away, or fixed by choosing an appropriate gauge.

Further assumptions can be made about energy conditions, imposing restrictions on what behaviors are acceptable for the stress-energy tensor of congruences associated with timelike or null congruences, to describe realistic matter distributions.

3.1.2 Open issues in general relativity

The mathematical formulation of general relativity proved itself to be powerful and predictive. However, it soon became clear that it ran into profound problems.

If for weak fields one could retrieve the well-known Newtonian theory of gravity, general relativity revealed its own limitations in other regimes. From the solution of the Einstein equations obtained by Schwarzschild, describing the static spherically symmetrical solution, it could be inferred that general relativity predicted the existence of singular regions: it was eventually understood that Schwarzschild's solution described black holes geometries, characterized by a trapping horizon surrounding a region where the curvature diverges. But a point where coordinate-independent quantities, *i.e.* physical quantities, appear to diverge cannot belong to the manifold, and therefore there is a region where the theory reaches its limits and is not defined, where it cannot describe events.

Indeed a more proper definition of singularity is associated with geodesic incompleteness: the singularity is a point that is reached by a geodesic of finite length beyond which the geodesic trajectories cannot be extended, *i.e.* that is reached for a finite value of the affine parameter.

As proved by the Penrose–Hawking theorems [16–18], in general relativity singularities cannot be avoided. The cosmic censorship conjecture has been proposed in order to get around the problem of naked singularities [19], where one would risk to encounter violations of the causality principle, but the generally unavoidable presence of singularities (like the curvature singularity of the Schwarzschild black hole, or the curved ring singularity of the rotating Kerr black hole) poses deep problems of predictivity: the theory of general relativity is incomplete. Also, general relativity has Cauchy horizons, which if not unstable could allow access to non-deterministic regions of spacetime.

This is not only a topological problem associated with a single point (or some points) missing from the manifold, it is generally and more properly a problem of how general relativity fails to describe the gravitational collapse of matter distributions, in terms of the classical fields involved in the theory. The singularity at the center of a black hole, in fact, is the result of the dynamic evolution of a gravitational system undergoing a collapse: this is consequential of the aforementioned Penrose–Hawking singularity theorems; a generic spacetime cannot remain regular for an indefinite time without producing singularities.

The reality principle would suggest that a singularity should be prevented by further physical phenomena other than those described by general relativity. It would be reasonable to expect that the high energy and density regimes achieved during the collapse of a distribution of matter would push the system out of the validity regime of general relativity: we already know that this theory, which is a classical field theory, will have to break down when we enter a regime in which the well known quantum nature of matter cannot be neglected. As long as matter is described in classical terms, we know that we are in fact considering an approximation of a more fundamental description.

We can therefore state that for energies higher than the Planck energy scale of 1.2209×10^{19} GeV (to compare with 3.2×10^{11} GeV which is the most energetic ultra-high-energy cosmic ray ever observed [20]), *i.e.* for length scales sufficiently short (the Planck length is 1.616255×10^{-35} m), it is appropriate to expect quantum corrections to the description to be needed.

Another issue associated with general relativity, again at the level of classical field theory, is that relative to dark energy. In the action of general relativity in equation (3.9) it is possible to introduce a term proportional to the volume of spacetime, with a proportionality constant Λ corresponding to the density of what we call dark energy; it is a cosmological constant identical in every point of spacetime. This term is unexceptionable from a mathematical point of view, and its existence is compatible with cosmological observations, although it is difficult to interpret physically what are its origins and causes. In particular if its origin is looked for in an ubiquitous vacuum energy, it looks like it should be huge — in contrast with observations — as the natural cutoff for such vacuum energy would be the Planck scale, or some other ultraviolet scale set by the theory considered.

Let us also add that cosmological observations actually present tensions, and it now seems that measurements of different physical observables and independent datasets provide incompatible values for the cosmological constant [21].

Various theories have been proposed to explain the nature of the cosmological constant and to suggest a solution to such experimentally detected tension. Some models propose solutions to this issue that introduce changes to either the geometric sector of the theory or to the matter sector. If instead of the Einstein–Hilbert action one considered a modified theory of gravity [22], this would directly affect our understanding of how the geometry of spacetime determines physical events. In alternative one could consider modifying the matter sector of the action, introducing new matter fields and new interaction terms: this would suggest a change at the basis of the description of classical matter, *i.e.* a change to the standard model, possibly with the introduction of dark energy and dark matter fields. In fact, a dark sector appears to be required to solve many long-standing puzzles in the theory and observation of cosmology and astrophysics.

3.1.3 Open issues between gravity and quantum theory

We have already introduced the fact that it is necessary to go beyond the description of gravity in purely classical terms: at high energies one comes to consider the fact that it is necessary to take into account the quantum nature of the fields that describe elementary particles. However, this quantum description of matter must be linked to the classical one of geometry.

On the one hand we could think of promoting the geometry to a quantum object, on the other we could consider maintaining a classical description of the matter fields, but introducing to the stress-energy tensor quantum corrections of ever-increasing order. In the first case we are talking about quantum geometry (or, more generally, quantum gravity, when matter is included in the formulation from the beginning), in the second we are talking about semiclassical gravity. In both cases we suggest that the fundamental nature of geometry and matter rests on a common quantum structure, compatible with general relativity in the classical limit.

In the case of quantum gravity, the problems are innumerable. What is the underlying structure of spacetime, characterized by quantum degrees of freedom associated with both geometry and matter? How is it possible to adequately characterize the Hilbert space, in a complete and consistent way?

Promoting general relativity to a quantum field theory through the quantization of the perturbations of the metric, we realize that the theory does not appear renormalizable: treated perturbatively, it would seem to require infinite counterterms in the quantum effective action to obtain predictions of physical quantities. The research program of asymptotic safety [23] is an attempt aimed at finding non-perturbatively a non-trivial fixed point of the theory which would provide a possible ultraviolet completion, in some measure preserving our understanding of space and time.

But various other models for quantum gravity have been proposed that require a paradigm shift, not only in considering the geometry as having a quantum nature, but also by describing it often by not immediately intuitive degrees of freedom. There is no universal agreement on what principles should be considered and on what physical assumptions should be made on the fundamental nature of gravity [24], except for the general requirement that there must be a unified theory for matter and geometry consistent with general relativity in the classical limit.

The theory of loop quantum gravity (or canonical quantum general relativity) [25,26], obtained by quantizing the canonical variables of Ashtekar's reformulation of general relativity, is a background independent description of gravity in terms of a quantum geometry of quantized holonomies interacting with matter.

Group field theory [27] is a related approach in which the theory of quantum geometry is investigated from first principles, and is required to emerge from the dynamical evolution of the states of a quantum field theory on a $SU(N)$ Lie group.

String theory [28] is a background dependent theory where gravity emerges from the dynamics of interacting quantum strings.

Considering the significant differences in the assumptions made in the different approaches we can say that there is still not a clear understanding of how the problems of the ultraviolet regime should be solved: while it is clear that the classical theory fails at high energies, as we know from the existence of singularities, it is not clear nor evident how a new theory should be formulated at those energy scales to ensure predictivity and completeness. Without an established theory of quantum gravity or a non-perturbative renormalization scheme, the theory continues to necessitate an ultraviolet completion for the high-energy regime. In the context of renormalization, the results of the programs of asymptotic safety and of loop quantum gravity are promising, but still there is not yet evidence to confirm any specific approach.

Instead, when we take into consideration semiclassical gravity, the first thing to highlight is the need to formulate the quantum field theory of matter fields in such a way that it is consistent with having as background a curved spacetime [29–31]. After it has been formulated consistently, a way can be sought to modify the expression of the stress-energy tensor in the Einstein equations by including quantum corrections, which then induce corrections to the metric [32,33]. Whether we solve the equations iteratively or perturbatively, in orders of $\hbar$, semiclassical gravity results from these assumptions.

It is possible to formulate quantum field theory in curved spacetime and it is particularly convenient to do so by considering the foliation of the spacetime $\mathcal{M}$ described in the ADM form, *i.e.* considering a foliation with respect to a parameter t in 3-dimensional spatial slices Σ_t [34]. In this way it is possible to pass naturally from quantum field theory in Minkowski spacetime to quantum field theory in curved spacetime: the canonical commutation relations are set on a spatial foliation, and then are evolved through the equations of motion. For example, in the case of a Klein–Gordon scalar field, the theory is obtained by directly promoting the partial derivatives into covariant derivatives and by appropriately setting the volume element:

$$S_{\mathrm{KG}} = \int \mathrm{d}^4x\sqrt{-g}\frac{1}{2}\left[-g^{\mu\nu}\left(\nabla_\mu\phi\right)\left(\nabla_\nu\phi\right) - m^2\phi^2\right] , \tag{3.13}$$

$$0 = \Box\phi - m^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi - m^2\phi\,, \tag{3.14}$$

$$\pi = \frac{\delta\mathcal{L}}{\delta\left(\nabla_t\phi\right)} = \sqrt{-g}g^{\mu t}\nabla_\mu\phi\,, \tag{3.15}$$

$$\left[\phi\left(t,\mathrm{x}\right), \pi\left(t,\mathrm{y}\right)\right] = \frac{i}{\sqrt{-g}}\delta^3\left(\mathrm{x},\mathrm{y}\right) , \tag{3.16}$$

$$\left[\phi\left(t,\mathrm{x}\right), \phi\left(t,\mathrm{y}\right)\right] = \left[\pi\left(t,\mathrm{x}\right), \pi\left(t,\mathrm{y}\right)\right] = 0\,. \tag{3.17}$$

The notion of internal product between elements of the Hilbert space of the theory is inherited from the Klein–Gordon equation itself and the notion of orthonormality between states is given by their product on a generic hypersurface Σ_t of the foliation of the manifold $\mathcal{M}$; the internal product is preserved along the flux of the parameter t.

$$\left\langle\phi_1\middle|\phi_2\right\rangle_{\Sigma_t} = i\int_{\Sigma_t}\mathrm{d}^3x\sqrt{h}n^\mu\left(\overline{\phi_1}\left(\nabla_\mu\phi_2\right) - \left(\nabla_\mu\overline{\phi_1}\right)\phi_2\right) , \tag{3.18}$$

$$\left\langle\phi_1\middle|\phi_2\right\rangle_{\Sigma_{t_1}} - \left\langle\phi_1\middle|\phi_2\right\rangle_{\Sigma_{t_2}} = i\int_{\mathcal{V}(t_1,t_2)}\mathrm{d}^4x\sqrt{-g}\nabla^\mu\left(\overline{\phi_1}\left(\nabla_\mu\phi_2\right) - \left(\nabla_\mu\overline{\phi_1}\right)\phi_2\right) = \tag{3.19}$$

$$= 0\,, \tag{3.20}$$

where h is the metric on the foliation Σ, and $\mathcal{V}\left(t_1, t_2\right)$ is the 4-dimensional volume enclosed by the 3-dimensional hypersurfaces Σ_{t_1} and Σ_{t_2}. The inner product allows to select a basis of orthonormal modes with positive norm, on which we explicitly expand the Klein–Gordon scalar field.

$$\left\langle f_i\middle|f_j\right\rangle = \delta_{ij}\,, \tag{3.21}$$

$$\left\langle\overline{f_i}\middle|\overline{f_j}\right\rangle = -\delta_{ij}\,, \tag{3.22}$$

$$\left\langle f_i\middle|\overline{f_j}\right\rangle = 0\,, \tag{3.23}$$

$$\phi\left(x\right) = \sum_i\left[f_i\left(x\right)a_i + \overline{f_i}\left(x\right)a_i^\dagger\right] . \tag{3.24}$$

In a similar way can be treated all the other fields, of the different representations of the Lorentz group. Among the various predictions of quantum field theory in curved spacetime, it is worth mentioning the Unruh effect, the cosmological particle creation and the Hawking radiation [30, 31, 35, 36].

It is possible to say in general that in quantum field theory in curved spacetime the strong equivalence principle plays a primary role: in the case of the Unruh effect we consider an accelerated observer, in the cases of Hawking radiation and cosmological particle creation we observe the role of the curvature of spacetime in proximity of a trapping horizon or during cosmological evolution. In the first case, an observer subject to acceleration will detect thermal radiation whereas an inertial observer would observe none, an effect comparable to the other two cases in which an observer moving in a curved spacetime could make similar detections, a gravitational field and an external force being indistinguishable. The definition of quantum vacuum therefore depends on the observer: it changes with acceleration as well as curvature, resulting in different measures for different observers.

Hawking radiation, in particular, opens up a whole series of interesting scenarios, on which we can start a separate discussion.

3.1.4 The issues arising from Hawking radiation

Hawking radiation is a phenomenon linked to the presence of a horizon that separates the spacetime in two regions. Radiation cannot leave the inner region due to the causal structure of spacetime, but in the outer region it is still possible to detect not only quantum particles going towards the horizon, passing through it, but also particles that move away from it, in particular radiation emitted from the black hole.

The origin of Hawking radiation is due to the non-equivalence of quantum states when measured by different observers in curved spacetime. We consider the formation of the black hole as a product of a gravitational collapse, and imagine an observer in free-fall together with the collapsing matter: this observer would not measure the creation of radiation, indeed it would be natural for him to define analytical modes along his own trajectory, defining a quantum vacuum state relatively to them. This is not the case for an observer at infinity, who sees the horizon and defines modes only in the region outside of it. An observer at infinity will then measure a different quantum state, in particular detecting thermal radiation associated with a temperature proportional to the surface gravity of the black hole, which corresponds to the gravitational acceleration experienced at the horizon, the Hawking temperature, which for the Schwarzschild black hole with horizon radius r_H is found to be

$$T_H = \frac{1}{4\pi r_H}\,. \tag{3.25}$$

Hawking radiation raises several problems. The transplanckian problem stems from the fact that infrared Hawking quanta observed at late times at infinity seems to require the extension of relativistic quantum field theories in curved spacetime well within the ultraviolet completion of the theory, *i.e.* the Hawking calculation seems to require a strong assumption about the structure of the theory at the Planck scale and beyond [37, 38].

The transplanckian problem is associated with the redshift of the radiation emitted by the black hole: Hawking radiation, which is thermal and follows the bosonic black body distribution, emits photons at every frequency. Although the number of high-energy quanta is exponentially suppressed — we do not encounter the ultraviolet catastrophe — tracing each photon back along its trajectory to the horizon one finds that its frequency, *i.e.* its energy, diverges: an observer at a fixed distance from the horizon measures, for a photon propagating radially towards the future null infinity, an energy ω' which depends on the radius at which the energy the measurement takes place. For the Schwarzschild black hole with horizon radius r_H, which has line element

$$\mathrm{d}s^2 = -\left(1 - \frac{r_H}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_H}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) , \tag{3.26}$$

and where the outgoing photon on a radial trajectory and a static observer located at radius r_{obs} have respectively four-velocity

$$k^\mu(r) = \left(\frac{\omega r}{r - r_H}, \omega, 0, 0\right)^\mu , \tag{3.27}$$

$$u^\mu = \left(\sqrt{\frac{r_{\mathrm{obs}}}{r_{\mathrm{obs}} - r_H}}, 0, 0, 0\right)^\mu , \tag{3.28}$$

the energy ω' measured by an observer at r_{obs} is

$$\omega' = \omega\sqrt{\frac{r_{\mathrm{obs}}}{r_{\mathrm{obs}} - r_H}} , \tag{3.29}$$

where ω is the energy measured at infinity. These formulas are valid for the Schwarzschild black hole, but can be generalized for other spacetimes by considering instead of $g_{tt} = -1 + r_H/r$ other metrics that have a horizon.

We can see that there is an obvious problem when r_{obs} is arbitrarily close to r_H, regardless of the energy needed by the observer to maintain that position: the observer seems to be able to detect photons of arbitrarily high energy.

That near the horizon the energy of photons diverges is a problem that cannot be overlooked: it is necessary to establish how it is possible that arbitrarily high-energy quanta, with arbitrarily small wavelengths, can exist and be detectable from the point of view of an observer. Close to the horizon there would be a bath of what would be experienced as ultra-high energy photons, the physics of which would be incompatible with the theoretical understanding of the observer.

Whether these ultra-high energy photons originate from the horizon or are reflected by it after coming from the past null infinity of the spacetime having an unknown physical origin, we understand that either an ultraviolet completion of quantum field theory in curved spacetime or a new description of geometry itself is needed. This problem arises even having taken care to preserve the local Lorentz invariance, and Lorentz violations in the high energy regime may have to be considered to solve this problem.

In any case, Hawking radiation describes a flow of energy that goes from the horizon to infinity. This flow, in terms of semiclassical gravity, necessarily entails the loss of mass by the black hole. The black hole then shrinks, and the temperature associated with the emitted radiation thus becomes higher and higher, increasing the rate of energy emitted: in this process a black hole evaporates completely, the mass is lost until there is no singularity and the horizon shrinks until it disappears.

This phenomenon is problematic because quantum fields carry information from the outer region into the region within the horizon, which can only be crossed in one direction: the causal structure of the spacetime does not allow quanta to leave the inner region. This property of the trapping horizon, defined instantaneously by the light cones determined by the metric, does not change during the evaporation process. When the evaporation is complete, the information that had been transferred inside the horizon during the evolution of the spacetime is eventually lost. So for example a field initially in a pure state at past null infinity could be found in a mixed, thermal, state at late times after the evaporation is completed.

The evaporation problem therefore implies the unitarity breaking of the quantum evolution, which largely depends on the fact that we are considering geometry and quantum fields as defined in two separate Hilbert spaces, and that there is no obvious way to incorporate quantum information into the geometric sector. The unitarity breaking is highlighted by the lack of one-to-one correspondence between the possible final states and the corresponding initial states, and the inability to preserve the purity of the state itself: a specific feature of Hawking radiation is that it connects pairs of states defined on the two sides of the horizon which have the same Killing frequency. The production of these pairs creates entanglement between the particles in the two regions, in a way that in principle would preserve purity. Indeed if we consider an initial pure state — which is characterized by an idempotent density matrix — the production of Hawking pairs does not change the purity, the state evolves unitarily. But after the evaporation, the states associated with particles that were in the inner region are traced away; this operation, which suppresses the information associated with part of the degrees of freedom of the system, leaves the particles in the outer region in a statistical superposition, is an operation of partial projection that makes the final density matrix mixed.

The information loss problem, sometimes referred to as the information loss paradox, is a thorny question of whether or not to allow the Hilbert space of the theory to be split into a geometric sector separated from a quantum sector. Various approaches can be proposed to solve this problem. One could modify the dispersion relations, allowing the emission of information in the final stages through superluminal quanta. Another option could be a mechanism for the purification of the state. Or the more elegant idea of unifying the two sectors of semiclassical gravity into a single fundamental theory of quantum gravity which could always preserve unitarity.

Finally, we mention how the laws of black hole thermodynamics — developed by Bekenstein [39], Bardeen, Carter and Hawking [40] — led in more recent times Susskind [41] and 't Hooft [42] to propose an approach to quantum gravity in which information itself is at the origin of the geometry. Like in black hole thermodynamics the role of entropy — which is a measure of the number of states, *i.e.* of the information, of the system — is played by the area of the black hole, they conjectured that, in general, the area that encloses a volume corresponds to the amount of information that it contains. In this context, the problem of information loss would be solved by considering a lower-dimensional unitary theory defined on surfaces. The holographic principle they proposed and developed would imply that the information is enclosed by strings and remains imprinted on the horizon of the black hole itself, and not lost. The correspondence between string theory in a $(d+1)$-dimensional Anti-de Sitter spacetime and conformal field theory on the d-dimensional boundary, proposed by Maldacena [43], has launched an extensive research program in string theory for a complete formulation of quantum gravity.

3.2 Analogue gravity

3.2.1 Theoretical motivations

Analogue gravity is a research program that investigates models that reproduce the typical phenomena of gravity, *i.e.* of the interplay between geometry and matter fields, in a controlled laboratory environment [44].

This program is based on the simple but powerful observation that it is possible to mimic the typical processes of matter fields in curved spacetime by studying small perturbations in various laboratory-viable systems, some of which are classical fluids, superfluids, superconducting circuits, ultracold atoms and optical systems. In these systems the dynamics of small perturbations are described by equations that reproduce, by analogy, those of quantum fields in curved spacetimes.

With analogue models it is obviously possible to preserve the physical principles of realism and causality — clearly also typical of the physics of laboratory systems — as well as the principles of completeness and coherence, inherited from the theory describing the experiments. What can therefore be simulated, through the dynamics of small perturbations, is the behavior of particles and waves in a gravitational field: the geodesic equation for free-fall, the invariance under local Lorentz transformation and the local position invariance are encoded in the equations of motion, which are recovered in a regime of approximation specific for each system considered. Nonetheless the gravitodynamical equations are rather different.

One of the most naturally broken assumptions is the local Lorentz invariance: the equations describing the perturbations typically deviate from those of the gravitational case when scales are involved at which the microscopic properties of the system cannot be neglected.

This would seem to place limits on the validity of the analogy at a minimum length scale set by the experiment. A scale which, however, would itself be analogous to the Planck scale. This intrinsic regularization of the ultraviolet behavior suggests a way to solve the transplanckian problem through the presence of a minimum scale, a natural cutoff fixed by physical constraints: the modified dispersion relations in analogue gravity suggest how a microscopic structure at the Planck scale could solve the transplanckian problem of gravity, pointing out investigable and quantifiable consequences [45].

By experimenting with analogue models, it is therefore possible to simulate phenomena that currently cannot be tested in gravitational settings, due to the scales involved — *e.g.* the shortest Planck length scale as well as the largest observed scales of cosmological times and distances — and the current lack of capability to make measurements of the production of radiation created by the curvature of spacetime, such as the Hawking radiation emitted by black holes.

The long-term goal of analogue gravity is to advance the understanding of gravity, especially regarding its open issues. While it is of course very valuable to simulate in the laboratory phenomena typical of quantum field theory in curved spacetime, even more interesting is the possibility of studying regimes analogous to semiclassical gravity, and potentially unveiling, by analogy, how semiclassical gravity itself might emerge from an underlying theory of quantum gravity.

Analogue gravity shows that with continuous media and with quantum systems it is possible to reproduce phenomena profoundly related to the physics of gravity. These phenomena therefore seem to have a significant degree of generality, being common to many branches of physics in which it is possible to define a notion of geometry.

The analogy can be analyzed in both directions, and if in the laboratory we can verify the predictions of the gravitational case, we must ask ourselves if gravity could have further points of contact with the known theories for the laboratory systems. The phenomena of semiclassical gravity dependent on curved metrics and the phenomena of analogue gravity dependent on effective analogue metrics belong to the same class: this suggests that just as analogue gravity emerges perturbatively from a known theory, gravitational phenomena can also be emergent from an underlying unified theory.

The connections of gravity with fluid thermodynamics [46] and with collective quantum phenomena [47–49] have often been highlighted with the aim, as already mentioned above, of developing a theory of quantum gravity.

The possibility of establishing and studying the analogy between gravity and laboratory systems is therefore much more than an curiosity, it is a way of investigating fundamental properties of gravity that it are reasonably expected, even if hidden.

3.2.2 Applications of analogue gravity

The similarities between the propagation of perturbations in media of various kinds and the propagation of fields in curved spaces has been known for a long time, Gordon in 1923 described the effective metric for the propagation of light in dielectric media [50], through an expression now known as the Gordon form

$$g_{\text{effective}\,\mu\nu} = \eta_{\mu\nu} + \left(1 - n^{-2}\right) V_\mu V_\nu \,, \tag{3.30}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, V_μ is a 1-form and n is the refractive index of the medium, dependent on the position.

A later work by White [51] introduced, with the study of the propagation of acoustic excitations in fluids, some of the ideas now associated with analogue gravity. But the beginning of the research program of analogue gravity is usually set in 1981, when Unruh demonstrated the analogy between the dynamics of acoustic perturbations in presence of an acoustic horizon — realized where the flow from subcritical becomes supercritical — and the equations for the propagation of fields in a black hole spacetime [52]. He proposed this model as a way to study Hawking radiation and the process of black holes evaporation, using a system whose physics was entirely known, *i.e.* without issues such as the transplanckian problem.

It was then shown that other classical systems can be used for the realization of analogue models: for example gravity waves formed at the surface of a flowing fluid in a shallow basin can be used to simulate in a laboratory a horizon and the behavior of fields in a black hole spacetime [53]. Experiments of this nature have been realized and have led to the study of the production of classical gravity waves analogous to Hawking radiation [54–58]. These systems have also been used to explore the superradiance effect in analogous rotating black holes [59–61].

Other classical models include linear and non-linear electrodynamical systems where the analogue geometry for the propagation of electromagnetic perturbations is determined by the Faraday tensor in the medium [62–66]. Scientific interest in electromagnetic analogues is ongoing, see for instance [67–72], and references therein.

The phenomena observed in these classical systems are all stimulated. Employing quantum systems it is instead possible to devise analogue models which allow the simulation of spontaneous quantum phenomena.

Very successful models of analogue gravity have been those realized with Bose–Einstein condensates, which have been recognized as natural candidates for the purposes of the program [44,73–75]. The transplanckian problem has an immediate solution, since the healing length, set by the density and by the strength of the interaction of the cold atoms, plays the role of the minimum length scale analogous to the Planck scale. The healing length defines a natural cutoff, preserving the soundness of the theory. These systems have therefore a well understood ultraviolet completion [38, 76, 77].

In a flowing condensate it is possible to define an acoustic metric, fully determined by the condensate wavefunction: the excited part of the system is described in terms of Bogoliubov quasi-particles, the quantum perturbation of the condensate wavefunction, which propagate following the acoustic metric determined by the velocity of the condensate and by its density. Through these propagating quantum excitations it is possible to reproduce the typically quantum spontaneous phenomena that would not obtainable in classical fluids.

The cosmological particle creation can be simulated in time-dependent spatially homogeneous condensates, where controlling the 2-body interaction via Feshbach resonance it is possible to reproduce an evolving cosmological background, and where valuable information on how to approach the problem of the cosmological constant can be obtained in terms of the back-reaction of quasi-particles [78–81].

But the Bose–Einstein condensates are most renowned, in the context of analogue gravity, for their contribution to the physics of Hawking radiation. The production of analogue Hawking radiation in condensates has been widely studied, theoretically and numerically. As for classical fluids, a condensate flowing from a subcritical region to a supercritical region creates an acoustic horizon. The Bogoliubov quasi-particles are produced in the near horizon region in agreement with Hawking's predictions. It has been understood that the analogue Hawking radiation can be tracked by studying the density-density correlation function in the condensates, and the production of pairs on the two sides of the horizon leaves a clear trace in the correlation function [82–90].

Great effort has been spent in trying to devise and realize an experiment which could provide confirmation of the prediction of Hawking radiation, and corroborate the soundness of the analogue gravity paradigm. In recent years, strong evidence of Hawking radiation has been obtained [91–95]: the experiment with which it was possible to make these observations was a condensate at low density and very low temperature. The lasing effect induced by the presence of two acoustic horizons resulted in an amplification of the radiation, and enabled the measurement of the density-density correlation function. The measured results showed the characteristic traces, largely confirming the theoretical expectations, and were then validated by further numerical analysis [96].

Further theoretical and numerical studies are currently aimed at the development of Bose–Einstein condensate models for rotating black holes, which would mimic the processes of superradiance and the Penrose effect [97, 98].

Other quantum analogue models that should be mentioned are those realized with superfluid ^{3}He which can simulate rotating black holes [99] and fiber-optics models [100], in which the refractive index can act as phase-velocity horizon inducing the creation of photons.

In what follows we shall look in more detail to some of the aforementioned analogue gravity frameworks.

3.3 Analogue models in classical fluids

3.3.1 Classical non-relativistic fluids

The model proposed by Unruh that started the analogue gravity program consisted in describing the perturbations propagating in a fluid medium, following an effective geometry described by an effective acoustic metric.

The description of a flowing classical fluid, with fixed equation of state, is made through its velocity field $\mathbf{v}$ and its local density ρ and pressure P. The dynamics of the flow is based on two equations: the continuity equation and the Euler equation, respectively

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{3.31}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla V_{\text{ext}}\,, \tag{3.32}$$

where V_{ext} is an external potential exercising a force on the fluid. The Euler equation (3.32) is valid for a perfect fluid without viscosity.

It is possible to specify these equations, making additional assumptions on the behavior of the fluid. At first it is convenient to modify the Euler equation, knowing the identity for the curl of a curl of a vector

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2}\nabla \mathbf{v}^2 - (\mathbf{v} \cdot \nabla)\,\mathbf{v}\,, \tag{3.33}$$

$$\Downarrow$$

$$\partial_t \mathbf{v} = -\frac{1}{\rho}\nabla P - \frac{1}{2}\nabla \mathbf{v}^2 + \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla V_{\text{ext}}\,. \tag{3.34}$$

The curl represents the rotation impressed on a fluid of a field. A flow is rotational when there is shear strain deformation of the fluid element which is due to tangential forces or shear stresses. While shear stresses are due to the viscosity of the fluid, also perfect fluids can have a rotational flow by external work or heat interaction, although the choice of a perfect fluid is necessary to have irrotationality.

When the fluid is irrotational the Euler equation is simplified, and the term depending on the curl of the velocity field is eliminated: if the fluid is irrotational the velocity field can be written as the gradient of a potential, an therefore its curl vanishes. Substituting the expression

$$\mathbf{v} = \nabla \phi\,, \tag{3.35}$$

in the continuity equation and in the Euler equation, we get

$$0 = \partial_t \rho + \nabla \cdot (\rho \nabla \phi)\ , \tag{3.36}$$

$$\nabla \partial_t \phi = -\frac{1}{\rho}\nabla P - \frac{1}{2}\nabla\,(\nabla \phi)^2 - \nabla V_{\text{ext}}\,. \tag{3.37}$$

Then we can make the last assumption, of barotropicity. When the pressure of the fluid is a function of the density $P = P\,(\rho)$, the pressure term in the Euler equation can be written as a gradient of a thermodynamical potential: the enthalpy. It is also convenient to point out already that the derivative of the pressure with respect to the density is the squared speed of sound, c_s:

$$\mathrm{d}h = \frac{\mathrm{d}P}{\rho} = c_s^2 \frac{\mathrm{d}\rho}{\rho}\,. \tag{3.38}$$

Under all these assumptions (no viscosity, irrotationality, barotropicity) the Euler equation can be integrated and, apart from an arbitrary integration constant, we get the Bernoulli equation

$$\partial_t \phi = -\,h - \frac{1}{2}\,(\nabla \phi)^2 - V_{\text{ext}}\,. \tag{3.39}$$

The idea of analogue gravity proposed by Unruh was formulated considering small perturbations to this equation together with the continuity equation (3.36), varying the velocity potential ϕ and the enthalpy h, *i.e.* varying the pressure and the density without breaking the assumption of barotropicity, by small quantities. We consider $\phi_1 \ll \phi$ and $\rho_1 \ll \rho$. The external potential plays no role in this perturbation. At first we perturb the Bernoulli equation (3.39), making use of equation (3.38) for the variation of the enthalpy, and we obtain an expression for ρ_1 as a conjugated variable to ϕ_1

$$\partial_t \phi_1 = -c_s^2 \frac{\rho_1}{\rho} - (\nabla\phi) \cdot (\nabla\phi_1) \,, \tag{3.40}$$

$$\Downarrow$$

$$\rho_1 = -\frac{\rho}{c_s^2} \left(\partial_t + (\nabla\phi) \cdot \nabla\right) \phi_1 \,. \tag{3.41}$$

Then, plugging this expression in the perturbation of the continuity equation, we get a second order differential equation for ϕ_1, the perturbation of the velocity potential

$$0 = \left(\partial_t + \nabla\left(\nabla\phi\right)\right) \rho_1 + \nabla \cdot \left(\rho \nabla\phi_1\right) = \tag{3.42}$$

$$= -\left(\partial_t + \nabla\left(\nabla\phi\right)\right) \frac{\rho}{c_s^2} \left(\partial_t + (\nabla\phi) \cdot \nabla\right) \phi_1 + \nabla \cdot \left(\rho \nabla\phi_1\right) \,. \tag{3.43}$$

This equation is clearly the same one would have found for a massless Klein–Gordon field in a curved spacetime, which generally would be

$$0 = \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi_1 \,. \tag{3.44}$$

Commonly one would include a prefactor $1/\sqrt{-g}$, giving the covariant expression of the equation, but it is not necessary and we can always easily factor it back in later, after finding the expression for the determinant of the metric.

It is immediate to find the elements of the inverse of the metric $g^{\mu\nu}$, using that the determinant of the inverse of the metric must be the inverse of the determinant of the metric

$$\sqrt{-g} = \frac{\rho^2}{c_s} \,, \tag{3.45}$$

$$g^{tt} = \frac{c_s}{\rho} \left(-\frac{1}{c_s^2}\right) \,, \tag{3.46}$$

$$g^{ti} = \frac{c_s}{\rho} \left(-\frac{1}{c_s^2} \left(\nabla\phi\right)^i\right) \,, \tag{3.47}$$

$$g^{ij} = \frac{c_s}{\rho} \left(\delta^{ij} - \frac{1}{c_s^2} \left(\nabla\phi\right)^i \left(\nabla\phi\right)^j\right) \,, \tag{3.48}$$

and then inverting these expressions we get the elements of the acoustic metric

$$g_{tt} = \frac{\rho}{c_s} \left(-c_s^2 + \left(\nabla\phi\right)^2\right) \,, \tag{3.49}$$

$$g_{ti} = \frac{\rho}{c_s} \left(-\left(\nabla\phi\right)_i\right) \,, \tag{3.50}$$

$$g_{ij} = \frac{\rho}{c_s} \delta_{ij} \,, \tag{3.51}$$

and the line element is the one typically associated with the acoustic metric, *i.e.*

$$\mathrm{d}s^2 = \frac{\rho}{c_s}\left(-c_s^2\mathrm{d}t^2 + \delta_{ij}\left(\mathrm{d}x^i - (\nabla\phi)^i\,\mathrm{d}t\right)\left(\mathrm{d}x^j - (\nabla\phi)^j\,\mathrm{d}t\right)\right)\,, \tag{3.52}$$

where ρ/c_s plays the role of a conformal factor. A conformal factor in general is not homogeneous and can depend on the position but does not modify the causal structure.

For an acoustic metric it is simple to recognize how to realize a horizon. The flow is subcritical as long as the fluid is slower than the speed of sound $\mathbf{v}^2 = (\nabla\phi)^2 < c^2$, and it is supercritical when if flows faster $\mathbf{v}^2 = (\nabla\phi)^2 > c^2$. At the interface, an acoustic horizon forms, that the perturbations cannot cross in both directions: with respect to fluid, the speed at which this acoustic perturbations propagate is the speed of sound, and therefore it is insufficient to cross the horizon going upstream from the supercritical region to the subcritical.

