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Generalised instanton counting in two, three and four complex dimensions

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ABSTRACT

In this thesis we study some topics related to instanton counting in (complex) dimension two, three and four. In dimension two, we introduce and study a class of surface defects in 4d $\mathcal{N} = 4$ Vafa-Witten theory on a product $\mathcal{C}_{g,k} \times T^2$ of a k-punctured Riemann surface by the two-torus, supporting BPS solutions which we call nested instantons. Exploiting supersymmetric localisation, we compute explicitly the partition function of the Vafa-Witten theory in presence of a general defect in this class and conjecture a specific polynomial behaviour for some nesting profiles. We also study the geometry of the moduli space of representations of the nested instantons quiver, and we prove that it's a quasi-projective variety isomorphic to the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$.

In dimension three, we extend to higher rank the vertex formalism developed by Maulik-Nekrasov-Okounkov-Pandharipande for Donaldson-Thomas theory. We use the critical structure of the quot scheme of points of $\mathbb{A}^3$ and the induced equivariant perfect obstruction theory to compute K-theoretic and cohomological higher rank DT invariants. We also prove two conjectures proposed by Awata-Kanno and Szabo about the plethystic expression of their generating function. Moreover, we define a chiral version of the virtual elliptic genus, and the corresponding elliptic DT invariants. This definition compares to results from string theory found in Benini-Bonelli-Poggi-Tanzini, and we argue that a conjecture about the partition function of elliptic DT invariants can be interpreted in terms of its behaviour in the modular parameter.

Finally, in dimension four we study solutions to an analogue to the self-duality equations in (real) four dimensions. We construct a Topological Field Theory describing the dynamics of D(-1)/D7 brane systems on Riemannian manifolds with Spin(7)-holonomy. The BPS-bound states counting reproduces the (cohomological) DT theory on four-folds. We also study the local model for eight-dimensional instantons and provide an ADHM-like construction. Finally we generalise the construction to orbifolds of $\mathbb{C}^4$ admitting a crepant resolution, and compute the corresponding partition function, which conjecturally encodes the corresponding orbifold DT invariants.

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INTRODUCTION

The interplay between string theory and geometry has been very fruitful in producing results of interest for both the parts at play. This is particularly true when talking about supersymmetric theories, because of their deep relation with the underlying geometrical objects. In particular, studying the BPS sector of string theories is often of great interest as it consists of quantities which are usually protected from quantum corrections, and which can sometimes be studied non-perturbatively. On the other hand, BPS state counting has been shown in many occasions to have a precise mathematical interpretation rooted in counting/classification problems in geometry. Moreover, the very structure of Quantum Field Theories enjoying supersymmetry makes them particularly suited to addressing this kind of problems. On the first place, one is led to consider supersymmetric (or conformal) QFTs as the traditional perturbative approach is insufficient for probing non-perturbative phenomena or the strong-coupling regime, and a way out is offered by those theories with sufficient symmetries so as to give a better analytical control. Supersymmetric theories come also equipped with a set of peculiar features making them stand out of the crowd of the other QFTs, and making it feasible to actually perform exact computations. One of these peculiarities is supersymmetric localisation, by which in principle one can reduce the problem of computing path integrals to ordinary integrations or even to summations over simple contributions determined only by those field configurations that are protected by the supersymmetry. One can indeed think to supersymmetric localisation as an infinite-dimensional analogue of previously known localisation theorems in mathematics [8; 92], a prototypical example of which being the Duistermaat-Heckman formula, which provides conditions for the saddle-point approximation of an oscillatory integral to be exact. The result of this is that we are sometimes able to evaluate the partition function exactly, thus capturing also all the non-perturbative effects. When taking into account the many existing connections to a plethora of different underlying structures, it becomes clear how supersymmetric gauge theories may be used as a powerful tool to approach several different problems both in physics and in mathematics, *e.g.* computing the spectrum/eigenfunctions of certain (quantum) integrable systems [179–181] or topological invariants of manifolds [4; 44; 45; 85; 135; 222].

One tool which often provides deep insight in the characterisation of both the behaviour of physical theories and the related mathematical structures is offered by the study of defects, *i.e.* embedded submanifolds on which a parabolic reduction of the gauge field is prescribed. Surface defects (*i.e.* codimension two defects) have been widely investigated in many contexts from various different perspectives. The study of their role in the classification of the phases of gauge theories was pioneered by 't Hooft [205]. Surface defects were also introduced by Kronheimer and Mrowka [145; 146] in the study of Donaldson invariants. The correspondence with two-dimensional conformal field theories [3] prompted a systematic analysis of surface defects and highlighted their relevance for quantum integrable systems [2; 174] and for the study of isomonodromic deformations and Painlevé equations [35; 39; 105]. Surface defects preserving half of the supersymmetry in $\mathcal{N} = 4$ four-dimensional theories were introduced in [114], and later works generalised the construction to $\mathcal{N} = 2$ theories, *e.g.* [51; 104; 113]. A simple example of surface defect is that of a $G = SU(N)$ gauge field A on $\mathbb{C}_z \times \mathbb{C}_w$, for which a divergent behaviour is prescribed at $w = 0$, as

$$A_\mu dx^\mu \sim \mathrm{diag}(\underbrace{\alpha_1, \dots, \alpha_1}_{n_1 \text{ times}}, \dots, \underbrace{\alpha_M, \dots, \alpha_M}_{n_M \text{ times}}) i d\theta,$$

thus breaking the gauge symmetry to the Levi subgroup $S[U(n_1) \times \cdots \times U(n_M)] = \mathbb{L} \subset SU(N)$ on the surface defined by $w = 0$. Corresponding to the divergent behaviour of the gauge field dictated by $(\alpha_1, \dots, \alpha_M)$ (together with their multiplicities) we also have a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}_\mathbb{C}$, generated by elements $x \in \mathfrak{g}_\mathbb{C}$ such that $[\alpha, x] = i\lambda x$, where $\lambda \geqslant 0$ and $\alpha \in \mathfrak{k}$ is an element of the Cartan subalgebra $\mathfrak{k} \subset \mathfrak{g}_\mathbb{C}$ such that $[\alpha, \mathbb{L}] = 0$. A surface operator in $\mathcal{N} = 4$ (resp. $\mathcal{N} = 2$) theories can then be understood as coupling the four-dimensional theory with a two-dimensional one on the surface, which might be taken as a non-linear sigma model with target $T^*(G_\mathbb{C}/\mathcal{P})$ (resp. $G_\mathbb{C}/\mathcal{P}$), were $\mathcal{P} \subset G_\mathbb{C}$ is the parabolic subgroup of $G_\mathbb{C}$ corresponding to the parabolic subalgebra $\mathfrak{p}$.

The study of brane dynamics, moreover, has provided over the years many interesting and deep results both in physics and in mathematics. Indeed it is often the case that brane systems are string theoretic realisations of interesting moduli spaces, and the contact with enumerative geometry is made through BPS counting. In principle one might start with a system of k Dp branes and N Dq-branes, with $p < q$, where even (resp. odd) p, q are allowed in type IIA (resp. type IIB) supersymmetric backgrounds. The low energy effective theory on the worldvolume of the Dq-branes is a $U(N)$ $d = q + 1$ supersymmetric Yang-Mills theory with a certain amount of supersymmetry preserved. Open strings can be suspended between Dp-Dp, Dp-Dq and Dq-Dq branes, where in general situations one might neglect the Dq-Dq modes. Adding a B-field along the Dq-branes can be useful to preserve part of the supersymmetry which is broken by the presence of the Dp-branes [225]. The scalars fields of the theory we are left with describe the fluctuations of the Dp-branes in the background of the Dq-branes, and they get combined with their fermionic partners in supersymmetric multiplets in some representation of $U(k)$. Then, from the point of view of the Dp-branes, we have a $(p + 1)$-dimensional Gauged Linear Sigma Model (GLSM, in the following) describing their dynamics. Frequently this GLSM carries a great deal of information about the geometry of the moduli space of vacua to which it flows in the infrared. A standard example of this procedure is the construction of the moduli space of ADHM instantons on $\mathbb{C}^2$ and ALE spaces via D(-1)/D3 brane systems [89–91; 223]. This approach has been successfully applied also to different settings. For example one might start with a type IIA supersymmetric background on $X \times \mathbb{C}^2$, X being a Calabi-Yau threefold, and consider a system D0/D2/D4/D6 branes wrapping holomorphic cycles of X. Compactifying on X, the resulting four-dimensional theory will have $\mathcal{N} = 2$ supersymmetry. Based on the specific brane configuration, various different moduli spaces can emerge, among which we have, for instance, the following two.

- In the case of a single D6-brane and no D4-branes, the partition function of the resulting theory will reproduce the Donaldson-Thomas theory of X. Indeed the low energy theory of a D6 in the presence of D0/D2 branes can be described as a U(1) gauge theory whose partition function localises on the solutions to the following BPS equations [13; 124]

$$
\begin{aligned}
&F_A^{2,0} = 0 = F_A^{0,2}, \\
&F_A^{1,1} \wedge \omega^2 = 0, \\
&d_A \Phi = 0,
\end{aligned}
$$

  where ω is the Kähler form of X, $F_A = dA$ is the curvature two-form and d_A is the covariant derivative acting on the complex scalar field Φ. The corresponding (Gieseker compactification of the) instanton moduli space is then identified with the moduli space of ideal sheaves on X, and it is stratified in components $\mathrm{Hilb}_{n,\beta}(X)$ given by the Hilbert scheme of points and curves of threefold X.

- If we turn off the D6-branes, and we consider instead N D4-branes wrapping a four-cycle $C \subset X$, the resulting gauge theory on the worldvolume of the D4 will be the four-dimensional $\mathcal{N} = 4$ $U(N)$ Vafa-Witten theory on C, [29; 212], with a topological twist needed in order to have the covariantly constant spinors necessary to define supersymmetry on curved spaces. In this case, the gauge theory localises on the solutions to the following equations

$$
\begin{aligned}
&F^+ + [B^+, B^+] + [B^+, \Phi] = 0, \\
&d_A \Phi + d_A^* B^+ = 0,
\end{aligned}
$$

  where F_A^+ is the self-dual part of the curvature F_A for the gauge connection A on a principal $U(N)$-bundle $P \to C$, $B^+ \in \Omega^{2,+}(C, \mathrm{ad}\, P)$ is a self-dual two-form, $\Phi \in \Omega^0(C, \mathrm{ad}\, P)$ a scalar field and $[B^+, B^+]_{ij} = [B_{ik}, B_{jl}]g^{kl}$, g being the metric of C.

Equivalent constructions can of course be given also in type IIB backgrounds, where, for instance, Donaldson-Thomas theory is reproduced by a system of D(-1)/D1/D5 branes. If there are no D1-branes, the effective theory on the D(-1)-branes, which in this case are 0-dimensional objects, is a quiver matrix model, which encodes the critical structure of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, where r and n are the number of D5 and D(-1) branes, respectively. This D-brane construction can be suitably generalised to get K-theoretic and elliptic versions of DT theory. One can study, for instance, a brane system of n D1-branes in the background of r D7-branes on the product of X by an elliptic curve, say $X \times T^2$, which provides the elliptic version of higher rank DT invariants, [22]. Generalisations of DT theory to higher dimension are also possible. Their study was initiated, on the mathematical side, by Donaldson-Thomas [88] and, on the physics side, they can be interpreted as the description of the brane dynamics of a D(-1)/D7 system [1; 33], stabilised by a non trivial B-field background [225]. One natural extension, as it was the case with threefolds, is to consider D0/D8/$\overline{\text{D8}}$ systems, where the D0 quantum mechanics realises the K-theory lift of DT theory on four-folds. Observables from the chiral ring can be included via descent equations, as in the four-dimensional case, [13], and explicit combinatorial computations have recently appeared in [173; 177].

Similarly to the DT case, many other problems from differential/enumerative geometry can be addressed by studying brane dynamics. For instance, partition functions of GLSMs on the two-sphere S^2 [23] are useful sources of information about moduli spaces of genus-zero pseudo-holomorphic maps to the target space, and are known to offer a convenient tool to extract Gromov-Witten invariants [135]. In this context, exact spherical partition functions of D1/D5 brane systems have been computed in [44; 45]. These particular models at low-energy are described by GLSMs with target Nakajima quiver varieties, and the spherical partition function links together their quantum cohomology with higher rank Donaldson-Thomas theory and certain integrable systems of hydrodynamic type.

In this thesis we are concerned, broadly speaking, with some topics in instanton counting, and with their sheaf theoretic counterpart. On one side we will study moduli spaces of generalised instantons supported by surface defects in Vafa-Witten theory. They are in turn engineered by specific systems of D-branes in type IIB supersymmetric backgrounds. On the other side, we will be interested in computing virtual invariants of moduli spaces of instantons, *i.e.* higher-rank Donaldson-Thomas invariants and some of their refinements, in three and four complex dimensions. The common thread shared by the different topics covered in this thesis is the technique of localisation, both from the mathematical perspective (in the forms of equivariant and virtual localisation) and from the physical one (in the form of supersymmetric localisation). Here is an outline of the content of this work.

THE FIRST TWO CHAPTERS are devoted to reviewing the main computational tools used in this thesis, namely equivariant and supersymmetric localisation. We start in §1.1 by reviewing briefly the construction of equivariant cohomology, which leads to the statement of equivariant localisation in §1.2. In §1.3 and §1.5 we give generalisations of the Atiyah-Bott localisation theorem (cf. Thm. (1.6)) to K-theory (Thm. 1.9) and the virtual setting (Thm. 1.12-1.13). In §2.3, instead, we review the localisation arguments commonly used in supersymmetric quantum field theories. As a geometric motivation to the computation of partition functions of supersymmetric theories we also recall in §2.1 the definition of the Witten index and its relation to topology of target spaces. This is generalised in §2.2 to refinements of the Witten index in connection to the χ_y-genus and elliptic genus of target spaces.

THE THIRD CHAPTER is the content of a joint work with Giulio Bonelli and Alessandro Tanzini, [36], and is concerned with the study new classes of surface defects in supersymmetric gauge theories. These are engineered as D7/D3 brane systems on local four-folds, and support BPS solutions which we call nested instantons and which we will describe mathematically in Chap. 4. The precise brane construction is carried out in §3.1, where it is also showed that a certain limit of the effective theory of the D3-branes is described by a quiver GLSM. In §3.2 we compute the complete partition function of the quiver GLSM above, which in some cases leads us to conjecture a certain polynomial behaviour in the quantum mechanical limit, studied in §3.2.4.

IN THE FOURTH CHAPTER we study the moduli space of representation of the nested instantons quiver underlying the constructions in chapter 3. The construction of the moduli space of representation of these quiver is carried out in §4.1 (Thm. 4.8), while in §4.2 we prove a scheme-theoretic isomorphism with a (suitably defined) moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$ (Thm. 4.15). Finally, in §4.3 we compute virtual K-theoretic invariants of flags of framed torsion-free sheaves on $\mathbb{P}^2$ using the virtual localisation theorem stated in §1.5. The material of this chapter is essentially contained in [37].

THE FIFTH CHAPTER is part of a joint work with Sergej Monavari and Andrea T. Ricolfi, [96], and it is devoted to developing a higher-rank generalisation of the equivariant vertex formalism for Donaldson-Thomas theory of points on the affine space $\mathbb{A}^3$. In §5.1 the local model of the quot scheme of points $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ is introduced, and it is proved that it is a critical locus (cf. Prop. 5.3) endowed with an induced equivariant perfect obstruction theory (cf. Lemma 5.10). This enables us to define the higher-rank vertex in §5.3 and to compute in §5.4 the K-theoretic DT partition function, which is expressed in a simple way as a plethystic exponential as in Thm. 5.17. In §5.5 we reduce the K-theoretic DT partition function to its cohomological counterpart, while §5.6 is devoted to defining a chiral version of the elliptic genus (Def. 5.27) and correspondingly an elliptic version of the DT generating function (Def. 5.29).

IN CHAPTER 6, which is the content of a joint work with Giulio Bonelli, Alessandro Tanzini and Yegor Zenkevich [38], we study the moduli space of solutions to the 8d version of the self-duality equations. In §6.1 we develop an equivariant Topological Field Theory on $\mathrm{Spin}(7)$ eight-manifolds, whose low-energy theory describes a D(-1)/D7 brane system. The BPS bound states counting reproduces the Donaldson-Thomas theory on four-folds. In §6.2 we describe an ADHM-like construction, which provides solutions to $\mathrm{SU}(4)$-invariant first order Yang-Mills equations in eight dimensions which generalise the self-duality equations in four dimensions. In §6.2.6 and §6.2.7 we explicitly solve the ADHM-like equations in the abelian case. Finally, in §6.3 we generalise the ADHM construction to orbifolds of $\mathbb{C}^4$ by the action of a finite subgroup G of $\mathrm{SU}(4)$ such that $\mathbb{C}^4/G$ admits a crepant resolution. The particular case of $\mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 2}$, resolving the orbifold singularity of $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$, is studied in §6.3.3 as an example.

1 LOCALISATION THEOREMS

Among the most powerful techniques we can exploit for getting information about geometric objects, we can certainly include localisation theorems. In a nutshell, they imply that much of the data characterizing certain spaces endowed with suitable group actions is encoded just in their fixed locus. The relevance of localisation results becomes even more manifest when taking into account the role they play also in theoretical physics, and in particular in string theory. In their simplest form, the algebraic statements about equivariant localisation can often be reduced to various previously known results in differential and symplectic geometry. A prototypical example of these is the Duistermaat-Heckman theorem [92], concerning the moment map $\mu : M \to \mathbb{R}^g$ for a symplectic manifold M equipped with the action of a torus T^g. On one side this theorem asserts that the push-forward under μ of the symplectic measure is a piece-wise polynomial measure on $\mathbb{R}^g$. On the other side, one can equivalently state the Duistermaat-Heckman theorem as an integration formula depending only on the fixed locus M^{T^g} under the action of T^g for the Fourier transform of the Liouville measure. In the case $T^g \cong S^1$ and M^{S^1} consists only of finite points, one has

$$\int_M e^{-it\mu}\frac{\omega^n}{n!} = \sum_{p\in M^{S^1}} \frac{e^{-it\mu(p)}}{(it)^n e(p)}, \tag{1.0.1}$$

where ω is the symplectic form on M and $e(p)$ are integers associated to the infinitesimal action of S^1 in a neighbourhood of p. In this framework, localisation is a statement about the exactness of the stationary phase approximation of the integral in (1.0.1), whenever the symplectic manifold M is equipped with a Hamiltonian circle action. This, as we will see, is indeed a general feature, as localisation theorems often enable one to reduce a computation to the fixed set of a certain group action. Many analogous results are moreover available in the context of equivariant Chow groups and equivariant K-theory, see, for instance, the Atiyah-Singer index theorem [10], which actually motivated the formulation of equivariant K-theoretic localisation. In this chapter we review the localisation results we will use the most in the rest of this thesis, starting from the simple framework of equivariant cohomology and the Atiyah-Bott theorem, following [8; 102; 189]. Later we will also state the K-theoretic and virtual versions of the localisation theorem, in §1.3 and §1.5 respectively. As an example we show in §1.4 how to compute the Hirzebruch χ_y-genera of the Hilbert scheme of points of smooth projective surfaces.

1.1 EQUIVARIANT COHOMOLOGY

The definition of the equivariant cohomology is made up in such a way to capture the topological data of a space equipped with the action of a group while still enjoying the usual functorial properties of ordinary cohomology. In a sense one can understand equivariant cohomology as an extension of ordinary cohomology, as its definition makes immediately clear.

Let then M be a topological space equipped with the action of a topological group G. If M is a smooth manifold and the action of G on M is free, we want the definition of equivariant cohomology to be $H^\bullet_G(M) = H^\bullet(M/G)$. If instead the action of G has fixed points on M, $H^\bullet_G(M)$ should capture the stabilizers of the fixed points. For instance, if $M = \{pt\}$, any G-action stabilizes the only point of M. Moreover M/G is also always a point, but the equivariant cohomology should keep track of the action of the group, and, as we want it to enjoy functorial properties, we should have pull-back maps to the equivariant cohomology $H^\bullet_H(M)$, for any subgroup $H \subset G$. The main idea here is to realize that there exist a contractible space EG on which G acts freely, and to define $H^\bullet_G(pt)$ to be the ordinary cohomology of the classifying space of G, *i.e.* $H^\bullet(EG/G) = H^\bullet(BG)$. In this case $H^\bullet_G(pt)$ is the so-called group cohomology, and it is also denoted $H^\bullet_G$. The existence of such a classifying space was proved by Milnor [160]. In particular, given any topological group G, there exists a universal G-principal bundle $EG \to BG$, such that any other G-principal bundle $P \to B$ is obtained as a pull-back from the former via the morphism $B \to BG$ in the homotopy category. The fact that any principal G-bundle $P \to B$ can be obtained from the classifying space via pull-back can be seen as follows. We have the following commuting diagram

$$\begin{tikzcd}
P \arrow[d] & P \times EG \arrow[l] \arrow[r] \arrow[d] & EG \arrow[d] \\
B & P_G \arrow[l, "p"'] \arrow[r] & BG,
\end{tikzcd}$$

and as p is a fibration with contractible fibre EG, B and P_G have the same homotopy type, while all the sections of p are fibre-wise homotopic. The composition of one of such sections with the projection $P_G \to BG$ produces a map $f : B \to BG$ such that $P \cong f^*(EG)$, which is moreover unique up to homotopy.

Let then M be a topological space equipped with a left G-action, or, for the sake of brevity, a G-space. We use the balanced product to form the space $M_G = EG \times^G M = (EG \times M)/G$, where we take the equivalence relation $(e \cdot g, p) \sim (e, g \cdot p)$. $EG \times M$ is homotopy equivalent to M, as EG is contractible. Moreover if M is a smooth manifold, the quotient $(EG \times M)/G$ is well defined also in the smooth category, as the action of G on $EG \times M$ is free.

Definition 1.1. *We define the G-equivariant cohomology of the G-space M to be the ordinary cohomology of M_G:*

$$H^\bullet_G(M) = H^\bullet(M_G).$$

Remark 1.1. The equivariant cohomology $H^\bullet_G(M)$ comes naturally equipped with the structure of a $H^\bullet_G$-module. Indeed $M_G \xrightarrow{p} BG$ is a bundle with fibre M, induced by the G-equivariant projection $EG \times M \to EG$. The module structure is then induced by the standard pull-back

$$p^* : H^\bullet_G \to H^\bullet_G(M).$$

In general, however, it is not true that $H^\bullet_G(M) \cong H^\bullet_G \otimes H^\bullet(M)$. While it is certainly true whenever the action of G is trivial, there are nevertheless examples of G-spaces, called equivariantly formal, whose equivariant cohomology is isomorphic to the tensor product of the ordinary cohomology by the group cohomology despite the action being non trivial (cf. Exampe 1.4). ◂

Remark 1.2. The inclusion $\iota : M \hookrightarrow M_G$ of M in M_G as the fibre over the basepoint of BG induces a homomorphism

$$\iota^* : H^\bullet_G(M) \to H^\bullet(M). \qquad ◂$$

Remark 1.3. Whenever $M = \{pt\}$ we recover the definition of group cohomology $H^\bullet_G(pt) = H^\bullet(pt_G) = H^\bullet(EG/G) = H^\bullet(BG)$. If the action of G on M is free, then $H^\bullet_G(M) \cong H^\bullet(M/G)$. Indeed we have a diagram

$$\begin{tikzcd}
EG \arrow[d] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\
BG & M_G \arrow[l] \arrow[r, "\sigma"] & M/G
\end{tikzcd}$$

where σ is not a fibering in general, as the fiber $\sigma^{-1}(Gp)$ over the orbit Gp is the quotient $EG/G_p \cong BG_p$. However, under reasonable assumptions (for example if G is a compact Lie group), σ induces a homotopy equivalence when the action of G is free, whence the assertion follows. ◂

An important point is that, by definition, a G-space may have non vanishing equivariant cohomology in degree higher than its dimension, as it is the case for instance for $H^\bullet_G = H^\bullet_G(pt)$. Let us also point out that though EG is only defined up to homotopy equivalence, however the construction of $H^\bullet_G(M)$ does not depend on the particular choice of classifying space $EG \to BG$.

Example 1.2. *If $G = \mathbb{R}$, then the classifying space can be chosen to be $\mathbb{R} \to pt$. If instead $G = \mathbb{Z}$, a good choice for the classifying space is $\mathbb{R} \to S^1$, mapping $x \mapsto e^{i\pi x}$.*

Example 1.3. *If $G = (\mathbb{C}^*)^n$, we may take the universal principal bundle over the classifying space as $(\mathbb{C}^\infty \setminus 0)^n \to (\mathbb{P}^\infty)^n$. If instead $G = (S^1)^n$, we can take in an analogous fashion $(S^\infty)^n \to (\mathbb{P}^\infty)^n$. At the level of cohomology we have that $H^\bullet(B(\mathbb{C}^*)^n) = H^\bullet((\mathbb{P}^\infty)^n) = H^\bullet_{(\mathbb{C}^*)^n} \cong \mathbb{Z}[t_1, \dots, t_n]$, where $t_i = c_1\left(\pi_i^* \mathscr{O}(-1)\right)$, and $\pi_i : (\mathbb{P}^\infty)^n \to \mathbb{P}^\infty$ is the projection onto the i-th factor.*

Example 1.4. *The 2-sphere $S^2 \subset \mathbb{R}^3$ with a circle action rotating S^2 around the z-axis is equivariantly formal, i.e. $H^\bullet_{S^1}(S^2) \cong H^\bullet_{S^1} \otimes H^\bullet(S^2)$. Indeed let $D_{1,2} \subset S^2$ the two disks depicted in Fig. 1. As both D_1 and D_2 are contractible (and S^1-invariant) $H^\bullet_{S^1}(D_1) \cong H^\bullet_{S^1}(D_2) \cong H^\bullet_{S^1} \cong \mathbb{Z}[t]$. Moreover $H^\bullet_{S^1} \cong \mathbb{Z}[t]$ is one-dimensional (resp. zero-dimensional) in even degree (resp. odd degree), while S^1*

acts freely on $D_1 \cap D_2$, *whence* $H^\bullet_{S^1}(D_1 \cap D_2) \cong H^\bullet((D_1 \cap D_2)/S^1) \cong H^\bullet(\mathrm{pt})$. *We then have the Mayer-Vietoris sequence in cohomology*

$$\cdots \longrightarrow H^{*-1}(\mathrm{pt}) \longrightarrow H^*_{S^1}(S^2) \longrightarrow H^*_{S^1} \oplus H^*_{S^1} \longrightarrow H^*(\mathrm{pt}) \longrightarrow \cdots,$$

and by comparing dimensions we conclude that S^2 *is* S^1*-equivariantly formal.*

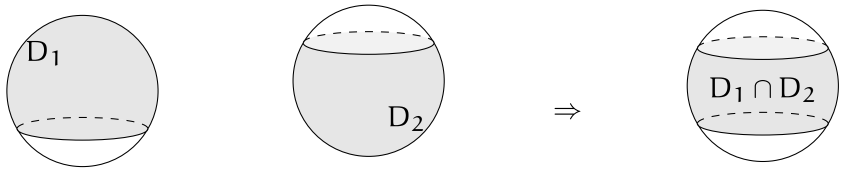


Figure 1: *Suitable covering of the 2-sphere for the Mayer-Vietoris sequence.*

Equivariant cohomology also enjoys functorial properties similarly to standard cohomology. For instance, if M is a G-space and N is a H-space, and we have maps $\varphi : G \to H$, $f : M \to N$, such that $f(g \cdot x) = \varphi(g) \cdot f(x)$, we are able to define a natural pull-back homomorphism $f^* : H^\bullet_H(N) \to H^\bullet_G(M)$. Indeed, we can construct a map $M_G \to N_H$, and taking its cohomology we naturally get the desired homomorphism f^*. The case in which both M and N are G-spaces and f is G-equivariant is particularly interesting, as it addresses G-equivariant vector bundles $E \to M$. Indeed we have the following diagram

$$\begin{tikzcd} E_G \arrow[d] & G \times E \arrow[l] \arrow[r] \arrow[d] & E \arrow[d] \\ M_G & G \times M \arrow[l] \arrow[r] & M, \end{tikzcd}$$

by which we have the vector bundle $V_E = E_G \to M_G$. This also enables us to define equivariant characteristic classes for equivariant vector bundles. For instance, G-equivariant Chern classes of E are defined to be

$$c_i^G(E) = c_i(V_E) \in H^{2i}_G(M).$$

Example 1.5. *A* G*-equivariant vector bundle over a point is simply a representation* $\rho : G \to GL(E)$, *which is a character* $\chi : G \to \mathbb{C}^*$ *in the case of a line bundle* $L \to \mathrm{pt}$. *Its weight is defined to be* $w_\chi = c_1^G(L) = c_1(V_\chi)$.

Although a push-forward map in equivariant cohomology $f_* : H^\bullet_G(M) \to H^\bullet_G(N)$ induced by a G-equivariant map $f : M \to N$ is more difficult to construct, it is still available whenever M and N are compact. In this case we cannot simply mimic the construction of the Gysin map in ordinary cohomology, as it relies on Poincaré duality and in general M_G and N_G can be infinite dimensional spaces. However one can use so-called approximation spaces to get around the issue and properly define the equivariant push-forward homomorphism. These form a system of smooth principal G-bundles $E_i \to B_i$ whose colimit recovers the universal principal bundle $EG \to BG$. Moreover if $M^i_G = E_i \times^G M$ and $N^i_G = E_i \times^G N$, one has that $H^p_G(M) \cong H^p(M^i_G)$ and $H^p_G(N) \cong H^p(N^i_G)$ for $i \geqslant p$. By using Poincaré duality at the level of approximation spaces it is then possible to define homomorphisms $f_*^p : H_G^p(M) \to H_G^{p-q}(N)$, where $q = \dim M - \dim N$.

$$\begin{tikzcd}
H^p(M_G^i) \arrow[r, "f_*^p"] \arrow[d, "\mathrm{PD}"'] & H^{p-q}(N_G^i) \\
H_{\dim M_G^i - p}(M_G^i) \arrow[r] & H_{\dim M_G^i - p}(N_G^i) \arrow[u, "\mathrm{PD}^{-1}"']
\end{tikzcd}$$

1.2 EQUIVARIANT LOCALISATION

Let $\iota : N \to M$ a closed embedding of compact manifolds of codimension d. In the context of ordinary cohomology we have an identity, which goes under the name of excess intersection formula

$$\iota^*\iota_*\mathbb{1} = e(N_{N/M}) \in H^d(X).$$

A similar formula is available also in the equivariant setting

$$\iota_G^*\iota_*^G\mathbb{1} = e_G(N_{N/M}),$$

as the normal bundle of the quotient $N_G \hookrightarrow M_G$ can be identified with the quotient $(N_{N/M})_G$ of the normal bundle $N_{N\times EG/M\times EG}$.

Suppose T is a torus and X is a smooth T-variety, so that its fixed locus $X^T \subset X$ is smooth. Let then ι be the inclusion $X^T \hookrightarrow X$ inducing the pushforward

$$\iota_* : H_T^\bullet(X^T) \to H_T^\bullet(X).$$

The content of the localisation theorem is essentially that that this homomorphism becomes an isomorphims after inverting a finite number of non-trivial characters. Indeed, let F_α be a component of the fixed locus X^T, and let $\iota_\alpha : F_\alpha \hookrightarrow X$ with normal bundle $N_\alpha = N_{F_\alpha/X}$. By virtue of the equivariant version of the excess intersection formula we have $\iota_\alpha^*\iota_{\alpha*}(-) = e_T(N_\alpha) \cap (-)$, and the equivariant Euler class $e_T(N_\alpha)$ is non vanishing whenever restricted to points of F_α. Indeed the T-action on $N_{x,\alpha} = T_xM/T_xF_\alpha$ is non-trivial, as $(T_xM)^T = T_xF_\alpha$, whence the Euler class of the normal bundle is non vanishing, being the product of non zero weights. Moreover, one can show that given a T-equivariant vector bundle $E \to M$, its restriction to F_α gets decomposed in T-characters $\mu : T \to \mathbb{C}^*$ as $E|_{F_\alpha} = \bigoplus E_{F,\mu}$. The equivariant Chern class $c_i^T(E_{F,\mu})$ is invertible in $H_T^{2i}(F)[\mu^{-1}]$. Thus, if F_α is a component of M^T of co-dimension d we, the Euler class $e_T(N_{F/M}) \in H_T^{2d}(F)$ is invertible in $H_T^\bullet(F)[\mu_1^{-1}, \ldots, \mu_s^{-1}]$, where $\mu_1, \ldots, \mu_s$ are the characters appearing in the decomposition of $N_{F/M}$. For convenience, in the following we will denote by $\mathcal{H}_T^\bullet = \operatorname{Frac} H_T^\bullet$ the field of fractions of $H_T^\bullet$.

Theorem 1.6 (Atiyah-Bott, [8]). *Let M be a compact smooth manifold equipped with the action of a torus T. The pushforward of the inclusion $\iota : M^T \hookrightarrow M$ gives an isomorphism*

$$\iota_* : H_T^\bullet(M^T) \otimes_{H_T^\bullet} \mathcal{H}_T^\bullet \xrightarrow{\cong} H_T^\bullet(M) \otimes_{H_T^\bullet} \mathcal{H}_T^\bullet.$$

The inverse of the isomorphism in the Atiyah-Bott theorem is given by

$$\psi \mapsto \sum_{\alpha} \frac{\iota_\alpha^* \psi}{e_T(N_\alpha)},$$

where the sum runs over the components of the fixed locus $M^T \subset M$. Rephrasing differently we might say that each element ψ in $H_T^\bullet(M) \otimes_{H_T^\bullet} \mathcal{H}_T^\bullet$ has a unique presentation as

$$\psi = \sum_{\alpha} \iota_{\alpha *} \frac{\iota_\alpha^* \psi}{e_T(N_\alpha)}.$$

Pushing forward the natural projection $\pi : M \to \mathrm{pt}$ we get the equivariant homomorphism $\pi_* : H_T^\bullet(M) \to H_T^\bullet$, which corresponds to integration along the fibers of $M_G \to BG$, over the base point of BG. The structure map $\pi_\alpha : F_\alpha \to \mathrm{pt}$ on any component of the fixed locus factors as $\pi_\alpha = \pi \circ \iota_\alpha$, and $\pi_{\alpha *} = \pi_* \circ \iota_{\alpha *}$. This finally leaves us with the following

Corollary 1.7 (Integration formula)**.** *Given* $\psi \in H_T^\bullet(M) \otimes_{H_T^\bullet} \mathcal{H}_T^\bullet$, *we have*

$$\int_M \psi = \pi_* \psi = \sum_\alpha \pi_{\alpha *} \frac{\iota_\alpha^* \psi}{e_T(N_\alpha)} = \sum_\alpha \int_{F_\alpha} \frac{\iota_\alpha^* \psi}{e_T(N_\alpha)}.$$

Part of the interest in the localisation formula lies in the fact that we can often trade an ordinary integral for an equivariant one, and apply localisation thereafter. Indeed there is a Cartesian square inducing a commutative diagram in cohomology,

$$\begin{array}{ccc} X & \xrightarrow{p} & \mathrm{pt} \\ \downarrow \iota & & \downarrow b \\ X_G & \xrightarrow{\pi} & BG \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} H^\bullet(X) & \xrightarrow{p_*} & \mathbb{Z} \\ \iota^* \uparrow & & b^* \uparrow \\ H_G^\bullet(X) & \xrightarrow{\pi_*} & H_G^\bullet \end{array}$$

which implies that given a cohomology class ψ, its pushforward $p_* \alpha$ can be computed, thanks to the commutativity of the diagram, as $b^* \pi_* \Psi$, where Ψ is an equivariant lift of ψ, *i.e.* $\iota^* \Psi = \psi$.

Example 1.8. *Let* M *be a smooth oriented compact manifold equipped with the action of a torus. Suppose that the fixed locus is comprised only of isolated points* $M^T = \{p_1, \dots, p_m\}$*. Then* $\chi(M) = m$*. Indeed*

$$\chi(M) = \int_M e(M) = \int_M e_T(TM) = \sum_{i=1}^m \frac{e_T(TM)|_{p_i}}{e_T(N_{p_i/M})} = \sum_{i=1}^m 1 = m.$$

1.3 A BRIEF K-THEORETIC INTERMEZZO

Localisation results analogous to those we examined in the context of equivariant cohomology are also available for equivariant K-theory. As this is often the correct framework to work in when comparing to string theory, we recall them here, following [48; 184; 210].

Let then X be a scheme acted upon by a reductive grouop G. By $K_0^G(X)$ (resp. $K_G^0(X)$) we denote the K-group of the category of G-equivariant coherent sheaves on X (resp. locally-free G-equivariant sheaves). The K-group of locally-free sheaves is equipped with a natural homomorphism $K_G^0(X) \to K_0^G(X)$, which is an isomorphism whenever X is non-singular. If X is a point the equivariant K-groups are identified with the representation ring, or character ring, of G, *i.e.* $K_0^G(\mathrm{pt}) \cong K_G^0(\mathrm{pt}) \cong R(G)$. Similarly to what happens in the case of equivariant cohomology, in general $K_G^0(X)$ is an algebra over $R(G)$, and the concept of equivariant formality can be extended to the context of K-theory as the case in which the natural map $K_G^0(X) \to K_G^0(\mathrm{pt}) \cong R(G)$ is surjective.

Push-forwards and pull-backs can be defined for equivariant morphisms of schemes $f : X \to Y$, under suitable assumptions. For example, if f is a flat morphism one can define the pull-back $f^* : K_0^G(Y) \to K_0^G(X)$ from the sheaf-theoretic inverse image functor. Indeed, to any G-equivariant coherent sheaf $\mathscr{F}$ on Y we can associate the coherent sheaf $f^*\mathscr{F}$ on X as

$$\mathscr{F} \mapsto f^*\mathscr{F} = \mathscr{O}_X \otimes_{f^*\mathscr{O}_Y} f^{-1}\mathscr{F}.$$

As $f^* : \mathrm{Coh}_G(Y) \to \mathrm{Coh}_G(X)$ is an exact functor it descends to equivariant K-theory. In the case of vector bundles $f^* : K_G^0(Y) \to K_G^0(X)$ can be simply defined by pulling them back. If $f : X \to Y$ is instead a proper morphism, we have the direct image functor $f_* : \mathrm{Coh}_G(X) \to \mathrm{Coh}_G(Y)$, which in general is not right exact. Given $[\mathscr{F}] \in K_0^G(X)$ we may then define the push-forward as

$$f_*[\mathscr{F}] = \sum (-1)^i [R^i f_* \mathscr{F}] \in K_0^G(Y).$$

Any short exact sequence of coherent sheaves $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ induces a long exact sequence

$$0 \longrightarrow f_*\mathscr{E} \longrightarrow f_*\mathscr{F} \longrightarrow f_*\mathscr{G} \longrightarrow R^1 f_*\mathscr{E} \longrightarrow R^1 f_*\mathscr{F} \longrightarrow R^1 f_*\mathscr{G} \longrightarrow \cdots$$

so we have that

$$f_*([\mathscr{F}] - [\mathscr{E}] - [\mathscr{G}]) = f_*[\mathscr{F}] - f_*[\mathscr{E}] - f_*[\mathscr{G}]$$

and f_* descends to a well defined morphism in K-theory. In the particular case f is the structure morphism $f : X \to \mathrm{pt}$, the direct image functor is really the global sections functor. Thus, given $[\mathscr{F}] \in K_0^G(X)$, we have

$$f_*[\mathscr{F}] = \sum (-1)^i [H^i(X, \mathscr{F})] \in R(G),$$

which coincides with the definition of the equivariant Euler characteristic in K-theory $\chi_G(X, -) : K_G^0(X) \to K_G^0(\mathrm{pt}) \cong R(G)$.

As the case of G being an algebraic torus will be the only one we will be interested in, from now on we will focus mainly on $G = T = (\mathbb{C}^*)^g$. The representation ring $R(T)$ is naturally the K-group of the category of T-modules. If V is a finite-dimensional T-representation, it will decompose naturally as the direct sum of one-dimensional T-modules, called the *weights* of V. Each weight of the T-module V is then associated to a character $\mu \in$

$\hat{T} = \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^g$. Moreover, we can associate each character μ to a monomial $t^\mu = t_1^{\mu_1} \cdots t_g^{\mu_g}$ in the coordinates $(t_1, \ldots, t_g)$ of T. There is a trace map

$$\mathrm{tr}_V(t) = \sum t^\mu$$

sending the class of a T-module to its decomposition into weight spaces, which provides an isomorphism

$$\mathrm{tr} : K_0^T(\mathrm{pt}) \xrightarrow{\cong} \mathbb{Z}\left[t^\mu \middle| \mu \in \hat{T}\right]. \tag{1.3.1}$$

Let us also notice that traces of finite dimensional T-modules can also be computed in the *localised K-theory* $K^0_{T,\mathrm{loc}}(\mathrm{pt})$, where some (or all) the elements $1 - t^\mu \in K_0^T(\mathrm{pt})$ are inverted

$$\mathbb{Z}[t^\mu | \mu \in \hat{T}] = K_0^T(\mathrm{pt}) \hookrightarrow K^0_{T,\mathrm{loc}}(\mathrm{pt}) = \mathbb{Z}\left[t^\mu, \frac{1}{1-t^\nu} \middle| \mu \in \hat{T}\right].$$

We can now turn to the formulation of the localisation theorem in equivariant K-theory. It is convenient to define the total antisymmetric algebra $\Lambda_t^\bullet(E)$ and the total symmetric algebra $\mathrm{Sym}_t^\bullet(E)$ of a vector bundle E over a scheme X, as

$$\Lambda_t^\bullet(E) = \sum_{i \geqslant 0} t^i [\Lambda^i E] \in K^0(X)[t], \qquad \mathrm{Sym}_t^\bullet(E) = \sum_{i \geqslant 0} t^i [\mathrm{Sym}^i E] \in K^0(X)[\![t]\!].$$

This definition can naturally be extended to G-equivariant K-theory, *i.e.* to any $E \in K^0_G(X)$. We now take a T-scheme X with fixed locus X^T, whose natural inclusion in X we denote by $\iota : X^T \hookrightarrow X$. As T is reductive, if X is smooth, so is $X^T \subset X$. Given any $\mathscr{F} \in K_0^T(X^T)$, we can state a result analogous to the excess intersection formula, which reads

$$\iota^* \iota_* \mathscr{F} = \mathscr{F} \otimes \Lambda^\bullet_{-1} N^\vee_{X^T/X}, \tag{1.3.2}$$

where $N_{X^T/X}$ is the normal bundle to the T-fixed locus. The localisation theorem in K-theory then states that the push-forward map ι_* in Eq. (1.3.2) becomes an isomorphism after inverting a finite number of terms of the form $(1 - t^\mu)$.

Theorem 1.9 ([210], Théorème 2.1/2.2)**.** *Let X be a scheme equipped with the action of a torus T. Let moreover X^T be the T-fixed locus in X and $\iota : X^T \hookrightarrow X$ the natural inclusion. The push-forward $\iota_* : K_0^T(X^T) \to K_0^T(X)$ is injective. Moreover there is an isomorphism*

$$\iota_* : S^{-1} K_0^T(X^T) \xrightarrow{\cong} S^{-1} K_0^T(X),$$

where S is the multiplicative set in $R(T)$ generated by elements of the form $1 - t^\mu$, with non-zero μ.

As we saw for the case of equivariant cohomology, the localisation theorem in K-theory enables us to compute push-forwards by only looking at the fixed locus of the torus action. Thus, for any T-equivariant vector bundle

V over a proper scheme X, we may compute the equivariant push-forward $\chi_T(X, V)$ as

$$\chi_T(X,V) = \chi_T\left(X^T, \frac{V|_{X^T}}{\Lambda^\bullet_{-1} N^\vee_{X^T/X}}\right) \in K^0_T(\mathrm{pt})\left[\frac{1}{1-t^\mu}\middle|\, \mu \in \hat{T}\right]. \qquad (1.3.3)$$

Remark 1.4. It may happen that sometimes we want to compute K-theoretic invariants for schemes which are not proper, say for quasi-projective varieties X. Though push-forwards in this cases are not well defined, we may nevertheless exploit the localisation theorem to *define* invariants to be the r.h.s. of Eq. (1.3.3), provided that the fixed locus X^T is proper. The same observation holds true also in the case of equivariant cohomology. ◀

1.4 AN EXAMPLE: $\chi_{-y}\left(S^{[n]}\right)$

If X is a proper scheme, we define the Hizebruch χ_{-y}-genus of X as

$$\chi_{-y}(X) = \chi(X, \Lambda^\bullet_{-y} T^\vee_X).$$

In this section, we want to compute the Hirzebruch χ_{-y}-genus of the Hilbert scheme of points of a projective surface S by exploiting the technique of equivariant localisation, along the lines of [153]. To this end, let Hilb(S) be defined as

$$\mathrm{Hilb}(S) = \coprod_{n \geqslant 0} S^{[n]} p^n.$$

We will exploit equivariant localisation in order to prove the notorious formula

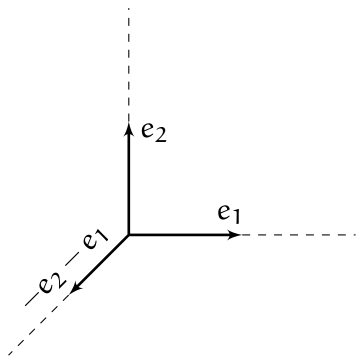
$$\chi_{-y}\left(\mathrm{Hilb}(S)\right) = \exp\left(\sum_{m>1} \frac{p^m}{m} \frac{\chi_{-y^m}(S)}{1-(yp)^m}\right), \qquad (1.4.1)$$

which was proved differently by Göttsche-Soergel [111], Cheah [67] and Ellingsrud-Göttsche-Lehn [94].

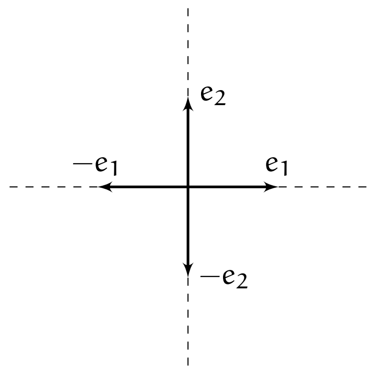
The key fact here is that any complex genus of Hilb(S) only depends on the cobordism class [S] in the complex cobordism ring with rational coefficients $\Omega = \Omega^U \otimes \mathbb{Q}$, [94]. Moreover, as Ω is the polynomial ring freely generated by the cobordism classes $[\mathbb{P}^i]$, it is sufficient to study the Hirzebruch genera of $\mathrm{Hilb}(\mathbb{P}^2)$ and of $\mathrm{Hilb}(\mathbb{P}^1 \times \mathbb{P}^1)$. Both of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are toric projective surfaces, so that their Hilbert schemes are smooth projective varieties carrying a natural torus action induced by the action of $T = (\mathbb{C}^*)^2$ on the toric charts.

If X is toric, we will denote by $\Delta(X)$ the set of vertices of its Newton polytope. Moreover, for any tuple of integers $\mathbf{n} = \{n_\alpha,\ \alpha \in \Delta(X)\}$ we define $|\mathbf{n}|$ to be $\sum_\alpha n_\alpha$. The T-fixed locus of $X^{[n]}$ is decomposed as

$$\mathrm{Hilb}^n(X)^T \cong \coprod_{|\mathbf{n}|=n} \prod_{\alpha \in \Delta(X)} \mathrm{Hilb}^{n_\alpha}(U_\alpha)^T.$$



(a) $\mathbb{P}^2$



(b) $\mathbb{P}^1 \times \mathbb{P}^1$

Figure 2: *Toric fans for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.*

Thus, the fixed points of the Hilbert scheme of X must have support contained in $\Delta(X)$, inducing a decomposition of the tangent space at a fixed point Z

$$T_Z X^{[n]} = \bigoplus_{\alpha\in\Delta(X)} T_{Z_\alpha} U_\alpha^{[n_\alpha]} \cong \bigoplus_{\alpha\in\Delta(X)} T_{Z_\alpha} (\mathbb{C}^2)^{[n_\alpha]}.$$

Let now X be a projective toric surface S. Then the decomposition of the tangent space at any fixed point, together with the localisation theorem, implies that

$$\chi_{-y}\left(S^{[n]}\right) = \sum_{|\mathbf{n}|=n} \prod_{\alpha\in\Delta(X)} \chi_{-y}\left(U_\alpha^{[n_\alpha]}\right) = \sum_{|\mathbf{n}|=n} \prod_{\alpha=0}^{\chi(X)-1} \chi_{-y}\left((\mathbb{C}^2)^{[n_\alpha]}\right),$$

with an abuse of notation justified in light of Remark 1.4.

As the complex cobordism ring is generated by the classes $[\mathbb{P}^2]$ and $[\mathbb{P}^1 \times \mathbb{P}^1]$, we will prove Eq. (1.4.1) for the χ_{-y}-genus of any projective surface by analysing the two cases above separately. This analysis will be made easy both by the decomposition of the tangent space at the fixed points and by the compactness of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, one can compute the χ_{-y}-genus equivariantly in the first place. Then, we can get information about the non-equivariant case by taking the limit of vanishing equivariant parameters, as the χ_{-y}-genus is rigid with respect to the torus action. We will exhibit the complete computation only for $\mathbb{P}^2$, as the case of $\mathbb{P}^1 \times \mathbb{P}^1$ is completely analogous.

1.4.1 Case I: $\mathbb{P}^2$

The projective plane $\mathbb{P}^2$ has three torus-fixed points, and correspondingly three toric charts. The equivariant parameters on the three affine patches can be read off the toric diagram in Fig. 2a as follows:

$$\begin{aligned} t_{1,(0)} &= t_1, & t_{2,(0)} &= t_2, \\ t_{1,(1)} &= 1/t_1, & t_{2,(1)} &= t_2/t_1, \\ t_{1,(2)} &= 1/t_2, & t_{2,(2)} &= t_1/t_2. \end{aligned} \tag{1.4.2}$$

Recall that fixed points of the Hilbert scheme of n points of the affine plane can be either characterized as monomial ideals or as partitions of n, [165]. Moreover, the tangent space at a fixed point Z of $\left(\mathbb{C}^2\right)^{[n]}$ corresponding to the partition $\mu \in \mathcal{P}(n)$ can be written as

$$T_Z\left(\mathbb{C}^2\right)^{[n]} = \sum_{s\in Y_\mu}\left(t_1^{l(s)+1}t_2^{-a(s)} + t_1^{-l(s)}t_2^{a(s)+1}\right) \in K_T^0(pt) \cong R(T), \quad (1.4.3)$$

where Y_μ is the Young diagram associated to the partition μ while the leg length $l(s)$ and the arm length $a(s)$ are defined as shown in Fig. 3. In particular, if s is a box in Y_μ with coordinates (i,j), then we define $a(s) = \mu'_j - i$ and $l(s) = \mu_i - j$, where μ' is the partition of n transposed to μ.

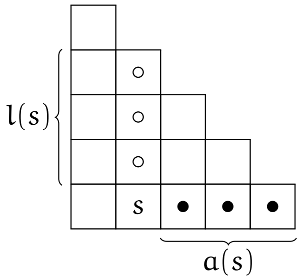


Figure 3: *Arm and leg length of a box in a Young diagram.*

From Eq. (1.4.3) we are immediately able to read off the contribution of each affine chart to the equivariant χ_{-y}-genus of any toric projective surface, which we might take as a definition by localisation of $\chi_{-y}\left((\mathbb{C}^2)^{[n]}\right)$:

$$\begin{aligned}\chi_{-y}\left(U_\alpha^{[n_\alpha]}\right) &= \chi_{-y}\left((\mathbb{C}^2)^{[n_\alpha]}\right)\\ &= \sum_{\mu_\alpha\in\mathcal{P}(n_\alpha)} N_{\mu_\alpha}^{-y}\\ &= \sum_{\mu_\alpha\in\mathcal{P}(n_\alpha)}\prod_{s\in Y_\mu}\frac{\left(1-yt_{1,\alpha}^{l(s)+1}t_{2,\alpha}^{-a(s)}\right)\left(1-yt_{1,\alpha}^{-l(s)}t_{2,\alpha}^{a(s)+1}\right)}{\left(1-t_{1,\alpha}^{l(s)+1}t_{2,\alpha}^{-a(s)}\right)\left(1-t_{1,\alpha}^{-l(s)}t_{2,\alpha}^{a(s)+1}\right)},\end{aligned} \quad (1.4.4)$$

where, of course, the equivariant parameters $t_{1,\alpha}$, $t_{2,\alpha}$ should be referred to the correct choice in the particular toric patch $U_\alpha \cong \mathbb{C}^2$. We are then interested in computing the generating function

$$\begin{aligned}\chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^2)\right) &= \sum_{n\geqslant 0}\chi_{-y}\left((\mathbb{P}^2)^{[n]}\right)p^n\\ &= \sum_{n\geqslant 0}p^n\sum_{|\mathbf{n}|=n}\prod_{\alpha\in\Delta(\mathbb{P}^2)}\chi_{-y}\left(U_\alpha^{[n_\alpha]}\right)\\ &= \prod_{\alpha\in\Delta(\mathbb{P}^2)}\sum_{n_\alpha\geqslant 0}\chi_{-y}\left(U_\alpha^{[n_\alpha]}\right)p^{n_\alpha},\end{aligned}$$

where everything is to be intended in the equivariant sense. By then studying the limit of vanishing equivariant parameters on each toric chart we get

$$\sum_{\mu_0\in\mathcal{P}(n_0)} N^{-y}_{\mu_0} \xrightarrow{t_{i,0}\longrightarrow 0} \sum_{\mu_0\in\mathcal{P}(n_0)} y^{n_0-r(\mu_0)}, \tag{1.4.5}$$

$$\sum_{\mu_1\in\mathcal{P}(n_1)} N^{-y}_{\mu_1} \xrightarrow{t_{i,1}\longrightarrow 0} \sum_{\mu_1\in\mathcal{P}(n_1)} y^{n_1-s(\mu_1)}, \tag{1.4.6}$$

$$\sum_{\mu_2\in\mathcal{P}(n_2)} N^{-y}_{\mu_2} \xrightarrow{t_{i,2}\longrightarrow 0} \sum_{\mu_2\in\mathcal{P}(n_2)} y^{n_2}, \tag{1.4.7}$$

where $r(\mu)$ denotes the number of rows in the partition μ, *i.e.* if $\mu = (\mu_1,\dots,\mu_k) \Rightarrow r(\mu) = k$, while

$$s(\mu) = \#\left\{p \in Y_{\mu'} : l(p) - 1 \leqslant a(p) \leqslant l(p)\right\}.$$

Indeed, let us show how to compute (1.4.5) as an example.

$$\lim_{t_1\to 0}\left(\frac{1-yt_1^{l(s)+1}t_2^{-a(s)}}{1-t_1^{l(s)+1}t_2^{-a(s)}}\right) = 1,$$

as $l(s) \geqslant 0$. Moreover

$$\lim_{t_1\to 0}\left(\frac{1-yt_1^{-l(s)}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)+1}}\right) = \begin{cases} y, & \text{if } l(s) > 0 \\ \dfrac{1-yt_2^{a(s)+1}}{1-t_2^{a(s)+1}}, & \text{if } l(s) = 0 \end{cases}$$

so that finally

$$\lim_{t_2\to 0}\lim_{t_1\to 0}\left(\frac{1-yt_1^{-l(s)}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)+1}}\right) = \begin{cases} y, & \text{if } l(s) > 0 \\ 1, & \text{if } l(s) = 0 \end{cases}$$

and putting together all the different contribution we recover Eq. (1.4.5).