Near the acoustic horizon the dynamics of the perturbations will be, as portrayed by the equation (3.43), analogous to that of a massless scalar field near the trapping horizon in a gravitational black hole spacetime.

3.3.2 Classical relativistic fluids

It is possible to generalize the above analogue model to relativistic barotropic perfect fluids, as extensively discussed in [101]. The fluid is described in a fully covariant form, and admits a non-relativistic limit which is compatible with the model seen previously.

In the relativistic setting we consider the barotropic perfect fluid to flow in a manifold with a geometry described by a generic metric tensor $g_{\mu\nu}$.

The assumption of barotropicity again implies that the pressure is a function of the energy density, with an equation of state $P = P(\rho)$, and the assumption that the fluid is a perfect implies that its stress-energy tensor takes the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + P\left(g^{\mu\nu} + u^\mu u^\nu\right)\,, \tag{3.53}$$

where u is the timelike four-velocity field of the fluid, which is properly normalized in every point of the manifold to $u^\mu u_\mu = -1$.

The equation of state of the barotropic fluid provides also an unambiguous definition of speed of sound as the derivative of the pressure with respect to the energy density

$$\mathrm{d}P = c_s^2\mathrm{d}\rho\,. \tag{3.54}$$

The fluid is described with the relativistic continuity equation and the relativistic Euler equation. Taken together, these two equations are equivalent to requiring the stress-energy tensor to be divergenceless, and they are obtained respectively projecting the equation $\nabla_\mu T^{\mu\nu} = 0$ on the direction of u^μ, and projecting it orthogonally to u. For the perfect fluid we have

$$0 = \nabla_\mu \rho u^\mu + P\nabla_\mu u^\mu\,, \tag{3.55}$$

$$0 = (\rho + P)\,u^\nu\nabla_\nu u^\mu + \left(g^{\mu\nu} + u^\mu u^\nu\right)\left(\nabla_\nu P\right)\,. \tag{3.56}$$

When the fluid is barotropic and the pressure is a function of the density it is easy to verify that the relativistic continuity equation (3.36) can be written in a more compact expression

$$0 = \nabla_\mu \left(\exp\left(\int_{P^{-1}(0)}^{\rho} \frac{\mathrm{d}\rho}{\rho + P(\rho)} \right) u^\mu \right) , \tag{3.57}$$

where P^{-1} is the inverse of the equation of state $P(\rho)$, and the lower limit in the integration is that value of density for which the pressure vanishes.

In this case, the exponential $\exp\left(\int \mathrm{d}\rho/(\rho + P)\right)$ can be interpreted as the local density of a conserved quantity: the number of particles predicted by the equation of state of the perfect fluid, and the limits of the integral are set in accordance to it. In the non-relativistic case it simply reduces to the local mass density, while in the relativistic case it is the energy density measured by an observer at rest with respect to the fluid.

In general relativity, requiring a fluid to be irrotational corresponds to requiring the timelike vector field describing its four-velocity to satisfy the equation

$$u_{[\mu} \nabla_\nu u_{\sigma]} = 0 . \tag{3.58}$$

Frobenius' theorem [13] states that a timelike vector-field satisfies this equation if and only if it is hypersurface orthogonal, *i.e.* if exists a foliation of the manifold made of spacelike hypersurfaces orthogonal to the vector field. This implies that the four-velocity field can be written as

$$u_\mu = \frac{\nabla_\mu \Theta}{\|\nabla\Theta\|} , \tag{3.59}$$

$$\|\nabla\Theta\| = \sqrt{-g^{\mu\nu} (\nabla_\mu \Theta)(\nabla_\nu \Theta)} . \tag{3.60}$$

With this definition of the vector field as the normalized four-gradient of a scalar function, we can manipulate the relativistic Euler equation (3.56) obtaining a relativistic Bernoulli equation

$$0 = \|\nabla\Theta\| - C(\Theta) \exp\left(\int \frac{\mathrm{d}P}{\rho + P} \right) = \tag{3.61}$$

$$= \|\nabla\Theta\| - C(\Theta) \exp\left(\int_{P^{-1}(0)}^{\rho} \frac{c_s^2 \mathrm{d}\rho}{\rho + P(\rho)} \right) , \tag{3.62}$$

where $C(\Theta)$ is a function that can always be set to 1 without losing generality: in principle one can always redefine Θ into a different $f(\Theta)$, without changing the relativistic Euler equation (3.56) which is obtained by projecting the gradient of the relativistic Bernoulli equation (3.62) on the direction orthogonal to u. But u is the field orthogonal to the foliation defined by Θ, which is the same of $f(\Theta)$ since the hypersurfaces of the foliation are the set of points where the scalar function Θ assumes a same value, property preserved by all the functions $f(\Theta)$. The same foliation gives the same normalized vector field u.

We can now apply a variation to both the relativistic Bernoulli equation (3.62) and the relativistic continuity equation (3.55), which give respectively

$$0 = \delta \|\nabla\Theta\| - \|\nabla\Theta\| \frac{c_s^2 \delta\rho}{\rho + P}, \tag{3.63}$$

$$0 = \nabla_\mu \left(\exp\left(\int \frac{\mathrm{d}\rho}{\rho + P} \right) \left(\left(\frac{\delta\rho}{\rho + P} \right) u^\mu + \delta u^\mu \right) \right). \tag{3.64}$$

The variations $\delta \|\nabla\Theta\|$ and δu are easily obtained as

$$\delta\rho = \frac{\rho + P}{c_s^2} \frac{\delta \|\nabla\Theta\|}{\|\nabla\Theta\|} = -\frac{\rho + P}{c_s^2} u^\nu \frac{\nabla_\nu \delta\Theta}{\|\nabla\Theta\|}, \tag{3.65}$$

$$\delta u^\mu = (g^{\mu\nu} + u^\mu u^\nu) \frac{\nabla_\nu \delta\Theta}{\|\nabla\Theta\|}. \tag{3.66}$$

We substitute these expressions in equation (3.63) and in equation (3.64), and then combining them we obtain a Klein–Gordon equation for the perturbation $\delta\Theta$ of the scalar function that defines the foliation

$$0 = \nabla_\mu \left(\Omega \left(g^{\mu\nu} + \left(1 - \frac{1}{c_s^2} \right) u^\mu u^\nu \right) \right) \nabla_\nu \delta\Theta, \tag{3.67}$$

$$\Omega = \frac{\exp\left(\int \frac{\mathrm{d}\rho}{\rho + P} \right)}{\|\nabla\Theta\|} = \exp\left(\int_{P^{-1}(0)}^{\rho} \frac{\left(1 - c_s^2\right) \mathrm{d}\rho}{\rho + P(\rho)} \right). \tag{3.68}$$

The equation (3.67) is clearly the analogue of a massless Klein–Gordon equation, but it is not immediate to recognize exactly the expression of the analogue metric. It is possible to verify that for a generic invertible symmetric d-dimensional tensor m and a generic 1-form v it holds that

$$\det\left(\alpha\left(m_{\mu\nu} + \beta v_\mu v_\nu\right)\right) = \alpha^d \left(\det m_{\mu\nu}\right) \left(1 + \beta \left(m^{-1}\right)^{\mu\nu} v_\mu v_\nu\right). \tag{3.69}$$

Therefore we obtain that the analogue metric g_{am} is such that

$$\sqrt{-g_{\mathrm{am}}}\, g_{\mathrm{am}}^{\mu\nu} = \Omega \left(g^{\mu\nu} + \left(1 - \frac{1}{c_s^2} \right) u^\mu u^\nu \right), \tag{3.70}$$

$$\Downarrow$$

$$\sqrt{-g_{\mathrm{am}}} = \frac{\Omega^2}{c_s \sqrt{-g}}, \tag{3.71}$$

$$g_{\mathrm{am}}^{\mu\nu} = \frac{c_s \sqrt{-g}}{\Omega} \left(g^{\mu\nu} + \left(1 - \frac{1}{c_s^2} \right) u^\mu u^\nu \right), \tag{3.72}$$

$$\Downarrow$$

$$g_{\mathrm{am}\,\mu\nu} = \frac{\Omega}{c_s \sqrt{-g}} \left(g_{\mu\nu} + \left(1 - c_s^2\right) u_\mu u_\nu \right). \tag{3.73}$$

It is possible to check the consistency of this expression with the analogue metric of the non-relativistic fluid. The make this check, consider the spacetime metric to be the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, and consider both the speed of sound and the velocity of the fluid to be much smaller than the speed of light.

Under this conditions, one finds that the conformal factor is $\Omega \approx \rho/c_s$ and the normalized future directed four-vector field of the fluid is $u^\mu \approx \left(1 + \mathbf{v}^2/2, \mathbf{v}\right)^\mu$. Considering the various elements of the analogue metric, applying these approximations and considering only the leading order we obtain the terms

$$\eta_{00} + \left(1 - c_s^2\right) u_0 u_0 \approx -c_s^2 + \mathbf{v}^2 \,, \tag{3.74}$$

$$\left(1 - c_s^2\right) u_0 u_i \approx -v_i \,, \tag{3.75}$$

$$\eta_{ij} + \left(1 - c_s^2\right) u_i u_j \approx \delta_{ij} \,, \tag{3.76}$$

which, together with the conformal factor, are exactly the tensor elements of the analogue metric in the non-relativistic fluid.

In analogue models realized with relativistic fluid, when the vector field u is timelike and normalized and when we consider the spacetime metric $g_{\mu\nu}$ to be flat, *i.e.* in a laboratory experiment it is expected that the spacetime is flat and described by the flat Minkowski metric $\eta_{\mu\nu}$, the analogue metric is of the type

$$g_{\mathrm{am}\,\mu\nu} = \eta_{\mu\nu} + \left(1 - c_s^2\right) u_\mu u_\nu \,, \tag{3.77}$$

which is indeed of the Gordon form in equation (3.30). Note that we shall use this denomination even if instead of $\eta_{\mu\nu}$ we consider some other expression for the flat metric, *i.e.* a metric $\overline{g}_{\mu\nu}$ obtainable from the Minkowski metric applying a coordinate transformation, and also when the vector field u does not satisfy the hypothesis of irrotationality.

3.3.3 Gravity waves

We have already mentioned that another important class of analogue models are those realized with classical fluids in a gravitational field, *i.e.* systems where it is possible to observe the waves created at the interface between two fluids as a result of a gravitational restoring force.

These setups are generally realized with a tank of shallow water in contact with air. Even if they have been widely explored, there is still remarkable scientific prolificacy of results in experiments aimed at observing analogue Hawking radiation, superradiance and related effects in both $(1 + 1)$-dimensional systems [54–58] and $(2 + 1)$-dimensional systems with vorticity [59–61].

The system consists of water and air, in which we consider the density of the second negligible compared to that of the first, which we describe as perfect, incompressible and irrotational fluids. In this case the potential flow is also divergenceless: again it holds that $\mathbf{v} = \nabla\phi$, but it is also assumed that it holds $\nabla \cdot \mathbf{v} = 0$. When a fluid is incompressible, it follows from the continuity equation (3.36) that its density is constant along the flow, $\mathrm{d}\rho/\mathrm{d}t = 0$; we can make a stronger but reasonable assumption requiring the flow to be stationary, and therefore we have that in every point the density is constant and homogeneous, *i.e.* we have both $\partial_t \rho = 0$ and $\nabla \rho = 0$.

Irrotationality allows again to study the system with the Bernoulli equation. We consider as external potential the Newtonian gravitational field, with gravity acceleration g. We therefore have two equations: the Bernoulli equation for the flow and the Laplacian equation for the velocity potential (which is the condition of incompressibility for an irrotational velocity field):

$$\partial_t \phi + \frac{1}{2} (\nabla \phi)^2 = -\frac{P}{\rho} - gz \,, \tag{3.78}$$

$$\nabla^2 \phi = 0 \,. \tag{3.79}$$

To describe gravity waves one considers the interface between water and air as the object of interest. In our $(3+1)$-dimensional laboratories, where we consider z as the vertical coordinate associated with the Newtonian gravitational potential, the interface is located at $z = h\,(x, y, t)$, which is a surface defined implicitly: the function h is the height at which for a point (x, y) on the plane, at a time t, one can identify the time-dependent surface where water and air are separated.

For any point on the plane, the depth of the water is therefore h, which is assumed to be a short length with respect to any wavelength of interest, while the bottom of the tank is at $z = 0$. The boundary conditions one must set are: the pressure must be continuous at the interface; the velocity field at the interface must satisfy Dini's implicit function theorem [102]; the velocity field at the bottom of the tank must be parallel to it.

$$\left[\partial_t \phi + \frac{1}{2} (\nabla \phi)^2 + gz \right]_h = 0 \,, \tag{3.80}$$

$$\left[\partial_z \phi - \partial_t h - \left(\nabla_\parallel \phi \right) \cdot \left(\nabla_\parallel h \right) \right]_h = 0 \,, \tag{3.81}$$

$$\left[\partial_z \phi \right]_0 = 0 \,, \tag{3.82}$$

where by $\nabla_\parallel$ we mean the operator of the two spatial derivatives with respect to the coordinates x and y, *i.e.* a 3-dimensional operator which is equal to the gradient projected on the horizontal plane, where the third component is identically null. The equation (3.80) stems from the fact that the density of air is negligible with respect to the density of water. In particular, we notice that the variations in height of the interface have to be negligible for the assumption of shallow water wave to hold, implying that only the components on the plane (x, y) should be considered, and the component $v_z = \partial_z \phi$ can be neglected with respect to the non-null components of $\nabla_\parallel \phi$.

The boundary conditions are equations on the horizontal plane which can be solved perturbatively. In particular we consider the decomposition $\phi = \phi_0 + \delta\phi$ and $h = h_0 + \delta h$.

First of all, considering equation (3.80), at the interface one must have that

$$0 = \left[\partial_t \phi + \frac{1}{2} (\nabla\phi)^2 + gz\right]_h = \tag{3.83}$$

$$= \partial_t \delta\phi + (\nabla\phi_0) \cdot (\nabla\delta\phi) + g\delta h = \tag{3.84}$$

$$= (\partial_t + \nabla (\nabla\phi_0)) \delta\phi + g\delta h = \tag{3.85}$$

$$= \left(\partial_t + \nabla_\parallel \left(\nabla_\parallel \phi_0\right)\right) \delta\phi + g\delta h \,, \tag{3.86}$$

where we have made use of the Laplacian equation for ϕ_0 and of the assumption that the vertical velocity, the spatial derivative $\partial_z \phi$, is negligible.

The two boundary conditions at the bottom of the tank and at the interface are modified, and are better understood considering a series expansion of $\delta\phi$ with respect to the coordinate z, which — in the considered limit of a shallow water basin — can be assumed to be small.

$$\delta\phi = \delta\phi_{(0)} + z\delta\phi_{(1)} + \frac{z^2}{2}\delta\phi_{(2)} + \dots \,, \tag{3.87}$$

$$\Downarrow$$

$$[\partial_z \delta\phi]_0 = \delta\phi_{(1)} = 0 \,, \tag{3.88}$$

$$[\partial_z \delta\phi]_h = h\delta\phi_{(2)} \left(1 + \mathcal{O}\left(\frac{h}{\lambda}\right)\right) \,. \tag{3.89}$$

where by λ we generically mean the wavelength of the perturbation. The Laplacian equation (3.79) for ϕ must be satisfied separately by ϕ_0 and by $\delta\phi$, and for the latter — considering the series expansion in z — it becomes

$$0 = \nabla_\parallel^2 \delta\phi + \delta\phi_{(2)} \left(1 + \mathcal{O}\left(\frac{h}{\lambda}\right)\right) \,, \tag{3.90}$$

where it holds that $\nabla_\parallel^2 = \partial_x^2 + \partial_y^2$. This equation holds exactly at the bottom of the tank, but it is a good approximation at every height. This is true also at the interface when the height h is small.

The boundary condition at the interface can therefore be written as an equation on the horizontal plane

$$0 = \left[\partial_z \delta\phi - \partial_t \delta h - \left(\nabla_\parallel \phi_0\right) \cdot \left(\nabla_\parallel \delta h\right) - \left(\nabla_\parallel h\right) \cdot \left(\nabla_\parallel \delta\phi\right)\right]_h = \tag{3.91}$$

$$= - \left(\partial_t + \left(\nabla_\parallel \phi_0\right) \cdot \nabla_\parallel\right) \delta h - \nabla_\parallel \cdot \left(h \nabla_\parallel \delta\phi\right) \,. \tag{3.92}$$

In conclusion, putting together equation (3.86) and equation (3.92), we obtain an equation for the fluctuation of the velocity potential which is analogous to the Klein–Gordon equation for a massless scalar field in a $(2+1)$-dimensional spacetime. The interface is the analogue of a manifold where the gravity waves propagate as a massless Klein–Gordon field:

$$0 = \left(\partial_t + \nabla_\parallel \left(\nabla_\parallel \phi_0\right)\right) \frac{1}{g} \left(\partial_t + \left(\nabla_\parallel \phi_0\right) \cdot \nabla_\parallel\right) \delta\phi - \nabla_\parallel \cdot \left(h \nabla_\parallel \delta\phi\right) \,, \tag{3.93}$$

corresponding to a $(2+1)$-dimensional acoustic metric determined by the horizontal components of the velocity field of the flowing water at the interface $\left[\nabla_{\parallel}\phi_0\right]_h$. The line element of this metric is

$$\mathrm{d}s^2 = \frac{h}{g}\left(-hg\mathrm{d}t^2 + \delta_{ij}\left(\mathrm{d}x^i - \left(\nabla_{\parallel}\phi_0\right)^i \mathrm{d}t\right)\left(\mathrm{d}x^j - \left(\nabla_{\parallel}\phi_0\right)^j \mathrm{d}t\right)\right) , \tag{3.94}$$

where the speed of propagation of the classical gravity waves is therefore $c_{\mathrm{cgw}} = \sqrt{hg}$.

The linearized equation (3.93) for the propagation of the perturbation $\delta\phi$ holds under the assumption of shallow waters and long wavelengths. For shorter wavelengths there are deviations from the linear dispersion relation. Higher order corrections will break the $(2+1)$-dimensional Lorentz invariance, and the modified dispersion relation will depend on an effective acceleration $\tilde{g}$ which would not be spatially homogeneous in the plane, substituting the gravity acceleration g. The dispersion relation is modified due to the fact that at sufficiently high momenta the depth of the tank cannot be neglected, and there are corrections that depend on the hydrodynamical properties of the fluid (density, speed of sound and surface tension) and on the configuration of the system (depth of tank, acceleration of gravity) and have been discussed in the literature [103].

3.4 Analogue gravity in Bose–Einstein condensates

Analogue models of gravity realized with Bose–Einstein condensates are of great importance in the general research program and study, as introduced in the previous chapter, the Bogoliubov excitations in condensates. Bogoliubov quasi-particles are an excellent example of perturbations propagating in a medium which is provided, in the case of Bose–Einstein condensates, by the condensate wavefunction itself, which can be treated as a classical background for the propagation of quasi-particles.

In this section we will highlight the many characteristics that these quantum systems have in common with the analogue models realized with classical fluids, both relativistic and non-relativistic. In particular, the excitations that we consider have a linear dispersion relation for large wavelengths and small momenta, are based on an underlying quantum atomic theory which is unitary and complete in the ultraviolet and which does not incur the transplanckian problem of short distances in the laboratory system.

The significant novelty of this system lies in the fact that the considered excitations are intrinsically quantum in nature, and will allow with even greater accuracy to describe the phenomena of quantum field theory in curved spacetime that generate the previously discussed problems at the semiclassical level.

3.4.1 The Madelung representation

We introduce the formalism usually adopted in analogue gravity, the Madelung representation, which is enlightening in the interpretation of the condensate as a quantum fluid where quasi-particles can propagate. The condensate wavefunction $\langle\phi\rangle$, a complex function, is rewritten in terms of two real functions phase and square modulus

$$\langle\phi\rangle = \langle\rho\rangle\, e^{\langle i\theta\rangle}\,, \tag{3.95}$$

where the functions $\langle\rho\rangle$ and $\langle\theta\rangle$ can be shown to have a straightforward hydrodynamical interpretation.

This formalism, applied in quantum mechanics to the single particle Schrödinger equation, can be extended to Bose–Einstein condensates to both the condensate wavefunction and the quantum perturbation, both in the relativistic and in the non-relativistic case.

In the mean-field approximation we apply the Madelung representation to both the mean-field $\langle\phi_0\rangle$ and to its perturbation $\delta\phi$.

For any theory we will have a different equation for the dynamics of the mean-field, approximating the condensate wavefunction, which we can reformulate in the new variables

$$\langle\phi_0\rangle = \langle\rho_0\rangle^{1/2}\, e^{i\langle\theta_0\rangle}\,, \tag{3.96}$$

and the expression for its complex conjugate follows immediately. This applies to any system, relativistic and non-relativistic, for different interaction terms: it is a change of variables always valid.

For example, we can consider the non-relativistic $\lambda\phi^4$ theory of the Hamiltonian (2.17) and apply the Madelung representation to the Gross–Pitaevskii equation (2.21) for the condensate wavefunction, and to the Bogoliubov–de Gennes equation (2.46) for the quantum perturbation, transforming appropriately the mean-field $\langle\phi_0\rangle$ and the quantum perturbation $\delta\phi$.

Applying the Madelung representation to the Gross–Pitaevskii equation (2.21) we can isolate a phase contribution from every term and then split the equations into two real differential equation for the new variables:

$$\partial_t\langle\rho_0\rangle = -\frac{1}{m}\nabla\left(\langle\rho_0\rangle\nabla\langle\theta_0\rangle\right)\,, \tag{3.97}$$

$$\partial_t\langle\theta_0\rangle = \langle\rho_0\rangle^{-1/2}\frac{\nabla^2}{2m}\langle\rho_0\rangle^{1/2} - \frac{1}{2m}\left(\nabla\langle\theta_0\rangle\right)\left(\nabla\langle\theta_0\rangle\right) - \lambda\langle\rho_0\rangle - V_{\rm ext}\,. \tag{3.98}$$

The first of these equations has an immediate interpretation as a continuity equation, were $\langle\rho_0\rangle$, the atom number density, is the local density of a conserved quantity — the total number of atoms — and $\langle\rho_0\rangle\nabla\langle\theta_0\rangle/m$ is its corresponding density current. The continuity equation is a reformulation of equation (2.24): the continuity equation implies, in absence of back-reaction, the global conservation of the number of atoms; the atoms are transported along the flow defined by the gradient of the phase.

The physical interpretation of the phase $\langle\theta_0\rangle$ is also straightforward: this function is the potential of a velocity field, as seen for the classical fluids, and equation (3.98) is the Bernoulli equation for this quantum fluid.

We can also apply the Madelung representation to the quantum fluctuation. If the mean-field is transformed as in equation (3.96), it is natural to write its quantum perturbation as

$$\frac{\delta\phi}{\langle\phi_0\rangle} = \left(\frac{\rho_1}{2\langle\rho_0\rangle} + i\theta_1\right) , \tag{3.99}$$

$$\rho_1 = \langle\rho_0\rangle\left(\frac{\delta\phi}{\langle\phi_0\rangle} + \frac{\delta\phi^\dagger}{\overline{\langle\phi_0\rangle}}\right) , \tag{3.100}$$

$$\theta_1 = -\frac{i}{2}\left(\frac{\delta\phi}{\langle\phi_0\rangle} - \frac{\delta\phi^\dagger}{\overline{\langle\phi_0\rangle}}\right) . \tag{3.101}$$

The new quantum operators θ_1 and ρ_1 inherit their commutation relations from those of previously established in equations (2.38)–(2.39).

$$[\theta_1(x), \rho_1(y)] = -i\delta^3(x,y) , \tag{3.102}$$

$$[\theta_1(x), \theta_1(y)] = 0 , \tag{3.103}$$

$$[\rho_1(x), \rho_1(y)] = 0 , \tag{3.104}$$

which show how these two quantum fields are one the conjugate of the other.

The equations describing their dynamics are obtained from the Bogoliubov–de Gennes and will depend on the Hamiltonian of the many-body system and on the configuration of the mean-field.

In the $\lambda\phi^4$ theory considered, we apply this transformation of the operators $\delta\phi$ and its Hermitian conjugate, using the Madelung representation for both the perturbation and the mean-field. The Bogoliubov–de Gennes equation (2.46) is reorganized in two coupled dynamical quantum equations for θ_1 and ρ_1

$$\left(\partial_t + \frac{(\nabla\langle\theta_0\rangle)}{m}\nabla\right)\theta_1 = \frac{1}{4m\langle\rho_0\rangle}\nabla\left(\langle\rho_0\rangle\nabla\frac{\rho_1}{\langle\rho_0\rangle}\right) - \lambda\rho_1 , \tag{3.105}$$

$$\left(\partial_t + \frac{(\nabla\langle\theta_0\rangle)}{m}\nabla\right)\frac{\rho_1}{\langle\rho_0\rangle} = -\frac{1}{m\langle\rho_0\rangle}\nabla\left(\langle\rho_0\rangle\nabla\theta_1\right) . \tag{3.106}$$

While further consideration should be made before defining an analogue gravity model, we can already recognize equations similar to those studied for classical fluids. The dynamics of the perturbation is naturally described by the total derivatives of the two fields taken along the flow of the condensate.

3.4.2 Non-relativistic condensates

We consider the analogue model realized in a bosonic many-body system described by the Hamiltonian (2.17). The description is made adopting the Madelung representation.

In this framework we can show how the Bogoliubov quasi-particles, *i.e* the linearized quantum perturbations of the condensate, are analogous to massless bosons propagating in curved spacetime.

The assumptions made are therefore that the system can be described in terms of a condensate wavefunction, which follows the Gross–Pitaevskii equation, and of its small quantum perturbations, which are described by the Bogoliubov–de Gennes equation.

The further assumption we have to make is the validity of the hydrodynamic approximation: we require that the quantum pressure term in equation (3.105) is negligible

$$\frac{1}{4\langle\rho_0\rangle}\nabla\left(\langle\rho_0\rangle\nabla\frac{\rho_1}{\langle\rho_0\rangle}\right)\ll\lambda m\rho_1=\frac{1}{\xi^2}\frac{\rho_1}{\langle\rho_0\rangle}\,. \tag{3.107}$$

This implies that the condensate wavefunction and the quantum perturbation change over length scales much larger than the healing length $\xi=\sqrt{1/\lambda\langle\rho_0\rangle m}$. When this approximation holds, the condensate can be treated as a continuum medium where the perturbations behave as phononic excitations with a linear dispersion relation, and are not affected by the microscopic structure of the many-body system.

Under these assumptions, the dynamical equation (3.105) is made significantly simpler, and provides an easily workable definition of ρ_1. For convenience we also reformulate the equation (3.106), making use of the continuity equation (3.97) for the mean-field

$$-\frac{1}{\lambda}\left(\partial_t+\frac{\nabla\langle\theta_0\rangle}{m}\nabla\right)\theta_1=\rho_1\,, \tag{3.108}$$

$$\left(\partial_t+\nabla\frac{(\nabla\langle\theta_0\rangle)}{m}\right)\rho_1=-\frac{1}{m}\nabla\left(\langle\rho_0\rangle\nabla\theta_1\right)\,. \tag{3.109}$$

Combining these two equations it is clear that together they define a massless Klein–Gordon equation for the quantum field θ_1

$$0=-\left(\partial_t+\nabla\frac{(\nabla\langle\theta_0\rangle)}{m}\right)\frac{m}{\lambda}\left(\partial_t+\frac{\nabla\langle\theta_0\rangle}{m}\nabla\right)\theta_1+\nabla\left(\langle\rho_0\rangle\nabla\theta_1\right)\,. \tag{3.110}$$

This is clearly a massless Klein–Gordon equation for an acoustic metric, analogue to that of a curved spacetime $0=\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\theta_1$. Apart for an irrelevant overall constant, the elements of the inverse of the metric tensor can be extracted directly. We do so rewriting the coupling λ in terms of the speed of sound $c_s=\sqrt{\lambda\langle\rho_0\rangle/m}$, which generally can depend on time and position

$$\sqrt{-g}=\frac{\langle\rho_0\rangle^2}{c_s}\,, \tag{3.111}$$

$$g^{tt}=\frac{c_s}{\langle\rho_0\rangle}\left(-\frac{1}{c_s^2}\right)\,, \tag{3.112}$$

$$g^{ti}=\frac{c_s}{\langle\rho_0\rangle}\left(-\frac{1}{c_s^2}\frac{\partial^i\langle\theta_0\rangle}{m}\right)\,, \tag{3.113}$$

$$g^{ij}=\frac{c_s}{\langle\rho_0\rangle}\left(\delta^{ij}-\frac{1}{c_s^2}\frac{\partial^i\langle\theta_0\rangle}{m}\frac{\partial^j\langle\theta_0\rangle}{m}\right)\,, \tag{3.114}$$

which corresponds to the metric tensor of components

$$g_{tt} = \frac{\langle\rho_0\rangle}{c_s}\left(-c_s^2 + \delta^{ij}\frac{\partial_i\langle\theta_0\rangle}{m}\frac{\partial_j\langle\theta_0\rangle}{m}\right), \tag{3.115}$$

$$g_{ti} = \frac{\langle\rho_0\rangle}{c_s}\left(-\frac{\partial_i\langle\theta_0\rangle}{m}\right), \tag{3.116}$$

$$g_{ij} = \frac{\langle\rho_0\rangle}{c_s}\delta_{ij}, \tag{3.117}$$

the line element associated with this acoustic metric is therefore

$$\mathrm{d}s^2 = \frac{\langle\rho_0\rangle}{c_s}\left(-c_s^2\mathrm{d}t^2 + \delta_{ij}\left(\mathrm{d}x^i - \frac{\partial^i\langle\theta_0\rangle}{m}\mathrm{d}t\right)\left(\mathrm{d}x^j - \frac{\partial^j\langle\theta_0\rangle}{m}\mathrm{d}t\right)\right). \tag{3.118}$$

Again we see that the condensate wavefunction produces an acoustic horizon for the propagation of the perturbation where the velocity of the condensate equals the speed of sound, and the Bogoliubov quasi-particles cannot cross it propagating from the supercritical region towards the subcritical region.

Finally we remark that the conformal factor $\langle\rho_0\rangle/c_s$ is in general position and time dependent, but it is a number when the condensate is stationary and the number density is homogeneous.

The case of homogeneous condensate, in which both $\langle\rho_0\rangle$ and $\langle\theta_0\rangle$ do not depend on position, is particularly interesting because it is the case in which we can calculate explicitly the deviations of the dispersion relation from linearity, *i.e.* in which we can study the system by Fourier transforming with respect to the coordinates.

In this case there is no need to neglect the quantum pressure term in the dynamics of the quantum perturbation. The equations (3.105) and (3.106) become

$$\rho_1 = -\left(\lambda + \frac{k^2}{4m\langle\rho_0\rangle}\right)^{-1}\partial_t\theta_1, \tag{3.119}$$

$$\partial_t\rho_1 = \frac{k^2\langle\rho_0\rangle}{m}\theta_1, \tag{3.120}$$

where the contribution of the quantum pressure introduces a correction to the coupling λ which modifies the dispersion relation at high momenta. If the condensate wavefunction is also stationary we recover that the linear dispersion relation $\omega = c_s|k| = k\sqrt{\lambda\langle\rho_0\rangle/m}$ is modified by a k-dependent factor

$$\omega = c_s|k|\sqrt{1 + \frac{\xi^2k^2}{4}}, \tag{3.121}$$

where we have made use of the healing length ξ. This is the same dispersion relation obtained previously in equation (2.52) considering the Bogoliubov quasi-particles, but this time derived in the Madelung representation: instead of diagonalizing the two Bogoliubov–de Gennes equations for the perturbation $\delta\phi$ and its Hermitian conjugate, which are two differential operator equations of first order in time and second in space derivatives, this is the equivalent derivation in terms of a single real differential operator equation of second order in time and fourth order in space derivatives. At high momenta, *i.e.* for the physics at shorter distances, the dispersion relation is modified and the phase velocity $\omega/|k|$ exceeds the speed of sound c_s the more the momentum k is high, giving to each mode a different effective horizon.

Another interesting case is that of a spherically symmetric analogue metric. When both the number density $\langle\rho_0\rangle$ and the phase $\langle\theta_0\rangle$ are only functions of radius and time, the equations for the mean-field are reduced to those of a $(1+1)$-dimensional problem

$$\partial_t \langle\rho_0\rangle = -\frac{1}{m}\partial_r \left(r^2 \langle\rho_0\rangle \partial_r \langle\theta_0\rangle\right) , \tag{3.122}$$

$$\partial_t \langle\theta_0\rangle = \frac{1}{2mr} \langle\rho_0\rangle^{-1/2} \partial_r^2 \left(r \langle\rho_0\rangle^{1/2}\right) - \frac{1}{2m} \left(\partial_r \langle\theta_0\rangle\right)^2 - \lambda \langle\rho_0\rangle - V_{\text{ext}} . \tag{3.123}$$

The quasi-particles follow the spherically symmetric acoustic metric whose line element is

$$\mathrm{d}s^2 = \frac{\langle\rho_0\rangle}{c_s} \left(-c_s^2 \mathrm{d}t^2 + \left(\mathrm{d}r - \left(\frac{\partial_r \langle\theta_0\rangle}{m}\right) \mathrm{d}t\right)^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) , \tag{3.124}$$

which shows a horizon at some radius r_H, where the radial velocity equals the speed of sound

$$\left(\frac{\partial_r \langle\theta_0\rangle}{m}\right)^2 (r_H) = c_s^2 (r_H) . \tag{3.125}$$

Considering the continuity equation (3.122) and the Bernoulli equation (3.123) in the radial case, we obtain that this metric can be stationary only if there is a relation of inverse proportionality between the density and the radial velocity of the condensate; in a $(3+1)$-dimensional setup we have

$$\partial_t \langle\rho_0\rangle \iff \langle\rho_0\rangle \propto \frac{1}{r^2 \partial_r \langle\theta_0\rangle} , \tag{3.126}$$

where some external potential V_{ext} must be fixed appropriately to satisfy the Bernoulli equation. Only one solution is possible that is stationary, has homogeneous number density and has an acoustic horizon separating an external region with subcritical flow from an internal one with supercritical flow: it is the case in which the phase of the condensate is $\langle\theta_0\rangle = mc_s r_H^2/r$, a configuration which we call the canonical analogue black hole.

Other systems will show an intrinsic dependence on time. Regardless of the behavior of the Bogoliubov quasi-particles, the condensate will undergo a transformation in time due to the continuity equation and the Bernoulli equation. Even it the condensate has an initial uniform number density but the condensate presents inhomogeneities, the evolution will propagate them in the density $\langle\rho_0\rangle$, and the initial configuration will be deformed in time. However, as long as $\nabla^2\theta_0$ is small, *i.e.* in the low momentum regime, the variations of $\langle\rho_0\rangle$ are small as well: in general the scale of the inhomogeneities will define a timescale for which one could safely assume stationarity.

3.4.3 Relativistic condensates

Another set of analogue models which can be described in the context of Bose–Einstein condensation are those realized in a relativistic setting. A detailed analysis of these systems can be found in [104].

As we have discussed previously, we can apply the mean-field approximation to a wide variety of systems. The requirement of condensation is in general consistent with the mean-field formulation, identifying the condensate wavefunction with the mean-field, approximated as the solution of the Gross–Pitaevskii equation.

The Gross–Pitaevskii equation can be defined for any quantum field theory, transforming the field operators into classical functions.

We consider the relativistic theory of a minimally coupled interacting massive scalar field, described by the Lagrangian density

$$\mathcal{L} = -g^{\mu\nu}\left(\partial_\mu\phi^\dagger\right)\left(\partial_\nu\phi\right) - m^2\phi^\dagger\phi - U\left(\phi^\dagger\phi, \lambda\right) , \tag{3.127}$$

where U is a many-body interaction potential, which we assume to be dependent only on the local density operator $\phi^\dagger\phi = \rho$ and by a set of couplings λ. We will assume the geometry of the system where the condensate is located to be the flat Minkowski spacetime, with $g^{\mu\nu} = \eta^{\mu\nu}$.

The operator U can in general contain many-body interactions of any order, but we require it to be an ultralocal operator, depending on a single point through powers of the density operator ρ.

Regardless of the expression of the interaction term, we want to write the equation of motion. The Lagrange equations for the Lagrangian density of equation (3.127) are

$$0 = \Box\phi\left(t,x\right) - m^2\phi\left(t,x\right) - \int \mathrm{d}^3y\left[\phi\left(t,y\right), U\left(t,x\right)\right] , \tag{3.128}$$

and its Hermitian conjugate.

From this equation we obtain the relativistic Gross–Pitaevskii equation, applying the transformation $\phi \to \langle\phi_0\rangle$, which gives

$$0 = \Box\left\langle\phi_0\right\rangle - m^2\left\langle\phi_0\right\rangle - U'\left\langle\phi_0\right\rangle , \tag{3.129}$$

which in the Madelung representation $\langle\phi_0\rangle = \langle\rho_0\rangle e^{i\langle\theta_0\rangle}$ corresponds to the two real equations

$$0 = \eta^{\mu\nu}\partial_\nu\left(\left\langle\rho_0\right\rangle\left(\partial_\mu\left\langle\theta_0\right\rangle\right)\right) , \tag{3.130}$$

$$0 = \left\langle\rho_0\right\rangle^{-1/2}\Box\left\langle\rho_0\right\rangle^{1/2} - \eta^{\mu\nu}\left(\partial_\mu\left\langle\theta_0\right\rangle\right)\left(\partial_\nu\left\langle\theta_0\right\rangle\right) - m^2 - U' . \tag{3.131}$$

The continuity equation describes the conservation of the Noether charge associated with the $U(1)$ symmetry under phase transformations, common to the relativistic and non-relativistic theory. The relativistic Bernoulli equation is more complicated to read, changes with the expression of the stress-energy tensor of the theory.

Again we can see that the four-gradient of the phase can be interpreted as the four-velocity of the condensate. We can define the properly normalized four-velocity of the fluid as

$$u^\mu = \frac{\eta^{\mu\nu}\partial_\nu \langle\theta_0\rangle}{\|\nabla\langle\theta_0\rangle\|}\,, \tag{3.132}$$

$$\|\nabla\langle\theta_0\rangle\| = \sqrt{-\eta^{\mu\nu}\left(\partial_\mu\langle\theta_0\rangle\right)\left(\partial_\nu\langle\theta_0\rangle\right)}\,. \tag{3.133}$$

We proceed as we have seen in the non-relativistic Bose–Einstein condensate: considering a perturbation of the continuity equation (3.130) and of the Bernoulli equation (3.131) we get the Bogoliubov–de Gennes equations of the system. In the Madelung representation it is immediate to obtain them

$$0 = \partial_\mu\left(\left(\partial^\mu\langle\theta_0\rangle\right)\rho_1\right) + \partial_\mu\left(\langle\rho_0\rangle\,\partial^\mu\theta_1\right)\,, \tag{3.134}$$

$$0 = \frac{1}{2\langle\rho_0\rangle}\eta^{\mu\nu}\left(\partial_\mu\langle\rho_0\rangle\left(\partial_\nu\frac{\rho_1}{\langle\rho_0\rangle}\right)\right) - 2\eta^{\mu\nu}\left(\partial_\mu\langle\theta_0\rangle\right)\left(\partial_\nu\theta_1\right) - U''\rho_1\,. \tag{3.135}$$

Also in this case there is a quantum pressure term, which we would like to neglect. Again, when we are considering low-momentum perturbations and when the condensate wavefunction is approximately homogeneous, the quantum pressure term is negligible with respect to the contribution of the interaction term, and we have

$$\rho_1 = -\frac{2}{U''}\eta^{\mu\nu}\partial_\mu\left(\langle\theta_0\rangle\,\partial_\nu\theta_1\right)\,, \tag{3.136}$$

which is once again a definition for quantum field ρ_1 that can be easily plugged in the equation for the quantum perturbation of the continuity equation, obtaining a massless Klein–Gordon equation for the field θ_1

$$0 = \partial_\mu\left(\eta^{\mu\nu}\langle\rho_0\rangle - \left(\partial^\mu\langle\theta_0\rangle\right)\frac{2}{U''}\left(\partial^\nu\langle\theta_0\rangle\right)\right)\partial_\nu\theta_1\,. \tag{3.137}$$

This expression is very similar to the one previously obtained in the case of the relativistic classic fluid. We can find the analogue metric $g_{\mu\nu}$ associated with it with some manipulation. We extract the determinant of the analogue metric as seen previously

$$\sqrt{-g}\,g^{\mu\nu} = \langle\rho_0\rangle\left(\eta^{\mu\nu} - \left(\partial^\mu\langle\theta_0\rangle\right)\frac{2}{U''\langle\rho_0\rangle}\left(\partial^\nu\langle\theta_0\rangle\right)\right)\,, \tag{3.138}$$

$$\Downarrow$$

$$\sqrt{-g} = \langle\rho_0\rangle^2\sqrt{1 - \frac{2\eta^{\mu\nu}\left(\partial_\mu\langle\theta_0\rangle\right)\left(\partial_\nu\langle\theta_0\rangle\right)}{U''\langle\rho_0\rangle}} = \frac{\langle\rho_0\rangle^2}{c_s}\,, \tag{3.139}$$

$$g^{\mu\nu} = \frac{c_s}{\langle\rho_0\rangle}\left(\eta^{\mu\nu} - \frac{2}{U''\langle\rho_0\rangle}\left(\partial^\mu\langle\theta_0\rangle\right)\left(\partial^\nu\langle\theta_0\rangle\right)\right)\,. \tag{3.140}$$

The definition of speed of sound follows naturally from the previous discussion on the relativistic classic fluids. It is therefore equivalent to use the speed of sound or the norm of the four-gradient of the phase of the condensate wavefunction

$$\eta^{\mu\nu} \left(\partial_\mu \langle\theta_0\rangle\right) \left(\partial_\nu \langle\theta_0\rangle\right) = -\frac{1-c_s^2}{c_s^2} \frac{U'' \langle\rho_0\rangle}{2} \,, \tag{3.141}$$

The analogue metric for this relativistic condensate is obtained inverting the expression in equation (3.140), which is a straightforward calculation

$$g_{\mu\nu} = \frac{\langle\rho_0\rangle}{c_s} \left(\eta_{\mu\nu} + \frac{2c_s^2}{U'' \langle\rho_0\rangle} \left(\partial_\mu \langle\theta_0\rangle\right) \left(\partial_\nu \langle\theta_0\rangle\right) \right) = \tag{3.142}$$

$$= \frac{\langle\rho_0\rangle}{c_s} \left(\eta_{\mu\nu} + \left(1-c_s^2\right) u_\mu u_\nu\right) \,. \tag{3.143}$$

The expression obtained is completely consistent with that of the relativistic fluid, and we also recover the expressions for the non-relativistic Bose–Einstein condensate when we consider the speed of sound being much less than the speed of light $c_s \ll 1$ and when the interaction potential to be approximately $U'' = 2\lambda m$, consistent with what is expected comparing the Lagrangian density with the many-body Hamiltonian studied in the non-relativistic case. The rest of the discussion would be equivalent to the classical fluids.

3.4.4 Analogue gravity with atom number conservation

We have discussed, in the previous chapter, the number conserving framework and its consistency with the mean-field description. We briefly discuss which assumptions should be required so that the number conserving formalism based on the time dependent natural orbits can be effective in analogue gravity to study the dynamics of the modes of an analogue massless scalar field.

We need want to verify that in the number conserving formalism it is possible to reproduce the Madelung representation and define Hermitian quantum operators describing the perturbations of the condensate.

These field operators, required for analogue gravity, will differ from those relative to the mean-field description, and they should be defined considering that we have by construction removed the contribution from the condensed 1-particle state f_0. The dynamical equation for the excited part in the number-conserving formalism as described in equation (2.121) appears to be the same as for the case of the Bogoliubov–de Gennes equation in the usual mean-field description. The difference is that instead of the field $\delta\phi$ one has $N_0^{-1/2} a_0^\dagger \phi_1$, where we remind that $N_0 = a_0^\dagger a_0$, and instead of the mean-field we consider the appropriately rescaled eigenfunction of the 2-point correlation function.

Using the Madelung representation, we redefine the real functions ρ_0 and θ_0 from the condensed wavefunction f_0 and the expectation value $\langle N_0\rangle$.

$$\langle N_0\rangle^{1/2} f_0 = \rho_0 e^{i\theta_0} \,. \tag{3.144}$$

These functions enter in the definition of the quantum operators θ_1 and ρ_1, which take a different expression from the usual Madelung representation when we employ the set of number-conserving ladder operators

$$\theta_1 = -\frac{i}{2} \langle N_0 \rangle^{-1/2} \sum_{I \neq 0} \left(\frac{f_I}{f_0} N_0^{-1/2} a_0^\dagger a_I - \frac{\overline{f_I}}{\overline{f_0}} a_I^\dagger a_0 N_0^{-1/2} \right) = \tag{3.145}$$

$$= -\frac{i}{2} \left(\frac{N_0^{-1/2} \phi_0^\dagger \phi_1 - \phi_1^\dagger \phi_0 N_0^{-1/2}}{\langle N_0 \rangle^{1/2} \overline{f_0} f_0} \right), \tag{3.146}$$

$$\rho_1 = \langle N_0 \rangle^{1/2} \sum_{I \neq 0} \left(\overline{f_0} f_I N_0^{-1/2} a_0^\dagger a_I + f_0 \overline{f_I} a_I^\dagger a_0 N_0^{-1/2} \right) = \tag{3.147}$$

$$= \langle N_0 \rangle^{1/2} \left(N_0^{-1/2} \phi_0^\dagger \phi_1 + \phi_1^\dagger \phi_0 N_0^{-1/2} \right). \tag{3.148}$$

From equations (3.145) and (3.147) we observe that the structure of the operators θ_1 and ρ_1 consists of a superposition of modes, each dependent on a different eigenfunction f_I of the 2-point correlation function, with a sum over the index $I \neq 0$.

The new fields θ_1 and ρ_1 do not satisfy the canonical commutation relations since the condensed wavefunction f_0 is treated separately by definition. However, these operators could be analyzed mode-by-mode, and therefore be compared in full extent to the modes of quantum fields in curved spacetime to which they are analogous. Their modes satisfy the relations

$$[\theta_I, \rho_J] = -i \overline{f_I} f_I \delta_{JI} \qquad \forall I, J \neq 0\,. \tag{3.149}$$

The equation (3.149) is a basis-dependent expression which can in general be found for the fields of interest. In the simplest case of homogeneous density of the condensate ρ_0, this commutation relations reduce to $-i\delta_{IJ}$, and the Fourier transform provides the tools to push the description to full extent where the indices labeling the functions are the momenta k.

The equations for analogue gravity are found under the usual assumptions regarding the quantum pressure, *i.e.* the space gradients of the atom densities are assumed to be small. We therefore require

$$\nabla \left(\rho_0^{-1} \left(\nabla \rho_0 \right) \right) \ll 4 m \lambda \rho_0 \,, \tag{3.150}$$

$$\nabla \left(\rho_0^{-1} \left(\nabla \rho_1 \right) \right) \ll 4 m \lambda \rho_1 \,. \tag{3.151}$$

Making the first assumption (3.150), the condensate and its speed of sound can be considered homogeneous. This means that all the possible inhomogeneities of the system are encoded in the velocity of the superfluid, the gradient of the phase of the condensate. As stated before, the continuity equation can induce inhomogeneities in the density if there are initial inhomogeneities in the phase, but for sufficiently short intervals of time the assumption is satisfied. Another effect of the first assumption (3.150) is that the term $\int \mathrm{d}x \overline{f_I f_0} f_0 f_0$ is negligible. The more ρ_0 is homogeneous, the closest this integral is to vanishing, making the description more consistent. The second assumption (3.151) is a general requirement in analogue gravity, needed to have local Lorentz symmetry, and therefore a proper Klein-Gordon equation for the field θ_1. When ρ_0 is homogeneous this approximation means considering only small momenta, for which we have the usual dispersion relation.

Under these assumptions, the usual equations for analogue gravity are obtained

$$\rho_1 = -\frac{1}{\lambda}\left((\partial_t\theta_1) + \frac{1}{m}(\nabla\theta_0)(\nabla\theta_1)\right) , \tag{3.152}$$

$$(\partial_t\rho_1) = -\frac{1}{m}\nabla\left(\rho_1(\nabla\theta_0) + \rho_0(\nabla\theta_1)\right) , \tag{3.153}$$

$$\Downarrow$$

$$\begin{aligned} 0 = {} & \partial_t\left(-\frac{1}{\lambda}(\partial_t\theta_1) - \frac{\delta^{ij}}{m\lambda}(\nabla_j\theta_0)(\nabla_i\theta_1)\right) + \\ & + \nabla_j\left(-\frac{\delta^{ij}}{m\lambda}(\nabla_i\theta_0)(\partial_t\theta_1) + \left(\frac{\delta^{ij}\rho_0}{m} - \frac{1}{\lambda}\frac{\delta^{il}}{m}\frac{\delta^{jm}}{m}(\nabla_l\theta_0)(\nabla_m\theta_0)\right)(\nabla_i\theta_1)\right) , \end{aligned} \tag{3.154}$$

so that θ_1 is the analogue of a scalar massless field in curved spacetime. However, the operator θ_1 is intrinsically unable to provide an exact full description of a massless field since it is missing the mode f_0. Therefore, the operator θ_1 is best handled when considering the propagation of its constituent modes, and relating them to those of the massless field.

The viability of this description as a good analogue gravity setup is ensured, ultimately, by the fact that the modes of θ_1, *i.e.* the operators describing the excited part of the atomic field, have a closed dynamics. The most important feature in the effective dynamics of the number-conserving operators $N_0^{-1/2}a_0^\dagger a_I$, as described in equation equation (2.121), is that its time derivative can be written as a composition of the same set of number operators, and this enables the analogue model.

In the following chapter we are going to stick to the case of a homogeneous condensate, which is arguably the most studied case in analogue gravity. The description is enormously simplified by the fact that the gradient of the condensate wavefunction vanishes, since in a volume V the condensate wavefunction is simply $f_0 = V^{-1/2}$, meaning that the condensed state is fully described by the state of zero momentum $k = 0$. In a homogeneous Bose–Einstein condensates all the time-dependent orbitals are labeled by the momenta they carry, and at every moment in time we can apply the same Fourier transform to transform the differential equations in the space of coordinates to algebraic equations in the space of momenta. We expect that the number-conserving treatment of the inhomogeneous condensate follows along the same lines, albeit being technically more complicated.

3.5 Final remarks

We have discussed the issues that, in our current understanding of gravity, remain open. The theory as we know it is incomplete: its limits are known and it is known that they are unsolvable without a paradigm shift. We still lack a complete understanding of what the interplay between geometry and quantum fields is, and how a description that unifies them can be formulated.

These open issues, even those that are already typical of general relativity, such as the presence of singularities, arise from the dynamic evolution of matter fields, objects that ultimately must be described within quantum field theory.

Analogue gravity thus presents itself as a tool for verifying the predictions of quantum field theory in curved spacetime and studying its problems: with analogue models it has already been possible, for example, to observe the analogues of Hawking radiation, cosmological particle creation and superradiance.

The aspiration is that analogue gravity can allow us to advance also in the understanding of semiclassical gravity and eventually provide us useful insights towards the development of quantum gravity, by equipping us with stronger paradigms with which to understand the nature of gravity and formulate a complete and predictive theory that describes it.

Of the many analogue gravity models, fluid models are of particular interest. Starting from Unruh's initial proposal, which considered classical perfect fluids, the research program has made many steps forward with the concrete realization of both classical and quantum systems.

Classical fluids show how it is possible to simulate typically gravitational phenomena in laboratory viable setups: the acoustic excitations in a fluid propagate perceiving an effective geometry described by an acoustic metric. These models allow to simulate curved spacetimes and can be used consistently in high speed regimes: the relativistic formalism demonstrates the soundness of the analogue description.

Although in presence of external stresses analogue models in classical fluids can simulate the phenomena under discussion, these models are effective for the study of stimulated, rather than spontaneous, effects. Classical models cannot provide a purely quantum description of phenomena such as the spontaneous production of radiation due to the properties of the quantum vacuum or to the inequivalence of the vacuum state for different observers.

We are therefore naturally led to study analogue systems in Bose–Einstein condensates. In these systems we have discussed how it is possible to reproduce the phenomenology of massless scalar fields, in a large class of geometries. These systems are characterized by perturbations that simulate fields in curved spacetime in a purely quantum formulation, valid in both cases of relativistic and non-relativistic condensates.

They hence provide a useful arena where we can investigate some of the open issues of gravity, as they share some feature with semiclassical gravity. For example also in Bose–Einstein condensates one would naturally assume the separation between classical spacetime and quantum perturbations. However, in this case there is the advantage that the underlying atomic theory is known and ultimately has the strength to provide solutions for the analogues of the aforementioned unsolved problems in gravitational systems.

In condensates we can therefore not only simulate the phenomena, but also try to understand how the properties of the underlying theory, which unifies the degrees of freedom that would seem separate between geometry and quasi-particles, blur in the emerging theory, and how in a gravitational context similar mechanisms may occur.

The completeness of the description made with condensates is by no means trivial; in an analogue model which is highly predictive and of wide interest such as that of gravity waves, for example, we are faced with the fact that the theory is incomplete: the study of a $(2+1)$-dimensional system obtained on the boundary of a $(3+1)$-dimensional system cannot predict the totality of the phenomena occurring in the concealed dimension, and observers living — in the case of gravity waves — in the interface, will encounter unpredictable phenomena that reveal the incompleteness of their description.

Bose–Einstein condensates, instead, are $(3+1)$-dimensional systems that allow to simulate gravitational systems of the same dimensionality, in which the analogue gravitational picture emerges from an underlying structure described by a unitary and complete quantum theory, as we conjecture — fundamentally — must be the final theory of gravity.

4 The analogue information loss problem and its resolution

4.1 The information loss problem

Of the open problems that can be investigated in analogue gravity, the information loss problem is one that most clearly touches the fundamental nature of spacetime and its interplay with matter.

It is usual to think of the two sectors of geometry and quantum matter separately, since they are described within two distinct frameworks: general relativity and quantum field theory. It is not until one tries to formulate a consistent theory for semiclassical gravity, *i.e.* a theory for the back-reaction of the quantum matter fields on the geometry, that one encounters the problem of information loss [31], and the separation between these two sectors is questioned.

General relativity is a dynamical theory of the geometry of spacetime formulated in terms of a classical field theory of the metric, a real $(0, 2)$ tensor defined on the tangent bundle of the spacetime manifold $\mathcal{M}$ [11, 12].

Quantum field theory, instead, describes the particles that make up matter. In a generic spacetime, particle fields can be studied promoting the usual standard model from a quantum field theory in Minkowski space to a quantum field theory in curved spacetime, defined on the Fock space obtained from the quantization of the $L^2(\mathcal{M})$ functions with respect to the norm induced by the Klein–Gordon equation [30].

The objects involved in these descriptions are mathematically profoundly different from each other, and it is not self-evident how a unified theory for the two sectors could be formulated. But some unification is certainly necessary, for consistency requirements: already in general relativity it is apparent that matter and geometry are strictly intertwined, as the classical stress-energy tensor is the source term in the Einstein equations. At the same time, it is also true that the classical stress-energy tensor must be determined by the quantum fields, since it must be consistent with the quantum stress-energy tensor when considered at scales at which the role of quantum corrections, in powers of $\hbar$, can be neglected.

The purpose of semiclassical gravity is exactly to obtain a coherent closed system of coupled equations for the dynamics of the metric tensor and the quantum fields, with the metric determining the propagation of the matter fields and the matter fields back-reacting on the geometry by means of the stress-energy tensor.

However, in semiclassical gravity, the state of the universe is still considered as a tensor product of a classical state for the geometry and some compatible quantum state for the matter fields, *i.e.* the state is considered as a couple of two distinct elements, defined in separate spaces. But this characterization of the states leads to problems due to the different nature of the two sectors.

A first problem emerges immediately by considering that there is no clear definition of a geometry in presence of an object in a superposition of two different quantum states of position [105]. Therefore, there is a problem in including consistently the notion of quantum entanglement in this description: we need to include quantum superposition in the matter sector, but we would not be able to define consistently a geometry under the hypothesis that the two sectors are separable; it would be necessary for the complete states to be, in their entirety, in a Hilbert space that admits quantum superposition. The same problem would emerge from the evolution of the quantum matter sector: when we consider the back-reaction from the matter fields on the geometry, and therefore as soon as we stop considering the geometry as a fixed background, the state would evolve into a quantum superposition of different states which could not all be compatible with a same classical geometry.

The quantum superposition of one sector should induce an appropriate response in the other, and we expect to find inconsistencies in the description, which could be avoided only with a common quantum description underlying both geometry and matter fields at the same time.

The most obvious example of the influence of the quantum fields on the geometry comes from the Hawking radiation, which is predicted in quantum field theory in curved spacetime: the emission of thermal radiation from the black hole is predicted already under the assumption of a fixed background, just by the presence of a horizon. But it is immediate to understand the effect that the Hawking radiation would have in terms of semiclassical gravity, as this process describes an outgoing flux of energy from the black hole to infinity. In a semiclassical picture the Hawking radiation implies the evaporation of the black hole itself [106].

The evaporation of a black hole means that the region of spacetime enclosed by the horizon shrinks until it vanishes, together with all the information contained inside it, as an apparent horizon cannot be crossed by propagating luminal and subluminal particles. The quantum information localized in the inner region is therefore lost; in particular, different initial states would result, after the evaporation process, in the same final state; the one-to-one correspondence between the initial and final state would be broken, and with the loss of the information inside the horizon, only the information initially outside it would play any role in the final state.

The natural endpoint of such process into a complete evaporation of the object leads to a thermal bath over a flat spacetime which appears to be incompatible with a unitary evolution of the quantum fields from the initial state to the final one. We are not considering here alternative solutions such as long-living remnants, as these are as well problematic in other ways [107–111], or they imply deviations from the black hole structure at macroscopic scale, see *e.g.* [112].

While the indistinguishability of the inner quantum state from the geometry leads to other important speculations on the thermodynamical nature of black holes and the holographic principle, we focus on the fact that if different initial states lead to a same thermal final state, their unitary evolution apparently needs to be broken. The information loss can therefore be understood as a unitarity breaking phenomenon which is due to the separation of geometry and matter into different sectors, and the impossibility to transfer information between the two in a semiclassical gravity framework.

The only way to solve the problem of information loss appears to be the introduction of a more fundamental theory, where the information can be exchanged between matter and geometry. This paradigm is what we could in general denote as quantum gravity, even without proposing a specific model.

The unitarity of the evolution, and therefore the conservation of information, would require the state of the universe to be in its entirety a quantum object, without separation of the Hilbert space into two sectors. We shall now see how analogue gravity in Bose–Einstein condensates therefore is a natural setting to address this idea.

4.2 Information in Bose–Einstein condensate analogues

Analogue gravity in Bose–Einstein condensates can provide an interesting perspective on the resolution of the information loss problem, albeit the back-reaction in analogue systems is not described by semiclassical Einstein equations. Even if the equations are different, the back-reaction by the Bogoliubov quasi-particles could still affect the acoustic metric similarly to the what happens in the case of the evaporation of gravitational black holes: the information associated to the quasi-particles could be lost in the deformation of the condensate wavefunction, as the problem of information loss is proper to every system with semiclassical interaction between a quantum and a classical sector, as described before.

But in analogue gravity in Bose–Einstein condensates there cannot be real information loss in the full Hilbert space, since the atomic theory is unitary all along the evolution of the system. Any information loss would only be apparent when considering those degrees of freedom of the full theory associated with the quasi-particles, misleadingly assumed separated from the acoustic geometry. This mimics the behavior one would expect in quantum gravity, *i.e.* in the unitary quantum theory assumed to underlie semiclassical gravity. The analogue systems realized in Bose–Einstein condensates can therefore provide natural toy models for understanding not so much quantum gravity itself as, in some measure, the mechanisms by which a single quantum theory could lead to the phenomenology of quantum field theory in curved spacetime without breaking the unitarity of the evolution when the back-reaction is included in a semiclassical gravity picture.

In particular one can expect the black hole evaporation to be a continuous process creating correlations between Hawking quanta and the microscopic quantum degrees of freedom of spacetime, implying so that a full quantum gravity treatment would resolve the information loss problem by transferring information unitarily within a full Hilbert space that incorporates all the degrees of freedom. During the evaporation, the information would therefore be transferred from the matter fields to more fundamental degrees of freedom that in semiclassical gravity are concealed behind the classical geometry.

By looking at the simpler problem of cosmological particle creation, we have shown in our work published in [7] that, in the context of analogue gravity in Bose–Einstein condensates, the emerging analogue geometry and the quasi-particles have correlations due to the quantum nature of the atomic degrees of freedom underlying the emergent spacetime. The quantum evolution is, of course, always unitary, but it is described on the whole Hilbert space, which cannot be exactly factorized *a posteriori* in geometry and quasi-particle components. These analogue systems can therefore not only reproduce Hawking radiation, but they can also provide a precious insight into the information loss problem.

We want to specify that, while the atomic system at the fundamental level cannot violate unitary evolution, one could conceive analogue black holes provided with singular regions for the emergent spacetime where the description of quasi-particles propagating on an analogue geometry fails. For example, one could describe flows characterized by regions where the hydrodynamical approximation fails even without necessarily having loss of atoms from the systems. In such cases, despite the full dynamics being unitary, it seems that a trace over the quasi-particle falling in these "analogue singularities" would be necessary, so leading to an apparent loss of unitarity from the analogue system point of view. But the scope of our investigation is to see how such unitarity evolution is preserved on the full Hilbert space.

This study explicitly requires the use of the number conserving formalism for condensates; it must be applied not only to the quasi-particles but also to the condensate 1-particle state. In such terms it is possible to keep track of the transfer of information from the quasi-particles to the quantum degrees of the condensate, corresponding to the acoustic geometry. To ensure the unitarity of the evolution of the analogue system it is necessary to keep track of the unitarity in terms of the atoms, and it is therefore necessary to include the notion of atomic number conservation in order not to break it.

In the number-conserving approach, it is possible to retain the quantum nature of the condensate operator as well as to describe the correlations with quasi-particles within an improved Bogoliubov description. In the simpler setting of a cosmological particle creation we shall describe the continuous generation of correlations between the condensate atoms and the quasi-particles. Such correlations are responsible for (and in turn consequence of) the non-factorizability of the Hilbert space and are assuring in any circumstances the unitary evolution of the full system.

4.3 Simulating cosmology in number-conserving analogue gravity

In homogeneous non-relativistic condensates we can simulate different cosmologies, with scale factor changing in time, by controlling and modifying in time the strength of the two-body interaction λ.

By doing so, we can simulate analogue cosmological systems, which we can use to verify the prediction of quantum field theory in curved spacetime that in an expanding universe one should observe a cosmological particle creation.

In this setup there is no ambiguity in approximating the mixed term of the interaction potential, as discussed previously, since the norm and the phase of the condensate wavefunction are homogeneous. The information relative to the evolution of the condensate wavefunction, *i.e.* the evolution of the analogue cosmology, is all contained in the time dependence of the phase of the condensate.

The system is described with the usual Madelung representation, for both the condensate wavefunction and the excitations; we use the number-conserving ladder operators introduced previously to describe the Bogoliubov excitations and the atoms in the condensed 1-particle state.

In this system the continuity equation and the Bernoulli equation describing the dynamics of the condensate wavefunction in the Madelung representation, under the hypothesis of homogeneity, are remarkably simple

$$\langle \phi_0 (t) \rangle = \langle \rho_0 \rangle \, e^{i \langle \theta_0 (t) \rangle} \, , \tag{4.1}$$

$$\partial_t \langle \rho_0 \rangle = 0 \, , \tag{4.2}$$

$$\partial_t \langle \theta_0 (t) \rangle = -\lambda (t) \langle \rho_0 \rangle - V_{\text{ext}} (t) \; . \tag{4.3}$$

In this chapter the function $\langle \phi_0 \rangle$ should be understood as a rescaling of the condensed 1-particle state defined from the 2-point correlation function, *i.e.* the normalized 1-particle state f_0, which is macroscopically occupied, times the square root of the occupation number of that state $\langle N_0 \rangle^{1/2}$. We thus remark that $\langle \phi_0 \rangle$ is not the expectation value of the field operator. As we have discussed introducing the number-conserving formalism, the mean-field formally is the expectation value of the field operator, and it strictly coincides with the condensate wavefunction — the eigenfunction of the 2-point correlation function — only when we are considering coherent states, but not in general.

To study the quantum phase fluctuations, *i.e.* the analogue scalar field of the theory described by the operator $\theta_1 (x)$, we need the basis of time-dependent orbitals, which in the case of a homogeneous condensate is given by the plane waves, the set of orthonormal functions which define the Fourier transform and are labeled by the momenta.

The number conserving operator ϕ_1 that describes the quantum excitations in the subspace orthogonal to the 1-particle condensed state is therefore simplified by the fact that the latter is the homogeneous state, with $k = 0$.

Therefore ϕ_1 is simply the superposition of the number-conserving operators associated with the plane waves, *i.e.* the ladder operators $\delta\phi_k$ associated with the different momenta k

$$\delta\phi_k \equiv \int \frac{\mathrm{d}x}{\sqrt{V}} e^{-ikx} N_0^{-1/2} a_0^\dagger \phi_1(x) = \tag{4.4}$$

$$= \int \frac{\mathrm{d}x}{\sqrt{V}} e^{-ikx} N_0^{-1/2} a_0^\dagger \sum_{q\neq 0} \frac{e^{iqx}}{\sqrt{V}} a_q = \tag{4.5}$$

$$= N_0^{-1/2} a_0^\dagger a_k \,, \tag{4.6}$$

$$\left[\delta\phi_k, \delta\phi_{k'}^\dagger\right] = \delta_{k,k'} \qquad\qquad \forall k, k' \neq 0\,. \tag{4.7}$$

Where by V we denote the volume in which the condensate is contained. We remind that by $a_0^\dagger$ and N_0 we mean respectively the creation operator and the number operator of the atoms in the 1-particle condensed state (the normalized $\langle\phi_0\rangle$, *i.e.* the homogeneous state), while by a_k we mean the destruction operator of the atoms in the plane wave state of momentum k.

We can use the same plane waves decomposition for the number-conserving operators corresponding to the modes of the analogue scalar field and of its conjugate, defining θ_k and ρ_k. These number-conserving operators are labeled with a non-zero momentum and act in the atomic Fock space, in a superposition of two operations, extracting momentum k from the state or introducing momentum $-k$ to it. All the following relations are defined for $k, k' \neq 0$

$$\theta_k = -\frac{i}{2}\left(\frac{\delta\phi_k}{\langle\phi_0\rangle} - \frac{\delta\phi_{-k}^\dagger}{\overline{\langle\phi_0\rangle}}\right) , \tag{4.8}$$

$$\rho_k = \langle\rho_0\rangle\left(\frac{\delta\phi_k}{\langle\phi_0\rangle} + \frac{\delta\phi_{-k}^\dagger}{\overline{\langle\phi_0\rangle}}\right) , \tag{4.9}$$

$$[\theta_k, \rho_{k'}] = -i\left[\delta\phi_k, \delta\phi_{-k'}^\dagger\right] = -i\delta_{k,-k'} \,, \tag{4.10}$$

$$\left\langle \delta\phi_k^\dagger(t)\, \delta\phi_{k'}(t)\right\rangle = \delta_{k,k'} \left\langle N_k\right\rangle \,. \tag{4.11}$$

We remark that in this number conserving formalism θ_k and ρ_k do not provide, through an inverse Fourier transform with respect to k of these operators, a pair of conjugated real fields $\theta_1(x)$ and $\rho_1(x)$ with the usual canonical commutation relations as in the coordinate space, because they are not relative to a set of functions that form a complete basis of the 1-particle Hilbert space, as the mode $k = 0$ is not included.

But these operators, describing each mode with $k \neq 0$, can be studied separately and they show the same behavior of the components of a quantum field in curved spacetime: the commutation relations in equation (4.10) are the same as those that are satisfied by the components of a quantum scalar field.

From the Bogoliubov–de Gennes we get the two coupled dynamical equations for θ_k

and ρ_k

$$\partial_t \theta_k = -\frac{1}{2}\left(\frac{k^2}{2m} + 2\lambda \langle \rho_0 \rangle\right) \frac{\rho_k}{\langle \rho_0 \rangle}\,, \tag{4.12}$$

$$\partial_t \frac{\rho_k}{\langle \rho_0 \rangle} = \frac{k^2}{m}\theta_k\,. \tag{4.13}$$

Combining these gives the analogue Klein-Gordon equation for each mode $k \neq 0$

$$\partial_t \left(-\frac{1}{\lambda \langle \rho_0 \rangle + \frac{k^2}{4m}} \left(\partial_t \theta_k\right)\right) = \frac{k^2}{m}\theta_k\,. \tag{4.14}$$

In this equation the term due to quantum pressure is retained for convenience, since the homogeneity of the condensed state makes it easy to maintain it in the description. It modifies the dispersion relation and breaks Lorentz symmetry, but the usual expression is found in the limit $\frac{k^2}{2m} \ll 2\lambda \langle \rho_0 \rangle$.