We are now able to combine the results in equations (1.4.5)-(1.4.7) to compute $\chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^2)\right)$:

$$\begin{aligned} \chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^2)\right) &= \left(\sum_{\mu} p^{|\mu|}y^{n+r(\mu)}\right)\left(\sum_{\mu} p^{|\mu|}y^{n-s(\mu)}\right)\left(\sum_{\mu} p^{|\mu|}y^{n}\right) \\ &= \left(\sum_{\mu} p^{|\mu|}y^{n+r(\mu)}\right)\left(\sum_{\mu} p^{|\mu|}y^{n-r(\mu)}\right)\left(\sum_{\mu} p^{|\mu|}y^{n}\right) \\ &= \prod_{\ell=0}^{2}\left(\sum_{\mu} p^{|\mu|}y^{n+(\ell-1)r(\mu)}\right), \end{aligned} \tag{1.4.8}$$

as we have the following

Lemma 1.10. *There is an identity*

$$\sum_{\mu\in\mathcal{P}(n)} y^{r(\mu)} = \sum_{\mu\in\mathcal{P}(n)} y^{s(\mu)}.$$

Proof. For a proof by induction, see [153, Lemma 3.1]. ∎

The following lemma will let us conclude

Lemma 1.11. *There is an identity*

$$\sum_{\mu} p^{|\mu|} y^{|\mu|+(\ell-1)r(\mu)} = \prod_{m\geqslant 1} \frac{1}{1-y^{m+\ell-1}p^m}.$$

Proof. The result follows after some simple manipulations. First of all let us point out that a partition μ of an integer n can be described by a sequence of integers $\{\lambda_j\}$, where only finitely many λ_j are non zero and $\sum_j j\lambda_j = n$. Then we have that $r(\mu) = \sum_j \lambda_j$ and

$$\begin{aligned}
\sum_{\mu} p^{|\mu|} y^{|\mu|+(\ell-1)r(\mu)} &= \sum_{n\geqslant 0} (py)^n \sum_{\mu\in\mathcal{P}(n)} y^{(\ell-1)r(\mu)} \\
&= \sum_{n\geqslant 0} (py)^n \sum_{|\lambda|=n} y^{(\ell-1)\sum_m \lambda_m} \\
&= \prod_{m\geqslant 1} \sum_{\lambda_m\geqslant 0} (p^m y^{m+\ell-1})^{\lambda_m} \\
&= \prod_{m\geqslant 1} \frac{1}{1-p^m y^{m+\ell-1}}.
\end{aligned}$$

∎

Using lemma 1.11 we are now able to rewrite Eq. (1.4.8) in a much simpler form

$$\begin{aligned}
\chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^2)\right) &= \prod_{\ell=0}^{2} \left(\sum_{\mu} p^{|\mu|} y^{n+(\ell-1)r(\mu)} \right) \\
&= \prod_{\ell=0}^{2} \prod_{m\geqslant 1} \frac{1}{1-p^m y^{m+\ell-1}} \\
&= \prod_{m\geqslant 0} \frac{1}{1-p^{m+1}y^m} \frac{1}{1-p^{m+1}y^{m+1}} \frac{1}{1-p^{m+1}y^{m+2}} \\
&= \prod_{m\geqslant 0} \exp\Big(-\log(1-p^{m+1}y^m) - \log(1-p^{m+1}y^{m+1}) \\
&\quad -\log(1-p^{m+1}y^{m+2}) \Big) \\
&= \exp\left(\sum_{n\geqslant 1} \frac{p^n}{n}(1+y^n+y^{2n}) \sum_{m\geqslant 0} (yp)^{nm} \right) \\
&= \exp\left(\sum_{n\geqslant 1} \frac{p^n}{n} \frac{1+y^n+y^{2n}}{1-y^n p^n} \right).
\end{aligned}$$

As $1+y^n+y^{2n}=\chi_{-y^n}(\mathbb{P}^2)$ we conclude that

$$\chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^2)\right)=\exp\left(\sum_{n\geqslant 1}\frac{p^n}{n}\frac{\chi_{-y^n}(\mathbb{P}^2)}{1-y^n p^n}\right).$$

1.4.2 Case II: $\mathbb{P}^1\times\mathbb{P}^1$

The equivariant parameters of the torus action on $\mathbb{P}^1\times\mathbb{P}^1$ can be chosen in the four affine patches according to Fig. 2b as follows

$$\begin{aligned}
t_{1,(00)}&=t_1, & t_{2,(00)}&=t_2,\\
t_{1,(01)}&=t_1, & t_{2,(01)}&=1/t_2,\\
t_{1,(10)}&=1/t_1, & t_{2,(10)}&=t_2,\\
t_{1,(11)}&=1/t_1, & t_{2,(11)}&=1/t_2.
\end{aligned}\tag{1.4.9}$$

Proceeding in the same way as we did for the $\mathbb{P}^2$ case, we need to compute the equivariant generating function

$$\begin{aligned}
\chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^1\times\mathbb{P}^1)\right)&=\sum_{n\geqslant 0}\chi_{-y}\left((\mathbb{P}^1\times\mathbb{P}^1)^{[n]}\right)p^n\\
&=\sum_{n\geqslant 0}p^n\sum_{|\mathbf{n}|=n}\prod_{\alpha\in\Delta(\mathbb{P}^1\times\mathbb{P}^1)}\chi_{-y}\left(U_\alpha^{[n_\alpha]}\right)\\
&=\prod_{\alpha\in\Delta(\mathbb{P}^1\times\mathbb{P}^1)}\sum_{n_\alpha\geqslant 0}\chi_{-y}\left(U_\alpha^{[n_\alpha]}\right)p^{n_\alpha},
\end{aligned}$$

and then take the vanishing limit of the equivariant parameters in each toric patch. A computation very similar to the one in the previous section yields

$$\chi_{-y}\left(\mathrm{Hilb}(\mathbb{P}^1\times\mathbb{P}^1)\right)=\exp\left(\sum_{n\geqslant 1}\frac{p^n}{n}\frac{\chi_{-y^n}(\mathbb{P}^1\times\mathbb{P}^1)}{1-y^n p^n}\right),\tag{1.4.10}$$

which finally completes the proof of Eq. (1.4.1).

1.5 VIRTUAL LOCALISATION

Up to this point we have been dealing only with non singular schemes. A natural question would then be how much of the localisation machinery we have been describing so far can be carried over to the singular case. It turns out that results analogous to the ones in Thm. 1.6 and Thm. 1.9 can be proved whenever the scheme X is equipped with the additional structure of an *equivariant perfect obstruction theory*.

In general, a *perfect obstruction theory* on a scheme X, as defined in [17; 152], is the datum of

- a two-term complex $\mathbb{E}=[E^{-1}\to E^0]$ on X;

- a morphism $\phi : \mathbb{E} \to \mathbb{L}_X$ in the derived category from $\mathbb{E}$ to the truncated cotangent complex $\mathbb{L}_X = \tau_{\geqslant -1} L_X^{\bullet}$, where $L_X^{\bullet} \in D^{[-\infty,0]}(X)$ is the full cotangent complex defined by Illusie [127]. The morphism ϕ is also required to satisfy the following two properties
  - $h^0(\phi)$ is an isomorphism;
  - $h^{-1}(\phi)$ is surjective.

Given a scheme X with a perfect obstruction theory $(\mathbb{E}, \phi)$, the integer $\mathrm{vd} = \operatorname{rk} \mathbb{E}$ is called the *virtual dimension* of X with respect to $(\mathbb{E}, \phi)$. a morphism

$$\phi \colon \mathbb{E} \to \mathbb{L}_X$$

Remark 1.5. If there exists an isomorphism $\theta \colon \mathbb{E} \sim \mathbb{E}^\vee[1]$ such that $\theta = \theta^\vee[1]$, the perfect obstruction theory defined by $(\mathbb{E}, \phi)$ is said to be *symmetric*. Such obstruction theories have virtual dimension 0, and moreover the *obstruction sheaf*, defined as $\mathrm{Ob} = h^1(\mathbb{E}^\vee)$, is canonically isomorphic to the cotangent sheaf Ω_X. In particular, one has the K-theoretic identity

$$\mathbb{E} = h^0(\mathbb{E}) - h^{-1}(\mathbb{E}) = \Omega_X - T_X \in K_0(X),$$

as the obstruction sheaf $h^0(\mathbb{E}) = h^1(\mathbb{E}^\vee)$ is canonically isomorphic to Ω_X (cf. [18, Prop. 1.14]) and $h^{-1}(\mathbb{E})$ is its dual. ◀

With the aid of the structure induced by a perfect obstruction theory on a scheme, one can define a *virtual fundamental class*, against which integration will be defined. Indeed, a perfect obstruction theory defines a cone $\mathfrak{C} \hookrightarrow (E^{-1})^\vee$, and the virtual fundamental class will be be defined via the refined intersection

$$[X]^{\mathrm{vir}} = 0^![\mathfrak{C}] \in A_{\mathrm{vd}}(X), \tag{1.5.1}$$

where $0 : X \hookrightarrow (E^{-1})^\vee$ is the zero section of the vector bundle $(E^{-1})^\vee$. This virtual fundamental class $[X]^{\mathrm{vir}}$ only depends on the K-theory class of $\mathbb{E}$, [202, Thm. 4.6]. Then, on a proper scheme, one can define virtual (enumerative) invariants as

$$\int_{[X]^{\mathrm{vir}}} \alpha \in \mathbb{Q},$$

where $\alpha \in A^i(X)$. These intersection numbers are going to vanish if $i \neq \mathrm{vd}$.

It also turns out that virtual invariants in a K-theoretic framework can be defined, as a perfect obstruction theory also induces a *virtual structure sheaf*, [17; 202]

$$\mathscr{O}_X^{\mathrm{vir}} = [\mathbf{L}0^* \mathscr{O}_{\mathfrak{C}}] \in K_0(X). \tag{1.5.2}$$

If the structure morphism $\pi : X \to \mathrm{pt}$ is proper, the K-theoretic push-forward defines the K-theoretic virtual invariants by

$$\chi^{\mathrm{vir}}(X, V) = \chi(X, V \otimes \mathscr{O}_X^{\mathrm{vir}}) \in K_0(\mathrm{pt}), \tag{1.5.3}$$

for any $V \in K^0(X)$.

Let now X be a scheme with a perfect obstruction theory $\mathbb{E}$ and the action of an algebraic torus G. Then $(\mathbb{E}, \phi)$ is a G-equivariant perfect obstruction theory if ϕ can be lifted to a morphism in the derived category of G-equivariant sheaves. As in the previous sections, we will mainly be interested in algebraic tori.

If X carries the trivial action of a torus $T \cong (\mathbb{C}^*)^g$, any T-equivariant coherent sheaf $\mathscr{E} \in \mathrm{Coh}_T(X)$ can be decomposed in eigensheaves according to the torus action: $\mathscr{E} = \bigoplus_\mu \mathscr{E}_\mu$, where, as usual, the sum runs over the characters of T, *i.e.* $\mu \in \hat{T}$. We can then define the fixed and moving part of $\mathscr{E}$ as

$$\mathscr{E}^{\mathrm{fix}} = \mathscr{E}_0, \qquad \mathscr{E}^{\mathrm{mov}} = \bigoplus_{\mu \neq 0} \mathscr{E}_\mu.$$

If X does not carry a trivial action of T, one can define the *virtual normal bundle* to a component F of its fixed locus $X^T \hookrightarrow X$, as the moving part of the restriction of virtual tangent bundle to F

$$N^{\mathrm{vir}}_{F/X} = T^{\mathrm{vir}}_X\Big|_F^{\mathrm{mov}} = \mathbb{E}^\vee\Big|_F^{\mathrm{mov}}. \tag{1.5.4}$$

This sets the stage for the virtual versions of the localisation theorems. In the virtual setting, *virtual localisation formulae* can be proved both in equivariant Chow theory, [112], and in equivariant K-theory, [95; 188].

Theorem 1.12 ([112], eq. (1))**.** *Let* X *be a scheme equipped with a* $\mathbb{C}^*$*-action and a* $\mathbb{C}^*$*-equivariant perfect obstruction theory. Then*

$$[X]^{\mathrm{vir}} = \iota_* \sum_i \frac{[X_i]^{\mathrm{vir}}}{e(N_i^{\mathrm{vir}})} \in A_*^{\mathbb{C}^*}(X) \otimes \mathbb{Q}\left[t, \frac{1}{t}\right],$$

where the sum ranges over the connected components $X_i \subset X^T \stackrel{\iota}{\hookrightarrow} X$ *with virtual normal bundle* N_i^{vir}.

Theorem 1.13 ([188, Thm. 3.3])**.** *Let* X *be a scheme equipped with the action of an algebraic torus* T *and a* T*-equivariant perfect obstruction theory. Let moreover* $\iota : X^T \to X$ *be the natural inclusion of the fixed locus. Then we have*

$$\mathscr{O}_X^{\mathrm{vir}} = \iota_* \left(\frac{\mathscr{O}^{\mathrm{vir}}_{X^T}}{\Lambda^\bullet_{-1} N^{\mathrm{vir},\vee}_{X^T/X}} \right) \in K_0^T(X) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}(t).$$

If X is also proper, we saw how it makes sense to consider the equivariant push-forward of the structure morphism. In this case Thm. (1.13) implies the following localisation formula

$$\chi^{\mathrm{vir}}(X, V) = \chi^{\mathrm{vir}}\left(X^T, \frac{V|_{X^T}}{\Lambda^\bullet_{-1} N^{\mathrm{vir},\vee}_{X^T/X}} \right) \in K_0^T(\mathrm{pt})\left[\frac{1}{1-t^\mu} \,\middle|\, \mu \in \hat{T} \right], \tag{1.5.5}$$

for any $V \in K_T^0(X)$.

2 SUSY AND GEOMETRY

2.1 THE WITTEN INDEX AND TOPOLOGY

As the Witten index and its generalisations will be the main objects we will be computing in the rest of this thesis, in the form of supersymmetric partition functions, this section will be devoted to reviewing its definition and its basic properties. We start by reviewing some facts about supersymmetric quantum mechanics, which moreover serves the purpose of better explaining the connection between supersymmetric theories and topology. By definition [216; 217], a supersymmetric quantum mechanics can be constructed starting from a $\mathbb{Z}_2$-graded Hilbert space $(\mathcal{H}, \langle -, - \rangle)$, with the $\mathbb{Z}_2$-grading realised by an operator $(-1)^F$. The Hilbert space must be further supplied with two odd nilpotent operators Q and $Q^\dagger$, *i.e.* the supercharges, whose anti-commutator defines a positive semi-definite even operator H, *i.e.* the Hamiltonian, as

$$\left\{Q, Q^\dagger\right\} = 2H,$$

so that $[H, Q] = 0 = [H, Q^\dagger]$. Under the $\mathbb{Z}_2$-grading, the Hilbert space can be decomposed in its even (bosonic) and odd (fermionic) components,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F,$$

and the supercharges act as operators exchanging them, *i.e.* $Q, Q^\dagger : \mathcal{H}_{B,F} \to \mathcal{H}_{F,B}$. A *ground state* will be a zero-energy state as $\langle \alpha, H\alpha \rangle \geqslant 0$ given $\alpha \in \mathcal{H}$, so it will be defined as $\alpha \in \mathcal{H}$ such that $H\alpha = 0$. By the commutation relations of the supercharges, it is easy to deduce that a state has zero energy if and only if it is annihilated by Q and $Q^\dagger$. Such a state is then preserved by supersymmetry, and it is called a *supersymmetric ground state*. If we also assume the Hamiltonian to have purely discrete spectrum $\sigma(H)$, the Hilbert space can be decomposed in eigenspaces of H

$$\mathcal{H} = \bigoplus_{n \in \sigma(H)} \mathcal{H}_{(n)},$$

and this decomposition is compatible with the $\mathbb{Z}_2$ grading, in the sense that

$$\mathcal{H} = \bigoplus_{n \in \sigma(H)} \mathcal{H}_{B,(n)} \oplus \mathcal{H}_{F,(n)}.$$

Moreover, as Q is nilpotent, we have a complex of vector spaces

$$\mathcal{H}_F \xrightarrow{Q} \mathcal{H}_B \xrightarrow{Q} \mathcal{H}_F \xrightarrow{Q} \mathcal{H}_B, \tag{2.1.1}$$

whose cohomology defines the Q-cohomology

$$H_B(Q) = \frac{\ker Q : \mathcal{H}_B \to \mathcal{H}_F}{\operatorname{Im} Q : \mathcal{H}_F \to \mathcal{H}_B}$$
$$H_F(Q) = \frac{\ker Q : \mathcal{H}_F \to \mathcal{H}_B}{\operatorname{Im} Q : \mathcal{H}_B \to \mathcal{H}_F}$$

which satisfies $H_{B/F}(Q) \cong \mathcal{H}_{B/F,(0)}$. If it so happens that there exists a hermitian operator F with integral eigenvalues, the *fermion number operator*, such that $e^{i\pi F} = (-1)^F$, the Hilbert space is $\mathbb{Z}$-graded by F, and the complex (2.1.1) splits into a $\mathbb{Z}$-graded one as

$$\cdots \xrightarrow{Q} \mathcal{H}_{p-1} \xrightarrow{Q} \mathcal{H}_p \xrightarrow{Q} \mathcal{H}_{p+1} \xrightarrow{Q} \cdots . \tag{2.1.2}$$

The cohomology of (2.1.2) $H^\bullet(Q)$ refines the Q-cohomology, in the sense that

$$H_B(Q) = \bigoplus_{n \text{ even}} H^p(Q), \qquad H_F(Q) = \bigoplus_{n \text{ odd}} H^p(Q).$$

One peculiar feature of supersymmetric quantum mechanics is that positive energy states are always paired, *i.e.* $\mathcal{H}_{B,(n)} \cong \mathcal{H}_{F,(n)}$ when $n > 0$. This implies the existence of a deformation invariant (where by deforming we mean deforming the spectrum of H), the *Witten index* $\operatorname{tr}(-1)^F e^{-\beta H}$, defined as

$$\begin{aligned}\operatorname{tr}(-1)^F e^{-\beta H} &= \dim \mathcal{H}_{B,(0)} - \dim \mathcal{H}_{F,(0)} \\ &= \dim \ker Q - \dim \operatorname{coker} Q \\ &= \dim \ker Q - \dim \ker Q^\dagger,\end{aligned}$$

which qualifies it naturally as the index of the Q operator.[1] We can also express the Witten index as the Euler characteristic of the complex (2.1.2), as

$$\begin{aligned}\operatorname{tr}(-1)^F e^{-\beta H} &= \dim \mathcal{H}_{B,(0)} - \dim \mathcal{H}_{F,(0)} \\ &= \dim H_B(Q) - \dim H_F(Q) \\ &= \sum_p (-1)^p \dim H^p(Q).\end{aligned}$$

The connection of the Witten index to the topology of, say, Riemannian manifolds is made manifest by analysing the path integral formulation of supersymmetric quantum mechanics. Indeed, quantum mechanics can be understood as a one-dimensional quantum field theory, or as a theory of maps $\varphi : X \to Y$, where X is one-dimensional, and is usually taken to be $\mathbb{R}$, an interval $I \subset \mathbb{R}$ or S^1. This formulation of quantum mechanics can be extended to the supersymmetric settings, where one needs to suitably extend the set of fields in order to include the fermionic partners to the bosonic fields $\varphi : X \to Y$. If, for instance, Y is an n-dimensional Riemannian manifold (M, g), the supersymmetric quantum mechanics with target space M will

1 In the case the spectrum $\sigma(H)$ of the Hamiltonian H is not purely discrete, the Witten index may gain anomalous contributions making it dependent on the parameter β.

consist of bosonic variables, which will be maps $\phi : X \to M$, and fermionic ones, which will be sections $\psi, \overline{\psi} \in \Gamma(X, \phi^* TM \otimes \mathbb{C})$. In the local coordinates x^i of M, they will be represented by collections of maps $\phi^i = x^i \circ \psi$, and similarly for the fermionic variables $\psi, \overline{\psi} \rightsquigarrow \{\psi^i, \overline{\psi}^i\}$. If t is the coordinate on X, by defining $D_t\psi^i = \partial_t\psi^i + \Gamma^i_{jk}\partial_t\phi^j\psi^k$, the Lagrangian for the theory is

$$\begin{aligned} L(\phi,\psi,\overline{\psi}) = \frac{g_{ij}}{2}\partial_t\phi^i\partial_t\phi^j + \frac{ig_{ij}}{2}\left(\overline{\psi}^i D_t\psi^j - D_t\overline{\psi}^i\psi^j\right) \\ -\frac{R_{ijkl}}{2}\psi^i\overline{\psi}^j\psi^k\overline{\psi}^l, \end{aligned} \tag{2.1.3}$$

and it is invariant under the supersymmetry

$$\begin{aligned} \delta_\epsilon\phi^i &= \epsilon\overline{\psi}^i - \overline{\epsilon}\psi^i, \\ \delta_\epsilon\psi^i &= \epsilon\left(i\partial_t\phi^i - \Gamma^i_{jk}\overline{\psi}^j\psi^k\right), \\ \delta_\epsilon\overline{\psi}^i &= -\overline{\epsilon}\left(i\partial_t\phi^i + \gamma^i_{jk}\overline{\psi}^j\psi^k\right), \end{aligned}$$

corresponding to the supersymmetric conserved charges $Q = ig_{ij}\overline{\psi}^i\partial_t\phi^j$ and $\overline{Q} = -ig_{ij}\psi^i\partial_t\phi^j$. The Witten index can then be understood as the finite-temperature partition function of this theory, at temperature β^{-1}

$$\operatorname{tr}(-1)^F e^{-\beta H} = \int_{\mathrm{PBC}} \left[\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\right] \exp\left(\int_0^\beta dt L(\phi,\psi,\overline{\psi})\right),$$

where periodic boundary conditions have to be imposed on fermions.

The Hilbert space for this supersymmetric quantum mechanics will be $\mathcal{H} = \Omega(M) \otimes \mathbb{C}$, and the grading of $\mathcal{H}$ by the form degree is identified with the $\mathbb{Z}$-grading by the fermion number operator F. On $\mathcal{H}$, the bosonic and fermionic fields are represented by operators acting as follows

$$\phi^i \rightsquigarrow M_{x^i}, \qquad \psi^i \rightsquigarrow g^{ij}\iota_{\partial/\partial x^i}, \qquad \overline{\psi}^i \rightsquigarrow dx^i \wedge,$$

where by M_{x^i} we denote the multiplication operator by x^i, and ι_v is the contraction by the vector field v. In this framework, we have $Q = d$, $\overline{Q} = \delta$ is the codifferential, and the Hamiltonian is the Laplace-Beltrami operator

$$H = \frac{1}{2}\{Q, \overline{Q}\} = \frac{1}{2}(d\delta + \delta d) = \frac{\Delta}{2}.$$

Thus, the zero energy states are harmonic forms $\mathcal{H}_{(0)} \cong \mathcal{H}(M, g)$, which implies at the level of Q-cohomology that $H^\bullet(Q) \cong H^\bullet_{dR}(M) \cong H^\bullet(M;\mathbb{R})$. We are then able to conclude that the Witten index of the supersymmetric quantum mechanics on a Riemannian manifold (M, g) computes the Euler characteristic of M, as

$$\operatorname{tr}(-1)^F e^{-\beta H} = \sum_p (-1)^p \dim H^p(Q) = \sum_p (-1)^p \dim H^p(M;\mathbb{R}) = \chi(M).$$

Remark 2.1. For the most part, similar considerations may be carried over to a supersymmetric quantum mechanics having as target space an n-dimensional

Kähler manifold. In that case, however, the model enjoys extended supersymmetry, in that one can find four conserved supercharges. Moreover, there are two different fermion number operators, the axial one F_A and the vector one F_V, such that $[F_V, F_A] = 0$. Thus the Hilbert space $\Omega(M) \otimes \mathbb{C}$ is $(\mathbb{Z} \oplus \mathbb{Z})$-graded as

$$\Omega(M) \otimes \mathbb{C} \cong \bigoplus_{p,q=1}^{n} \Omega^{p,q}(M).$$

Correspondingly, the harmonic forms $\mathcal{H}(M, g)$ representing the zero energy states are bigraded, and at the cohomological level one gets the well-known isomorphisms for Dolbeault cohomology

$$H^{p,q}_{\partial}(M) \cong \mathcal{H}^{p,q}(M, g) \cong H^{p,q}_{\bar{\partial}}(M).$$ ◀

2.2 REFINING THE WITTEN INDEX: ELLIPTIC GENERA

In passing from Riemannian manifolds to complex manifolds, one is often led to defining certain refinements of topological invariants capable to capture more detailed features. A simple example is offered by the *Hirzebruch χ_y-genus*, which is defined for a holomorphic vector bundle E over a compact complex manifold X as

$$\chi_y(X, E) = \sum_{p,q \geqslant 0} (-1)^q y^p \dim H^q\left(X, \mathscr{O}_X(E) \otimes \Lambda^p T_X^\vee\right),$$

where $\mathscr{O}_X(E)$ is the sheaf of holomorphic sections of E, while $T_X^\vee$ is the cotangent bundle of X. By defining the Euler-Poincaré characteristic of E as

$$\chi(X, E) = \sum_{p \geqslant 0} (-1)^p \dim H^p\left(X, \mathscr{O}_X(E)\right),$$

we see that the Hirzebruch genus can be also written as

$$\chi_y(X, E) = \sum_{p \geqslant 0} y^p \chi\left(X, E \otimes \Lambda^p T_X^\vee\right).$$

Whenever $E \cong \mathscr{O}_X$ is the trivial line bundle, we have

$$\chi_y(X) = \chi_y(X, \mathscr{O}_X) = \sum_{p,q \geqslant 0} (-1)^q y^p \dim H^{p,q}(X),$$

and we immediately see that under suitable choices of the value of y, the Hirzebruch genus $\chi_y(X)$ reduces to other known invariants:

- if $y = -1$ it reduces to the topological Euler characteristic of X, *i.e.* $\chi_{-1}(X) = e(X)$;
- if $y = 0$, $\chi_y(X)$ reduces to the so-called *arithmetic genus* $\chi(X)$;
- if $y = 1$ we get the signature $\sigma(X)$.

From the point of view of supersymmetric theories, the Hirzebruch χ_y-genus of a compact Kähler manifold (M, g, ω) can be interpreted as the Witten index of a supersymmetric quantum mechanics with target space M, whose action is a *deformation* of (2.1.3) by a *mass-like term* of the form

$$L_{\text{def}}(\psi) = -\frac{1}{2} m \psi^i \omega_{ij} \psi^j + c,$$

where ω is the Kähler form of M and m plays the role of the mass parameter, [126]. Moreover, the mass parameter m is related to the y parameter of the χ_y-genus as $y = \exp(-\beta m)$. Thus in the decoupling limit $m \to +\infty$ the partition function of the supersymmetric quantum mechanics on the Kähler manifold M reproduces the arithmetic genus $\chi(M)$.

In a similar sense, when dealing with a supersymmetric quantum field theory, one can often study modifications in the definition of the Witten index so as to capture more information in the partition function. This can be done, for example, if one is able to find a set of *fugacities* q_i associated to *conserved charges* C_i, such that $[Q, C_i] = 0 = [Q^\dagger, C_i]$. In this case one can define the index as

$$\mathcal{I}(q_i) = \text{tr}_{\mathcal{H}}(-1)^F e^{-\beta H} \prod_i q_i^{C_i}.$$

For instance, we can consider a two-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on a torus. The partition function of the theory reproduces the so-called *elliptic genus* of the target space, which can be identified with a refinement of the Witten index of the kind above. Indeed, if the supersymmetry content is that of a 2d $\mathcal{N} = (2, 2)$ theory with flavour symmetry group K and $U(1)$ R-symmetry J, we can compute the partition function as the index

$$\mathcal{I}_{\text{EG}}(\tau, z, u_\alpha) = \text{tr}_{\text{RR}}(-1)^F q^{H_L} \overline{q}^{H_R} y^J \prod_\alpha x_\alpha^{K_\alpha}, \tag{2.2.1}$$

where K_α are the Cartan generators of K while τ, z and u are defined by $q = \exp(2\pi i \tau)$, $y = \exp(2\pi i z)$ and $x_\alpha = \exp(2\pi i u_\alpha)$ respectively. The trace is to be computed over the Ramond-Ramond sector, *i.e.* fermions are given periodic boundary conditions, and the left- and right-moving hamiltonians $H_{L,R}$ are defined in terms of the Hamiltonian H and momentum P as $2H_L = H + iP$ and $2H_R = H - iP$. Then we see that, as $q^{H_L} \overline{q}^{H_R} = \exp(-2\pi \Im\tau H - 2\pi \Re\tau P)$, we can actually compute the index (2.2.1) as the partition function of a theory on a torus $w \sim w + 1 \sim w + \tau$ with complex structure τ. We are however not limited to theories with $\mathcal{N} = (2, 2)$ supersymmetry, and in the case of a two-dimensional $\mathcal{N} = (0, 2)$ theory we can indeed define an index analogous to (2.2.1). In the $\mathcal{N} = (0, 2)$ case we do not have a $U(1)$ R-symmetry, so we define

$$\mathcal{I}_{\text{EG}}(\tau, z, u_\alpha) = \text{tr}_{\text{RR}}(-1)^F q^{H_L} \overline{q}^{H_R} \prod_\alpha x_\alpha^{K_\alpha}, \tag{2.2.2}$$

with the same notation as before. These indices are known in physics to be computing the elliptic genus of target space, and were first introduced in the 80's, [93; 149; 186; 196; 197; 218; 219]. Later it was realised that the elliptic genus of a Landau-Ginzburg model could be computed by localisation [26; 27; 139; 140; 221]. More recently, in [24; 25] the application of the localisation argument (see §2.3 for a very brief review) to the path integral formulation of the elliptic genus made the computation possible for general two-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. These localisation formulae are particularly useful as they can be exploited to compute partition functions of GLSMs, that is to say gauge theories of vector multiplets with additional matter, possibly supplied with superpotential interactions. These are known to flow at low energy to sigma models having as target space the locus cut out by the vanishing of the superpotential terms, whose elliptic genus is actually computed through the GLSM partition function.

From a mathematical point of view, instead, the indices (2.2.1) and (2.2.2) are computing a refinement of the χ_y-genus which we described at the beginning of this section. In particular, for any vector bundle E over a scheme X we define the formal power series

$$\mathcal{E}(E; q, y) = \bigotimes_{n \geqslant 1} \left(\Lambda^\bullet_{-yq^n}(E^\vee) \otimes \Lambda^\bullet_{-y^{-1}q^n}(E) \otimes \mathrm{Sym}^\bullet_{q^n}(E \oplus E^\vee) \right),$$

so that $\mathcal{E}(E; q, y) \in K^0(X)[y, y^{-1}][\![t]\!]$ and if X is a smooth projective scheme, while $V \in K^0(X)$ a vector bundle over it, we are able to define its elliptic genus as

$$\begin{aligned} \mathrm{Ell}(X, V; y, q) &= y^{-\dim X/2} \chi_{-y}(X, \mathcal{E}(T_X) \otimes V) \\ &= y^{-\dim X/2} \chi(X, V \otimes \Lambda^\bullet_{-y} T_X^\vee \otimes \mathcal{E}(T_X)). \end{aligned} \tag{2.2.3}$$

We immediately see that the elliptic genus so defined is indeed a refinement of the χ_{-y}-genus, to which it reduces for $q = 0$, as $\mathrm{Ell}(X, V; y, q)|_{q=0} = y^{-\dim X/2} \chi_{-y}(X, V)$.

Remark 2.2. Even though all the definitions given so far are valid for smooth manifolds/non singular varieties, some generalisations to the case of singular schemes are available [95]. In particular, whenever X is a possibly singular scheme equipped with a perfect obstruction theory, a virtual version of the holomorphic Euler characteristic, and correspondingly also of the χ_y-genus and elliptic genus, can be defined (cf. §1.5). One may then be tempted to assume that the elliptic genus, in its virtual version, can still be interpreted as the partition function of a two-dimensional supersymmetric gauge theory whose target space is the scheme X. While this is true for 2d $\mathcal{N} = (2, 2)$ theories, it may fail at the equivariant level for purely $\mathcal{N} = (0, 2)$ theories, as it is the case, for instance, for the low energy effective theory of a D1/D7 system, [22]. In this case the index (2.2.2) computes a "chiral" version of the elliptic genus (cf. §5.6.1). ◀

2.3 SUSY LOCALISATION

As we already pointed out, one of the advantages of studying supersymmetric quantum field theories lies in the availability of localisation techniques. Much like in the mathematical setting of equivariant cohomology

[8; 92], and inspired by it, localisation arguments were first introduced in physics for computing partition functions for twisted theories on compact manifolds, [220; 224]. Later localisation arguments were extended to theories on both non compact manifolds with the so-called Ω-background [162; 170; 175] and non-twisted theories on certain classes compact manifolds or manifold with boundaries. In this section we will review supersymmetric localisation arguments, while in the next section we will briefly see how these can be exploited in order to compute the elliptic genus of two-dimensional $\mathcal{N} = (0,2)$ GLSMs.

Let then $\{\varphi\}$ be the set of fields for a quantum field theory with action $S[\{\varphi\}]$. As usual, fields in $\{\varphi\}$ can be divided in two families according to their statistics, the even ones are bosons, the odd ones fermions. We will assume the theory to enjoy a Grassmann-odd (fermionic) symmetry $\mathcal{Q}$, *i.e.* $\mathcal{Q}S[\{\varphi\}] = 0$ and we say $S[\{\varphi\}]$ to be $\mathcal{Q}$-closed. If the quantum field theory we are considering is supersymmetric, $\mathcal{Q}$ will be identified with a supercharge. We will also assume this symmetry to be non-anomalous, *i.e.* the path integral measure is $\mathcal{Q}$-invariant, and nilpotent or, more generally, we will assume it to square to a Grassmann-even (bosonic) symmetry B, *i.e.* $\mathcal{Q}^2 = B$. The objects of interest of quantum field theory are usually encoded in the partition function

$$Z = \int_{\mathcal{F}} [\mathcal{D}\{\varphi\}]\, e^{-S[\{\varphi\}]},$$

and in vacuum expectation values of observables $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} [\mathcal{D}\{\varphi\}]\, \mathcal{O} e^{-S[\{\varphi\}]},$$

where, in both cases, we denote by $\mathcal{F}$ the whole space of field configurations. Of course, $\mathcal{F}$ is generally speaking an infinite-dimensional space, which can be identified with the space of sections of the bundles defining the relevant fields in the theory, and the path integral is ill-defined. As we will see, however, the localisation argument often provides a way of reducing the ill-defined path integral to a well-defined ordinary integration. To this end, we will be interested in the particular set of BPS observables $\mathcal{O}_{\text{BPS}}$, *i.e.* local or non-local observables preserved by the fermionic symmetry $\mathcal{Q}\mathcal{O}_{\text{BPS}} = 0$.

Let then $\mathcal{G}$ be the symmetry group associated to $\mathcal{Q}$. If $\mathcal{G}$ was to act freely on $\mathcal{F}$ we would have had

$$\begin{aligned} \langle \mathcal{O}_{\text{BPS}} \rangle &= \int_{\mathcal{F}} [\mathcal{D}\{\varphi\}]\, \mathcal{O}_{\text{BPS}} e^{-S[\{\varphi\}]} \\ &= \text{vol}(\mathcal{G}) \int_{\mathcal{F}/\mathcal{G}} [\mathcal{D}\{\varphi\}]\, \mathcal{O}_{\text{BPS}} e^{-S[\{\varphi\}]}. \end{aligned}$$

However, as we chose $\mathcal{Q}$ to be a fermionic symmetry, $\text{vol}(\mathcal{G}) = 0$, so the action of $\mathcal{G}$ on $\mathcal{F}$ can't be free. In particular, the fixed locus of $\mathcal{G}$ corresponds to the BPS locus

$$\mathfrak{M}_{\text{BPS}} = \{\{\varphi\} \in \mathcal{F} : \mathcal{Q}\{\varphi\} = 0\}$$

of $\mathcal{Q}$-preserved field configurations. The path integral localises to $\mathfrak{M}_{\text{BPS}}$. As this is often finite-dimensional, the path integral gets reduced to an ordinary integral, and expectation values of BPS observables can be computed exactly, thus also capturing all the non-perturbative phenomena of the theory.

A slightly different localisation argument goes as follows. Consider a deformed version of the expectation value of a BPS operator by a $\mathcal{Q}$-exact term

$$\langle \mathcal{O}_{\mathrm{BPS}} \rangle(t) = \int_{\mathcal{F}} [\mathcal{D}\{\varphi\}]\, \mathcal{O}_{\mathrm{BPS}} e^{-S[\{\varphi\}]-t\mathcal{Q}V[\{\varphi\}]},$$

where $V[\{\varphi\}]$ is a fermionic operator invariant under B, so that $\mathcal{Q}^2 V = BV = 0$. It turns out that the expectation value $\langle \mathcal{O}_{\mathrm{BPS}} \rangle(t)$ is actually independent of t as long as V doesn't change the asymptotics of the integrand at infinity. Indeed

$$\begin{aligned} \frac{d}{dt}\langle \mathcal{O}_{\mathrm{BPS}} \rangle(t) &= \int_{\mathcal{F}} [\mathcal{D}\{\varphi\}]\, \mathcal{O}_{\mathrm{BPS}} \mathcal{Q}V[\{\varphi\}] e^{-S[\{\varphi\}]-t\mathcal{Q}V[\{\varphi\}]} \\ &= \int_{\mathcal{F}} [\mathcal{D}\{\varphi\}]\, \mathcal{Q}\left(\mathcal{O}_{\mathrm{BPS}} V[\{\varphi\}] e^{-S[\{\varphi\}]-t\mathcal{Q}V[\{\varphi\}]}\right) = 0, \end{aligned}$$

so that the expectation value (and also the partition function) only depends on the $\mathcal{Q}$-cohomology class of the action. Thus it is convenient to deform the action by a positive definite term, so as to be able to compute $\langle \mathcal{O}_{\mathrm{BPS}} \rangle$ by taking the limit $t \to +\infty$ in the deformed expectation value $\langle \mathcal{O}_{\mathrm{BPS}} \rangle(t)$. More precisely, it is sufficient for the bosonic part of V to be positive semi-definite for the path integral to localise to the set of so-called BPS vacua (or minima) $\mathfrak{M}_{\mathrm{vac}} \subset \mathcal{F}$, where

$$\mathfrak{M}_{\mathrm{vac}} = \left\{ \{\varphi\} \in \mathcal{F} : \mathcal{Q}V[\{\varphi\}]\big|_{\mathrm{bos}} = 0 \right\}.$$

In general there is no guarantee for $\mathfrak{M}_{\mathrm{BPS}}$ to coincide with $\mathfrak{M}_{\mathrm{vac}}$. Though for certain choices of the deformation $\mathcal{Q}V[\{\varphi\}]$ this can certainly happen, in general situations the path integral localises to $\mathfrak{M}_{\mathrm{loc}} = \mathfrak{M}_{\mathrm{BPS}} \cap \mathfrak{M}_{\mathrm{vac}}$.

The approach to localising the path integral by $\mathcal{Q}$-exact deformations can be used in order to extract explicit localisation formulae for the partition function, or, more generally, for expectation values of BPS operators. Indeed, let us consider for simplicity the case of the partition function, and expand the fields $\{\varphi\}$ around the minima $\{\varphi_0\}$ as $\varphi = \varphi_0 + t^{-1/2}\delta\varphi$. As the semi-classical *1-loop expansion* of the action is exact at $t \to \infty$, we have

$$\lim_{t\to+\infty} Z(t) = \int_{\mathfrak{M}_{\mathrm{vac}}} [\mathcal{D}\{\varphi_0\}]\, e^{-S[\{\varphi_0\}]} \int [\mathcal{D}\{\delta\varphi\}]\, e^{-\langle \delta\varphi, \mathrm{Hess}(\mathcal{Q}V)[\{\varphi_0\}]\delta\varphi\rangle/2},$$

where

$$\langle \delta\varphi, \mathrm{Hess}(\mathcal{Q}V)[\{\varphi_0\}]\delta\varphi\rangle = \iint \delta\varphi\, \frac{\delta^2 S[\{\varphi\}]}{\delta\phi^2}\bigg|_{\{\varphi\}=\{\varphi_0\}} \delta\varphi.$$

Even though the second integral is in principle infinite-dimensional, it only involves a Gaussian integral containing the Hessian of the localising deformation $\mathcal{Q}V[\{\varphi\}]$, which can be factorized into bosonic and fermionic contributions

$$\langle \delta\varphi, \mathrm{Hess}(\mathcal{Q}V)[\{\varphi_0\}]\delta\varphi\rangle = \langle \delta\varphi_f, K_f[\{\varphi_0\}]\delta\varphi_f\rangle + \langle \delta\varphi_b, K_b[\{\varphi_0\}]\delta\varphi_b\rangle.$$

The inverse of the super-determinant (*i.e.* the ratio of bosonic and fermionic *1-loop determinants*) of the localising deformation is then the result of the integration over $\{\delta\varphi\}$, so that in the end one gets

$$\begin{aligned}\lim_{t\to+\infty} Z(t) &= \int_{\mathfrak{M}_{\mathrm{vac}}} [\mathcal{D}\{\varphi_0\}]\, e^{-S[\{\varphi_0\}]} \left(\frac{\det K_b(\{\varphi_0\})}{\det K_f(\{\varphi_0\})}\right)^{1/2} \\ &= \int_{\mathfrak{M}_{\mathrm{vac}}} [\mathcal{D}\{\varphi_0\}]\, e^{-S[\{\varphi_0\}]}\, \mathrm{Sdet}^{-1}\left(\mathrm{Hess}(\mathcal{Q}V)\right)(\{\varphi_0\}).\end{aligned}$$

Let us end this section with a couple of remarks. First we want to point out that the choice of the localising deformation $\mathcal{Q}V$ is not unique, and it determines the so-called *localisation scheme*. Different localisation schemes will naturally produce different BPS vacua and different 1-loop determinants, but the end result will nevertheless be the same, as $Z(t)$ and $\langle \mathcal{O}_{\mathrm{BPS}}\rangle(t)$ are independent of t, hence also of the particular deformation chosen. In some of these localisation schemes $\mathfrak{M}_{\mathrm{vac}}$ may degenerate to constant field configurations, in which case the path integral is reduced to a finite dimensional integral, *i.e.* a so-called *matrix model*. Moreover, when computing the super-determinant of the "Hessian" of $\mathcal{Q}V$ one has to be careful about possible bosonic/fermionic zero-modes, which, whenever present, have to be discarded from the computation.

2.4 ELLIPTIC GENERA VIA SUSY LOCALISATION

We saw in §2.2 how the generalisations of the Witten index (2.2.1)-(2.2.2) compute the elliptic genus of the target space of a two-dimensional supersymmetric gauge theory on a torus. In many cases the computation of this partition function can be carried out by means of localisation, and in particular localisation formulae can be derived in the case of GLSMs, [24; 25]. Throughout chapter 3 we will be interested in computing partition functions of low energy effective theories of a certain 4d $\mathcal{N}=4$ topologically twisted Vafa-Witten theory. As these effective theories are 2d GLSMs on a torus, we briefly review here the localisation formula for the partition function of such GLSMs. A detailed analysis of the problem is given in [24; 25].

A 2d $\mathcal{N}=(0,2)$ GLSM is determined by the following data

- A gauge group G, with Lie algebra $\mathfrak{g}$. The corresponding gauge field A_μ will find its place in a $\mathfrak{g}$-valued vector multiplet $V=(A_\mu,\lambda^+,\overline{\lambda}^+,D)$, where λ^+ is a right-moving Weyl fermion and D is a real scalar.
- Charged matter fields assembled in Fermi and chiral multiplets Λ and Φ, transforming in given representations ρ_Λ and ρ_Φ of G. A general chiral multiplet $\Phi=(\phi,\psi^-)$ is comprised of a complex scalar ϕ and of a left-moving Weyl spinor ψ^-. A generic Fermi multiplet $\Lambda=(\psi^+,G)$ consists instead of a right-moving Weyl spinor ψ^+ and an auxiliary complex scalar G. The supersymmetric variation of a Fermi multiplet involves the choice of a chiral multiplet $\mathcal{E}(\Phi)=(E,\psi_E^-)$, which is a holomorphic function of the fundamental chiral multiplets in the theory.

- Holomorphic functions $J(\Phi)$ of the chiral multiplets expressing generic interactions.

The most general Lagrangian density one can write with the ingredients above will then be

$$L(V,\Phi,\Lambda,J) = L_V + L_\Phi + L_\Lambda + L_J, \tag{2.4.1}$$

with

$$L_V = \operatorname{tr}\left(F_{12}^2 + D^2 - 2\overline{\lambda}^+ D_{\bar z}\lambda^+\right), \tag{2.4.2}$$

$$L_\Phi = D_\mu\overline{\phi}D^\mu\phi + i\overline{\phi}D\phi + 2\overline{\psi}^- D_z\psi^- - \overline{\psi}^-\lambda^+\phi + \overline{\phi\lambda}^+\psi^-, \tag{2.4.3}$$

$$L_\Lambda = -2\overline{\psi}^+ D_{\bar z}\psi^+ + \overline{E}E + \overline{G}G + \overline{\psi}^+\psi_E^- - \overline{\psi}_E^-\psi^+, \tag{2.4.4}$$

$$L_J = \sum_\alpha \left(G_\alpha J + i\psi_\alpha^+\psi_J^{-\alpha}\right), \tag{2.4.5}$$

where we used the complex coordinates $z = x^1 + ix^2$, $\bar z = x^1 = ix^2$. We will also correspondingly write the action as a sum

$$S[V,\Phi,\Lambda,J] = S_V + S_\Phi + S_\Lambda + S_J.$$

A great simplification is offered by the fact the Lagrangian (2.4.1) is $\mathcal{Q}$-exact, as

$$L_V = \mathcal{Q}\operatorname{tr}(\lambda^+ D + i\lambda^+ F_{12}), \qquad L_\Phi = \mathcal{Q}(2i\overline{\phi}D_z\psi^- i\overline{\phi}\lambda^+\phi),$$
$$L_\Lambda = \mathcal{Q}(\overline{\psi}^+ G - i\overline{E}\psi^+), \qquad L_J = \mathcal{Q}(\psi_\alpha^+ J^\alpha).$$

In order to compute the partition function via localisation, one needs to choose a suitable localising action, and the $\mathcal{Q}$-exactness of the Lagrangian suggests that a good choice might be

$$S_{\text{loc}}[V,\Phi,\Lambda] = \frac{1}{e^2}S_V + \frac{1}{g^2}(S_\Phi + S_\Lambda),$$

so that localisation might be performed by taking the limits $e, g \to 0$. As we can easily see from (2.4.2), the localisation locus for the vector multiplet coincides with flat connections $F_{12} = 0$ (up to gauge transformations), whereas D needs to vanish. Whenever the gauge group G has connected and simply-connected non-abelian part the space of flat connections $F_{12} = 0$ can be identified with the quotient of a torus $T^{2\,\mathrm{rk}\,G}$ by the Weyl subgroup $W \subset G$. All the other fields in the chiral and Fermi multiplet must instead vanish in order to ensure the vanishing of the localising action.

One can moreover compute the 1-loop determinants of the vector, Fermi, chiral multiplets around $\mathfrak{M}_{\text{loc}}$, taking care of the necessary regularisations. The result is the following

$$Z^{1-\text{loop}}_{\Phi,\rho_\Phi}(\tau,u) = \prod_{\rho\in\rho_\Phi} i\frac{\eta(q)}{\theta_1(q,x^\rho)}, \tag{2.4.6}$$

$$Z^{1-\text{loop}}_{\Lambda,\rho_\Lambda}(\tau,u) = \prod_{\rho\in\rho_\Lambda} i\frac{\theta_1(q,x^\rho)}{\eta(q)}, \tag{2.4.7}$$

$$Z_V^{1-\text{loop}}(\tau,u) = \left(\frac{2\pi\eta^2(q)}{i}\right)^{\text{rk}\,G} \prod_{\alpha\in R_G} i\frac{\theta_1(q,x^\alpha)}{\eta(q)} \prod_{a=1}^{\text{rk}\,G} du_a, \qquad (2.4.8)$$

where $\rho \in \rho_\bullet$ denotes the weights of the representation $\rho_\bullet$ of G, while R_G is the root lattice of G. We also use the shorthand notation $x^\rho = e^{2\pi i\rho(u)}$, where we identify u as an element $u \in \mathfrak{k}$ in the Cartan subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and a weight ρ with an element $\rho \in \mathfrak{k}^\vee$. If the theory also has flavour group K, with Cartan generators $y_\alpha = e^{2\pi i z_\alpha}$, the matter multiplets will be decorated by a representation of $G \times K$, and the 1-loop determinants (2.4.6)-(2.4.8) can be made to keep track also of the action of K as in (2.2.2) by performing the substitution $x^\rho \rightsquigarrow x^{\rho_G} y^{\rho_K}$.

After denoting collectively by $\{\Phi\}$ and $\{\Lambda\}$ the chiral and Fermi multiplets of the theory, we define

$$Z_{1-\text{loop}}(\tau,u) = Z_V^{1-\text{loop}}(\tau,u) \prod_{(\Phi,\rho_\Phi)\in\{\Phi\}} Z_{\Phi,\rho_\Phi}^{1-\text{loop}}(\tau,u) \prod_{(\Lambda,\rho_\Lambda)\in\{\Lambda\}} Z_{\Lambda,\rho_\Lambda}^{1-\text{loop}}(\tau,u).$$

We see that the 1-loop determinant of a chiral multiplet Φ transforming in the representation ρ_Φ of $G \times K$ develops poles along the hyperplane

$$H_{\rho_\Phi} = \{\rho_G(u) + \rho_K(z) = 0 \mod \mathbb{Z} + \tau\mathbb{Z}\}.$$

We define the union of all the possible singular hyperplanes as $\mathfrak{M}_{\text{sing}} = \bigcup_{\{\Phi\}} H_{\rho_\Phi}$, which contains

$$\mathfrak{M}^*_{\text{sing}} = \left\{u^* \in \mathfrak{M}_{\text{sing}} : \text{at least } \text{rk}\,G \text{ linearly independent } H_{\rho_\Phi} \text{ meet at } u^*\right\}.$$

All in all, a careful analysis of the localisation procedure reveals that the T^2 partition function of an $\mathcal{N} = (0,2)$ GLSM takes the form (2.4.9) (cf. [24; 25])

$$Z_{T^2}(\tau,z) = \frac{1}{|W|} \sum_{u^*\in\mathfrak{M}^*_{\text{sing}}} \text{JK-Res}_{u=u^*}(Q(u^*),\eta) Z_{1-\text{loop}}(\tau,z,u), \qquad (2.4.9)$$

where JK-Res is a linear functional on the space of meromorphic rk G-forms known as the Jeffrey-Kirwan residue, [131; 132]. The JK residue depends on the representations ρ_Φ associated to the hyperplanes meeting at u^*, which we assemble in the *charge matrix* $Q(u^*)$. More precisely, if ℓ hyperplanes meet at u^*, corresponding to weights $\rho_i \in \mathfrak{k}^\vee$ chosen among the 1-loop determinants of the chiral multiplets $\{\Phi\}$, we define $Q(u)$ to be the $(\text{rk}\,G \times \ell)$-matrix whose columns are given by the transposed covectors ρ_i^T, *i.e.* $Q(u^*) = \{\rho_i^T\}_{i=1}^{\ell}$. The residue in (2.4.9) also depends *a priori* on the choice of a generic vector $\eta \in \mathfrak{k}^\vee$, but it's locally constant in η.

Even though the JK residue is, generally speaking, the correct framework for computing partition functions of higher-rank gauge theories, it's actually equivalent to an integration along a suitable cycle over $\mathfrak{k}$. This will be indeed the route we will take when dealing with computations of this kind in Section 3.2.2.

3 SURFACE DEFECTS AND NESTED INSTANTONS

In this chapter, we study the low-energy effective theory of a D7/D3 system on a local four-fold embedded in the ten dimensional IIB superstring supersymmetric background. The D3 branes effective theory is the topologically twisted Vafa-Witten (VW) theory [212] with two extra chiral multiplets in the fundamental describing the D7/D3 open string sector. The D7 branes gauge theory is related to (equivariant) Donaldson-Thomas theory [87] on the fourfold. Actually, we consider these theories in a non-trivial Ω-background corresponding to the equivariant parameters associated to rotations along the non-compact directions of the fourfold. This lead to a refinement of the above mentioned gauge theories. We focus on the case $S = T^2 \times \mathcal{C}$, the last being a Riemann surface with punctures $\{p_i\}$. Surface operators of this four-dimensional gauge theory are real codimension two defects located at $T^2 \times \{p_i\}$.

The effective theory describing the dynamics of such surface defects is obtained in the limit of small area of $\mathcal{C}$ and turns out to be a quiver gauged linear sigma model which flows in the infrared to a non-linear sigma model of maps from T^2 to the moduli space of *nested instantons*. This is a generalisation of the usual ADHM instanton moduli space, structured on the decomposition of the gauge connection at the surface defect. It is obtained from the usual ADHM instanton moduli space by implementing a suitable orbifold action which generates the fractional fluxes of the gauge field at the defect. The partition function of the D7/D3 effective theory computes the equivariant (virtual) elliptic genus of this moduli space in presence of matter content dictated by the topology of $\mathcal{C}$, which, for genus g amounts to g-hypermultiplets in the adjoint representation. Their contribution is encoded in a bundle $\mathscr{V}_g$ over the moduli space of nested instantons. The general formula for the elliptic genus is (3.1.16) which, in the particular case $r = 1$ and $k = 1$, calculates the virtual elliptic genus of the bundle $\mathscr{V}_g$ over the nested Hilbert scheme of points on $\mathbb{C}^2$. The explicit combinatorial expression of (3.1.16) is given by (3.2.34) in terms of nested partitions.

We also study the circle reduction of this system, which leads to a T-dual D6/D2 quantum mechanics. In this case, we find that the generating function of the defects, obtained by summing over all possible decompositions of the connection at the puncture, or in other terms over all possible tails of the quiver, displays a very nice polynomial structure in the equivariant parameters.

The method we used to compute the partition function of the D-brane system is twofold. One, worked out in §3.2.1, makes use of super-localisation formulae [52] directly leading to a sum over fixed points with weights computed from the character of the torus action on the nested instanton moduli space. An alternative derivation is performed in §3.2.2, where the T^2 partition function is evaluated via a higher dimensional contour integral *à la* [170].

This can be also prescribed via Jeffrey-Kirwan residue method [24; 25], as it was used in the study of D1/D7 BPS bound state counting on $\mathbb{C}^3$ in [22]. We remark that although the residue method is computationally more demanding, it has the advantage of allowing the study of wall-crossing among spaces with different stability conditions by changing the integration contour [7; 45].

When one considers a single D7 brane, the nested instanton moduli space reduces to the nested Hilbert scheme of points on $\mathbb{C}^2$. Our brane construction provides a conjectural description of this space as the moduli space of representations of the quiver considered in §3.1.7. Moreover, in this case the summation over the tails of the quiver gives rise to polynomials related to the modified Macdonald polynomials, and the whole partition function is related to the generating function introduced in [119] to describe the cohomology of character varieties. The analogue result for the full T^2 partition function gives rise to special combinations of elliptic functions which can be regarded as an elliptic lift of these polynomials. We display few examples in equations (3.2.48),(3.2.49),(3.2.50). These formulae should encode the elliptic cohomology of character varieties and can be viewed as an elliptic virtual refinement of the generating function of [119]. We remark that the D6/D2 quantum mechanical system and its relation with [119] was studied in [69] via a different approach based on topological string amplitudes on orbifold Calabi-Yau.

The relation with character varieties can be understood from the fact that Vafa-Witten theory on $S = T^2 \times \mathcal{C}$ is known to reduce in the small $\mathcal{C}$ limit to a GLSM on T^2 with target space the Hitchin moduli space over $\mathcal{C}$ [28]. This in turn is homeomorphic [203] to the character variety of $\mathcal{C}$, namely the moduli space of representations of the first fundamental group of $\mathcal{C}\backslash\{p_i\}$ into $GL_n(\mathbb{C})$ with fixed semi-simple conjugacy classes at the punctures.