When the quantum pressure is neglected, the analogue metric tensor is

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \sqrt{\frac{\langle \rho_0 \rangle}{m\lambda}} \left(-\frac{\lambda \langle \rho_0 \rangle}{m}\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j\right)\,. \tag{4.15}$$

This metric tensor is clearly analogous to that of a cosmological spacetime, where the evolution is given by the time dependence of the coupling constant λ. This low-momenta limit is the regime in which we are mostly interested, because when these conditions are realized the quasi-particles, the excitations of the field θ_k, behave most similarly to particles in a curved spacetime with local Lorentz symmetry.

4.3.1 Cosmological particle production

We now consider a setup for which the coupling varies from an initial value λ to a final value λ' through a transient phase. The coupling λ is assumed asymptotically constant for both $t \to \pm\infty$. This setup has been studied in the Bogoliubov approximation in [75, 78, 84, 113, 114] and can be experimentally realized with *e.g.* via Feshbach resonance. For one-dimensional Bose gases, where significant corrections to the Bogoliubov approximation are expected far from the weakly interacting limit, a study of the large time evolution of correlations was presented in [115]. Here our aim is to study the effect of the variation of the coupling constant in the number-conserving framework.

There will be particle creation and the field in general takes the expression

$$\theta_k(t) = \frac{1}{\mathcal{N}_k(t)}\left(e^{-i\Omega_k(t)}c_k + e^{i\Omega_{-k}(t)}c^\dagger_{-k}\right)\,, \tag{4.16}$$

where the operators c_k are the creation and destruction operators for the quasi-particles at $t \to -\infty$. For the time $t \to +\infty$ there will be a new set of number conserving operators c'

$$\theta_k(t \to -\infty) = \frac{1}{\mathcal{N}_k}\left(e^{-i\omega_k t}c_k + e^{i\omega_k t}c^\dagger_{-k}\right) , \quad (4.17)$$

$$\theta_k(t \to +\infty) = \frac{1}{\mathcal{N}'_k}\left(e^{-i\omega'_k t}c'_k + e^{i\omega'_k t}c'^\dagger_{-k}\right) . \quad (4.18)$$

From these equations, in accordance with equation (4.12), we obtain

$$\rho_k = -\frac{2\langle\rho_0\rangle}{\frac{k^2}{2m} + 2\lambda\langle\rho_0\rangle}\partial_t\theta_k , \quad (4.19)$$

and the two following asymptotic expressions for ρ_k:

$$\rho_k(t \to -\infty) = \frac{2i\omega_k\langle\rho_0\rangle}{\frac{k^2}{2m} + 2\lambda\langle\rho_0\rangle}\frac{1}{\mathcal{N}_k}\left(e^{-i\omega_k t}c_k - e^{i\omega_k t}c^\dagger_{-k}\right) , \quad (4.20)$$

$$\rho_k(t \to +\infty) = \frac{2i\omega'_k\langle\rho_0\rangle}{\frac{k^2}{2m} + 2\lambda'\langle\rho_0\rangle}\frac{1}{\mathcal{N}'_k}\left(e^{-i\omega'_k t}c'_k - e^{i\omega'_k t}c'^\dagger_{-k}\right) . \quad (4.21)$$

With the previous expressions for θ_k and ρ_k and imposing the commutation relations in equation (4.10), we retrieve the energy spectrum $\omega_k = \sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda\langle\rho_0\rangle\right)}$ as expected and the (time-dependent) normalization prefactor $\mathcal{N}$:

$$\mathcal{N}_k = \sqrt{4\langle\rho_0\rangle\sqrt{\frac{\frac{k^2}{2m}}{\frac{k^2}{2m} + 2\lambda\langle\rho_0\rangle}}} . \quad (4.22)$$

The expected commutation relations for the operators c and c' are found (again not including the mode $k = 0$):

$$0 = \left[c_k, c_{k'}\right] = \left[c'_k, c'_{k'}\right] , \quad (4.23)$$

$$\delta_{k,k'} = \left[c_k, c^\dagger_{k'}\right] = \left[c'_k, c'^\dagger_{k'}\right] . \quad (4.24)$$

The two sets of operators are connected by a Bogoliubov transformation

$$c'_k = \cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k}c^\dagger_{-k} , \quad (4.25)$$

with

$$\cosh\Theta_k = \cosh\Theta_{-k} , \quad (4.26)$$

$$\sinh\Theta_k e^{i\varphi_k} = \sinh\Theta_{-k}e^{i\varphi_{-k}} . \quad (4.27)$$

The initial state in which we are interested is the vacuum of quasi-particles, so that each quasi-particle destruction operators c_k annihilates the initial state [1]:

$$c_k \left|in\right\rangle \equiv 0 \qquad\qquad \forall k \neq 0\,. \tag{4.28}$$

To realize this initial condition we should impose constraints, in principle, on every correlation function. We focus on the 2-point correlation functions $\left\langle \delta\phi^\dagger \delta\phi \right\rangle$ and $\left\langle \delta\phi\delta\phi \right\rangle$. In particular, the first of the two determines the number of atoms with momentum k in the initial state:

$$\left\langle \delta\phi_k^\dagger \delta\phi_k \right\rangle = \left\langle a_k^\dagger a_0 N_0^{-1} a_0^\dagger a_k \right\rangle = \left\langle a_k^\dagger a_k \right\rangle = \left\langle N_k \right\rangle \,. \tag{4.29}$$

In order for the state to be condensed with respect to the state with momentum 0, the atom occupation numbers must be $\langle N_k \rangle \ll \langle N_0 \rangle = \langle \rho_0 \rangle V$. When the vacuum condition of equation (4.28) holds, the 2-point correlation functions can be easily evaluated to be:

$$\left\langle \delta\phi_k^\dagger \delta\phi_{k'} \right\rangle = \left(\frac{1}{2} \frac{\frac{k^2}{2m} + \lambda \langle \rho_0 \rangle}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda \langle \rho_0 \rangle\right)}} - \frac{1}{2} \right) \delta_{k,k'} = \tag{4.30}$$

$$\approx \frac{1}{4} \sqrt{\frac{2\lambda \langle \rho_0 \rangle}{\frac{k^2}{2m}}} \delta_{k,k'} \,, \tag{4.31}$$

$$\left\langle \delta\phi_{-k} \delta\phi_{k'} \right\rangle = -\frac{e^{2i\langle \theta_0 \rangle}}{4} \frac{2\lambda \langle \rho_0 \rangle}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 2\lambda \langle \rho_0 \rangle\right)}} \delta_{k,k'} = \tag{4.32}$$

$$\approx -e^{2i\langle \theta_0 \rangle} \left\langle \delta\phi_k^\dagger \delta\phi_{k'} \right\rangle \,, \tag{4.33}$$

where in the last line we have used $\frac{k^2}{2m} \ll 2\lambda \langle \rho_0 \rangle$, the limit in which the quasi-particles propagate in accordance with the analogue metric of equation (4.15), and one has to keep into account that the phase of the condensate is time dependent and consequently the last correlator is oscillating.

We now see that the conditions of condensation $\langle N_k \rangle \ll \langle N_0 \rangle$ and of low-momenta translate into

$$\frac{2\lambda \langle \rho_0 \rangle}{16 \langle N_0 \rangle^2} \ll \frac{k^2}{2m} \ll 2\lambda \langle \rho_0 \rangle \,. \tag{4.34}$$

The range of momenta that should be considered is therefore set by the number of condensate atoms, the physical dimension of the atomic system and the strength of the two-body interaction.

[1] To make contact with the standard Bogoliubov approximation, if there one denotes by γ_k the quasi-particles one has that the γ_k are a combination of the atom operators a_k, a_{-k} of the form $\gamma_k = u_k a_k + v_k a_{-k}^\dagger$ [3]. Correspondingly, in the number-conserving formalism the quasi-particle operators c_k are a combination of the atom operators $\delta\phi_k \equiv \alpha_k$, $\delta\phi_{-k} \equiv \alpha_{-k}$.

The operators θ_k satisfying equation (4.14) — describing the excitations of quasi-particles over a Bose–Einstein condensate — are analogous to the components of a scalar quantum field in a cosmological spacetime. In particular, if we consider a cosmological metric given in the usual form

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}\tau^2 + a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{4.35}$$

the analogy is realized for a specific relation between the coupling $\lambda\,(t)$ and the scale factor $a\,(\tau)$, which then induces the relation between the laboratory time t and the cosmological time τ. These relations are given by

$$a\,(\tau\,(t)) = \left(\frac{\langle\rho_0\rangle}{m\lambda\,(t)}\right)^{1/4}\frac{1}{C}\,, \tag{4.36}$$

$$\mathrm{d}\tau = \frac{\langle\rho_0\rangle}{ma\,(\tau\,(t))}\frac{1}{C^2}\mathrm{d}t\,, \tag{4.37}$$

for an arbitrary constant C.

In cosmology the evolution of the scale factor leads to the production of particles by cosmological particle creation, as implied by the Bogoliubov transformation relating the operators which, at early and late times, create and destroy the quanta we recognize as particles. The same happens for the quasi-particles over the condensate, as discussed in section 4.3, because the coupling λ is time-dependent and the definition itself of quasi-particles changes from initial to final time. The ladder operators associated to these quasi-particles are related to each other by the Bogoliubov transformation introduced in equation (4.25), fully defined by the parameters Θ_k and φ_k (which must also satisfy the equations (4.26)-(4.27)).

4.3.2 Scattering operator

The exact expressions of Θ_k and φ_k depend on the behavior of $\lambda\,(t)$, which is a function of the cosmological scale parameter and is therefore different for each cosmological model. They can in general be evaluated with the well established methods used in quantum field theory in curved spacetimes [30]. In general it is found that $\cosh\Theta_k > 1$, as the value $\cosh\Theta_k = 1$ (*i.e.* $\sinh\Theta_k = 0$) is restricted to the case in which λ is a constant for the whole evolution, and the analogue spacetime is simply flat.

The unitary operator describing the evolution from initial to final time is $U\,(t_{out}, t_{in})$; when $t_{out} \to +\infty$ and $t_{in} \to -\infty$, the operator U, is the scattering operator S. This is exactly the operator acting on the quasi-particles, defining the Bogoliubov transformation in which we are interested

$$c'_k = S^{\dagger}c_kS\,. \tag{4.38}$$

The behavior of c'_k, describing the quasi-particles at late times, can therefore be understood from the behavior of the initial quasi-particle operators c_k when the expression of the scattering operator is known. In particular, the phenomenon of cosmological particle creation is quantified considering the expectation value of the number operator of quasi-particles at late times in the vacuum state as defined by early times operators [30].

Consider as initial state the vacuum of quasi-particles at early times, satisfying the condition in equation (4.28). It is analogue to a Minkowski vacuum, and the evolution of the coupling $\lambda(t)$ induces a change in the definition of quasi-particles. We find that, of course, the state is not a vacuum with respect to the final quasi-particles c'. It is

$$S^\dagger c_k^\dagger c_k S = c_k'^\dagger c_k' = \left(\cosh\Theta_k c_k^\dagger + \sinh\Theta_k e^{-i\varphi_k} c_{-k}\right)\left(\cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k} c_{-k}^\dagger\right) , \tag{4.39}$$

and

$$\left\langle S^\dagger c_k^\dagger c_k S\right\rangle = \sinh^2\Theta_k \left\langle c_{-k} c_{-k}^\dagger\right\rangle = \sinh^2\Theta_k > 0 . \tag{4.40}$$

We are interested in the effect that the evolution of the quasi-particles have on the atoms. The system is fully characterized by the initial conditions and the Bogoliubov transformation: we have the initial occupation numbers, the range of momenta which we should consider and the relation between initial and final quasi-particles.

What is most significant is that the quasi-particle dynamics affects the occupation number of the atoms. Considering that for t sufficiently large we are already in the final regime, the field takes the following values:

$$\delta\phi_k(t\to-\infty) = i\langle\rho_0\rangle^{1/2} e^{i\langle\theta_0(t)\rangle}\frac{1}{\mathcal{N}_k}\left(\left(\mathcal{F}_k+1\right)e^{-i\omega_k t}c_k - \left(\mathcal{F}_k-1\right)e^{i\omega_k t}c_{-k}^\dagger\right) , \tag{4.41}$$

$$\delta\phi_k(t\to+\infty) = i\langle\rho_0\rangle^{1/2} e^{i\langle\theta_0(t)\rangle}\frac{1}{\mathcal{N}_k'}\left(\left(\mathcal{F}_k'+1\right)e^{-i\omega_k' t}c_k' - \left(\mathcal{F}_k'-1\right)e^{i\omega_k' t}c_{-k}'^\dagger\right) , \tag{4.42}$$

where $\mathcal{F}_k \equiv \frac{\omega_k}{\frac{k^2}{2m}+2\lambda\langle\rho_0\rangle}$ and $\mathcal{F}_k' \equiv \frac{\omega_k'}{\frac{k^2}{2m}+2\lambda'\langle\rho_0\rangle}$, with $\omega_k' = \sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m}+2\lambda'\langle\rho_0\rangle\right)}$. One finds

$$\left\langle\delta\phi_k^\dagger(t)\,\delta\phi_k(t)\right\rangle = \frac{\frac{k^2}{2m}+\lambda'\langle\rho_0\rangle}{2\omega_k'}\cosh(2\Theta_k) - \frac{1}{2} + \frac{\lambda'\langle\rho_0\rangle\sinh(2\Theta_k)}{2\omega_k'}\cos\left(2\omega_k' t - \varphi_k\right) . \tag{4.43}$$

In equation (4.43) the last term is oscillating symmetrically around 0 — meaning that the atoms will leave and rejoin the condensate periodically in time — while the first two are stationary.

An increase in the value of the coupling λ has therefore deep consequences. It appears explicitly in the prefactor and more importantly it affects the hyperbolic functions $\cosh\Theta_k > 1$, which implies that the mean value is larger than the initial one, differing from the equilibrium value corresponding to the vacuum of quasi-particles.

This result is significant because it explicitly shows that the quasi-particle dynamics influences the underlying structure of atomic particles. Even assuming that the back-reaction of the quasi-particles on the condensate is negligible for the dynamics of the quasi-particles themselves, the mechanism of extraction of atoms from the condensate fraction is effective and increases the depletion (as also found in the standard Bogoliubov approach). This extraction mechanism can be evaluated in terms of operators describing the quasi-particles, that can be defined *a posteriori*, without notion of the operators describing the atoms.

The fact that analogue gravity can be reproduced in condensates independently from the use of coherent states enhances the validity of the discussion. It is not strictly necessary that we have a coherent state to simulate the effects of curvature with quasi-particles, but in the more general case of condensation, the condensate wavefunction provides a support for the propagation of quasi-particles. From an analogue gravity point of view, its intrinsic role is that of seeding the emergence of the analogue scalar field [44].

4.4 Squeezing and quantum state structure

The Bogoliubov transformation in equation (4.25) leading to the quasi-particle production describes the action of the scattering operator on the ladder operators, relating the operators at early and late times. The linearity of this transformation is obtained by the linearity of the dynamical equation for the quasi-particles, which is particularly simple in the case of homogeneous condensate.

The scattering operator S is unitary by definition, as it is easily checked by its action on the operators c_k. Its full expression can be found from the Bogoliubov transformation, finding the generators of the transformation when the arguments of the hyperbolic functions, the parameters Θ_k, are infinitesimal:

$$S^\dagger c_k S = c'_k = \cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k} c^\dagger_{-k}\,. \tag{4.44}$$

It follows

$$S = \exp\left(\frac{1}{2}\sum_{k\neq 0}\left(-e^{-i\varphi_k} c_k c_{-k} + e^{i\varphi_k} c^\dagger_k c^\dagger_{-k}\right)\Theta_k\right)\,. \tag{4.45}$$

The scattering operator is particularly simple, and takes the peculiar expression that is required for producing squeezed states. This is the general functional expression which is found in cosmological particle creation and in its analogue gravity counterparts, whether they are realized in the usual Bogoliubov framework or in its number-conserving reformulation. As discussed previously, the number-conserving formalism is more general, reproduces the usual case when the state is an eigenstate of the destruction operator a_0, and includes the notion that the excitations of the condensate move condensate atoms to the excited part.

The expression in equation (4.45) has been found under the hypothesis that the mean value of the operator N_0 is macroscopically larger than the other occupation numbers. Instead of using the quasi-particle ladder operators, S can be rewritten easily in terms of the atom operators. In particular, we remind that the time-independent operators c_k depend on the condensate operator a_0 and can be defined as compositions of number-conserving atom operators $\delta\phi_k(t)$ and $\delta\phi^\dagger_{-k}(t)$ defined in equation (4.6). At any time there will be a transformation from a set of operators to the other. It is significant that the operators c_k commute with the operators $N_0^{-1/2} a_0^\dagger$ and $a_0 N_0^{-1/2}$, which are therefore conserved in time (as long as the linearized dynamics for $\delta\phi_k$ is a good approximation)

$$\left[\phi_1 (x) \, , N_0^{-1/2} (t) \, a_0^\dagger (t)\right] = 0 \, , \tag{4.46}$$

$$\Downarrow$$

$$\left[c_k, N_0^{-1/2} (t) \, a_0^\dagger (t)\right] = 0 \, , \tag{4.47}$$

$$\Downarrow$$

$$\left[S, N_0^{-1/2} (t) \, a_0^\dagger (t)\right] = 0 \, . \tag{4.48}$$

The operator S cannot have other terms apart for those in equation (4.45), even if it is defined for its action on the operators c_k, and therefore on a set of functions which is not a complete basis of the 1-particle Hilbert space. Nevertheless the notion of number conservation implies its action on the condensate and on the operator a_0.

One could investigate whether it is possible to consider a more general expression with additional terms depending only on a_0 and $a_0^\dagger$, *i.e.* assuming the scattering operator to be

$$S = \exp \left(\frac{1}{2} \sum_{k \neq 0} \left(Z_k c_k c_{-k} + c_k^\dagger c_{-k}^\dagger Z_{-k}^\dagger \right) + G_0 \right) \, , \tag{4.49}$$

where we could assume that the coefficients of the quasi-particle operators are themselves depending on only a_0 and $a_0^\dagger$, and so G_0. But the requirement that S commutes with the total number of atoms N implies that so do its generators, and therefore Z and G_0 must be functionally dependent on N_0, and not on a_0 and $a_0^\dagger$ separately, since they do not conserve the total number. Therefore it must hold

$$0 = \left[\left(\frac{1}{2} \sum_{k \neq 0} \left(Z_k c_k c_{-k} + c_k^\dagger c_{-k}^\dagger Z_{-k}^\dagger \right) + G_0 \right) , N \right] \, . \tag{4.50}$$

The only expressions in agreement with the linearized dynamical equation for $\delta\phi$ imply that Z and G_0 are multiple of the identity, otherwise they would modify the evolution of the operators $\delta\phi_k = N_0^{-1/2} a_0^\dagger a_k$, as they do not commute with N_0. This means that that corrections to the scattering operator are possible only involving higher-order corrections (in terms of $\delta\phi$).

The fact that the operator S as in equation (4.45) is the only number-conserving operator satisfying the dynamics is remarkable because it emphasizes that the production of quasi-particles is a phenomenon which holds only in terms of excitations of atoms from the condensate to the excited part, with the number of transferred atoms evaluated in the previous subsection. The expression of the scattering operator shows that the analogue gravity system produces states in which the final state presents squeezed quasi-particle states, but the occurrence of this feature in the emergent dynamics happens only introducing correlations in the condensate, with each quanta of the analogue field θ_1 entangling atoms in the condensate with atoms in the excited part.

The quasi-particle scattering operator obtained in the number-conserving framework is functionally equivalent to that in the usual Bogoliubov description, and the difference between the two appears when considering the atom operators, depending on whether a_0 is treated for its quantum nature or it is replaced with the number $\langle N_0 \rangle^{1/2}$. This reflects that the dynamical equations are functionally the same when the expectation value $\langle N_0 \rangle$ is macroscopically large.

There are no requirements on the initial density matrix of the state, and it is not relevant whether the state is a coherent superposition of infinite states with different number of atoms or it is a pure state with a fixed number of atoms in the same 1-particle state. The quasi-particle description holds the same and it provides the same predictions. This is useful for implementing analogue gravity systems, but also a strong hint in interpreting the problem of information loss. When producing quasi-particles in analogue gravity one can, in first approximation, reconstruct the initial expectation values of the excited states, and push the description to include the back-reaction on the condensate. What we are intrinsically unable to do is reconstruct the entirety of the initial atom quantum state, *i.e.* how the condensate is composed.

We know that in analogue gravity the evolution is unitary, the final state is uniquely determined by the initial state. Knowing all the properties of the final state we could reconstruct the initial state, and yet the intrinsic inability to infer all the properties of the condensate atoms from the excited part shows that the one needs to access the full correlation properties of the condensate atoms with the quasi-particles to fully appreciate (and retrieve) the unitarity of the evolution.

4.4.1 Correlations

In the previous section we made the standard choice of considering as initial state the quasi-particle vacuum. To characterize it with respect to the atomic degrees of freedom, the quasi-particle ladders operators have to be expressed as compositions of the number-conserving atomic operators, manipulating the equations (4.8)-(4.16).

By definition, at any time, both sets of operators satisfy the canonical commutation relations (4.11) and (4.24) $\forall k, k' \neq 0$. Therefore, it must exist a Bogoliubov transformation linking the quasi-particle and the number-conserving operators which will in general be written as

$$c_k = e^{-i\alpha_k} \cosh \Lambda_k \delta\phi_k + e^{i\beta_k} \sinh \Lambda_k \delta\phi^\dagger_{-k} \,. \tag{4.51}$$

The transformation is defined through a set of functions Λ_k, constant in the stationary case, and the phases α_k and β_k, inheriting their time dependence from the atomic operators. These functions can be obtained from the equations (4.8)-(4.16):

$$\cosh \Lambda_k = \left(\frac{\omega_k + \left(\frac{k^2}{2m} + 2\lambda \langle \rho_0 \rangle \right)}{4\omega_k} \right) \frac{\mathcal{N}_k}{\langle \phi_0 \rangle} \,. \tag{4.52}$$

If the coupling changes in time, the quasi-particle operators during the transient are defined knowing the solutions of the Klein-Gordon equation. With the Bogoliubov transformation of equation (4.51), it is possible to find the quasi-particle vacuum-state $|\emptyset\rangle_{qp}$ in terms of the atomic degrees of freedom

$$|\emptyset\rangle_{qp} = \prod_k \frac{e^{-\frac{1}{2}e^{i(\alpha_k+\beta_k)}\tanh\Lambda_k \delta\phi_k^\dagger \delta\phi_{-k}^\dagger}}{\cosh\Lambda_k} |\emptyset\rangle_a = \tag{4.53}$$

$$= \exp\sum_k \left(-\frac{1}{2}e^{i(\alpha_k+\beta_k)}\tanh\Lambda_k \delta\phi_k^\dagger \delta\phi_{-k}^\dagger - \ln\cosh\Lambda_k\right)|\emptyset\rangle_a \ , \tag{4.54}$$

where $|\emptyset\rangle_a$ should be interpreted as the vacuum of excited atoms.

From equation (4.54) it is clear that, in the basis of atom occupation number, the quasi-particle vacuum is a complicated superposition of states with different number of atoms in the condensed 1-particle state (and a corresponding number of coupled excited atoms, in pairs of opposite momenta). Every correlation function is therefore dependent on the entanglement of this many-body atomic state.

This feature is enhanced by the dynamics, as can be observed from the scattering operator in equation (4.45) relating early and late times. The scattering operator acts on atom pairs and the creation of quasi-particles affects the approximated vacuum differently depending on the number of atoms occupying the condensed 1-particle state. The creation of more pairs modifies further the superposition of the entangled atomic states depending on the total number of atoms and the initial number of excited atoms.

We can observe this from the action of the condensed state ladder operator, which does not commute with the the creation of coupled quasi-particles $c_k^\dagger c_{-k}^\dagger$, which is described by the combination of the operators $\delta\phi_k^\dagger\delta\phi_k$, $\delta\phi_k^\dagger\delta\phi_{-k}^\dagger$ and $\delta\phi_k\delta\phi_{-k}$. The ladder operator $a_0^\dagger$ commutes with the first, but not with the others:

$$\left(\delta\phi_k^\dagger\delta\phi_k\right) a_0^\dagger = a_0^\dagger \left(\delta\phi_k^\dagger\delta\phi_k\right) \ , \tag{4.55}$$

$$\left(\delta\phi_k^\dagger\delta\phi_{-k}^\dagger\right)^n a_0^\dagger = a_0^\dagger \left(\delta\phi_k^\dagger\delta\phi_{-k}^\dagger\right)^n \left(\frac{N_0+1}{N_0+1-2n}\right)^{1/2} \ , \tag{4.56}$$

$$\left(\delta\phi_k\delta\phi_{-k}\right)^n a_0^\dagger = a_0^\dagger \left(\delta\phi_k\delta\phi_{-k}\right)^n \left(\frac{N_0+1}{N_0+1+2n}\right)^{1/2} \ . \tag{4.57}$$

The operators a_0 and $a_0^\dagger$ do not commute with the number-conserving atomic ladder operators, and therefore the creation of pairs and the correlation functions, up to any order, will present corrections of order $1/N$ to the values that could be expected in the usual Bogoliubov description. Such corrections appear in correlation functions between quasi-particle operators and for correlations between quasi-particles and condensate atoms. This is equivalent to saying that a condensed state, which is generally not coherent, will present deviations from the expected correlation functions predicted by the Bogoliubov theory due to both the interaction and the features of the initial state itself (through contributions coming from connected expectation values).

4.4.2 Entanglement structure in number-conserving formalism

As discussed previously, the mean-field approximation for the condensate is most adequate for states close to coherence, thus allowing a separate analysis for the mean-field. The field operator is split in the mean-field function $\langle\phi\rangle$ and the fluctuation operator $\delta\phi$ which is assumed not to affect the mean-field through back-reaction. Therefore the states in this picture can be written as

$$|\langle\phi\rangle\rangle_{mf} \otimes \left|\delta\phi, \delta\phi^\dagger\right\rangle_{a\,Bog} , \tag{4.58}$$

meaning that the state belongs to the product of two Hilbert spaces: the mean-field defined on one and the fluctuations on the other, with $\delta\phi$ and $\delta\phi^\dagger$ ladder operators acting only on the second. The Bogoliubov transformation from atom operators to quasi-particles allows to rewrite the state as shown in equation (4.54). The transformation only affects its second part

$$|\langle N\rangle\rangle_{mf} \otimes |\emptyset\rangle_{qp\,Bog} = |\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a_{lr} \, |l, r\rangle_{a\,Bog} . \tag{4.59}$$

With such transformation the condensed part of the state is kept separate from the superposition of coupled atoms (which here are denoted l and r for brevity) forming the excited part, a separation which is maintained during the evolution in the Bogoliubov description. Also the Bogoliubov transformation from early-times quasi-particles to late-times quasi-particles affects only the second part

$$|\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a_{lr} \, |l, r\rangle_{a\,Bog} \Rightarrow |\langle N\rangle\rangle_{mf} \otimes \sum_{lr} a'_{lr} \, |l, r\rangle_{a\,Bog} . \tag{4.60}$$

In the number-conserving framework there is not such a splitting of the Fock space, and there is no separation between the two parts of the state. In this case the best approximation for the quasi–particle vacuum is given by a superposition of coupled excitations of the atom operators, but the total number of atoms cannot be factored out:

$$|N; \emptyset\rangle_{qp} \approx \sum_{lr} a_{lr} \, |N - l - r, l, r\rangle_a . \tag{4.61}$$

The term in the RHS is a superposition of states with N total atoms, of which $N - l - r$ are in the condensed 1-particle state and the others occupy excited atomic states and are coupled with each other analogously to the previous equation (4.59) (the difference being the truncation of the sum, required for a sufficiently large number of excited atoms, implying a different normalization).

The evolution does not split the Hilbert space, and the final state will be a different superposition of atomic states

$$\sum_{lr} a_{lr} \, |N, l, r\rangle_a \Rightarrow \sum_{lr} a'_{lr} \left(1 + \mathcal{O}\left(N^{-1}\right)\right) |N - l - r, l, r\rangle_a . \tag{4.62}$$

We remark that in the RHS the final state must include corrections of order $1/N$ with respect to the Bogoliubov prediction, due to the fully quantum behavior of the condensate ladder operators. These are small corrections, but we expect that the difference from the Bogoliubov prediction will be relevant when considering many-point correlation functions.

Moreover, these corrections remark the fact that states with different number of atoms in the condensate are transformed differently. If we consider a superposition of states of the type in equation (4.61) with different total atom numbers so to reproduce the state in equation (4.59), therefore replicating the splitting of the state, we would find that the evolution produces a final state with a different structure, because every state in the superposition evolves differently. Therefore, also assuming that the initial state could be written as

$$\sum_N \frac{e^{-N/2}}{\sqrt{N!}} \left|N;\emptyset\right\rangle_{qp} \approx \left|\langle N\rangle\right\rangle_{mf} \otimes \left|\emptyset\right\rangle_{qp\,Boq} \,, \tag{4.63}$$

anyway the final state would unavoidably have different features

$$\sum_N \frac{e^{-N/2}}{\sqrt{N!}} \sum_{lr} a'_{lr} \left(1+\mathcal{O}\left(N^{-1}\right)\right) \left|N-l-r,l,r\right\rangle_a \neq \left|\langle N\rangle\right\rangle_{mf} \otimes \sum_{lr} a'_{lr} \left|l,r\right\rangle_{a\,Bog} \,. \tag{4.64}$$

We remark that our point is qualitative. Indeed it is true that also in the weakly interacting limit the contribution coming from the interaction of Bogoliubov quasi-particles may be quantitatively larger than the $\mathcal{O}\left(N^{-1}\right)$ term in equation (4.64). However, even if one treats the operator a_0 as a number disregarding its quantum nature, then one cannot have the above discussed entanglement. In that case, the Hilbert space does not have a sector associated to the condensed part and no correlation between the condensate and the quasi-particles is present. To have them one has to keep the quantum nature of a_0, and its contribution to the Hilbert space.

Alternatively, let us suppose to have an interacting theory of bosons for which no interactions between quasi-particles are present (as in principle one could devise and engineer similar models based on solvable interacting bosonic systems [116]). Even in that case one would have a qualitative difference (and the absence or presence of the entanglement structure here discussed) if one retains or not the quantum nature of a_0 and its contribution to the Hilbert space. Of course one could always argue that in principle the coupling between the quantum gravity and the matter degrees of freedom may be such to preserve the factorization of an initial state. This is certainly possible in principle, but it would require a surprisingly high degree of fine tuning at the level of the fundamental theory.

In conclusion, in the Bogoliubov description the state is split in two sectors, and the total density matrix is therefore a product of two contributions, of which the one relative to the mean-field can be traced away without affecting the other. The number-conserving picture shows instead that unavoidably the excited part of the system cannot be manipulated without affecting the condensate. Tracing away the quantum degrees of freedom of the condensate would imply a loss of information even without tracing away part of the pairs created by analogue curved spacetime dynamics. In other words, when one considers the full Hilbert space and the full dynamics, the final state ρ_{fin} is obtained by an unitary evolution. But now, unlike the usual Bogoliubov treatment, one can trace out in ρ_{fin} the condensate degrees of freedom of the Hilbert space, an operation that we may denote by "$\mathrm{Tr}_0[\ldots]$". So

$$\rho_{fin}^{reduced} = \mathrm{Tr}_0[\rho_{fin}]\,, \tag{4.65}$$

is not pure, as a consequence of the presence of the correlations. So one has $\mathrm{Tr}[\rho_{fin}^2] = 1$, at variance with $\mathrm{Tr}[(\rho_{fin}^{reduced})^2] \neq 1$. The entanglement between condensate and excited part is an unavoidable feature of the evolution of these states.

4.5 Final remarks

We have considered analogue gravity in Bose–Einstein condensates, where the condensate wavefunction defines an acoustic geometry for the propagation of the quantum fluctuations of the phase of the condensate, which is analogous to a massless scalar field. The analogue metric and the analogue scalar quantum field are defined in different Hilbert spaces, and one would not be able to find correlations between the two in a standard quantum calculation: the condensate wavefunction is a classical function treated separately from the quantum excitations. An analogous separation holds in semiclassical gravity between the classical metric tensor and the quantum matter fields.

The usual description of the condensates considered in analogue gravity is made in the mean-field approximation for the condensate wavefunction and the Bogoliubov approximation for the quantum fluctuation. The mean-field approximation is valid for coherent states, while in a more general case the condensate wavefunction is defined from the 2-point correlation function. The use of coherent states requires to identify the condensate wavefunction with the expectation value of the field operator, and implies that the usual description of the Bogoliubov excitations is made in terms of the translation of the field operator; this leads to a formalism which is not explicitly number-conserving, thus making it more difficult to keep track of the underlying atomic dynamics.

With the number-conserving formalism instead we have gone beyond the usual description and we have made explicit use of the underlying atomic theory. In this improved description we were able to investigate the correlations that are produced between the quantum degrees of freedom that describe the analogue geometry and the analogue scalar field, *i.e.* the condensate and excited part.

In the case of an analogue cosmology we have used this formalism to show that the cosmological evolution — realized by varying the coupling of the two-body interaction from an initial value to a final one — induces the creation of quasi-particles and generates correlation between the excited part and the atoms in the condensed 1-particle state. In terms of the number conserving formalism it is clear that the excitations of the Bogoliubov quasi-particles, in this case of analogue cosmology, contribute to the depletion of atoms from the condensate to the excited part.

We have shown that also in the number-conserving formalism one can define a unitary scattering operator, and thus the Bogoliubov transformation from early-times to late-times quasi-particles. The scattering operator provided in equation (4.45) not only shows the nature of quasi-particle creation as a squeezing process of the initial quasi-particle vacuum, but also that the evolution process as a whole is unitary precisely because it entangles the quasi-particles with the condensate atoms constituting the geometry over which the former propagate.

The correlation between the quasi-particles and the condensate atoms is a general feature, it is not realized just in a regime of high energies — analogous to the late stages of a black hole evaporation process or to sudden cosmological expansion — but it happens during all the evolution, albeit they are suppressed in the number of atoms, N, relevant for the system and are hence generally negligible.

When describing the full Fock space there is not unitarity breaking, and the purity of the state is preserved: it is not retrieved at late times, nor it is spoiled in the transient of the evolution. Nonetheless, such a state after particle production will not factorize into the product of two states — a condensate (geometrical) and quasi-particle (matter) one — but, as we have seen, it will be necessarily an entangled state. This implies, as we have discussed at the end of the previous section, that an observer unable to access the condensate (geometrical) quantum degrees of freedom would define a reduced density matrix (obtained by tracing over the latter) which would no more be compatible with an unitary evolution.

This is the main feature which cannot be appreciated in the usual mean-field description: assuming the initial state to be coherent with respect to the condensed 1-particle state, *i.e.* to the action of a_0, and assuming that the evolution of the mean-field is described separately by the Gross–Pitaevskii equation, it is impossible to lose the factorization of the state.

The Bogoliubov approximation corresponds to taking the quantum degrees of freedom of the geometry as classical. This is not *per se* a unitarity violating operation, as it is equivalent to effectively recover the factorization of the above mentioned state. Indeed the squeezing operator so recovered (which corresponds to the one describing particle creation on a classical spacetime) is unitarity preserving. However, the two descriptions are no longer practically equivalent when a region of quantum gravitational evolution is somehow simulated. In this case, having the possibility of tracking the quantum degrees of freedom underlying the background enables to describe the full evolution, while in the analogue of quantum field theory in curved spacetime a trace over the ingoing Hawking quanta is necessary with the usual problematic implications for the preservation of unitary evolution.

In the analogue gravity picture, the above alternatives would correspond to the fact that the number-conserving evolution can keep track of the correlations between the atoms and the quasi-particles that cannot be accounted for in the standard Bogoliubov framework.

5

Back-reaction in canonical analogue black holes

5.1 Interplay of Hawking radiation and geometry

Bose–Einstein condensates have been among the most successful analogue models due to their intrinsic quantum nature, simplicity and experimental realizability [44, 73–75]. In these systems, the dynamics of Bogoliubov quasi-particles simulates the phenomenology of massless quantum fields in curved spacetime.

Recent experimental results have given strong evidence of Hawking radiation in Bose–Einstein condensates [91–95], largely confirming the theoretical expectations (at least in some suitable regime of the analogue black hole dynamics). These results corroborated the experimental viability of analogue gravity and validated its techniques, through the detection of quantum effects which we are not capable of measuring in a gravitational setting: Hawking radiation is basically impossible to measure for astrophysical objects, but it can be investigated in analogue gravity through the analogy between quasi-particles and quantum fields in curved spacetime.

The experimental results also supported the robustness of Hawking radiation in an ultraviolet complete theory, and paved the way for new studies to address the various open issues of gravity. With quantum analogue systems it becomes possible to study the interplay of the microscopical (the analogue quantum gravitational) dynamics and the emergent phenomenology of spacetime and quantum fields.

In particular, with analogue Hawking radiation in Bose–Einstein condensates we can investigate the information loss problem, as we have discussed in the previous chapter and in [7]: in an isolated Bose–Einstein condensate system the unitarity of the evolution must always be preserved at the microscopical level; the correlations between the atoms underlying the emergent spacetime and the continuously generated Hawking quanta incorporate the information which is missing in the standard mean-field description, analogous to the semiclassical picture.

By devising a system that reproduces the phenomenology of the evaporation of black holes, we could therefore have a complete analogue framework in which to study two complementary aspects: both how the quantum degrees of freedom from which the analogue geometry emerges and the quasi-particles develop their correlations, and how geometry is modified by the dynamics of quantum fields.

In our work [117], from which we take this chapter, we have studied the problem of evaporation, investigating the back-reaction associated with the emission of Hawking radiation by an acoustic canonical analogue black hole in a Bose–Einstein condensate.

Although in analogue gravity the back-reaction of the quantum fields on the acoustic geometry would not follow the Einstein equations, the observation of evaporation in analogue black holes could provide information over the interplay of quantum fields and classical geometry within a semiclassical scheme in a broad class of quantum gravity scenarios. If one sees this analogue gravity system as a toy model for emergent gravity, it could then give a physical intuition of how Hawking radiation can be pictured as a feature emerging from an underlying full quantum theory.

So, seeking this kind of theoretical insights, as well as the possibility to test this understanding of the back-reaction in experimental realizations, attracted an increasing interest on the nature of the back-reaction in analogue Hawking radiation [7, 61, 118, 119]. For example, in recent experiments with surface waves on a draining vortex water flow [61], it has been shown that the back-reaction is indeed observable, and it is possible to measure the exchange of energy and angular momentum between the background flow and the waves incident on the horizon. In Bose–Einstein condensates, a next step in this direction should be achievable as it should be possible to observe the back-reaction of the Hawking Bogoliubov quasi-particles on the acoustic geometry — *i.e.* on the substratum of condensate atoms — and interpret it through the lens of the underlying atomic theory.

With this aim in mind we focus our attention on the back-reaction of the Hawking radiation on a canonical analogue black hole within an analogue system in a Bose–Einstein condensate characterized by a $\lambda\phi^4$ interaction [120–122]. The canonical analogue black hole is for our study a remarkable geometry because it is the spherically symmetric stationary solution of the Gross–Pitaevskii equation with homogeneous atom number density. The spherical symmetry allows to reduce the problem to a time-radius problem that presents a horizon at the radius where the velocity of the stationary ingoing superfluid equals the speed of sound.

In what follows we first describe the acoustic metric in this system and how the solutions of the Klein–Gordon equation behave at the horizon, and we calculate the Hawking radiation. These are the basis for showing how the back-reaction exerted by the Hawking radiation can be studied and how it affects the geometry of the black hole. We then propose to study a regime in which the Hawking radiation would lead to a sizeable evaporation of the analogue black hole. Finally, we show how this evaporation corresponds, in the evolution of the system, to the depletion of atoms from the condensate to the excited part.

5.2 Analogue black holes in Bose–Einstein condensates

We have discussed previously how analogue gravity models can be realized with Bose–Einstein condensates, in which the dynamics of the condensate wavefunction and of the quantum fluctuations are described respectively with the Gross–Pitaevskii equation (2.21) and with the Bogoliubov–de Gennes equation (2.46).

Adopting the Madelung representation, the quantum fluctuations of the phase of the condensate are described by the field θ_1 as massless quasi-particles that propagate following an effective metric determined by the condensate wavefunction.

The analogue field θ_1 and its conjugate ρ_1 follow the canonical commutation relations, which we recall from equations (3.102)–(3.103)–(3.104)

$$[\theta_1(x), \rho_1(y)] = -i\delta^3(x,y) \; , \tag{5.1}$$

$$[\theta_1(x), \theta_1(y)] = 0 \, , \tag{5.2}$$

$$[\rho_1(x), \rho_1(y)] = 0 \, , \tag{5.3}$$

where the definition of ρ_1 depends on the theory considered: for the $\lambda\phi^4$ theory, in the hydrodynamic approximation of negligible quantum pressure, it is

$$\left(\partial_t + \frac{(\nabla \langle\theta_0\rangle)}{m}\nabla\right)\theta_1 = -\lambda\rho_1 \, . \tag{5.4}$$

The analogue field satisfies the Klein–Gordon equation $\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\theta_1 = 0$, where $g_{\mu\nu}$ is the effective acoustic metric determined by the condensate wavefunction $\langle\rho_0\rangle^{1/2} e^{i\langle\theta_0\rangle}$, with line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \tag{5.5}$$

$$= \frac{\langle\rho_0\rangle}{c_s}\left(-c_s^2\mathrm{d}t^2 + \delta_{ij}\left(\mathrm{d}x^i - \left(\frac{\partial^i\langle\theta_0\rangle}{m}\right)\mathrm{d}t\right)\left(\mathrm{d}x^j - \left(\frac{\partial^j\langle\theta_0\rangle}{m}\right)\mathrm{d}t\right)\right) \, . \tag{5.6}$$

Acoustic metrics of this kind can have acoustic horizons, where the velocity of the flowing condensate reaches the speed of sound $\|\nabla\langle\theta_0\rangle\|/m = c_s$. The points where this condition is verified form an interface that separates a subcritical region from a supercritical region. The quasi-particles, which propagate at the speed of sound, cannot cross the horizon from the supercritical to the subcritical region. When there is spherical symmetry, it is natural to pass to spherical coordinates, and the line element is the one in equation (3.124), which we recall

$$\mathrm{d}s^2 = \frac{\langle\rho_0\rangle}{c_s}\left(-c_s^2\mathrm{d}t^2 + \left(\mathrm{d}r - \left(\frac{\partial_r\langle\theta_0\rangle}{m}\right)\mathrm{d}t\right)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right) \, . \tag{5.7}$$

In presence of a horizon in a spherically symmetric system, we denote the radius of the a horizon with r_H, where it holds that $(\partial_r\langle\theta_0\rangle/m)^2(r_H) = c_s^2(r_H)$.

A horizon causes the quasi-particles to follow either horizon-crossing trajectories, but only in the ingoing direction, or trajectories which can be traced to the horizon and peel off from it.

The behavior of quasi-particles in presence of acoustic horizons is therefore analogous to that of massless particles in black hole spacetime. The acoustic metric in equation (5.6) is analogous to the metric of a $(3+1)$-dimensional spacetime in Painlevé–Gullstrand coordinates

$$\mathrm{d}s^2 = -\left(1 - \delta_{ij} V^i V^j\right) c^2 \mathrm{d}t^2 + 2\delta_{ij} V^i c \mathrm{d}t \mathrm{d}x^i + \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \,, \tag{5.8}$$

up to the conformal factor $\langle\rho_0\rangle / c_s$. In a Bose–Einstein condensate the role of the velocity V^i is played by the superfluid velocity

$$V^i = -\delta^{ij} \frac{\partial_j \langle\theta_0\rangle}{m} \,. \tag{5.9}$$

In the case of spherical symmetry with $\partial_r \langle\theta_0\rangle < 0$ we are in the case of ingoing Painlevé–Gullstrand coordinates, which allow to study horizon penetrating trajectories and the behavior outside a black hole horizon. The solutions for the null radial geodesics are one regular ingoing solution and one outgoing solution that peels off the horizon (at the radius where $|V| = 1$)

$$0 = -\left(1 - V^2\right) u^{t2} + 2\,|V|\, u^r u^t + u^{r2} \,, \tag{5.10}$$

$$\Downarrow$$

$$u^t = -\frac{u^r}{|V| \mp 1} \,. \tag{5.11}$$

In the following we will make use the useful notation

$$h\,(r) = 1 - V^2\,(r) \,, \tag{5.12}$$

since it is most convenient to describe the behavior at the horizon.

5.2.1 Canonical analogue black holes

Among the acoustic metrics simulated with Bose–Einstein condensates, it is convenient for our investigation to consider the canonical analogue black hole.

A canonical analogue black hole could be obtained as the stationary solution of the Gross–Pitaevskii equation with spherical symmetry and with homogeneous atom number density. The stationarity of the background metric, of the mean-field, means that the deviations from this configuration should be understood as the effect of the phenomenology previously neglected: the subleading effect of the quasi-particles and their back-reaction. The homogeneity of the atom density $\langle\rho_0\rangle$ means that the conformal factor $\langle\rho_0\rangle / c_s$ is a number, and can trivially be transformed away.

Taking a constant and homogeneous number density $\langle\rho_0\rangle$, and assuming $\langle\theta_0\rangle$ to be a function of the radius only, from the equations (3.97)–(3.98) we obtain the stationary spherically symmetric solution of the Gross–Pitaevskii equation. The superfluid velocity V_0 and the external potential setting the corresponding velocity profile are respectively

$$-\frac{\partial_r \langle\theta_0\rangle}{m} = c_s \frac{r_H^2}{r^2} = V_0 \,, \tag{5.13}$$

$$V_{\mathrm{ext}} = -m c_s^2 \left(1 + \frac{1}{2}\frac{r_H^4}{r^4}\right) \,, \tag{5.14}$$

with an ingoing flux of atoms crossing every closed surface containing the origin at a rate of $4\pi c_s r_H^2 \langle\rho_0\rangle$ atoms per unit time, which must be pumped in the system and move towards the singularity in the origin. The radius of the horizon r_H is set by the external potential $V_{\rm ext}$.

Omitting the conformal factor, the analogue metric implied by equation (5.13) is therefore

$$\mathrm{d}s^2 = -\left(1-\frac{r_H^4}{r^4}\right)c_s^2\mathrm{d}t^2 + 2\frac{r_H^2}{r^2}c_s\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\,. \tag{5.15}$$

The singularity in the origin poses a problem that would be unavoidable in devising an experiment: one should either remove the atoms reaching the origin and put them back in the system at large radius [123], or should consider only the region in which the hydrodynamical limit holds with a good approximation, at a distance from the origin and for a period of time such that it is not affected by atoms which flowed towards the origin. [1]

These are however practical concerns that we shall assume dealt with in our theoretical study. In addition to them, we observe that the possible removal of atoms from the system would affect the atomic correlation functions and introduce statistical mixture, but the dynamics of the quasi-particles in the near-horizon region, on which we focus in the following discussion, would not be sensibly affected.

5.2.2 Klein–Gordon equation and field modes

The massless scalar field living in our analogue spacetime can be described in terms of its modes, found by solving the Klein–Gordon equation. For a stationary spherically symmetric spacetime in Painlevé–Gullstrand coordinates, the equation to solve is

$$\begin{aligned}0 &= \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu f\left(t,r,\Omega\right) = \\ &= \left(-\partial^2_{c_st} + \sqrt{1-h}\partial_r\partial_{c_st} + \frac{1}{r^2}\partial_r r^2\sqrt{1-h}\partial_{c_st} + \frac{1}{r^2}\partial_r r^2 h\partial_r - \frac{L^2}{r^2}\right) f\left(t,r,\Omega\right)\,,\end{aligned} \tag{5.16}$$

where Ω is a compact notation for the solid angle and L^2 is the usual angular momentum operator

$$L^2 = -\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial^2_\phi\,. \tag{5.17}$$

We can find a complete set of solutions looking for those which preserve the symmetries of the system [121, 122], *i.e.* the eigenfunctions of both the time-derivative and of the angular momentum operator

$$f_{\omega lm}\left(t,r,\Omega\right) = e^{-i\omega t}Y_{lm}\left(\Omega\right)f_{\omega l}\left(r\right)\,, \tag{5.18}$$

[1] To implement the latter, one could devise a $3D$ hard wall trapping potential (possibly spherical), plus a variation in the central region of the scattering length and suitable external potential there added via a superimposed laser beam.

$$\partial_t f_{\omega lm}(t,r,\Omega) = -i\omega f_{\omega lm}(t,r,\Omega) \ , \tag{5.19}$$
$$L^2 f_{\omega lm}(t,r,\Omega) = l(l+1) f_{\omega lm}(t,r,\Omega) \ , \tag{5.20}$$
$$\partial_\phi f_{\omega lm}(t,r,\Omega) = im f_{\omega lm}(t,r,\Omega) \ , \tag{5.21}$$

where the functions $Y_{lm}(\Omega)$ are the usual spherical harmonics.

So equation (5.16) becomes

$$0 = \left(\frac{\omega^2}{c_s^2} - i\frac{\omega}{c_s}\sqrt{1-h}\partial_r - i\frac{\omega}{c_s}\frac{1}{r^2}\partial_r r^2\sqrt{1-h} + \frac{1}{r^2}\partial_r r^2 h \partial_r - \frac{l(l+1)}{r^2} \right) f_{\omega l}(r) = \tag{5.22}$$
$$= \frac{e^{i\frac{\omega}{c_s}\int^r \frac{\sqrt{1-h}}{h}\mathrm{d}r}}{rh} \left(h\partial_r h\partial_r + \frac{\omega^2}{c_s^2} - \frac{h(\partial_r h)}{r} - \frac{hl(l+1)}{r^2} \right) r e^{-i\frac{\omega}{c_s}\int^r \frac{\sqrt{1-h}}{h}\mathrm{d}r} f_{\omega l}(r) \ . \tag{5.23}$$

We have written equation (5.23) reorganizing the various contribution to have a clearer view of the problem. The operator $h\partial_r$ can be understood as the directional derivative with respect to the tortoise radial coordinate. The term $h\partial_r h/r$ is a potential which induces mode mixing. This second order differential equation will have two linearly independent solutions for each set of eigenvalues, an outgoing mode and an ingoing mode. Splitting the solutions in norm and phase we get two coupled real differential equations

$$f_{\omega l}(r) = \sqrt{n_{\omega l}(r)} e^{i\phi_{\omega l}(r)} e^{i\frac{\omega}{c_s}\int^r \frac{\sqrt{1-h}}{h}\mathrm{d}r} \ , \tag{5.24}$$
$$0 = \frac{1}{r}\left(h\partial_r h\partial_r + \frac{\omega^2}{c_s^2} - \frac{h(\partial_r h)}{r} - \frac{l(l+1)h}{r^2} \right) r\sqrt{n_{\omega l}(r)} e^{i\phi_{\omega l}(r)} \ , \tag{5.25}$$
$$\Downarrow$$
$$\partial_r \phi_{\omega l}(r) = \frac{C}{r^2 h n_{\omega l}(r)} \ , \tag{5.26}$$
$$0 = \frac{1}{r}\left(h\partial_r h\partial_r + \frac{\omega^2}{c_s^2} - \frac{h(\partial_r h)}{r} - \frac{l(l+1)h}{r^2} \right) r\sqrt{n_{\omega l}} - \frac{C^2}{r^4}\frac{1}{\sqrt{n_{\omega l}^3(r)}} \ , \tag{5.27}$$

Different integrating constants C set the sign of the derivative of the phase and give the different solutions.

5.2.3 Near-horizon behavior

We are interested in the solutions in proximity of the horizon, for a small distance $\delta r = r - r_H$, and such that $n_{\omega l}(r)$ is regular. The horizon-crossing solutions will have a regular behavior in phase, and the horizon-tangent solutions will be the remaining linearly independent solutions. In presence of a horizon, h vanishes at r_H and so do all the terms depending on it in the Klein–Gordon equation. We can therefore set the phase-defining constant C of each solution with regular norm by requiring it to satisfy the Klein–Gordon equation at the horizon

$$h\left(r_H+\delta r\right)=h^{(1)}\left(r_H\right)\delta r+\mathcal{O}\left(\delta r^2\right)\,, \tag{5.28}$$

$$n_{\omega l}\left(r_H+\delta r\right)=n_{\omega l}\left(r_H\right)+\mathcal{O}\left(\delta r\right)\,, \tag{5.29}$$

$$C=\pm\frac{\omega}{c_s}r_H^2 n_{\omega l}\left(r_H\right)\,. \tag{5.30}$$

Considering the phase of the function $f_{\omega l}\left(r\right)$ we get, in the specific case of the canonical analogue black hole,

$$\partial_r \arg\left(f_{\omega l}\left(r\right)\right)=\frac{C}{r^2 h n_{\omega l}\left(r\right)}+\frac{\omega}{c_s}\frac{\sqrt{1-h}}{h}= \tag{5.31}$$

$$=\pm\frac{\omega}{c_s h}\left(\frac{r_H^2 n_{\omega l}\left(r_H\right)}{r^2 n_{\omega l}\left(r\right)}\pm\sqrt{1-h}\right)= \tag{5.32}$$

$$=\pm\frac{\omega r_H}{4c_s\delta r}\left(1\pm 1+\mathcal{O}\left(\delta r\right)\right)\,. \tag{5.33}$$

When the constant C takes positive values, the above expression describes solutions that peel off from the horizon and are relevant for the production of Hawking quanta, while for negative values of C it describes horizon-crossing ingoing solutions.

From equation (5.33) it follows that the phase of each mode with positive C diverges logarithmically, and on the two sides of the horizon we have two independent solutions. In the outer region these are outgoing modes, while in the inner region they peel off the horizon towards the singularity. Considering the radial part of the modes, we have defined them either for $r>r_H$ or for $0<r<r_H$. These functions can be analytically continued in the complex plane only assuming the presence of a branch cut.

In conclusion, the various modes are the horizon crossing modes $f_{\omega lm}^{\mathrm{HC}}$, the outgoing modes in the outer region $f_{\omega lm}^{\mathrm{ext}}$ and their counterparts in the inner region $f_{\omega lm}^{\mathrm{int}}$.

5.2.4 Far-field behavior

At large radii the equation (5.27) is such that $h\approx 1$ and the mode-mixing potential $h\partial_r h/r$ is negligible, and the modes can be approximated with linear combination of Bessel functions of the first and of the second kind.

At infinity even the angular momentum potential can be neglected and the properly normalized solutions take the form

$$f_{\omega l}\left(r\right)\approx\sqrt{\frac{\lambda}{4\pi\omega}}\frac{1}{r}e^{i\phi_{\omega l}^{\infty}}\left(\cos\Theta_{\omega l}e^{-i\omega r}+\sin\Theta_{\omega l}e^{i\omega r}\right)\,, \tag{5.34}$$

where ingoing and outgoing modes are mixed at infinity, as expressed by the phase $\Theta_{\omega l}$ due to the potential encountered along the radial propagation.

In this region, apart from the overall phase which is given by the limit $\lim_{r\to\infty}\int^r\frac{\sqrt{1-h}}{h}\mathrm{d}r$ and by the conditions at the horizon, the radial part of the outgoing and the ingoing modes are one the conjugate of the other, with

$$n_{\omega l} = \frac{\lambda}{4\pi\omega}\frac{1}{r^2}\left(1+\sin 2\Theta_{\omega l}\cos\left(\omega r\right)\right) , \tag{5.35}$$

$$\partial_r \phi_{\omega l} = \frac{C}{r^2 n_{\omega l}} = \pm\frac{4\pi\omega^2}{\lambda c_s}\frac{r_H^2 n_{\omega l}\left(r_H\right)}{\left(1+\sin 2\Theta_{\omega l}\cos\left(\omega r\right)\right)} . \tag{5.36}$$

5.2.5 Mode decomposition of the scalar field

In conclusion one obtains the mode decomposition of the scalar field, as a superposition of modes with fixed angular momentum and frequency

$$\begin{aligned}\theta_1\left(t,r,\Omega\right) = &\int_0^\infty \mathrm{d}\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(f^{\mathrm{HC}}_{\omega lm}\left(t,r,\Omega\right)a^{\mathrm{HC}}_{\omega lm} + \overline{f^{\mathrm{HC}}_{\omega lm}}\left(t,r,\Omega\right)a^{\mathrm{HC}\dagger}_{\omega lm}\right) + \\ &+\int_0^\infty \mathrm{d}\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(f^{\mathrm{ext}}_{\omega lm}\left(t,r,\Omega\right)a^{\mathrm{ext}}_{\omega lm} + \overline{f^{\mathrm{ext}}_{\omega lm}}\left(t,r,\Omega\right)a^{\mathrm{ext}\dagger}_{\omega lm}\right) + \\ &+\int_0^\infty \mathrm{d}\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(f^{\mathrm{int}}_{\omega lm}\left(t,r,\Omega\right)a^{\mathrm{int}}_{\omega lm} + \overline{f^{\mathrm{int}}_{\omega lm}}\left(t,r,\Omega\right)a^{\mathrm{int}\dagger}_{\omega lm}\right) ,\end{aligned} \tag{5.37}$$

$$\rho_1\left(t,r,\Omega\right) = -\frac{1}{\lambda}\left(\partial_t - V\left(r\right)\partial_r\right)\theta_1\left(t,r,\Omega\right) . \tag{5.38}$$

Depending on the radius, only one set of modes between f^{ext} and f^{int} appears in the definition of the field.

The field operators follow the canonical commutation relations

$$\left[\theta_1\left(t,r,\Omega\right),\theta_1\left(t,r',\Omega'\right)\right] = 0 , \tag{5.39}$$

$$\left[\theta_1\left(t,r,\Omega\right),\rho_1\left(t,r',\Omega'\right)\right] = \delta\left(\Omega,\Omega'\right)\frac{\delta\left(r,r'\right)}{rr'} , \tag{5.40}$$

and the ladder operators relative to the modes are defined to follow the usual commutation relations in the quasi-particle Fock space

$$\left[a^{\mathrm{HC}}_{\omega lm}, a^{\mathrm{HC}\dagger}_{\omega' l' m'}\right] = \delta_{ll'}\delta_{mm'}\delta\left(\omega,\omega'\right) , \tag{5.41}$$

$$\left[a^{\mathrm{ext}}_{\omega lm}, a^{\mathrm{ext}\dagger}_{\omega' l' m'}\right] = \delta_{ll'}\delta_{mm'}\delta\left(\omega,\omega'\right) , \tag{5.42}$$

$$\left[a^{\mathrm{int}}_{\omega lm}, a^{\mathrm{int}\dagger}_{\omega' l' m'}\right] = \delta_{ll'}\delta_{mm'}\delta\left(\omega,\omega'\right) , \tag{5.43}$$

and all the other commutation relations between ladder operators vanish identically.

5.3 Hawking radiation

We discuss now the creation of Hawking quanta. The presence of the horizon induces the creation of Hawking radiation, and the velocity profile defines the temperature of the canonical analogue black hole. Then we study the properties of the initial state, the Unruh vacuum state.

Until now we have focused on the region outside the horizon, where for every positive ω the modes $f^{\text{ext}}_{\omega lm}$ and $f^{\text{HC}}_{\omega lm}$ — with positive norm with respect to the Klein–Gordon internal product — must have the time dependence $e^{-i\omega t}$ usual for stationary spacetimes, while in the inner region the time dependence is inverted for the modes with positive norm $f^{\text{int}}_{\omega lm}$, because of the change of sign of the function $h\left(r\right)$.

Moreover, each of the modes peeling off from the horizon has a phase that diverges logarithmically, as can be deduced from equation (5.33), implying that on the two sides of the horizon the two independent sets of modes $f^{\text{ext}}_{\omega lm}$ and $\overline{f^{\text{int}}_{\omega lm}}$ can be analytically continued and put in superposition with each other.

The operation of mixing destruction operators of the modes defined in the outer region with creation operators of the modes defined in the inner region is described by a Bogoliubov transformation.

In particular we are interested in the mixing between modes at the horizon with those associated to a vacuum state for static observers at past null infinity.

Near the horizon, at the two sides, the modes are (dropping the eigenvalues from the notation)

$$f^{\text{ext}} = \sqrt{n^{\text{ext}}} e^{-i\omega t} e^{i\frac{\omega r_H}{2c_s}\ln\frac{|\delta r|}{r_H}}\Theta\left(\delta r\right) , \tag{5.44}$$

$$\overline{f^{\text{int}}} = \sqrt{n^{\text{int}}} e^{-i\omega t} e^{i\frac{\omega r_H}{2c_s}\ln\frac{|\delta r|}{r_H}}\Theta\left(-\delta r\right) . \tag{5.45}$$

Linear combinations of these modes define new solutions, in particular we are interested in the analytic continuations of the radial part on the real r-axis. One solution can be extended in the upper half-plane — with a branch cut in the lower, and one in the lower half-plane — with a branch cut in the upper.

$$f_+ = \alpha_1 f^{\text{ext}} + \beta_1 \overline{f^{\text{int}}} , \tag{5.46}$$

$$f_- = \beta_2 f^{\text{ext}} + \alpha_2 \overline{f^{\text{int}}} . \tag{5.47}$$

Due to the logarithmic term in the phase of the functions, passing from one side of the horizon to the other, the analytic continuations gain a phase of $\pm\pi\omega r_H/2c_s$, and therefore we can write

$$\alpha_1 \sqrt{n^{\text{ext}}} = e^{\frac{\pi\omega r_H}{2c_s}} \beta_1 \sqrt{n^{\text{int}}} , \tag{5.48}$$

$$\beta_2 \sqrt{n^{\text{ext}}} = e^{-\frac{\pi\omega r_H}{2c_s}} \alpha_2 \sqrt{n^{\text{int}}} . \tag{5.49}$$

The coefficients α and β of the Bogoliubov transformation can be found imposing that the field can be rewritten in terms of these new modes and imposing the canonical commutation relations for the new ladder operators:

$$f_+ a_+ + f_- a_-^\dagger = f^{\text{ext}} a^{\text{ext}} + \overline{f^{\text{int}}} a^{\text{int}\dagger} , \tag{5.50}$$

$$\left[a_+, a_+^\dagger\right] = \left[a_-, a_-^\dagger\right] = \left[a^{\text{ext}}, a^{\text{ext}\dagger}\right] = \left[a^{\text{int}}, a^{\text{int}\dagger}\right] . \tag{5.51}$$

Therefore it is found that, apart from irrelevant overall phases, the Bogoliubov transformation is given by

$$|\alpha_1|^2 = |\alpha_2|^2 = \frac{e^{\frac{\pi\omega r_H}{c_s}}}{e^{\frac{\pi\omega r_H}{c_s}} - 1} \,, \tag{5.52}$$

$$|\beta_1|^2 = |\beta_2|^2 = \frac{1}{e^{\frac{\pi\omega r_H}{c_s}} - 1} \,. \tag{5.53}$$

The Unruh vacuum $|\emptyset\rangle$ is defined such that

$$a_+ |\emptyset\rangle = 0 \,, \tag{5.54}$$

$$a_- |\emptyset\rangle = 0 \,, \tag{5.55}$$

$$a^{\mathrm{HC}} |\emptyset\rangle = 0 \,, \tag{5.56}$$

so equations (5.52)–(5.53) imply that an observer outside the horizon will observe a thermal distribution of radiation coming from the black hole horizon. Indeed, through the Bogoliubov transformation, the occupation of the modes f^{ext} is given by

$$\left\langle a^{\mathrm{ext}\dagger}_{\omega lm} a^{\mathrm{ext}}_{\omega lm} \right\rangle = \frac{1}{e^{\frac{\pi\omega r_H}{c_s}} - 1} \delta_{ll'} \delta_{mm'} \delta\left(\omega, \omega'\right) \,. \tag{5.57}$$

This occupation number corresponds to the thermal Bose–Einstein distribution associated to the temperature

$$T = \frac{c_s}{\pi r_H} = \frac{c_s \kappa}{2\pi} \,, \tag{5.58}$$

$$\kappa = \frac{2}{r_H} \,, \tag{5.59}$$

where κ is the surface gravity of the canonical analogue black hole, which is half the radial derivative of h evaluated at the horizon

$$\kappa = \left[\frac{1}{2}\partial_r h\right]_{r_H} \,. \tag{5.60}$$

5.4 Back-reaction

When the Gross–Pitaevskii equation is modified by introducing the terms of anomalous mass and density, it becomes possible to study the back-reaction that the analogous scalar field induces on the condensate wavefunction.

The study of the back-reaction in analogue systems enables to address various issues, such as the problem of the evaporation of black holes and in general the dynamics of analogue horizon. To study the dynamics of the horizon it is necessary to express the anomalous mass and density as correlation functions of the analogous scalar field and its conjugate. While the anomalous density is strictly real, the anomalous mass contains a real and an imaginary part.

It is particularly convenient to study these quantities in dimensionless terms:

$$n = \frac{\left\langle \delta\phi^\dagger \delta\phi \right\rangle}{\overline{\langle \phi_0 \rangle} \langle \phi_0 \rangle} = \frac{1}{4} \left\langle \frac{\rho_1}{\rho_0} \frac{\rho_1}{\rho_0} \right\rangle + \langle \theta_1 \theta_1 \rangle \ , \tag{5.61}$$

$$m_R = \frac{1}{2} \left(\frac{\langle \delta\phi\delta\phi \rangle}{\langle \phi_0 \rangle \langle \phi_0 \rangle} + \frac{\left\langle \delta\phi^\dagger \delta\phi^\dagger \right\rangle}{\overline{\langle \phi_0 \rangle \langle \phi_0 \rangle}} \right) = \frac{1}{4} \left\langle \frac{\rho_1}{\rho_0} \frac{\rho_1}{\rho_0} \right\rangle - \langle \theta_1 \theta_1 \rangle \ , \tag{5.62}$$

$$m_I = \frac{1}{2i} \left(\frac{\langle \delta\phi\delta\phi \rangle}{\langle \phi_0 \rangle \langle \phi_0 \rangle} - \frac{\left\langle \delta\phi^\dagger \delta\phi^\dagger \right\rangle}{\overline{\langle \phi_0 \rangle \langle \phi_0 \rangle}} \right) = \frac{1}{2} \left\langle \left\{ \frac{\rho_1}{\rho_0}, \theta_1 \right\} \right\rangle = \tag{5.63}$$

$$= -\frac{1}{2} \frac{1}{\lambda \langle \rho_0 \rangle} \left(\partial_t + \frac{\nabla \langle \theta_0 \rangle}{m} \nabla \right) \langle \theta_1 \theta_1 \rangle \ . \tag{5.64}$$

We observe that in equation (5.64), in the hydrodynamic limit, the imaginary part of the anomalous density is the derivative along the flow of the $\langle \theta_1 \theta_1 \rangle$ correlation function, the vacuum polarization. Note that in equation (5.61) we have regularized the definition of anomalous density, thus eliminating a $\delta^3(0)$ term. Formally this is done defining the correlation function as the limit for vanishing distance of the 2-point correlation function.

To describe the mean-field dynamics with the inclusion of the anomalous terms we modify the Gross–Pitaevskii equation as discussed previously in equation (2.66) and apply the Madelung representation. [2]

We study separately the square norm of the wavefunction — the atom density of the condensate — and the phase:

$$\partial_t \langle \rho \rangle = -\frac{1}{m} \nabla \left(\langle \rho \rangle \left(\nabla \langle \theta \rangle \right) \right) + 2\lambda \langle \rho \rangle^2 m_I \ , \tag{5.65}$$

$$\partial_t \langle \theta \rangle = \frac{1}{2m} \langle \rho \rangle^{-1/2} \nabla^2 \langle \rho \rangle^{1/2} - \frac{1}{2m} \left(\nabla \langle \theta \rangle \right) \left(\nabla \langle \theta \rangle \right) - \lambda \langle \rho \rangle (1 + 2n + m_R) - V_{\text{ext}} \ . \tag{5.66}$$

The equation for the dynamics of the atom density function is affected by the imaginary part of the anomalous mass, while the equation for the dynamics of the phase of the condensate wavefunction is modified by a combination of anomalous density and real part of the anomalous mass.

We study the regime in which these classical perturbations to the condensate wavefunction are small deviations from the stationary solution of the Gross–Pitaevskii equation that do not break the hydrodynamic approximation.