There are some open questions to be discussed about the above construction. Actually, the 2d $(2,0)$ D3/D7 quiver gauge theory that we consider is *anomalous*, the D3/D7 open string modes breaking $(2,2)$ to $(2,0)$ and generating an R-symmetry anomaly. Indeed instantons in the D7 brane gauge theory are sourced from D3 branes. The mathematical counterpart is that Donaldson-Thomas (DT) theory on fourfolds has positive virtual dimension and requires the insertion of observables to produce the appropriate measure on the moduli space [57; 59]. To cure this, we introduce new fields with opposite representations with respect to the gauge group and global symmetries. These are sources of the insertion of suitable observables which compensate the R-symmetry anomaly. Actually, the extra fields we consider can be thought as arising from coupling of D3 branes to $\overline{\text{D7}}$-branes. It was recently conjectured [200], that $\overline{\text{D7}}$/D7 system undergoes tachyon condensation leaving behind D3-branes. This proposal is a generalisation of the known condensation [199] of $\overline{\text{D5}}$/D5 into D3s. Indeed in our calculations we find that, at special values of the equivariant parameters, the contribution of the D3/$\overline{\text{D7}}$ and D3/D7 modes to the elliptic genus cancels out, in line with the above expectations. It would be extremely interesting to further analyse a possible application of our technique to the string field theoretic description of D-branes/anti D-branes annihilation.

The mathematical implication of all this is that DT theory on the local surface four-fold should reduce to VW theory on the complex surface S itself, the corresponding partition function providing conjectural formulae for VW invariants on S in presence of surface defects. We aim to further investigate this reduction in the future and to analyse the elliptic genus of the nested instanton moduli space and in particular of the nested Hilbert scheme of points on toric surfaces. This can be obtained via gluing the contributions of the local patches [30; 31; 169]. Let underline that our computations concern a *refined* version of VW theory, a refinement being given by the mass m of the adjoint hypermultiplet. Therefore, by studying the limit at $m \to \infty$, with appropriately rescaled gauge coupling, we reduce to pure twisted $\mathcal{N} = 2$ gauge theory computing higher rank equivariant Donaldson invariants of S. Moreover, while in this work we considered $S = T^2 \times \mathcal{C}$, non trivial elliptic fibrations or other product geometries can be studied. In this way our approach could be used to generalise the results on Donaldson invariants of [154; 156]. The general modular properties of these generating functions are worth to be analysed [155; 212].

Finally, the relation of our results to representation theory and quantum integrable systems should be explored, in particular investigating whether the cohomology of the nested instanton moduli space hosts representations of suitable infinite dimensional Lie algebrae, generalising the results of [159; 163; 198]. Also the characterisation of the polynomials appearing in the quantum mechanical limit is to be worked out, by studying recurrence relations and/or difference equations they satisfy. This would possibly open a window on the relation with quantum integrable systems. For example, in [43; 45], the relation between D1/D5 systems on $\mathbb{P}^1$ and quantum Intermediate Long Wave hydrodynamics was studied, finding that the mirror of the associated GLSM provides the Bethe ansatz equations of the latter. Analogous relations between the mirror of the 2d comet-shaped quiver gauge theories and suitable integrable systems are worth to be explored.

The chapter is structured in two main Sections. In the first one we provide the general brane set-up and a detailed derivation of the comet-shaped quiver from D-branes on orbifolds. We then discuss the reduction to quiver quantum mechanics and the relation to character varieties. In the second, we perform explicit computations of the relevant partition functions and the relation with modified Macdonald polynomials of the reduced quantum mechanical quiver theory.

3.1 D-BRANES, GEOMETRY AND QUIVERS

3.1.1 Preliminaries

Let us start by discussing the geometric D-branes set-up.

We consider a Type IIB supersymmetric general background built as the total space of a rank three complex vector bundle V_S^3 on a complex surface S

$$X_5 = \mathrm{tot}\left(V_S^3\right) \tag{3.1.1}$$

where supersymmetry requires the Calabi-Yau condition $\det V_S^3 = \mathcal{K}_S$, where $\mathcal{K}_S$ is the canonical bundle over S. To place a D3-D7 system in such a background, we assume that V_S^3 has the following reduced structure

$$V_S^3 = \mathcal{K}_S \otimes \det^{-1} V_S^2 \oplus V_S^2$$

where the rank two bundle V_S^2 is otherwise unconstrained.

Let us therefore consider the theory of N D3-branes wrapping the complex surface S in the background of r D7-branes along the local surface fourfold $\mathrm{tot}\left(V_S^2\right)$.

The low-energy dynamics of the N D3-branes can be obtained as usual by dimensional reduction of the $\mathcal{N} = 1$ $D = 10$ supersymmetric Yang-Mills theory on X_5 down to their world-volume. This produces a topologically twisted version of the $\mathcal{N} = 4$ $D = 4$ theory on S [29] whose boson content is given by the gauge connection $\mathcal{A}$, a section $\Phi_{\mathcal{L}}$ of the line bundle $\mathcal{L} = \mathcal{K}_S \otimes \det^{-1} V_S^2$ and a doublet Φ_{V^2} which is a section of V_S^2, the latter describing the transverse motion of the D3-branes in the ambient X_5. All these fields are in the adjoint representation of the $U(N)$ gauge group. The above set-up reduces to the Vafa-Witten topologically twisted $\mathcal{N} = 4$ $D = 4$ on S if the rank two vector bundle $V_S^2 = \mathbb{C}^2$ is trivial and therefore $X_5 = \mathrm{tot}\,(\mathcal{K}_S) \oplus \mathbb{C}^2$. In this case, the above construction indeed gives the gauge connection $\mathcal{A}$ on S, a complex $(2,0)$-form Φ_S valued in the fiber of $\mathcal{K}_S$ describing the transverse D3-branes motion within the local surface $X_3 = \mathrm{tot}\,(\mathcal{K}_S)$, while the motion along the remaining $\mathbb{C}^2$ transverse directions is described by two other complex scalars B_i, with $i = 1, 2$.

The effect of the additional r background D7-branes on the D3-branes is kept into account by a further set of two complex scalars I and J in the bifundamental $N \times \bar{r}$ and $r \times \bar{N}$ of the gauge symmetry group $U(N)$ and flavour global $U(r)$ group. These are sections respectively of $\mathcal{O}_S$ and $\det V_S^2$ in general. This follows from the fact that these fields are in the positive chiral spinorial representation of the transverse $SO(4)$ and are therefore sections of $\mathcal{S}_+ \sim \Lambda^{(even,0)}(V_S^2)$, for S a Kahler surface.

The continuous symmetries of this geometric set-up in the transverse directions to the D3-branes are the $(\mathbb{C}^*)^3$-action on the $\mathbb{C}^3$ fiber of V_S^3 with respective weights $(\epsilon_1, \epsilon_2, m)$. These are the global symmetries of the gauge theory on S which can be uses to define the relevant Ω-background after turning on the relative background gauge fields. The parameter m introduces a mass for the adjoint hypermultiplet of the four dimensional theory inducing the supersymmetry breaking from the $\mathcal{N} = 4$ Vafa-Witten theory to its $\mathcal{N} = 2^*$ refined version.

In the following, we will study the above general system in the case in which the complex surface is in the product form $S = T^2 \times \mathcal{C}$, where $\mathcal{C}$ is a Riemann surface and V_S^2 is trivial. In this case, the canonical bundle over S reduces to the holomorphic cotangent bundle over $\mathcal{C}$ and

$$X_5 = \mathrm{tot}\,(T^*\mathcal{C}) \times T^2 \times \mathbb{C}^2 \,.$$

In order to introduce surface defects in the gauge theory, we're going to generalise the above set-up to the case in which $\mathcal{C}$ is punctured at the points where the defects are located. More precisely, the parabolic reduction of the gauge bundle at the punctures is encoded in an orbibundle structure. The effective two dimensional field theory describing the dynamics of the defect is obtained from the above set-up in the chamber of small $\mathcal{C}$ volume leading to a quiver gauged linear sigma model describing the relevant open string modes. In the IR this reduces to a non-linear sigma model of maps from T^2 to the moduli space of representation of the quiver above.

3.1.2 D-branes on the orbicurve and defects

Let us now generalise the above setup to the case in which $\mathcal{C}$ is an orbicurve, that is a Riemann surface with elliptic singular points. This means that the local geometry at some marked points $\{P_\alpha\}$ of $\mathcal{C}$ is that of the $\mathbb{Z}_{s_\alpha}$ quotient of a disk D acted by $z_\alpha \to \omega_\alpha z_\alpha$ with $\omega_\alpha^{s_\alpha} = 1$.

Placing D-branes on an orbicurve consists in excising a regular cylinder out of the total space of the corresponding regular vector bundle and prescribing new local transition functions defining the lift of the discrete group action to the total space of the vector bundle. This operation extends the vector bundle to an orbibundle.

Let us therefore consider the geometry of the D-branes in the vicinity of a marked point P of order s with local coordinate z. The action on the D-brane Chan-Paton factors induces a modification of the gauge symmetry due to D-branes fractionalisation [91]. Let γ_ℓ be the number of D-branes in the ℓ^{th} sector, namely the one corresponding to the charge ℓ representation z^ℓ of $\mathbb{Z}_s$. This corresponds to prescribe the new transition function at the excised disk as

$$g_P = \bigoplus_{\ell=0}^{s-1} z^\ell \mathbf{1}_{\gamma_\ell}$$

and, correspondingly the local behaviour of the gauge connection as

$$\mathcal{A}_P = g_P^{-1} dg_P = \left(\frac{dz}{z}\right) \bigoplus_{\ell=0}^{s-1} \ell \; \mathbf{1}_{\gamma_\ell} = \left(\frac{d\tilde{z}}{\tilde{z}}\right) \bigoplus_{\ell=0}^{s-1} \frac{\ell}{s} \mathbf{1}_{\gamma_\ell} \,,$$

where $\tilde{z} = z^s$. This finally induces the local prescription on the curvature $\mathcal{F} = d\mathcal{A}$ as

$$\mathcal{F}_P/(2\pi) = \sqrt{-1}\delta(\tilde{z}) d\tilde{z} \wedge d\bar{\tilde{z}} \bigoplus_{\ell=0}^{s-1} \left(\frac{\ell}{s}\right) \mathbf{1}_{\gamma_\ell}$$

which implements the realisation of the real co-dimension two defect in the four-dimensional gauge theory. Let us remark that from the algebraic geometry viewpoint this corresponds to study sheaves on root stacks, which is a natural framework were fractional Chern classes appear [54].

One can better describe the resulting gauge theory structure of the local D-brane configuration from the viewpoint of the geometry of the covering disk with local coordinate $\tilde{z} = z^s$.

This is the s-covering of the quotient disk, that is given by the collection of s consecutive Riemann sheets. The γ_ℓ D-branes in the ℓ^{th} sector and their images span ℓ Riemann sheets. As a consequence the ℓ^{th} Riemann sheet is spanned by an overall number of $n_\ell = \sum_{\ell'=\ell}^{s-1} \gamma_{\ell'}$ D-branes. Let us notice

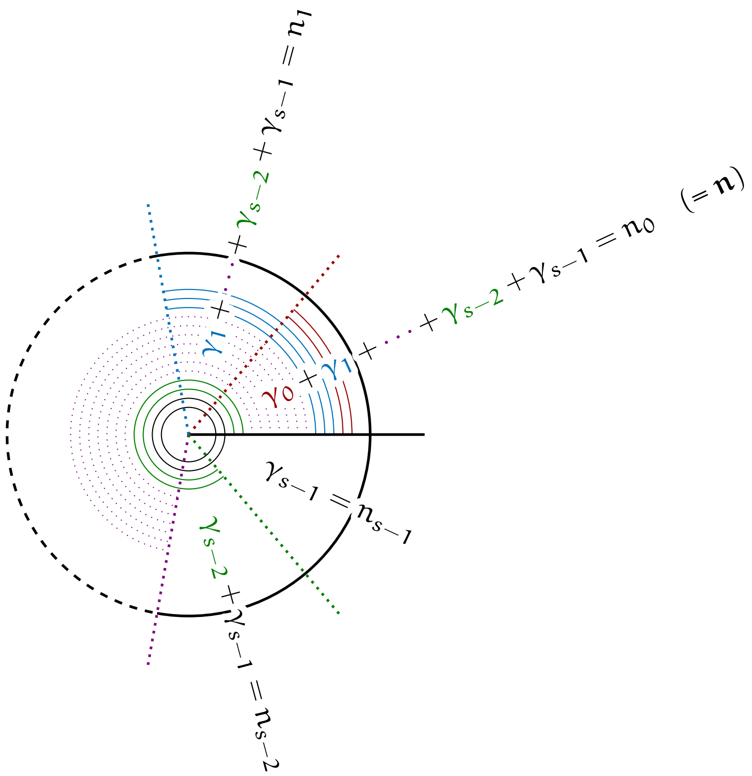


Figure 4: *The "brane cake" describing the covering structure on the local orbifold disk.*

that the outward of the quotient disk is joined to the rest of the Riemann surface by the first Riemann sheet which is consistently covered by all the $n_0 = \sum_{\ell'=0}^{s-1} \gamma_{\ell'} = N$ D-branes.

3.1.3 Two dimensional quiver GLSM from the reduction to small $\mathcal{C}$ volume: bulk part

Let us consider now the reduction to small $\mathcal{C}$ volume of the system above. This leaves behind a gauge theory on the leftover T^2 world volume whose spectrum can be computed by harmonic analysis. We denote by g the genus of $\mathcal{C}$.

Let us first discuss the reduction on a regular Riemann surface and then the more general situation in which $\mathcal{C}$ is an orbicurve.

The complex scalars I and J get simple dimensional reduction and stay scalars in the bifundamental, the gauge connection $\mathcal{A}$ on $S = \mathcal{C} \times T^2$ leaves behind the gauge connection A on T^2 and g complex scalars in the adjoint, while other g complex scalars in the adjoint arise from the reduction of the transverse field Φ_S. These will be denoted as $B_3^{(i)}$ and $B_4^{(i)}$, where $i = 1, \dots, g$.

The other two complex scalar fields in the adjoint, namely B_1 and B_2, get simply dimensionally reduced.

This field content results in the quiver of Fig. 5.

The relations of this quiver can be read from the reduction of the F-term equations in the Appendix A (3.A.1) and (3.A.3) by expanding in harmonic

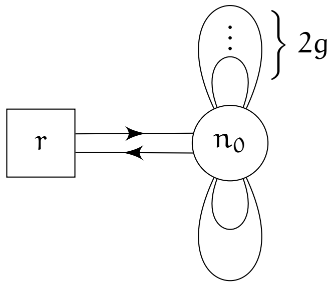


Figure 5: *Quiver gauge theory arising from the compactification on* $\mathcal{C}_{g,0}$.

modes along the curve $\mathcal{C}$. More explicitely, the Φ_S field and the component of the gauge connection $A_{\mathcal{C}}$ along $\mathcal{C}$ give rise to the g hypers in the adjoint representation $(B_3^{(i)}, B_4^{(i)})$, where $i = 1, \ldots, g$, obeying the BPS equations

$$\begin{aligned}
&[B_1, B_2] + IJ = 0\,,\ [B_3^i, B_4^j] = 0 \\
&[B_1, B_3^i] = 0\,,\ [B_1, B_4^i] = 0\,,\ [B_2, B_3^i] = 0\,,\ [B_2, B_4^i] = 0 \\
&B_3^i I = 0\,,\ JB_3^i = 0\,,\ B_4^i I = 0\,,\ JB_4^i = 0
\end{aligned} \tag{3.1.2}$$

The above equations are equivalent to g commuting copies of the ADHM equations for gauge theory with one adjoint hypermultiplet [52], as it can be shown by a simple squaring argument.

In the general Ω-background the supersymmetry of the D3-brane system reduced on T^2 is $(2, 2)$ while the combined D3/D7-brane system reduced on T^2 has $(0, 2)$ supersymmetry due to the presence of the chiral fields I and J and the above field content, augmented by the relevant fermions, form the corresponding multiplets.

Let us underline that this theory itself suffers of a $U(1)_R$-symmetry anomaly due to its chiral unbalanced field content. This can be immediately understood from the fact that the D3-branes profile produces an instanton background in the D7-brane gauge theory inducing chiral symmetry breaking. From the mathematical viewpoint it is known that the Donaldson-Thomas theory on fourfolds has positive virtual dimension which implies that one has to introduce observables matching the dimension counting. We propose that the suitable set of observables is given by a compensating sector of opposite charges – given by $\bar{I}$, $\bar{J}$ and other fields associated to the g-hypers to be specified later – which cancels the anomaly. This sector may be interpreted as a background antiD7-brane system.

3.1.4 Two dimensional GLSM of the defect: the nested instanton quiver

When the curve $\mathcal{C}$ is extended to an orbicurve, at each orbifold point the gauge symmetry is reduced and further 2D degrees of freedom are present. These correspond to the open strings stretching between the twisted D-branes and, from the gauge theory viewpoint, to the degrees of freedom defining the codimension two defect prescribed by the singular behaviour of the gauge curvature at the orbifold points.

To obtain the effective low energy quiver description, we excise a disk around each puncture of $\mathcal{C}$ and discuss the the local behaviour of the D-branes system at the orbifold points computing the associated low energy quiver gauge theory. We then glue back the disks to the bulk Riemann surface obtaining the full description of the gauge theory with defects reduced to two dimensions by the small $\mathcal{C}$-volume limit. This procedure is pictorially described in Fig. 6.

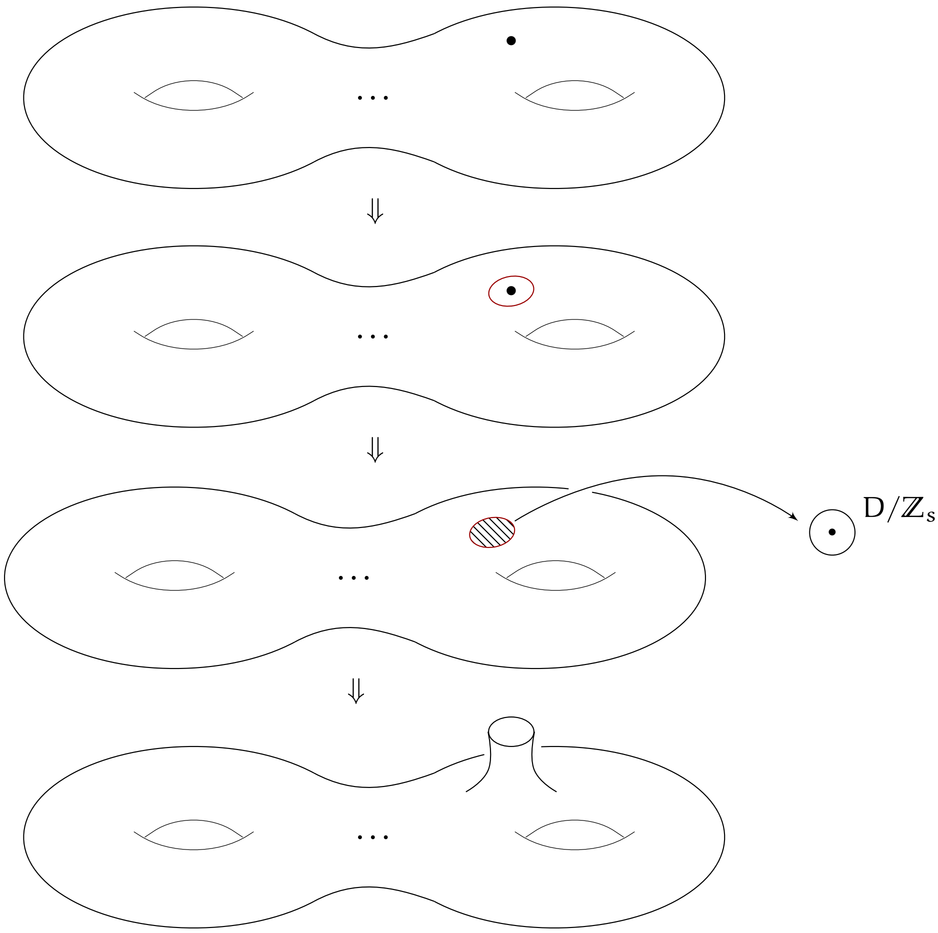


Figure 6: *Disk excision and gluing*

The relevant open strings degrees of freedom can be inferred from the D-branes distribution as in the above Fig. 4. More precisely, see for example [101; 209], the Chan-Paton space of the D-brane system decomposes into irreducible representations R_ℓ of the local discrete group $\mathbb{Z}_s$ as

$$\mathbb{V} = \sum_{\ell=0}^{s-1} \mathbb{V}_\ell \otimes R_\ell \tag{3.1.3}$$

$$\mathbb{W} = \sum_{\ell=0}^{s-1} \mathbb{W}_\ell \otimes R_\ell \tag{3.1.4}$$

where each of the D3 and D7 -brane charged sectors is denoted as

$$\mathbb{V}_\ell = \mathbb{C}^{\gamma_\ell}, \, \mathbb{W}_\ell = \mathbb{C}^{\beta_\ell}. \tag{3.1.5}$$

As depicted in Fig. 4, the ℓ-th Riemann sheet of the covering hosts a net number of $n_j \equiv \sum_{\ell=j}^{s-1} \gamma_\ell$ D3-branes and of $r_j \equiv \sum_{\ell=j}^{s-1} \beta_\ell$ D7-branes so that the open string degrees of freedom are represented as linear maps among the spaces

$$V_j = \sum_{\ell=j}^{s-1} \mathbb{V}_\ell \tag{3.1.6}$$

$$W_j = \sum_{\ell=j}^{s-1} \mathbb{W}_\ell \tag{3.1.7}$$

Let us now discuss the corresponding quiver gauge theory. This consists of a $(0,2)$ quiver gauge theory on T^2 with gauge group $\otimes_{j=0,\dots,s-1} U(n_j)$, each node being coupled to two chiral multiplets in the adjoint $B_1^j, B_2^j \in \mathrm{End}V_j$ and each pair of successive nodes by a chiral in the bifundamental $F^j \in \mathrm{Hom}\left(V_j, V_{j+1}\right)$ for $j = 0, \dots, s-1$. The D3-D7 open strings modes are described by the linear maps $I^j \in \mathrm{Hom}\left(V_j, W_j\right)$ and $J^j \in \mathrm{Hom}\left(W_j, V_j\right)$. Summarizing, the local D3-D7 system is effectively described by

$$\begin{aligned} &B_1^j, B_2^j \in \mathrm{End}V_j\,, F^j \in \mathrm{Hom}\left(V_j, V_{j+1}\right) \\ &I^j \in \mathrm{Hom}\left(V_j, W_j\right) \text{ and } J^j \in \mathrm{Hom}\left(W_j, V_j\right) \end{aligned} \tag{3.1.8}$$

As is shown in the Appendix A these fields obey the relations

$$[B_1^j, B_2^j] + I^j J^j = 0\,, \quad B_1^j F^j - F^j B_1^{j+1} = 0 \quad B_2^j F^j - F^j B_2^{j+1} = 0\,, \quad J^j F^j = 0. \tag{3.1.9}$$

Therefore, the resulting quiver describing the local D3-D7 system at the defect is given in Fig. 7.

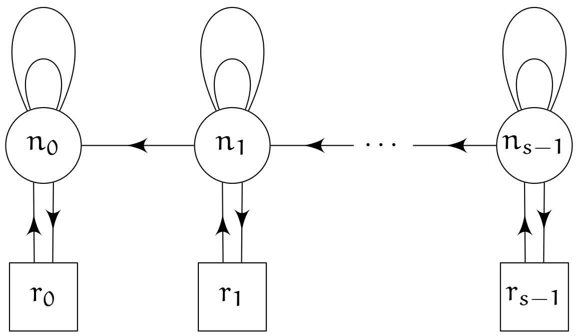


Figure 7: *Quiver gauge theory arising from the compactification on $\mathcal{C}_{g,1}$.*

The moduli space $\mathcal{N}_{r,\lambda,n,\mu}$ of its representations describes *nested instantons*. Indeed the n D3 branes realise an n-instanton profile for the $U(r)$ D7 gauge fields, preserving the flag structure at the puncture. The partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots)$ of r and $\mu = (\mu_1 \geqslant \mu_2 \geqslant \dots)$ of n describe respectively the decomposition of the D7 and D3 Chan-Paton vector spaces into representations of the $\mathbb{Z}_s$ group. More precisely, as shown in fig. 7, one gets the quiver of the flag manifold realised by the Chan Paton vector spaces of the D3 branes $V_{s-1} \subset V_{s-2} \subset \dots \subset V_0$ with dimensions $n_j = n_0 - \sum_{l=1}^{j} \mu_l$ *framed* by the D7 branes vector spaces with dimensions $r_j = r_0 - \sum_{l=1}^{j} \lambda_l$.

The heights of the columns of each partition is obtained from an ordering of the data of the dimensions vector spaces β_ℓ and γ_ℓ of (3.1.5). Indeed, these can be ordered by using Weyl symmetry of D3 and D7 branes gauge groups such that $\gamma_0 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_{s-1}$ and $\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_{s-1}$.

The moduli space of nested instantons has a natural projection to the standard ADHM instanton moduli space $\mathscr{M}_{r,n}$

$$\pi : \mathscr{N}_{r,\lambda,n,\mu} \to \mathscr{M}_{r,n} \tag{3.1.10}$$

which is realised by setting all the open string twisted sectors to be empty, namely by setting to zero all the fields F^j, $j = 0, \ldots, s-1$ and (I^j, J^j, B_1^j, B_2^j) for $j = 1, \ldots, s-1$.

3.1.5 Relation to other quiver defect theories

Some comments are in order regarding the quiver theory of the defect we obtain in our construction with respect to other quiver defect theories. The quiver we study is derived from a Dp/Dp+4 system via an orbifold action which affects a *transverse* direction to both the brane types. In this respect, it is different from the chain-saw quiver describing affine Laumon spaces [137], were the orbifold acts instead on the coordinates B_1, B_2 describing the motion of Dp branes *inside* the Dp+4. This induces a different quiver with a different set of relations. A quiver which relates to the one in [137] can be obtained by considering a different specialization of the general geometric background for the D3/D7 system described in §3.1. More precisely, one can consider $T^2 \times X_6 \times \mathbb{C}_{\epsilon_1}$, where $X_6 = \mathrm{tot}\,[\mathscr{O}(p) \oplus \mathscr{O}(-p+2g-2)]_{\mathcal{C}_{g,k}}$ is the total space of a sum of two line bundles of the compensating degree on the orbicurve. In such a geometry we can consider the D3-branes along $T^2 \times \mathcal{C}_{g,k}$ and the D7 say along $T^2 \times Y_4 \times \mathbb{C}_{\epsilon_1}$, where $Y_4 = \mathrm{tot}\,[\mathscr{O}(p)]_{\mathcal{C}_{g,k}}$ and the fiber still hosts the torus action corresponding to the ϵ_2-parameter of the Omega-background. For $p > 0$ in the vicinity of the orbifold points the geometry in the fiber direction is sensitive to the orbifold group. As a consequence, the corresponding modes in the open string sectors get twisted and the quiver changes by loosing an adjoint multiplet per node which gets a bifundamental, as well as the flavoured fields J_i will now point from the gauge node to the nearby framing node. The resulting local quiver at the defect is then the chain-saw quiver. Correspondingly, the comet shaped quiver would in this case display tails given by chain-saw quivers. This can be also obtained from a D1/D5 system with both D1 and D5 wrapping $\mathcal{C}_{g,k}$ via a double T-duality along transverse directions to both.

Since on the other hand both quivers are describing the parabolic reduction of the gauge connection on a surface defect, it is conceivable to expect that a relation can be found between the associated partition functions at least in some limit or suitable parametrisation. This could require non-trivial combinatorial identities on the partition functions themselves, similarly to what dicussed in [133] concerning the relation between orbifold and vortex-like defects.

Moreover, when decoupling the D7 branes by setting $I^j = 0, J^j = 0$, the description of the D3 branes at the defect lead to the quiver of fig. 8 which describes a flag manifold with extra adjoint hypers at each node.

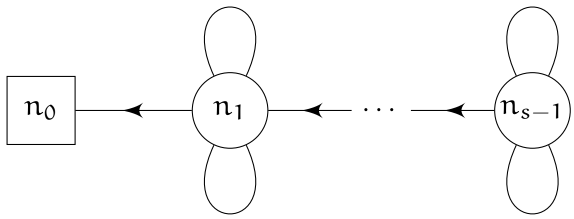


Figure 8: *Quiver gauge theory for D3 branes at a single puncture.*

We notice that also the TSU[N] quivers for defects studied [46; 47; 51; 72; 97; 167] are based on flag manifold quivers but display a different field content. It should be possible to compare the two kind of defect gauge theories in suitable limits by finding an appropriate dictionary.

3.1.6 Nested Hilbert scheme of points

The nested instanton moduli space is expected to reduce for a single D7-brane $r = r_0 = 1$ to the moduli space of the nested Hilbert scheme of points on $\mathbb{C}^2$, $\mathrm{Hilb}^{n,\mu}(\mathbb{C}^2)$. In this particular case the quiver described in the previous subsection reduces to the one of fig. 9 with relations

$$\begin{aligned} &[B_1^0, B_2^0] + I^0 J^0 = 0, \quad [B_1^j, B_2^j] = 0 \ , \quad j \geqslant 1 \\ &B_1^j F^j - F^j B_1^{j+1} = 0 \quad B_2^j F^j - F^j B_2^{j+1} = 0, \quad J^0 F^0 = 0. \end{aligned} \tag{3.1.11}$$

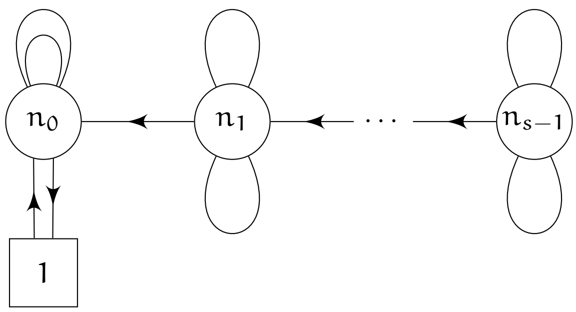


Figure 9: *Nested Hilbert scheme quiver.*

The moduli space of representations of this quiver is expected to provide an explicit description of $\mathrm{Hilb}^{n,\mu}(\mathbb{C}^2)$. This has been indeed proven for the particular case of two-step nested Hilbert scheme $n_j = 0$ for $j \geqslant 2$ in [213], where it is also shown that this variety is smooth for $n_1 = 1$. Indeed it is known that for $n_1 > 1$ the two-step nested Hilbert scheme is singular. Moreover, nested Hilbert schemes with more than two steps are always singular, and *a fortiori* also the nested instanton moduli space. The D3/D7 partition functions we will evaluate via localisation will then compute *virtual* invariants of these moduli spaces, since a perfect obstruction theory for them is expected to exist.

3.1.7 Comet shaped quiver

Finally, the description of the D3/D7 system on the full geometry gives rise to the *comet shaped quiver* of fig. 10.

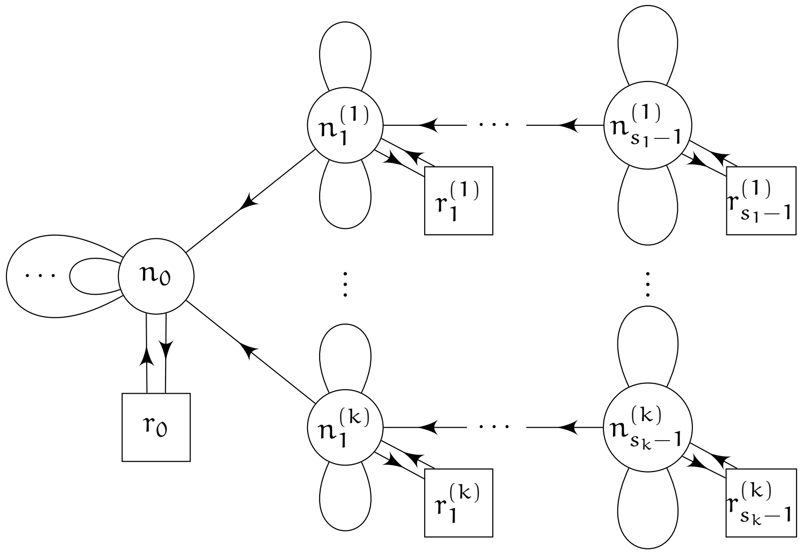


Figure 10: *The comet-shaped quiver.*

This is obtained by gluing the nested instanton moduli quivers describing the decompositions of the branes at the defects to the bulk quiver of Fig. 5. The number of tails in the comet quiver is equal to the number of punctures of the Riemann surface, while their length is related to the flag structure due to the parabolic reduction of the connection at each puncture. All in all, the effective theory describing the D3-D7 system on T^2 reduces to a GLSM with target space the total space of the bundle

$$\mathscr{V}_g \equiv \pi^* \left((T^*\mathscr{M}_{r,n})^{\oplus g} \otimes (\det \mathscr{T})^{1-g} \right) \tag{3.1.12}$$

over the moduli space of nested instantons $\mathscr{N}_{r,\underline{\lambda},n,\underline{\mu}}$, where the collection of partitions $\underline{\lambda} = (\lambda^1, \dots, \lambda^k)$ and $\underline{\mu} = (\mu^1, \dots, \mu^k)$ describe the decomposition of D7 and D3 branes respectively under the cyclic groups $\mathbb{Z}_{s_i}$, $i = 1, \dots k$ acting at the punctures. The physical interpretation of the above bundle is the following: the first factor is simply the contribution of the g hypermultplets in the adjoint representation of the bulk theory described in subsection 3.1.3. Regarding the second factor, let us remark that the couplings of the D3/D7 brane system turns on a background line bundle describing the determinant bundle of the Dirac zero modes in the instanton background. This is given by the determinant of the tautological bundle $\mathscr{T}$ over $\mathscr{M}_{r,n}$. The power $(1-g)$ is due to the multiplicity of fermionic zero-modes on the Riemann surface $\mathcal{C}$. In the limit of degeneration of the T^2 to a circle this leads to a Chern-Simons interaction term for the resulting D2/D6 system. This term is essential in the comparison with results on character varieties and will be discussed in detail in §3.1.9, while in the next one we will briefly recall some basic definitions about character varieties that will be useful for the subsequent discussion.

3.1.8 Character varieties

Given a Riemann surface $\mathcal{C}$ of genus g with k punctures $D = \sum_{i=1}^{k} p_i$, one defines the $GL_n(\mathbb{C})$ *character variety* as the moduli space of representations of the first fundamental group of $\mathcal{C}\backslash D$ into $GL_n(\mathbb{C})$

$$\mathcal{G}_\sigma = \{\rho \in \mathrm{Hom}\left(\pi_1\left(\mathcal{C}\backslash D\right), GL_n(\mathbb{C})\right) | \rho(\gamma_i) \in C_i\} // PGL_n(\mathbb{C}) \qquad (3.1.13)$$

where $C_1, \ldots, C_k \subset GL_n(\mathbb{C})$ are semisimple conjugacy classes of type $\sigma^1, \ldots, \sigma^k$, namely the parts of the partition σ^i, $(\sigma^i_1 \geqslant \sigma^i_2 \geqslant \ldots)$, describe the multiplicities of the eigenvalues of any matrix in the conjugacy class C_i.

When non empty (3.1.13) is a smooth projective variety of dimension

$$d_\sigma = n^2(2g - 2 + k) - \sum_{i,j} \left(\sigma^i_j\right)^2 + 2$$

To describe the cohomology of (3.1.13), Hausel-Letellier-Rodriguez-Villegas [119] introduced the *k- puntures, genus* g *Cauchy function*

$$\Omega(z, w) = \sum_{\sigma \in \mathcal{P}} \mathcal{H}_\sigma(z, w) \prod_{i=1}^{k} \tilde{H}^i_\sigma(\mathbf{x}_i; z^2, w^2) \qquad (3.1.14)$$

where $\mathcal{P}$ is the set of partitions, $\tilde{H}^i_\sigma(\mathbf{x}_i; z^2, w^2)$ are refined Macdonald polynomials and

$$\mathcal{H}_\sigma(z, w) = \prod_{s \in \sigma} \frac{\left(z^{2a(s)+1} - w^{2l(s)+1}\right)^{2g}}{\left(z^{2a(s)+2 - w^{2l(s)}}\right)\left(z^{2a(s)} - w^{2l(s)+2}\right)}. \qquad (3.1.15)$$

where $a(s), l(s)$ are respectively the arm and leg length of the s box of the Young diagram σ representing the partition (see Fig. 3). Eq.(3.1.14) turns out to be the generating function of the cohomology polynomials of $GL_n(\mathbb{C})$ character varieties, summed over n.

Let us now outline the connection with the brane construction described in the previous subsections. The dynamics of D3 branes on the local surface S is refined Vafa-Witten theory. When $S = T^2 \times \mathcal{C}$, this reduces in the limit of small area of $\mathcal{C}$ to a gauged linear sigma model from T^2 to Hitchin's moduli space on $\mathcal{C}$ [28]. On the other hand, in [203] it was proved that (3.1.13) is homeomorphic to the moduli space of Higgs bundles with parabolic reduction on the divisor $D = \sum_{i=1}^{k} p_i$. In presence of D7 branes, the non-perturbative effects on their dynamics are obtained by summing over the D3-branes partition functions. One then naturally obtains a generating function of the elliptic cohomology of $GL_n(\mathbb{C})$ character varieties. Summarising the T^2 partition function of the D3-D7 comet shaped quiver reads

$$Z_{T^2} = \sum_{n} \sum_{\underline{\mu} \in \mathcal{P}(n)^k} (\mathbf{q}^{\underline{\mu}})^{\mathbf{r}}\, \mathrm{Ell}^{\mathrm{vir}}\left(\mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}, \mathcal{V}_g\right), \qquad (3.1.16)$$

with $(\mathbf{q}^{\underline{\mu}})^{\mathbf{r}} = \prod_{i=1}^{k} \prod_{\alpha=0}^{s_i-1} \left(q_{i,\alpha}^{|\mu^i_\alpha|}\right)^{r^i_\alpha}$ and the $\mathrm{Ell}^{\mathrm{vir}}(\mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}, \mathcal{V}_g)$ is defined as in §2.2. For a single D7 brane $\mathbf{r} = r_0 = 1$, the above formulae can be understood as an elliptic virtual generalisation of the generating function introduced by HLRV. Indeed, we will show in the following that in the limit of degeneration of T^2 to a circle, one obtains HLRV formulae, or more precisely a virtual refinement of them.

3.1.9 Reduction to quantum mechanics, Chern-Simons term and HLRV formulae

In this subsection we summarise the reduction of the D3/D7 system on T^2 to a quantum mechanical system in a T-dual picture. More precisely, if the two torus factorises as $T^2 = S^1 \times S^1$ and one of the two circles is taken to be very small, our D-brane system can be T-dualised along the small circle and reduced to a corresponding D2/D6 system on $\mathcal{C} \times S^1$. This corresponds to the quantum mechanics of the comet shaped quiver with a Chern-Simons coupling, given by a phase factor $e^{im\int CS(A,F)} = e^{im\int dx^\mu \mathcal{A}_\mu}$ so that the particle is coupled to an external vector potential. Let us briefly recall how this works in the standard ADHM case [206] in order to then generalise it to the nested instanton moduli space. The partition function is the equivariant index

$$Z_{S^1} = \sum_n q^{nr} \operatorname{Ind}\left(\mathscr{M}_{r,n}, \mathcal{L}^{\otimes m}\right), \tag{3.1.17}$$

where $\mathcal{L}$ is the determinant line bundle $\mathcal{L} = \operatorname{Det} \not{D}$, whose fiber on the space of connections $\mathcal{A}/\mathcal{G}$ is $(\det\ker \not{D}_A)^* \otimes (\det\ker \not{D}_A^\dagger)$. By making use the ADHM construction for the moduli space of ASD connections, the $n-$dimensional vector space V_0 is actually the space of fermionic zero-modes. In order to compute the Chern-Simons level, we make use of the Atiyah-Singer index theorem for a vector bundle $E \to M$

$$\operatorname{Ind}(M, E) = \operatorname{Ind}(\not{D}) = \int_M \hat{A}(TM) \wedge \operatorname{ch} E, \tag{3.1.18}$$

which gives the index of the Dirac operator twisted by E, *ie* $D : S \otimes E \to S \otimes E$, S being the spin bundle over M. To compute the CS level in the case at hand one has to consider the geometric background $S^1 \times T^*\mathcal{C} \times \mathbb{C}^2 \times \mathbb{R}$. Because of the twisting of the supersymmetric theory along $\mathcal{C}$, the $\not{D}$ operator along $\mathcal{C}$ reduces to the $\bar{\partial}$ operator and the roof genus $\hat{A}(TM)$ to the Todd class. Thus, when we take the effective theory obtained by shrinking the size of $\mathcal{C}$, $\operatorname{Ind}(\bar{\partial})_{\mathcal{C}}$ gives the muliplicity of the fermionic zero modes, according to the decomposition $\Psi^{(0)} = \psi^{(0)}_{\mathcal{C}} \otimes \psi^{(0)}_{\mathbb{C}^2}$. The index theorem along $\mathcal{C}$ reads

$$\operatorname{Ind}(\bar{\partial})_{\mathcal{C}} = \int_{\mathcal{C}} \operatorname{Td}(T\mathcal{C}) = 1 - g, \tag{3.1.19}$$

which determines the level of the Chern-Simons interaction to be $m = 1 - g$. Finally, the partition function is given by the following equivariant (virtual) index

$$Z_{S^1} = \sum_n \sum_{\mu \in \mathcal{P}(n)} (\mathbf{q}^\mu)^r \operatorname{Ind}\left(\mathscr{N}_{r,\lambda,n,\mu}, \operatorname{Det}(\not{D})^{\otimes(1-g)}\right), \tag{3.1.20}$$

where we use the notation $\mathbf{q}^{\mu} = q_0^{n_0(\mu)} \cdots q_{s-1}^{n_{s-1}(\mu)}$ and

$$\mathrm{Ind}\left(\mathscr{N}, \mathrm{Det}(\not{D})^{\otimes(1-g)}\right) = \hat{A}(T^{\mathrm{vir}}\mathscr{N})\,\mathrm{ch}\left(\mathrm{Det}(\not{D})^{\otimes(1-g)}\right) \cap [\mathscr{N}]^{\mathrm{vir}}\,. \quad (3.1.21)$$

In the quiver representation of the nested instanton moduli space, the $\not{D}$ operator on $\mathbb{C}^2$ appearing in the above equation is given by the pull-back of the tautological bundle $\mathscr{T}$ on the ADHM moduli space $\mathscr{M}_{r,n}$, so that its determinant line bundle coincides with the one of $\mathscr{T}$, which will be used in the equivariant localisation formulae.

In the following §3.2 we will show that the above partition function, when computed for the particular case of the nested Hilbert scheme of points on $\mathbb{C}^2$, gives a virtual generalization of HLRV formulae and reduces precisely to them when the nested Hilbert scheme is smooth. Let us remark that the quantum mechanical system of the nested Hilbert scheme of points and its relation with HLRV formulae has been studied in [69] via a different approach based on topological string amplitudes on orbifold Calabi-Yaus.

3.2 PARTITION FUNCTIONS

In this section we proceed to the evaluation of the partition function of the effective quiver gauge theories of the D3/D7-system discussed in the previous section in the limit of small volume of the wrapped curve $\mathcal{C}$. This is performed by making use of supersymmetric localisation which is a version of equivariant localisation formulae [187] for super-manifolds which allows a generalisation to supersymmetric path integrals in quantum field theories. The only configurations contributing to the latter are the fixed loci of the supersymmetry transformations. When these are isolated points, the path integral reduces to a sum over them weighted by one-loop super-determinants of the tangent bundle T at those points, that is

$$\sum_{x \in \{FP\}} \frac{e^{-S(x)}}{\mathrm{Sdet} T_x} \quad (3.2.1)$$

where $\{FP\}$ is the set of fixed points, $S(x)$ is the value of the action at $x \in \{FP\}$ and $T_x = T|_x$ is the restriction of T at x.

In the following we will implement the above computational scheme by calculating the above data for the relevant quiver gauge theories on T^2. We will first focus on the contribution of a single defect on the sphere encoding the parabolic reduction of the connection at a given point, which is described by a single legged quiver. Then, we will consider the case of higher genus Riemann surface and combine all the contributions in the comet-shaped quiver theory partition function.

3.2.1 Contribution of a single surface defect on the sphere

The matter content of the GLSM we are interested in is the one summarized in table 1, where $G = U(n_0) \times U(n_1) \times \cdots \times U(n_{s-1})$ and $\square_i$ de-

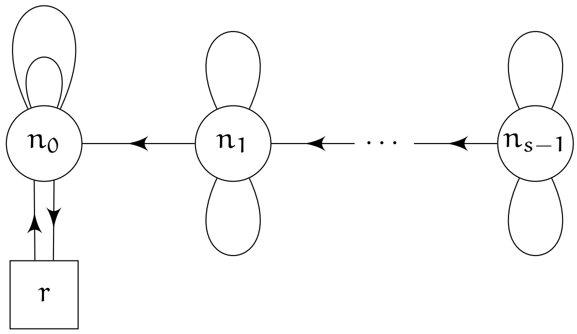


Figure 11: *Low energy GLSM quiver in the case of* $g = 0$, $k = 1$.

notes the Young diagram corresponding to the fundamental representation of $U(n_i)$. The relations satisfied by the quiver GLSM are enforced by the

	gauge G	flavour $U(1) \times U(1)^2$	twisted mass	$R-$charge
B_1^i	$\overline{\Box}_i \otimes \Box_i$	$\mathbb{1}_{(1,0)}$	$-\epsilon_1$	q
B_2^i	$\overline{\Box}_i \otimes \Box_i$	$\mathbb{1}_{(0,1)}$	$-\epsilon_2$	q
I	$\Box_0$	$\overline{\Box}_{(0,0)}$	$-a$	$q+p$
J	$\overline{\Box}_0$	$\Box_{(1,1)}$	$a-\epsilon$	$q-p$
F^i	$\overline{\Box}_i \otimes \Box_{i-1}$	$\mathbb{1}_{(0,0)}$	0	0
χ^i	$\overline{\Box}_i \otimes \Box_i$	$\mathbb{1}_{(-1,-1)}$	ϵ	$-2q$
$\chi_i^{B_1}$	$\Box_i \otimes \overline{\Box}_{i-1}$	$\mathbb{1}_{(-1,0)}$	ϵ_1	$-q$
$\chi_i^{B_2}$	$\Box_i \otimes \overline{\Box}_{i-1}$	$\mathbb{1}_{(0,-1)}$	ϵ_2	$-q$
χ_{JF}	$\Box_1$	$\Box_{(1,1)}$	$\epsilon-a$	$p-q$

Table 1: *Field content for quiver 11*

superpotential $\mathcal{W}$ in (3.2.2).

$$\begin{aligned}\mathcal{W} =& \operatorname{tr}_0\left[\chi_0([B_1^0, B_2^0] + IJ)\right] + \sum_{i=1}^{N} \operatorname{tr}_i\Big[\chi_i[B_1^i, B_2^i] + \chi_i^{B_1}(B_1^{i-1}F^i - F^i B_1^i) + \\ &+ \chi_i^{B_2}(B_2^{i-1}F^i - F^i B_2^i)\Big] + \chi_{JF} J F^1 .\end{aligned} \tag{3.2.2}$$

Let us notice that, as we already pointed out, the locus cut out by $\mathcal{W}$ through the $D-$term equations is overdetermined. Thus we still have to introduce $s-1$ additional chiral fields Q_i, $i = 1, \dots, s-1$ taking care of the relations over the constraints. These additional fields will transform in the $\overline{\Box}_i \otimes \Box_{i-1}$ representation of $U(n_i) \times U(n_{i-1})$. We will assign them $R-$charge $2q$ and they will be charged under the $U(1)^2$ flavour symmetry with charge $(1, 1)$. The relations over the constraints induced by these chirals is

$$\begin{aligned}0 =& [B_1^{i-1}, B_2^{i-1}]F^i + B_2^{i-1}(B_1^{i-1}F^i - F^i B_1^i) - (B_1^{i-1}F^i - F^i B_1^i)B_2^i + \\ &+ (B_2^{i-1}F^i - F^i B_2^i)B_1^i - B_1^{i-1}(B_2^{i-1}F^i - F^i B_2^i) - F^i[B_1^i, B_2^i],\end{aligned}$$

when $i > 1$, while

$$\begin{aligned}0 =& ([B_1^0, B_2^0] + IJ)F^1 + B_2^0(B_1^0 F^1 - F^1 B_1^1) - (B_1^0 F^1 - F^1 B_1^1)B_2^1 + \\ &+ (B_2^0 F^1 - F^1 B_2^1)B_1^1 - B_1^0(B_2^0 F^1 - F^1 B_2^1) - I(JF^1) - F^1[B_1^1, B_2^1]\end{aligned}$$

covers the remaining case $i = 1$.

The chiral supersymmetry transformations of the above fields are

$$\begin{aligned} QI = \mu_I\,, \quad Q\mu_I &= D_A I + \phi^0 I - Ia \\ QJ = \mu_J\,, \quad Q\mu_J &= D_A J - J\phi^0 + aJ - \epsilon J \\ QB^i_l = M^i_l\,, \quad QM^i_l &= D_A B^i_l + [\phi^i, B^i_l] - \epsilon_l B^i_l \\ Q\psi^i_F = F^i\,, \quad QF^i &= D_A\psi^i_F - \phi^i\psi^i_F + \psi^i_F\phi^{i+1} \\ Q\chi_i = h_i\,, \quad Qh_i &= D_A\chi_i + [\phi^i, h_i] + \epsilon h_i \\ Q\chi_{JF} = h_{JF}\,, \quad Qh_{JF} &= D_A\chi_{JF} + [\phi^0, \chi_{JF}] + (\epsilon - a)\chi_{JF} \\ Q\chi^{B_l}_i = h^{B_l}_i\,, \quad Qh^{B_l}_i &= D_A\chi^{B_l}_i + \phi^i\chi^{B_l}_i - \chi^{B_l}_i\phi^{i+1} + \epsilon_l\chi^{B_l}_i \\ Q\chi_{Q_i} = h_{Q_i}\,, \quad Qh_{Q_i} &= D_A\chi_{Q_i} + \phi^{i-1}\chi_{Q_i} - \chi_{Q_i}\phi^i + \epsilon\chi_{Q_i} \\ Q\bar{A} = \eta\,, \quad Q\eta = F_A\,, \quad QA &= 0 \end{aligned}$$

where $(A, \bar{A})$ is the connection on T^2 in holomorphic coordinates and F_A its curvature two-form, ϵ_l, $l = 1, 2$ are the equivariant weights of the $U(1)^2$ rotation group acting on $\mathbb{C}^2$ and $\epsilon = \epsilon_1 + \epsilon_2$. Moreover ϕ^i, $i = 0, \ldots, s-1$ are the zero modes of the A-connection implementing global $U(n_i)$ gauge transformations of the i^{th}-node. The fixed points of the above supersymmetry transformation impose that the connection $(A, \bar{A})$ is flat. Then by a standard squaring argument one can show that the other fields must be constant so that the supersymmetry fixed locus reduces to the fixed locus of the $U(1)^{(r+2)}$-torus action on the nested instanton moduli space, where $U(1)^r$ is the Cartan torus of the $U(r)$ gauge group with equivariant parameters a_b, $b = 1, \ldots, r$.

As we already discussed at the end of §3.1.3, the $(0, 2)$ D3/D7-branes theory displays a $U(1)_R$ anomaly whose compensation can be obtained via the insertion of suitable observables. To this end we introduce a sector of additional degrees of freedom $\bar{I}$ and $\bar{J}$ with opposite gauge global charges w.r.t. I and J which, once integrated out, produces the insertion of the observables. These will be properly taken into account in the following computations.

The characterization of the fixed locus of the torus action on the moduli space of nested instantons $\mathcal{N}(r, n_0, \ldots, n_{s-1}) \simeq \mathcal{N}_{r,[r^1],n,\mu(n)}$, which will be carried out in detail in §4.3.1, is most easily understood by describing it as the moduli space of (suitably defined) stable representations of the quiver in Fig. 11. In this setting we associate to the quiver 11 the vector spaces W and V_i, in addition to the space

$$\mathbb{X} = \mathrm{End}(V_0)^{\oplus 2} \oplus \mathrm{Hom}(V_0, W) \oplus \mathrm{Hom}(W, V_0) \oplus \\ \oplus \left[\bigoplus_{i=1}^{s-1} \left(\mathrm{End}(V_i)^{\oplus 2} \oplus \mathrm{Hom}(V_i, V_i - 1) \right) \right]$$

of the morphisms of the quiver corresponding to the matter fields $B^i_{1,2}$, F^i, I and J. In this language, the quiver in Fig. 11 would be represented graphically as the one in Fig. 12. On $\mathbb{X}$ we have a natural action of $\mathcal{G} = GL(V_0) \times \cdots \times GL(V_{s-1})$, which preserves the subscheme $\mathbb{X}_0$ of those points satisfying the relations (3.1.11). Then, given a framed representation $(W, V_0, \ldots, V_{s-1}, X)$, $X \in \mathbb{X}_0$ of the quiver 12, one can prove that there is a suitable definition of

$$
\begin{tikzcd}
V_{s-1} \arrow[loop above, "B_1^{s-1}"] \arrow[loop below, "B_2^{s-1}"'] \arrow[r, "F^{s-1}"] & \cdots \arrow[r, "F^2"] & V_1 \arrow[loop above, "B_1^1"] \arrow[loop below, "B_2^1"'] \arrow[r, "F^1"] & V_0 \arrow[loop above, "B_1^0"] \arrow[loop below, "B_2^0"'] \arrow[r, shift left, "J"] & W \arrow[l, shift left, "I"]
\end{tikzcd}
$$

Figure 12: *General representation of quiver 11.*

stability such that, in a particular chamber of the parameters at play, semi-stability is equivalent to stability (also as a GIT quotient), so that it makes sense to talk about the moduli space of stable framed representations of the quiver 12 without any further specification. This space will be denoted by $\mathscr{N}(r, n_0, \dots, n_{s-1}) := \mathbb{X}_0 /\!\!/_\chi \mathcal{G}$, for some suitable choice of an algebraic character χ of $\mathcal{G}$.

By means of this construction one can show that there is a sum decomposition $V_0 = V_i \oplus \tilde{V}_i$ and $V_i = V_{i+1} \oplus \hat{V}_{i+1}$, such that $\tilde{V}_i = \hat{V}_i \oplus \tilde{V}_{i-1}$. This splitting also induces the following block matrix decomposition of the morphisms $B^0_{1,2}$, I and J in (3.2.3),

$$
B_1^0 = \begin{pmatrix} B_1^i & B_1'^i \\ 0 & \tilde{B}_1^i \end{pmatrix}, \quad B_2^0 = \begin{pmatrix} B_2^i & B_2'^i \\ 0 & \tilde{B}_2^i \end{pmatrix}, \quad I = \begin{pmatrix} I'^i \\ \tilde{I}^i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \tilde{J}^i \end{pmatrix}. \tag{3.2.3}
$$

such that $(W, \tilde{V}_i, \tilde{B}_1^i, \tilde{B}_2^i, \tilde{I}^i, \tilde{J}^i)$ is a stable ADHM datum.

Once an equivariant action of a torus $T \curvearrowright \mathscr{N}(r, n_0, \dots, n_{s-1})$ is introduced in the natural way suggested by the SUSY construction of the quiver (11), the previous observations makes it possible to characterize the T-fixed locus of $\mathscr{N}(r, n_0, \dots, n_{s-1})$ in terms of those of some moduli spaces of stable ADHM data. However, in order to give a combinatorial description of the T-fixed points of $\mathscr{N}(r, n_0, \dots, n_{s-1})$ we need the following definitions.

Definition 3.1. *Let $\mathcal{P}$ be the set of non-increasing finite sequences of positive integers. A partition of an integer $n \in \mathbb{Z}_{>0}$ is an element $\mu = (\mu^1, \dots, \mu^m) \in \mathcal{P}$ such that $\mu^1 + \dots + \mu^m = n$ (of course we may extend the definition to include the case $n = 0$ by defining its only partition to correspond to the empty sequence). We define the size of μ as the integer $|\mu| = n$. Similarly, given $r \in \mathbb{Z}_{>0}$, an r-coloured partition $\boldsymbol{\mu}$ of n is an r-tuple of ordinary partitions $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r) \in \mathcal{P}^r$ such that $|\mu_1| + \dots + |\mu_r| = n$. We will sometimes refer to the i-th component μ_i of $\boldsymbol{\mu}$ as the component labelled by the i-th colour. The set of partitions (resp. r-coloured partitions) of an integer n form a finite subset $\mathcal{P}(n) \subset \mathcal{P}$ (resp. $\mathcal{P}^r(n) \subset \mathcal{P}^r$).*

Definition 3.2. *Two partitions $\mu_1 = (\mu_1^1, \dots, \mu_1^{m_1}) \in \mathcal{P}$ and $\mu_2 = (\mu_2^1, \dots, \mu_2^{m_2}) \in \mathcal{P}$ form a nested couple $\mu_1 \subseteq \mu_2$ if $\mu_1 \leqslant \mu_2$ and $\mu_1^i \leqslant \mu_2^i$ for $i = 1, \dots, m_1$. Similarly, a couple of r-coloured partitions $\boldsymbol{\mu}_1 \in \mathcal{P}^r$ and $\boldsymbol{\mu}_2 \in \mathcal{P}^r$ will be a nested couple $\boldsymbol{\mu}_1 \subseteq \boldsymbol{\mu}_2$ if $|\boldsymbol{\mu}|_1 \leqslant |\boldsymbol{\mu}|_2$ and if $\mu_{1,i} \subseteq \mu_{2,i}$ for all the colours $i = 1, \dots, r$. The nesting relation makes $(\mathcal{P}, \subseteq)$ and $(\mathcal{P}^r, \subseteq)$ into partially ordered sets.*

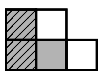
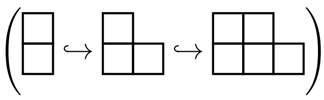
We now have all the needed ingredients to state the following proposition

Proposition 3.3. *The T-fixed locus $[\mathscr{N}(r, n_0, \dots, n_{s-1})]^T$ of $\mathscr{N}(r, n_0, \dots, n_{s-1})$ is described by s−tuples of nested coloured partitions $\boldsymbol{\mu}_1 \subseteq \dots \subseteq \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$, with $|\boldsymbol{\mu}_0| = n_0$ and $|\boldsymbol{\mu}_{i>0}| = n_0 - n_i$.*

Example 3.4. *As an example, consider the moduli space* $\mathcal{N}(2,3,2,1)$*. Its fixed point locus will be described by the following couples of nested partitions,*

$$[\mathcal{N}(2,3,2,1)]^{T} \leftrightarrow \begin{cases} (1^1,2^1,3^1;\emptyset)\,,(1^1,2^1,2^1;1^1)\,,(1^1,2^1;1^1,1^1,1^1)\,,\\ (1^1,1^1,2^1;1^1,1^1)\,,(1^1;1^1,2^1,2^1)\,,(1^1,1^1,1^1;1^1,2^1)\,,\\ (1^1,1^1;1^1,1^1,2^1)\,,(\emptyset,1^1,2^1,3^1)\,,(1^1,2^1,2^11^1;\emptyset)\,,\\ (\emptyset;1^1,2^1,2^11^1)\,,(1^1,1^2,2^11^1;\emptyset)\,,(\emptyset;1^1,1^2,2^11^1)\,,\\ (1^1,1^2,1^3;\emptyset)\,,(1^1,1^2,1^2;1^1)\,,(1^1,1^2;1^1,1^1)\,,\\ (1^1,1^1,1^2;1^1,1^1)\,,(1^1;1^1,1^2,1^2)\,,(1^1,1^1,1^1;1^1,1^2)\,,\\ (1^1,1^1;1^1,1^1,1^2)\,,(\emptyset;1^11^21^3)\,, \end{cases}$$

where each term on the r.h.s. has to be interpreted as a couple of nested partitions, e.g.

$(1^2,2^11^1,3^12^1;\emptyset) \longleftrightarrow$  $\oplus\,\emptyset \longleftrightarrow$  $\oplus\,\emptyset.$

The notation we use for a partition $\mu \in \mathcal{P}$ *is descriptive of its corresponding Young diagram in the following sense:* $[\mu_1^{i_1}\cdots\mu_j^{i_j}\cdots]$ *denotes the partition*

$$\mathcal{P} \ni [\mu_1^{i_1}\cdots\mu_j^{i_j}\cdots] = (\underbrace{\mu_1,\dots,\mu_1}_{i_1},\dots,\underbrace{\mu_j,\dots,\mu_j}_{i_j},\dots),$$

or, in other words, i_j *counts the number of rows of length* μ_j *stacked one over the other.*

The super determinant weighting the contribution of each fixed point can be computed from the character decomposition of the torus action on the (virtual) tangent space:

$$\begin{aligned} T_Z^{\mathrm{vir}}\mathcal{N} = {} & \mathrm{End}(V_0)\otimes(Q-1-\Lambda^2Q)+\mathrm{Hom}(W,V_0)+\mathrm{Hom}(V_0,W)\otimes\Lambda^2Q+\\ & -\mathrm{Hom}(V_1,W)\otimes\Lambda^2Q+\\ & +\sum_{\ell=1}^{s-1}[\mathrm{End}(V_\ell)-\mathrm{Hom}(V_\ell,V_{\ell-1})]\otimes(Q-1-\Lambda^2Q) \end{aligned} \tag{3.2.4}$$

where Q denotes the representation T_1+T_2 in the representation ring $R(T)$. In the previous presentation of the virtual tangent space, the first line accounts for the standard ADHM quiver (B_1^0, B_2^0, I, J) and their constraints, the second line for the constraint $JF^1 = 0$ and the third line for the maps in the tail, their constraints and the relations among them.

By decomposing the vector spaces V_i in terms of characters of the torus action $T \curvearrowright \mathcal{N}_{r,[r^1],n,\mu}$ we can then study the character decomposition of the virtual tangent space to the moduli space of nested instantons

$$\begin{aligned} T^{\mathrm{vir}}_Z \mathcal{N}_{r,[r^1],n,\mu} =& T_{\tilde{Z}} \mathcal{M}_{r,n_0} + \sum_{a,b=1}^{r} \sum_{i=1}^{M_0^{(a)}} \sum_{j=1}^{N_0^{(b)}} R_b R_a^{-1} \left(T_1^{i-\mu_{1,j}^{(b)}} - T_1^i \right) \\ & \left(T_2^{-j+\mu_{1,i}^{(a)\prime}+1} - T_2^{-j+\mu_{0,i}^{(a)\prime}+1} \right) - \sum_{i=1}^{M_0^{(a)}} \sum_{j=1}^{\mu_{0,i}^{(a)\prime}-\mu_{1,i}^{(a)\prime}} T_1^i T_2^{j+\mu_{1,i}^{(a)\prime}} + \\ & + \sum_{k=2}^{s-1} \left[\sum_{a,b=1}^{r} \sum_{i=1}^{M_0^{(a)}} \sum_{j=1}^{N_0^{(b)}} R_b R_a^{-1} \left(T_1^{i-\mu_{k,j}^{(b)}} - T_1^{i-\mu_{k-1,j}^{(b)}} \right) \right. \\ & \left. \left(T_2^{-j+\mu_{k,i}^{(a)\prime}+1} - T_2^{-j+\mu_{0,i}^{(a)\prime}+1} \right) \right] + (s-1)(T_1 T_2), \end{aligned} \tag{3.2.5}$$

where the fixed point Z is to be identified with a choice of a sequence of coloured nested partitions $\boldsymbol{\mu}_1 \subseteq \boldsymbol{\mu}_{N-1} \subseteq \cdots \subseteq \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$, as in Prop. 3.3, $\tilde{Z} \leftrightarrow \boldsymbol{\mu}_0$, μ' denotes the partition transposed to μ and T_i, R_a are the generators of the torus action of $U(1)^{r+1}$. Let us also point out that the last term, namely $(s-1)(T_1 T_2)$, has been added in order to take into account the over-counting in the relations $[B_1^i, B_2^i] = 0$ due to the commutator being automatically traceless.

Having the character decomposition of the virtual tangent space to the moduli space of nested instantons enables us to easily compute the 2d $\mathcal{N} = (0,2)$ partition functions of the low energy GLSM of §3.1.4 in terms of the eigenvalues of the torus action, which we will do in the particular case of $r = 1$ for the sake of simplicity. The partition function we want to compute on the sphere $\mathcal{C}_0 = S^2$ with 1 marked point will take the form

$$\mathcal{Z}_1^{\mathrm{ell}}(S^2; \mathbf{q}) = \sum_{\mu_1 \subseteq \cdots \subseteq \mu_0} q_0^{|\mu_0|} q_1^{|\mu_0 \setminus \mu_1|} \cdots q_{s-1}^{|\mu_0 \setminus \mu_{s-1}|} Z^{\mathrm{ell}}_{(\mu_0, \mu_1, \ldots, \mu_{s-1})}, \tag{3.2.6}$$

with $\mathbf{q} = (q_0, \ldots, q_{s-1})$, $|\mu_i \setminus \mu_j| = |\mu_i| - |\mu_j|$ denoting the number of boxes in the skew Young diagram $Y_{\mu_i \setminus \mu_j}$, while $Z^{\mathrm{ell}}_{(\mu_0, \ldots, \mu_{s-1})}$ is the contribution at a fixed instanton profile.

In particular, once we fix an instanton configuration by choosing a sequence of nested partitions $\mu_1 \subseteq \cdots \subseteq \mu_{s-1} \subseteq \mu_0$ we can write the torus partition function as

$$Z^{\mathrm{ell}}_{(\mu_0, \mu_1, \ldots, \mu_{s-1})} = \mathcal{L}^{\mathrm{ell}}_{\mu_0} \mathcal{N}^{\mathrm{ell}}_{\mu_0} \overline{\mathcal{N}}^{\mathrm{ell}}_{\mu_0} \mathcal{T}^{\mathrm{ell}}_{\mu_0, \mu_1} \overline{\mathcal{T}}^{\mathrm{ell}}_{\mu_0, \mu_1} \mathcal{W}^{\mathrm{ell}}_{\mu_0, \ldots, \mu_{s-1}},$$

where

$$\mathcal{L}^{\mathrm{ell}}_{\mu_0} = \prod_{s \in Y_{\mu_0}} \exp\left[-\mathrm{vol}(T^2) \left(\phi(s) - \xi \right) \right], \tag{3.2.7}$$

$$\mathcal{N}^{\mathrm{ell}}_{\mu_0} = \prod_{s \in Y_{\mu_0}} \frac{1}{\theta_1(\tau | E(s)) \theta_1(\tau | E(s) - \epsilon)}, \tag{3.2.8}$$

$$\overline{\mathcal{N}}^{\text{ell}}_{\mu_0} = \prod_{s \in Y_{\mu_0} \setminus \square} \theta_1(\tau|\phi(s) - \tilde{a})\theta_1(\tau|\phi(s) - \tilde{a} + \epsilon), \tag{3.2.9}$$

$$\mathcal{T}^{\text{ell}}_{\mu_0,\mu_1} = \prod_{i=1}^{M_0} \prod_{j=1}^{\mu_{0,i}-\mu_{1,i}} \theta_1(\tau|\epsilon_1 i + \epsilon_2(j + \mu'_{1,i})), \tag{3.2.10}$$

$$\overline{\mathcal{T}}^{\text{ell}}_{\mu_0,\mu_1} = \prod_{s \in Y_{\mu_0} \setminus \mu_1} \frac{1}{\theta_1(\tau|\phi(s) - \tilde{a} + \epsilon)}, \tag{3.2.11}$$

$$\begin{aligned} \mathcal{W}_{\mu_0,\ldots,\mu_{s-1}} = \prod_{k=0}^{s-2} \Bigg[& \prod_{i=1}^{M_0}\prod_{j=1}^{N_0} \frac{\theta_1(\tau|\epsilon_1(i+\mu_{k,j}) + \epsilon_2(\mu'_{k+1,i} - j + 1))}{\theta_1(\tau|\epsilon_1(i-\mu_{k+1,j}) + \epsilon_2(\mu'_{k+1,i} - j + 1))} \\ & \prod_{i=1}^{M_0}\prod_{j=1}^{N_0} \frac{\theta_1(\tau|\epsilon_1(i-\mu_{k+1,j}) + \epsilon_2(\mu'_{0,i} - j + 1))}{\theta_1(\tau|\epsilon_1(i-\mu_{k,j}) + \epsilon_2(\mu'_{0,i} - j + 1))} \Bigg], \end{aligned} \tag{3.2.12}$$

and for any box s in a Young diagram Y_μ we defined $\phi(s)$ to be the quantity (3.2.13)

$$\phi(s) = a + (i-1)\epsilon_1 + (j-1)\epsilon_2, \tag{3.2.13}$$

and

$$E(s) = -\epsilon_1 a(s) + \epsilon_2(l(s) + 1), \tag{3.2.14}$$

with $a(s)$ and $l(s)$ being respectively the arm and leg length of s in Y_μ.

Notice that $\mathcal{N}^{\text{ell}}_{\mu_0}$ is the elliptic analogue of the Nekrasov partition function, while $\overline{\mathcal{N}}^{\text{ell}}_{\mu_0}$ is its $\overline{\text{ADHM}}$ analogue due to the $\overline{\text{D7}}$ coupling, and by $\tilde{a}$ we mean its Coulomb parameter. As in [177; 225] the $\overline{\text{D7}}$-branes get stabilised in presence of a B-field flux, so that their presence doesn't break supersymmetry and the low energy description of the D7/$\overline{\text{D7}}$ system is that of a $U(N|N)$ gauge theory, [84; 142; 185]. Moreover the contributions from the functions $\mathcal{T}^{\text{ell}}_{\mu_0,\mu_1}$, $\overline{\mathcal{T}}^{\text{ell}}_{\mu_0,\mu_1}$ and $\mathcal{W}_{\mu_0,\ldots,\mu_{s-1}}$ altogether encodes the contribution of the surface defect insertion. Finally, $\mathcal{L}^{\text{ell}}_{\mu_0}$ encodes the CS-like term we discussed in §3.1.7 and 3.1.9. This is interpreted as a CS-term contribution when the limit to QM is taken, and a 5d partition function on $\mathbb{R}^4 \times S^1$ is retrieved. In any case, it comes from the coupling to a background connection on the determinant line bundle Det $\not{D}$ encoding fermionic zero modes. This background connection is mirrored by the presence of ξ in (3.2.7), which is intended to be later specialized to $\xi \to a$.

Because of the previous observations it is instructive to perform the summation over all the sequences of s nested partitions in two steps. First we sum over all the smaller partitions $\mu_1 \subseteq \cdots \subseteq \mu_{s-1} \subseteq \mu_0$ at fixed $\mu_0 \in \mathcal{P}$. It will prove useful for what we will do later to define the rational function $P^{\text{ell}}_{\mu_0}$ as in (3.2.15).

$$P^{\text{ell}}_{\mu_0} = \sum_{\mu_1 \subseteq \cdots \subseteq \mu_{s-1}} \mathcal{T}^{\text{ell}}_{\mu_0,\mu_1} \overline{\mathcal{T}}^{\text{ell}}_{\mu_0,\mu_1} \mathcal{W}^{\text{ell}}_{\mu_0,\ldots,\mu_{s-1}} q_1^{|\mu_0 \setminus \mu_1|} \cdots q_{s-1}^{|\mu_0 \setminus \mu_{s-1}|}. \tag{3.2.15}$$

Finally, by summing also over the μ_0 partitions we can rewrite the full partition function as in (3.2.16),

$$\mathcal{Z}^{\text{ell}}_1(S^2; q_0, \ldots, q_{s-1}) = \sum_{\mu_0} q_0^{|\mu_0|} \mathcal{Y}^{\text{ell}}_{\mu_0} P^{\text{ell}}_{\mu_0}, \tag{3.2.16}$$

where we defined

$$\mathcal{Y}^{\text{ell}}_{\mu_0} = \mathcal{L}^{\text{ell}}_{\mu_0}\mathcal{N}^{\text{ell}}_{\mu_0}\overline{\mathcal{N}}^{\text{ell}}_{\mu_0}, \tag{3.2.17}$$

and $P^{\text{ell}}_{\mu_0}$ are particular elliptic functions which can be regarded as an elliptic virtual uplift of modified Macdonald polynomials. The first few examples are listed in (3.2.48),(3.2.49),(3.2.50). As a useful remark, we want to point out that by taking the limit $q_{i>0} \to 0$ we can effectively switch off the tail of the quiver, since $P^{\text{ell}}_{\mu_0} \xrightarrow{q_{i>0}\to 0} 1$, and we recover the partition function on the sphere with one puncture of trivial holonomy, $\mathcal{Z}^{\text{ell}}_0(S^2; q_0)$.

3.2.2 An alternative derivation: contour integral formulae

In this section we will be explicitly computing the partition functions of the low energy theory coming from the D3/D7 system described in §3.1.4 by reducing the supersymmetric path integral to a contour integral via supersymmetric localization [24; 25].

The model we are interested in gives rise to a 2d $\mathcal{N} = (0, 2)$ GLSM on T^2. The mechanism of supersymmetry breaking from the maximal amount to $\mathcal{N} = (0, 2)$ in the reduction to the low energy theory leaves us with a matter content comprised of chiral fields corresponding to the morphisms in the representation theory of quiver 9 in the category of vector spaces, and Fermi fields implementing the Lagrange multipliers in the superpotential. Let us first study the partition function for the quiver GLSM of Fig. 11, having fixed the numerical type of the quiver to $(1, n_0, \dots, n_{s-1})$. In this case the localization formula is given by

$$Z_{T^2} = \frac{1}{(2\pi i)^N} \oint_C Z_{T^2,1-\text{loop}}(\tau, z, x) \tag{3.2.18}$$

where C is a real N-dimensional cycle in the moduli space of flat connections on T^2 to be fixed with the Jeffrey-Kirwan prescription, x denotes the collection of the (exponentiated) coordinates we are integrating over and

$$\begin{aligned}
\hat{Z}_{T^2,1-\text{loop}}(\tau, z, x) = \tilde{\mathcal{Z}} &\left(\prod_{i\neq j}^{n_0} \frac{\theta_1(\tau|u^0_{ij})\theta_1(\tau|u^0_{ij} - zq + \epsilon)}{\theta_1(\tau|u^0_{ij} + zq/2 - \epsilon_1)\theta_1(\tau|u^0_{ij} + zq/2 - \epsilon_2)} \right.\\
&\left. \prod_{i=1}^{n_0} \frac{1}{\theta_1(\tau|u^0_i + z(q+p)/2 - a)\theta_1(\tau|u^0_i - z(q-p)/2 - a + \epsilon)} \right)\\
&\prod_{k=1}^{s-1} \left(\prod_{i\neq j}^{n_k} \frac{\theta_1(\tau|u^k_{ij})\theta_1(\tau|u^k_{ij} - zq + \epsilon)}{\theta_1(\tau|u^k_{ij} + zq/2 - \epsilon_1)\theta_1(\tau|u^k_{ij} + zq/2 - \epsilon_2)} \right.\\
&\left. \prod_{i=1}^{n_k}\prod_{j=1}^{n_{k-1}} \frac{\theta_1(\tau|u^k_i - u^{k-1}_j - zq/2 + \epsilon_1)\theta_1(\tau|u^k_i - u^{k-1}_j - zq/2 + \epsilon_2)}{\theta_1(\tau|u^{k-1}_j - u^k_i)\theta_1(\tau|u^{k-1}_j - u^k_i + zq - \epsilon)} \right)\\
&\prod_{i=1}^{n_1} \theta_1(\tau|u^1_i + z(p-q)/2 - a + \epsilon),
\end{aligned} \tag{3.2.19}$$

with

$$\tilde{\mathcal{Z}} = \prod_{i=0}^{s-1}\left[\frac{1}{n_i!}\left(\frac{2\pi\eta^2(\tau)\theta_1(\tau|-zq+\epsilon)}{\theta_1(\tau|zq/2-\epsilon_1)\theta_1(\tau|zq/2-\epsilon_2)}\right)\right]\frac{(\eta^2(\tau))^{n_0}}{(i\eta(\tau))^{n_1}}.$$

As was already pointed out in §3.1.3, the coupling of the D3-branes to the D7-branes makes the theory anomalous. This chiral anomaly is encoded in the contributions dependent on the fields coupled to the framing, namely I and J, which break a chiral half of the original $\mathcal{N}=(2,2)$ supersymmetry. From the point of view of the localization formula this is most easily made manifest by studying the transformation properties of the integrand under shifts along the generators of the torus. Let us the recall that the Jacobi $\theta_1(\tau|z)$ function is defined in terms of the exponentiated modular parameter $q=e^{2\pi i\tau}$, $\Im\tau\geqslant 0$, and $y=e^{2\pi i z}$ as

$$\theta_1(\tau|z) = q^{1/8}y^{-1/2}(q,q)_\infty\theta(\tau|z),$$

where $\theta(\tau|z)=(y,q)_\infty(qy^{-1},q)_\infty$ and $(a,q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$ is the $q-$Pochhammer symbol. By this definition it is easy to see that the Jacobi function $\theta_1(\tau|z)$ is odd in z, *i.e.* $\theta_1(\tau|-z)=-\theta_1(\tau|z)$, and that it is quasi-periodic under shifts $z\to z+a+b\tau$, $a,b\in\mathbb{Z}$:

$$\theta_1(\tau|z+a+b\tau) = (-1)^{a+b}e^{-2\pi i bz}e^{-i\pi b^2\tau}\theta_1(\tau|z), \qquad \forall a,b\in\mathbb{Z}.$$

The anomaly then comes from the fact the integrand is unbalanced in terms of the theta functions, exactly due to the presence of I and J (indeed their contribution appears in the second and last lines in Eq. (3.2.19)). The part of the $1-$loop determinant coming from adjoint and bifundamental fields does not contribute to the gauge anomaly, as it comes from an $\mathcal{N}=(2,2)$ multiplet. As we already explained in §3.1.3, we take care of this anomaly by introducing extra Fermi fields $\bar{I}$ and $\bar{J}$, which we think can be interpreted as accounting for interactions with $\overline{\text{D7}}-$branes. In this way we get that the T^2 partition function is corrected by the presence of the $\overline{\text{D7}}$ as

$$\begin{aligned}\hat{Z}^{D3/D7/\overline{D7}}_{T^2,1-\text{loop}}(\tau,z,x) = \hat{\mathcal{Z}}&\left(\prod_{i\neq j}^{n_0}\frac{\theta_1(\tau|u^0_{ij})\theta_1(\tau|u^0_{ij}-zq+\epsilon)}{\theta_1(\tau|u^0_{ij}+zq/2-\epsilon_1)\theta_1(\tau|u^0_{ij}+zq/2-\epsilon_2)}\right.\\ &\left.\prod_{i=1}^{n_0}\frac{\theta_1(\tau|u^0_i+zR_{\bar{I}}/2-\bar{a})\theta_1(\tau|u^0_i+zR_{\bar{J}}/2-\bar{a}+\epsilon)}{\theta_1(\tau|u^0_i+z(q+p)/2-a)\theta_1(\tau|u^0_i-z(q-p)/2-a+\epsilon)}\right)\\ &\prod_{k=1}^{s-1}\left(\prod_{i\neq j}^{n_k}\frac{\theta_1(\tau|u^k_{ij})\theta_1(\tau|u^k_{ij}-zq+\epsilon)}{\theta_1(\tau|u^k_{ij}+zq/2-\epsilon_1)\theta_1(\tau|u^k_{ij}+zq/2-\epsilon_2)}\right.\\ &\left.\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k-1}}\frac{\theta_1(\tau|u^k_i-u^{k-1}_j-zq/2+\epsilon_1)\theta_1(\tau|u^k_i-u^{k-1}_j-zq/2+\epsilon_2)}{\theta_1(\tau|u^{k-1}_j-u^k_i)\theta_1(\tau|u^{k-1}_j-u^k_i+zq-\epsilon)}\right)\\ &\prod_{i=1}^{n_1}\frac{\theta_1(\tau|u^1_i+z(p-q)/2-a+\epsilon)}{\theta_1(\tau|u^1_i+zR_{\bar{J}F}/2-\bar{a}+\epsilon)},\end{aligned} \tag{3.2.20}$$

with

$$\hat{\mathcal{Z}} = (-1)^{n_1} \prod_{i=0}^{s-1} \left[\frac{1}{n_i!} \left(\frac{2\pi\eta^2(\tau)\theta_1(\tau|-zq+\epsilon)}{\theta_1(\tau|zq/2-\epsilon_1)\theta_1(\tau|zq/2-\epsilon_2)} \right)^{n_i} \right].$$

Two observations are due here.

1. An appropriate choice of the R−charges $R_{\bar{I}}$ and $R_{\bar{J}}$ (which also determines the R−charge $R_{\bar{J}F}$ relative to the multiplier for the constraint $JF = 0$) makes it possible to overcome completely the anomaly issue in the integration variables and in the $U(1)_R$ fugacity. However, asking for the double periodicity of the integrand forces us also to impose a constraint on the twisted masses a and $\bar{a}$, namely $\tilde{a} = a - \bar{a} \in \mathbb{Z}$. This condition is responsible for the fact that introducing the extra fields needed to cure the anomaly doesn't change the fixed point structure of the localization computation. The procedure we adopted has one additional beneficial side-effect. In fact, even though the theory involving the $\overline{D7}$ branes is different from the one we started with, however it is still an interesting quantity, as it should compute a generating functions for insertions of observables, as it was proposed in the $D8/\overline{D8}$ case by Nekrasov in [173].

2. As for the second remark, it is interesting to study the QM limit ($\tau \to i\infty$) of the partition function at hand. In fact when we shrink one S^1 in T^2 to a point, we can decouple the contribution of the $\overline{D7}$ branes by taking very large values of $\bar{a}$ and by then rescaling the relevant gauge coupling. By doing this we recover the 5d partition function one can independently compute on $\mathbb{R}^4 \times S^1$, apart from an overall normalization factor. This will give us the equivariant Euler number of the nested Hilbert scheme of points on $\mathbb{C}^2$, possibly twisted by a power of the determinant line bundle of the Dirac $\not{D}$ operator.

Now, in order to explicitly compute the partition function we need to remember that the Jacobi $\theta_1(\tau|z)$ function does not have any pole, however it has simple zeros on the lattice $z \in \mathbb{Z} + \tau\mathbb{Z}$. Moreover it is simple to verify that $\theta(\tau|z)^{-1}$ has residue in $z = \alpha + \beta\tau$ given by the following formula

$$\frac{1}{2\pi i} \oint_{z=\alpha+\beta\tau} \frac{1}{\theta_1(\tau|z)} = \frac{(-1)^{\alpha+\beta} e^{i\beta^2\tau}}{2\pi\eta^3(\tau)}.$$

In general a careful analysis of singularities would be needed in order to understand which poles are giving a non-vanishing contribution to the computation of the partition function on T^2. In our particular case the poles contributing to the residue computation will be classified in terms of nested partitions $\mu_1 \subseteq \cdots \subseteq \mu_{s-1} \subseteq \mu_0$. In principle this result could be obtained via the systematic approach of Jeffrey-Kirwan. Here we follow an alternative procedure by giving a suitable imaginary part to the twisted masses (for example through the R−charges via a redefinition of the relevant parameters, as in [44]) and by closing the integration contour in the lower-half plane. In this particular setting we take care of redefining a, ϵ_i in such a way that $\Im a, \Im\epsilon_i < 0$ and $\Im a > \Im\epsilon$. By the requirement on the Cartan parameters of the D7-branes, namely $a - \bar{a} \in \mathbb{Z}$, we also have $\Im \bar{a} = \Im a < 0$. It is sufficient to study the pole structure of the first two integrations (namely $\{u_j^0\}$ and $\{u_j^1\}$) in (3.2.20), whose poles and zeros are schematically shown in table 2. The

Poles	Zeros
$u_j^0 = a$	$u_j^0 = \bar{a}$
$u_j^0 = a - \epsilon$	$u_j^0 = \bar{a} - \epsilon$
$u_{ij}^0 = \epsilon_1$	$u_{ij}^0 = 0$
$u_{ij}^0 = \epsilon_2$	$u_{ij}^0 = -\epsilon$
$u_j^1 = u_i^0$	$u_j^1 = u_i^0 - \epsilon_1$
$u_j^1 = u_i^0 - \epsilon$	$u_j^1 = u_i^0 - \epsilon_2$
$u_j^1 = \bar{a} - \epsilon$	$u_j^1 = a - \epsilon$
$u_{ij}^1 = \epsilon_1$	$u_{ij}^1 = 0$
$u_{ij}^1 = \epsilon_2$	$u_{ij}^1 = -\epsilon$

Table 2: *Poles and zeros for $\{u_j^0\}$ and $\{u_j^1\}$ in (3.2.20).*

integration over the $\{u_j^0\}$ is standard, as it is has the same pole structure of the standard Nekrasov partition function [44; 170], and the poles contributing to the residue computation will be described by partitions μ_0. Each box in μ_0 will then encode the position of a pole for the first n_0 integrations. As for the integrations over the $\{u_j^1\}$ variables we first point out that the 1-loop determinant due to the $\overline{\text{D7}}$—brane, as $a - \bar{a} \in \mathbb{Z}$ and the corresponding pole falls out of the integration contour. In the same way also poles of the 1-loop determinant of Q_i give a vanishing contributions, because of one out of two different reasons: either the singularity falls out of the integration contour or its contribution is annihilated by a zero coming from the determinants of $\chi_1^{B_i}$. Any pole that might fall outside the Young diagram associated to μ_0 must also be excluded from the computations, because of the flag structure of the quiver in Fig. 11. These considerations lead us to the classification of poles of the $\{u_j^1\}$ integrations in terms of partitions as follows: by choosing the order of the integration to be $u_1^1, u_2^1, \ldots, u_{n_1}^1$ poles are chosen by successively picking outer corners of Y_{μ_0} so that the complement in Y_{μ_0} is still a Young diagram corresponding to a partition μ_1, with $|\mu_1| = n_0 - n_1$. The procedure we just described is depicted in Fig. 13.

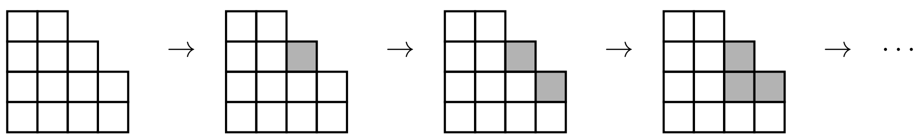


Figure 13: *Procedure for picking poles of $\{u_j^1\}$ from Y_{μ_0}.*

Any successive integration is done in the same way, and the poles contributing to the integration are classified by sequences of nested partitions, as we discussed in §3.3.

Boxes in the skew partitions $\mu_0 \setminus \mu_j$ will denote positions for poles in the j—th integration, according to the following rule: a box of $Y_{\mu_0 \setminus \mu_k}$ located at position (i, j) inside Y_{μ_0} (this is required by the nesting phenomenon) corresponds to the coordinate $u_l^{(k)} = a + (i-1)\epsilon_1 + (j-1)\epsilon_2$. One thing to be pointed out is that the assignment of a certain Young diagram configuration do in fact specify a particular pole only up to Weyl permutations of the coordinates: because of this we choose a particular ordering of the coordinates and neglect the counting factor $(n_0!\cdots n_{s-1}!)^{-1}$ in $\hat{\mathcal{Z}}$.

The partition function $Z^{D3/D7/\overline{D7}}_{T^2}$ will then take the following form

$$Z^{D3/D7/\overline{D7}}_{T^2} = \hat{\mathcal{Z}}_{\text{res}} \sum_{\mu_1\subseteq\cdots\mu_{s-1}\subseteq\mu_0} \Bigg(Z_{\mu_0}(\epsilon_1,\epsilon_2,\overline{a}) Z^{JF}_{\mu_1,\mu_0}(\epsilon_1,\epsilon_2,\overline{a}) \prod_{i=0}^{s-2} Z_{\mu_{i+1},\mu_i}(\epsilon_1,\epsilon_2,\overline{a}) \Bigg),$$

with

$$\hat{\mathcal{Z}}_{\text{res}} = (-1)^{n_1} \prod_{i=0}^{s-1} \left[\frac{1}{n_i!} \left(\frac{\theta_1(\tau|-zq+\epsilon)}{\theta_1(\tau|zq/2-\epsilon_1)\theta_1(\tau|zq/2-\epsilon_2)\eta(\tau)} \right)^{n_i} \right]$$

$$Z_{\mu_0}(\epsilon_1,\epsilon_2,\overline{a}) = \prod_{s\in\mu_0\setminus\square} \frac{\theta_1(\tau|\phi(s)-\overline{a})\theta_1(\tau|\phi(s)-\overline{a}+\epsilon)}{\theta_1(\tau|\phi(s))\theta_1(\tau|\phi(s)+\epsilon)} \prod_{\substack{s\neq s'\\ s,s'\in\mu_0}} \left(\frac{\theta_1(\tau|\phi(s)-\phi(s'))}{\theta_1(\tau|\phi(s)-\phi(s')-\epsilon_1)} \frac{\theta_1(\tau|\phi(s)-\phi(s')+\epsilon)}{\theta_1(\tau|\phi(s)-\phi(s')-\epsilon_2)} \right)$$

$$Z_{\mu_{k+1},\mu_k}(\epsilon_1,\epsilon_2,\overline{a}) = \prod_{\substack{s\neq s'\\ s,s'\in\mu_0\setminus\mu_{k+1}}} \left(\frac{\theta_1(\tau|\phi(s)-\phi(s'))}{\theta_1(\tau|\phi(s)-\phi(s')-\epsilon_1)} \frac{\theta_1(\tau|\phi(s)-\phi(s')+\epsilon)}{\theta_1(\tau|\phi(s)-\phi(s')-\epsilon_2)} \right) \prod_{\substack{s\in\mu_0\setminus\mu_{k+1}\\ s'\in\mu_0\setminus\mu_k}} \left(\frac{\theta_1(\tau|\phi(s)-\phi(s')+\epsilon_1)}{\theta_1(\tau|\phi(s')-\phi(s))} \frac{\theta_1(\tau|\phi(s)-\phi(s')+\epsilon_2)}{\theta_1(\tau|\phi(s')-\phi(s)-\epsilon)} \right)$$

$$Z^{JF}_{\mu_1,\mu_0}(\epsilon_1,\epsilon_2,\overline{a}) = \prod_{\substack{s\in\mu_0\setminus\mu_1\\ s'\in\mu_0}} \frac{\theta_1(\tau|\phi(s)+\epsilon)}{\theta_1(\tau|\phi(s)-\overline{a}+\epsilon)}$$

These formulae are to be compared with the contribution of a quiver with fixed numerical type to $\mathcal{Z}^{\text{ell}}_1(S^2; q_0,\ldots,q_{s-1})$, in particular the contribution at each fixed point will be the same as $Z^{\text{ell}}_{(\mu_0,\ldots,\mu_{s-1})}$, which was defined in §3.2.1, but in principle one could use the same technique in order to compute partition functions in the more general case of a genus g Riemann surface $\mathcal{C}_g$.

3.2.3 General Riemann Surfaces

When we switch from the genus 0 case to a generic Riemann surface $\mathcal{C}_g$ with 1 puncture, we are effectively turning on a matter bundle corresponding to the contribution of g adjoint hypermultiplets, and the quiver in Fig. 11 describing the GLSM we are studying gets modified into quiver 14.

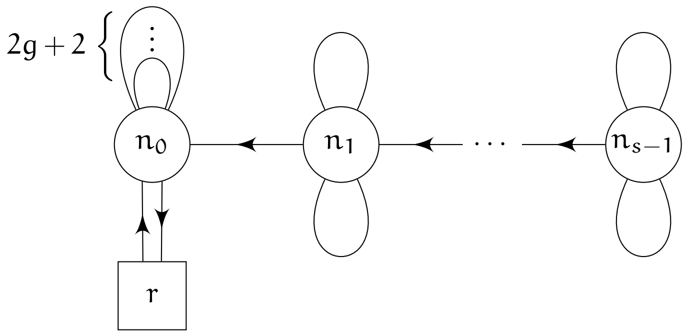


Figure 14: *Low energy GLSM quiver for a general $\mathcal{C}_{g,1}$.*

This GLSM encodes the ADHM construction of $\mathscr{N}_{r,[r^1],n,\mu}$ with additional g hypermultiplets in the adjoint representation, all of them with twisted mass m, which reproduces an $\mathcal{N} = (0,2)^*$ theory.[1] In the same spirit as in [52], from the point of view of the matter fields this consists in introducing $2g$ adjoint chirals and $2g$ fundamental chirals, with appropriate relations dictated by the brane system. As it was the case for the theory without any adjoint hypermultiplet, each of the fundamentals we introduce makes the theory anomalous by breaking a chiral half of the supersymmetry, and this phenomenon can be cured by insertion of observables, encoded in $\overline{D7}$ contributions. The additional field content to 1 is summarized in table 3, while the ADHM relation on the n_0 node must be modified

	gauge G	flavour $U(1) \times U(1)^2$	twisted mass	R−charge
B_3^i	$\overline{\square}_0 \otimes \square_0$	$\mathbb{1}_{(0,0)}$	m	$-q+t$
B_4^i	$\overline{\square}_0 \otimes \square_0$	$\mathbb{1}_{(-1,-1)}$	$\epsilon - m$	$-q-t$
K_i	$\overline{\square}_0$	$\overline{\square}_{(0,0)}$	$a - m$	$-p-t$
L_i	$\square_0$	$\square_{(1,1)}$	$-a+\epsilon-m$	$p-t$
$\chi_0^{(3),i}$	$\overline{\square}_0 \otimes \square_0$	$\mathbb{1}_{(0,1)}$	$\epsilon_1 - m$	$-t$
$\chi_0^{(4),i}$	$\overline{\square}_0 \otimes \square_0$	$\mathbb{1}_{(-1,0)}$	$m - \epsilon_2$	t

Table 3: *Hypermultiplet additional fields for quiver 14*

$$[B_1^0, B_2^0] + \sum_{i=1}^{g} [B_3^{i\dagger}, B_4^{i\dagger}] + IJ = 0 \tag{3.2.21}$$

and the relations (3.2.22)-(3.2.25) must be enforced through $\chi_0^{(3,4),i}$, K_i and L_i.

$$\mathcal{E}^{adj}_{3,i} = [B_1, B_3^i] - [B_2^\dagger, B_4^{i\dagger}] \tag{3.2.22}$$

$$\mathcal{E}^{adj}_{4,i} = [B_1, B_4^i] - [B_2^\dagger, B_3^{i\dagger}] \tag{3.2.23}$$

1 Strictly speaking we are dealing with an $\mathcal{N} = (0,2)^*$ theory only in the case in which $\mathcal{C}$ is a $g = 1$ Riemann surface. In the same spirit we might want to point out that the 5d partition function to which the elliptic index is reduced in the QM limit is not really computing the equivariant virtual χ_y−genus of the vector bundle $\mathscr{V}_g$, but rather the equivariant virtual Euler characteristic of an antisymmetric power of $\mathscr{V}_g$. This also means that the torus partition function is not, strictly speaking, the (equivariant virtual) elliptic genus as it is defined in [95], as it is instead an elliptic generalization of the virtual Euler characteristic.

$$\mathcal{E}^{\text{fun}}_{K_i} = B^i_3 I - B^{i\dagger}_4 J^\dagger \tag{3.2.24}$$

$$\mathcal{E}^{\text{fun}}_{L^\dagger_i} = B^i_4 I + B^{i\dagger}_3 J^\dagger \tag{3.2.25}$$

The partition function for a general genus g riemann surface $\mathcal{C}_g$ with one puncture will now read (we take the $r = 1$ case for the sake of simplicity)

$$\mathcal{Z}^{\text{ell}}_1(\mathcal{C}_g; \mathbf{q}) = \sum_{\mu_1 \subseteq \cdots \subseteq \mu_0} q_0^{|\mu_0|} q_1^{|\mu_0 \setminus \mu_1|} \cdots q_{s-1}^{|\mu_0 \setminus \mu_{s-1}|} Z^{\text{ell},g}_{(\mu_0,\mu_1,\dots,\mu_{s-1})}, \tag{3.2.26}$$

with $\mathbf{q} = (q_0, \dots, q_{s-1})$ and

$$Z^{\text{ell},g}_{(\mu_0,\mu_1,\dots,\mu_{s-1})} = \mathcal{L}^{\text{ell}}_{\mu_0} \mathcal{N}^{\text{ell}}_{\mu_0} \overline{\mathcal{N}}^{\text{ell}}_{\mu_0} \mathcal{E}^{\text{ell}}_{g,\mu_0} \overline{\mathcal{E}}^{\text{ell}}_{g,\mu_0} \mathcal{T}^{\text{ell}}_{\mu_0,\mu_1} \overline{\mathcal{T}}^{\text{ell}}_{\mu_0,\mu_1} \mathcal{W}^{\text{ell}}_{\mu_0,\dots,\mu_{s-1}},$$

where we defined

$$\mathcal{E}^{\text{ell}}_{g,\mu_0} = \prod_{s \in Y_{\mu_0}} \theta_1^g(\tau | E(s) - m) \theta_1^g(\tau | E(s) - \epsilon + m),$$

$$\overline{\mathcal{E}}^{\text{ell}}_{g,\mu_0} = \prod_{s \in \mu_0 \setminus \square} \frac{1}{\theta_1^g(\tau | \phi(s) - \tilde{a} - m) \theta_1^g(\tau | \phi(s) - \tilde{a} + \epsilon - m)}.$$

We remark that by setting $g = 0$ we readily recover the function $Z^{\text{ell}}_{(\mu_0,\dots,\mu_{s-1})}$ which is needed in order to compute the partition function $\mathcal{Z}^{\text{ell}}_1(S^2; q_0, \dots, q_{s-1})$.

In the same way as we did in §3.2.1, we can compute the full partition function $\mathcal{Z}^{\text{ell}}_1(\mathcal{C}_g; q_0, \dots, q_{s-1})$ by first summing over the nested partitions $\mu_1 \subseteq \cdots \subseteq \mu_{s-1}$ and use the definition (3.2.15) of P^{ell} in order to get

$$\mathcal{Z}^{\text{ell}}_1(\mathcal{C}_g; q_0, \dots, q_{s-1}) = \sum_{\mu_0} q_0^{|\mu_0|} \mathcal{Y}^{\text{ell}}_{g,\mu_0} P^{\text{ell}}_{\mu_0}, \tag{3.2.27}$$

with the following definition of $\mathcal{Y}^{\text{ell}}_{g,\mu_0}$

$$\mathcal{Y}^{\text{ell}}_{g,\mu_0} = \mathcal{L}^{\text{ell}}_{\mu_0} \mathcal{N}^{\text{ell}}_{\mu_0} \overline{\mathcal{N}}^{\text{ell}}_{\mu_0} \mathcal{E}^{\text{ell}}_{g,\mu_0} \overline{\mathcal{E}}^{\text{ell}}_{g,\mu_0}. \tag{3.2.28}$$

Again we remark that $P^{\text{ell}}_{\mu_0} \xrightarrow{q_{i>0} \to 0} 1$ so that $\mathcal{Z}^{\text{ell}}_1(\mathcal{C}_g; q_0, \dots, q_{s-1}) \xrightarrow{q_{i>0} \to 0} \mathcal{Z}^{\text{ell}}_1(\mathcal{C}_g; q_0)$.

Comet shaped quiver

Finally, we are interested in computing the partition function on a Riemann surface $\mathcal{C}_g$ with k punctures of generic holonomy, whose low energy GLSM is in general described by the quiver in Fig. 15.

We will start from the case of $\mathcal{C}_0 = S^2$, which will take the form (3.2.29)

$$\begin{aligned}\mathcal{Z}^{\text{ell}}_k(S^2; q_0, \{q^i_1, \dots, q^i_{s-1}\}) = \sum_{\mu_0} q_0^{|\mu_0|} \sum_{\{\mu^i_1 \subseteq \cdots \subseteq \mu^i_{s-1}\}_{i=1}^k} \prod_{j=1}^k \Big(q_1^{|\mu_0 \setminus \mu^j_1|} \cdots \\ \cdots q_{s-1}^{|\mu_0 \setminus \mu^j_{s-1}|} \Big) Z^{\text{ell}}_{(\mu_0, \{\mu^i_1, \dots, \mu^i_{s-1}\})}.\end{aligned} \tag{3.2.29}$$

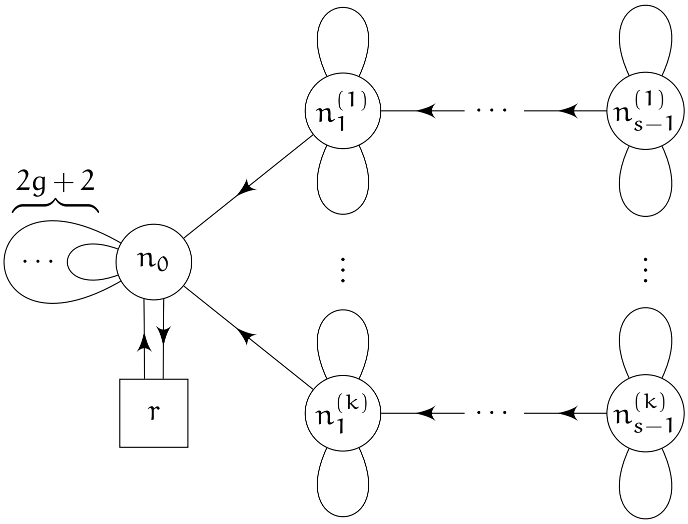


Figure 15: *Comet-shaped quiver.*

In this case the virtual tangent space to $\mathcal{N}_{r,[r^1],n,\mu}$ in (3.2.4) will be modified to be of the form (3.2.30).

$$\begin{aligned} T_Z^{\mathrm{vir}}\mathcal{N}(r,\{n_0^i,\dots,n_{s-1}^i\}) = {} & \mathrm{End}(V_0)\otimes(Q-1-\Lambda^2 Q)+\mathrm{Hom}(W,V_0) \\ & +\mathrm{Hom}(V_0,W)\otimes\Lambda^2 Q \\ & -\sum_{i=1}^{k}\mathrm{Hom}(V_1^{(k)},W)\otimes\Lambda^2 Q \\ & +\sum_{i=1}^{k}\sum_{\ell=1}^{s-1}\Bigg[\Big(\mathrm{End}(V_\ell^{(k)})-\mathrm{Hom}(V_\ell^{(k)},V_{\ell-1}^{(k)})\Big) \\ & \quad \otimes(Q-1-\Lambda^2 Q)\Bigg]. \end{aligned} \tag{3.2.30}$$

By a simple generalization of the computations leading to (3.2.5) it is possible to see that $\mathcal{Z}_k^{\mathrm{ell}}(S^2;q_0,\{q_1^i,\dots,q_{s-1}^i\})$ takes a form similar to (3.2.16), as is shown in (3.2.31)

$$\mathcal{Z}_k^{\mathrm{ell}}(S^2;q_0,\{q_1^i,\dots,q_{s-1}^i\}) = \sum_{\mu_0} q_0^{|\mu_0|}\mathcal{Y}_{\mu_0}^{\mathrm{ell}}\prod_{i=1}^{k}P_{\mu_0}^{\mathrm{ell},i}, \tag{3.2.31}$$

with

$$P_{\mu_0}^{\mathrm{ell},i} = \sum_{\mu_1^i\subseteq\cdots\subseteq\mu_{s-1}^i}\mathcal{T}_{\mu_0,\mu_1^i}^{\mathrm{ell}}\overline{\mathcal{T}}_{\mu_0,\mu_1^i}^{\mathrm{ell}}\mathcal{W}_{\mu_0,\dots,\mu_{s-1}^i}^{\mathrm{ell}}\,(q_1^i)^{|\mu_0\setminus\mu_1^i|}\cdots(q_{s-1}^i)^{|\mu_0\setminus\mu_{s-1}^i|}, \tag{3.2.32}$$

and the functions $\mathcal{T}_{\mu_0,\mu_1^i}^{\mathrm{ell}}$, $\overline{\mathcal{T}}_{\mu_0,\mu_1^i}^{\mathrm{ell}}$ and $\mathcal{W}_{\mu_0,\dots,\mu_{s-1}^i}$ take the same form as in equations (3.2.7)-(3.2.12).

By a completely analogous procedure we can get that partition function of the low energy theory relative to a general Riemann surface of genus g, possibly $g = 0$. By using the results of §3.2.3, we easily see that

$$\mathcal{Z}_k^{\mathrm{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) = \sum_{\mu_0} q_0^{|\mu_0|} \sum_{\{\mu_1^i \subseteq \dots \subseteq \mu_{s-1}^i\}_{i=1}^k} \prod_{j=1}^k \left(q_1^{|\mu_0 \setminus \mu_1^j|} \dots \right.$$
$$\left. \dots q_{s-1}^{|\mu_0 \setminus \mu_{s-1}^j|} \right) Z^{\mathrm{ell},g}_{(\mu_0, \{\mu_1^i, \dots, \mu_{s-1}^i\})}. \tag{3.2.33}$$

By turning on the matter bundle described in §3.2.3 on the moduli space of nested instantons $\mathcal{N}(r, n_0, \{n_1^i, \dots, n_{s-1}^i\})$, whose virtual tangent space is given in Eq. (3.2.30) as an element of the representation ring of the torus $R(T)$, the suspersymmetric localization theorem (or equivalently the equivariant one) gives us (3.2.34),

$$\mathcal{Z}_k^{\mathrm{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) = \sum_{\mu_0} q_0^{|\mu_0|} \mathcal{Y}^{\mathrm{ell}}_{g,\mu_0} \prod_{i=1}^k P^{\mathrm{ell},i}_{\mu_0}, \tag{3.2.34}$$

where $P^{\mathrm{ell},i}_{\mu_0}$ is defined in (3.2.32) and $\mathcal{Y}^{\mathrm{ell}}_{g,\mu_0}$ is the same one as in Eq. (3.2.28).

A couple of final remarks are due here. First of all we notice that we can switch off any number of the contributions of the tails of the comet shaped quiver 15 by taking the limit to 0 of the respective instanton counting parameters. Then, given any $k' < k$ we have that

$$\mathcal{Z}_k^{\mathrm{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) \xrightarrow[\substack{i=1,\dots,s-1 \\ j=k'+1,\dots,k}]{q_i^j \to 0} \mathcal{Z}_{k'}^{\mathrm{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}).$$

Moreover, we expect our partitions functions to be computing the equivariant elliptic cohomology of the moduli spaces of stable representations of quivers 11-15, as in [193].

3.2.4 Limit to supersymmetric quantum mechanics

We now want to study a particular dimensional reduction of the 2d $\mathcal{N} = (0,2)$ system we studied on T^2 in the previous subsections. By reducing on a circle we get the Witten index of an $\mathcal{N} = 2$ SQM. This dimensional reduction can be obtained from the elliptic case we just studied by taking the limit $e^{2\pi i \tau} \to 0$. In this scaling limit we can use the fact that $\theta_1(\tau|z) \to 2q^{1/8}\sin(\pi z)$ as $q = e^{2\pi i \tau} \to 0$. In the resulting theory on S^1 we can decouple the $\overline{\mathrm{D7}}$ branes by taking very large values of the Cartan parameter $\overline{a}$ and then rescaling the gauge coupling. Moreover the quantum mechanical partition function can be obtained by itself via localisation, and the result agrees with the decoupling procedure we just described. Indeed, the $\overline{\mathrm{D7}}$-branes only act as a source of observables matching the anomaly, so they do not give rise to new poles in the localisation integral. Moreover the observables they generate do not contribute in the dimensionally reduced theory, whose moduli space is zero-dimensional, *i.e.* it doesn't display unbalanced fermionc zero modes. As we already anticipated, we will see how the results we obtain by this procedure compute particular equivariant virtual invariants of the bundle $\mathscr{V}_g$ over the moduli space of nested instantons $\mathscr{N}_{r,[r^1],n,\mu}$, which is described by the stable representations of the quiver in Fig. 14. A bit of care is required in order to take the correct scaling limit, and in particular one has to require that $q \to 0$ while $\mathrm{vol}(T^2) \to \beta = r_{S^1}$. Moreover, one should take into account that in the S^1 theory twisted masses are also rescaled by β, so that the result may be expressed in terms of $q_1 = e^{\beta\epsilon_1/2}$, $q_2 = e^{\beta\epsilon_2/2}$ and $y = e^{-\beta m}$.

The geometric interpretation of the Witten index of the quiver gauge theories described in the previous section is the equivariant (virtual) Euler characteristic of a given bundle over the moduli space of nested instantons. Then, computing the Witten index geometrically amounts to studying the stable representations in the category of vector spaces of the quiver 7 under suitable stability conditions. This procedure has the advantage of letting us compute the weight decomposition of the virtual tangent space $T_Z^{\mathrm{vir}}\mathscr{N}_{r,[r^1],n,\mu}$ at the fixed points Z in the representation ring of the torus. The way in which this is done is very briefly described in §3.2.1. As it is shown in §3.2.1, the fixed locus of the torus action consists only of isolated points, which are characterized in terms of $s-$tuples of nested coloured partitions $\boldsymbol{\mu}_1 \subseteq \cdots \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$, such that $|\boldsymbol{\mu}_0| = n_0 = n$, while $|\boldsymbol{\mu}_j| = n_0 - n_j$.

Once the fixed point locus has been completely characterized and a weight decomposition of the virtual tangent space is at hand, one can in full generality define an $s-$parameter family of partition functions on $\mathscr{N}_{r,[r^1],n,\mu}$, with parameters $\mathbf{p} = (p_0, p_1, \dots, p_{s-1}) \in \mathbb{Z}^s$. In terms of the quiver vector spaces $(W, V_0, \dots, V_{s-1})$ one can introduce $(s+1)-$tautological bundles $\mathcal{W}$ and $\mathcal{V}_i$, $i = 0, \dots, s-1$, with $\mathcal{W} = \mathscr{O}_{\mathscr{N}_{r,[r^1],n,\mu}}$. We can then define $\mathcal{L}_i = \det \mathscr{V}_i$, $\mathcal{L}_{\mathbf{p}} = \bigotimes_i \mathcal{L}_i^{\otimes p_i}$ and compute the virtual Euler characteristic of the bundle $S \otimes \mathcal{L}_{\mathbf{p}}$ over $\mathscr{N}_{r,[r^1],n,\mu}$, with S an arbitrary irreducible representation of T. The generating function of the virtual Euler characteristics of the moduli space of nested instantons in (3.2.35) will then reproduce the QM partition function 3.1.20, when $\mathbf{p} = (1-g, 0, \dots, 0)$.[2]

$$Z_{\mathbf{p}}^{\mathrm{vir}}(q_1, q_2, \mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^s_{\geqslant 0}} \mathrm{ch}_T\, \chi_T^{\mathrm{vir}}\left(\mathscr{N}_{r,[r^1],n,\mu}, \mathcal{L}_{\mathbf{p}}\right) \prod_{i=1}^{s} x_i^{n_i}. \tag{3.2.35}$$

In the following we use the notation ch_T to denote the $T-$equivariant Chern character of a vector bundle, which has a very convenient representation in the representation ring $R(T)$. The usual Chern character is defined as follows: if E is rank r vector bundle over X, with Chern roots $x_1, \dots, x_r$, then one defines

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{x_i},$$

2 It is interesting to compare the role of this line bundle $\mathcal{L}$ to the way in which the Cern-Simons term was introduced in §3.1.9. In particular it turns out that the vector space V_0 can be recognized to be the space of fermionic zero modes, [141; 166; 206], so that the identification of $\mathcal{L}_{(1,0,\dots,0)} = \det \mathscr{V}_0$ with $\mathrm{Det}\, \not{D}$ is in fact quite natural.

which can be equivariantly extended to a ring homomorphism $\mathrm{ch}_G : K^i_G(X) \to H^i_G(\hat{X}, \mathbb{C})$, where $\hat{X} = \{(x,g) \in X \times G | xg = x\} = \coprod_g X^g$ and $H^i_G(\hat{X}, \mathbb{C}) \simeq \left[\bigoplus_g H^i(X^g, \mathbb{C})\right]^G$. The effect of ch_G can be concretely characterized as follows: if E is a $G-$equivariant vector bundle on X, for each $x \in X^g$, we can compute the eigenvalues (supposed to be distinct) $\lambda_1, \dots, \lambda_r$ of the $G-$action, and the corresponding eigenspaces $E^1_x, \dots, E^r_x$, so that $E|_{X^g}$ can be represented as the direct sum of vector bundles

$$E_{X^g} = E^1 \oplus \cdots \oplus E^r.$$

Finally one defines $\mathrm{ch}_g(E) = \sum_i \lambda_i \, \mathrm{ch}(E^i)$, so that

$$\mathrm{ch}_G(E) = \bigoplus_{g \in G} \mathrm{ch}_g(E) \in \left[\bigoplus_{g \in G} H^{\mathrm{ev}}(X^g, \mathbb{C})\right]^G.$$

The Chern character, and also the equivariant Chern character, satisfies some important properties which we will use extensively in the following:

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}\, E + \mathrm{ch}\, F, \qquad \mathrm{ch}(E \otimes F) = \mathrm{ch}\, E \, \mathrm{ch}\, F.$$

If we restrict to the case $\mathbf{p} = (p_0, 0, \dots, 0)$, the fiber of $\mathcal{L}_{\mathbf{p}}$ at a fixed point $Z \leftrightarrow \boldsymbol{\mu}_1 \subseteq \cdots \subseteq \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$ will be given by (3.2.36),

$$\mathcal{L}_Z = \mathcal{L}_{\mu_0} = \left(\prod_{\alpha=1}^{r} \prod_{i=1}^{M_0^{(a)}} \prod_{j=1}^{\mu_{0,i}^{(a)\prime}} T_{a_\alpha} T_1^{-i+1} T_2^{-j+1} \right)^{p_0}, \tag{3.2.36}$$

where $(T_1, T_2, T_{a_1}, \dots, T_{a_r})$ denote the fundamental characters of $T \times (\mathbb{C}^*)^r$ acting on $\mathcal{N}_{r,[r^1],n,\mu}$ and $M_0^{(a)} = \mu_{0,1}^{(a)\prime}$, $N_0^{(a)} = \mu_{0,1}^{(a)}$.