Therefore, we neglect the terms that are suppressed by the healing length scale $\xi = 1/\sqrt{\lambda \langle \rho \rangle m}$, because they are only relevant at a high energy scale where the interactions between the individual atoms become dominant in determining the phenomenology. In the equations (5.65)–(5.66) we neglect the contributions to the anomalous density and to the real part of the anomalous mass which goes as $\langle \rho_1 \rho_1 \rangle$. Since the definition of the

[2] Let us stress that while these are definitely not the Einstein equations, they can nonetheless be cast in the form of a modified Poisson equation [81].

conjugate field ρ_1 depends on the healing length scale as

$$\frac{\rho_1}{\langle\rho_0\rangle} = -\frac{1}{\lambda\langle\rho_0\rangle}\left(\partial_t + \frac{\nabla\langle\theta_0\rangle}{m}\nabla\right)\theta_1 = -\frac{\xi}{c_s}\left(\partial_t + \frac{\nabla\langle\theta_0\rangle}{m}\nabla\right)\theta_1\,, \tag{5.67}$$

the anomalous density in equation (5.61) and the real part of the anomalous mass in equation (5.62) can be approximated to the contributions of the correlation function $\langle\theta_1\theta_1\rangle$:

$$n \approx -m_R \approx \langle\theta_1\theta_1\rangle\ . \tag{5.68}$$

Under these assumptions, from equation (5.65) and equation (5.66) we proceed linearizing the equations for the mean-field for small perturbations — of the same order of magnitude of the anomalous terms — of the solution of the Gross–Pitaevskii equation, getting

$$\langle\rho\rangle = \langle\rho_0\rangle + \langle\delta\rho\rangle\ , \tag{5.69}$$

$$\langle\theta\rangle = \langle\theta_0\rangle + \langle\delta\theta\rangle\ , \tag{5.70}$$

$$\left(\partial_t + \frac{\nabla\langle\theta_0\rangle}{m}\nabla\right)\left(\frac{\langle\delta\rho\rangle}{\langle\rho_0\rangle} + \langle\theta_1\theta_1\rangle\right) = -\frac{1}{m\langle\rho_0\rangle}\nabla\left(\langle\rho_0\rangle\nabla\langle\delta\theta\rangle\right)\,, \tag{5.71}$$

$$\left(\partial_t + \frac{\nabla\langle\theta_0\rangle}{m}\nabla\right)\langle\delta\theta\rangle = -\lambda\langle\rho_0\rangle\left(\frac{\langle\delta\rho\rangle}{\langle\rho_0\rangle} + \langle\theta_1\theta_1\rangle\right)\,. \tag{5.72}$$

We can see that the classical fluctuation of the phase of the condensate $\langle\delta\theta\rangle$ evolves like a Klein–Gordon mode, with conjugate momentum $\langle\delta\rho\rangle / \langle\rho_0\rangle + \langle\theta_1\theta_1\rangle$.

These equations allow to study the dynamics of the classical perturbation, and are differential equations that can solved after the source term $\langle\theta_1\theta_1\rangle$ and the initial conditions for $\langle\delta\rho\rangle$ and $\langle\delta\theta\rangle$ are provided. Unless broken by the initial conditions, the evolution of the classical perturbation will preserve the spherical symmetry and the time translation symmetry.

5.4.1 Dynamics of the horizon

The production of Hawking radiation emitted by a black hole formed by gravitational collapse is a process that transfers energy towards the future null infinity. The outgoing flux of Hawking quanta induces a loss of mass of the black hole, whose apparent horizon gradually shrinks, until its eventual evaporation. In this non-stationary process, the gravitational collapse triggers the production of Hawking radiation quanta, consequently exerting a back-reaction on the classical metric tensor.

In the case of the canonical analogue black hole we can study the dynamics of the horizon assuming the condensate to be initially in the stationary condensate configuration described in section 5.2.1. Then, turning on the production of quasi-particles of the field θ_1, we observe a back-reaction on the wavefunction.

In general, having spherical symmetry, the horizon radius r_H is defined as the radius at which the radial velocity of the condensate equals the speed of sound. In the unperturbed case we have

$$0 = \left[c_s^2 - V_0^2\right]_{r_H} . \tag{5.73}$$

If the condensate is in a state which is a small perturbation of the stationary solution of the Gross–Pitaevskii equation, the density and the phase of the condensate are modified as described in the equations (5.69)–(5.70). Consequently the velocity of the condensate and the speed of sound change and, under this variation, the equation (5.73) is satisfied at a different radius $r_H + \delta r_H$, where r_H is the unperturbed horizon radius:

$$V_0 \to V = V_0 + \delta V = -\frac{\partial_r \langle \theta_0 \rangle}{m} - \frac{\partial_r \langle \delta\theta \rangle}{m} , \tag{5.74}$$

$$c_s^2 \to c_s^2 \left(1 + \frac{\langle \delta\rho \rangle}{\langle \rho_0 \rangle}\right) , \tag{5.75}$$

$$r_H \to r_H + \delta r_H = r_H - \left[\frac{\delta \left(V^2/c_s^2\right)}{\partial_r \left(V_0^2/c_s^2\right)}\right]_{r_H} . \tag{5.76}$$

The denominator in equation (5.76) is different for each spherically symmetric system and is proportional to the surface gravity of the horizon, *i.e.* to the temperature of the black hole in equation (5.58)

$$\left[-\partial_r \left(V^2/c_s^2\right)\right]_{r_H} = 2\kappa > 0 . \tag{5.77}$$

The radius of the perturbed horizon is smaller or larger than r_H depending on the sign of the variation $\delta \left(V^2/c_s^2\right)$.

For the canonical analogue black hole we substitute the expression of V_0 of equation (5.13) in equation (5.76) and we get

$$\delta r_H = -\frac{r_H}{4} \left[\frac{\langle \delta\rho \rangle}{\langle \rho_0 \rangle} - 2\frac{\delta V}{c_s}\right]_{r_H} . \tag{5.78}$$

Deriving the variation of the horizon radius with respect to time we observe which quantities determine the dynamics of δr_H

$$\partial_t \delta r_H = -\left[\frac{\partial_t \delta \left(V^2/c_s^2\right)}{\partial_r \left(V^2/c_s^2\right)}\right]_{r_H} - \delta r_H \left[\partial_t \ln \partial_r \left(V^2/c_s^2\right)\right]_{r_H} . \tag{5.79}$$

For a stationary spacetime the second term vanishes.

We consider the case of the canonical analogue black hole substituting from equation (5.71) and equation (5.72) the expressions for δV and $\langle \delta\rho \rangle$:

$$\partial_t \delta r_H = \frac{r_H c_s}{4} \left[\partial_r \left(\frac{\langle \delta\rho \rangle}{\langle \rho_0 \rangle} + \langle \theta_1 \theta_1 \rangle + \frac{\delta V}{c_s}\right)\right]_{r_H} + \left[-\frac{3}{2}\delta V\right]_{r_H} . \tag{5.80}$$

The source term on the RHS is a quantity that depends on the quantum state of the field θ_1 — through the correlation function $\langle\theta_1\theta_1\rangle$ — and on the state of the classical perturbation. It does not have definite sign: different initial data of the classical perturbation determine different regimes of the black hole, of expansion or contraction. This is expected, as equation (5.80) applies to every possible regime of the black hole.

Nonetheless, such different initial conditions for the classical perturbation do not affect, the contribution from the quantum field θ_1, which remains always the same. In the hierarchy of the equations, the analogue scalar field affects the perturbation of the analogue metric, not *vice versa.* A self-consistent semiclassical approach would require including the perturbation of the wavefunction in the Bogoliubov–de Gennes equation, explicitly or by iteratively recalculating $\langle\theta_1\theta_1\rangle$ and the pair $(\langle\delta\rho\rangle, \langle\delta\theta\rangle)$.

The effect of the analogue Hawking radiation is understood when $\langle\theta_1\theta_1\rangle$ is the contribution leading the phenomenology. We therefore assume the perturbation terms $\langle\delta\theta\rangle$ and $\langle\delta\rho\rangle$ and their radial derivatives to be negligible. In the analogy between the canonical analogue black hole and the gravitational black hole, this corresponds to assuming the initial perturbation of the metric to be negligible, and then to excite it through the presence of Hawking radiation. This reflects the evolution of a gravitational black hole, where the free falling matter induces the formation of the horizon, which triggers the production of Hawking quanta slowly driving the spacetime far from the stationary configuration.

Therefore we consider equation (5.80) and assume the perturbation of the wavefunction to be null and the corresponding terms to vanish. Moreover, we can assume the time derivative of the Hawking radiation term to be negligible. The Hawking quanta are produced in pairs of equal frequency, and in the correlation functions their time dependent phases cancel each other out. The term $\partial_t\langle\theta_1\theta_1\rangle$ is therefore suppressed.

In this regime, we are left with

$$\partial_t \delta r_H \approx \frac{r_H c_s}{4} \left[\partial_r \langle\theta_1\theta_1\rangle\right]_{r_H} . \tag{5.81}$$

The quantity $\langle\theta_1\theta_1\rangle$ is of paramount interest in the investigation of the back-reaction that quantum fields exert on curved geometries. Together with the stress-energy tensor, the vacuum polarization is a quantity that in literature is studied in association with the production of Hawking quanta in gravitational systems [124, 125].

As described in [32, 126] this function is obtained through the coincidence limit of the Green function of the Klein–Gordon operator, *i.e.* the limit for $x' \to x$ and $t' \to t$ of $G(t, x; t', x')$. Different boundary conditions determine different expressions for $\langle\theta_1\theta_1\rangle$: the occupation numbers of the modes of the field, which are associated at the horizon with the Hawking radiation and with the horizon-crossing quanta, determine the behavior of this function. In particular, the occupation numbers are zero for the Boulware vacuum; are thermal for the outgoing modes and null for the ingoing modes for the Unruh vacuum (as seen in equations (5.54)–(5.55)–(5.56)); and are all thermal for the Hartle–Hawking vacuum. To obtain the expression for $\langle\theta_1\theta_1\rangle$ in the various cases one must subtract different renormalization counterterms. Renormalizing is strictly necessary as the Green function generally includes divergent terms. In the coincidence limit already in the

(3 + 1)-dimensional flat Minkowski spacetime it is found that

$$G\left(t+\epsilon, x; t, x\right) \propto -\frac{1}{\epsilon^2}\,. \tag{5.82}$$

The most commonly used renormalization scheme is based on the point splitting regularization method [29, 32, 127] which removes the divergence by splitting the point in which the Green functions is evaluated in two nearby by points characterized by their geodesic distance, so regularizing the vacuum polarization as measured along geodesic trajectories.

For Hadamard states [30] the resulting divergent structure in the coincidence limit is universal in curved spacetimes: one gets the above mentioned divergent terms together with other logarithmically divergent terms typical of the Hadamard structure. The universality of such ultraviolet divergences, of the same functional form of those obtainable in Minkowski spacetime, allows to safely discard them. However, other irregular behaviors may nonetheless arise from the peculiarities of the curved geometry and the vacuum state. In particular, in presence of a horizon, the Boulware vacuum gives a divergent vacuum polarization due to the vanishing of the time-time element of the metric h. As argued by Candelas, in proximity of the horizon the Green function is divergent as $G = -1/h\epsilon^2$ [126]. Also, let us mention that while the above mentioned regularization schemes have been mostly applied in Ricci flat spacetimes, in non-Ricci flat spacetimes like ours, they will generically include an extra contribution, which however we can expect to provide at the horizon at most a prefactor of order unity, which can be therefore neglected for our considerations.

When we consider the Bogoliubov transformations, the Green function depends on the occupation numbers of the quasi-particle states. For states different from the Boulware vacuum, in presence of non-null occupation numbers, the vacuum polarization is given by the limit

$$\langle\theta_1\theta_1\rangle \propto \frac{1}{h}\lim_{\epsilon\to 0^*}\int_0^\infty \mathrm{d}\omega\, e^{-i\omega\epsilon}\omega\left(1+2n_\omega\right)\,, \tag{5.83}$$

where the first contribution is fixed for every state.

To appropriately regularize the contribution of the Hawking radiation, it is therefore necessary to subtract the quadratic divergence that defines the Boulware vacuum and consider the occupation numbers n_ω. The Unruh vacuum is associated with the occupation numbers of equation (5.57), for which the integral expression in equation (5.83) gives

$$\frac{2}{h}\int_0^\infty \mathrm{d}\omega\,\omega n_\omega = \frac{2}{h}\int_0^\infty \mathrm{d}\omega\,\frac{\omega}{e^{\frac{2\pi\omega}{c_s\kappa}}-1} = \frac{c_s^2\kappa^2}{6h}\,. \tag{5.84}$$

This quantity is divergent at the horizon, but is regularized by subtraction of the second term in the series expansion of the geodesic distance $\frac{c_s^2}{6h}\left(\frac{h'}{2}\right)^2$, which removes the divergence due to h in the limit $r \to r_H$.

Reintroducing the proper numerical factors, we therefore obtain that in proximity of the horizon the leading effect of the Hawking radiation induces the perturbative expression

$$\langle\theta_1\theta_1\rangle \approx \frac{\lambda}{c_s}\frac{1}{48\pi^2 h}\left(\kappa^2 - \left(\frac{h'(r)}{2}\right)^2\right) = \tag{5.85}$$

$$= \frac{\lambda}{c_s}\frac{1}{96\pi^2}\left(-h^{(2)}(r_H) - \frac{1}{2}h^{(3)}(r_H)\,\delta r + \mathcal{O}\left(\delta r^2\right)\right) = \tag{5.86}$$

$$= \frac{\lambda}{c_s}\frac{1}{96\pi^2}\frac{20}{r_H^2}\left(1 - 3\frac{\delta r}{r_H} + \mathcal{O}\left(\delta r^2\right)\right) . \tag{5.87}$$

This approximation does not hold for every radius, *e.g.* it includes not integrable divergences in the origin, but is such that it is consistent with the expected behavior of the Rindler spacetime — the near-horizon region [128, 129]. In equation (5.87) we retain the term of order δr, as we are interested in the radial derivative of this expression.

The adimensionality of the field is a feature that differs from the usual notation, as in quantum field theory in $(3+1)$-dimensional spacetime the Klein–Gordon scalar field is generally not dimensionless, but it has the dimension of an inverse length. In equation (5.85) we introduced the length scale $\sqrt{\lambda/c_s}$, which is the proper quantity needed to replicate the usual relations

$$S = -\int \mathrm{d}^4x \frac{1}{2}\sqrt{g}g^{\mu\nu}\left(\partial_\mu\theta\right)\left(\partial_\nu\theta\right) , \tag{5.88}$$

$$\pi = -\frac{\delta\mathcal{L}}{\delta\partial_0\theta} = \sqrt{g}g^{0\mu}\left(\partial_\mu\theta\right) , \tag{5.89}$$

$$\left[\theta(x), \pi(y)\right] = -i\delta^3(x,y) . \tag{5.90}$$

Considering the definition of ρ_1 given in equation (5.4), a straightforward calculation gives the length scale required.

Therefore we obtain the estimate for the time derivative of the horizon radius, and from the expression of the surface gravity we can also make an estimate of its rate of change

$$\partial_t\delta r_H \approx \frac{r_H c_s}{4}\partial_r\langle\theta_1\theta_1\rangle = \tag{5.91}$$

$$= -\frac{5}{32\pi^2}\frac{\lambda}{r_H^2} = \tag{5.92}$$

$$= -\frac{5}{32\pi^2}\frac{c_s}{\xi r_H^2\langle\rho\rangle} , \tag{5.93}$$

$$\frac{1}{c_s}\frac{\dot{\kappa}}{\kappa^2} \approx \frac{5}{64\pi^2}\frac{1}{\xi r_H^2\langle\rho\rangle} \ll 1 . \tag{5.94}$$

These derivatives are very small: while typically the atom separation — equal to $\rho_0^{-1/3}$ — is of an order comparable to the healing length, the radius of the horizon can safely be assumed being much larger than both.

Given that the rate of change of the surface gravity is very small, the system can be assumed to evolve adiabatically along the evolution: the hydrodynamic approximation is broken before the rate of change of the curvature in equation (5.94) becomes comparable to 1.

We can therefore take equation (5.92) to provide a rough approximate description of the evolution of the black hole, promoting r_H on the RHS to be the dynamical horizon radius. We can now see that the expected lifetime of the black hole is long and proportional to the inverse of the interaction coupling λ

$$r_H(t) = \left(r_H^3(t_0) - \frac{15\lambda}{32\pi^2} t \right)^{1/3} , \tag{5.95}$$

$$t_{\text{fin}} = \frac{32\pi^2 r_H^3(t_0)}{15\lambda} . \tag{5.96}$$

Let us note that the cubic dependence on r_H in equation (5.96) resembles that of a Schwarzschild black hole, and follows from the proportionality between $\delta \dot{r}_H$ and r_H^{-2}. To make this prediction, it is necessary to assume the regime in which the Hawking radiation, *i.e.* the contribution of the Bogoliubov quasi-particles obtained from the solution of the Bogoliubov–de Gennes equation, is dominant over the perturbation obtained by back-reaction in the modified Gross–Pitaevskii equation.

We briefly point out that another regime which could be of interest is that of an analogue black hole in equilibrium with the analogue scalar field, where the back-reaction is not neglected but the classical perturbation is stationary. This would be the analogue of the Hartle–Hawking vacuum state for our system. In this case a solution can be found without changing the speed profile, *i.e.* with $\delta V \approx 0$, but with a perturbation of the number density $\langle \delta\rho \rangle$ such that it counterbalances $\langle \theta_1 \theta_1 \rangle$:

$$0 = \frac{\langle \delta\rho \rangle}{\langle \rho_0 \rangle} + \langle \theta_1 \theta_1 \rangle , \tag{5.97}$$

$$0 = \langle \delta\theta \rangle . \tag{5.98}$$

In this case the horizon for the stationary configuration is found at a radius larger than r_H, but this solution will be driven out of equilibrium by the terms of order ξ^2 and the non-linearities previously neglected. In this case the density-density correlation function differs from the unperturbed case by a negative quantity on the diagonal (when the density operators are evaluated at the same position):

$$\langle \rho\rho \rangle - \langle \rho_0 \rangle \langle \rho_0 \rangle \approx 2 \langle \rho_0 \rangle \langle \delta\rho \rangle = -2 \langle \rho_0 \rangle^2 \langle \theta_1 \theta_1 \rangle . \tag{5.99}$$

If in a realization of the system the evolution is kept under control allowing only adiabatic transformations between near equilibrium configurations, it would be reasonable to assume these initial conditions for the system; but this equilibrium configuration would not be analogous to the Unruh vacuum.

5.4.2 Dynamics of the number of atoms in the condensate

From equation (5.71) we observe that the dynamics of the perturbation of the condensate affects not only the position of the horizon, but also the global properties of the system, such as the number of atoms in the condensate state.

Let us consider the dynamics of $\langle \delta\rho \rangle$. The integral of this quantity in space gives the rate at which the atoms leave the condensate and move to the excited part. It is a global process which can already be described at the level of the Bogoliubov–de Gennes equations, considering the derivative with respect to time of the total number of atoms in the condensate 1-particle state, and assuming the stationarity of the unperturbed condensate wavefunction:

$$\partial_t N = \cancel{\partial_t N_{\mathrm{TOT}}} - \partial_t \int \mathrm{d}x \left\langle \delta\phi^\dagger \delta\phi \right\rangle . \tag{5.100}$$

The same derivative can be expressed as the derivative of the integral of the number density, *i.e.* of the squared norm of the condensate wavefunction. Since the unperturbed configuration is assumed stationary, the only contribution is made by the classical perturbation, *i.e.* by the back-reaction of the analogue quantum field. From equation (5.71) we get

$$\partial_t N = \partial_t \int \mathrm{d}x \left\langle \delta\rho \right\rangle = \tag{5.101}$$

$$= - \int \mathrm{d}x \left\langle \rho_0 \right\rangle \left(\frac{\nabla \left\langle \theta_0 \right\rangle}{m} \right) \nabla \left\langle \theta_1 \theta_1 \right\rangle - \cancel{\frac{1}{m} \int \mathrm{d}x \nabla \left(\left\langle \delta\rho \right\rangle \nabla \left\langle \theta_0 \right\rangle + \left\langle \rho_0 \right\rangle \nabla \left\langle \delta\theta \right\rangle \right)} = \tag{5.102}$$

$$= 4\pi c_s r_H^2 \left\langle \rho_0 \right\rangle \int_0^\infty \mathrm{d}r \partial_r \left\langle \theta_1 \theta_1 \right\rangle = \tag{5.103}$$

$$= -3 N_{BH} T \left[\left\langle \theta_1 \theta_1 \right\rangle \right]_0 . \tag{5.104}$$

We observe that the time derivative of the number of atoms in the condensate — in the entire occupied volume — depends on three factors: the expectation number of condensate atoms in the region within the horizon

$$N_{BH} = \frac{4\pi}{3} \left\langle \rho_0 \right\rangle r_H^3 ; \tag{5.105}$$

the Hawking temperature of the black hole, as defined in equation (5.58), that goes with the inverse of the horizon radius (plus subleading corrections describing the imperfection in the adiabaticity of the evolution); the value that the vacuum polarization $\langle \theta_1 \theta_1 \rangle$ assumes at the origin, which is effectively a dimensionless structure factor depending on the velocity profile. The vacuum polarization requires a proper renormalization and it is generally difficult to extend it down to the singularity in the origin, not only when studying the canonical analogue black hole but also, *e.g.* for the Schwarzschild black hole [130].

Anyhow, in the integral form as in equation (5.103) it is clear that it is the radial derivative of the vacuum polarization, on both sides of the horizon, that determines locally the contribution to the depletion.

5.5 Final remarks

We have studied the canonical analogue black hole as a particular realization of analogue gravity with Bose–Einstein condensates.

The linearized quantum fluctuation over the background — the mean-field described by the Gross–Pitaevskii equation — propagates as a massless scalar quantum field in the curved spacetime of a black hole. In the limit of negligible quantum pressure the Bogoliubov–de Gennes equation can be reorganized to show how the quantum fluctuation of the phase is governed by an acoustic metric, defined by the condensate wavefunction, that presents a horizon. The calculation of the Hawking radiation produced in this acoustic geometry follows closely that of a massless scalar field near a gravitational black hole, and we have set the conditions for the modes of the field to replicate the Unruh vacuum, the state expected in a black hole spacetime formed through gravitational collapse.

Being interested in the back-reaction that the quasi-particles exert on the condensate, we have considered the modified Gross–Pitaevskii equation which includes the anomalous terms, and thus observed how the acoustic metric is modified by the quasi-particle dynamics. The study of the quasi-particle back-reaction pushes analogue gravity towards an understanding of semiclassical gravity, where the back-reaction of the quantum fields is expected to be included in Einstein's equations.

We have provided expressions for the back-reaction in analogue gravity with Bose–Einstein condensates, in an approximation based on the suppression of the terms depending on the healing length scale — subleading with respect to scale set by the Hawking temperature — that leads to an equation that focuses on the vacuum polarization of the analogue scalar field.

From this general result, we have specialized to the canonical analogue black hole and have obtained an expression for the dynamics of the horizon and for how it is affected by the state of the quantum field. In particular, we have argued that in a regime in which the Hawking radiation is the leading contribution in the dynamics of the acoustic geometry (that would otherwise be stationary) it leads to a shrinking of the horizon radius.

This process of evaporation induces the depletion of atoms from the condensate to the excited part, and the rate of depletion is driven by the radial derivative of the vacuum polarization. The exchange of atoms and therefore the exchange of information between the two sectors of the system happens not only at the horizon, but also in the black hole region enclosed by it. In particular we have obtained an expression which ties together the rate of depletion to the number of atoms in the region within the the horizon, the temperature of the black hole and the value of the vacuum polarization near the singularity. These are all results in agreement with the conclusions reported in [7] with regards to the nature of a possible resolution of the information loss problem associated to black hole evaporation.

What is peculiar of the presented calculation is the role played by the vacuum polarization in the inner region. In the gravitational case of the Schwarzschild black hole there is neither access to the information in the inner region nor knowledge of an underlying quantum gravity structure from which the classical geometry would emerge. Instead, in the canonical analogue black whole there are both these features. Not only in an experimental realization one has access to the whole space involved in the dynamics, but it is now clear that the role played by the vacuum polarization of the analogue field, the Bogoliubov quasi-particles, is fundamental in understanding the exchange of information between the condensate and the excited part of the system.

6

Rotating analogue black holes: the Kerr metric

6.1 Rotating geometries in analogue gravity

In general relativity, black holes are characterized by the extensive quantities of mass, angular momentum and electric charge: the no-hair theorem states that these three quantities are sufficient to completely characterize the solution of the Einstein equations for the geometry of a black hole spacetime [131]. They also completely define the thermodynamical state of a black hole at equilibrium [132]. Understanding how to simulate these features in analogue gravity is therefore necessary to properly devise a configuration, *i.e.* an acoustic geometry for the propagation of the perturbations, that is more comprehensive in its analogy to gravitational systems.

In Bose–Einstein condensates, we have shown how the acoustic geometry in which the Bogoliubov quasi-particles propagate is set by the condensate wavefunction, the solution of the Gross–Pitaevskii equation [44, 133]. It is well established that in these systems the acoustic geometry can be made to include acoustic horizons, where the velocity of the condensate equals the speed of sound, *i.e.* the speed of the Bogoliubov quasi-particles. These analogue black holes are realized controlling the atom density and the velocity profile by means of the external potential applied to the system, which can be set appropriately in order to simulate a specific gravitational system.

The size of the acoustic black hole, analogue of the black hole mass, and the speed of sound, analogue of the speed of light, are typically the main features that can be controlled in analogue gravity.

While in a spherically symmetric stationary system we can realize an acoustic horizon and observe the analogue Hawking radiation [83, 92, 93], the inclusion of the angular momentum also allows to simulate other interesting physical phenomena, such as superradiance and the ergosphere instability [98, 134–136].

Superradiance is the enhancement effect of the radiation incident on the rotating black hole: when the rotating black hole is irradiated it loses energy and angular momentum, which is gained by the perturbations, whether they are classic or quantum. For an extensive overview of the phenomenon, see [137]. This phenomenon has been studied theoretically in various models of analogue gravity [59, 138, 139] and has been observed in experiments, by means of the surface gravity waves propagating in a configuration of water flowing in a vortex [60, 140, 141]. The focus has generally been on simulating the equatorial plane of a rotating black hole, both with classical fluids and with Bose–Einstein condensates.

Through superradiance, the perturbation of the geometry, gravitational or analogue, extracts energy and angular momentum from the background [61]. In both semiclassical gravity and Bose–Einstein condensates, energy and momentum are extracted respectively from the classical geometry or from the condensate wavefunction and are transferred to the quantized excitations propagating in the system.

Superradiance is therefore, together with analogue Hawking radiation and analogue cosmological quasi-particle creation, another remarkable example of the interaction between the classical geometry and the quantum fields. In the context of analogue gravity in Bose–Einstein condensates we are interested in the mutual interaction between the condensate and the Bogoliubov quasi-particles as a model for the behavior of rotating black hole spacetimes in a semiclassical gravity regime.

In this chapter we discuss our study on analogue rotating black holes and vorticity, published in the two companion papers [142, 143]. In these papers we have focused on the Kerr metric, which is of particular interest being the renown vacuum solution of the Einstein equations for the uncharged rotating black hole in $3+1$ dimensions.

The Kerr spacetime (discovered in 1963 [144]) continues to attract considerable interest and provide unexpected new discoveries [144–149]. The Kerr black hole is among the most studied objects in general relativity, and simulating it would be in itself of great interest for all the field of gravity research.

One specific and relatively simple form of the Kerr metric given in [146] is this

$$\begin{aligned} \mathrm{d}s^2 = &- \mathrm{d}t^2 + \mathrm{d}r^2 + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi + \left(r^2 + a^2\cos^2\theta\right)\,\mathrm{d}\theta^2 + \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\phi^2 + \\ &+ \frac{2Mr}{r^2 + a^2\cos^2\theta}\left(-\mathrm{d}t \mp \mathrm{d}r \mp a\,\sin^2\theta\,\mathrm{d}\phi\right)^2 . \end{aligned} \tag{6.1}$$

Here $a = J/M$ is as usual the angular momentum per unit mass of the black hole. Several other forms for the Kerr metric are known [145–147]. Indeed, only relatively recently (2000) was the Doran form of the Kerr spacetime developed [148, 149]; this seeming to be as close as one can get to putting the Kerr metric into Painlevé–Gullstrand "acoustic geometry" form. Now, historically it has been found that every significantly new form of the Kerr solution has lead to advances in our understanding, and it is still possible (though maybe not entirely likely) that the Kerr solution could be greatly simplified by writing it in some particularly clean form [146].

Let us state that while the Kerr metric is the only uncharged rotating vacuum solution of the Einstein equations, it is not, however, the only metric displaying an ergosphere. Other rotating black holes spacetimes can also be simulated in analogue gravity. In these analogue systems the rotation, *i.e.* the angular momentum of the black hole, corresponds to the vorticity of the flowing medium. See for instance [59,60,138,140,141,145,146,150].

To build an analogue system with vorticity corresponding to the Kerr spacetime, it is necessary to write the Kerr metric in an expression which conforms to the descriptiveness of analogue gravity.

In our work we therefore set the basis for finding a Gordon form for the Kerr geometry. The Gordon form is explicitly given in terms of a flat background, a 4-velocity and a "refractive index" of an effective medium, elements that can be reproduced in analogue systems. This a formalism is almost a century old [44,151–153], and Gordon's paper [50] and was explored describing what would now be called electromagnetic analogues.

A spacetime metric is said to be in Gordon form if it can organized as the sum of the flat metric and a 1-form in tensor product with itself:

$$g_{\mu\nu} = \eta_{\mu\nu} + (1 - c_*^2) V_\mu V_\nu \,. \tag{6.2}$$

Here $\eta_{\mu\nu}$ is some background metric (typically taken to be flat Minkowski space, but any coordinate change other than a Poincaré transformation of the coordinates would give a different, but equally valid, expression), while V_μ is some 4-velocity (properly normalized to $\eta^{\mu\nu} V_\mu V_\nu = -1$ in terms of the background metric), and c_* can be interpreted as the speed of light in the medium (so in terms of the refractive index $c_* = 1/n$). In situations discussed below (where we might not necessarily want to adopt the moving medium interpretation) c_* can still be interpreted as the coordinate speed of light at spatial infinity. This Gordon form for the spacetime metric has much deeper implications and a significantly wider range of applicability than the original context in which it was developed [62–64, 70–72, 154–158], though only relatively recently (2004) has it become clear that the theoretically important Schwarzschild spacetime can be put into this form [135, 159].

Gordon forms are not relevant only in electromagnetic analogues (see particularly [101], or more generally [52, 120, 160] and [44, 74, 75, 151, 153, 161].), but are obtained also in the case of analogue gravity in relativistic Bose–Einstein condensates: the Bogoliubov quasi-particles propagate in accordance with an acoustic metric which, in the hydrodynamical limit, is naturally given in an expression which is conformal to a Gordon form. The 4-velocity appearing in the Gordon form corresponds therefore to the 4-velocity of the condensate itself, and has to include all the relevant information for the analogy: the radius of the horizon, where the velocity of the flow reaches the speed of sound, and the vorticity of the condensate, corresponding to the angular momentum.

While the conformal factor affects the local geometry, the causal structure of the analogue geometry is the same. In any case, we can control the behavior of the conformal factor: in principle it is not strictly necessary to find a Gordon form of the Kerr metric, but a conformally Gordon form for the Kerr metric would be equally useful.

In this chapter we show how the Gordon form of the Kerr metric can be searched for by manipulating other known expressions of the metric, applying the appropriate changes of coordinates. The transformations should be such that the metric is split into some expression for the flat metric summed to a tensor product of a 1-form with itself. We obtained approximate expressions for the Gordon form in two cases: the first for slowly rotating black holes, consistent with the well known Lense–Thirring metric; the second for the near null 1-form, corresponding to an ultrarelativistic superfluid.

In conclusion, we explore the role of vorticity in Bose–Einstein condensates, and the physical constraints set on to the analogy by the atomic structure of Bose–Einstein condensates and their magnetic properties.

6.2 The Gordon form: two introductory examples

To set the stage, let us first present two simple results, before developing a general algorithm for implementing infinitesimal coordinate changes.

6.2.1 Gordon form of Schwarzschild spacetime

The Gordon form of the Schwarzschild metric [135, 159] is less well-known than perhaps is should be. Consider the line element

$$\mathrm{d}s^2 = \left(\eta_{\mu\nu} + \left(1 - c_*^2\right) V_\mu V_\nu\right) \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{6.3}$$

$$V = -\sqrt{1 + \frac{2\tilde{M}}{r}}\, \mathrm{d}t + \sqrt{\frac{2\tilde{M}}{r}}\, \mathrm{d}r \,. \tag{6.4}$$

In spherical coordinates this is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \left(1 - c_*^2\right)\left(-\sqrt{1 + \frac{2\tilde{M}}{r}}\, \mathrm{d}t + \sqrt{\frac{2\tilde{M}}{r}}\, \mathrm{d}r\right)^2 . \tag{6.5}$$

The corresponding metric is spherically symmetric and easily checked to be Ricci flat — so by Birkhoff's theorem it must be Schwarzschild spacetime in disguise. Here c_* is an arbitrary constant $c_* \in (0, 1)$, which at spatial infinity can be viewed, as anticipated, as the coordinate speed of light. Furthermore V_a is a 4-velocity, (normalized in the background metric, $\eta_{\mu\nu} V^\mu V^\nu = -1$), and the parameter $\tilde{M}$ is proportional to the physical mass of the Schwarzschild spacetime. By noting that

$$g_{tt} = -1 + (1 - c_*^2)(1 + \frac{2\tilde{M}}{r}) = -c_*^2 + (1 - c_*^2)\frac{2\tilde{M}}{r}, \tag{6.6}$$

and comparing to the asymptotic behavior of Schwarzschild in the usual curvature coordinates, we identify the physical mass of the black hole M as

$$M = \frac{(1 - c_*^2)\, \tilde{M}}{c_*^2} = (c_*^{-2} - 1)\, \tilde{M}. \tag{6.7}$$

6.2.2 Gordon form of Lense–Thirring slow-rotation spacetime

Let us remind ourselves of the quite standard version of the Lense–Thirring slow-rotation spacetime (in the usual Schwarzschild curvature coordinates). The line element is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\left(\mathrm{d}\phi - \frac{2Ma}{r^3}\mathrm{d}t\right)^2 . \tag{6.8}$$

This represents a metric which is Schwarzschild (in curvature coordinates) plus $\mathcal{O}(a)$ modifications, and for this metric one can easily check that $R_{ab} = \mathcal{O}(a^2)$; all components of the Ricci tensor are $\mathcal{O}(a^2)$.

This $\mathcal{O}(a^2)$ behavior for the Ricci tensor is *what we mean* by saying that the Lense–Thirring spacetime is an approximate solution to the vacuum Einstein equations corresponding to a slowly rotating spacetime. The spacetime has angular momentum $J = Ma$.

For current purposes we could equally well ignore the $\mathcal{O}(a^2)$ term *in the metric* and write the simplified Lense–Thirring line element as:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 - \frac{4Ma\sin^2\theta}{r}\mathrm{d}t\mathrm{d}\phi\,. \tag{6.9}$$

This simplified line element represents a metric which is still Schwarzschild (in curvature coordinates) plus $\mathcal{O}(a)$ modifications, and for this metric we still get $R_{ab} = \mathcal{O}(a^2)$; all components of the Ricci tensor are $\mathcal{O}(a^2)$. That is, the spacetime is still Ricci flat up to terms quadratic in a.