Then supersymmetric localization (equiv. equivariant localization) can be exploited in order to compute partition functions (equiv. virtual equivariant Euler characteristics). If we start from the case $g = 0$ we get

$$\begin{aligned}
\mathrm{ch}_T \left[\chi_T^{\mathrm{vir}}\left(\mathcal{N}(r, \mathbf{n}), \mathcal{L}_{\mathbf{p}}\right)\right] &= \sum_{Z \in \mathcal{N}^T_{r,[r^1],n,\mu}} \frac{\mathrm{ch}_T \, \mathcal{L}_Z}{\Lambda_{-1}\left[T_Z^{\mathrm{vir}} \mathcal{N}^\vee_{r,[r^1],n,\mu}\right]} \\
&= \sum_{\mu_1 \subseteq \cdots \subseteq \mu_{s-1} \subseteq \mu_0} \left(\frac{\mathcal{L}_{\mu_0}(q_1, q_2)}{\Lambda_{-1}\left[T_{\tilde{Z}} \mathcal{M}^\vee_{r,n_0}\right]} \right. \\
&\qquad \left. \mathcal{T}_{\mu_0,\mu_1}(q_1, q_2) \mathcal{W}_{\mu_0,\dots,\mu_{s-1}}(q_1, q_2) \right),
\end{aligned}$$

with $\Lambda_t(E) = \sum_{i \geqslant 0} t^i \Lambda^i E$ for any (equivariant) vector bundle E on $\mathcal{N}_{r,[r^1],n,\mu}$, while $\mathcal{L}_{\mu_0}(q_1, q_2)$, $\mathcal{W}_{\mu_0,\dots,\mu_{s-1}}(q_1, q_2)$ and $\mathcal{T}_{\mu_0,\mu_1}(q_1, q_2)$ are given by equations (3.2.37)-(3.2.39),

$$\mathcal{L}_{\mu_0}(q_1, q_2) = \left(\prod_{a=1}^{r} \prod_{i=1}^{M_0^{(a)}} \prod_{j=1}^{\mu_{0,i}^{(a)\prime}} \rho_a q_1^{i-1} q_2^{j-1} \right)^{p_0} \tag{3.2.37}$$

$$\mathcal{T}_{\mu_0,\mu_1}(q_1,q_2) = \prod_{a=1}^{r}\prod_{i=1}^{M_0^{(a)}}\prod_{j=1}^{\mu_{0,i}^{(a)}-\mu_{1,i}^{(a)}}\left(1-\rho_a q_1^{-i} q_2^{-j-\mu_{1,i}^{(a)\prime}}\right) \tag{3.2.38}$$

$$\begin{aligned}\mathcal{W}_{\mu_0,\dots,\mu_{s-1}}(q_1,q_2) = \prod_{k=0}^{s-2}\prod_{a,b=1}^{r}\prod_{i=1}^{M_0^{(a)}}\prod_{j=1}^{N_0^{(b)}}&\left(\frac{(1-\rho_a\rho_b^{-1} q_1^{\mu_{k,j}^{(b)}-i} q_2^{j-\mu_{k+1,i}^{(a)\prime}-1})}{(1-\rho_a\rho_b^{-1} q_1^{\mu_{k+1,j}^{(b)}-i} q_2^{j-\mu_{k+1,i}^{(a)\prime}-1})}\right.\\ &\left.\frac{(1-\rho_a\rho_b^{-1} q_1^{\mu_{k+1,j}^{(b)}-i} q_2^{j-\mu_{0,i}^{(a)\prime}-1})}{(1-\rho_a\rho_b^{-1} q_1^{\mu_{k,j}^{(b)}-i} q_2^{j-\mu_{0,i}^{(a)\prime}-1})}\right)\end{aligned} \tag{3.2.39}$$

with $\rho_i = \operatorname{ch} T_{a_i}$ and similarly $q_i = \operatorname{ch} T_i$.

The generalization to the case of a general Riemann surface $\mathcal{C}_g$ of genus g is immediate, as it only amounts to computing the "virtual Hirzebruch χ_y−genus" of the bundle $\pi^*\mathscr{V}_g \to \mathscr{N}_{r,[r^1],n,\mu}$. This is obviously the same as turning on a matter bundle relative to additional g adjoint hypermultiplets, whose twisted mass m is naturally identified with y in the Hirzebruch genus by exponentiation.

$$\begin{aligned}\operatorname{ch}_T \chi_y^{T,\mathrm{vir}}(\pi^*\mathscr{V}_g,\mathscr{N}) = \sum_{\mu_1\subseteq\cdots\subseteq\mu_{s-1}\subseteq\mu_0}&\left(\frac{\operatorname{ch}_T(\mathscr{L}_{\mu_0})\operatorname{ch}_T\Lambda_y[(T_{\mu_0}\mathscr{M}_{r,n_0}^\vee)^{\oplus g}]}{\operatorname{ch}_T\Lambda_{-1}[T_{\mu_0}\mathscr{M}_{r,n_0}^\vee]}\cdot\right.\\ &\left.\cdot\mathcal{T}_{\mu_1,\mu_0}(q_1,q_2)\prod_{i=0}^{s-2}\mathcal{W}^{\mu_0}_{\mu_{i+1},\mu_i}(q_1,q_2)\right).\end{aligned} \tag{3.2.40}$$

Surprisingly enough, explicit computations suggest that the partition function of each choice of numerical type for the nested instantons quiver should consists of a usual Nekrasov partition function multiplied by a polynomial in the torus characters. This observation is summarized in the following conjecture.

Conjecture 3.5. *The function* $\sum_{\mu_{i>0}}\mathcal{T}_{\mu_0,\mu_1}(q_1,q_2)\mathcal{W}_{\mu_0,\dots,\mu_{s-1}}(q_1,q_2)$ *is a polynomial in* $q=q_1^{-1}$ *and* $t=q_2^{-1}$ *with rational coefficients in the* $\{\rho_i\}_{1\leqslant i\leqslant r}$, *while it is a polynomial with integer coefficients when* $r=1$.

3.2.5 Comparison to HLRV formulae

The Nekrasov partition function on $\mathbb{R}^4\times S^1$ is known to have the following form

$$Z_{k,N}^{\mathbb{R}^4\times S^1} = \sum_{\mathbf{Y}_k}\prod_{\lambda,\tilde\lambda=1}^{N}\prod_{s\in Y_\lambda}\frac{\sinh\left[\frac{\beta}{2}(E(s)-m))\right]\sinh\left[\frac{\beta}{2}(E(s)-\epsilon+m)\right]}{\sinh\left[\frac{\beta}{2}E(s)\right]\sinh\left[\frac{\beta}{2}(E(s)-\epsilon)\right]}, \tag{3.2.41}$$

where $E(s) = a_{\lambda\tilde\lambda} - \epsilon_1 h(s) + \epsilon_2(v(s)+1)$, and given two Young diagrams $Y_\lambda, Y_{\tilde\lambda}\in\mathbf{Y}_k$ the quantities $h(s)$ and $v(s)$ are defined to be $h(s) = \nu_{i_\lambda} - j_\lambda$ and $v(s) = \tilde\nu'_{j_\lambda} - i_\lambda$. We will be interested in the specialization of the Nekrasov partition function to the case $N=1$, so that $h(s)$ and $v(s)$ will become respectively the arm length $a(s)$ and leg length $l(s)$ for the box s in the

Young tableaux classifying a given pole configuration. Now, following the conventions of [119], let $\mathbf{x}_1 = \{x_{1,1}, x_{1,2}, \dots\}$ and $\mathbf{x}_k = \{x_{k,1}, x_{k,2}, \dots\}$ be k infinite sets of variables and let moreover $\Lambda(\mathbf{x}_1), \dots, \Lambda(\mathbf{x}_k)$ be the corresponding rings of symmetric functions. Given a partition λ, $\tilde{H}_\lambda(\mathbf{x}; q, t) \in \Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}[q, t]$ will denote the modified Macdonald symmetric function. The $k-$point genus g Cauchy function $\Omega(z, w)$, with coefficients in $\mathbb{Q}[z, w] \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k)$, is defined as follows

$$\Omega(z,w) = \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2),$$

with

$$\mathcal{H}_\lambda(z,w) = \prod_{s \in \lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})^{2g}}{(z^{2a(s)+2-w^{2l(s)}})(z^{2a(s)} - w^{2l(s)+2})}. \tag{3.2.42}$$

The modified Macdonald polynomials $\tilde{H}_\lambda(\mathbf{x}; q, t)$ are defined as

$$\tilde{H}_\lambda(\mathbf{x}; q, t) = \sum_\mu \tilde{K}_{\mu\lambda}(q,t) s_\mu(\mathbf{x}),$$

where $s_\lambda(\mathbf{x})$ are the usual Schur functions, while $\tilde{K}_{\lambda\mu}(q, t)$ denotes the modified Kostka polynomials, which are expressed in terms of the usual Kostka polynomials as

$$\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q, t^{-1}),$$

with $n(\mu) = \sum_{i=1}^{l(\mu)} \mu_i(i-1)$, and $K_{\lambda\mu}(q, t)$ can be interpreted as being a deformation of the Kostka coefficients $K_{\lambda\mu}$ appearing in the expansion of the Schur polynomials in terms of the monomial symmetric functions:

$$s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda\mu} m_\mu(\mathbf{x}).$$

Moreover, the modified Macdonald polynomials can be viewed as a q-deformation of the standard Hall-Littlewood polynomials, and are related in a non trivial way to the Macdonald polynomials $P_\mu(\mathbf{x}; q, t)$, which are eigenfunctions of the trigonometric Ruijsenaars-Schneider Hamiltonian [106; 144]:

$$\tilde{H}_\lambda[X; q, t] = t^{n(\lambda)} J_\lambda \left[\frac{X}{1 - 1/t}; q, 1/t \right],$$

where X denotes the plethystic substitution $X = x_1 + x_2 + x_3 + \cdots$, the square brackets are to be intended as a plethystic insertion and

$$J_\lambda(\mathbf{x}; q, t) = \prod_{s \in \lambda} \left(1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}\right) P_\lambda(\mathbf{x}; q, t).$$

The modified Macdonald polynomials are also eigenfunctions of a linear operator Δ, [118], which acts on a symmetric function f as

$$\Delta f = f\left[X + \frac{(1-q)(1-t)}{z}\right] \Omega[-zX]\Big|_{z^0},$$

where $\Omega[X] = \sum_{n=0}^{\infty} h_n(X)$ and $(\bullet)|_{z^0}$ denotes the constant part in z.

We will think to $\Omega(z,w)$ as being a function associated to a genus g Riemann surface with k punctures. Moreover, if we are give $\mu = (\mu^1, \dots, \mu^k) \in \mathcal{P}^k$ we can define the following function

$$\mathbb{H}_{\mu}(z,w) = (z^2-1)(1-w^2)\langle \mathrm{PL}\,\Omega(z,w), h_{\mu}\rangle,$$

where $h_{\mu} = h_{\mu^1}(\mathbf{x}_1)\cdots h_{\mu^k}(\mathbf{x}_k) \in \Lambda(\mathbf{x}_1,\dots,\mathbf{x}_k)$ are the complete symmetric functions, and $\langle\cdot,\cdot\rangle$ is an extension of the Hall pairing. The interest in $\mathbb{H}_{\mu}(z,w)$ lays in the fact that it encodes information both about $GL_n(\mathbb{C})$ character varieties $\mathscr{M}_{\mu}$ of $k-$punctured genus g Riemann surfaces with generic semisimple conjugacy classes of type μ at the punctures and about comet-shaped quivers $\mathcal{Q}_{\mu}$ with g loops and k tails of length defined by μ. It is in fact conjectured that through the knowledge of $\mathbb{H}_{\mu}(z,w)$ we can get the mixed Hodge polynomial and the $E-$polynomial (and thus the Euler characteristic) of both these character varieties and quiver varieties.

If we now study the particular case of comet-shaped quivers with $k=1$, $l(\mu)=1$ and $g=1$, whose corresponding quiver is the Jordan quiver, we can specialize $\mathbf{x} = (T,0,\dots)$ for some variable T and $\tilde{H}_{\lambda}(T,0,\dots;z,w) = T^{|\lambda|}$, so that

$$\Omega(z,w) = \sum_{k}\sum_{|\lambda|=k}\prod \frac{\left(z^{2a(s)+1}-w^{2l(s)+1}\right)^2}{\left(z^{2a(s)+2}-w^{2l(s)}\right)\left(z^{2a(s)}-w^{2l(s)+2}\right)}T^{|\lambda|}. \tag{3.2.43}$$

If we now compare (3.2.43) to (3.2.41) in the case $N=1$, with $m=\epsilon/2$, we can immediately see how closely $\Omega(z,w)$ resembles to $\sum_k Z_{k,1}^{\mathbb{R}^4\times S^1} q^k$ as long as we make the identifications $z^2 = e^{\beta\epsilon_1}$, $w^2 = e^{\beta\epsilon_2}$ and $T=q$, q being the instanton counting parameter.

If we next take g to be arbitrary, but still take $k=1$ and $l(\mu)=1$ a generalization of our previous observations is straightforward. In fact, as we already pointed out in the previous sections, adding loops to the Jordan quiver has the net effect of introducing $2g+2$ matter fields B_1, B_2, $B_3^{(i)}$, $B_4^{(i)}$ (with $i=1,\dots,g$) transforming in the adjoint representation of the gauge group $U(k)$. The role played by each of the $B_3^{(i)}$, $B_4^{(i)}$ fields is analogous to the one of B_3 and B_4 in the ADHM linear sigma model with adjoint matter. Since all of these fields do not contribute with poles to the residue computation of the localization formula, if we choose their twisted masses and $R-$charges to be the same as the ones for B_3 and B_4 their net effect will be that of introducing a $g-$th power to the numerator of (3.2.41) (which really is the meaning of turning on a matter bundle for g adjoint hypermultiplets twisted by their mass m).

Actually one needs to turn on a Chern-Simons coupling in order to exactly reproduce $\Omega(z,w)$ starting from a gauge theory. In fact we can rewrite (3.2.43) as

$$\begin{aligned}\Omega(z,w) = \sum_{k}\sum_{|\lambda|=k}\prod_{s\in\lambda}\Bigg[&(-1)^{g-1}\frac{(z^{2a(s)+1}w^{2l(s)+1})^g}{z^{2a(s)+2}w^{2l(s)+2}}\cdot\\ &\cdot\frac{(1-z^{-2a(s)-1}w^{2l(s)+1})^g(1-z^{2a(s)+1}w^{-2l(s)+1})^g}{(1-z^{-2a(s)-2}w^{2l(s)})(1-z^{2a(s)}w^{-2l(s)-2})}T^{|\lambda|}\Bigg]\end{aligned}$$

and we can easily see that

$$\begin{aligned}\prod_{s\in\lambda}\frac{(z^{2a(s)+1}w^{2l(s)+1})^g}{z^{2a(s)+2}w^{2l(s)+2}} &= \frac{1}{(zw)^{|\lambda|}}\prod_{s\in\lambda}(z^{2a(s)+2}w^{2l(s)+2})^{g-1}\\ &= \frac{1}{(zw)^{|\lambda|}}\left(z^{2\sum_s(a(s)+1)}w^{2\sum_s(l(s)+1)}\right)^{g-1}\\ &= \frac{1}{(zw)^{|\lambda|}}\left(z^{2\sum_s i(s)}w^{2\sum_s j(s)}\right)^{g-1}\\ &= \frac{(zw)^{|\lambda|(2g-2)}}{e^{a(g-1)|\lambda|}(zw)^{|\lambda|}}\prod_{s\in\lambda}\left(e^a z^{2(i(s)-1)}w^{2(j(s)-1)}\right)^{g-1},\end{aligned}$$

which, apart from a harmless overall normalization, is the contribution of a Chern-Simons interaction at level $1-g$, [206]. Thus we conclude that the partition function for the 5d $\mathcal{N}=1^*$ ADHM quiver theory with g adjoint hypermultiplets and a Chern-Simons term at level $1-g$ reproduces the Cauchy function (3.2.44) when resummed over all the instanton sectors (see also [71]).[3]

$$\Omega(z,w)=\sum_k\sum_{|\lambda|=k}\prod\frac{\left(z^{2a(s)+1}-w^{2l(s)+1}\right)^{2g}}{\left(z^{2a(s)+2}-w^{2l(s)}\right)\left(z^{2a(s)}-w^{2l(s)+2}\right)}T^{|\lambda|}. \quad (3.2.44)$$

As it was shown in [54; 69], one interesting thing to point out in Eq. (3.2.44) is that it computes a generating function for a geometric index. It is actually known that the moduli space of stable representations for the ADHM data (3.2.45) is isomorphic to the Hilbert scheme of $\dim(V)=n$ points in $\mathbb{C}^2$ when $\dim(W)=1$.

$$\begin{tikzcd} V \arrow[loop above, "B_1"] \arrow[loop below, "B_2"] \arrow[r, shift left, "J"] & W \arrow[l, shift left, "I"] \end{tikzcd}, \qquad [B_1,B_2]+IJ=0 \quad (3.2.45)$$

Then $\Omega_\lambda(q_1,q_2)$ such that $\Omega(z,w)=\sum_k\Omega_\lambda(z^2,w^2)T^{|\lambda|}$ is computing the Hirzebruch χ_y-genus of a vector bundle over $(\mathbb{C}^2)^{[n]}$. In particular we have [69; 70]

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}(n)}\Omega_\lambda(q_1,q_2,y) &= \mathrm{ch}_T\,\chi_y\left[\left(T^\vee(\mathbb{C}^2)^{[n]}\right)^{\oplus g}\otimes(\det\mathscr{T})^{1-g},(\mathbb{C}^2)^{[n]}\right]\\ &= \sum_{\lambda\in\mathcal{P}(n)}\frac{\mathrm{ch}_T\,(\det\mathscr{T})^{1-g}\,\mathrm{ch}_T\,\Lambda_y\left[(T^\vee_\lambda(\mathbb{C}^2)^{[n]})^{\oplus g}\right]}{\mathrm{ch}_T\,\Lambda_{-1}\left[T^\vee_\lambda(\mathbb{C}^2)^{[n]}\right]},\end{aligned}$$

where $\det\mathscr{T}$ denotes the determinant line bundle on $(\mathbb{C}^2)^{[n]}$ and $y=e^{-m}$.

It was proved in [69] that a similar result holds true also for the genus g Cauchy function relative to punctured Riemann surfaces with non-trivial

3 The 5d $\mathcal{N}=1*$ theory here is intended to be abelian (as we focused only on the case $N=1$ after Eq. (3.2.41)) and it amounts to a free theory of g adjoint hypermultimplets, as displayed in Eq. (3.2.42).

holonomy around the punctures. In the case of a single puncture (assumed to be generic) of type μ, the Cauchy function at fixed $|\lambda| = n$ computes the residual equivariant Hirzebruch genus of a vector bundle over a nested Hilbert scheme of n points $\mathscr{N}_{1,[1^1],n,\mu}$ on $\mathbb{C}^2$:

$$\sum_{\lambda\in\mathcal{P}(n)} \mathcal{H}_\lambda(z,w)\tilde{H}_\lambda(\mathbf{x};z^2,w^2) = \mathrm{ch}_T\,\chi_y\left[\pi^*\mathscr{V}_g, \mathscr{N}_{1,[1^1],n,\mu}\right], \tag{3.2.46}$$

where $\pi : \mathscr{N}_{1,[1^1],n,\mu} \to (\mathbb{C}^2)^{[n]}$ is the natural projection of the nested Hilbert scheme to the underlying Hilbert scheme of n points on $\mathbb{C}^2$, and $\mathscr{V}_g = \left(T^\vee(\mathbb{C}^2)^{[n]}\right)^{\oplus g} \otimes (\det \mathscr{T})^{1-g}$. Moreover the rhs of (3.2.46) can be computed in terms only of characters of vector bundles over $(\mathbb{C}^2)^{[n]} = \mathrm{Hilb}^n(\mathbb{C}^2)$ due to a result by Haiman, [69; 117], and we have that

$$\mathrm{ch}_T\,\chi_y\left[\pi^*\mathscr{V}_g, \mathscr{N}_{1,[1^1],n,\mu}\right] = \sum_{\lambda\in\mathcal{P}(n)} \frac{\mathrm{ch}_T\,\Lambda_y\left[(T^\vee_\lambda(\mathbb{C}^2)^{[n]})^{\oplus g}\right]}{\mathrm{ch}_T\,\Lambda_{-1}\left[T^\vee_\lambda(\mathbb{C}^2)^{[n]}\right]}$$
$$\mathrm{ch}_T\left(\det\mathscr{T}\right)^{1-g}\mathrm{ch}_T(\mathcal{P}^\gamma_\lambda),$$

where $\mathcal{P}^\gamma$ is a vector bundle over $(\mathbb{C}^2)^{[n]}$ whose fibers over closed points $[I] \in (\mathbb{C}^2)^{[n]}$ are isomorphic to permutation representations of $\mathcal{S}_n$.

By virtue of what we showed in subsection 3.2.4, we expect our results to give a virtual refinement of the formulae found in [69; 119]. For the sake of simplicity, let us start from studying the case of a quiver consisting of only two gauge nodes and $r = 1$, corresponding to a complex curve $\mathcal{C}$ of genus $g = 0$. We already computed in §3.2.4 the partition function relative to any generic quiver of the type shown in Fig. 7, with $(r_0, r_1, \ldots, r_{s-1}) = (r, 0, \ldots, 0)$. We will then be computing the generating function

$$\begin{aligned} Z^{(p_0,p_1)}_{\mathrm{vir}} &= \sum_{\mathbf{n}\in\mathbb{Z}^2_{\geqslant 0}} Z^{(p_0,p_1)}_{\mathbf{n}} \prod_{i=0}^{1} x_i^{n_i} \\ &= \sum_{\mathbf{n}\in\mathbb{Z}^2_{\geqslant 0}} \mathrm{ch}_T\,\chi^{\mathrm{vir}}_T\left(\mathscr{N}_{1,[1^1],n,\gamma(\mathbf{n})}, \mathcal{L}_{(p_0,p_1)}\right) \prod_{i=0}^{1} x_i^{n_i}, \end{aligned}$$

where $\gamma(\mathbf{n})$ is the ordered sequence determined by n_i determining the relevant quiver variety of numerical type $(1, \hat{n}_0, \hat{n}_1)$.

We will restrict our attention to $\mathbf{p} = (p_0, 0)$, in which case the restriction $\mathcal{L}_Z$ of $\mathcal{L}_{(p_0,0)}$ to the fixed point under $T \curvearrowright \mathscr{N}_{1,[1^1],n,\gamma(\mathbf{n})}$ is

$$\mathcal{L}_Z = \left(\prod_{i=1}^{M_1}\prod_{j=1}^{\nu'_i} T_1^{-i+1}T_2^{-j+1}\right)^{p_0}.$$

The result obtained in §3.2.4 by means of SUSY localization then specializes in this case to the form (3.2.47):

$$
\begin{aligned}
Z_{\mathbf{n}}^{(p_0,0)} &= \sum_{\substack{Z=(\nu,\mu)\\(|\nu|,|\mu|)=\gamma(\mathbf{n})}} \frac{\mathrm{ch}_T\,\mathcal{L}_Z}{\Lambda_{-1}\left[T_Z^{\mathrm{vir}}\mathcal{N}^{\vee}_{1,[1^1],n,\gamma(\mathbf{n})}\right]} \\
&= \sum_{\substack{Z=(\nu,\mu)\\(|\nu|,|\mu|)=\gamma(\mathbf{n})}} \frac{\mathcal{L}_\nu(q_1,q_2)\tilde{\mathcal{W}}_{(\nu,\mu)}(q_1,q_2)}{\Lambda_{-1}\left[T_{\tilde{Z}}\mathcal{M}^{\vee}_{1,n_0}\right]},
\end{aligned}
\tag{3.2.47}
$$

with

$$
\mathcal{L}_\nu(q_1,q_2) = \left(\prod_{i=1}^{M_1}\prod_{j=1}^{\nu'_i} q_1^{i-1}q_2^{j-1}\right)^{p_0},
$$

and

$$
\begin{aligned}
\tilde{\mathcal{W}}_{(\nu,\mu)} = &\prod_{i=1}^{M_1}\prod_{j=1}^{N_1} \frac{(1-q_1^{\mu_j-i}q_2^{j-\nu'_i-1})(1-q_1^{-i}q_2^{j-\mu'_i-1})}{(1-q_1^{\mu_j-i}q_2^{(j-\mu'_i-1)})(1-q_1^{-i}q_2^{j-\nu'_i-1})} \\
&\prod_{i=1}^{M_1}\prod_{j=1}^{\nu'_i-\mu'_i} \frac{(1-q_1^{-i}q_2^{-j-\mu'_i})}{(1-q_1^{-1}q_2^{-1})},
\end{aligned}
$$

where, as usual, $q_1 = \mathrm{ch}_T\, T_1$ and $q_2 = \mathrm{ch}_T\, T_2$.

In order to support our conjecture that the quiver we studied so far do indeed provide an ADHM-type construction for the nested Hilbert scheme of points on $\mathbb{C}^2$ we will show some relevant examples in the following. In the two-steps quiver case this is true by a result of [213], which moreover implies that the non-abelian quiver provides an ADHM description for the moduli space of framed torsion-free flags of sheaves on $\mathbb{P}^2$. A very brief review of the result of [213] which are useful for what follows can be found in appendix 3.B. Even in the two-steps case we can still compare the results coming from direct localization computations to the formulae in [69; 119]. In particular, since the nested Hilbert scheme of points is known to be non smooth except for the case $(n_0,n_1)=(n,1)$ or $(n_0,n_1)=(n,0)$, the polynomials we get multiplied by the Nekrasov partition function order by order are expected to reproduce the modified Macdonald polynomials $\tilde{H}_\lambda(\mathbf{x};q,t)$ when $n_1=1$. For the sake of ease of comparison, in what follows we will use the notation $\mathcal{N}(r,n_0,\ldots,n_{s-1})$, which is found in [69; 213], instead of $\mathcal{N}_{r,[r^1],n,\mu}$.

Example 3.6. *If $\mathbf{n}=(n,0)$ we need to compute the partition function for $\mathcal{N}(1,n,0)$, and obviously the partition function reproduces the result in Eq. (3.2.44), for $g=0$.*

Example 3.7. *Take $\mathbf{n}=(1,1)$, so that $\mathcal{F}(1,1,1)\simeq\mathcal{N}(1,2,1)\simeq\mathrm{Hilb}^{(1,2)}(\mathbb{C}^2)$, [213]. We have two different choices for the fixed points:*

$$
(\nu,\mu) = \blacksquare\square = (2^1,1^1) \qquad \text{or} \qquad (\nu,\mu) = \begin{smallmatrix}\square\\ \blacksquare\end{smallmatrix} = (1^2,1^1)
$$

and we have for the partition function

$$
\begin{aligned}
Z_{\mathbf{n}}^{(1-g,0)}(\mathbf{x};q,t) &= \sum_{\nu} Z_{\mathbf{n},\nu}^{(1-g,0)}(\mathbf{x};q,t) \\
&= x_0 x_1 \left(\sum_{(\nu,\mu)} \frac{\mathcal{L}_{\nu}(q^{-1},t^{-1})\tilde{\mathcal{W}}_{(\nu,\mu)}(q^{-1},t^{-1})}{\Lambda_{-1}\left[T_{\tilde{Z}}\mathscr{M}_{1,n_0}^{\vee}\right]} \right)
\end{aligned}
$$

with

$$
\begin{cases}
Z_{\mathbf{n},2^1}^{(1-g,0)}(\mathbf{x};q,t) = \dfrac{\mathcal{L}_{2^1}(q^{-1},t^{-1})}{\Lambda_{-1}\left[T_{2^1}\mathscr{M}_{1,n_0}^{\vee}\right]}(1+q)x_0x_1 \\[2ex]
Z_{\mathbf{n},1^2}^{(1-g,0)}(\mathbf{x};q,t) = \dfrac{\mathcal{L}_{1^2}(q^{-1},t^{-1})}{\Lambda_{-1}\left[T_{1^2}\mathscr{M}_{1,n_0}^{\vee}\right]}(1+t)x_0x_1
\end{cases}
$$

By putting together with the previous example, we have that

$$
\begin{aligned}
Z_{|\mathbf{n}|=2}^{1-g,0} &= \sum_{\nu\in\mathcal{P}(2)} \frac{\mathcal{L}_{\nu}(q^{-1},t^{-1})}{\Lambda_{-1}\left[T_{\nu}\mathscr{M}_{1,n_0}^{\vee}\right]}\tilde{H}_{\nu}(x_0,x_1;q,t) \\
&= \sum_{\nu\in\mathcal{P}(2)} \mathcal{H}_{\nu}^{g=0}(z,w)\tilde{H}_{\nu}(x_0,x_1;z^2,w^2)
\end{aligned}
$$

We want to point out that the elliptic counterpart to the polynomials determined by $\tilde{\mathcal{W}}_{(\nu,\mu)}$ are the following

$$
\begin{cases}
P^{\mathrm{ell}}_{\ydiagram{2}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0x_1} = \dfrac{\theta_1(\tau|2\epsilon_1)}{\theta_1(\tau|\epsilon_1)}, \\[2ex]
P^{\mathrm{ell}}_{\ydiagram{1,1}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0x_1} = \dfrac{\theta_1(\tau|2\epsilon_2)}{\theta_1(\tau|\epsilon_2)},
\end{cases}
\tag{3.2.48}
$$

which obviously reduce to the corresponding modified Macdonald polynomials coefficients when $\tau \to i\infty$.

Example 3.8. *Let's now consider $\mathbf{n}$ to be such that $n_0 + n_1 = 3$. The only quantity we need to compute is related to $\mathbf{n} = (2,1)$, which corresponds to $\mathscr{N}(1,3,1)$. We have the following possibilities for the fixed points:*

$$
\{(\nu,\mu)\} = \left\{ \ydiagram[*(gray)]{2}*{3},\ \ydiagram[*(gray)]{0,2}*{1,2},\ \ydiagram[*(gray)]{1,1}*{1,2},\ \ydiagram[*(gray)]{0,1,1}*{1,1,1} \right\}
$$

and

$$
\begin{cases}
\tilde{\mathcal{W}}_{\ydiagram{3}}(q^{-1},t^{-1}) = (1+q+q^2) \\[1ex]
\tilde{\mathcal{W}}_{\ydiagram{1,2}}(q^{-1},t^{-1}) + \mathcal{W}_{\ydiagram{1,2}}(q^{-1},t^{-1}) = (1+q+t) \\[1ex]
\tilde{\mathcal{W}}_{\ydiagram{1,1,1}}(q^{-1},t^{-1}) = (1+t+t^2)
\end{cases}
$$

As in the previous example, we can exhibit explicitly the elliptic counterparts to these modified Macdonald polynomials, which read:

$$
\begin{cases}
P^{\mathrm{ell}}_{\ydiagram{3}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^2x_1} = \dfrac{\theta_1(\tau|3\epsilon_1)}{\theta_1(\tau|\epsilon_1)}, \\
P^{\mathrm{ell}}_{\ydiagram{1,2}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^2x_1} = \left(\dfrac{\theta_1(\tau|2\epsilon_1-\epsilon_2)}{\theta_1(\tau|\epsilon_1-\epsilon_2)} + \dfrac{\theta_1(\tau|2\epsilon_2-\epsilon_1)}{\theta_1(\tau|\epsilon_2-\epsilon_1)}\right), \\
P^{\mathrm{ell}}_{\ydiagram{1,1,1}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^2x_1} = \dfrac{\theta_1(\tau|3\epsilon_2)}{\theta_1(\tau|\epsilon_2)}.
\end{cases}
\tag{3.2.49}
$$

Example 3.9. *As a final example of a smooth nested Hilbert scheme of points we will take $\mathscr{N}(1,4,1)$, so that the fixed points will be*

$$
\{(\nu,\mu)\} = \left\{ \ydiagram[*(gray)]{3}*{4},\ \ydiagram[*(gray)]{0,3}*{1,3},\ \ydiagram[*(gray)]{1,1}*{1,3},\ \ydiagram[*(gray)]{1,2}*{2,2},\ \ydiagram[*(gray)]{1,1,1}*{1,1,2},\ \ydiagram[*(gray)]{0,1,2}*{1,1,2},\ \ydiagram[*(gray)]{0,1,1,1}*{1,1,1,1} \right\}
$$

by which we get

$$
\begin{cases}
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{3}*{4}}(q^{-1},t^{-1}) = (1+q+q^2+q^3) \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{0,3}*{1,3}}(q^{-1},t^{-1}) + \tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,2}*{1,3}}(q^{-1},t^{-1}) = (1+q+q^2+t) \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,2}*{2,2}}(q^{-1},t^{-1}) = (1+q+t+qt) \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,1,1}*{1,1,2}}(q^{-1},t^{-1}) + \tilde{\mathcal{W}}_{\ydiagram[*(gray)]{0,1,2}*{1,1,2}}(q^{-1},t^{-1}) = (1+t+t^2+q) \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{0,1,1,1}*{1,1,1,1}}(q^{-1},t^{-1}) = (1+t+t^2+t^3)
\end{cases}
$$

which again reproduce modified Macdonald polynomials which can be found tabulated in the mathematical literature. Their elliptic counterpart is now given by:

$$
\begin{cases}
P^{\mathrm{ell}}_{\ydiagram{4}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^3x_1} = \dfrac{\theta_1(\tau|4\epsilon_1)}{\theta_1(\tau|\epsilon_1)}, \\
P^{\mathrm{ell}}_{\ydiagram{3,1}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^3x_1} = \left(\dfrac{\theta_1(\tau|2\epsilon_1)}{\theta_1(\tau|\epsilon_1)}\dfrac{\theta_1(\tau|3\epsilon_1-\epsilon_2)}{\theta_1(\tau|2\epsilon_1-\epsilon_2)} \right. \\
\qquad\qquad \left. +\dfrac{\theta_1(\tau|2\epsilon_2-2\epsilon_1)}{\theta_1(\tau|\epsilon_2-2\epsilon_1)}\right), \\
P^{\mathrm{ell}}_{\ydiagram{2,2}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^3x_1} = \left(\dfrac{\theta_1(\tau|2\epsilon_1)}{\theta_1(\tau|\epsilon_1)}\dfrac{\theta_1(\tau|2\epsilon_2)}{\theta_1(\tau|\epsilon_2)}\right), \\
P^{\mathrm{ell}}_{\ydiagram{2,1,1}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^3x_1} = \left(\dfrac{\theta_1(\tau|2\epsilon_2)}{\theta_1(\tau|\epsilon_2)}\dfrac{\theta_1(\tau|3\epsilon_2-\epsilon_1)}{\theta_1(\tau|2\epsilon_2-\epsilon_1)} \right. \\
\qquad\qquad \left. +\dfrac{\theta_1(\tau|2\epsilon_1-2\epsilon_2)}{\theta_1(\tau|\epsilon_1-2\epsilon_2)}\right), \\
P^{\mathrm{ell}}_{\ydiagram{1,1,1,1}}(\mathbf{x};\epsilon_1,\epsilon_2)\Big|_{x_0^3x_1} = \dfrac{\theta_1(\tau|4\epsilon_2)}{\theta_1(\tau|\epsilon_2)}.
\end{cases}
\tag{3.2.50}
$$

The following is the easiest example of a non smooth nested Hilbert scheme, namely $\mathscr{N}(1,4,2)$, and we can see how in this case our computation doesn't reproduce the χ_y genus of [69], hence the formulae of [119], giving instead their virtual generalization.

Example 3.10. *Take* $(n_0,n_1)=(4,2)$*. The prescription for the fixed points gives us*

$$
\{(\nu,\mu)\} = \left\{ \ydiagram[*(gray)]{2}*{4},\ \ydiagram[*(gray)]{2}*{3,1},\ \ydiagram[*(gray)]{1,1}*{3,1},\ \ydiagram[*(gray)]{1,1}*{2,2},\ \ydiagram[*(gray)]{2}*{2,2},\ \ydiagram[*(gray)]{1,1}*{2,1,1},\ \ydiagram[*(gray)]{2}*{2,1,1},\ \ydiagram[*(gray)]{1,1}*{1,1,1,1} \right\}
$$

by which we get

$$
\begin{cases}
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{2}*{4}}(q^{-1},t^{-1}) = 1+q+2q^2+q^3+q^4-q^2t-q^3t-2q^4t-q^5t-q^6t \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{2}*{3,1}}(q^{-1},t^{-1}) + \tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,1}*{3,1}}(q^{-1},t^{-1}) = 1+q+2q^2+t+qt-q^2t-q^3t \\
\qquad\qquad -q^4t-qt^2-q^2t^2-q^3t^2 \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,1}*{2,2}}(q^{-1},t^{-1}) + \tilde{\mathcal{W}}_{\ydiagram[*(gray)]{2}*{2,2}}(q^{-1},t^{-1}) = 1+q+q^2+t+qt+t^2-q^2t-qt^2 \\
\qquad\qquad -2q^2t^2-q^3t^2-q^2t^3 \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,1}*{2,1,1}}(q^{-1},t^{-1}) + \tilde{\mathcal{W}}_{\ydiagram[*(gray)]{2}*{2,1,1}}(q^{-1},t^{-1}) = 1+q+t+qt+2t^2-q^2t-qt^2 \\
\qquad\qquad -q^2t^2-qt^3-q^2t^3-qt^4 \\
\tilde{\mathcal{W}}_{\ydiagram[*(gray)]{1,1}*{1,1,1,1}}(q^{-1},t^{-1}) = 1+t+2t^2+t^3+t^4-qt^2-qt^3-2qt^4-qt^5-qt^6
\end{cases}
.
$$

The polynomials above contain the coefficients for the modified Macdonald polynomials which in this case read

$$\begin{cases} \tilde{H}_{\yng(4)}(q,t)|_{x_0^2x_1^2} = 1+q+2q^2+q^3+q^4 \\ \tilde{H}_{\yng(1,3)}(q,t)|_{x_0^2x_1^2} = 1+q+2q^2+t+qt \\ \tilde{H}_{\yng(2,2)}(q,t)|_{x_0^2x_1^2} = 1+q+q^2+t+qt+t^2 \\ \tilde{H}_{\yng(1,1,2)}(q,t)|_{x_0^2x_1^2} = 1+q+t+qt+2t^2 \\ \tilde{H}_{\yng(1,1,1,1)}(q,t)|_{x_0^2x_1^2} = 1+t+2t^2+t^3+t^4 \end{cases}$$

As a final remark let us point out that, even though the GLSM partition function is naturally computing virtual invariants, as the moduli space $\mathscr{N}(r, n_0, n_1)$ is in general a singular quasi-projective variety, [68], however one should be able to use equivariant localization to compute usual topological invariants also for singular varieties [214; 215].

3.A LOW ENERGY THEORY FOR D3/D7

Let us here sketch a derivation of the low energy effective theory of the D3-D7 system at an orbifold point by studying the equations of motion reduced on $T^2 \times \mathcal{C}$. This amounts to solve the BPS equations

$$F^{(2,0)} = 0, \quad \partial_A \Phi_S = 0, \quad \partial_A B_i = 0, \quad \partial_A I = 0, \quad \partial_A J = 0 \tag{3.A.1}$$

$$\omega \cdot F + [B_i, B_i^\dagger] + [\Phi_S, \Phi_S^\dagger] + I^\dagger I - JJ^\dagger = \zeta \mathbf{1}_N \tag{3.A.2}$$

while we minimise the super potential

$$\mathcal{W} = \mathrm{Tr}\{\Phi_S\left([B_1, B_2] + IJ\right)\}\,. \tag{3.A.3}$$

Let us now focus in the vicinity of the orbifold point, where the local geometry of $\mathcal{C}$ is $\mathbb{C}/\mathbb{Z}_s$ and that of $T^*\mathcal{C}$ is the ALE quotient $\mathbb{C}^2/\mathbb{Z}_s$. There the Chan-Paton bundle of the open string modes decomposes in $\mathbb{Z}_s$-representations as already discussed in §3.1. (3.A.1) admit vortex solutions centered at the orbifold point, whose vorticity is fixed by the order of the cyclic group. On the vortex background, the gauge field along $\mathbb{C}/\mathbb{Z}_s$ becomes massive due to the Higgs mechanism and decouples from the low energy spectrum.

Unpacking the open strings moduli in the V_j twisted sectors one gets the degrees of freedom in (3.1.8) and the relations (3.1.9). Let us now discuss how these arise. The modes B_1^j and B_2^j come from the $\mathbb{Z}_s$ representation of the B_1 and B_2 fields and analogously I^j and J^j from I and J. The further degrees of freedom arise from Φ_S, that is the one-form in the adjoint. Since these are describing open string modes in twisted directions under the $\mathbb{Z}_s$ group, the fields which arise from Φ_S are homomorphisms between nearby twisted sectors. Explicitly from the reduction of Φ_S one gets the bifundamental modes $F^j \in \mathrm{Hom}\left(V_j, V_{j+1}\right)$.

The BPS vacua equations of this system therefore are obtained from the reduction to the constant modes of (3.A.1) and the minimization of the super potential

$$[B_1^j, B_2^j] + I^j J^j = 0, \quad B_1^j F^j - F^j B_1^{j+1} = 0 \quad B_2^j F^j - F^j B_2^{j+1} = 0, \quad J^j F^j = 0.$$

3.B FIXED POINTS AND VIRTUAL DIMENSION

The characterization of the fixed points we described in §3.2.1 makes it clear that the T−fixed locus in $\mathcal{N}_{r,[r^1],n,\mu}$ consists only of isolated non-degenerate points. Moreover through a simple computation it's now very easy to compute the virtual dimension of $\mathcal{N}(r, n_0, \dots, n_{s-1})$. Altogether these facts get summarized by the following proposition, which for the sake of simplicity we state in the simple case of the two-steps quiver.

Proposition 3.11. *The* T*-fixed locus of the moduli space* $\mathcal{N}(r, n_0, n_1)$ *consists only of isolated non-degenerate points, which are into* $1-1$ *correspondence with* r*-tuples of coloured nested partitions. Moreover* $\mathrm{vd}_{\mathcal{N}(r,n_0,n_1)} = 2rn_0 - rn_1 + 1$.

Proof. A very brief sketch of how to prove the statement about the fixed points was previously given in §3.2.1, so now we will only focus on computing the virtual dimension of $\mathcal{N}(r, n_0, n_1)$. Using the description provided by quiver 12 we see that the number of variables involved in the computation is $\#\mathrm{var} = 2n_0^2 + 2n_1^2 + 2n_0 r + n_0 n_1$, with $r = \dim W$. Moreover, the number of constraints we need to implement is $\#\mathrm{constr} = n_0^2 + n_1^2 - 1 + n_0 n_1 + n_1 r$, where we also took into account that the constraints are not independent. Finally we account for the fact that we take the GIT quotient by the action of $GL(n_0) \times GL(n_1)$, which contributes by $\#\mathrm{symm} = n_0^2 + n_1^2$. Then

$$\text{“}\dim \mathcal{N}(r, n_1, n_2)\text{”} = \#\mathrm{var} - \#\mathrm{constr} - \#\mathrm{symm} = 2n_0 r - n_1 r + 1.$$

In order to directly compute the virtual dimension of the nested Hilbert scheme of points on $\mathbb{C}^2$, we use the character decomposition of $T_Z^{\mathrm{vir}}\mathcal{N}(1, n_0, n_1)$ at a generic fixed point under the torus action. Then

$$\begin{aligned}
\mathrm{vd}_{\mathcal{N}(1,n_0,n_1)} = \lim_{T_i \to 1} \Bigg[& T_{\tilde{Z}}\mathcal{M}(1, n_0) \\
& + \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} (T_1^{i-\mu_j} - T_1^i)(T_2^{-j+\mu_i'+1} - T_2^{-j+\nu_i'+1}) \\
& - \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_1^i T_2^{j+\mu_i'} + T_1 T_2 \Bigg] \\
= 2n_0 - n_1 + 1, &
\end{aligned}$$

which, in the case of a smooth nested Hilbert scheme of points, coincides with the computation of [213]. A completely analogous computation can be carried out in the generic (non necessarily smooth) case, by using the character decomposition we computed for $T_Z^{\mathrm{vir}}\mathcal{N}(r, n_0, n_1)$, which in turn coincides with the representation of the virtual tangent space to the nested Hilbert scheme of points (when $r = 1$) given in [107]. ∎

4 FLAGS OF SHEAVES ON $\mathbb{P}^2$

This chapter is devoted to studying the representations of the *nested instantons* quiver we introduced in §3.1.4. In particular, we focus our attention to the case in which the dimension vector for the framing is $\mathbf{r} = (r, 0, \ldots, 0)$ where r is the dimension of the rightmost framing node. We also study its relation to *flags of framed torsion-free sheaves* on $\mathbb{P}^2$ and *nested Hilbert schemes*, and compute some relevant virtual invariants via equivariant localisation.

We want to point out that the moduli space we are studying seems to be analogous to the Filt-scheme studied in [161] in the case of smooth projective curves. The importance of studying these moduli spaces on (smooth projective) surfaces lies in their application to the computation of monopole contributions to Vafa-Witten invariants defined in [207; 208]. In fact these monopole contributions to Vafa-Witten invariants are expressed in terms of invariants of flags of sheaves, which in some cases reduce to nested Hilbert schemes, see [116; 148] for computations in this case. The deformation-obstruction theory and virtual cycle for the components of the monopole branch in Vafa-Witten theory giving rise to flags of higher rank sheaves were explicitly constructed in [201]. Nested Hilbert schemes on surfaces were interpreted in terms of degeneracy loci in [109; 110], where they are also shown to be equipped with a perfect obstruction theory. Similarly nested Hilbert schemes of points were also studied in [107], and a perfect obstruction theory and virtual cycles are explicitly constructed. Their application to reduced DT and PT invariants are also discussed in [83; 107; 108].

In the following we give a summary of the result of this chapter. In §4.1 we start our analysis by proving the following

Theorem. *The moduli space $\mathcal{N}(r, \mathbf{n})$ of stable representation of the nested instantons quiver of numerical type $(r, \mathbf{n})$ is a virtually smooth quasi-projective variety over $\mathbb{C}$ equipped with a natural action of $\mathbb{T} = T \times (\mathbb{C}^*)^r$, $T = (\mathbb{C}^*)^2$, and a perfect obstruction theory.*

We also prove that $\mathcal{N}(r, \mathbf{n})$ embeds into a smooth hyperkähler variety $\mathcal{M}(r, \mathbf{n})$, see §4.1.3.

In §4.2, we construct the moduli space $\mathcal{F}(r, \boldsymbol{\gamma})$ of flags of framed torsion free sheaves on $\mathbb{P}^2$ and prove the existence of an isomorphism with $\mathcal{N}(r, \mathbf{n})$. As a particular case, we have

Theorem. *The moduli space of nested instantons $\mathcal{N}(1, \mathbf{n})$ is isomorphic to the nested Hilbert scheme of points on $\mathbb{C}^2$, namely*

$$\mathcal{N}(1, \mathbf{n}) = \mathbb{X}_0 /\!\!/_\chi \mathcal{G} \simeq \mathrm{Hilb}^{\hat{\mathbf{n}}}(\mathbb{C}^2).$$

The moduli space of flags of sheaves is constructed by means of a functor

$$\mathsf{F}_{(r,\boldsymbol{\gamma})} : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Sets}$$

parametrizing flags of torsion-free sheaves on $\mathbb{P}^2$ in

Proposition. *The moduli functor $\mathsf{F}_{(r,\gamma)}$ is representable. The (quasi-projective) variety representing $\mathsf{F}_{(r,\gamma)}$ is the moduli space of flags of framed (coherent) torsion-free sheaves on $\mathbb{P}^2$, denoted by $\mathscr{F}(r,\boldsymbol{\gamma})$.*

while its isomorphism with $\mathscr{N}(r,\mathbf{n})$ is proven in

Theorem. *The moduli space of stable representations of the nested instantons quiver is a fine moduli space isomorphic to the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$: $\mathscr{F}(r,\boldsymbol{\gamma}) \simeq \mathscr{N}(r,\mathbf{n})$, as schemes, where $n_i = \gamma_i + \cdots + \gamma_N$.*

The ADHM construction of a particular class of flags of sheaves on $\mathbb{P}^2$ was given in [168], where their connection to shuffle algebras on K−theory is also studied. Moreover the construction of the functor $\mathsf{F}_{(r,\gamma)}$ shows that the moduli space of nested instantons is isomorphic to a relative Quot−scheme. Perfect obstruction theories on Quot−schemes and the description of their local model in terms of a quiver is discussed in [14; 192].

In §4.3 we proceed to the evaluation of the relevant virtual invariants via equivariant localisation. The classification of the T-fixed locus of $\mathscr{N}(r,\mathbf{n})$ is presented in the

Proposition. *The T−fixed locus of $\mathscr{N}(r,n_0,\ldots,n_{s-1})$ can be described by s−tuples of nested coloured partitions $\boldsymbol{\mu}_1 \subseteq \cdots \subseteq \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$, with $|\boldsymbol{\mu}_0| = n_0$ and $|\boldsymbol{\mu}_{i>0}| = n_0 - n_i$.*

In 4.3.2 we compute the generating function of the virtual Euler characteristics of $\mathscr{N}(1,\mathbf{n})$, see eq.(4.3.10) for the explicit combinatorial formula. We conjecture that, by summing over the nested partitions, this generating function is expressed in terms of polynomials:

Conjecture. *The generating function*

$$\chi^{\mathrm{vir}}(\mathscr{N}(1,n_0,\ldots,n_N); q_1^{-1}=q, q_2^{-1}=t) = \sum_{\mu_0} P_{\mu_0}(q,t)/N_{\mu_0}(q,t)$$

is such that

$$P_{\mu_0}(q,t) = \frac{Q_{\mu_0}(q,t)}{(1-qt)^N},$$

with $Q_{\mu_0}(q,t) \in \mathbb{Z}[q,t]$.

For specific profiles of the nesting, these polynomials are conjectured to compute sums of (q,t)−Kostka polynomials:

Conjecture. *When $|\mu_0| = |\mu_N| + 1 = |\mu_{N-1}| + 2 = \cdots = |\mu_1| + N$ we have*

$$\begin{aligned} Q_{\mu_0}(q,t) &= \left\langle h_{\mu_0}(\mathbf{x}), \widetilde{H}_{\mu_0}(\mathbf{x};q,t)\right\rangle \\ &= \left\langle h_{\mu_0}(\mathbf{x}), \sum_{\lambda,\nu\in\mathcal{P}(n_0)} \widetilde{K}_{\lambda,\mu_0}(q,t) K_{\mu_0,\nu} m_\nu(\mathbf{x})\right\rangle \\ &= \sum_{\substack{\lambda\in\mathcal{P}(n_0)\\ m_\lambda(\mathbf{x})\neq 0}} \widetilde{K}_{\lambda,\mu_0}(q,t), \end{aligned}$$

where the Hall pairing $\langle -,- \rangle$ *is such that* $\langle h_\mu, m_\lambda \rangle = \delta_{\mu,\lambda}$ *and* $\widetilde{H}_\mu(\mathbf{x};q,t)$, $\widetilde{K}_{\lambda,\mu}(q,t)$ *are the modified Macdonald polynomials and the modified Kostka polynomials, respectively.*

In 4.3.3 we compute the generating function of the virtual χ_{-y}-genus of $\mathcal{N}(1,\mathbf{n})$, see eq.(4.3.13), and of $\mathcal{N}(r,\mathbf{n})$, see eq.(4.3.14). We also show that, by specialising at $y=1$, one gets that the generating function of nested partitions of arbitrary length is the Macmahon function as expected, see Eq. (4.3.15).

In 4.3.4 we compute the generating function of the virtual elliptic genus of $\mathcal{N}(1,\mathbf{n})$, see eq.(4.3.16), and of $\mathcal{N}(r,\mathbf{n})$, see eq.(4.3.17).

Finally, in §4.4, we extend our results to $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ in the case of $\chi_{-y}-$genera, see formulae (4.4.2) and (4.4.3) respectively. Notice that the choice of computing $\chi_{-y}-$genera was due to the expected simple polynomial dependence in y. Everything which was done in this context is however completely general and holds for any complex genus.

4.1 THE NESTED INSTANTONS QUIVER

4.1.1 Quiver representations and stability

In the following we will mainly be interested in studying the following quiver, which will be called the *nested instantons quiver*

$$\begin{tikzcd} V_N \arrow[loop above, "\alpha_N"] \arrow[loop below, "\beta_N"] \arrow[r, shift left, "\phi_N"] & \cdots \arrow[l, shift left, "\gamma_N"] \arrow[r, shift left, "\phi_2"] & V_1 \arrow[l, shift left, "\gamma_2"] \arrow[loop above, "\alpha_1"] \arrow[loop below, "\beta_1"] \arrow[r, shift left, "\phi_1"] & V_0 \arrow[l, shift left, "\gamma_1"] \arrow[loop above, "\alpha_0"] \arrow[loop below, "\beta_0"] \arrow[r, shift left, "\eta"] & W \arrow[l, shift left, "\xi"] \end{tikzcd} \tag{4.1.1}$$

with relations

$$[\alpha_0,\beta_0]+\xi\eta=0,\quad [\alpha_i,\beta_i]=0,\quad \alpha_i\phi_i-\phi_i\alpha_{i+1}=0=\beta_i\phi_i-\phi_i\beta_{i+1}$$
$$\gamma_i\alpha_i-\alpha_{i+1}\gamma_i=0=\gamma_i\beta_i-\beta_{i+1}\gamma_i,\quad \phi_i\gamma_i=0,\quad \eta\phi_1=0,\quad \gamma_1\xi=0$$

Given

$$\begin{aligned}\mathbb{X}=&\operatorname{End}V_0^{\oplus 2}\oplus\operatorname{Hom}(V_0,W)\oplus\operatorname{Hom}(W,V_0)\oplus\operatorname{End}(V_1)^{\oplus 2}\oplus\operatorname{Hom}(V_1,V_0)\\&\oplus\operatorname{Hom}(V_0,V_1)\oplus\cdots\oplus\operatorname{End}(V_N)^{\oplus 2}\oplus\operatorname{Hom}(V_N,V_{N-1})\oplus\operatorname{Hom}(V_{N-1},V_N)\end{aligned}$$

a representation of numerical type $(r,\mathbf{n})$ of (4.1.1) in the category of vector spaces will be given by the datum of $X=(\mathbb{W},h)$, with $\mathbb{W}=(W,V_0,\ldots,V_N)$, with $\dim W=r$ and $\dim V_i=n_i$, and

$$\mathbb{X}\ni h=(B_1^0,B_2^0,I,J,B_1^1,B_2^1,F^1,G^1,\ldots),$$

satisfying

$$\begin{aligned}
&[B_1^0, B_2^0] + IJ = 0, \quad [B_1^i, B_2^i] = 0,\\
&B_1^i F^i - F^i B_1^{i+1} = 0 = B_2^i F^i - F^i B_2^{i+1},\\
&G^i B_1^i - B_2^{i+1} G^i = 0 = G^i B_2^i - B_2^{i+1} G^i,\\
&F^i G^i = 0, \quad JF^1 = 0, \quad G^1 I = 0
\end{aligned} \tag{4.1.2}$$

which we will call *nested ADHM equations*. In the following we need to address the problem of King stability for representations of the nested instantons quiver.