Based on these observations, to find a Gordon form for Lense–Thirring we simply take the Gordon form of Schwarzschild and make the ansatz

$$V = V_\mu\,\mathrm{d}x^\mu \;\to\; -\sqrt{1 + \frac{2\tilde{M}}{r}}\,\mathrm{d}t + \sqrt{\frac{2\tilde{M}}{r}}\,\mathrm{d}r + \frac{2\tilde{M}\tilde{a}\sin^2\theta}{r\sqrt{1 + \frac{2\tilde{M}}{r}}}\mathrm{d}\phi\,. \tag{6.10}$$

That is, we consider the metric ansatz represented by the line element

$$\begin{aligned}\mathrm{d}s^2 = &-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \\ &+ \left(1 - c_*^2\right)\left(-\sqrt{1 + \frac{2\tilde{M}}{r}}\,\mathrm{d}t + \sqrt{\frac{2\tilde{M}}{r}}\mathrm{d}r + \frac{2\tilde{M}\tilde{a}\sin^2\theta}{r\sqrt{1 + \frac{2\tilde{M}}{r}}}\mathrm{d}\phi\right)^2 .\end{aligned} \tag{6.11}$$

Here we have again $c_* \in (0,1)$, the parameter $\tilde{M}$ is proportional to the physical mass of the Lense–Thirring spacetime, and $\tilde{a}$ is proportional to a. Note that (in the background metric) $||V||^2 = -1 + \mathcal{O}(\tilde{a}^2)$, so that V is approximately a unit timelike 4-vector. Furthermore, since obviously $\tilde{a} = \mathcal{O}(a)$, to first order in a this metric ansatz is the just Gordon form of Schwarzschild plus an $\mathcal{O}(a)$ perturbation. Finally, a brief computation verifies that $R_{ab} = \mathcal{O}(a^2)$, the metric is Ricci-flat to $\mathcal{O}(a^2)$. This observation justifies calling this metric the Gordon form of Lense–Thirring spacetime. That is, for slow rotation, we can approximate the Kerr spacetime to arbitrary accuracy by a metric that is of the Gordon form.

To see how the parameters $\tilde{a}$ and $\tilde{M}$ are related to the physical parameters a and M, note that at very large r we have $g_{tt} \to -c_*^2$, while at all values of r we have $g_{t\phi} = 2(1 - c_*^2)\tilde{M}\tilde{a}\sin^2\theta/r$. Comparing this to the equivalent results for the usual form of the Lense–Thirring line element, (where at very large r we have $g_{tt} \to -1$, while at all values of r we have $g_{t\phi} = 2Ma\sin^2\theta/r$), we see that:

$$J = Ma = \frac{(1-c_*^2)\tilde{M}\tilde{a}}{c_*} = \frac{(1-c_*^2)\tilde{M}}{c_*^2} \times (c_*\,\tilde{a}) = M \times (c_*\,\tilde{a}). \tag{6.12}$$

That is, $a = c_*\,\tilde{a}$, while $M = (c_*^{-2} - 1)\tilde{M}$.

6.3 General algorithm

Now let us try to make these observations more systematic by presenting a general algorithm for searching for the Gordon form (if it exists).

6.3.1 Non-normalized Gordon and Kerr–Schild forms

Both Gordon and Kerr–Schild forms of the metric express the metric tensor as the sum of a Riemann-flat background metric $\overline{g}_{\mu\nu}$ and a 1-form v_μ in tensor product with itself. Let us adopt the notation

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + v_\mu v_\nu\,. \tag{6.13}$$

Here v_μ is not normalized; this lack of normalization is useful in some explicit computations. If v is timelike (with respect to the background metric) then the Gordon form expression of equation (6.2) can be recovered normalizing $v_\mu = ||v||V_\mu$. If v is null then we call this a Kerr–Schild form for the metric tensor. (The remaining case where v is spacelike does not seem to be particularly interesting.) In general, letting $\overline{g}^{\mu\nu}$ denote the inverse of the flat background metric, which here we do not necessarily presume has to be in the form $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$, the inverse of the full metric is

$$g^{\mu\nu} = \overline{g}^{\mu\nu} - \frac{\overline{g}^{\mu\alpha}\,\overline{g}^{\nu\beta}\;v_\alpha\,v_\beta}{1+\overline{g}^{\rho\sigma}\;v_\rho\,v_\sigma}\,. \tag{6.14}$$

The specific choice of coordinates is manifestly irrelevant for this description: as long as the metric tensor can be put into a Gordon form, every coordinate transformation *that acts on both sides* will provide an equivalent expression for the same decomposition. It is in principle possible to find inequivalent Gordon forms for the same spacetime if, choosing a common flat background metric, different 1-forms provide different full metric tensors which are equivalent through coordinate transformations.

6.3.2 How to find analytic expressions for Gordon and Kerr–Schild forms

Knowing an expression for the full metric in a certain set of coordinates $g_{\mu\nu}$, and an expression for the flat metric in a generally different set of coordinates $\overline{g}_{\alpha\beta}$, we look for possible inequivalent Gordon forms of the metric by applying a coordinate transformation of the form

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)\;, \tag{6.15}$$

and then noting

$$g_{\mu\nu}(x)\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \to g_{\mu\nu}\left(x'\right)\mathrm{d}x'^{\mu}\mathrm{d}x'^{\nu} = \tag{6.16}$$

$$= g_{\mu\nu}\left(x+\xi\right)\left(\delta^{\mu}_{\alpha} + \frac{\partial \xi^{\mu}}{\partial x^{\alpha}}\right)\left(\delta^{\nu}_{\beta} + \frac{\partial \xi^{\nu}}{\partial x^{\beta}}\right)\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}\,. \tag{6.17}$$

The RHS of equation (6.17) is a new expression for the spacetime metric which depends on the chosen local translations ξ defining the coordinate transformations of equation (6.15). This expression can be written in a Gordon form if it is possible to find a timelike 1-form v satisfying

$$g_{\mu\nu}\left(x+\xi\right)\left(\delta^{\mu}_{\alpha} + \partial_{\alpha}\xi^{\mu}\right)\left(\delta^{\nu}_{\beta} + \partial_{\beta}\xi^{\nu}\right) - \overline{g}_{\alpha\beta}\left(x\right) = v_{\alpha}v_{\beta}\,. \tag{6.18}$$

It is a straightforward algebraic exercise to extract — up to an overall sign — the expressions for the functions v_{α} in terms of the functions ξ^{μ} and their derivatives, from four of these ten equations. The remaining six equations provide a system of highly non-trivial and non-linear partial differential equations for the functions ξ^{μ} and the initially chosen tensors g and $\overline{g}$.

Ultimately the problem of finding a Gordon form for a metric is that of finding an appropriate coordinate transformation, *i.e.*, solving the differential equations for the ξ^{μ}, such that the initial system equation (6.18) admits a solution. In particular, it will be operationally convenient to investigate a class of coordinate transformations which is general enough to find a solution, but possibly without spoiling the explicit symmetries of the metric. The Schwarzschild spacetime is a remarkable example of a system where this problem admits an explicit solution, and we first use this to present a specific implementation of the general algorithm.

6.3.3 Checking Schwarzschild in Kerr–Schild form

We first apply this procedure to recover the well known Kerr–Schild form of the Schwarzschild metric, describing a static black hole of physical mass M. The usual expression for the metric obviously requires a transformation of coordinates to be put in a Gordon form since when choosing spherical coordinates for the flat background (mildly abusing notation by conflating metrics with their line elements)

$$\overline{g}_{\text{spherical}} = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\,, \tag{6.19}$$

we have

$$g = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 = \tag{6.20}$$

$$\neq \overline{g}_{\text{spherical}} + v \otimes v\,. \tag{6.21}$$

We need to apply a coordinate transformation which does not spoil the explicit time translation symmetry and the explicit spherical symmetry. The spatial coordinates are better left untouched since the angular part of both the full metric and the spherical flat background is the same: v must have vanishing angular components, making a rotation completely irrelevant; moreover, r^2 appears as the scale factor of the angular part of both metric tensors, and therefore r cannot be transformed. So we shall initially consider the simple coordinate transformation $t \to t + f(r)$.

Applying such transformation to the Schwarzschild metric, and choosing as flat background the spherical flat metric $\overline{g}_{\text{spherical}}$, the system in equation (6.18) admits a solution for $f'(r) = \mp\frac{2M}{r-2M}$. That is, we obtain the Kerr–Schild expression for the Schwarzschild spacetime, as equation (6.13), with

$$g = \overline{g}_{\text{spherical}} + v \otimes v \,, \tag{6.22}$$

$$v = -\sqrt{\frac{2M}{r}}\, \mathrm{d}t \mp \sqrt{\frac{2M}{r}}\, \mathrm{d}r \,. \tag{6.23}$$

The 1-form v defines a Kerr–Schild decomposition since it is a null 1-form, as is easily checked by verifying $\overline{g}^{\mu\nu} v_\mu v_\nu = 0$. The overall sign is chosen in such a way that the dual of this 1-form is a future-directed vector field.

In this case the process of solving the system of equations (6.18) only requires the expression for $f'(r)$, which can be obtained algebraically: the analytical expression for $f(r)$ itself is not needed; it is enough to know that $f'(r)$ is integrable to be sure that the coordinate transformation is properly defined, as it would imply explicitly that the new coordinate exists and admits the differential $\mathrm{d}t + f'(r)\,\mathrm{d}r$. This is a special case in which the differential of the time coordinate is mapped into an exact 1-form.

When the differential of a coordinate is mapped into a 1-form $\mathrm{d}t \to \omega$, and the 1-form ω is exact, it means that it can be written as the differential of a function of the initial coordinates, and therefore it represents a well defined coordinate change (assuming it is possible to invert the transformation without ambiguity, *i.e.* having a non-singular Jacobian).

By Poincaré's lemma [162], necessary and sufficient condition for a differentiable 1-form to be exact in a contractible domain is to be a closed 1-form. Therefore in our treatment, encountering a globally closed 1-form, we can be certain that in any domain that does not include a singularity it is possible to define the required coordinate change.

The computation was made particularly easy by an appropriate choice of background: the flat metric in spherical coordinates explicitly contains the same symmetries as the full spacetime, and this background is simply the limit of the full initial metric for vanishing black hole mass $M \to 0$. (This can also be considered as the limiting case of vanishing angular momentum for the known solution of the Kerr–Schild form of the Kerr metric.)

6.3.4 Checking Schwarzschild in Gordon form

A second application of the general algorithm allows us to recover the known result of the Gordon form of the Schwarzschild metric: by considering a somewhat more general class of coordinate transformations it is possible to find inequivalent non-null 1-forms reproducing the Schwarzschild metric as in the Gordon form of equation (6.13).

The reasoning presented above suggested that it would be profitable to consider a translation of the t coordinate by a function of the radial coordinate r only, so that the explicit symmetries of the metric were preserved. However, more generally we note that the time translation symmetry is still explicitly preserved if the t coordinate is deformed by rescaling. So a wider class of coordinate transformations to apply to the standard Schwarzschild metric equation (6.20) is this

$$t \to \sqrt{1-\zeta}\, t + f(r) \ , \tag{6.24}$$

$$\mathrm{d}t \to \sqrt{1-\zeta}\, \mathrm{d}t + f'(r)\, \mathrm{d}r \, . \tag{6.25}$$

We will consider the rescaling factor $\sqrt{1-\zeta} = c_*$ many times in the following discussion, and we will usually refer to ζ as the deformation parameter. This class of coordinate transformations modifies the appearance of the metric tensor, and the metric can be written in Gordon form with respect to the flat spherical background, (that is, a solution for the system of equations (6.18) with $\overline{g} = \overline{g}_{\text{spherical}}$ exists), if and only if

$$f'(r) = \mp \frac{2M}{r-2M} \sqrt{1 + \zeta \frac{r-2M}{2M}} \, . \tag{6.26}$$

Since $f'(r)$ is integrable this describes a proper coordinate transformation.

In conclusion, the Schwarzschild metric can be cast in a Gordon form with

$$g = \overline{g}_{\text{spherical}} + v \otimes v \, , \tag{6.27}$$

$$v = -\sqrt{\zeta + \frac{2M}{r}(1-\zeta)}\, \mathrm{d}t \mp \sqrt{\frac{2M}{r}(1-\zeta)}\, \mathrm{d}r \, . \tag{6.28}$$

The 1-form v is in general non-null, since $\overline{g}^{\mu\nu} v_\mu v_\nu = -\zeta$; the limit $\zeta \to 0$ reproduces the Kerr–Schild form. The original expression of the Gordon form of equation (6.2), or equation (6.5), is obtained by rewriting these expressions in terms of the speed of light in the medium and the normalized 4-velocity:

$$c_*^2 = 1 - \zeta \, , \tag{6.29}$$

$$\zeta = 1 - c_*^2 \, , \tag{6.30}$$

$$V = \frac{v}{\sqrt{\zeta}} = -\sqrt{1 + \frac{2\tilde{M}}{r}}\, \mathrm{d}t \mp \sqrt{\frac{2\tilde{M}}{r}}\, \mathrm{d}r \, , \tag{6.31}$$

$$\tilde{M} = M\, \frac{1-\zeta}{\zeta} \, . \tag{6.32}$$

Here M is again the physical mass, while $\tilde{M}$ is a convenient shorthand. The parameter ζ is bounded from both sides: in order to transform the t coordinate we must have $\zeta < 1$, otherwise we wouldn't be able to consider the square root $\sqrt{1-\zeta}$. From equation (6.26) we also understand that it must be required that the parameter ζ be non-negative: in order for the square root $\sqrt{1 + \zeta \frac{r-2M}{2M}}$ to exist in the external region $r > 2M$, we must have $\zeta \geq 0$. In conclusion, the deformation parameter is bounded within a finite interval $\zeta \in [0, 1)$, which corresponds to $c_*^2 \in (0, 1]$. This means that in an analogue gravity system the speed of light in the medium should be real, non-negative, and bounded by the speed of light in vacuum.

6.3.5 Gordon form of a spherically symmetric stationary metric

Here we consider the Gordon form of a spherically symmetric stationary metric, with a brief and instructive generalization of the previous case of the Schwarzschild black hole which was not included in [142].

It is useful to consider the Gordon forms associated with generic stationary spherically symmetric spacetimes, to understand how the coordinate changes affect the form of the metric and how Gordon forms are related to the Kerr–Schild forms even in cases slightly more complex than the Schwarzschild spacetime.

If we take the line element of a generic spherically symmetric stationary spacetime

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) , \tag{6.33}$$

with $\partial_t g_{\mu\nu} = 0$, we can expect to put it in a Gordon form by selecting an appropriate expression for the flat metric — the flat metric in spherical coordinates seems the most reasonable choice — and finding a 1-form of the kind $V_t\mathrm{d}t + V_r\mathrm{d}r$. But it is evident that we need a coordinate change, as the line element can be rewritten as a Gordon form only if one between g_{tt} and g_{rr} is equal to 1.

We can make a coordinate change being careful not to break the symmetries of the system. This means that we do not want to modify the spatial sector, and we only want to change the time coordinate. To preserve the explicit time translation symmetry of the metric, the time coordinate should be redefined only by dilatation and translation by a function of the radial coordinate.

$$\mathrm{d}t \to \sqrt{1-\zeta}\mathrm{d}t + f'(r)\,\mathrm{d}r\,, \tag{6.34}$$

$$\begin{aligned}\mathrm{d}s^2 \to & -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \\ & + \left(1 + g_{tt}(1-\zeta)\right)\mathrm{d}t^2 + 2g_{tt}f'\sqrt{1-\zeta}\mathrm{d}t\mathrm{d}r - \left(1 - g_{rr} - g_{tt}f'^2\right)\mathrm{d}r^2\,.\end{aligned} \tag{6.35}$$

This metric can be put in Gordon form without further manipulation, meaning that the second line in equation (6.35) is the tensor product of a 1-form with itself, only when the coordinate change is set by

$$f' = \pm\sqrt{-\frac{1}{g_{tt}}\left(1 + g_{tt}(1-\zeta)\right)\left(g_{rr} - 1\right)}\,. \tag{6.36}$$

This gives that the 1-form and its norm are

$$v = -\sqrt{1 + g_{tt}(1-\zeta)}\mathrm{d}t \mp \sqrt{-g_{tt}(1-\zeta)(g_{rr}-1)}\mathrm{d}r\,, \tag{6.37}$$

$$\eta^{\mu\nu}v_\mu v_\nu = -1 - g_{tt}g_{rr}(1-\zeta)\,. \tag{6.38}$$

We can make a few observations. When $g_{rr} = -g_{tt}^{-1}$, such as in the Schwarzschild case, the metric admits a natural Kerr–Schild form for $\zeta = 0$, *i.e.* without dilatation of the time coordinate. For any other value of ζ, the metric is in an explicit Gordon form.

The role of ζ, as the time dilatation parameter, becomes immediately evident as the necessary step in general for passing from a Kerr–Schild form to a Gordon form.

On the other hand, a metric does not present naturally a Kerr–Schild form unless there is the inverse proportionality between these two elements of the metric tensor: it must be $g_{rr} \propto -g_{tt}^{-1}$ for it to be possible that the norm of the 1-form vanishes everywhere. This is due to the fact that ζ is strictly a parameter, cannot depend on the position, otherwise the 1-form in equation (6.34) would not be a closed form, *i.e.* the redefinition of the time coordinate would not hold as a coordinate change.

In any case, when missing the above mentioned inverse proportionality between the elements of the metric, it is always possible to redefine the radial coordinate, such that one can extract a conformal factor: one can find a new radial coordinate $r'(r)\,\Omega(r)$ such that the line element becomes

$$\mathrm{d}s^2 = \Omega^2 \left(g'_{tt}\mathrm{d}t^2 - \frac{1}{g'_{tt}}\mathrm{d}r'^2 + r'^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) . \tag{6.39}$$

Following the same reasoning, this metric can be put in a conformal Gordon form that can be transformed in a conformal Kerr–Schild form by dilatation only of the time coordinate.

6.4 The Kerr spacetime

We now consider the more interesting case of rotating black holes, described with the Kerr metric, which we would like to express in Gordon form.

It is well-known that the Kerr metric can be written in Kerr–Schild form, which we could obtain following an analogous procedure to the one seen for the Schwarzschild spacetime. For our purposes, it is most convenient to consider the expression of the metric tensor as presented in Kerr's original derivation [144]. Again slightly abusing notation by conflating the metric with its corresponding line element we have

$$\begin{aligned} g = &\left(r^2 + a^2\cos^2\theta\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + 2\left(\mathrm{d}u + a\sin^2\theta\mathrm{d}\phi\right)\left(\mathrm{d}r + a\sin^2\theta\mathrm{d}\phi\right) + \\ &- \left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)\left(\mathrm{d}u + a\sin^2\theta\mathrm{d}\phi\right)^2 , \end{aligned} \tag{6.40}$$

where u should be read as a modified time coordinate (the advanced/retarded time). Applying the transformation $u \to \pm t + r$ to this metric we easily obtain the Kerr–Schild form of the Kerr metric, making use of a non-trivial representation of the flat background

$$g = \overline{g}_{\text{spheroidal}} + v \otimes v . \tag{6.41}$$

Here

$$\begin{aligned} \overline{g}_{\text{spheroidal}} = &- \mathrm{d}t^2 + \mathrm{d}r^2 + 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi + \left(r^2 + a^2\cos^2\theta\right)\,\mathrm{d}\theta^2 + \\ &+ \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\phi^2 , \end{aligned} \tag{6.42}$$

is a non-trivial non-diagonal implementation of spheroidal coordinates, and the null 1-form v is given by

$$v = \sqrt{\frac{2Mr}{r^2 + a^2 \cos^2\theta}} \left(-\mathrm{d}t \mp \mathrm{d}r \mp a \sin^2\theta \, \mathrm{d}\phi\right) . \tag{6.43}$$

As expected, the limit $a \to 0$ of $\bar{g}_{\text{spheroidal}}$ yields the spherical polar flat metric, while the limit $a \to 0$ of v reproduces the 1-form of equation (6.23) for the Kerr–Schild form of the Schwarzschild metric, as presented in equation (6.22). This decomposition is therefore a general description for Kerr, of which the Schwarzschild version is a particular case. The norm of v can be shown — after computing $\bar{g}^{\mu\nu}_{\text{spheroidal}}$ — to be vanishing, proving that this expression is indeed of Kerr–Schild form.

The flat background metric (6.42) considered here is the limit $M \to 0$ of the full metric; it is indeed Riemann-flat since it is obtained from the usual spherical flat metric equation (6.19) through a (somewhat non-obvious) coordinate transformation

$$r^2 \to r^2 + a^2 \sin^2\theta \, , \tag{6.44}$$

$$\sin^2\theta \to \frac{\left(r^2 + a^2\right)\sin^2\theta}{r^2 + a^2 \sin^2\theta} \, , \tag{6.45}$$

$$\phi \to \phi + \arctan\left(\frac{r}{a}\right) . \tag{6.46}$$

Note that not transforming the coordinate ϕ as done in equation (6.46) would have resulted in obtaining the somewhat more usual diagonal form of oblate spheroidal coordinates (without the $\mathrm{d}r\,\mathrm{d}\phi$ cross term).

To apply the general procedure described in subsection (6.3.2) and to search for a Gordon form of the Kerr metric, we now need to manipulate the metric tensor with a sufficiently wide class of coordinate transformations. Indeed, this will be considerably less trivial than for the Schwarzschild metric, simply because the Kerr spacetime has fewer symmetries. This implies that, while previously we could assume transformations preserving spherical symmetry, now in the Kerr spacetime we can only make weaker assumptions, as only an axial symmetry is left in the spatial sector. Expressing the coordinate transformation in terms of local translations as in equation (6.15), we can expect the translation to depend neither on ϕ, nor on t, apart from again possibly rescaling t by the factor $\sqrt{1-\zeta} = c_*$. Accordingly we consider

$$x^\mu \to x^\mu + \left(\sqrt{1-\zeta} - 1\right) \delta_t{}^\mu \, t + G^\mu\left(r, \theta\right) . \tag{6.47}$$

In general all four coordinates should now be transformed.

As discussed in subsection (6.3.2), the full resolution of the system of equations (6.18) requires solving nonlinear (quadratic) partial differential equations. Therefore finding the exact Gordon form for the Kerr metric seems (at least for now) to be a step too far. But we can certainly investigate this system perturbatively: we can expect that for a small rotation parameter a the perturbative expression of the Gordon form of the Kerr spacetime is a perturbation of the Gordon form of the Schwarzschild spacetime. Moreover, for a small deformation of the time coordinate — thereby considering a description at first order in ζ — we can also expect the Gordon form of the Kerr spacetime to be a perturbation of the known Kerr–Schild form of the Kerr metric.

In the following analysis we present these two different perturbative approaches, and the general expressions resulting from the resolution of the system (6.18) within the two separate approximations of slow-rotation and a near-null 1-form.

6.4.1 Slow rotation

When considering the slow-rotation regime, we can adopt approximate descriptions of the Kerr metric equivalent to the Lense–Thirring metric (6.9), which can be interpreted as a perturbation of the Schwarzschild metric. So in this subsection we shall consider a perturbative expansion for small angular momentum: we look for a Gordon form approximating the Kerr metric at first order in a. This is a common assumption in the literature, which is physically reasonable since astrophysical rotating black holes must certainly have $a < M$, (albeit they can become almost extreme due to accretion processes). Initially we discuss how to obtain the full solution of the system of equation (6.18) in the case of the Kerr metric approximated at order a; what we obtain is the most general first-order (in a) approximation to the Gordon form of the Kerr metric. Then we make a consistency check with the Gordon form of the Lense–Thirring metric as expressed in equation (6.11).

General case of slow-rotating Kerr in Gordon form

To obtain the most general Gordon form of the Kerr metric at order a, one should proceed as described previously in section (6.3.2), transforming the first-order approximation of the Kerr metric, (this is simply equations (6.41)–(6.42)–(6.43) with the 1-form v approximated to first order in a)

$$\overline{g}_{\text{spheroidal}} + \left(-\sqrt{\frac{2M}{r}} \mathrm{d}t \mp \sqrt{\frac{2M}{r}} \mathrm{d}r \mp a\sqrt{\frac{2M}{r}} \sin^2\theta \mathrm{d}\phi \right)^2 + \mathcal{O}\left(a^2\right) , \tag{6.48}$$

with the most general coordinate transformation which preserves the explicit axial symmetry and time translation symmetry,

$$t \to \sqrt{1-\zeta}\, t + \tilde{f}\left(r\right) + a\, G^t\left(r,\theta\right) + \mathcal{O}\left(a^2\right) , \tag{6.49}$$

$$r \to r + a\, G^r\left(r,\theta\right) + \mathcal{O}\left(a^2\right) , \tag{6.50}$$

$$\theta \to \theta + \frac{a}{2M}\, G^\theta\left(r,\theta\right) + \mathcal{O}\left(a^2\right) , \tag{6.51}$$

$$\phi \to \phi + \frac{a}{2M}\, G^\phi\left(r,\theta\right) + \mathcal{O}\left(a^2\right) . \tag{6.52}$$

Here the function $\tilde{f}$ in equation (6.49) must (see equation (6.26)) have derivative

$$\tilde{f}'(r) = \pm \frac{2M}{r-2M}\left(1-\sqrt{1+\zeta\frac{r-2M}{2M}}\right) , \tag{6.53}$$

for consistency with what we already know is needed, in the case of vanishing a, to put the Kerr–Schild form of the Schwarzschild geometry into Gordon form.

In order to find the most general Gordon form of the Kerr metric at order a, the functions G^μ generating the coordinate transformation must be solutions of the system of equations (6.18) — after linearization with respect to a — together with an appropriate 1-form v. The solution at zeroth order a^0 is known, since it will simply be the Gordon form of the Schwarzschild metric. At next higher order the 1-form will include a correction of order a for every component. So we will have

$$v = -\sqrt{\zeta + \frac{2M}{r}(1-\zeta)}\mathrm{d}t \mp \sqrt{\frac{2M}{r}(1-\zeta)}\mathrm{d}r + a\,\delta v + \mathcal{O}\left(a^2\right) . \tag{6.54}$$

Taking the first order approximation, the nonlinear system of equations (6.18) is reduced to a system of first-order partial differential equations. This system can be solved patiently, step by step — first obtaining the expressions for the components δv_μ in terms of the functions G^μ and their derivatives, and then solving the system, finding their explicit expressions. The integration constants should be fixed in such a way that for vanishing ζ the 1-form obtained ultimately reduces to that defining the Kerr–Schild metric (6.48).

Here is the full expression obtained for the Gordon form of the Kerr spacetime, written in the form $g = \overline{g}_{\text{spheroidal}} + v \otimes v$, at first order in a:

$$\begin{aligned} v_t = & -\sqrt{\frac{2M}{r}\left(1+\zeta\frac{r-2M}{2M}\right)}+ \\ & -\frac{a\kappa}{2r}\sqrt{\frac{2M}{r}\frac{(1-\zeta)^2}{1+\zeta\frac{r-2M}{2M}}}\left(1-\sqrt{1+\zeta\frac{r-2M}{2M}}\right)\cos\theta + \mathcal{O}\left(a^2\right) , \end{aligned} \tag{6.55}$$

$$\begin{aligned} v_r = & \mp\sqrt{\frac{2M}{r}}(1-\zeta)+ \\ & \mp\frac{a\kappa}{2r}\sqrt{\frac{2M}{r}}(1-\zeta)\left(\left(1-\sqrt{1+\zeta\frac{r-2M}{2M}}\right)+\frac{\zeta r^2}{4M^2\sqrt{1+\zeta\frac{r-2M}{2M}}}\right)\cos\theta + \mathcal{O}\left(a^2\right) , \end{aligned} \tag{6.56}$$

$$v_\theta = \mp a\kappa\sqrt{\frac{2M}{r}}(1-\zeta)\left(\left(1-\sqrt{1+\zeta\frac{r-2M}{2M}}\right)-\frac{\zeta r(r-2M)}{8M^2\sqrt{1+\zeta\frac{r-2M}{2M}}}\right)\sin\theta + \mathcal{O}\left(a^2\right) , \tag{6.57}$$

$$v_\phi = \mp a\sqrt{\frac{2M}{r}\frac{1-\zeta}{1+\zeta\frac{r-2M}{2M}}}\sin^2\theta + \mathcal{O}\left(a^2\right) . \tag{6.58}$$

Here κ is a dimensionless residual integration constant one finds from the coordinate transformation described by equations (6.49)–(6.52), when the integration constants are chosen to be independent both from a and ζ. This Gordon form correctly describes the Kerr spacetime up to order a^2, *i.e.* it can be verified that this Gordon form produces a vanishing Ricci tensor up to $\mathcal{O}(a^2)$. This was expected since we simply considered coordinate transformations of the Kerr metric; with this check the formalism used is therefore proven to be consistent.

We observe that in general the norm of the 1-form v is non-trivial: for non-vanishing κ, it has a contribution of order a which strongly depends on the angular and radial position

$$\overline{g}^{\mu\nu} v_\mu v_\nu = -\,\zeta \left(1 + \frac{a\kappa}{2M} \frac{1-\zeta}{\sqrt{1+\zeta\frac{r-2M}{2M}}} \cos\theta \right) + \mathcal{O}\left(a^2\right) \,. \tag{6.59}$$

In the limit of null deformation parameter $\zeta \to 0$, in which case the 1-form v reproduces the Kerr–Schild case — the norm vanishes identically, for any value of κ.

That is, the first order in a Gordon form of the Kerr metric has been obtained as a perturbation of the Gordon form of the Schwarzschild metric, (which is the limiting case for vanishing a). It should be noted that in this more general case the deformation parameter ζ is again bounded within the same interval, $\zeta \in [0,1)$. If the integration constant κ is not neglected the norm is highly point-dependent; in particular it may be possible that the norm changes sign due to the presence of the factor $\cos\theta$ in the term of order a. To avoid this change of sign, a bound on κ should be imposed. To define this bound we make the assumptions $0 \le \zeta < 1$ and $\left|\frac{a}{M}\right| < 1$, and we consider as the region of interest that with $r > 2M$ (implying $1 < (1+\zeta\frac{r-2M}{2M})^{1/2} < \infty$). With such assumptions we find that $|\kappa| < 2$ always prevents a change of sign of the norm of the 1-form. In general, for a given a/M ratio, one needs $|\kappa| < |2M/a|$.

Consistency check with Lense–Thirring

We have found a general expression for the Gordon form of the slow rotating Kerr spacetime. We can now easily make a consistency check to prove that this Gordon form, defined in terms of the spheroidal flat background and the 1-form of components (6.55)–(6.58), is equivalent to the Lense–Thirring Gordon form introduced in equations (6.10) and (6.11).

This can be done assuming a vanishing integration constant κ and making use of the properly redefined parameters — as discussed previously — of the speed of light in the medium and rescaled mass and angular momentum parameters

$$\kappa = 0\,, \tag{6.60}$$

$$c_*^2 = 1 - \zeta + \mathcal{O}\left(a^2\right)\,, \tag{6.61}$$

$$\tilde{M} = M\frac{1-\zeta}{\zeta}\,, \tag{6.62}$$

$$\tilde{a} = \frac{a}{\sqrt{1-\zeta}}\,. \tag{6.63}$$

Extracting the normalization of the 1-form v (for vanishing κ its norm is simply $\sqrt{\zeta}$) and substituting the parameters, we obtain the same expression for v as equation (6.10)

$$V = \frac{v}{\sqrt{\zeta}} = -\sqrt{1+\frac{2\tilde{M}}{r}}\mathrm{d}t \mp \sqrt{\frac{2\tilde{M}}{r}}\mathrm{d}r \mp \frac{2\tilde{M}\tilde{a}\sin^2\theta}{r\sqrt{1+\frac{2\tilde{M}}{r}}}\mathrm{d}\phi + \mathcal{O}(\tilde{a}^2)\,. \tag{6.64}$$

Actually, this is not yet quite enough to verify the equivalence between the two Gordon forms because this 1-form is referred to the spheroidal flat background, while the Lense–Thirring Gordon form in equation (6.11) is referred to the spherical flat metric. However with a final simple coordinate transformation it is possible to prove that these Gordon forms are completely equivalent: The inverse of the transformation from the spherical flat metric to the spheroidal flat metric — approximated at order a, and therefore transforming only the coordinate ϕ with the inverse of the transformation (6.46) — is indeed what is needed, since it properly transforms the flat background and does not modify the 1-form up to order $\mathcal{O}(a^2)$. This transformation is

$$\mathrm{d}\phi \to \mathrm{d}\phi - \frac{a}{r^2}\mathrm{d}r + \mathcal{O}\left(a^2\right)\,, \tag{6.65}$$

$$\overline{g}_{\text{spheroidal}} \to \overline{g}_{\text{spherical}} + \mathcal{O}\left(a^2\right)\,, \tag{6.66}$$

while V (and so v) is the same. We have therefore proved the equivalence between these two Gordon forms, they can be obtained one from the other through a coordinate transformation.

6.4.2 Near-null Gordon form of Kerr spacetime

In this section we consider a different approach to the problem of the description of the Kerr metric in a Gordon form. We have already seen that — given a fixed flat background — inequivalent Gordon forms of the metric can be obtained through a class of coordinate transformations which include a rescaling of the time coordinate. To describe such deformations we have used the rescaling term $\sqrt{1-\zeta}$. Rescaling the time coordinate, one can obtain new 1-forms with non-null norm, passing from a Kerr–Schild form to a Gordon form. The parameter ζ is therefore strictly related to the norm and becomes the instrument to explore the space of possible inequivalent Gordon forms of the Kerr spacetime.

In the slow-rotation case, the order a approximation has allowed us to consider Gordon forms where the rescaling parameter ζ was free to explore the whole range of allowed values: that is, the same interval $\zeta \in [0, 1)$ which was acceptable in the Schwarzschild case. While the small a result is already of great interest, it rules out the whole regime of rapidly rotating spacetimes. We now wish to explore that region of parameter space, and we want to do so by making a different approximation: we consider a small deformation parameter ζ, and we obtain an expression for the Gordon form of the Kerr metric at first order in ζ.