Definition 4.1. *Let $\Theta = (\boldsymbol{\theta}, \theta_\infty) \in \mathbb{Q}^{s+1}$ be such that $\Theta(X) = \mathbf{n} \cdot \boldsymbol{\theta} + r\theta_\infty = 0$. We will say that a framed representation X of (4.1.1) is Θ-semistable if*

- *$\forall 0 \neq \tilde{X} \subset X$ of numerical type $(0, \tilde{\mathbf{n}})$ we have $\Theta(\tilde{X}) = \boldsymbol{\theta} \cdot \tilde{\mathbf{n}} \leqslant 0$;*
- *$\forall 0 \neq \tilde{X} \subset X$ of numerical type $(\tilde{r}, \tilde{\mathbf{n}})$ we have $\Theta(\tilde{X}) = \boldsymbol{\theta} \cdot \tilde{\mathbf{n}} + \tilde{r}\theta_\infty \leqslant 0$.*

If strict inequalities hold X is said to be Θ-stable.

In [51; 213] the two node case, namely $N = 1$ was considered and we can here generalize their result to the more general nested instantons quiver (4.1.1).

Proposition 4.2. *Let X be a representation of (4.1.1) of numerical type $(r, \mathbf{n}) \in \mathbb{N}_{>0}^{N+2}$, then choose $\theta_i > 0$, $\forall i > 0$ and θ_0 s.t. $\theta_0 + n_1\theta_1 + \cdots n_{s-1}\theta_{s-1} < 0$. The following are equivalent:*

(i) *X is Θ-stable;*

(ii) *X is Θ-semistable;*

(iii) *X satisfies the following conditions:*

s1 *$F^i \in \mathrm{Hom}(V_{i+1}, V_i)$ is injective, $\forall i \geqslant 1$;*

s2 *the ADHM datum $\mathcal{A} = (W, V_0, B_1^0, B_2^0, I, J)$ is stable.*

Proof. $(\mathbf{i}) \Rightarrow (\mathbf{ii})$ This is immediately true, as any Θ-stable representation is also Θ-semistable.

$(\mathbf{ii}) \Rightarrow (\mathbf{iii})$ Let us first take a Θ-semistable representation X having at least one of the F^i not injective. Without loss of generality let F^k be the only one to be such a map. Then, if $v_k \in \ker F^k \Rightarrow B_2^{k+1} v_k \in \ker F^k$, due to the nested ADHM equations, and $B_2^{k+1}(\ker F^k) \subset \ker F^k$ (the same is obviously true for B_1^{k+1}. Now

$$\tilde{X} = (0, \ldots, 0, \ker F^k, 0, \ldots, F^k, B_1^{k+1}|_{\ker F^k}, B_2^{k+1}|_{\ker F^k}, 0, \ldots, 0)$$

is a subrepresentation of X of numerical type $(0, \ldots, 0, \dim \ker F^k, 0, \ldots, 0)$. Thus

$$\tilde{\mathbf{n}} \cdot \boldsymbol{\theta} + \tilde{r}\theta_\infty = \theta_k \dim \ker F^k > 0,$$

which contradicts the hypothesis of X being Θ-semistable.

If instead we take X to be Θ–semistable and suppose **S2** to be false, then $\exists 0 \subset S \subset V_1$ s.t. $B_1^0(S)$, $B_2^0(S)$, $\mathrm{Im}(I) \subseteq S$. In this case

$$\tilde{X} = (W, S, V_1, \dots, B_1^0|_S, B_2^0|_S, I, J|_S, \dots)$$

is a subrepresentation of X of numerical type $(r, \dim S, n_1, \dots)$ but, since $\mathbf{n} \cdot \theta + r\theta_\infty = 0$ having $\theta_{i>0} > 0$ and $\theta_0 - n_1\theta_1 - \dots < 0$, we have

$$\dim S\theta_0 + n_1\theta_1 + \dots + r\theta_\infty = (\dim S - n_0)\theta_0 > 0,$$

which again leads to a contradiction.

$(\mathbf{iii}) \Rightarrow (\mathbf{i})$ If we take a proper subrepresentation $\tilde{X}$ of numerical type $(\tilde{r}, \tilde{\mathbf{n}})$, we just need to check the cases $\tilde{r} = 0$ and $\tilde{r} = r$.

- If $\tilde{r} = r$ then $\tilde{W} = W$, which in turn implies that $I \neq 0$, otherwise the ADHM datum (B_1^0, B_2^0, I, J) would not be stable. Since $\tilde{X}$ is proper the following diagram commutes

$$\begin{array}{ccc} W & \xrightarrow{I} & V_0 \\ {\scriptstyle \mathbb{1}_W}\uparrow & & \uparrow{\scriptstyle i} \\ W & \xrightarrow{\tilde{I}} & \tilde{V}_0 \end{array} \quad \Rightarrow i \circ \tilde{I} = I \circ \mathbb{1}_W$$

so that $\tilde{n}_0 > 0$, otherwise we would have $I = 0$. Moreover the following diagram also commutes (and so does the analogous one for B_2^0)

$$\begin{array}{ccc} V_0 & \xrightarrow{B_1^0} & V_0 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \tilde{V}_0 & \xrightarrow{\tilde{B}_1^0} & \tilde{V}_0 \end{array} \quad \Rightarrow i \circ \tilde{B}_1^0 = B_1^0 \circ i \Rightarrow B_1^0(\tilde{V}_0) \subset \tilde{V}_0,$$

leading to a contradiction with the stability of $(W, V_0, B_1^0, B_2^0, I, J)$. Since we are interested in proper subrepresentations of X, at least one $\tilde{n}_{i>0}$ is not zero, and at least one of these non-zero $\tilde{n}_k < n_k$, so that $\theta \cdot \tilde{\mathbf{n}} + \theta_\infty r < 0$, and X is stable.

- Let now $\tilde{r} = 0$. Since we are interested in proper subrepresentations we must choose $\tilde{n}_0 > 0$, otherwise $\tilde{V}_{k>0} = 0$ by virtue of the injectivity of F_k. In the same way as in the previous case the only option is $\tilde{n}_0 = n_0$. Following the same steps we previously carried out $\theta \cdot \tilde{\mathbf{n}} = \sum_{k>0} \theta_k(\tilde{n}_k - n_k) - \theta_\infty r < 0$. ∎

Corollary 4.3. *If* X *is a stable representation of the nested instantons quiver,* $G^k = 0, \forall k$.

Proof. By the previous proposition, due to the injectivity of F^k, $F^kG^k = 0 \Rightarrow G^k = 0$. ∎

4.1.2 The nested instantons moduli space

We want now to discuss the construction of the moduli space of stable representations of the quiver (4.1.1), and its connection to GIT theory and stability. First of all we define the space of the nested ADHM data to be the space $\mathbb{X}$ we defined previously, and an element $X \in \mathbb{X}$ is called an nested ADHM datum. On $\mathbb{X}$ we have a natural action of $\mathcal{G} = GL(V_0) \times \cdots \times GL(V_N)$ defined by

$$\begin{aligned} \Psi : (g_0, g_1, \ldots, g_N, X) \longmapsto (&g_0 B_1^0 g_0^{-1}, g_0 B_2^0 g_0^{-1}, g_0 I, J g_0^{-1}, \\ &g_1 B_1^1 g_1^{-1}, g_1 B_2^1 g_1^{-1}, g_0 F^1 g_1^{-1}, g_1 G^1 g_0^{-1}, \\ &\cdots \\ &g_N B_1^N g_N^{-1}, g_N B_2^N g_N^{-1}, g_{N-1} F^N g_N^{-1}, g_N G^N g_{N-1}^{-1}) \end{aligned}$$

This action of $\mathcal{G}$ on $\mathbb{X}$ is free on the stable points of $\mathbb{X}$. In fact if $\mathbf{g} \in \mathbb{G}$ is such that $\mathbf{g} \cdot X = X$, $\forall X \in \mathbb{X}$, we claim that $\mathbf{g} = (\mathbb{1}_{V_0}, \ldots, \mathbb{1}_{V_N})$. In order to see this, let $S = \ker(g_0 - \mathbb{1}_{V_0})$. Since $\mathbf{g} \cdot X = X$ it follows that $g_0 I = I$, which means $\operatorname{Im} I \subset S$. Moreover $g_0 B_1^0 = B_1^0 g_0$ and $g_0 B_2^0 = B_2^0 g_0$, but if $v \in S \Rightarrow (g_0 - \mathbb{1}_{V_0})v = 0 \Rightarrow g_0 v = v$, thus implying that $B_1^0(S), B_2^0(S) \subset S$. The stability of $(W, V_0, B_1^0, B_2^0, I, J)$ then force $S = V_0$. Finally since, $g_0 = \mathbb{1}_{V_0}$, $F^1(\mathbb{1}_{V_1} - g_1^{-1}) = 0 \Rightarrow g_1 = \mathbb{1}_{V_1}$ by the injectivity of F^1. By using this procedure then one can prove by iteration that $g_k = \mathbb{1}_{V_k}$, $\forall k$, thus $\mathbf{g} \cdot X = X$ $\forall X \in \mathbb{X} \Leftrightarrow \mathbf{g} = \mathbb{1}$. This proves that the action of $\mathcal{G}$ is free on the stable points of $\mathbb{X}$, and it is easy to prove that it preserves $\mathbb{X}_0$, which denotes the space of nested ADHM data satisfying the relations of quiver (4.1.1).

Now if $\chi : \mathcal{G} \to \mathbb{C}^*$ is an algebraic character for the algebraic reductive group $\mathcal{G}$, we can produce the moduli space of χ-semistable orbits following a construction due to [143], $\mathcal{N}_\chi^{ss}(r, \mathbf{n})$, which is a quasi-projective scheme over $\mathbb{C}$ and is defined as

$$\mathcal{N}_\chi^{ss}(r, \mathbf{n}) = \mathbb{X}_0 /\!\!/_\chi \mathcal{G} = \operatorname{Proj}\left(\bigoplus_{n \geqslant 0} A(\mathbb{X}_0(r, \mathbf{n}))^{\mathcal{G}, \chi^n}\right)$$

with

$$A(\mathbb{X}_0(r, \mathbf{n}))^{\mathcal{G}, \chi^n} = \{f \in A(\mathbb{X}_0(r, \mathbf{n})) | f(\mathbf{h} \cdot X) = \chi(\mathbf{h})^n f(X), \forall \mathbf{h} \in \mathcal{G}\}.$$

The scheme $\mathcal{N}_\chi^{ss}(r, \mathbf{n})$ contains an open subscheme $\mathcal{N}_\chi^{s}(r, \mathbf{n}) \subset \mathcal{N}_\chi^{ss}(r, \mathbf{n})$ encoding χ-stable orbits. It turns out that also in this framed case there is a relation between χ-stability and Θ-stability, as it was shown in [143] in the non framed setting.

Proposition 4.4. *Let $\Theta = (\theta_0, \theta_1, \ldots, \theta_N) \in \mathbb{Z}^{N+1}$ and define $\chi_\Theta : \mathcal{G} \to \mathbb{C}^*$ the character*

$$\chi_\Theta(\mathbf{h}) = \det(h_0)^{-\theta_0} \cdots \det(h_N)^{-\theta_N}.$$

A representation X of the nested ADHM quiver (4.1.1) is χ_Θ-(semi)stable iff it is Θ-(semi)stable.

Since the proof for Prop. 4.4 deeply relies on some known results about equivalent characterizations of χ-stability, we will first recall them. In full generality, let V be a vector space over $\mathbb{C}$ equipped with the action of a connected subgroup G of $U(V)$, whose complexification is denoted by $G^{\mathbb{C}}$. Then if $\chi : G \to U(1)$ is a character of G, we can extend it to form its complexification $\chi : G^{\mathbb{C}} \to \mathbb{C}^*$. We then form the trivial line bundle $V \times \mathbb{C}$, which carries an action of $G^{\mathbb{C}}$ via χ:

$$g \cdot (x, z) = (g \cdot x, \chi(g)^{-1} z), \qquad g \in G, (x, z) \in V \times \mathbb{C}.$$

Definition 4.5. *An element $x \in V$ is*

1. *χ−semistable if there exists a polynomial $f \in A(V)^{G^{\mathbb{C}}, \chi^n}$, with $n \geqslant 1$ such that $f(x) \neq 0$;*
2. *χ−stable if it satisfies the previous condition and if*
   a) *$\dim(G^{\mathbb{C}} \cdot x) = \dim(G^{\mathbb{C}}/\Delta)$, where $\Delta \subseteq G^{\mathbb{C}}$ is the subgroup of $G^{\mathbb{C}}$ acting trivially on V;*
   b) *the action of $G^{\mathbb{C}}$ on $\{x \in V : f(x) \neq 0\}$ is closed.*

Given the previous definition, the next lemma due to King [143] gives an alternative characterization of χ−(semi)stable points under the $G^{\mathbb{C}}$−action.

Lemma 4.6 (Lemma 2.2 and Prop. 2.5 in [143]). *Given the character $\chi : G^{\mathbb{C}} \to \mathbb{C}^*$ for the action of $G^{\mathbb{C}}$ on the vector space V, and the lift of this action to the trivial line bundle $V \times \mathbb{C}$, a point $x \in V$ is*

1. *χ−semistable iff $\overline{G^{\mathbb{C}} \cdot (x, z)} \cap (V \times \{0\}) = \emptyset$, for any $z \neq 0$;*
2. *χ−stable iff $G^{\mathbb{C}} \cdot (x, z)$ is closed and the stabilizer of (x, z) contains Δ with finite index.*

Equivalently, a point $x \in V$ is

1. *χ−semistable iff $\chi(\Delta) = \{1\}$ and $\chi(\lambda) \geqslant 0$ for any 1−parameter subgroup $\lambda(t) \subseteq G^{\mathbb{C}}$ for which $\lim_{t\to 0} \lambda(t) \cdot x$ exists;*
2. *χ−stable iff the only $\lambda(t)$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists and $\chi(\lambda) = 0$ are in Δ.*

With these notations, if $V^{ss}(\chi)$ denotes the set of χ−semistable points of V, $V/\!\!/_{\chi} G^{\mathbb{C}}$ can be identified with $V^{ss}(\chi)/\sim$, where $x \sim y$ in $V^{ss}(\chi)$ iff $\overline{G^{\mathbb{C}} \cdot x} \cap \overline{G^{\mathbb{C}} \cdot y} \neq \emptyset$ in $V^{ss}(\chi)$.

Proof of Prop. 4.4. Take a θ−semistable representation $X \in \mathbb{X}$ and assume it doesn't satisfy χ_θ−semistability. Then there exists a 1−parameter subgroup $\lambda(t)$ of $\mathcal{G}$ such that $\lim_{t\to 0} \lambda(t) \cdot X$ exists and $\chi(\lambda) < 0$. However each such 1−parameter subgroup λ determines a filtration $\cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots$ of subrepresentations of X, [143], and

$$\chi_\theta(\lambda) = -\sum_{n\in\mathbb{Z}} \theta(X_n) \geqslant 0,$$

thus proving one side of the proposition, as the part concerning stability is obvious from the fact that trivial subrepresentations of X correspond to subgroups in Δ.

Conversely, if X is a χ_θ−semistable representation, we want to show that it is also a θ−semistable one. We only need to consider two cases, corresponding to subrepresentations $\tilde{X}$ of X with $\tilde{r} = r$ or $\tilde{r} = 0$. Each vector space in

X, say V_i will have then a direct sum decomposition $V_i = \widetilde{V}_i \oplus \widehat{V}_i$. We will then take a $1-$parameter subgroup $\lambda(t)$ such that

$$\lambda_i(t) = \begin{bmatrix} t\mathbb{1}_{\widetilde{V}_i} & 0 \\ 0 & \mathbb{1}_{\widehat{V}_i} \end{bmatrix}.$$

Then one can easily compute

$$\begin{aligned} \chi_\theta(\lambda(t)) \cdot z &= \left[\det(\lambda_0(t))^{-\theta_0} \cdots \det(\lambda_N(t))^{-\theta_N}\right]^{-1} \cdot z \\ &= t^{\tilde{\mathbf{n}}\cdot\theta} z \end{aligned}$$

It is then a matter of a simple computation to verify that, if X wasn't $\theta-$semistable, then one would have had $\lim_{t\to 0} \lambda(t) \cdot X \in \mathbb{X} \times \{0\}$, thus contradicting the $\chi_\theta-$semistability. A completely analogous computation can be carried over when $\tilde{r} = r$, taking

$$\lambda_0(t) = \begin{bmatrix} \mathbb{1}_{\widetilde{V}_0} & 0 \\ 0 & t^{-1}\mathbb{1}_{\widehat{V}_0} \end{bmatrix}, \qquad \lambda_i(t) = \begin{bmatrix} \mathbb{1}_{\widetilde{V}_i} & 0 \\ 0 & t^{-1}\mathbb{1}_{\widehat{V}_i} \end{bmatrix},\ i > 0,$$

and since $(\tilde{\mathbf{n}} - \mathbf{n}) \cdot \theta > 0$ if X is supposed not to be $\theta-$semistable, this would still lead to a contradiction.

Finally, if X was to be $\chi_\theta-$stable but not $\theta-$stable, the $1-$parameter subgroups previously described would have stabilized the pair (X, z), $z \neq 0$, in the two different cases $\tilde{r} = 0$ and $\tilde{r} = r$ respectively, thus again giving rise to a contradiction. ∎

Corollary 4.7. *Given a representation of the nested instantons quiver* (4.1.1) *of numerical type* $(r, \mathbf{n})$*, there exists a chamber in* $\mathbb{Q}^{N+1} \ni (\theta, \theta_\infty) = \Theta$ *in which* $\theta_{i>0} > 0$ *and* $\theta_0 + n_1\theta_1 + \cdots + n_{s-1}\theta_{s-1} < 0$ *such that the following are equivalent:*

1. *X is* $\Theta-$*semistable;*
2. *X is* $\Theta-$*stable;*
3. *X is* $\chi_\Theta-$*semistable;*
4. *X is* $\chi_\Theta-$*stable;*
5. *X satisfies* **S1** *and* **S2** *in Prop.* 4.2*.*

Because of the previous corollary, in the stability chamber defined by Prop. 4.2 all notions of stability are actually the same, so that a representation satisfying anyone of the conditions in corollary 4.7 will be called stable, and the corresponding $\mathcal{N}^{ss}_{\chi_\Theta}(r, \mathbf{n}) = \mathcal{N}(r, \mathbf{n}) \simeq \mathcal{N}_{r,[r^1],n,\mu}$ (with the notations of [36]) will be addressed to as the moduli space of stable representations of (4.1.1) or, equivalently, as the moduli space of nested instantons. Altogether, the previous considerations prove the following theorem.[1]

1 We thank Valeriano Lanza for pointing out to us a correction to the original proof for the two-nodes quiver found in [213].

Theorem 4.8. *The moduli space $\mathscr{N}(r, \mathbf{n})$ of stable representation of the nested instantons quiver of numerical type $(r, \mathbf{n})$ is a virtually smooth quasi-projective variety equipped with a natural action of $\mathbb{T} = T \times (\mathbb{C}^*)^r$, $T = (\mathbb{C}^*)^2$, and a perfect obstruction theory. The moduli space $\mathscr{N}(r, \mathbf{n})$ can thus be identified in a suitable stability chamber with the moduli space of nested instantons $\mathscr{N}_{r,[r^1],n,\mu}$.*

Proof. The first part of the proof has already been proved. Consider then the following complex

$$\mathcal{C}(X): \ \mathcal{C}^0(X) \xrightarrow{d_0} \mathcal{C}^1(X) \xrightarrow{d_1} \mathcal{C}^2(X) \xrightarrow{d_2} \mathcal{C}^3(X) \tag{4.1.3}$$

with

$$\begin{aligned}
\mathcal{C}^0(X) =& \bigoplus_{i=0}^{N} \mathrm{End}(V_i),\\
\mathcal{C}^1(X) =& \mathrm{End}(V_0)^{\oplus 2} \oplus \mathrm{Hom}(W, V_0) \oplus \mathrm{Hom}(V_0, W) \oplus \\
& \oplus \left[\bigoplus_{i=1}^{N} \mathrm{End}(V_i)^{\oplus 2} \oplus \mathrm{Hom}(V_i, V_{i-1} \right],\\
\mathcal{C}^2(X) =& \mathrm{End}(V_0) \oplus \mathrm{Hom}(V_1, W) \oplus \left[\bigoplus_{i=1}^{N} \mathrm{Hom}(V_i, V_{i-1})^{\oplus 2} \oplus \mathrm{End}(V_i) \right]\\
\mathcal{C}^3(X) =& \bigoplus_{i=1}^{N} \mathrm{Hom}(V_i, V_{i-1}),
\end{aligned}$$

while the morphisms d_i are defined as:

$$\begin{aligned}
d_0(\mathbf{h}) =& \left([h_0, B_1^0], [h_0, B_2^0], h_0 I, -J h_0, [h_1, B_1^1], [h_1, B_2^1], h_0 F^1 - F^1 h^1, \dots \right)^T,\\
d_1 \left(b_1^0, b_2^0, i, j, b_1^1, b_2^1, f^1, \dots \right)^T =& \left([b_1^0, B_2^0] + [B_1^0, b_2^0] + iJ + Ij, jF^1 + Jf^1, \right.\\
& B_1^0 f^1 + b_1^0 F^1 - F^1 b_1^1 - f^1 B_1^1, B_2^0 f^1 + b_2^0 F^1 - F^1 b_2^1 - f^1 B_2^1,\\
& \left. \dots, [b_1^1, B_2^1] + [B_1^1, b_2^1], \dots, [b_1^N, B_2^N] + [B_1^N, b_2^N] \right)^T,\\
d_2 \left(c_1, \dots, c_{3N+2} \right)^T =& \left(c_1 F^1 + B_2^0 c_3 - c_3 B_2^1 + c_4 B_1^1 - B_1^0 c_4 - I c_2 - F^1 c_{2N+3}, \right.\\
& \dots, c_{2N+2+i} F^i + B_2^i c_{2+2i} - c_{2+2i} B_2^1 +\\
& \left. + c_{3+2i} B_1^1 - B_1^0 c_{3+2i} - F^i c_{2N+3+i}, \dots \right)^T.
\end{aligned}$$

Notice that the maps d_0 and d_1 are the linearisation of the action of $\mathcal{G}$ on $\mathbb{X}$ and of the nested instantons quiver relations (neglecting G^i, since $G^i = 0, \forall i$), respectively. The morphism d_2 is instead signalling the fact that the quiver relations are not all independent.

Our claim is then that $H^0(\mathcal{C}(X)) = H^3(\mathcal{C}(X)) = 0$, and that $\mathcal{C}(X)$ is an explicit representation of the perfect obstruction theory complex, so $H^1(\mathcal{C}(X))$ will be identified with the Zariski tangent to $\mathscr{N}(r, \mathbf{n})$, while $H^2(\mathcal{C}(X))$ will encode the obstructions to its smoothness. In fact elements of $H^1(\mathcal{C}(X))$ parametrize infinitesimal displacements at given points, up to the $\mathcal{G}$-action, so $H^1(\mathcal{C}(X))$ provides a local model for the Zariski tangent space to $\mathscr{N}(r, \mathbf{n})$. In the same way $H^2(\mathcal{C}(X))$ is interpreted to be the local model for the obstructions as its elements encode the linear dependence of the nested ADHM equations. Actually one might explicitly determine the truncated cotangent complex $\tau_{\geqslant -1}L^\bullet_{\mathcal{N}(r,\mathbf{n})}$ by a standard computation in deformation theory along the lines of [82] and compute extension and obstruction classes in terms of the cohomology of the complex $\mathcal{C}(X)$.

In order to see that indeed the 0–th and 3–rd cohomology of $\mathcal{C}(X)$ does indeed vanish, we construct three other complexes $\mathcal{C}(\mathcal{A})$, $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{A},\mathcal{B})$:

$$\mathcal{C}(\mathcal{A}):\ \mathcal{C}(\mathcal{A})^0 \xrightarrow{d_0^{\mathcal{A}}} \mathcal{C}(\mathcal{A})^1 \xrightarrow{d_1^{\mathcal{A}}} \mathcal{C}(\mathcal{A})^2,$$

$$\mathcal{C}(\mathcal{B}):\ \mathcal{C}(\mathcal{B})^0 \xrightarrow{d_0^{\mathcal{B}}} \mathcal{C}(\mathcal{B})^1 \xrightarrow{d_0^{\mathcal{B}}} \mathcal{C}(\mathcal{B})^2,$$

$$\mathcal{C}(\mathcal{A},\mathcal{B}):\ \mathcal{C}(\mathcal{A},\mathcal{B})^0 \xrightarrow{d_0^{\mathcal{A},\mathcal{B}}} \mathcal{C}(\mathcal{A},\mathcal{B})^1 \xrightarrow{d_1^{\mathcal{A},\mathcal{B}}} \mathcal{C}(\mathcal{A},\mathcal{B})^2,$$

with

$$\begin{aligned}
\mathcal{C}(\mathcal{A})^0 &= \mathcal{C}(\mathcal{A})^2 = \mathrm{End}(V_0),\\
\mathcal{C}(\mathcal{A})^1 &= \mathrm{End}(V_0)^{\oplus 2} \oplus \mathrm{Hom}(V_0, W) \oplus \mathrm{Hom}(W, V_0),\\
\mathcal{C}(\mathcal{B})^0 &= \mathcal{C}(\mathcal{B})^2 = \bigoplus_{i=1}^{N} \mathrm{End}(V_i),\\
\mathcal{C}(\mathcal{B})^1 &= \bigoplus_{i=1}^{N} \mathrm{End}(V_i)^{\oplus 2},\\
\mathcal{C}(\mathcal{A},\mathcal{B})^0 &= \mathcal{C}(\mathcal{A},\mathcal{B})^2 = \bigoplus_{i=1}^{N} \mathrm{Hom}(V_i, V_{i-1}),\\
\mathcal{C}(\mathcal{A},\mathcal{B})^1 &= \bigoplus_{i=1}^{N} \mathrm{Hom}(V_i, V_{i-1})^{\oplus 2} \oplus \mathrm{Hom}(V_1, W)
\end{aligned}$$

and

$$\begin{aligned}
&d_0^{\mathcal{A}}(h_0) = \left([h_0, B_1^0], [h_0, B_2^0], h_0 I, -J h_0\right)^T,\\
&d_1^{\mathcal{A}}\left(b_1^0, b_2^0, i, j\right)^T = [b_1^0, B_2^0] + [B_1^0, b_2^0] + Ij + iJ,\\
&d_0^{\mathcal{B}}\left(h_1, \dots, h_N\right)^T = \left([h_1, B_1^1], [h_1, B_2^1], \dots, [h_N, B_1^N], [h_N, B_2^N]\right)^T\\
&d_1^{\mathcal{B}}\left(b_1^1, b_2^1, \dots, b_1^N, b_2^N\right)^T = \left([b_1^1, B_2^1] + [B_1^1, b_2^1], \dots, [b_1^N, B_2^N] + [B_1^N, b_2^N]\right)^T,\\
&d_0^{\mathcal{A},\mathcal{B}}\left(f^1, \dots, f^N\right)^T = \left(-B_1^0 f^1 + f^1 B_1^1, -B_2^0 f^1 + f^1 B_2^1, \dots\right.\\
&\qquad\qquad \left.\dots, -B_1^{N-1} f^N + f^N B_1^N, -B_2^{N-1} f^N + f^N B_2^N, -J f^1\right)^T,\\
&d_1^{\mathcal{A},\mathcal{B}}\left(c_3, \dots, c_{2N+2}, c_2\right)^T = \left(-B_2^0 c_3 + c_3 B_2^1 - c_4 B_1^1 + B_1^0 c_4 + I c_2, \dots,\right.\\
&\qquad\qquad \left.\dots, B_2^{N-1} c_{2N+2} - c_{2N+2} B_2^1 + c_{2N+3} B_1^1 - B_1^0 c_{2N+3}\right)^T.
\end{aligned}$$

Then one can prove that there exists a distinguished triangle

$$\mathcal{C}(X) \longrightarrow \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \xrightarrow{\rho} \mathcal{C}(\mathcal{A},\mathcal{B})\ , \tag{4.1.4}$$

as $\mathcal{C}(X)[1]$ is a cone for $\rho = (\rho_0, \rho_1, \rho_2)$, where

$$\begin{aligned}
&\rho_0 (h_0, \dots, h_N)^T = \left(-h_0 F^1 + F^1 h_1, \dots, -h_{N-1} F^N + F^N h_N\right)^T, \\
&\rho_1 \left(b_1^0, b_2^0, i, j, b_1^1, b_2^1, \dots, \right)^T = \left(-b_1^0 F^1 + F^1 b_1^1, -b_2^0 F^1 + F^1 b_2^1, \dots \right. \\
&\qquad\qquad \left. \dots,, -b_1^{N-1} F^N + F^N b_1^N, -b_2^{N-1} F^N + F^N b_2^N, -jF^1\right)^T, \\
&\rho_2 (c_1, c_{2N+3}, \dots, c_{3N+2})^T = \left(-c_1 F^1 + F^1 c_{2N+3}, \dots, -c_{3N+1} F^N + F^N c_{3N+2}\right)^T.
\end{aligned}$$

By the triangle (4.1.4) one gets the long sequence in cohomology:

$$\begin{array}{l}
0 \longrightarrow H^0(\mathcal{C}(X)) \longrightarrow H^0(\mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B})) \xrightarrow{H^0(\rho)} H^0(\mathcal{C}(\mathcal{A}, \mathcal{B})) \\
\hookrightarrow H^1(\mathcal{C}(X)) \longrightarrow H^1(\mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B})) \xrightarrow{H^1(\rho)} H^1(\mathcal{C}(\mathcal{A}, \mathcal{B})) \\
\hookrightarrow H^2(\mathcal{C}(X)) \longrightarrow H^2(\mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B})) \longrightarrow H^2(\mathcal{C}(\mathcal{A}, \mathcal{B})) \\
\hookrightarrow H^3(\mathcal{C}(X)) \longrightarrow 0,
\end{array}$$

and, since $\mathcal{A}$ is a stable representation of the standard ADHM quiver, $H^0(\mathcal{C}(\mathcal{A})) = H^2(\mathcal{C}(\mathcal{A})) = 0$. Then $H^0(\mathcal{C}(X)) = 0$ by the injectivity of $H^0(\rho) : H^0(\mathcal{C}(\mathcal{B})) \to H^0(\mathcal{C}(\mathcal{A}, \mathcal{B}))$. In fact we have

$$H^0(\rho) \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} F^1 h_1 \\ \vdots \\ F^N h_N \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} = 0,$$

since F^i is injective. Moreover the stability of X implies that $d_1 : \mathcal{C}(\mathcal{A}, \mathcal{B})^1 \to \mathcal{C}(\mathcal{A}, \mathcal{B})^2$ is surjective: this in turn means that $H^2(\mathcal{C}(\mathcal{A}, \mathcal{B})) = 0$, which implies $H^3(\mathcal{C}(X)) = 0$. In fact let's take $d_1^\vee$:

$$d_1^\vee(\boldsymbol{\phi}) = d_1^\vee \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} = \begin{pmatrix} B_2^1 \phi_1 - \phi_1 B_2^0 \\ -B_1^1 \phi_1 + \phi_1 B_1^1 \\ \vdots \\ B_2^N \phi_N - \phi_N B_2^{N-1} \\ -B_1^N \phi_N + \phi_N B_1^{N-1} \\ \phi_1 I \end{pmatrix},$$

and if $\boldsymbol{\phi} \in \ker(d_1^\vee)$ then $\ker(\phi_1)$ would be a (B_1^0, B_2^0)−invariant subset of V_0 containing $\mathrm{Im}(I)$ which contradicts the stability of X, by which we conclude that $\ker(\phi_1) = V_0$. Similar statements hold also for each other component of $\boldsymbol{\phi}$, which we then conclude to be $\boldsymbol{\phi} = 0$.

The only thing left to prove is that the moduli space $\mathcal{N}(r, \mathbf{n})$ of stable representations of the nested instantons quiver is embedded in a smooth variety which is obtained as an hyperkähler quotient. We will leave this for §4.1.3. ■

For future reference we want now to exhibit some morphisms between different nested instantons moduli spaces and between them and usual moduli spaces of instantons, which are moduli spaces of framed torsion-free sheaves on $\mathbb{P}^2$. We obviously have iterative forgetting projections

$$\eta_i : \mathscr{N}(r, n_0, \ldots, n_i) \to \mathscr{N}(r, n_0, \ldots, n_{i-1}).$$

Moreover we also have other morphisms to underlying Hilbert schemes of points on $\mathbb{C}^2$, which are summarized by the commutative diagram in Fig. 4.1.1. In order to see that these maps do indeed exist, take a stable

$$\begin{tikzcd}
\mathscr{N}(r,n_0,\ldots,n_N) \arrow[r, "\mathfrak{p}_N^{(N)}"] \arrow[d, "\eta_N"] \arrow[dr, "\mathfrak{p}_{N-1}^{(N)}"] \arrow[dddr, "\mathfrak{p}_i^{(N)}"] \arrow[ddd, bend right, "\eta_{N,i}"'] & \mathscr{M}(r,n_0-n_N) \arrow[d, "\tilde{\eta}_{N-1}"] \arrow[ddd, bend left] \\
\mathscr{N}(r,n_0,\ldots,n_{N-1}) \arrow[r] \arrow[d] \arrow[dd, bend right] \arrow[ddr] & \mathscr{M}(r,n_0-n_{N-1}) \arrow[d] \arrow[dd, bend left] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathscr{N}(r,n_0,\ldots,n_i) \arrow[r] \arrow[d] & \mathscr{M}(r,n_0-n_i) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Figure 4.1.1: *Morphisms between moduli spaces of sheaves.*

representation $[X]$ of the nested instantons quiver. The fact that $[X]$ is stable implies that the morphisms F^i are injective, so that we can construct the stable ADHM datum $(W, \tilde{V}_i, \tilde{B}_1^i, \tilde{B}_2^i, \tilde{I}^i, \tilde{J}^i)$ as follows. Let $\tilde{V}_i$ be $V_0/\operatorname{Im}(F^1 \cdots F^i)$ and choose a basis of V_i in such a way that

$$F^1 \cdots F^i = \begin{pmatrix} \mathbb{1}_{V_i} \\ 0 \end{pmatrix}, \qquad F^1 \circ F^2 \circ \cdots \circ F^i : V_i \to V_0,$$

whence $V_0 = V_i \oplus \tilde{V}_i$. Then define the projections $\pi_i = V_0 \to V_i$ and $\tilde{\pi}_i : V_0 \to \tilde{V}_i$ as $\pi_i(v, \tilde{v}) = v$ and $\tilde{\pi}_i(v, \tilde{v}) = \tilde{v}$, with $v \in V_i$, $\tilde{v} \in \tilde{V}_i$. We can then show how $\tilde{V}_i$ inherits an ADHM structure by its embedding in V_0. Indeed if we define $\tilde{B}_1^i = B_1^0|_{\tilde{V}_i}$, $\tilde{B}_2^i = B_2^0|_{\tilde{V}_i}$, $\tilde{I}^i = \tilde{\pi}_i \circ I$ and $\tilde{J}^i = J|_{\tilde{V}_i}$ the datum $(W, \tilde{V}_i, \tilde{B}_1^i, \tilde{B}_2^i, \tilde{I}^i, \tilde{J}^i)$ satisfies the ADHM Eq. (4.1.5).

$$[\tilde{B}_1^i, \tilde{B}_2^i] + \tilde{I}^i \tilde{J}^i = [B_1^0|_{\tilde{V}_i}, B_2^0|_{\tilde{V}_i}] + \tilde{\pi}_i \circ I \circ J|_{\tilde{V}_i} = \left([B_1^0, B_2^0] + IJ \right)\Big|_{\tilde{V}_i} = 0. \quad (4.1.5)$$

This new ADHM datum is moreover stable, as if it would exist $0 \subset \tilde{S}_i \subset \tilde{V}_i$ such that $\tilde{B}_{1,2}^i(\tilde{S}_i), \tilde{I}^i(W) \subset \tilde{S}_i$ it would imply that also the ADHM datum $(W, V_0, B_1^0, B_2^0, I, J)$ wouldn't be stable. In fact in that case we could take $0 \subset V_i \oplus \tilde{S}_i \subset V_0$ and it would be such that $B_1^0(V_i \oplus \tilde{S}_i), B_2^0(V_i \oplus \tilde{S}_i), I(W) \subset$

$V_i \oplus \tilde{S}_i$. In fact if we take any $(v, \tilde{s}) \in V_i \oplus \tilde{S}_i$ it happens that $B_1^0(v, s) = (B_1^0|_{V_i}(v), B_1^0|_{\tilde{V}_i}\tilde{s}) = (B_1^0|_{V_i}, \tilde{B}_1^i(\tilde{s})) \in V_i \oplus \tilde{S}_i$, $B_2^0(v, s) = (B_2^0|_{V_i}(v), B_2^0|_{\tilde{V}_i}\tilde{s}) = (B_2^0|_{V_i}, \tilde{B}_2^i(\tilde{s})) \in V_i \oplus \tilde{S}_i$ and $I(W) = I(W) \cap V_i \oplus I(W) \cap \tilde{V}_i = (\pi_i \circ I)(W) \oplus (\tilde{\pi}_i \circ I)(W) \subset V_i \oplus \tilde{S}_i$. Thus we constructed a map $\mathfrak{p}_i^{(N)} : \mathscr{N}(r, n_0, \dots, n_N) \to \mathscr{M}(r, n_0 - n_i)$.

4.1.3 Hyperkähler embedding

In this section we exhibit an embedding of the moduli space of nested instantons into a smooth projective variety, which is moreover hyperkähler. In the following vector space

$$\begin{aligned}\mathbb{X} = \operatorname{End}(V_0)^{\oplus 2} \oplus \operatorname{Hom}(W, V_0) \oplus \operatorname{Hom}(V_0, W) \bigoplus_{k=1}^{N} \Bigg[& \operatorname{End}(V_k)^{\oplus 2} \\ & \oplus \operatorname{Hom}(V_{k-1}, V_k) \oplus \operatorname{Hom}(V_k, V_{k-1}) \Bigg]\end{aligned} \tag{4.1.6}$$

we will introduce a family of relations:

$$[B_1^0, B_2^0] + IJ + F^1 G^1 = 0, \tag{4.1.7}$$

$$[B_1^i, B_2^i] - G^i F^i + F^{i+1} G^{i+1} = 0, \quad i = 1, \dots, N. \tag{4.1.8}$$

Then an element $(B_1^0, B_2^0, I, J, \{B_1^i, B_2^i, F^i, G^i\}) = X \in \mathbb{X}$ is called stable if it satisfies conditions **S1** and **S2** in Prop. 4.2. With these conventions we will define $\mathbb{M}(r, \mathbf{n})$ to be the space of stable elements of $\mathbb{X}$ satisfying the relations (4.1.7)-(4.1.8):

$$\mathbb{M}(r, \mathbf{n}) = \{X \in \mathbb{X} : X \text{ is stable and satisfies } (4.1.7), (4.1.8)\}.$$

Exactly in the same way as we did before we can easily see that there is a natural action of $\mathcal{G} = GL(V_0) \times \cdots \times GL(V_N)$ which is free on $\mathbb{M}(r, \mathbf{n})$ and preserves the equations (4.1.7)-(4.1.8): the same is then true for the natural $\mathcal{U}$–action on $\mathbb{M}(r, \mathbf{n})$, with $\mathcal{U} = U(V_0) \times \cdots \times U(V_N)$. Thus a moduli space $\mathscr{M}(r, \mathbf{n})$ of stable $\mathcal{U}$–orbits in $\mathbb{M}(r, \mathbf{n})$ can be defined by means of GIT theory, as it was the case for $\mathscr{N}(r, \mathbf{n})$ in the previous sections. Moreover any stable point of $\mathbb{X}$ satisfying the nested ADHM equations automatically satisfies (4.1.7) and (4.1.8). Indeed, a stable representation of quiver (4.1.1) satisfies, among other relations, the following equations

$$[B_1^0, B_2^0] + IJ = 0, \qquad [B_1^{i>0}, B_2^{i>0}] = 0,$$

while $G^i = 0$, for $i = 1, \dots, N$, by Corollary 4.3. Thus, any stable representation of quiver (4.1.1) with relations (4.1.2) also satisfies the relations (4.1.7)-(4.1.8), so that $\mathscr{N}(r, \mathbf{n}) \hookrightarrow \mathscr{M}(r, \mathbf{n})$ via the natural inclusion.

Next let us point out that on each $T\operatorname{Hom}(V_i, V_k)$ we can introduce an hermitean metric by defining

$$\langle X, Y \rangle = \frac{1}{2} \operatorname{tr}\left(X \cdot Y^\dagger + X^\dagger \cdot Y \right), \qquad \forall X, Y \in \operatorname{Hom}(V_i, V_k),$$

which in turn can be linearly extended to a hermitean metric $\langle -,- \rangle : T\mathbb{M}(r,\mathbf{n}) \times T\mathbb{M}(r,\mathbf{n}) \to \mathbb{C}$. Finally we can introduce some complex structures on $T\mathbb{M}(r,\mathbf{n})$: given $X \in T\mathbb{M}(r,\mathbf{n})$ these are defined as the following $I, J, K \in \mathrm{End}(T\mathbb{M}(r,\mathbf{n}))$

$$\begin{aligned}
&I(X) = \sqrt{-1}X,\\
&J(X) = (-b_2^{0\dagger}, b_1^{0\dagger}, -j^\dagger, i^\dagger, \{-b_2^{i\dagger}, b_1^{i\dagger}, -g^{i\dagger}, f^{i\dagger}\}),\\
&K(X) = I \circ J(X),
\end{aligned}$$

with $X = (b_1^0, b_2^0, i, j, \{b_1^i, b_2^i, f^i, g^i\})$. These three complex structures make the datum of

$$(\mathbb{M}(r,\mathbf{n}), \langle -,- \rangle, I, J, K)$$

a hyperkähler manifold, as one can readily verify. It is a standard fact that once we fix a particular complex structure, say I, and its respective Kähler form, ω_I, the linear combination $\omega_{\mathbb{C}} = \omega_J + \sqrt{-1}\omega_K$ is a holomorphic symplectic form for $\mathbb{M}(r,\mathbf{n})$. The thing we finally want to prove is that the hyperkähler structure on $\mathbb{M}(r,\mathbf{n})$ induce a hyperkähler structure on the GIT quotient $\mathscr{M}(r,\mathbf{n})$, which will be moreover proven to be smooth. This is made possible by the fact that the natural $\mathcal{U}-$action on $\mathbb{M}(r,\mathbf{n})$ preserves the hermitean metric and the complex structures we introduced. Then, letting $\mathfrak{u}$ be the Lie algebra of the group $\mathcal{U}$, we need to construct a moment map

$$\mu : \mathbb{M}(r,\mathbf{n}) \to \mathfrak{u}^* \otimes \mathbb{R}^3,$$

satisfying

1. $\mathcal{G}-$equivariance: $\mu(g \cdot X) = \mathrm{Ad}^*_{g^{-1}}\, \mu(X)$;
2. $\langle d\mu_i(X), \xi \rangle = \omega_i(\xi^*, X)$, for any $X \in T\mathbb{M}(r,\mathbf{n})$ and $\xi \in \mathfrak{u}$ generating the vector field $\xi^* \in T\mathbb{M}(r,\mathbf{n})$.

If then $\zeta \in \mathfrak{u}^* \otimes \mathbb{R}^3$ is such that $\mathrm{Ad}^*_g(\zeta_i) = \zeta_i$ for any $g \in \mathcal{U}$, $\mu^{-1}(\zeta)$ is $\mathcal{U}-$invariant and it makes sense to consider the quotient space $\mu^{-1}(\zeta)/\mathcal{U}$. It is known, [123], that if $\mathcal{U}$ acts freely on $\mu^{-1}(\zeta)/\mathcal{U}$, the latter is a smooth hyperkähler manifold, with complex structures and metric induced by those of $\mathbb{M}(r,\mathbf{n})$.

Our task of finding a moment map $\mu : \mathbb{M}(r,\mathbf{n}) \to \mathfrak{u}^* \otimes \mathbb{R}^3$ then translates into the following. Define $(\mu_1^0, \dots, \mu_1^N) = \mu_1 : \mathbb{M}(r,\mathbf{n}) \to \mathfrak{u}$

$$\begin{cases}
\mu_1^0(X) = \dfrac{\sqrt{-1}}{2}\left([B_1^0, B_1^{0\dagger}] + [B_2^0, B_2^{0\dagger}] + II^\dagger - J^\dagger J + F^1F^{1\dagger} - G^{1\dagger}G^1\right)\\
\mu_1^1(X) = \dfrac{\sqrt{-1}}{2}\left([B_1^1, B_1^{1\dagger}] + [B_2^1, B_2^{1\dagger}] - F^{1\dagger}F^1 + G^1G^{1\dagger} + F^2F^{2\dagger} - G^{2\dagger}G^2\right)\\
\qquad\vdots\\
\mu_1^N(X) = \dfrac{\sqrt{-1}}{2}\left([B_1^N, B_1^{N\dagger}] + [B_2^N, B_2^{N\dagger}] - F^{N\dagger}F^N + G^NG^{N\dagger}\right),
\end{cases} \tag{4.1.9}$$

with $X = (B_1^0, B_2^0, I, J, \{B_1^i, B_2^i, F^i, G^i\}) \in \mathbb{M}(r, \mathbf{n})$. In addition to μ_1 we also define a map $\mu_{\mathbb{C}} : \mathbb{M}(r, \mathbf{n}) \to \mathfrak{g}$, with $\mathfrak{g} = \mathfrak{gl}(V_0) \times \cdots \times \mathfrak{gl}(V_N)$:

$$\begin{cases} \mu_{\mathbb{C}}^0(X) = [B_1^0, B_2^0] + IJ + F^1 G^1 \\ \mu_{\mathbb{C}}^1(X) = [B_1^1, B_2^1] - G^1 F^1 + F^2 G^2 \\ \qquad \vdots \\ \mu_{\mathbb{C}}^N(X) = [B_1^N, B_2^N] - G^N F^N, \end{cases} \tag{4.1.10}$$

by means of which we define $\mu_{2,3} : \mathbb{M}(r, \mathbf{n}) \to \mathfrak{u}$ as $\mu_{\mathbb{C}}(X) = (\mu_2 + \sqrt{-1}\mu_3)(X)$. Notice that in absence of B_i^j and I, J the complex moment map we defined would reduce to the Crawley-Boevey moment map in [76]. We then claim that $\mu = (\mu_1, \mu_2, \mu_3)$ is a moment map for the $\mathcal{U}-$action on $\mathbb{M}(r, \mathbf{n})$. If this is true and χ is the algebraic character we introduced in §4.1.2, the space

$$\begin{aligned} \widetilde{\mathscr{M}}(r, \mathbf{n}) &= \frac{\mu_1^{-1}(\sqrt{-1}d\chi) \cap \mu_{\mathbb{C}}^{-1}(0) \cap \mathbb{M}(r, \mathbf{n})}{\mathcal{U}} \\ &= \frac{\mu^{-1}(\sqrt{-1}d\chi, 0, 0) \cap \mathbb{M}(r, \mathbf{n})}{\mathcal{U}} \end{aligned}$$

is a smooth hyperkähler manifold which, by an analogue of Kempf-Ness theorem is also isomorphic to $\mathscr{M}(r, \mathbf{n})$. In fact it is known, due to a result of [143; 165] and the characterization of $\chi-$(semi)stable points we gave in the previous sections, that there exists a bijection between $\mu_1^{-1}(\sqrt{-1}d\chi)$ and the set of $\chi-$(semi)stable points in $\mu_{\mathbb{C}}^{-1}(0)$. Then, in order to prove that μ is actually a moment map, we will first compute the vector field ξ^* generated by a generic $\xi \in \mathfrak{u}$. Let then $X = (b_1^0, b_2^0, i, j, \{b_1^i, b_2^i, f^i, g^i\})$ be a vector in $T\mathbb{M}(r, \mathbf{n})$ and $\Psi_X : \mathcal{U} \to \mathbb{M}(r, \mathbf{n})$ the action of $\mathcal{U}$ onto $X \in \mathbb{M}(r, \mathbf{n})$: the fundamental vector field generated by $\xi \in \mathfrak{u}$ is

$$\xi^*|_X = d\Psi_X(\mathbb{1}_{\mathcal{U}})(\xi) = \frac{d}{dt}\left(\Psi_X \circ \gamma\right)|_{t=0},$$

where γ is a smooth curve $\gamma : (-\epsilon, \epsilon) \to \mathcal{U}$ such that $\gamma(0) = \mathbb{1}_{\mathcal{U}}$ and $\dot{\gamma}(0) = \xi$. Thus we can compute

$$\begin{aligned} \xi^*|_X = \big([\xi_0, b_1^0], [\xi_0, b_2^0], \xi_0 i, -j\xi_0, [\xi_1, b_1^1], [\xi_1, b_2^1], \\ \xi_0 f^1 - f^1 \xi_1, \xi_1 g^1 - g^1 \xi_0, \dots \\ \dots, [\xi_N, b_1^N], [\xi_N, b_2^N], \\ \xi_{N-1} f^N - f^N \xi_N, \xi_N g^N - g^N \xi_{N-1}\big). \end{aligned}$$

Then if $\pi_i : \mathbb{M}(r, \mathbf{n}) \to \mathbb{M}(r, \mathbf{n})$ denotes the projection on the $i-$th component of the direct sum decomposition induced by (4.1.6) so that i runs over the index set $\mathcal{I}$, by inspection one can see that ω_1 is exact, and in particular $\omega_1 = d\lambda_1$, with

$$\lambda_1 = \frac{\sqrt{-1}}{2} \operatorname{tr}\left(\sum_{i \in \mathcal{I}} \pi_i \wedge \pi_{i*}^{\dagger}\right).$$

This implies that

$$\langle \mu_1(x), \xi \rangle = \imath_{\xi^*} \lambda_1,$$

and it is easy to verify that $\mu_1 : \mathbb{M}(r, \mathbf{n}) \to \mathfrak{u}^*$ thus defined indeed matches with the definition (4.1.9). Similarly one can realize that

$$\lambda_2 = \Re \left[\operatorname{tr} \left(\sum_{i \in 2\mathbb{Z} \cap \mathcal{I}} \pi_i \wedge \pi_{1+1*} \right) \right], \tag{4.1.11}$$

$$\lambda_3 = -\sqrt{-1}\Im \left[\operatorname{tr} \left(\sum_{i \in 2\mathbb{Z} \cap \mathcal{I}} \pi_i \wedge \pi_{1+1*} \right) \right]. \tag{4.1.12}$$

and the moment map components satisfying $\langle \mu_i(x), \xi \rangle = \iota_{\xi^*} \lambda_i$ agree with the combination $\mu_2 + \sqrt{-1}\mu_3 = \mu_{\mathbb{C}}$ we gave previously in Eq. (4.1.10).

4.2 FLAGS OF FRAMED TORSION-FREE SHEAVES ON $\mathbb{P}^2$

We give in this paragraph the construction of the moduli space of flags of framed torsion-free sheaves of rank r on the complex projective plane. We also show that there exists a natural isomorphism between the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$ and the stable representations of the nested instantons quiver. In the rank $r = 1$ case our definition reduces to the nested Hilbert scheme of points on $\mathbb{C}^2$, as it is to be expected. By this reason we first want to carry out the analysis of the simpler $r = 1$ case, which also has the advantage of providing us with a new characterization of nested Hilbert schemes of points on $\mathbb{C}^2$, analogous to that of [56].

4.2.1 $\mathrm{Hilb}^{\hat{n}}(\mathbb{C}^2)$ and $\mathcal{N}(1, \mathbf{n})$

Before delving into the analysis of the relation between nested instantons moduli spaces and flags of framed torsion-free sheaves on $\mathbb{P}^2$, we want to show a special simpler case. In particular we will prove the existence of an isomorphism between the nested Hilbert scheme of points in $\mathbb{C}^2$ and the nested instantons moduli space $\mathcal{N}(1, n_0, \ldots, n_N)$. This effectively gives us the ADHM construction of a general nested Hilbert scheme of points on $\mathbb{C}^2$, which will serve as a local model for more general nested Hilbert schemes of points on, say, toric surfaces S. In order to see this, we first recall the definition of a nested Hilbert scheme of points.

Definition 4.9. *Let S be a complex (projective) surface and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ a sequence of integers. The nested Hilbert scheme of points on S is defined as*

$$\mathrm{Hilb}^{(n_1, \ldots, n_k)}(S) = \{ I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \mathcal{O}_S : \mathrm{length}\,(\mathcal{O}_S / I_i) = n_i \}.$$

Alternatively, if X is a quasi-projective scheme over the complex numbers, we can equivalently define the nested Hilbert scheme $\mathrm{Hilb}^{(n_1, \ldots, n_k)}(X)$ as

$$\mathrm{Hilb}^{(n_1, \ldots, n_k)}(X) = \left\{ (Z_1, \ldots, Z_k) \,\middle|\, \begin{array}{l} Z_i \in \mathrm{Hilb}^{n_i}(X), \\ Z_i \text{ is a subscheme of } Z_j \text{ if } i < j \end{array} \right\}.$$

Before actually exhibiting the isomorphism we are interested in, we want to prove an auxiliary result, which gives an alternative definition for the nested Hilbert schemes over the affine plane, analogously to the case of Hilbert schemes studied in [165]. For convenience, in the following we will sometimes use the shorthand notation $(\mathbf{b}, \mathbf{f}) = (b_1^1, b_2^1, f_1, \ldots, b_1^k, b_2^k, f_k)$.

Proposition 4.10. *Let $\mathbb{k}$ be an algebraically closed field, and $\mathbf{n}$ a sequence of integers $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_k$. Define $\hat{\mathbf{n}}$ to be the sequence of integers $\hat{n}_0 = n_0 \geqslant \hat{n}_1 = n_0 - n_k \geqslant \cdots \geqslant \hat{n}_k = n_0 - n_1$, then there exists an isomorphism*

$$\mathrm{Hilb}^{\hat{\mathbf{n}}}(\mathbb{A}^2) \simeq \left\{ (b_1^0, b_2^0, i, \mathbf{b}, \mathbf{f}) \;\middle|\; \begin{array}{l} \text{(i) } [b_1^i, b_2^i] = 0 \\ \text{(ii) } b_{1,2}^{i-1} f_i - f_i b_{1,2}^i = 0 \\ \text{(iii) } \nexists S \subset \mathbb{k}^{n_0} : b_{1,2}^0(S) \subset S \text{ and} \\ \qquad\qquad \mathrm{Im}(i) \subset S \\ \text{(iv) } f_i : \mathbb{k}^{n_i} \to \mathbb{k}^{n_{i-1}} \text{ is injective} \end{array} \right\} \Big/ \mathcal{G}_{\mathbf{n}},$$

where $\mathcal{G}_{\mathbf{n}} = GL_{n_0}(\mathbb{k}) \times \cdots \times GL_{n_k}(\mathbb{k})$, $b_{1,2}^i \in \mathrm{End}(\mathbb{k}^{n_i})$, $i \in \mathrm{Hom}(\mathbb{k}, \mathbb{k}^{n_0})$ and $f_i \in \mathrm{Hom}(\mathbb{k}^{n_i}, \mathbb{k}^{n_{i-1}})$. The action of $\mathcal{G}_{\mathbf{n}}$ is given by

$$\begin{aligned} \mathbf{g} \cdot (b_1^0, b_2^0, i, \ldots, b_1^k, b_2^k, f_k) = (&g_0 b_1^0 g_0^{-1}, g_0 b_2^0 g_0^{-1}, g_0 i, \ldots \\ &\ldots, g_k b_1^k g_k^{-1}, g_k b_2^k g_k^{-1}, g_{k-1} f_k g_k^{-1}). \end{aligned}$$

Proof. Suppose we have a sequence of ideals $I_0 \subseteq I_1 \subseteq \cdots I_k \in \mathrm{Hilb}^{\hat{\mathbf{n}}}(\mathbb{A}^2)$. Let's first define $V_0 = \mathbb{k}[z_1, z_2]/I_0$, $b_{1,2}^0 \in \mathrm{End}(V_0)$ to be the multiplication by $z_{1,2} \mod I_0$, and $i \in \mathrm{Hom}(\mathbb{k}, V_0)$ by $i(1) = 1 \mod I_0$. Then obviously $[b_1^0, b_2^0] = 0$ and condition (iii) holds since 1 multiplied by products of z_1 and z_2 spans the whole $\mathbb{k}[z_1, z_2]$. Then define $\tilde{V}_i = \mathbb{k}[z_1, z_2]/I_i$ and, since $I_0 \subseteq I_i$ for any $i > 0$, complete $\tilde{V}_i$ to V_0 as $V_0 = \tilde{V}_i \oplus V_i$, so that $V_i \simeq \mathbb{k}^{n_i}$. The restrictions of $b_{1,2}^0$ to V_i then yield homomorphisms $b_{1,2}^i \in \mathrm{End}(V_i)$ naturally satisfying $[b_1^i, b_2^i] = 0$, while the inclusion of the ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k$ implies the existence of an embedding $f_i : V_i \hookrightarrow V_{i-1}$ such that condition (ii) holds by construction.

Conversely, let $(b_1^0, b_2^0, i, \ldots, b_1^k, b_2^k, f_k)$ be given as in the proposition. In the first place one can define a map $\phi_0 : \mathbb{k}[z_1, z_2] \to \mathbb{k}^{n_0}$ to be $\phi_0(f) = f(b_1^0, b_2^0)i(1)$. This map is surjective, so that $I_0 = \ker \phi$ is an ideal for $\mathbb{k}[z_1, z_2]$ of length n_0. Then, since $f_i \in \mathrm{Hom}(\mathbb{k}^{n_i}, \mathbb{k}^{n_{i-1}})$ is injective we can embed $\mathbb{k}^{n_i}$ into $\mathbb{k}^{n_0}$ though $F_i = f_1 \circ \cdots \circ f_{i-1} \circ f_i$ in such a way that $b_{1,2}^i = b_{1,2}^0|_{\mathbb{k}^{n_i} \hookrightarrow \mathbb{k}^{n_0}}$, which is a simple consequence of condition (ii). Then we have the direct sum decomposition $\mathbb{k}^{n_0} = \mathbb{k}^{n_0 - n_i} \oplus \mathbb{k}^{n_i}$,the restrictions $\tilde{b}_{1,2}^i = b_{1,2}^0|_{\mathbb{k}^{n_0 - n_i}}$ and the projection $\tilde{\imath}_i = \pi_i \circ i$, with $\pi_i = \mathbb{k}^{n_0} \to \mathbb{k}^{n_0 - n_i}$, satisfying $[\tilde{b}_1^i, \tilde{b}_2^i] = 0$ and a stability condition analogous to (iii). Thus we define $\phi_i : \mathbb{k}[z_1, z_2] \to \mathbb{k}^{n_0 - n_i}$ by $\phi_i(f) = f(\tilde{b}_1^i, \tilde{b}_2^i)\tilde{\imath}(1)$. This map is surjective, just like ϕ_0, so that $I_j = \ker(\phi_j)$ is an ideal for $\mathbb{k}(z_1, z_2)$ of length $n_0 - n_i$. Finally, due to the successive embeddings $\mathbb{k}^{n_k} \hookrightarrow \mathbb{k}^{n_{k-1}} \hookrightarrow \cdots \hookrightarrow \mathbb{k}^{n_0}$ we have the inclusion of the ideals $I_j \subset I_{j-1}$. ∎

One can readily notice that the description given by the previous proposition of the nested Hilbert scheme of points doesn't really coincide with the quiver we were studying throughout this section. However we can very easily overcome this problem by using the fact that if (b_1^0, b_2^0, i, j) is a stable ADHM datum with $r = 1$, then $j = 0$, [165]. This proves the following proposition.

Proposition 4.11. *With the same notations of proposition 4.10, we have that*

$$\mathrm{Hilb}^{\hat{n}}(\mathbb{A}^2) \simeq \left\{ (b_1^0, b_2^0, i, \mathbf{b}, \mathbf{f}) \;\middle|\; \begin{array}{l} \text{(a) } [b_1^0, b_2^0] + ij = 0 \\ \text{(a}') \ [b_1^i, b_2^i] = 0,\ i > 0 \\ \text{(b) } b_{1,2}^{i-1} f_i - f_i b_{1,2}^i = 0 \\ \text{(c) } jf_1 = 0 \\ \text{(d) } \nexists S \subset \mathbb{k}^{n_0} : b_{1,2}^0(S) \subset S \text{ and} \\ \qquad\qquad \mathrm{Im}(i) \subset S \\ \text{(e) } f_i : \mathbb{k}^{n_i} \to \mathbb{k}^{n_{i-1}} \text{ is injective} \end{array} \right\} \Big/ \mathcal{G}_{\mathbf{n}}.$$

All the previous observations, together with corollary 4.7, immediately prove the following theorem.

Theorem 4.12. *The moduli space of nested instantons $\mathscr{N}_{r,\lambda,n,\mu}$ is isomorphic to the nested Hilbert scheme of points on $\mathbb{C}^2$ when $r = 1$ and $\lambda = [1^1]$.*

$$\mathscr{N}(1, \mathbf{n}) = \mathbb{X}_0 /\!\!/_{\chi} \mathcal{G} \simeq \mathrm{Hilb}^{\hat{n}}(\mathbb{C}^2).$$

4.2.2 $\mathscr{F}(r, \boldsymbol{\gamma})$ and $\mathscr{N}(r, \mathbf{n})$

A more general result relates the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$ to the moduli space of nested instantons. In the case of the two-step quiver this result was proved in [213], here we give a generalization of their theorem in the case of the moduli space $\mathscr{N}_{r,[r^1],n,\mu}$ represented by a quiver with an arbitrary number of nodes.

Definition 4.13. *Let $\ell_\infty \subset \mathbb{P}^2$ be a line and F a coherent sheaf on $\mathbb{P}^2$. A framing ϕ for F is then a choice of an isomorphism $\phi : F|_{\ell_\infty} \xrightarrow{\simeq} \mathscr{O}_{\ell_\infty}^{\oplus r}$, with $r = \mathrm{rk}\, F$. An $(N+2)$–tuple $(E_0, E_1, \ldots, E_N, \phi)$ is a framed flag of sheaves on $\mathbb{P}^2$ if E_0 is a torsion-free (coherent) sheaf on $\mathbb{P}^2$ framed at ℓ_∞ by ϕ, and $E_{j>0}$ form a flag of subsheaves $E_N \subseteq \cdots \subseteq E_0$ of E_0 s.t. the quotients E_i/E_j, $i < j$, are supported away from ℓ_∞.*

By the framing condition we get that $c_1(E_0) = 0$, while the quotient condition on the subsheaves of E_0 naturally implies that the quotients E_j/E_N are 0–dimensional subsheaves and $c_1(E_{j>0}) = 0$. Then a framed flag of sheaves on $\mathbb{P}^2$ is characterized by the set of integers $(r, \boldsymbol{\gamma})$, where $r = \mathrm{rk}\, E_0 = \cdots = \mathrm{rk}\, E_N$, $c_2(E_0) = \gamma_0$, $h^0(E_0/E_j) = \gamma_1 + \cdots + \gamma_j$ so that $c_2(E_{j>0}) = \gamma_0 + \cdots + \gamma_j$.

We now define the moduli functor

$$\mathsf{F}_{(r,\boldsymbol{\gamma})} : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Sets},$$

by assigning to a $\mathbb{C}$–scheme S the set of isomorphism classes of $(2N+2)$-tuples $(F_S, \varphi_S, Q_S^1, g_S^1, \ldots, Q_S^N, g_S^N\}$, with

- F_S a coherent sheaf over $\mathbb{P}^2 \times S$ flat over S and such that $F_S|_{\mathbb{P}^2\times\{s\}}$ is a torsion-free sheaf for any closed point $s \in S$, $\operatorname{rk} F_S = r$, $c_1(F_S) = 0$ and $c_2(F_S) = \gamma_0$;
- $\varphi_S : F_S|_{\ell_\infty \times S} \to \mathscr{O}_{\ell_\infty\times S}^{\oplus r}$ is an isomorphism of $\mathscr{O}_{\ell_\infty\times S}$–modules;
- Q_S^i is a coherent sheaf on $\mathbb{P}^2 \times S$, flat over S and supported away from $\ell_\infty \times S$, such that $h^0(Q_S^i|_{\mathbb{P}^2\times\{s\}}) = \gamma_1 + \cdots + \gamma_i$, for any closed point $s \in S$;
- $g_S^i : F_S \to Q_S^i$ is a surjective morphism of $\mathscr{O}_{\mathbb{P}^2\times S}$–modules.

Two tuples $(F_S, \varphi_S, Q_S^1, g_S^1, \dots, Q_S^N, g_S^N)$ and $(F_S', \varphi_S', Q_S^{1\prime}, g_S^{1\prime}, \dots, Q_S^{N\prime}, g_S^{N\prime})$ are said to be isomorphic if there exist isomorphisms of $\mathscr{O}_{\mathbb{P}^2\times S}$–modules $\Theta_S : F_S \to F_S'$ and $\Gamma_S^i : Q_S^i \to Q_S^{i\prime}$ such that the following diagrams commute

$$
\begin{tikzcd}
F_S|_{\ell_\infty\times S} \arrow[r, "\varphi_S"] \arrow[d, "\Theta_S|_{\ell_\infty\times S}"'] & \mathscr{O}_{\ell_\infty\times S}^{\oplus r} \\
F_S'|_{\ell_\infty\times S} \arrow[ur, "\varphi_S'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F_S \arrow[r, "g_S^i"] \arrow[d, "\Theta_S"] & Q_S^i \arrow[d, "\Gamma_S^i"] \\
F_S' \arrow[r, "g_S^{i\prime}"] & Q_S^{i\prime}
\end{tikzcd}
$$

If this functor is representable, the variety representing it will called the moduli space of flags of framed torsion-free sheaves on $\mathbb{P}^2$.

What we want to show next is that the moduli space of flags of torsion free sheaves on $\mathbb{P}^2$ is a fine moduli space, and that it is indeed isomorphic (as a scheme) to the moduli space of nested instantons we defined previously. First of all we will focus our attention on proving the following statement.

Proposition 4.14. *The moduli functor $F_{(r,\gamma)}$ is represented by a (quasi-projective) variety $\mathscr{F}(r,\gamma)$ isomorphic to a relative quot-scheme.*

Proof. We base our proof on the concept of Quot functor, so let us recall its constuction and basic properties. First of all let us take the universal framed sheaf $(U^{(0)}, \varphi_0)$ on $\mathbb{P}^2 \times \mathscr{M}(r,\gamma_0)$, with $\varphi_0 : U^{(0)}_{\ell_\infty\times\mathscr{M}(r,\gamma_0)} \xrightarrow{\simeq} \mathscr{O}^{\oplus r}_{\ell_\infty\times\mathscr{M}(r,\gamma_0)}$ an isomorphism of $\mathscr{O}_{\ell_\infty\times\mathscr{M}(r,\gamma_0)}$–modules. We then define

$$\mathrm{Quot}_{(U^{(0)},\gamma_1)} : \mathrm{Sch}^{\mathrm{op}}_{\mathscr{M}(r,\gamma_0)} \to \mathrm{Sets}$$

by

$$\mathrm{Quot}_{(U^{(0)},\gamma_1)}\left(S \xrightarrow{\pi} \mathscr{M}(r,\gamma_0)\right) = \left\{\text{isomorphism classes of } (Q_S, q_S)\right\}$$

where

- Q_S is a coherent sheaf on $\mathbb{P}^2 \times S$, flat over S, supported away from $\ell_\infty \times S$ and such that $h^0(Q_S|_{\ell_\infty\times\{s\}}) = \gamma_1$, for any $s \in S$ closed;
- $q_S : U_S^{(0)} \to Q_S$ is a surjective morphism of $\mathscr{O}_{\mathbb{P}^2\times S}$–modules, where $U_S^{(0)}$ is the pull-back of $U^{(0)}$ to $\mathbb{P}^2 \times S$ via

$$(\mathbb{1}_{\mathbb{P}^2} \times \pi) : \mathbb{P}^2 \times S \to \mathbb{P}^2 \times \mathscr{M}(r,\gamma_0).$$

By Grothendieck theory this is a representable functor and it was proved in [213] to be isomorphic to the moduli functor of flags of couples of framed torsion-free sheaves on $\mathbb{P}^2$. In fact there exist a natural forgetting map $F_{(r,\gamma_0,\gamma_1)} \to \mathrm{Quot}_{(\mathcal{U}^{(0)},\gamma_1)}$ which act as $(F_S, \varphi_S, Q^1_S, g^1_S) \mapsto (Q^1_S, g^1_S)$. This map also has an inverse given by setting $F_S = \ker(g^1|_S)$, which has a framing φ_S at $\ell_\infty \times S$ induced by the framing φ_0 of $\mathcal{U}^{(0)}$ at $\ell_\infty \times \mathscr{M}(r,\gamma_0)$. The variety representing $F_{(r,\gamma_0,\gamma_1)}$ is then the quot scheme $\mathrm{Quot}^{\gamma_1}(\mathcal{U}^{(0)})$ relative to $\mathscr{M}(r,\gamma_0)$. We can then construct a universal framed sheaf $(\mathcal{U}^{(1)},\varphi_1)$ on $\mathbb{P}^2 \times \mathscr{F}(r,\gamma_1,\gamma_2)$ with $\varphi_0 : \mathcal{U}^{(1)}_{\ell_\infty\times\mathscr{F}(r,\gamma_1,\gamma_2)} \xrightarrow{\simeq} \mathscr{O}^{\oplus r}_{\ell_\infty\times\mathscr{F}(r,\gamma_0,\gamma_1)}$ an isomorphism of $\mathscr{O}_{\ell_\infty\times\mathscr{F}(r,\gamma_0,\gamma_1)}$−modules. One can then use the quot functor

$$\mathrm{Quot}_{(\mathcal{U}^{(1)},\gamma_2)} : \mathrm{Sch}^{\mathrm{op}}_{\mathscr{F}(r,\gamma_1,\gamma_2)} \to \mathrm{Sets},$$

in order to show that $F_{(r,\gamma_0,\gamma_1,\gamma_2)}$ is isomorphic to $\mathrm{Quot}_{(\mathcal{U}^{(1)},\gamma_2)}$, exactly in the same way as before and since the latter is representable so is the former. By iterating this procedure we can finally show that our moduli functor $F_{(r,\boldsymbol{\gamma})}$ is indeed representable, being isomorphic to a quot functor $\mathrm{Quot}_{(\mathcal{U}^{(N-1)},\gamma_N)}$. Then $\mathscr{F}(r,\boldsymbol{\gamma})$ is a fine moduli space isomorphic to the relative quot-scheme $\mathrm{Quot}^{\gamma_N}(\mathcal{U}^{(N-1)})$. ∎

Remark 4.1. The previous description of the moduli space of framed flags of sheaves on $\mathbb{P}^2$ suggests we could also take a slightly different perspective on $\mathscr{F}(r,\boldsymbol{\gamma})$, namely as the moduli of the sequence of quotients

$$Z_N \hookrightarrow \cdots \hookrightarrow Z_1 \hookrightarrow F \twoheadrightarrow Q_1 \twoheadrightarrow \cdots \twoheadrightarrow Q_N,$$

where F is a vector bundle. In this sense $\mathscr{F}(r,\boldsymbol{\gamma})$ seems to be analogous to the Filt-scheme studied by Mochizuki in [161] in the case of curves. ◂

Now that we proved that the definition of moduli space of framed flags of sheaves on $\mathbb{P}^2$ is indeed a good one we are ready to tackle the problem of showing that there exists an isomorphism between this moduli space and the space of stable representation of the nested instantons quiver we studied in the previous sections. First of all let us point out that our definition of flags of framed torsion-free sheaves reduce in the rank 1 case to the nested Hilbert scheme of points on $\mathbb{C}^2$, and the isomorphism we are interested in was showed to exist in Thm. 4.12 of §4.2.1. This is in fact compatible with the statement of the following Thm. 4.15.

Theorem 4.15. *The moduli space of stable representations of the nested ADHM quiver is a fine moduli space isomorphic to the moduli space of flags of framed torsion-free sheaves on* $\mathbb{P}^2$*:* $\mathscr{F}(r,\boldsymbol{\gamma}) \simeq \mathscr{N}(r,\mathbf{n})$*, as schemes, where* $n_i = \gamma_i + \cdots + \gamma_N$.

Proof. We first want to show how, starting from an element of $\mathscr{N}(r,n_0,\ldots,n_N)$ one can construct a flag of framed torsion-free sheaves on $\mathbb{P}^2$. As we showed previously, to each (V_i, B^i_1, B^i_2, F^i) in the datum of $X \in \mathscr{N}(r,n_0,\ldots,n_N)$ we can associate a stable ADHM datum $(W, \widetilde{V}_i, \widetilde{B}_1^i, \widetilde{B}_2^i, \tilde{I}^i, \tilde{J}^i)$, fitting in the diagram (4.2.1)

$$
\begin{tikzcd}
V_1 \arrow[rr, "F^1"] \arrow[dr, shift left] & & V_0 \arrow[rr] \arrow[dd] \arrow[dr, shift left] & & \widetilde{V}_1 \arrow[dd] \arrow[dr, shift left] & \\
& \{0\} \arrow[ul, shift left] \arrow[rr, dashed] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[rr, dashed] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[dd, dashed] \\
V_2 \arrow[uu, "F^2"] \arrow[rr] \arrow[dr, shift left] & & V_0 \arrow[rr] \arrow[dd] \arrow[dr, shift left] & & \widetilde{V}_2 \arrow[dd] \arrow[dr, shift left] & \\
& \{0\} \arrow[ul, shift left] \arrow[rr, dashed] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[rr, dashed] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[dd, dashed] \\
\vdots \arrow[uu, "F^3"] \arrow[rr] \arrow[dr, shift left] & & \vdots \arrow[rr] \arrow[dd] \arrow[dr, shift left] & & \vdots \arrow[dd] \arrow[dr, shift left] & \\
& \{0\} \arrow[ul, shift left] \arrow[rr, dashed] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[rr, dashed] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[dd, dashed] \\
V_N \arrow[uu, "F^N"] \arrow[rr] \arrow[dr, shift left] & & V_0 \arrow[rr] \arrow[dr, shift left] & & \widetilde{V}_N \arrow[dr, shift left] & \\
& \{0\} \arrow[ul, shift left] \arrow[rr, dashed] & & W \arrow[ul, shift left] \arrow[rr, dashed] & & W \arrow[ul, shift left]
\end{tikzcd}
\tag{4.2.1}
$$

where we suppressed all of the endomorphisms $B_{1,2}^i$, $\widetilde{B}_{1,2}^j$. We will then call $\mathbf{Z}_i$, $\mathbf{S}$ and $\mathbf{Q}_i$ the representations of the ADHM data $(\{0\}, V_i, B_1^i, B_2^i)$, $(W, V_0, B_1^0, B_2^0, I, J)$ and $(W, \widetilde{V}_i, \widetilde{B}_1^i, \widetilde{B}_2^i, \tilde{I}^i, \tilde{J}^i)$, respectively. The the diagram (4.2.1) can be restated in the following form:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbf{Z}_1 \arrow[r] & \mathbf{S} \arrow[r] \arrow[d] & \mathbf{Q}_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbf{Z}_2 \arrow[r] \arrow[u] & \mathbf{S} \arrow[r] \arrow[d] & \mathbf{Q}_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & \mathbf{Z}_N \arrow[r] \arrow[u] & \mathbf{S} \arrow[r] & \mathbf{Q}_N \arrow[r] & 0
\end{tikzcd}
\tag{4.2.2}
$$

Moreover, if $E^\bullet_{\mathbf{Z}_i}$, $E^\bullet_{\mathbf{S}}$ and $E^\bullet_{\mathbf{Q}_i}$ denotes the ADHM complex corresponding to $\mathbf{Z}_i$, $\mathbf{S}$ and $\mathbf{Q}_i$ the diagram (4.2.2) induces the following

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & E^\bullet_{\mathbf{Z}_1} \arrow[r] & E^\bullet_{\mathbf{S}} \arrow[r] \arrow[d] & E^\bullet_{\mathbf{Q}_1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & E^\bullet_{\mathbf{Z}_2} \arrow[r] \arrow[u] & E^\bullet_{\mathbf{S}} \arrow[r] \arrow[d] & E^\bullet_{\mathbf{Q}_2} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & E^\bullet_{\mathbf{Z}_N} \arrow[r] \arrow[u] & E^\bullet_{\mathbf{S}} \arrow[r] & E^\bullet_{\mathbf{Q}_N} \arrow[r] & 0
\end{tikzcd}
\tag{4.2.3}
$$

Then, since $\mathbf{S}$ and $\mathbf{Q}_i$ are stable one has that $H^p(E^\bullet_{\mathbf{S}}) = H^p(E^\bullet_{\mathbf{Q}_i}) = 0$, for $p = -1, 1$, so that for each line in (4.2.3) the long exact sequence for the cohomology associated to it reduces to:

$$
0 \longrightarrow H^0(E^\bullet_{\mathbf{S}}) \longrightarrow H^0(E^\bullet_{\mathbf{Q}_i}) \longrightarrow H^1(E^\bullet_{\mathbf{Z}_i}) \longrightarrow 0,
$$

and by the ADHM construction $(H^0(E^\bullet_{\mathbf{Q}_i}),\varphi)$ is a rank r framed torsion-free sheaf on $\mathbb{P}^2$, with framing $\varphi : H^0(E^\bullet_{\mathbf{Q}_i})|_{\ell_\infty} \xrightarrow{\simeq} W\otimes\mathscr{O}_{\ell_\infty}$. Moreover $H^0(E^\bullet_{\mathbf{S}})$ is a subsheaf of $H^0(E^\bullet_{\mathbf{Q}_i})$, and $H^1(E^\bullet_{\mathbf{Z}_i})$ is a quotient sheaf

$$H^1(E^\bullet_{\mathbf{Z}_i}) \simeq H^0(E^\bullet_{\mathbf{Q}_i})/H^0(E^\bullet_{\mathbf{S}}),$$

which is 0−dimensional and supported away from $\ell_\infty \subset \mathbb{P}^2$. Finally one can immediately see from (4.2.3) that $H^0(E^\bullet_{\mathbf{Q}_i})$ is a subsheaf of $H^0(E^\bullet_{\mathbf{Q}_{i+1}})$. One can moreover check that the numerical invariants classifying flags of sheaves do agree with the statement of the theorem.

Conversely let $(E_0,\dots,E_N,\varphi)$ be a flag of framed torsion-free sheaves on $\mathbb{P}^2$ such that $\operatorname{rk} E_j = r$, $c_2(E_0)=\gamma_0$, $h^0(E_0/E_{j>0})=\gamma_1+\cdots+\gamma_j$. By definition each (E_j,φ) defines a stable ADHM datum $\mathbf{Q}_j=(\widetilde{W}_j,\widetilde{V}_j,\widetilde{B}^j_1,\widetilde{B}^j_2,\widetilde{I}^j,\widetilde{J}^j)$ (with the convention of calling $\mathbf{S}=\mathbf{Q}_N$), since it can be identified with a framed torsion-free sheaf on $\mathbb{P}^2$, with $\operatorname{rk} E_j = r$, $c_2(E_j)=\gamma_0+\cdots+\gamma_j$. Moreover we have the inclusion $E_0 \hookrightarrow E_j$, which induces an epimorphism $\Psi_j : \mathbf{S}\to\mathbf{Q}_j$. In fact, we can construct vector spaces V_0, $\widetilde{V}_j$, W and $\widetilde{W}_j$ as in [165], so that

$$V_0 \simeq H^0(E_N(-1)),\quad \widetilde{V}_j \simeq H^0(E_j(-1)),\quad W\simeq H^0\left(E_N|_{\ell_\infty}\right),\quad \widetilde{W}_j \simeq H^0\left(E_j|_{\ell_\infty}\right),$$

and by the fact that the quotient sheaf E_j/E_N is 0−dimensional and supported away from ℓ_∞ we can construct an isomorphism

$$\Psi_{j,2} : H^0\left(E_N|_{\ell_\infty}\right) \xrightarrow{\simeq} H^0\left(E_j|_{\ell_\infty}\right).$$

Finally we have the exact sequence

$$0\longrightarrow E_N \longrightarrow E_j \longrightarrow E_j/E_N \longrightarrow 0,$$

which induces the following exact sequence of cohomology, thanks to the fact that $H^0(E_j(-1))=0$, being that E_j is a framed torsion-free μ−semistable sheaf with $c_1(E_j)=0$ (due to the standard ADHM construction), while $H^1(E_j/E_N(-1))=0$, since the quotient sheaf E_j/E_N is 0−dimensional,

$$0\longrightarrow H^0(E_j/E_N(-1))\longrightarrow H^1(E_N(-1)) \xrightarrow{\Psi_{j,1}} H^1(E_j(-1))\longrightarrow 0.$$

The morphism $\Psi_j=(\Psi_{j,1},\Psi_{j,2})$ is then an epimorphism, since both $\Psi_{j,1}$ and $\Psi_{j,2}$ are surjective. Taking into account the flag structure of the datum $(E_0,\dots,E_N,\varphi)$, the sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\Psi_{N-1} & \longrightarrow & \mathbf{S} & \longrightarrow & \mathbf{Q}_1 & \longrightarrow & 0\\
\downarrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & \ker\Psi_{N-2} & \longrightarrow & \mathbf{S} & \longrightarrow & \mathbf{Q}_2 & \longrightarrow & 0\\
\downarrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow\\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots\\
\downarrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & \ker\Psi_0 & \longrightarrow & \mathbf{S} & \longrightarrow & \mathbf{Q}_N & \longrightarrow & 0
\end{array}$$

give us $(N+1)$ stable ADHM data fitting in the following diagram.

$$
\begin{tikzcd}
V_1 \arrow[rr] \arrow[dr, shift left] & & V_0 \arrow[rr, "\Psi_{N-1,1}"] \arrow[dd] \arrow[dr, shift left] & & \tilde{V}_1 \arrow[dd] \arrow[dr, shift left] & \\
& \{0\} \arrow[rr, dashed] \arrow[ul, shift left] \arrow[dd, dashed] & & W \arrow[rr, dashed, "\Psi_{N-1}"] \arrow[ul, shift left] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[dd, dashed] \\
V_2 \arrow[uu] \arrow[rr] \arrow[dr, shift left] & & V_0 \arrow[rr, "\Psi_{N-2,1}"] \arrow[dd] \arrow[dr, shift left] & & \tilde{V}_2 \arrow[dd] \arrow[dr, shift left] & \\
& \{0\} \arrow[rr, dashed] \arrow[ul, shift left] \arrow[dd, dashed] & & W \arrow[rr, dashed, "\Psi_{N-2}"] \arrow[ul, shift left] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[dd, dashed] \\
\vdots \arrow[uu] \arrow[rr] \arrow[dr, shift left] & & \vdots \arrow[rr] \arrow[dd] \arrow[dr, shift left] & & \vdots \arrow[dd] \arrow[dr, shift left] & \\
& \{0\} \arrow[rr, dashed] \arrow[ul, shift left] \arrow[dd, dashed] & & W \arrow[rr, dashed] \arrow[ul, shift left] \arrow[dd, dashed] & & W \arrow[ul, shift left] \arrow[dd, dashed] \\
V_N \arrow[uu] \arrow[rr] \arrow[dr, shift left] & & V_0 \arrow[rr, "\Psi_{0,1}"] \arrow[dr, shift left] & & \tilde{V}_N \arrow[dr, shift left] & \\
& \{0\} \arrow[rr, dashed] \arrow[ul, shift left] & & W \arrow[rr, dashed, "\Psi_{0,2}"] \arrow[ul, shift left] & & W \arrow[ul, shift left]
\end{tikzcd}
$$

Finally, we need to show that there exists a scheme-theoretic isomorphism between the moduli space of flags of sheaves $\mathscr{F}(r,\gamma)$ and the nested ADHM moduli space $\mathscr{N}(r,\mathbf{n})$. We may then notice that diagram (4.2.1) enables us to think to (4.2.3) as sequences of complexes parameterized by $\mathscr{N}(r,\mathbf{n})$, *i.e.* complexes of sheaves on $\mathbb{P}^2\times\mathscr{N}(r,\mathbf{n})$. Thus passing to cohomology we get a family of framed torsion-free sheaves also parameterized by $\mathscr{N}(r,\mathbf{n})$, and the isomorphism we were just describing may be regarded as an element of $F_{r,\gamma}(\mathscr{N}(r,\mathbf{n}))$, to which we may associate a unique morphism of schemes $\mathscr{N}(r,\mathbf{n})\to\mathscr{F}(r,\gamma)$ by the representability of $F_{r,\gamma}$. Conversely, suppose we are given coherent sheaves $(F_S,Q_S^1,\dots,Q_S^N)$ on $\mathbb{P}^2\times S$, flat over S, defining a family of flags of framed torsion free-sheaves $(F_S,\varphi_S,Q_S^1,g_S^1,\dots,Q_S^N,g_S^N)$. We can associate to this family of sheaves a family of representations of the nested ADHM quiver parameterized by S, *i.e.* a morphism $S\to\mathscr{N}(r,\mathbf{n})$. Then, corresponding to the universal family we get a morphism $\mathscr{F}(r,\gamma)\to\mathscr{N}(r,\mathbf{n})$. ∎

4.3 VIRTUAL INVARIANTS

In this section we study fixed points under the action of a torus on the moduli space of framed stable representations of fixed numerical type of the nested instantons quiver. By doing this we are then able to apply virtual equivariant localization and compute certain relevant virtual topological invariants. On the physics side this is equivalent to the computation of certain partition functions of some suitable quiver GLSM theory by means of the SUSY localization technique.

4.3.1 Equivariant torus action and localization

We begin the analysis of the fixed locus under a certain toric action on the moduli space of nested instantons with a brief recall of the results obtained in [213] and show how they enable us to fully characterize the T–fixed locus of the two-step nested instantons quiver. In this case we take $\mathsf{T} = (\mathbb{C}^*)^2$ acting on $\mathscr{N}(r, n_0, n_1)$ by

$$(t_1, t_2) \cdot (V_0, V_1, W, B_1^0, B_2^0, I, J, B_1^1, B_2^1, F) = (V_0, V_1, W, t_1 B_1^0, t_2 B_2^0, I, t_1 t_2 J, \\ t_1 B_1^1, t_2 B_2^1, F),$$

and we define $Q = Q_1 + Q_2$, where Q_i is the one-dimensional representation of T having character t_i. The main result we want to recall is the following:

Theorem 4.16 (von Flach-Jardim, [213]). *The moduli space $\mathscr{N}(r, n_0, n_1) \simeq \mathscr{F}(r, n_0 - n_1, n_1)$ of stable representations of the nested ADHM quiver is a quasi-projective variety equipped with a perfect obstruction theory. The T–equivariant lift of the deformation complex is the following*

$$\mathcal{C}(X)^0 \xrightarrow{d_0} \mathcal{C}(X)^1 \xrightarrow{d_1} \mathcal{C}(X)^2 \xrightarrow{d_2} \mathcal{C}(X)^3,$$

with

$$\begin{aligned}
\mathcal{C}(X)^0 &= \mathrm{End}(V_0) \otimes \mathrm{End}(V_1), \\
\mathcal{C}(X)^1 &= Q \otimes \mathrm{End}(V_0) \oplus \mathrm{Hom}(W, V_0) \oplus \Lambda^2 Q\, \mathrm{Hom}(V_0, W) \oplus \\
&\quad \oplus Q \otimes \mathrm{End}(V_1) \oplus \mathrm{Hom}(V_1, V_0), \\
\mathcal{C}(X)^2 &= \Lambda^2 Q \otimes \mathrm{End}(V_0) \oplus Q \otimes \mathrm{Hom}(V_1, V_0) \oplus \Lambda^2 Q \otimes \mathrm{Hom}(V_1, W) \oplus \\
&\quad \oplus \Lambda^2 Q \otimes \mathrm{End}(V_1), \\
\mathcal{C}(X)^3 &= \Lambda^2 Q \otimes \mathrm{Hom}(V_1, V_0)
\end{aligned}$$

and

$$\begin{cases}
d_0(h_0, h_1) = \big([h_0, B_1^0], [h_0, B_2^0], h_0 I, -J h_0, [h_1, B_1^1], [h_1, B_2^1], h_0 F - F h_1\big) \\
d_1(b_1^0, b_2^0, i, j, b_1^1, b_2^1, f) = \big([b_1^0, B_2^1] + [B_1^0, b_2^0] + iJ + Ij, B_1^0 f + b_1^0 F - F b_1^1 - f B_1^1, \\
\qquad\qquad B_2^0 f + b_2^0 F - F b_2^1 - f B_2^1, jF + Jf, [b_1^1, B_2^1] + [B_1^1, b_2^1]\big) \\
d_2(c_1, c_2, c_3, c_4, c_5) = c_1 F + B_2^0 c_2 - c_2 B_2^1 + c_3 B_1^0 - B_1^1 c_3 - I c_4 - F c_5
\end{cases}$$

Thus the infinitesimal deformation space and the obstruction space at any X will be isomorphic to $H^1[\mathcal{C}(X)]$ and $H^2[\mathcal{C}(X)]$, respectively. $\mathscr{N}(r, n_1, n_2)$ is smooth iff $n_1 = 1$ ([68]).

Moreover, it turns out, [213], that there exists a surjective morphism

$$q : (B_1^0, B_2^0, I, J, B_1^1, B_2^1, F) \mapsto (B_1', B_2', I', J')$$

mapping the nested ADHM data of type (r, n_0, n_1) to the ADHM data of numerical type $(r, n_0 - n_1)$. Thus we have two different maps sending the moduli space of stable representations of the nested ADHM quiver to the moduli space of stable representations of ADHM data. The situation is depicted by the following commutative diagram

$$\begin{tikzcd}
\mathscr{N}(r,n_0,n_1) \arrow[r, "\eta"] \arrow[dr, "q"'] & \mathscr{M}(r,n_0) \\
& \mathscr{M}(r,n_0-n_1) \arrow[u, "\tilde{f}"']
\end{tikzcd}$$

by means of which one can characterize T−fixed points of $\mathscr{N}(r,n_0,n_1)$ by means of fixed points of $\mathscr{M}(r,n_0)$ and $\mathscr{M}(r,n_0-n_1)$. In particular we can first take the decomposition $V_0 = V \oplus V_1$, then decompose the vector spaces V_0, V with respect to the action of T: if $\lambda_0 : T \to U(V_0)$ and $\lambda : T \to U(V)$ are morphisms for the toric action on V_0, V, we have

$$\begin{cases} V = \bigoplus_{k,l} V(k,l) = \bigoplus_{k,l} \{v \in V | \lambda(t)v = t_1^k t_2^l v\} \\ V_0 = \bigoplus_{k,l} V_0(k,l) = \bigoplus_{k,l} \{v_0 \in V_0 | \lambda_0(t)v_0 = t_1^k t_2^l v_0\} \end{cases}$$

Thus if $X = (W, V, B_1', B_2', I', J')$, $X_0 = (W, V_0, B_1^0, B_2^1, I, J)$ are fixed points for this torus action, the very well known results about the classification of fixed points for ADHM data leads us to the following commutative diagram.

$$\begin{tikzcd}
& V(k-1,l) \arrow[rr, "B_2'"] & & V(k-1,l-1) \\
V_0(k-1,l) \arrow[ur] \arrow[rr] & & V_0(k-1,l-1) \arrow[ur] & \\
& V(k,l) \arrow[uu] \arrow[rr] & & V(k,l-1) \arrow[uu, "B_1'"'] \\
V_0(k,l) \arrow[uu, "B_1^0"] \arrow[ur] \arrow[rr, "B_2^0"] & & V_0(k,l-1) \arrow[uu] \arrow[ur, "\tilde{f}"] &
\end{tikzcd} \tag{4.3.1}$$

Proposition 4.17. *Let $X \in \mathbb{X}_0$ be a fixed point of the toric action. The following statements hold:*

1. *If* $k > 0$ *or* $l > 0$, *then* $V_0(k,l) = 0$, $V(k,l) = 0$;
2. $\dim V_0(k,l) \leqslant 1$, $\forall k,l$ *and* $\dim V(k,l) \leqslant 1$, $\forall k,l$;
3. *If* $k,l \leqslant 0$, *then*
   a) $\dim V_0(k,l) \geqslant \dim V_0(k-1,l)$,
   b) $\dim V_0(k,l) \geqslant \dim V_0(k,l-1)$,
   c) $\dim V(k,l) \geqslant \dim V(k-1,l,$
   d) $\dim V(k,l) \geqslant \dim V(k,l-1)$,
   e) $\dim V_0(k,l) \geqslant \dim V(k,l)$.

The previous propositions give us an easy way of visualizing fixed points of the T−action on the nested ADHM data. If we suitably normalize each non-zero map to 1 by the action of $\prod_{k,l} GL(V_0(k,l)) \times \prod_{k',l'} GL(V(k',l'))$

each critical point point can be put into one-to-one correspondence with nested Young diagrams $Y_\mu \subseteq Y_\nu$. Thus the fixed points of the original nested ADHM data are classified by couples $(\nu, \nu \setminus \mu)$, where $\mu \subset \nu$ and $\nu \setminus \mu$ is the skew Young diagram constructed by taking the complement of μ in ν.

If we now take a fixed point $Z = (\nu, \mu)$ and define $\nu_i = \sum_k \dim V_0(k, 1-i)$, $\nu'_j = \sum_l \dim V_0(1-j, l)$ and similarly $\mu_i = \sum_k \dim V(k, 1-i)$, $\mu'_j = \sum_l \dim V(1-j, l)$, we can regard V_0 and V as T−modules and write them as

$$\begin{cases} V_0 = \bigoplus_{k,l} V_0(k,l) = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu'_i} T_1^{-i+1} T_2^{-j+1} = \sum_{j=1}^{N_1} \sum_{j=1}^{\nu_j} T_1^{-i+1} T_2^{-j+1} \\ V = \bigoplus_{k,l} V(k,l) = \sum_{i=1}^{M_2} \sum_{j=1}^{\mu'_i} T_1^{-i+1} T_2^{-j+1} = \sum_{j=1}^{N_2} \sum_{j=1}^{\mu_j} T_1^{-i+1} T_2^{-j+1} \end{cases}$$

with $M_1 = \nu_1$, $M_2 = \mu_1$, $N_1 = \nu'_1$, $N_2 = \mu'_1$. If we now take $V_0 = V \oplus V_1$, then

$$V_1 = \sum_{(i,j) \in \nu \setminus \mu} T_1^{-i+1} T_2^{-j+1} = \sum_{i=1}^{M_1} \sum_{j=1}^{\nu'_i - \mu'_i} T_1^{-i+1} T_2^{-\mu'_i - j+1}$$

The virtual tangent space $T_Z^{\mathrm{vir}} \mathscr{N}(1, n_0, n_1)$ to $\mathscr{N}(1, n_0, n_1)$ at Z can be regarded as a T−module, so that

$$\begin{aligned} T_Z^{\mathrm{vir}} \mathscr{N}(1, n_0, n_1) &= \mathrm{End}(V_0) \otimes (Q - 1 - \Lambda^2 Q) \\ &\quad + \mathrm{End}(V_1) \otimes (Q - 1 - \Lambda^2 Q) \\ &\quad + \mathrm{Hom}(W, V_0) + \mathrm{Hom}(V_0, W) \otimes \Lambda^2 Q \\ &\quad - \mathrm{Hom}(V_1, W) \otimes \Lambda^2 Q \\ &\quad + \mathrm{Hom}(V_1, V_0)(1 + \Lambda^2 Q - Q) \\ &\cong (V_1 \otimes V_0^* + V_1 \otimes V_1^*) \otimes (Q - 1 - \Lambda^2 Q) \\ &\quad - V_1^* \otimes V_0 \otimes (Q - 1 - \Lambda^2 Q) \\ &\quad + V_0 + V_0^* \otimes \Lambda^2 Q - V_2^* \otimes \Lambda^2 Q. \end{aligned}$$

In the first place we might recognize the term $V_0^* \otimes V_0 \otimes (Q - \Lambda^2 Q - 1) + V_0 + V_0^* \otimes \Lambda^2 Q$ in the sum as being the tangent space at the moduli space of stable representation of the ADHM quiver $T_{\tilde{Z}} \mathscr{M}(1, n_0)$, with $\tilde{Z} = (\nu)$. Thus we have

$$\begin{aligned} T_Z^{\mathrm{vir}} \mathscr{N}(1, n_0, n_1) &= T_{\tilde{Z}} \mathscr{M}(1, n_0) \\ &\quad + (V_1 \otimes V_1^* - V_1^* \otimes V_0) \otimes (Q - 1 - \Lambda^2 Q) \\ &\quad - V_1^* \otimes \Lambda^2 Q. \end{aligned} \tag{4.3.2}$$

We have

$$\begin{aligned}
V_1^* \otimes (Q - 1 - \Lambda^2 Q) &= (T_1 - 1)(1 - T_2) \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_1^{i-1} T_2^{\mu_i' + j - 1} \\
&= (T_1 - 1) \sum_{i=1}^{M_1} T_1^{i-1} T_2^{\mu_i' - 1} (1 - T_2) \sum_{j=1}^{\nu_i' - \mu_i'} T_2^j \\
&= (T_1 - 1) \sum_{i=1}^{M_1} T_1^{i-1} T_2^{\mu_i' - 1} (1 - T_2) \left(\frac{1 - T_2^{\nu_i' - \mu_i' + 1}}{1 - T_2} - 1 \right) \\
&= (T_1 - 1) \sum_{i=1}^{M_1} T_1^{i-1} T_2^{\mu_i' - 1} (1 - T_2) \left(\frac{T_2 - T_2^{\nu_i' - \mu_i' + 1}}{1 - T_2} \right) \\
&= (T_1 - 1) \sum_{i=1}^{M_1} T_1^{i-1} (T_2^{\mu_i'} - T_2^{\nu_i'}),
\end{aligned}$$

so that

$$\begin{aligned}
V_1^* \otimes V_1 \otimes (Q - 1 - \Lambda^2 Q) &= (T_1 - 1) \sum_{j=1}^{N_1} \sum_{j'=1}^{\nu_j - \mu_j} T_1^{-\mu_j - j' + 1} T_2^{-j+1} \sum_{i=1}^{M_1} T_1^{i-1} (T_2^{\mu_i'} - T_2^{\nu_i'}) \\
&= \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^{i - \mu_j} (T_2^{-j + \mu_i' + 1} - T_2^{-j + \nu_i' + 1}) (T_1 - 1) \sum_{j'=1}^{\nu_j - \mu_j} T_1^{-j'} \\
&= \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} (T_1^{i - \mu_j} - T_1^{i - \nu_j}) (T_2^{-j + \mu_i' + 1} - T_2^{-j + \nu_i' + 1}),
\end{aligned}$$

while we have

$$\begin{aligned}
V_1^* \otimes V_0 \otimes (Q - 1 - \Lambda^2 Q) &= (T_1 - 1) \sum_{j=1}^{N_1} \sum_{j'=1}^{\nu_j} T_1^{-j'+1} T_2^{-j+1} \sum_{i=1}^{M_1} T_1^{i-1} (T_2^{\mu_i'} - T_2^{\nu_i'}) \\
&= \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} T_1^{i} (T_2^{-j + \mu_i' + 1} - T_2^{-j + \nu_i' + 1}) (T_1 - 1) \sum_{j'=1}^{\nu_j} T_1^{-j'} \\
&= \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} (T_1^{i} - T_1^{i - \nu_j}) (T_2^{-j + \mu_i' + 1} - T_2^{-j + \nu_i' + 1}),
\end{aligned}$$

and

$$\begin{aligned}
V_1^* \otimes \Lambda^2 Q &= T_1 T_2 \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_1^{i-1} T_2^{\mu_i' + j - 1} \\
&= \sum_{i=1}^{M_1} \sum_{j=1}^{\nu_i' - \mu_i'} T_1^{i} T_2^{\mu_i' + j}.
\end{aligned}$$

Putting everything together we finally get that

$$\begin{aligned} T^{\mathrm{vir}}_Z \mathcal{N}(1,n_0,n_1) = T_{\tilde{Z}}\mathcal{M}(1,n_0) + \sum_{i=1}^{M_1}\sum_{j=1}^{N_1} \Big[(T_2^{-j+\mu_i'+1} - T_2^{-j+\nu_i'+1}) \\ (T_1^{i-\mu_j} - T_1^{i}) \Big] - \sum_{i=1}^{M_1}\sum_{j=1}^{\nu_i'-\mu_i'} T_1^i T_2^{j+\mu_i'}. \end{aligned} \tag{4.3.3}$$

As an immediate generalization of (4.3.3) we can easily see that

$$\begin{aligned} T^{\mathrm{vir}}_Z \mathcal{N}(r,n_0,n_1) = T_{\tilde{Z}}\mathcal{M}(r,n_0) + \sum_{a,b=1}^{r}\sum_{i=1}^{M_1^{(a)}}\sum_{j=1}^{N_1^{(b)}} \Bigg[R_b R_a^{-1} \left(T_1^{i-\mu_j^{(b)}} - T_1^{i} \right) \\ \left(T_2^{-j+\mu_i^{(a)\prime}+1} - T_2^{-j+\nu_i^{(a)\prime}+1} \right) \Bigg] \\ - \sum_{i=1}^{M_1^{(a)}} \sum_{j=1}^{\nu_i^{(a)\prime}-\mu_i^{(a)\prime}} T_1^i T_2^{j+\mu_i^{(a)\prime}}, \end{aligned}$$

where $(T_1, T_2, R_a), a = 1, \ldots, r$ are the canonical generators of the representation ring of $T \curvearrowright \mathcal{N}(r, n_0, n_1)$.

Remark 4.2. It turns out that the character representation for the virtual tangent $T^{vir}_Z\mathcal{N}$ can be computed by exploiting deformation theory techniques. These techniques may also be employed to compute the virtual fundamental class and (**T**−character of) the virtual tangent bundle at fixed points of nested Hilbert schemes on surfaces, as it's done in [107].

In particular, if one takes $(\mathbb{C}^2)^{[N_0 \geqslant N_1]}$ to be the nested Hilbert scheme of points on $\mathbb{C}^2 = \mathrm{Spec}(R)$, with $R = \mathbb{C}[x_0, x_1]$, by lifting the natural torus action on $\mathbb{C}^2$ to $(\mathbb{C}^2)^{[N_0 \geqslant N_1]}$, it is proved in [107] that the T−fixed locus is isolated and given by the inclusion of monomial ideals $I_0 \subseteq I_1$, which is equivalent to the assignment of couples of nested partitions $\mu \subseteq \nu$. Then the virtual tangent space at a fixed point is given by

$$T^{vir}_{I_0 \subseteq I_1} = -\chi(I_0, I_0) - \chi(I_1, I_1) + \chi(I_0, I_1) + \chi(R, R),$$

with $\chi(-,-) = \sum_{i=0}^{2} (-1)^i \,\mathrm{Ext}^i_R(-,-)$. Then the T−representation of $T^{vir}_{I_0 \subseteq I_1}$ can be explicitly written in terms of Laurent polynomials in the torus characters t_1, t_2 of T. Then in terms of the characters Z_0, Z_1 of the T−fixed 0−dimensional subschemes $Z_1 \subseteq Z_0 \subset \mathbb{C}^2$ corresponding to $I_0 \subseteq I_1$ one has (see Eq. (29) in [107])

$$\mathrm{tr}\, T^{vir}_{I_0 \subseteq I_1} = Z_0 + \frac{\overline{Z}_1}{t_1 t_2} + \left(\overline{Z}_0 Z_1 - \overline{Z}_0 Z_0 - \overline{Z}_1 Z_1 \right) \frac{(1-t_1)(1-t_2)}{t_1 t_2}.$$

If we now make the necessary identifications $t_i = T_i^{-1}$, $Z_0 = V_0$ and $Z_1 = V$ we can see that Eq. (29) of [107] exactly agrees with our prescription for the character representation (4.3.2) of the virtual tangent space $T^{vir}_Z\mathcal{N}(1, n_0, n_1)$, with $n_0 = N_0$ and $n_1 = N_0 - N_1$. ◀

We now move on studying the fixed locus of the more general nested instantons moduli space $\mathscr{N}(r, n_0, \ldots, n_N)$. However, similarly to the previous case we first want to show that the moduli space of stable representations of the nested ADHM quiver is equivalently described by the datum of $(N+1)$ moduli spaces of framed torsion-free sheaves on $\mathbb{P}^2$, namely $\mathscr{M}(r, n_0), \mathscr{M}(r, n_0 - n_1), \ldots, \mathscr{M}(r, n_0 - n_{s-1})$. In order to do this we want to know if it is possible to recover the structure of the nested ADHM quiver given a set of stable ADHM data. First of all we can notice that, as F^i is injective $\forall i$, we have the sum decomposition $V_0 = V_i \oplus \tilde{V}_i$, but also $V_i = V_{i+1} \oplus \hat{V}_{i+1}$, with $\hat{V}_{i+1} = V_i / \operatorname{Im} F_i$, so that $V_0 = V_i \oplus \hat{V}_i \oplus \tilde{V}_{i-1}$, thus $\tilde{V}_i = \hat{V}_i \oplus \tilde{V}_{i-1}$.

Let us first focus on the vector spaces V_0 and V_1. It can be shown as in [51; 213] that once we fix a stable ADHM datum $(W, \tilde{V}_1, \tilde{B}^1_1, \tilde{B}^1_2, \tilde{I}^1, \tilde{J}^1)$ and the endomorphisms $B^1_1, B^1_2 \in \operatorname{End} V_1$ it is always possible to reconstruct the stable ADHM datum $(W, V_0, B^0_1, B^0_2, I, J)$ as

$$B^0_1 = \begin{pmatrix} B^1_1 & B'^1_1 \\ 0 & \tilde{B}^1_1 \end{pmatrix}, \quad B^0_2 = \begin{pmatrix} B^1_2 & B'^1_2 \\ 0 & \tilde{B}^1_2 \end{pmatrix}, \quad I = \begin{pmatrix} I'^1 \\ \tilde{I}^1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \tilde{J}^1 \end{pmatrix}$$

together with the morphism $F^1 = \mathbb{1}_{V_1}$ such that $[B^1_1, B^1_2] = 0$, $B^0_1 F^1 - F^1 B^1_1 = B^0_2 F^1 - F^1 B^1_2 = 0$ and $JF^1 = 0$. The same can obviously be done for any of the stable ADHM data $(W, \tilde{V}_i, \tilde{B}^i_1, \tilde{B}^i_2, \tilde{I}^i, \tilde{J}^i)$ we constructed previously, and we would have

$$B^0_1 = \begin{pmatrix} B^i_1 & B'^i_1 \\ 0 & \tilde{B}^i_1 \end{pmatrix}, \quad B^0_2 = \begin{pmatrix} B^i_2 & B'^i_2 \\ 0 & \tilde{B}^i_2 \end{pmatrix}, \quad I = \begin{pmatrix} I'^i \\ \tilde{I}^i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \tilde{J}^i \end{pmatrix}$$

together with the morphism $f^i = \mathbb{1}_{V_i}$ such that $[B^i_1, B^i_2] = 0$, $B^0_1 f^i - f^i A_i = B^0_2 f^i - f^i B^i_2 = 0$ and $Jf^i = 0$. If we now fix

$$F^i = \begin{pmatrix} \mathbb{1}_{V_i} \\ 0 \end{pmatrix}, \qquad F^i : V_i \to V_{i-1},$$

which is clearly injective, then obviously $f^i = F^1 F^2 \cdots F^i$, where F^j now stands for the linear extension to V_0, and $B^0_1 f^i - f^i B^i_1 = 0$ (resp. $B^0_2 f^i - f^i B^i_2 = 0$) is equivalent to $B^0_1 F^1 F^2 \cdots F^{i-1} F^i - F^1 F^2 \cdots F^i B^i_1 = B^{i-1}_1 F^i - F^i B^i_1 = 0$ (resp. $B^{i-1}_2 F^i - F^i B^i_2 = 0$), and $Jf^i = JF^1 F^2 \cdots F^i = 0$. This construction makes it possible to us to classify the $T-$fixed locus of $\mathscr{N}(r, n_0, \ldots, n_{s-1})$ in terms of the $T-$fixed loci of $\mathscr{M}(r, n_0)$ and $\{\mathscr{M}(r, n_0 - n_i)\}_{i>0}$. In particular the $T-$fixed locus of $\mathscr{M}(r, k)$ is into $1-1$ correspondence with coloured partitions $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^r$ such that $|\boldsymbol{\mu}| = |\mu^1| + \cdots + |\mu^r| = k$ (cf. Def. 3.1). This fact and the inclusion relations between the vector spaces V_i prove the following

Proposition 4.18. *The $T-$fixed locus of $\mathscr{N}(r, n_0, \ldots, n_{s-1})$ can be described by $s-$tuples of nested coloured partitions $\boldsymbol{\mu}_1 \subseteq \cdots \subseteq \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$ (cf. Def. 3.2), with $|\boldsymbol{\mu}_0| = n_0$ and $|\boldsymbol{\mu}_{i>0}| = n_0 - n_i$.*

In the same way as we did in a previous section, we can read the virtual tangent space to $\mathcal{N}(r, n_0, \dots, n_{s-1})$ off the following equivariant lift of the complex (4.1.3)

$$\mathcal{C}(X): \qquad \mathcal{C}(X)^0 \xrightarrow{d_0} \mathcal{C}(X)^1 \xrightarrow{d_1} \mathcal{C}(X)^2 \xrightarrow{d_2} \mathcal{C}(X)^3, \tag{4.3.4}$$

where

$$\begin{aligned}
\mathcal{C}(X)^0 &= \bigoplus_{i=0}^{N} \operatorname{End}(V_i), \\
\mathcal{C}(X)^1 &= Q \otimes \operatorname{End}(V_0) \oplus \operatorname{Hom}(W, V_0) \oplus \Lambda^2 Q \otimes \operatorname{Hom}(V_0, W) \\
&\qquad \oplus \left[\bigoplus_{i=1}^{N} (Q \otimes \operatorname{End}(V_i) \oplus \operatorname{Hom}(V_i, V_{i-1}) \right] \\
\mathcal{C}(X)^2 &= \Lambda^2 Q \otimes (\operatorname{End}(V_0) \oplus \operatorname{Hom}(V_1, W)) \oplus \left[\bigoplus_{i=1}^{N} Q \otimes \operatorname{Hom}(V_i, V_{i-1}) \right. \\
&\qquad \left. \oplus \Lambda^2 Q \otimes \operatorname{End}(V_i) \right] \\
\mathcal{C}(X)^3 &= \bigoplus_{i=1}^{N} \Lambda^2 Q \otimes \operatorname{Hom}(V_i, V_{i-1}).
\end{aligned}$$

Then, the cohomology of the complex (4.3.4) reproduces the virtual tangent space, which takes the form (4.3.5).

$$\begin{aligned}
T_Z^{\mathrm{vir}} \mathcal{N}(1, n_1, n_2) = &\operatorname{End}(V_0) \otimes (Q - 1 - \Lambda^2 Q) \\
&+ \operatorname{Hom}(W, V_0) + \operatorname{Hom}(V_0, W) \otimes \Lambda^2 Q \\
&+ \operatorname{End}(V_1) \otimes (Q - 1 - \Lambda^2 Q) \\
&- \operatorname{Hom}(V_1, W) \otimes \Lambda^2 Q + \\
&+ \operatorname{Hom}(V_1, V_0) \otimes (1 + \Lambda^2 Q - Q) + \\
&+ \operatorname{End}(V_2) \otimes (Q - 1 - \Lambda^2 Q) \\
&+ \operatorname{Hom}(V_2, V_1) \otimes (1 + \Lambda^2 Q - 1) + \\
&\dots \\
&+ \operatorname{End}(V_{s-1}) \otimes (Q - 1 - \Lambda^2 Q) \\
&+ \operatorname{Hom}(V_{s-1}, V_{s-2}) \otimes (1 + \Lambda^2 Q - Q).
\end{aligned} \tag{4.3.5}$$

By decomposing the vector spaces V_i in terms of characters of the torus T we can also rewrite the representation of (4.3.5) in R(T) as (4.3.6)

$$\begin{aligned}
T_Z^{\mathrm{vir}} \mathcal{N}(r, \mathbf{n}) = &T_{\tilde{Z}} \mathcal{M}(r, n_0) + \sum_{a,b=1}^{r} \sum_{i=1}^{M_0^{(a)}} \sum_{j=1}^{N_0^{(b)}} \left[R_b R_a^{-1} \left(T_1^{i - \mu_{1,j}^{(b)}} - T_1^{i} \right) \right. \\
&\left. \left(T_2^{-j + \mu_{1,i}^{(a)\prime} + 1} - T_2^{-j + \mu_{0,i}^{(a)\prime} + 1} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{M_0^{(a)}}\sum_{j=1}^{\mu_{0,i}^{(a)\prime}-\mu_{1,i}^{(a)\prime}} T_1^i T_2^{j+\mu_{1,i}^{(a)\prime}}\\
&+\sum_{k=2}^{s-1}\left[\sum_{a,b=1}^{r}\sum_{i=1}^{M_0^{(a)}}\sum_{j=1}^{N_0^{(b)}} R_b R_a^{-1}\left(T_1^{i-\mu_{k,j}^{(b)}}-T_1^{i-\mu_{k-1,j}^{(b)}}\right)\right.\\
&\left.\left(T_2^{-j+\mu_{k,i}^{(a)\prime}+1}-T_2^{-j+\mu_{0,i}^{(a)\prime}+1}\right)\right],
\end{aligned}
\tag{4.3.6}
$$

where the fixed point Z is to be identified with a choice of a sequence of coloured nested partitions $\mu_1 \subseteq \mu_{N-1} \subseteq \cdots \subseteq \mu_{s-1} \subseteq \mu_0$ as in Prop. 4.18 and $\tilde{Z} \leftrightarrow \mu_0$.