Infinitesimal local translation of the Kerr metric

We want to apply the procedure described in the introductory subsection (6.3.2), transforming the Kerr–Schild form of the Kerr metric (6.41) with an infinitesimal transformation. Again we want this coordinate transformation to include a deformation of the time coordinate; since we want this deformation to be infinitesimal, we can approximate at order ζ the rescaling term, and consider ζ to be arbitrary small

$$\sqrt{1-\zeta} = 1 - \frac{\zeta}{2} + \mathcal{O}\left(\zeta^2\right). \tag{6.67}$$

The other coordinates should be transformed infinitesimally too, and the rest of the transformation should be assumed not to spoil the explicit axial and time translation symmetries. So we consider

$$t \to \left(1 - \frac{\zeta}{2}\right) t + \zeta\, F^t\,(r,\theta) + \mathcal{O}\left(\zeta^2\right)\,, \tag{6.68}$$

$$r \to r + \zeta\, F^r\,(r,\theta) + \mathcal{O}\left(\zeta^2\right)\,, \tag{6.69}$$

$$\theta \to \theta + \zeta\, F^\theta\,(r,\theta) + \mathcal{O}\left(\zeta^2\right)\,, \tag{6.70}$$

$$\phi \to \phi + \zeta\, F^\phi\,(r,\theta) + \mathcal{O}\left(\zeta^2\right)\,. \tag{6.71}$$

This coordinate transformation is easily brought back to the formalism of infinitesimal local translations as presented in section (6.3.2). The coordinates are transformed with $x^\mu \to x^\mu + \xi^\mu\,(x)$, and the translation vector ξ defining this coordinate transformation is clearly of order ζ. This means that for an arbitrarily small ζ, the Kerr metric (6.41) is moved infinitesimally along the vector field ξ, and for this infinitesimal transformation it can be written in terms of the Lie derivative along this vector field

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu} + \left(\mathcal{L}_\xi \overline{g}_{\mu\nu} + v_\mu \mathcal{L}_\xi v_\nu + \mathcal{L}_\xi v_\mu v_\nu\right)\,, \tag{6.72}$$

where we are now dropping the subscript in the flat background metric, writing $\overline{g}$ instead of $\overline{g}_{\text{spheroidal}}$ for simplicity. Now of course the relevant question is: can the transformed expression of the Kerr metric be written in Gordon form? This will be the focus of the next section.

Solution in modified components

The infinitesimal translation transforms the Kerr–Schild form metric (6.41) into the modified metric tensor (6.72) through the Lie derivative; both are metric tensors describing the Kerr spacetime with the same set of coordinates. The new expression can be put in a Gordon form if it is possible to find a new 1-form which solves, together with vector field ξ, the system of equations (6.18).

The new 1-form must differ from that defining the initial Kerr–Schild form in equation (6.43) by a term of order ζ, such that the system admits a solution valid for every small value of the deformation parameter; *i.e.* if the Kerr–Schild form was given (with respect to the spheroidal flat background) by the 1-form

$$v = \Phi \left(-\mathrm{d}t \mp \mathrm{d}r \mp a \sin^2 \theta \mathrm{d}\phi \right) , \tag{6.73}$$

$$\Phi = \sqrt{\frac{2Mr}{r^2 + a^2 \cos^2 \theta}} , \tag{6.74}$$

where the function Φ is a shorthand defined for convenience. Then the new solution must be of the form $v + \delta v$, with δv being the correction of order ζ. With this assumption for the modified metric tensor and the 1-form, equation (6.18) is expressed at order ζ as

$$g_{\mu\nu} + \left(\mathcal{L}_\xi \overline{g}_{\mu\nu} + v_\mu \, \mathcal{L}_\xi v_\nu + \mathcal{L}_\xi v_\mu \, v_\nu \right) - \overline{g}_{\mu\nu} = (v_\mu + \delta v_\mu)(v_\nu + \delta v_\nu) , \tag{6.75}$$

$$\Downarrow$$

$$\mathcal{L}_\xi \overline{g}_{\mu\nu} + v_\mu \, \mathcal{L}_\xi v_\nu + \mathcal{L}_\xi v_\mu \, v_\nu = v_\mu \, \delta v_\nu + \delta v_\mu \, v_\nu + \mathcal{O}\left(\zeta^2\right) . \tag{6.76}$$

It is easier to solve this problem by change of dependent variables, redefining the 1-form of interest. If we consider the modified 1-form correction

$$\delta v' = \delta v - \mathcal{L}_\xi v, \tag{6.77}$$

then equation (6.18), which we have already reduced to (6.76), takes the simplified form

$$\mathcal{L}_\xi \overline{g}_{\mu\nu} = v_\mu \, \delta v'_\nu + \delta v'_\mu \, v_\nu + \mathcal{O}\left(\zeta^2\right) . \tag{6.78}$$

For this system, similarly to what has been done for the general case of the Gordon form in the slowly rotating approximation for Kerr spacetime as presented in section (6.4.1), the solution is found step by step. First we obtain the expressions for the components of $\delta v'$ in terms of the components — and their derivatives — of the translation vector field ξ. Then the system is solved by sequentially finding the functions ξ^μ one after the other. The integration constants should be chosen such that the coordinate transformation reduces to the identity for vanishing ζ, ensuring that no trivial translations in the coordinates t and ϕ are introduced. Doing all this, the components of the translation vector field are found (to order $\mathcal{O}(\zeta)$) to be

$$\xi^t = \left(-\frac{1}{2}t \mp \frac{r^2 + a^2\cos^2\theta}{2r}\right)\zeta\,, \tag{6.79}$$

$$\xi^r = \left(\lambda\frac{\left(r^2+a^2\right)\cos\theta}{r^2+a^2\cos^2\theta} + \frac{a^2}{2}\left(\frac{\cos^2\theta}{r} - \frac{r\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\right)\zeta\,, \tag{6.80}$$

$$\xi^\theta = \left(-\lambda\frac{r\sin\theta}{r^2+a^2\cos^2\theta} - \frac{a^2}{2}\frac{\sin\theta\cos\theta}{r^2+a^2\cos^2\theta}\right)\zeta\,, \tag{6.81}$$

$$\xi^\phi = \left(-\lambda\frac{a\cos\theta}{r^2+a^2\cos^2\theta} + \frac{a}{2}\frac{r}{r^2+a^2\cos^2\theta}\right)\zeta\,. \tag{6.82}$$

Here λ is a dimensionful integration constant. Inserting these back into the expressions for the $\delta v'$, and evaluating the Lie derivative of the 1-form v, we obtain the solutions (to order $\mathcal{O}(\zeta)$) for the components of the modified 1-form $v+\delta v$ on the spheroidal flat background $\overline{g} = \overline{g}_{\text{spheroidal}}$. We first note

$$v_\mu + \delta v_\mu = v_\mu + \delta v'_\mu + \mathcal{L}_\xi v_\mu = v_\mu + \delta v'_\mu + \xi^\sigma\partial_\sigma v_\mu + v_\sigma\partial_\mu\xi^\sigma\,. \tag{6.83}$$

The components of $\delta v'$ are easily found by substitution after solving equation (6.78). We find

$$\delta v' = \frac{\zeta}{2\Phi}\left(-\mathrm{d}t \pm \frac{a^2\cos^2\theta}{r^2}\mathrm{d}r \pm \frac{a^2\sin 2\theta}{r}\mathrm{d}\theta \pm a\sin^2\theta\mathrm{d}\phi\right) + \mathcal{O}\left(\zeta^2\right)\,. \tag{6.84}$$

The other contribution to δv comes from the Lie derivatives of the 1-form v with respect to the translation vector field ξ, and in general depends on the integration constant λ

$$\begin{aligned}\mathcal{L}_\xi v = {} & (\mathcal{L}_\xi\Phi)\left(-\mathrm{d}t \mp \mathrm{d}r \mp a\sin^2\theta\mathrm{d}\phi\right) + \\ & + \frac{\Phi}{2}\left(\mathrm{d}t \pm \mathrm{d}r \pm 2\frac{a^2 r\cos\theta\sin\theta}{r^2+a^2\cos^2\theta}\mathrm{d}\theta \pm 2\frac{a^3\cos^2\theta\sin^2\theta}{r^2+a^2\cos^2\theta}\mathrm{d}\phi\right) + \\ & \pm \lambda\Phi\left(\frac{r^2-a^2\cos^2\theta}{r^2+a^2\cos^2\theta}\mathrm{d}\theta + 2\frac{ar\cos\theta\sin\theta}{r^2+a^2\cos^2\theta}\mathrm{d}\phi\right)\,,\end{aligned} \tag{6.85}$$

(6.86)

where

$$\begin{aligned}\mathcal{L}_\xi\Phi = & -\Phi\frac{\left(2r^4a^2\cos(2\theta) + 3r^2a^4\sin(2\theta) - 2a^6\cos^6\theta\right)}{8r^2\left(a^2\cos^2\theta + r^2\right)^2} + \\ & - \lambda\Phi\frac{\left(r^4 + 3r^2a^2\sin^2\theta - a^4\cos^2\theta\right)}{2r\left(a^2\cos^2\theta + r^2\right)^2}\cos\theta\,.\end{aligned} \tag{6.87}$$

It is interesting to note that the components $\delta v'$ can be interpreted as providing the components of the same 1-form as δv, when the chosen background is not $\overline{g}$ but $\overline{g} - \mathcal{L}_\xi\overline{g}$. This tensor, differing from the flat background by its Lie derivative is still an acceptable approximately flat background (up to order ζ in the metric and order ζ^2 in the Riemann tensor) in order to put the Kerr metric in a Gordon form. Manipulating the expression of the metric (6.72) with the solution found from equation (6.78), we can verify that

$$g_{\mu\nu} = \left(\bar{g}_{\mu\nu} - \mathcal{L}_\xi \bar{g}_{\mu\nu}\right) + \left(v_\mu + \delta v'_\mu\right)\left(v_\nu + \delta v'_\nu\right) \ . \tag{6.88}$$

This means that the Kerr metric in Kerr–Schild form can be manipulated with Lie derivatives with respect to the vector field ξ we obtained, in order to write it down as a Gordon form with a modified flat background and a modified 1-form. It is in this setup that is more convenient to evaluate the norm of the 1-form, which can be evaluated straightforwardly by the known inverse of the metric

$$g^{\mu\nu}\left(v_\mu + \delta v'_\mu\right)\left(v_\nu + \delta v'_\nu\right) = -\zeta\, \frac{r^2 + a^2\cos^2\theta}{r^2} + \mathcal{O}\left(\zeta^2\right) \ , \tag{6.89}$$

proving that this is really a Gordon form for a near null 1-form.

6.5 Vorticity and applications to analogue spacetimes in Bose–Einstein condensates

As said, interest in the Gordon form of spacetime metrics is due (among other things) to potential applications in the analogue spacetime program. See specifically [101, 104], and more generally [44, 74, 151, 153, 163–166]. Indeed, the Gordon form, (or something conformal to the Gordon form), generically describes the acoustic metric experienced by a linearized perturbation on a relativistic fluid [101, 104]. That one might want vorticity in analogue systems is clear from references [59, 60, 134, 135, 138, 140, 141, 150]. Very often, however, in theoretical analyses of these analogue systems the inclusion of vorticity is tricky [167] — most typically the four velocity of the fluid considered is by construction hypersurface orthogonal (implying that it can be written as being proportional to the gradient of some scalar function) and as such — by the Frobenius theorem — it is vorticity free (in the relativistic sense that $V \wedge \mathrm{d}V = 0$). This is potentially a problem for an experimental simulation of a true Kerr geometry given that one can very easily realize that the Kerr–Schild and Kerr–Gordon forms of the Kerr metric found in this work always require a four velocity (or equivalently a one-form) which is not vorticity free. This fact can be seen most easily by looking at the simplest case, the four velocity obtained in the Gordon form of the Lense–Thirring spacetime,

$$V = -\sqrt{1 + \frac{2\tilde{M}}{r}}\mathrm{d}t \mp \sqrt{\frac{2\tilde{M}}{r}}\mathrm{d}r \mp \tilde{a}\sqrt{\frac{2\tilde{M}}{r}\frac{2\tilde{M}}{r + 2\tilde{M}}}\sin^2\theta \mathrm{d}\phi + \mathcal{O}\left(a^2\right). \tag{6.90}$$

For this four velocity we can compute the 4-vorticity

$$\omega = \star (V \wedge \mathrm{d}V) = V_{\nu_1} V_{\nu_3,\nu_2} \ \star (\mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\nu_2} \wedge \mathrm{d}x^{\nu_3}) = \tag{6.91}$$

$$= V_{\nu_1} V_{\nu_3,\nu_2} \ \epsilon^{\nu_1\nu_2\nu_3\nu_4} \overline{g}_{\nu_4\mu_4} \ \mathrm{d}x^{\mu_4} = \tag{6.92}$$

$$= \sqrt{\overline{g}} V_{\nu_1} V_{\nu_3,\nu_2} \ \overline{g}^{\nu_1\mu_1} \overline{g}^{\nu_2\mu_2} \overline{g}^{\nu_3\mu_3} \ \hat{\epsilon}_{\mu_1\mu_2\mu_3\mu_4} \ \mathrm{d}x^{\mu_4} = \tag{6.93}$$

$$= - \frac{4\tilde{a}\tilde{M}\cos\theta}{r^3} \sqrt{\frac{2\tilde{M}}{r + 2\tilde{M}}} \mathrm{d}t \mp \frac{4\tilde{a}\tilde{M}\cos\theta}{r^3} \mathrm{d}r \mp \frac{2\tilde{a}\tilde{M}\sin\theta}{r\left(2\tilde{M} + r\right)} \mathrm{d}\theta \,. \tag{6.94}$$

The 4-vorticity 1-form ω is evaluated with respect to the usual flat background. This result is non-vanishing at first order in a, and is valid for arbitrary deformation parameter $\zeta \in (0, 1)$. This is enough to imply that also for the near-null Gordon form it is impossible for the 4-vorticity to vanish. Indeed, the expression of the 4-vorticity of the near null Gordon form will depend on the parameter a; at first order, it must be consistent with the non-vanishing expression in equation (6.94). It will therefore also be impossible for the 4-vorticity in the near-null case to vanish identically.

Hence we conclude that any analogue model of the Kerr geometry will have to necessarily include vorticity in the background flow.

6.6 Vorticity in Bose–Einstein condensates

For what we have discussed in this chapter, in order to simulate a rotating black hole, being the Kerr black hole or a different spacetime with an ergosphere, it is necessary to have vorticity in the experimental realization.

When considering Bose–Einstein condensates, a simple way to introduce the vorticity in the description is by coupling the bosonic field to a gauge theory, *e.g.* considering charged Bose–Einstein condensates [134, 135]. In this case, the conserved current which appears in the continuity equation for the background flow is the sum of the gradient of the phase of the condensate and of the vector potential.

The acoustic metric for the propagation of quasi-particles will be defined by this new flow, and can therefore, in general, include vorticity.

While it is clear how in principle the vorticity can be included in the background flow, one has to understand if this leads to a stable configuration: the Meissner effect can lead to the expulsion of the magnetic flux from the condensate. We therefore have to check whether it is possible to have a sufficiently large London penetration depth, while at the same time keeping the healing length sufficiently small.

In this section we will consider both non-relativistic and relativistic condensates [143]; we will slightly change the notation from the previous sections, and we will use a to denote the scattering length.

6.6.1 Charged non-relativistic BECs

When considering a charged Bose–Einstein condensate [134,135], the line element of the metric is given in the usual acoustic form

$$\mathrm{d}s^2 = \frac{\langle\rho_0\rangle}{c_s}\left(-c_s^2\mathrm{d}t^2 + \delta_{ij}\left(\mathrm{d}x^i - v^i\mathrm{d}t\right)\left(\mathrm{d}x^j - v^j\mathrm{d}t\right)\right) , \tag{6.95}$$

where the 3-velocity of the effective Madelung fluid also depends on the vector potential:

$$v_i = \frac{\partial_i\langle\theta_0\rangle}{m} - \frac{eA_i}{m} . \tag{6.96}$$

The condensate wavefunction is $\langle\phi_0\rangle = \langle\rho_0\rangle^{1/2} e^{i\langle\theta_0\rangle}$, while c_s is the speed of sound in the condensate, and the purely spatial 3-vector $v \propto \nabla\langle\theta_0\rangle - eA$ is gauge invariant.

The key point is that $\partial_i\langle\theta_0\rangle - eA_i$ makes sense only if one has a condensate. This would in principle allow the background flow to have some vorticity while keeping the perturbations irrotational. Specifically the vorticity would be

$$\omega = \frac{eB}{m}. \tag{6.97}$$

Unfortunately one also has $v \propto j_{\mathrm{London}}$, the so-called London current that is central to the analysis of the Meissner effect. Indeed, there is widespread agreement within the condensed matter community that *any* charged Bose–Einstein condensate, (not just a BCS superconductor, where formation of the Cooper pairs, and condensation of the Cooper pairs are essentially simultaneous), will exhibit the Meissner effect — magnetic flux expulsion. See for instance [168, 169]. This would naively seem to confine any vorticity to a thin layer of thickness comparable to the London penetration depth λ_L. However there is a trade-off between the healing length (which controls the extent to which one can trust the effective metric picture) and the London penetration depth.

Let us be more quantitative about this: the London penetration depth and healing length are given by:

$$\lambda_L = \sqrt{\frac{m}{\mu_0\rho_0 q^2}}, \qquad \xi = \frac{1}{\sqrt{8\pi\rho_0 a}} . \tag{6.98}$$

Here m is the mass of the atoms making up the charged condensate, μ_0 is the magnetic permeability in vacuum, ρ_0 is the number density of atoms in the condensate; $q = Qe$ is the charge of each atom, and a is the scattering length. In particular, for the ratio of penetration depth to healing length the number density ρ_0 cancels and using $\epsilon_0\mu_0 = 1$ we have

$$\frac{\lambda_L}{\xi} = \sqrt{\frac{8\pi a m}{\mu_0 q^2}} = \sqrt{\frac{8\pi\epsilon_0 a m}{q^2}} . \tag{6.99}$$

Write $q = Qe$ and $m = Nm_p$, where N is the atomic mass number. Then

$$\frac{\lambda_L}{\xi} = \frac{\sqrt{N}}{Q}\sqrt{\frac{2am_p}{\alpha}} . \tag{6.100}$$

Then in terms of the Bohr radius, $a_0 = 1/(\alpha m_e)$, one has

$$\frac{\lambda_L}{\xi} = \frac{\sqrt{N}}{Q}\frac{1}{\alpha}\sqrt{\frac{2m_p}{m_e}}\sqrt{\frac{a}{a_0}}\,. \tag{6.101}$$

We can in principle make this ratio large, simply by tuning to a Feshbach resonance. Let us write $a = (a/a_*)\, a_*$, where a_* is the zero-field scattering length before tuning to a Feshbach resonance. We know that $\sqrt{2m_p/m_e} \approx 60$. For a heavy atom charged condensate $\sqrt{N}/Q \approx 9$ and typically $a_* \approx 100a_0$. Then

$$\frac{\lambda_L}{\xi} \approx 750000\sqrt{\frac{a}{a_*}}. \tag{6.102}$$

So there is a significant separation of scales between healing length and London penetration depth, which can be made even larger by tuning to a Feshbach resonance.

The net outcome of this discussion is that despite potential problems due to the Meissner effect there is a parameter regime in which we can simultaneously have vorticity penetrate deep into the bulk and still trust the effective metric formalism.

6.6.2 Charged relativistic Bose–Einstein condensates

As we have already discussed, in the case of relativistic Bose–Einstein condensates the expression of the metric naturally presents in a conformal Gordon form [134, 135]:

$$(g_{\text{eff}})_{\mu\nu} = \frac{\langle\rho_0\rangle}{c_s}\left(\eta_{\mu\nu} + \left(1 - c_s^2\right) V_\mu V_\nu\right). \tag{6.103}$$

The same holds for the charged relativistic Bose–Einstein condensate, with the properly normalized 4-velocity of the flowing condensate being

$$V_\mu = \frac{\nabla_\mu \langle\theta_0\rangle - eA_\mu}{||\nabla \langle\theta_0\rangle - eA||}, \tag{6.104}$$

again including the vector potential in the definition, and where the condensate wavefunction is written in terms of the Madelung representation $\langle\phi_0\rangle = \langle\rho_0\rangle^{1/2} e^{i\langle\theta_0\rangle}$, while c_s is the speed of sound in the condensate, and the 4-velocity $V \propto \nabla\Phi - eA$ is gauge invariant. This (formally) allows the background flow to have some vorticity while keeping the perturbations irrotational. Specifically for the 4-vorticity we have

$$\epsilon_{\mu\nu\rho\sigma}\omega^\sigma = V_{[\mu}V_{\nu,\rho]} = e\,\frac{V_{[\mu}F_{\nu\rho]}}{||\nabla \langle\theta_0\rangle - eA||}. \tag{6.105}$$

Working in the rest frame of the fluid we see

$$||\omega|| = \frac{e\,||B||}{||\nabla \langle\theta_0\rangle - eA||}. \tag{6.106}$$

With $\nabla_\mu \langle\theta_0\rangle - eA_\mu$ is a gauge invariant 4-vector field that makes sense only if one has a condensate.

The same potentially problematic issue regarding the Meissner effect also arises in this relativistic setting. The 4-velocity now satisfies $V \propto \nabla\Phi - eA \propto J_{\text{London}}$, where this is now the London 4-current $J_{\text{London}} = (\rho_{\text{London}}, j_{\text{London}})$. Naively, the magnetic field (and hence the vorticity) will be confined to a thin transition layer of thickness comparable to the London penetration depth. However the same parameter regime as was considered for the non-relativistic case will still apply in the full relativistic setting: drive the London penetration depth large while holding the healing length constant.

6.7 Final remarks

The study of vorticity in Bose–Einstein condensates is motivated by the aim of simulating, in an analogue gravity picture, the metric of rotating black hole spacetimes.

In particular, the formalism of analogue gravity in relativistic Bose–Einstein condensates requires to put the metric of the spacetime in Gordon form, up to a conformal factor. Such achievement would be valuable *per se* as new forms of the Kerr metric have always improved our understanding and technical mastery of the solution.

We have presented two approximate results for the Gordon form of the Kerr metric, describing the regimes of small rotation and near-null 1-form; they were obtained respectively by perturbing the Gordon form of the Schwarzschild metric, and by infinitesimally deforming the Kerr–Schild form of the Kerr metric.

These two results plausibly suggest the existence of an (as yet unknown) full analytical expression for the Gordon form of the Kerr metric. Such an expression would be the full solution of the system of equations (6.18), possibly found through the algorithm presented in subsection (6.3.2).

We also remark that in finding these results we showed that a proper choice of flat background is of crucial importance for the resolution of the problem — both in terms of the final expression for the Gordon form, and in computation time, and this should be considered in any future approach to this problem.

While the theoretical and mathematical study of the Kerr metric is in itself an interest topic in the field of gravity, we are interested in the possibility to realize an analogue to the Kerr black hole in a laboratory setup. An experiment of superradiance in an analogue rotating black hole realized in Bose–Einstein condensates would allow to investigate the exchange of information between the classical acoustic metric and the quantized radiation enhanced in the process.

Introducing vorticity into analogue models is subtle, due to the practical difficulties in managing charged Bose–Einstein condensates due to the Meissner effect. Nevertheless, we have shown how systems where the London penetration depth is considerably larger than the healing length in principle allow to study a rotating analogue black holes realized in a condensate in the hydrodynamical limit.

There are of course many additional relevant articles on related topics from within the astrophysics, condensed matter, and optics communities. See for instance [170] and the extensive list of references in [44]. We have unavoidably had to be somewhat selective in our selection of references. Relatively recent developments include the notions of “quantum vorticity” [171–173] and “holographic vorticity” [174–177].

Taken as a whole, these observations collectively give us confidence that it is likely to be possible to mimic the Kerr solution at the wave optics or wave acoustics level — presumably through some conformal Gordon form of the metric.

7

Discussion and conclusions

In this thesis we have studied models of analogue gravity, focusing on Bose–Einstein condensates. These systems are described by a bosonic many-body theory that has a hydrodynamical limit in which the analogy to quantum field theory in curved spacetime emerges naturally: a classical fluid (described in terms of the condensate wavefunction) is the background for the propagation of linearized quantum perturbations (the Bogoliubov quasi-particles) which follow an analogue metric determined by the fluid. Studying mesoscopic scales it is possible to define local hydrodynamical quantities (typically we are interested in the density and in the speed of sound), while at the same time neglecting the details of the underlying microscopic structure, which has a substantial role only when considering high-momenta: in that case the microscopic structure of the system modifies the dispersion relation of the quasi-particles and ensures that the ultraviolet behavior is regularized, avoiding naturally what would be the analogue of the gravitational transplanckian problem in the condensate.

Just as analogue systems suggested to look for the solution of the transplanckian problem in the Lorentz symmetry breaking and in some microscopic structure of spacetime (and so showed the robustness of Hawking radiation), we have investigated how other features of analogue models provide hints on how to solve other open issues of gravity, focusing in particular on the evaporation and information loss phenomena. Making use of the conceptual symmetry of the principle of analogy we can not only use the models of analogue gravity to test the predictions of quantum field theory in curved spacetime, but we can also explore the limits of the analogue models and the features of the systems in which they are realized to possibly get insights on how to go beyond semiclassical gravity, which is an intermediate step in the unification of the description of geometry and quantum field theory.

In particular we have decided to adopt the number conserving formalism, applied to the natural orbitals, to go beyond the standard formulation of analogue gravity. Instead of using the usual mean-field approximation for the condensate wavefunction, we have retained the full quantum behavior of the atoms in the condensate. In this way we have shown that, while the dynamics of the condensate wavefunction is generally described in good approximation by the Gross–Pitaevskii equation, the paradigm of the mean-field description can be revisited. In order to derive the equations for analogue gravity it is not strictly necessary to consider coherent states (for which are simultaneously true that the mean-field description is most accurate, and that the contributions of the condensate are factored out from every correlation function): analogue gravity models can also be obtained considering condensate states which can be studied retaining the complexity of the many-body description. [1]

For coherent states the information on how the atoms in the condensate are correlated with the atoms in the excited part is lost, because by factoring out the condensate one would split the Hilbert space, and the analogue geometry would therefore be defined in a space separated from the Fock space of the analogue quantum fields. This is analogous to the separation of the Hilbert spaces of geometry and matter which is generally assumed in quantum field theory in curved spacetime, and which is at the origin of the information loss problem: in semiclassical gravity the back-reaction of the quantum fields on the geometry is a modification of the dynamics that relates them but does not unify the Hilbert spaces, and in the evaporation of a black hole it is impossible to transfer quantum information to the classical geometry.

Instead, in the number conserving formalism that we have discussed it is possible to keep track of the correlations between the condensate and the excited atoms, which are correlations that can be interpreted, within the analogy, as correlations between the quasi-particles and quantum degrees of freedom of spacetime which are hidden in the usual description.

In particular we have shown that retaining the quantum nature of the atoms in the condensate it is still possible to formulate a model of analogue gravity [7] (which we have used to study the analogue cosmological particle creation): the condensate wavefunction can still be described by the Gross–Pitaevskii equation, and the linearized dynamics of the quantum perturbations, *i.e.* the modes of the Bogoliubov quasi-particles, propagate following the same equations. In the explicitly number conserving formalism we see that not only it is not necessary to split the Hilbert space, but it is natural to interpret the quasi-particles as dynamical exchanges of atoms between the condensate and the excited part. It is therefore an emergent phenomenon of a purely quantum system, the hydrodynamical description (meaning the description of the quasi-particle dynamics in an analogue geometry) does not require at any point to break the unitarity of the evolution.

In conclusion, the unitarity of the evolution of the bosonic system is not lost in the analogue description, as long as the the creation of quasi-particles is a process which is studied in the full Hilbert space. During the evolution, the unitary scattering operator transforms the initial many-body atomic state and squeezes it creating the quasi-particles. These are excitations of the atomic state in which the condensate and the

[1] Moreover, let us notice that the relation between several quantum gravity scenarios and analogue gravity in Bose–Einstein condensates appears to be even stronger than expected: in many of these models a classical spacetime is recovered by considering an expectation value of the geometrical quantum degrees of freedom over a global coherent state the same way that the analogue metric is introduced by taking the expectation value of the field on a coherent ground state (see e.g. [178, 179]).

excited part are not treated separately. The creation of quasi-particles, described in terms of the action of number conserving atomic operators, shows how the presence of correlations between condensate atoms and excited atoms is a necessary consequence of the unitarity of the evolution of the full atomic system.

With respect to the mean-field approximation, we have discussed how the number conserving formalism induces corrections on the correlation functions, depending on the number of atoms in the condensate. The many-body states that in the mean-field approximation would have been considered indistinguishable, develop corrections of order N^{-1} that allow them to be differentiated from each other. These corrections become more noticeable when the depletion effects are large and drive the dynamics of the system out of the approximation of linearized perturbations.

The creation of quasi-particles affects the structure of the many-body state due to the correlations between quasi-particles and geometry: the quasi-particle dynamics is linear, but it is described by composite operators, and therefore the dynamics of the atomic modes is non-linear. This implies that, in the atomic Fock space, the quantum states develop entanglement between condensate and excited part, and in any case the sectors of geometry and matter can never preserve a separation even if provided initially.

In general the back-reaction of the quasi-particle on the system becomes increasingly more noticeable when the number of atoms in the condensate does not overwhelm the number of the depleted atoms and the system is not — at least locally — well described as a condensate. In this case the deviations from the mean-field picture become larger, and it is no longer adequate to describe systems in which the quantum nature becomes evident and non-negligible.

In the black hole case a finite region of spacetime is associated to the particle creation, hence N is not only finite but decreases as a consequence of the evaporation making the correlators between geometry and Hawking quanta more and more non-negligible in the limit in which one simulates a black hole at late stages of its evaporation. This implies that tracing over the quantum geometry degrees of freedom, *i.e.* suppressing the quantum nature of the condensate, could lead to non-negligible violation of unitary even for regular black hole geometries (*i.e.* for geometries without inner singularities, see *e.g.* [180–182])

The Bogoliubov quasi-particles in the analogue system have the dual function of establishing correlations and producing depletion. We therefore studied the effects they have on the analogue geometry, it is in fact the back-reaction of the fields on the geometry that allows us to analyze their properties and show their mutual relation. The study of the back-reaction on the condensate wavefunction is analogous to the study of semiclassical gravity, but in the analogue system we can not only test how the effects predicted by quantum field theory in curved spacetime modify the classical metric tensor, but we know exactly the quantum theory from which this description emerges, and how to account for the quantum behavior of the substrate.

In particular, we studied canonical analogue black holes [117], which are interesting examples as they enable us to study the back-reaction of Hawking radiation on a geometry with an acoustic horizon. In the case of the analogue cosmology, it may not be immediately noticeable how the depletion influences the geometry of the condensate wavefunction, since its homogeneity is not broken (in these systems the back-reaction has been related to effects such as analogue dark energy [81]), while the action of the Hawking radiation dynamically influences the geometry, and in particular the horizon. We have therefore studied the effect of the analogue Hawking radiation in the near-horizon region and how to devise a configuration that allows to study the evaporation process.

In this system we have made a complete calculation of the Hawking radiation, showing how the outgoing modes can be easily studied in the near horizon region directly obtaining the thermal distribution of the radiation and the temperature of the canonical analogue black hole (which differs from the case of the Schwarzschild black hole due to the different dependence of the gravitational field on the radius).

We have also shown how the gravitodynamic equations of the analogue metric, for condensates described by the $\lambda\phi^4$ theory, are modified by anomalous terms that can be interpreted as the back-reaction of the analogue quantum field. The condensate wavefunction is thus modified by a perturbation (which is also a classical function) generated by the anomalous terms. The Bogoliubov quasi-particles affect the analogue metric through the vacuum polarization $\langle\theta_1\theta_1\rangle$, appropriately renormalized, which determines the dynamical behavior of the horizon. The mathematical description provided has validity beyond the case of the canonical analogue black hole, and can be employed for other analogue models in Bose–Einstein condensates.

Moreover, the equations we obtained describe the dynamics of the acoustic horizons in the condensates in dynamic regimes. We are interested in the case in which the initial conditions are set so that the perturbation is initially zero, similarly to what is expected to result from a gravitational collapse. In this system, by fixing the initial state of the analogue field as the vacuum of Unruh, we have shown how we can recognize a contribution to the evaporation of the analogous black hole that can be traced back to the effect of the production of Hawking radiation.

We also stress that the equations we obtained describe the acoustic horizons in Bose–Einstein condensates for generic dynamic regimes. We are interested in the case in which the initial conditions are set so that the perturbation is initially zero, analogously to what is expected to result from a gravitational collapse. In this system, by fixing the initial state of the analogue field as the Unruh vacuum, we have shown how we can recognize a contribution to the evaporation of the analogue black hole that can be traced back to back-reaction of Hawking radiation.

We have also studied the feasibility of systems with vorticity [142, 143] as examples allowing to further develop the analogue gravity program, and possible implementations of the Kerr geometry. Most notably by providing it in an analogue gravity (Gordon) form for the slow-rotating and nearly luminal velocity flows.

The study of the back-reaction in these systems allows to study by analogy the gravitational phenomena of evaporation of black holes and superradiance, processes in which geometry exchanges physical quantities (mass and angular momentum) with quantum fields. These physical quantities, together with the electric charge, describe the geometry and thermodynamics of a black hole. Therefore the phenomena of evaporation and superradiance allow to deepen the understanding of the connections between spacetime geometry and quantum physics, which is fundamental for developing a microscopic theory from which relativity can emerge as a classical limit.

The recently obtained experimental evidence of analogue Hawking radiation in Bose–Einstein condensates, makes now mandatory the goal of developing more complex, sound and effectual analogue models able to reproduce the aforementioned effects and their back-reaction on the geometry. Our work also suggests that the research program should focus on further aspects: we expect that from the study of many-point correlation functions it will generally be possible to recognize the deviation from the mean-field theory, and to verify and quantify the transfer of information from the condensate to the excited part of the system during the process of quasi-particle creation; moreover, it is interesting to study the regimes at the limits of validity of the analogue gravity framework, where it would be possible to understand how the full quantum behavior of the condensate begins to become non-negligible and how this can reveal the underlying microscopic quantum structure.

We can then conclude that a deeper understanding of the dynamics of acoustic horizons in analogue gravity in Bose–Einstein condensates, and in particular of the associated phenomenology of depletion, could give — in spite of the different form of the gravitodynamic equations — precious insights towards a deeper understanding of the role of back-reaction in semiclassical gravity in the entanglement between quantum matter and quantum spacetime degrees of freedom. An entanglement which might be key for understanding the compatibility of Hawking radiation with unitarity and hence for understanding black holes as thermodynamic objects.

Analogue gravity suggests that a solution to the information loss problem should be sought in terms of a process of continuous creation of correlations between the quantum degrees of freedom of matter and geometry that have the same physical origin. The means by which the information loss problem is avoided in analogue gravity therefore suggest that a definitive solution to the problem could only come from the theory of quantum gravity and not from an approach of semiclassical gravity, through the description of a unitary evolution in a unified and complete Hilbert space [183].

Furthermore, analogue gravity indicates that in the gravitational setting these quantum effects should largely remain hidden, except in the regimes where there is no possibility of having a consistent classical hydrodynamic description: the purely quantum nature of the underlying unified theory can emerge in a regime in which the geometry and matter cease to have a clear distinction (fact which in the case of Bose–Einstein condensates is quantified in terms of the occupation numbers of the 1-particle states).

That it is possible to establish a bridge between the way in which analogue gravity emerges from atomic theory and the way in which general relativity could be an emergent description, as a classical limit, from a theory of quantum gravity is in itself suggestive and fascinating; and in the absence of a direct experimental approach to quantum gravity, the development of the techniques of analogue gravity establishes itself as a valid research field in the near future to explore this idea.

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