4.3.2 Virtual equivariant holomorphic Euler characteristic

The first virtual invariant we are going to study is the holomorphic virtual equivariant Euler characteristic of the moduli space of nested instantons. We will exploit the virtual localisation theorem we described in §1.5. In particular, if X is proper and $V \in K^0(X)$ we can apply the virtual Riemann-Roch theorem [95, Corollary 3.6] to get that

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(V)\cdot \mathrm{td}(T_X^{\mathrm{vir}}), \tag{4.3.7}$$

where $[X]^{\mathrm{vir}}$ is the virtual fundamental class of X, $[X]^{\mathrm{vir}} \in A_{\mathrm{vd}}(X)$. Clearly, if we are interested in $\chi^{\mathrm{vir}}(X) = \chi^{\mathrm{vir}}(X, \mathscr{O}_X)$ then the eq. (4.3.7) reduces to

$$\chi^{\mathrm{vir}}(X) = \int_{[X]^{\mathrm{vir}}} \mathrm{td}(T_X^{\mathrm{vir}}). \tag{4.3.8}$$

Equations (4.3.7) and (4.3.8) can be made even more explicit. In fact if we take $n = \mathrm{rk}\, E_0$, $m = \mathrm{rk}\, E_1$, so that $\mathrm{vd} = n - m$, and define $x_1, \ldots, x_n$ and $u_1, \ldots, u_m$ to be respectively the Chern roots of E_0 and E_1, then (4.3.8) becomes

$$\chi^{\mathrm{vir}}(X) = \int_{[X]^{\mathrm{vir}}} \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \prod_{j=1}^{m} \frac{1-e^{-u_j}}{u_j},$$

while for (4.3.7) we have

$$\chi^{\mathrm{vir}}(X, V) = \int_{[X]^{\mathrm{vir}}} \left(\sum_{k=1}^{r} e^{\nu_k}\right) \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \prod_{j=1}^{m} \frac{1-e^{-u_j}}{u_j},$$

since we can consider $V \in K^0(X)$ to be a vector bundle on X with Chern roots $\nu_1, \ldots, \nu_r$.

Now, if we have a proper scheme X equipped with an action of a torus $(\mathbb{C}^*)^N$ and an equivariant 1−perfect obstruction theory we can apply virtual equivariant localization in order to compute virtual invariants of X. Let us then recall that, if $p : X \to \mathrm{pt}$ is the structure morphism, one has that (cf. §1.5 and [95, Prop. 6.3])

$$\begin{aligned}\chi^{\mathrm{vir}}(X, \tilde{V}; \varepsilon_1, \dots, \varepsilon_N) &= \sum_i p^{\mathrm{vir}}_{i*}\left(\tilde{V}_i / \Lambda_{-1}(N^{\mathrm{vir}}_i)^\vee\right) \\ &= \sum_i p^{\mathrm{vir}}_{i*}\left(\tilde{V}_i / \Lambda_{-1}(T^{\mathrm{vir}}_X|^{\mathrm{mov}}_{X_i})^\vee\right)\end{aligned} \tag{4.3.9}$$

belongs to $\mathbb{Q}[\![\varepsilon_1, \dots, \varepsilon_N]\!]$ and the virtual holomorphic Euler characteristic is $\chi^{\mathrm{vir}}(X, V) = \chi^{\mathrm{vir}}(X, \tilde{V}; 0)$. Here we denote by $\tilde{V}$ an equivariant lift of V, and by $\tilde{V}_i$ its restriction to the i-th component in the fixed locus.

Of course, as we will be computing virtual invariants of quasi-projective variety, we will take the r.h.s. of (4.3.9) as the *definition* of the virtual Euler characteristic. Computations are now made very easy by the fact that we represented the virtual tangent space to the $T = (\mathbb{C}^*)^2-$fixed points to the moduli space of nested instantons in the representation ring $R(T)$ of the torus $(\mathbb{C}^*)^2$. In this way $T^{\mathrm{vir}}_{X_i}$ is decomposed as a direct sum of line bundles which are moreover eigenbundles of the torus action. Then we can use the following properties

$$\begin{aligned}\mathrm{ch}(E \oplus F) &= \mathrm{ch}\, E + \mathrm{ch}\, F, \\ \Lambda_t(E \oplus F) &= \Lambda_t(E) \cdot \Lambda_t(F), \\ S_t(E \oplus F) &= S_t(E) \cdot S_t(F)\end{aligned}$$

and Eq. (4.3.9) in order to compute the equivariant holomorphic Euler characteristic of the moduli space of nested instantons in terms of the fundamental characters $q_{1,2}$ of the torus T. These will be related to the equivariant parameters by $q_i = e^{\beta \varepsilon_i}$, with β being a parameter having a very clear meaning in the physical framework modelling the moduli space of nested instantons as a low energy effective theory. In this framework is very easy to explicitly compute the virtual equivariant holomorphic Euler characteristic of the moduli space of nested instantons as we already described the $T-$fixed locus of $\mathcal{N}(r, n_0, \dots, n_N)$ as being $0-$dimensional and non-degenerate. As we saw in §4.3.1 the fixed points of $\mathcal{N}(r, n_0, \dots, n_N)$ are completely described by $r-$tuples of nested coloured partitions $\mu_1 \subseteq \cdots \subseteq \mu_N \subseteq \mu_0$, with $\mu_j \in \mathcal{P}^r$, in such a way that $|\mu_0| = \sum_j |\mu_0^j| = n_0$ and $|\mu_0 \setminus \mu_{i>0}| = n_{i>0}$. In the simplest case of $r = 1$ we get

$$\begin{aligned}&\chi^{\mathrm{vir}}(\mathcal{N}, \tilde{V}; q_1, q_2) = \\ &\quad = \sum_{\substack{\mu_1 \subseteq \cdots \subseteq \mu_0 \\ |\mu_0 \setminus \mu_j| = n_j}} \frac{T_{\mu_0,\mu_1}(q_1, q_2) W_{\mu_0,\dots,\mu_N}(q_1, q_2)}{N_{\mu_0}(q_1, q_2)}) \; [\tilde{V}]|_{\mu_0,\dots,\mu_N},\end{aligned} \tag{4.3.10}$$

where $a(s)$ and $l(s)$ denote the arm length and the leg length of the box s in the Young diagram Y_μ associated to μ, respectively. We moreover defined

$$N_{\mu_0}(q_1, q_2) = \prod_{s \in Y_{\mu_0}} \left(1 - q_1^{-l(s)-1} q_2^{a(s)}\right)\left(1 - q_1^{l(s)} q_2^{-a(s)-1}\right),$$

$$T_{\mu_0,\mu_1}(q_1,q_2) = \prod_{i=1}^{M_0} \prod_{j=1}^{\mu'_{0,i}-\mu'_{1,i}} \left(1 - q_1^{-i} q_2^{-j-\mu'_{1,i}}\right),$$

$$W_{\mu_0,\dots,\mu_N}(q_1,q_2) = \prod_{k=1}^{N} \prod_{i=1}^{M_0} \prod_{j=1}^{N_0} \frac{\left(1 - q_1^{\mu_{k,j}-i} q_2^{j-\mu'_{0,i}-1}\right)\left(1 - q_1^{\mu_{k-1,j}-i} q_2^{j-\mu'_{k,i}-1}\right)}{\left(1 - q_1^{\mu_{k,j}-i} q_2^{j-\mu'_{k,i}-1}\right)\left(1 - q_1^{\mu_{k-1,j}-i} q_2^{j-\mu'_{0,i}-1}\right)}$$

A very interesting and surprising fact can be observed if we rearrange the expression the holomorphic virtual Euler characteristic of $\mathcal{N}(1,n_0,\dots,n_N)$. In fact if we perform the summation over the smaller partitions $\mu_1 \subseteq \cdots \subseteq \mu_N$ and redefine $q = q_1^{-1}$, $t = q_2^{-1}$, we get

$$\chi^{\mathrm{vir}}(\mathcal{N}(1,n_0,\dots,n_N);q_1,q_2) = \sum_{\mu_0} \frac{P_{\mu_0}(q,t)}{N_{\mu_0}(q,t)}$$

and the unexpected fact is that we think $P_{\mu_0}(q,t)$ to be a polynomial in q,t except for a factor $(1-qt)^{-1}$.

Conjecture 4.19. $P_{\mu_0}(q,t)$ *is a a function of the form:*

$$P_{\mu_0}(q,t) = \frac{Q_{\mu_0}(q,t)}{(1-qt)^N}, \tag{4.3.11}$$

with $Q_{\mu_0}(q,t)$ *a polynomial in the* (q,t)*—variables.*

Sometimes the polynomials in (4.3.11) can be given an interpretation in terms of some known symmetric polynomials. In fact, let us define the following generating function

$$Z_{MD}(q,t;x_0,\dots,x_N) = \sum_{n_0 \geqslant \cdots \geqslant n_N} \chi^{\mathrm{vir}}(\mathcal{N}(1,\tilde{n}_0,\dots,\tilde{n}_N);q,t) \prod_{i=0}^{N} x_i^{m_i},$$

where $m_i = n_i - n_{i+1}$ and the integers $\tilde{n}_i$ form a sequence obtained from n_i by asking the integers $\tilde{m}_i = \tilde{n}_i - \tilde{n}_{i+1}$ to be ordered. By construction $Z_{MD}(q,t;x_0,\dots,x_N) \in \mathbb{Q}[q,t] \otimes_{\mathbb{Z}} \Lambda(\mathbf{x})$, *i.e.* it is a symmetric function in $\{x_i\}_{i=0}^N$ with coefficients in $\mathbb{Q}[q,t]$. By conjecture 4.19 we have

$$\begin{aligned} Z_{MD}(q,t;x_0,\dots,x_N) &= \sum_{n_0 \geqslant \cdots \geqslant n_N} \sum_{\mu_0 \in \mathcal{P}(n_0)} \frac{Q_{\mu_0}(q,t)}{(1-qt)^N N_{\mu_0}(q,t)} \prod_{i=0}^{N} x_i^{m_i} \\ &= \sum_{\mu \in \mathcal{P}} \frac{Q_{\mu}(q,t)}{(1-qt)^N N_{\mu}(q,t)} m_\mu(\mathbf{x}). \end{aligned}$$

Conjecture 4.20. *When* $|\mu_0| = |\mu_N| + 1 = |\mu_{N-1}| + 2 = \cdots = |\mu_1| + N$ *we have*

$$\begin{aligned} Q_{\mu_0}(q,t) &= \left\langle h_{\mu_0}(\mathbf{x}), \widetilde{H}_{\mu_0}(\mathbf{x};q,t) \right\rangle \\ &= \left\langle h_{\mu_0}(\mathbf{x}), \sum_{\lambda,\nu \in \mathcal{P}(n_0)} \widetilde{K}_{\lambda,\mu_0}(q,t) K_{\mu_0,\nu} m_\nu(\mathbf{x}) \right\rangle \\ &= \sum_{\substack{\lambda \in \mathcal{P}(n_0) \\ m_\lambda(\mathbf{x}) \neq 0}} \widetilde{K}_{\lambda,\mu_0}(q,t), \end{aligned}$$

where the Hall pairing $\langle -,- \rangle$ *is such that* $\langle h_\mu, m_\lambda \rangle = \delta_{\mu,\lambda}$ *and* $\widetilde{H}_\mu(\mathbf{x}; q, t)$, $\widetilde{K}_{\lambda,\mu}(q,t)$ *are the modified Macdonald polynomials and the modified Kostka polynomials, respectively.*

We checked the previous conjectures up to $n_0 = 10$.

If instead $\mathscr{N} = \mathscr{N}(r > 1, n_0, \ldots, n_N)$ we get a more complicated result, even though its structure is the same as we had previously

$$\chi^{\mathrm{vir}}(\mathscr{N}, \tilde{V}; q_1, q_2, \{t_i\}) = \\ = \sum_{\substack{\mu_1 \subseteq \cdots \subseteq \mu_0 \\ |\mu_0 \setminus \mu_j| = n_j}} \frac{T^{(r)}_{\mu_0,\mu_1}(q_1,q_2) W^{(r)}_{\mu_0,\ldots,\mu_N}(q_1,q_2)}{N^{(r)}_{\mu_0}(q_1,q_2)} [\tilde{V}]\big|_{\mu_0,\ldots,\mu_N}, \quad (4.3.12)$$

with

$$N^{(r)}_{\mu_0}(q_1,q_2) = \prod_{a,b=1}^{r} \prod_{s \in Y_{\mu_0^{(a)}}} \left[\left(1 - t_{ab} q_1^{-l_a(s)-1} q_2^{a_b(s)}\right) \left(1 - q_1^{l_a(s)} q_2^{-a_b(s)-1}\right) \right],$$

$$T^{(r)}_{\mu_0,\mu_1}(q_1,q_2) = \prod_{a,b}^{r} \prod_{i=1}^{M_0^{(a)}} \prod_{j=1}^{\mu_{0,i}^{(a)\prime} - \mu_{1,i}^{(a)\prime}} \left(1 - t_{ab} q_1^{-i} q_2^{-j-\mu_{1,i}^{(a)\prime}}\right),$$

$$W^{(r)}_{\mu_0,\ldots,\mu_N}(q_1,q_2) = \prod_{k=1}^{N} \prod_{a,b}^{r} \prod_{i=1}^{M_0^{(a)}} \prod_{j=1}^{N_0^{(b)}} \left[\frac{\left(1 - t_{ab} q_1^{\mu_{k,j}^{(b)}-i} q_2^{j-\mu_{0,i}^{(a)\prime}-1}\right)}{\left(1 - t_{ab} q_1^{\mu_{k,j}^{(b)}-i} q_2^{j-\mu_{k,i}^{(a)\prime}-1}\right)} \frac{\left(1 - t_{ab} q_1^{\mu_{k-1,j}^{(b)}-i} q_2^{j-\mu_{k,i}^{(a)\prime}-1}\right)}{\left(1 - t_{ab} q_1^{\mu_{k-1,j}^{(b)}-i} q_2^{j-\mu_{0,i}^{(a)\prime}-1}\right)} \right],$$

where now $t_{ab} = t_a t_b^{-1}$ and $\{t_i\}$ are the fundamental characters of $(\mathbb{C}^*)^r$ in $\mathcal{G} = (\mathbb{C}^*)^r \times T$, and $a_b(s)$ denotes the arm length of the box s with respect to the Young diagram $Y_{\mu^{(b)}}$ associated to the partition $\mu^{(b)}$ of $\boldsymbol{\mu}$ (with an analogous definition for the leg length).

4.3.3 Virtual equivariant χ_{-y}−genus

The first refinement of the equivariant holomorphic Euler characteristic we are going to study is the virtual equivariant χ_{-y}−genus, as defined in [95]. We will denote by Ω_X^{vir} the virtul cotangent bundle of X, *i.e.* $\Omega_X^{\mathrm{vir}} = (T_X^{\mathrm{vir}})^\vee$. If X is a proper scheme equipped with a perfect obstruction theory of virtual dimension d, the virtual χ_{-y}−genus of X is defined by

$$\chi_{-y}^{\mathrm{vir}}(X) = \chi^{\mathrm{vir}}(X, \Lambda_{-y}\Omega_X^{\mathrm{vir}}) = \sum_{i \geqslant 0} (-y)^i \chi^{\mathrm{vir}}(X, \Omega_X^{i,\mathrm{vir}}),$$

while, if $V \in K^0(X)$, the virtual χ_{-y}−genus of X with values in V is

$$\chi^{\mathrm{vir}}_{-y}(X,V) = \chi^{\mathrm{vir}}(X, V \otimes \Lambda_{-y}\Omega^{\mathrm{vir}}_X) = \sum_{i\geqslant 0}(-y)^i \chi^{\mathrm{vir}}(X, V \otimes \Omega^{i,\mathrm{vir}}_X).$$

Though in principle one would expect $\chi^{\mathrm{vir}}_{-y}(X,V)$ to be an element of $\mathbb{Z}[\![t]\!]$, it is in fact true that $\chi^{\mathrm{vir}}_{-y}(X,V) \in \mathbb{Z}[t]$, [95].

The virtual Hirzebruch-Riemann-Roch theorem implies that [95, Eq. 2]

$$\begin{aligned}
\chi^{\mathrm{vir}}_{-y}(X) &= \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(\Lambda_{-y}T^{\mathrm{vir}}_X)\cdot \mathrm{td}(T^{\mathrm{vir}}_X) \\
&= \int_{[X]^{\mathrm{vir}}} \mathcal{X}_{-y}(X), \\
\chi^{\mathrm{vir}}_{-y}(X,V) &= \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(\Lambda_{-y}T^{\mathrm{vir}}_X)\cdot \mathrm{ch}(V)\cdot \mathrm{td}(T^{\mathrm{vir}}_X) \\
&= \int_{[X]^{\mathrm{vir}}} \mathcal{X}_{-y}(X)\cdot \mathrm{ch}(V),
\end{aligned}$$

which, in terms of the Chern roots of E_0, E_1 and V become

$$\begin{aligned}
\chi^{\mathrm{vir}}_{-y}(X) &= \int_{[X]^{\mathrm{vir}}} \prod_{i=1}^{n} x_i \frac{1-ye^{-x_i}}{1-e^{-x_i}} \prod_{j=1}^{m} \frac{1}{u_j}\frac{1-e^{-u_j}}{1-ye^{-u_j}}, \\
\chi^{\mathrm{vir}}_{-y}(X,V) &= \int_{[X]^{\mathrm{vir}}} \left(\sum_{k=1}^{r} e^{\nu_k}\right) \prod_{i=1}^{n} x_i \frac{1-ye^{-x_i}}{1-e^{-x_i}} \prod_{j=1}^{m} \frac{1}{u_j}\frac{1-e^{-u_j}}{1-ye^{-u_j}}.
\end{aligned}$$

Finally one can define the virtual Euler number $e^{\mathrm{vir}}(X)$ and the virtual signature $\sigma^{\mathrm{vir}}(X)$ of X as $e^{\mathrm{vir}}(X) = \chi^{\mathrm{vir}}_{-1}(X)$ and $\sigma^{\mathrm{vir}}(X) = \chi^{\mathrm{vir}}_{1}(X)$. Whenever $y = 0$ one recovers the holomorphic virtual Euler characteristic instead.

By extending the definition of χ_{-y}−genus to the equivariant case in the obvious way and by lifting V to an equivariant vector bundle $\tilde{V}$, one has (cf. [95, Corollary 6.6])

$$\chi^{\mathrm{vir}}_{-y}(X,\tilde{V};\varepsilon_1,\dots,\varepsilon_N) = \sum_i p^{\mathrm{vir}}_{i*}\left(\tilde{V}_i \otimes \Lambda_{-y}(\Omega^{\mathrm{vir}}_X|_{X_i})/\Lambda_{-1}(N^{\mathrm{vir}}_i)^\vee\right),$$

whence $\chi^{\mathrm{vir}}_{-y}(X,V) = \chi^{\mathrm{vir}}_{-y}(X,\tilde{V};0,\dots,0)$.

A simple computation in equivariant localization gives us the following result for $\mathcal{N} = \mathcal{N}(1,n_0,\dots,n_N)$:

$$\begin{aligned}
&\chi^{\mathrm{vir}}_{-y}(\mathcal{N},\tilde{V};q_1,q_2) = \\
&\qquad = \sum_{\substack{\mu_1\subseteq\cdots\subseteq\mu_0 \\ |\mu_0\setminus\mu_j|=n_j}} \frac{T^{-y}_{\mu_0,\mu_1}(q_1,q_2) W^{-y}_{\mu_0,\dots,\mu_N}(q_1,q_2)}{N^{-y}_{\mu_0}(q_1,q_2)} [\tilde{V}]\big|_{\mu_0,\dots,\mu_N},
\end{aligned} \tag{4.3.13}$$

with

$$N^{-y}_{\mu_0}(q_1,q_2) = \prod_{s\in Y_{\mu_0}} \left[\frac{\left(1-q_1^{-l(s)-1}q_2^{a(s)}\right)}{\left(1-yq_1^{-l(s)-1}q_2^{a(s)}\right)}\right.$$

$$\left.\frac{\left(1-q_1^{l(s)}q_2^{-a(s)-1}\right)}{\left(1-yq_1^{l(s)}q_2^{-a(s)-1}\right)}\right],$$

$$T^{-y}_{\mu_0,\mu_1}(q_1,q_2)=\prod_{i=1}^{M_0}\prod_{j=1}^{\mu'_{0,i}-\mu'_{1,i}}\frac{\left(1-q_1^{-i}q_2^{-j-\mu'_{1,i}}\right)}{\left(1-yq_1^{-i}q_2^{-j-\mu'_{1,i}}\right)},$$

$$W^{-y}_{\mu_0,\dots,\mu_N}(q_1,q_2)=\prod_{k=1}^{N}\prod_{i=1}^{M_0}\prod_{j=1}^{N_0}\left[\frac{\left(1-q_1^{\mu_{k,j}-i}q_2^{j-\mu'_{0,i}-1}\right)}{\left(1-yq_1^{\mu_{k,j}-i}q_2^{j-\mu'_{0,i}-1}\right)}\frac{\left(1-q_1^{\mu_{k-1,j}-i}q_2^{j-\mu'_{k,i}-1}\right)}{\left(1-yq_1^{\mu_{k-1,j}-i}q_2^{j-\mu'_{k,i}-1}\right)}\frac{\left(1-yq_1^{\mu_{k,j}-i}q_2^{k-\mu'_{k,i}-1}\right)}{\left(1-q_1^{\mu_{k,j}-i}q_2^{k-\mu'_{k,i}-1}\right)}\frac{\left(1-yq_1^{\mu_{k-1,j}-i}q_2^{j-\mu'_{0,i}-1}\right)}{\left(1-q_1^{\mu_{k-1,j}-i}q_2^{j-\mu'_{0,i}-1}\right)}\right].$$

The limit $y\to 0$ manifestly reverts to the case of the equivariant holomorphic Euler characteristic of the moduli space of nested instantons.

A similar result holds also for the general case $r>1$:

$$\chi^{\mathrm{vir}}_{-y}(\mathscr{N},\tilde{V};q_1,q_2,\{t_i\})=$$
$$=\sum_{\substack{\mu_1\subseteq\cdots\subseteq\mu_0\\|\mu_0\setminus\mu_j|=n_j}}\frac{T^{(r),y}_{\mu_0,\mu_1}(q_1,q_2)W^{(r),y}_{\mu_0,\dots,\mu_N}(q_1,q_2)}{N^{(r),y}_{\mu_0}(q_1,q_2)}[\tilde{V}]\big|_{\mu_0,\dots,\mu_N},\quad(4.3.14)$$

with

$$N^{(r),y}_{\mu_0}(q_1,q_2)=\prod_{a,b=1}^{r}\prod_{s\in Y_{\mu_0^{(a)}}}\left[\frac{\left(1-t_{ab}q_1^{-l_a(s)-1}q_2^{a_b(s)}\right)}{\left(1-yt_{ab}q_1^{-l_a(s)-1}q_2^{a_b(s)}\right)}\frac{\left(1-t_{ab}q_1^{l_a(s)}q_2^{-a_b(s)-1}\right)}{\left(1-yt_{ab}q_1^{l_a(s)}q_2^{-a_b(s)-1}\right)}\right],$$

$$T^{(r),y}_{\mu_0,\mu_1}(q_1,q_2)=\prod_{a,b}^{r}\prod_{i=1}^{M_0^{(a)}}\prod_{j=1}^{\mu^{(a)\prime}_{0,i}-\mu^{(a)\prime}_{1,i}}\frac{\left(1-t_{ab}q_1^{-i}q_2^{-j-\mu^{(a)\prime}_{1,i}}\right)}{\left(1-yt_{ab}q_1^{-i}q_2^{-j-\mu^{(a)\prime}_{1,i}}\right)},$$

$$W^{(r),y}_{\mu_0,\dots,\mu_N}(q_1,q_2)=\prod_{k=1}^{N}\prod_{a,b}^{r}\prod_{i=1}^{M_0^{(a)}}\prod_{j=1}^{N_0^{(b)}}\left[\frac{\left(1-t_{ab}q_1^{\mu^{(b)}_{k,j}-i}q_2^{j-\mu^{(a)\prime}_{0,i}-1}\right)}{\left(1-yt_{ab}q_1^{\mu^{(b)}_{k,j}-i}q_2^{j-\mu^{(a)\prime}_{0,i}-1}\right)}\right.$$

$$\frac{\left(1-t_{ab}q_1^{\mu^{(b)}_{k-1,j}-i}q_2^{j-\mu^{(a)\prime}_{k,i}-1}\right)}{\left(1-yt_{ab}q_1^{\mu^{(b)}_{k-1,j}-i}q_2^{j-\mu^{(a)\prime}_{k,i}-1}\right)}\frac{\left(1-yt_{ab}q_1^{\mu^{(b)}_{k,j}-i}q_2^{k-\mu^{(a)\prime}_{k,i}-1}\right)}{\left(1-t_{ab}q_1^{\mu^{(b)}_{k,j}-i}q_2^{k-\mu^{(a)\prime}_{k,i}-1}\right)}\frac{\left(1-yt_{ab}q_1^{\mu^{(b)}_{k-1,j}-i}q_2^{j-\mu^{(a)\prime}_{0,i}-1}\right)}{\left(1-t_{ab}q_1^{\mu^{(b)}_{k-1,j}-i}q_2^{j-\mu^{(a)\prime}_{0,i}-1}\right)}\Bigg],$$

with the same notations of the previous section.

Virtual Euler number

As we already pointed out previously, two specifications of the value of y in the $\chi_{-y}-$genus, namely $y=\pm 1$, give back two interesting topological invariants of a given nested instantons moduli space. Let us consider first the simpler case of rank 1. We can easily see, in the case of the virtual Euler number, that taking the specialization $y=+1$ amounts to counting the number of nested partitions of a given size. Then, if we assemble everything in a single generating function we have

$$\begin{aligned}M(q_1,\dots,q_N)&=\sum_{j=0}^{\infty}\sum_{n_0,\dots,n_j}e^{\mathrm{vir}}\left(\mathscr{N}(1,n_0,\dots,n_j)\right)q_0^{n_0}\cdots q_j^{n_j}\\&=\sum_{j=0}^{\infty}\sum_{n_0,\dots,n_j}\#\left\{\mu_1\subseteq\cdots\subseteq\mu_j\subseteq\mu_0\right\}q_0^{n_0}\cdots q_j^{n_j},\end{aligned}$$

for which a closed formula is not available to our knowledge. Of course if we focus our attention on smooth nested Hilbert schemes only (*i.e.* $n_{i>0}=0$ or $n_1=1$, $n_{i>1=0}$), the generating function of the virtual Euler number of smooth nested Hilbert schemes is easily expressed in terms of standard generating functions of partitions:

$$\begin{aligned}\sum_{n\geqslant 0}e^{\mathrm{vir}}(\mathscr{N}(1,n,1))q^n&=\prod_{k=0}^{\infty}\left(\frac{1}{1-q^k}\right)\\&=(\phi(q))^{-1}=\sum_{n\geqslant 0}\chi(\mathscr{M}(1,n))q^n,\end{aligned}$$

which, in the case of higher rank becomes

$$\begin{aligned}\sum_{n\geqslant 0}e^{\mathrm{vir}}(\mathscr{N}(r,n,1))q^n&=\prod_{k=0}^{\infty}\left(\frac{1}{1-q^k}\right)^r\\&=(\phi(1))^{-r}=\sum_{n\geqslant 0}\chi(\mathscr{M}(r,n))q^n.\end{aligned}$$

We also notice that whenever $q_0 = q_1 = \cdots = q_N$, generating function of Euler numbers is actually accounting for the enumeration of plane partition, whose generating function is known to be the Macmahon function $\Phi(q)$:

$$M(q,\ldots,q) = \sum_{j=0}^{\infty} \sum_{n_0,\ldots,n_j} e^{\mathrm{vir}}(\mathscr{N}(1,n_0,\ldots,n_j))q^{n_0+\cdots+n_j} = \Phi(q). \quad (4.3.15)$$

4.3.4 Virtual equivariant elliptic genus

A further refinement of the virtual χ_{-y}−genus is finally given by the virtual elliptic genus. In this case, if F is any vector bundle over X, we define

$$\mathcal{E}(F) = \bigotimes_{n\geqslant 1} \left(\Lambda_{-yq^n}F^\vee \otimes \Lambda_{-y^{-1}q^n}F \otimes S_{q^n}(F \oplus F^\vee)\right) \in 1 + q\cdot K^0(X)[y, y^{-1}][\![q]\!],$$

so that the virtual elliptic genus $\mathrm{Ell}^{\mathrm{vir}}(X; y, q)$ of X is defined by (cd. §2.2)

$$\mathrm{Ell}^{\mathrm{vir}}(X; y, q) = y^{-d/2}\chi^{\mathrm{vir}}_{-y}(X, \mathcal{E}(T^{\mathrm{vir}}_X)) \in \mathbb{Q}(\!(y^{1/2})\!)[\![q]\!],$$

or, equivalently

$$\mathrm{Ell}^{\mathrm{vir}}(X, V; y, q) = y^{-d/2}\chi^{\mathrm{vir}}_{-y}(X, \mathcal{E}(T^{\mathrm{vir}}_X) \otimes V),$$

By applying the virtual Riemann-Roch theorem again one can see that the virtual elliptic genus admits an integral representation [95, Eq. 6]

$$\mathrm{Ell}^{\mathrm{vir}}(X; y, q) = \int_{[X]^{\mathrm{vir}}} \mathcal{E}\ell\ell(T^{\mathrm{vir}}_X; y, q),$$
$$\mathrm{Ell}^{\mathrm{vir}}(X, V; y, q) = \int_{[X]^{\mathrm{vir}}} \mathcal{E}\ell\ell(T^{\mathrm{vir}}_X; y, q)\cdot \mathrm{ch}(V),$$

with

$$\mathcal{E}\ell\ell(F; y, q) = y^{-\operatorname{rk} F/2}\,\mathrm{ch}(\Lambda_{-y}F^\vee)\cdot \mathrm{ch}(\mathcal{E}(F))\cdot \mathrm{td}(F) \in A^*(X)[y^{-1/2}, y^{1/2}][\![q]\!].$$

It is also interesting to study how the virtual elliptic genus is described in terms of the usual Chern roots x_i, u_j, v_k, as its formula involves the Jacobi theta function $\theta(z, \tau)$ defined as

$$\theta(z,\tau) = q^{1/8}\frac{y^{1/2} - y^{-1/2}}{i}\prod_{l=1}^{\infty}(1-q^l)(1-q^l y)(1-q^l y^{-1}),$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$. In fact if F is any vector bundle over X with Chern roots $\{f_i\}$, one can prove [50] that

$$\mathcal{E}\ell\ell(F; z, \tau) = \prod_{i=1}^{\operatorname{rk} F} f_i \frac{\theta(f_i/2\pi i - z, \tau)}{\theta(f_i/2\pi i, \tau)},$$

so that finally

$$\mathrm{Ell}^{\mathrm{vir}}(X; y, q) = \int_{[X]^{\mathrm{vir}}} \prod_{i=1}^{n} x_i \frac{\theta(x_i/2\pi i - z, \tau)}{\theta(x_i/2\pi i, \tau)} \prod_{j=1}^{m} \frac{1}{u_j} \frac{\theta(u_j/2\pi i, \tau)}{\theta(u_j/2\pi i - z, \tau)}.$$

Moreover we can extend the definition of virtual elliptic genus to the equivariant setting, and apply the virtual localisation theorem, thus getting [95, Corollary 7.3]

$$\begin{aligned}\mathrm{Ell}^{\mathrm{vir}}(X, \tilde{V}, z, \tau; \varepsilon_1, \dots, \varepsilon_N) &= \\ &= y^{-\mathrm{vd}/2} \sum_i p^{\mathrm{vir}}_{i*} \left(\tilde{V}_i \otimes \mathcal{E}(T^{\mathrm{vir}}_X \otimes \Lambda_{-y}(\Omega^{\mathrm{vir}}_X|_{X_i})/\Lambda_{-1}(N^{\mathrm{vir}}_i)^\vee \right)\end{aligned}$$

and $\mathrm{Ell}^{\mathrm{vir}}(X, V) = \mathrm{Ell}^{\mathrm{vir}}(X, \tilde{V}; 0, \dots, 0)$ (cf. [95, Corollary 6.6]). In particular we get in rank 1

$$\begin{aligned}\mathrm{Ell}^{\mathrm{vir}}(\mathcal{N}, \tilde{V}; \varepsilon_1, \varepsilon_2) &= \\ &= \sum_{\substack{\mu_1 \subseteq \cdots \subseteq \mu_0 \\ |\mu_0 \setminus \mu_j| = n_j}} \frac{\mathcal{T}^{z,\tau}_{\mu_0,\mu_1}(\varepsilon_1, \varepsilon_2) \mathcal{W}^{z,\tau}_{\mu_0,\dots,\mu_N}(\varepsilon_1, \varepsilon_2)}{\mathcal{N}^{z,\tau}_{\mu_0}(\varepsilon_1, \varepsilon_2)}) \left[\tilde{V}\right]\Big|_{\mu_0,\dots,\mu_N}, \end{aligned} \tag{4.3.16}$$

with

$$\begin{aligned}\mathcal{N}^{z,\tau}_{\mu_0}(\varepsilon_1, \varepsilon_2) &= \prod_{s \in Y_{\mu_0}} \left[\frac{\theta(\epsilon_1(l(s)+1) - \epsilon_2 a(s), \tau)}{\theta(\epsilon_1(l(s)+1) - \epsilon_2 a(s) - z, \tau)} \right. \\ &\qquad \left. \frac{\theta(-\epsilon_1 l(s) + \epsilon_2(a(s)+1), \tau)}{\theta(-\epsilon_1 l(s) + \epsilon_2(l(s)+1) - z, \tau)} \right], \\ \mathcal{T}^{z,\tau}_{\mu_0,\mu_1}(\varepsilon_1, \varepsilon_2) &= \prod_{i=1}^{M_0} \prod_{j=1}^{\mu'_{0,i} - \mu'_{1,i}} \frac{\theta(\epsilon_1 i + \epsilon_2(j + \mu'_{1,i}) - z, \tau)}{\theta(\epsilon_1 i + \epsilon_2(j + \mu'_{1,i}), \tau)}, \\ \mathcal{W}^{z,\tau}_{\mu_0,\dots,\mu_N}(\varepsilon_1, \varepsilon_2) &= \prod_{k=1}^{N} \prod_{i=1}^{M_0} \prod_{j=1}^{N_0} \left[\frac{\theta(\epsilon_1(i - \mu_{k,j}) + \epsilon_2(1 + \mu'_{0,i} - j), \tau)}{\theta(\epsilon_1(i - \mu_{k,j}) + \epsilon_2(1 + \mu'_{0,i} - j) - z, \tau)} \cdot \right. \\ &\qquad \frac{\theta(\epsilon_1(i - \mu_{k-1,j}) + \epsilon_2(1 + \mu'_{k,i} - j), \tau)}{\theta(\epsilon_1(i - \mu_{k-1,j}) + \epsilon_2(1 + \mu'_{k,i} - j) - z, \tau)} \\ &\qquad \frac{\theta(\epsilon_1(i - \mu_{k,j}) + \epsilon_2(1 + \mu'_{k,i} - j) - z, \tau)}{\theta(\epsilon_1(i - \mu_{k,j}) + \epsilon_2(1 + \mu'_{k,i} - j), \tau)} \\ &\qquad \left. \frac{\theta(\epsilon_1(i - \mu_{k-1,j}) + \epsilon_2(1 + \mu'_{0,i} - j) - z, \tau)}{\theta(\epsilon_1(i - \mu_{k-1,j}) + \epsilon_2(1 + \mu'_{0,i} - j), \tau)} \right],\end{aligned}$$

with $\epsilon_i = \varepsilon_i/2\pi i$. One can easily see that the virtual elliptic genus we just computed is indeed a Jacobi form, and that its limit $\tau \to i\infty$ reproduces the $\chi_{-y}-$genus. Moreover by taking the limit $y \to 0$ in the $\chi_{-y}-$genus one can recover the virtual equivariant holomorphic Euler characteristic.

Finally, if we study the virtual equivariant elliptic genus in the more general case of rank $r \geqslant 1$, we get

$$\mathrm{Ell}^{\mathrm{vir}}(\mathcal{N}, \tilde{V}; \varepsilon_1, \varepsilon_2, \{a_i\}) =$$

$$= \sum_{\substack{\mu_1\subseteq\cdots\subseteq\mu_0 \\ |\mu_0\setminus\mu_j|=n_j}} \frac{\mathcal{T}^{z,\tau}_{\mu_0,\mu_1}(\varepsilon_1,\varepsilon_2)\mathcal{W}^{z,\tau}_{\mu_0,\dots,\mu_N}(\varepsilon_1,\varepsilon_2)}{\mathcal{N}^{z,\tau}_{\mu_0}(\varepsilon_1,\varepsilon_2)}) \; [\tilde{V}]\big|_{\mu_0,\dots,\mu_N}, \quad (4.3.17)$$

with

$$\mathcal{N}^{z,\tau}_{\mu_0}(\varepsilon_1,\varepsilon_2) = \prod_{a,b=1}^{r}\prod_{s\in Y_{\mu_0}}\left[\frac{\theta(a_{ab}+\epsilon_1(l(s)+1)-\epsilon_2 a(s),\tau)}{\theta(a_{ab}+\epsilon_1(l(s)+1)-\epsilon_2 a(s)-z,\tau)}\right.$$
$$\left.\frac{\theta(a_{ab}+-\epsilon_1 l(s)+\epsilon_2(a(s)+1),\tau)}{\theta(a_{ab}+-\epsilon_1 l(s)+\epsilon_2(l(s)+1)-z,\tau)}\right],$$

$$\mathcal{T}^{z,\tau}_{\mu_0,\mu_1}(\varepsilon_1,\varepsilon_2) = \prod_{a,b=1}^{r}\prod_{i=1}^{M_0^{(a)}}\prod_{j=1}^{\mu_{0,i}^{(a)\prime}-\mu_{1,i}^{(a)\prime}}\frac{\theta(a_{ab}+\epsilon_1 i+\epsilon_2(j+\mu_{1,i}^{(a)\prime}),\tau)}{\theta(a_{ab}+\epsilon_1 i+\epsilon_2(j+\mu_{1,i}^{(a)\prime})-z,\tau)},$$

$$\mathcal{W}^{z,\tau}_{\mu_0,\dots,\mu_N}(\varepsilon_1,\varepsilon_2) = \prod_{k=1}^{N}\prod_{a,b=1}^{r}\prod_{i=1}^{M_0^{(a)}}\prod_{j=1}^{N_0^{(b)}}\left[\frac{\theta(a_{ab}+\epsilon_1(i-\mu_{k,j}^{(b)})+\epsilon_2(1+\mu_{0,i}^{(a)\prime}-j),\tau)}{\theta(a_{ab}+\epsilon_1(i-\mu_{k,j}^{(b)})+\epsilon_2(1+\mu_{0,i}^{(a)\prime}-j)-z,\tau)}\right.$$
$$\frac{\theta(a_{ab}+\epsilon_1(i-\mu_{k-1,j}^{(b)})+\epsilon_2(1+\mu_{k,i}-j)^{(a)\prime},\tau)}{\theta(a_{ab}+\epsilon_1(i-\mu_{k-1,j}^{(b)})+\epsilon_2(1+\mu_{k,i}^{(a)\prime}-j)-z,\tau)}$$
$$\frac{\theta(a_{ab}+\epsilon_1(i-\mu_{k,j}^{(b)})+\epsilon_2(1+\mu_{k,i}^{(a)\prime}-j)-z,\tau)}{\theta(a_{ab}+\epsilon_1(i-\mu_{k,j}^{(b)})+\epsilon_2(1+\mu_{k,i}^{(a)\prime}-j),\tau)}$$
$$\left.\frac{\theta(a_{ab}+\epsilon_1(i-\mu_{k-1,j}^{(b)})+\epsilon_2(1+\mu_{0,i}^{(a)\prime}-j)-z,\tau)}{\theta(a_{ab}+\epsilon_1(i-\mu_{k-1,j}^{(b)})+\epsilon_2(1+\mu_{0,i}^{(a)\prime}-j),\tau)}\right].$$

Notice that by knowing the equivariant virtual elliptic genus one is able to recover both the virtual equivariant holomorphic Euler characteristic and $\chi_{-y}-$ genus. In fact the limit $\tau \to i\infty$ of (4.3.17) recovers exactly the $\chi_{-y}-$genus found in (4.3.14) and a successive limit $y \to 0$ gives us back the virtual equivariant holomorphic Euler characteristic (4.3.12).

4.4 TORIC SURFACES

In this section we will generalize the results we got in the previous ones to the case of nested Hilbert schemes on toric surfaces, and in particular we will be interested in $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$. This is because one might expect any complex genus of $\mathrm{Hilb}^{(\mathbf{n})}(S)$ to depend only on the cobordism class of S, as it was the case for $\mathrm{Hilb}^n(S)$, [94], and the complex cobordism ring $\Omega = \Omega^U\otimes\mathbb{Q}$ with rational coefficients was showed by Milnor to be a polynomial algebra freely generated by the cobordism classes $[\mathbb{P}^n]$, $n>0$. Then in the case of complex projective surfaces any case can be reduced to $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ by the fact that $[S]=a[\mathbb{P}^2]+b[\mathbb{P}^1\times\mathbb{P}^1]$. The advantage given by having an ADHM-like construction for the nested punctual Hilbert scheme on the affine plane is that it provides us with the local model of the more general case of smooth projective surfaces. In particular, whenever S is toric, one can construct it starting from its toric fan by appropriately gluing the affine patches (*e.g.* Fig. 2a for $\mathbb{P}^2$ and 2b for $\mathbb{P}^1\times\mathbb{P}^1$), and computation of topological invariants can still be easily carried out by means of equivariant (virtual) localization. In general, given the toric fan describing the patches which glued together makes up a toric surface S, each patch U_i will be $U_i \simeq \mathbb{C}^2$, with a natural action of $T = (\mathbb{C}^*)^2$. Moreover, if $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $Z \in \mathrm{Hilb}^{(\mathbf{n})}(S)$ is a fixed point of the $T-$action, its support must be contained in $\{P_0, \dots, P_{\chi(S)-1}\}$ (as a consequence of [66]) with P_i corresponding to the vertices of the polytope associated to the fan, so that one can write in general that $Z = Z_0 \cup \dots \cup Z_{\chi(S)-1}$, with Z_i being supported in P_i. This also induces a decomposition of the representation in $R(T)$ of the virtual tangent space at the fixed points:

$$T_Z^{\mathrm{vir}}\left(\mathrm{Hilb}^{(\mathbf{n})}(S)\right) = \bigoplus_{\ell=0}^{\chi(S)-1} T_{Z_\ell}^{\mathrm{vir}}\left(\mathrm{Hilb}^{(\mathbf{n}_\ell)}(U_\ell)\right).$$

Let us call then $\chi_{-y}^{\mathrm{vir}}(P_\ell) = \chi_{-y}^{\mathrm{vir}}(\mathcal{N}(1, n_0^{(\ell)}, \dots, n_N^{(\ell)}); q_{1,(\ell)} q_{2,(\ell)})$: we will see how we will be able to compute $\chi_{-y}^{\mathrm{vir}}(\mathrm{Hilb}^{(\mathbf{n})}(\mathbb{P}^2))$ and $\chi_{-y}^{\mathrm{vir}}(\mathrm{Hilb}^{(\mathbf{n})}(\mathbb{P}^1 \times \mathbb{P}^1))$ in terms of $\chi_{-y}^{\mathrm{vir}}(P_\ell)$.

4.4.1 Case 1: $S = \mathbb{P}^2$

We will be interested in the following generating function

$$\sum_{\mathbf{n}} \chi_{-y}^{\mathrm{vir}}\left(\mathrm{Hilb}^{(\hat{\mathbf{n}})}(\mathbb{P}^2)\right) \mathbf{q}^{\mathbf{n}} = \prod_{\ell=0}^{2} \left(\sum_{\mathbf{n}_\ell \geqslant 0} \chi_{-y}^{\mathrm{vir}}(P_\ell) \mathbf{q}^{\mathbf{n}_\ell} \right), \tag{4.4.1}$$

with $\hat{\mathbf{n}}$ defined as in §4.2.1, and since the left-hand side doesn't depend on $q_{1,2}$, we can perform the computation by taking the iterated limits $q_1 \to +\infty$, $q_2 \to +\infty$ or $q_1 \to 0x$, $q_2 \to 0$. In each one of the three affine patches the weights of the torus action will be

$$\begin{aligned} q_{1,(0)} &= q_1 & q_{2,(0)} &= q_2 \\ q_{1,(1)} &= 1/q_1 & q_{2,(1)} &= q_2/q_1 \\ q_{1,(2)} &= 1/q_2 & q_{2,(2)} &= q_1/q_2 \end{aligned}$$

We will study separately the three patches $\ell = 0, 1, 2$. First of all we notice that since the $\chi_{-y}-$genus is multiplicative, the first contribution coming from $N_{\mu_0}^{y}(q_1, q_2)$ coincides with the same contribution arising in the context of standard Hilbert schemes. It wash shown in [153] (see §1.4 for a brief review) that

$$\begin{aligned} \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1}{N_{\mu_0,\mu_1}^{-y}(q_{1,(0)}, q_{2,(0)})} &= y^{|\mu_0| - M_0}, \\ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1}{N_{\mu_0,\mu_1}^{-y}(q_{1,(1)}, q_{2,(1)})} &= y^{|\mu_0|}, \\ \lim_{q_2 \to +\infty} \lim_{q_1 \to +\infty} \frac{1}{N_{\mu_0,\mu_1}^{-y}(q_{1,(2)}, q_{2,(2)})} &= y^{|\mu_0| + s(\mu_0)}, \end{aligned}$$

where $s(\mu_0) = \#\{s \in Y_{\mu_0'} : a(s) \leqslant l(s) \leqslant a(s)+1\}$. We are thus left with the evaluation of the other contributions only. Starting from $T^{-y}_{\mu_0,\mu_1}$ we get

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\left[\prod_{i=1}^{M_0}\prod_{j=1}^{\mu'_{0,i}-\mu'_{1,i}}\frac{\left(1-q_{1,(0)}^{-i}q_{2,(0)}^{-j-\mu'_{1,i}}\right)}{\left(1-yq_{1,(0)}^{-i}q_{2,(0)}^{-j-\mu'_{1,i}}\right)}\right]=1,$$

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\left[\prod_{i=1}^{M_0}\prod_{j=1}^{\mu'_{0,i}-\mu'_{1,i}}\frac{\left(1-q_{1,(1)}^{-i}q_{2,(1)}^{-j-\mu'_{1,i}}\right)}{\left(1-yq_{1,(1)}^{-i}q_{2,(1)}^{-j-\mu'_{1,i}}\right)}\right]=y^{-1},$$

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\left[\prod_{i=1}^{M_0}\prod_{j=1}^{\mu'_{0,i}-\mu'_{1,i}}\frac{\left(1-q_{1,(2)}^{-i}q_{2,(2)}^{-j-\mu'_{1,i}}\right)}{\left(1-yq_{1,(2)}^{-i}q_{2,(2)}^{-j-\mu'_{1,i}}\right)}\right]=1,$$

whence

$$\begin{aligned}
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}T^{-y}_{\mu_0,\mu_1}(q_{1,(0)},q_{2,(0)})=1,\\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}T^{-y}_{\mu_0,\mu_1}(q_{1,(1)},q_{2,(1)})=y^{-|\mu_0\setminus\mu_1|},\\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}T^{-y}_{\mu_0,\mu_1}(q_{1,(2)},q_{2,(2)})=1.
\end{aligned}$$

Finally we need to take care of the limit involving $W^{-y}_{\mu_0,\dots,\mu_N}(q_1,q_2)$ and in order to tackle let us first point out that we can rewrite $W^{-y}_{\mu_0,\dots,\mu_N}$ in the following simpler form:

$$\begin{aligned}
W^{-y}_{\mu_0,\dots,\mu_N}(q_1,q_2)=\prod_{k=1}^{N}\prod_{s\in Y_{\mu_0^{\mathrm{rec}}}}&\frac{\left(1-q_1^{l_k(s)}q_2^{-a_0(s)-1}\right)\left(1-q_1^{l_{k-1}(s)}q_2^{-a_k(s)-1}\right)}{\left(1-yq_1^{l_k(s)}q_2^{-a_0(s)-1}\right)\left(1-yq_1^{l_{k-1}(s)}q_2^{-a_k(s)-1}\right)}\cdot\\
&\cdot\frac{\left(1-yq_1^{l_k(s)}q_2^{-a_k(s)-1}\right)\left(1-yq_1^{l_{k-1}(s)}q_2^{-a_0(s)-1}\right)}{\left(1-q_1^{l_k(s)}q_2^{-a_k(s)-1}\right)\left(1-q_1^{l_{k-1}(s)}q_2^{-a_0(s)-1}\right)},
\end{aligned}$$

where μ_0^{rec} is the smallest rectangular partition containing μ_0 and $a_k(s)$ (resp. $l_k(s)$) denotes the arm length (resp. leg length) of the box s with respect to Y_{μ_k}. Then, by recalling that the partitions labelling the T−fixed points are included one into the other as $\mu_1 \subseteq \cdots \subseteq \mu_N \subseteq \mu_0 \subseteq \mu_0^{\mathrm{rec}}$ it's easy to realize that, in the case $\ell = 0$, one gets

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\frac{1-q_1^{l_k(s)}q_2^{-a_0(s)-1}}{1-yq_1^{l_k(s)}q_2^{-a_0(s)-1}}=\begin{cases}1 & \text{for } l_k(s)\leqslant 0\\ y^{-1} & \text{for } l_k(s)>0\end{cases},$$

and similarly in every other case:

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\frac{1-q_1^{l_{k-1}(s)}q_2^{-a_k(s)-1}}{1-yq_1^{l_{k-1}(s)}q_2^{-a_k(s)-1}}=\begin{cases}1 & \text{for } l_{k-1}(s)\leqslant 0\\ y^{-1} & \text{for } l_{k-1}(s)>0\end{cases},$$

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\frac{1-yq_1^{l_k(s)}q_2^{-a_k(s)-1}}{1-q_1^{l_k(s)}q_2^{-a_k(s)-1}}=\begin{cases}1 & \text{for } l_k(s)\leqslant 0\\ y & \text{for } l_k(s)>0\end{cases},$$

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}\frac{1-yq_1^{l_{k-1}(s)}q_2^{-a_0(s)-1}}{1-q_1^{l_{k-1}(s)}q_2^{-a_0(s)-1}}=\begin{cases}1 & \text{for } l_{k-1}(s)\leqslant 0\\ y & \text{for } l_{k-1}(s)>0\end{cases},$$

so that finally

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}W^{-y}_{\mu_0,\dots,\mu_N}(q_{1,(0)},q_{2,(0)})=1.$$

It is easy to see that the same holds true also for $\ell=2$:

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}W^{-y}_{\mu_0,\dots,\mu_N}(q_{1,(2)},q_{2,(2)})=1,$$

while the case $\ell=1$ is more difficult, even though the analysis of the different cases can be carried out exactly in the same way. We then introduce the following notation:

$$s(\mu_{i_1},\mu_{i_2})=\#\left\{s\in Y_{\mu_0^{\mathrm{rec}}}: l_{i_1}(s)>a_{i_2}(s)+1\vee l_{i_1}(s)=a_{i_2}(s)+1, a_{i_2}(s)<-1\right\},$$

and we get

$$\lim_{q_2\to+\infty}\lim_{q_1\to+\infty}W^{-y}_{\mu_0,\dots,\mu_N}(q_{1,(1)},q_{2,(1)})=$$
$$=\prod_{k=1}^{N}y^{s(\mu_k,\mu_k)+s(\mu_{k-1},\mu_0)-s(\mu_k,\mu_0)-s(\mu_{k-1},\mu_k)}.$$

Finally, by putting everything together, we have an explicit expression for (4.4.1).

$$\begin{aligned}\sum_{\mathbf{n}}\chi^{\mathrm{vir}}_{-y}\left(\mathrm{Hilb}^{(\hat{\mathbf{n}})}(\mathbb{P}^2)\right)\mathbf{q}^{\mathbf{n}}=&\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}}y^{|\mu_0|+M_0}\right)\\ &\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}}y^{|\mu_0|-|\mu_0\setminus\mu_1|}\prod_{k=1}^{N}y^{s(\mu_k,\mu_k)+s(\mu_{k-1},\mu_0)}\right.\\ &\left.y^{-s(\mu_k,\mu_0)-s(\mu_{k-1},\mu_k)}\right)\\ &\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}}y^{|\mu_0|-s(\mu_0)}\right).\end{aligned}\tag{4.4.2}$$

4.4.2 Case 2: $S=\mathbb{P}^1\times\mathbb{P}^1$

Similarly to previous case, we are interested in studying the following generating function

$$\sum_{\mathbf{n}}\chi^{\mathrm{vir}}_{-y}\left(\mathrm{Hilb}^{(\hat{\mathbf{n}})}(\mathbb{P}^1\times\mathbb{P}^1)\right)\mathbf{q}^{\mathbf{n}}=\prod_{\ell=0}^{3}\left(\sum_{\mathbf{n}_\ell\geqslant 0}\chi^{\mathrm{vir}}_{-y}(P_\ell)\mathbf{q}^{\mathbf{n}_\ell}\right),$$

and we can still perform the computation by taking the successive limits $q_1 \to +\infty$, $q_2 \to +\infty$ or $q_1 \to 0$, $q_2 \to 0$. The four patches are now indexed by $\ell = (00), (01), (10), (11)$, and the characters $q_{i,(\ell)}$ can be identified to be in this case

$$\begin{aligned} q_{1,(00)} &= q_1 & \qquad q_{2,(00)} &= q_2 \\ q_{1,(01)} &= q_1 & q_{2,(01)} &= 1/q_2 \\ q_{1,(10)} &= 1/q_1 & q_{2,(10)} &= q_2 \\ q_{1,(11)} &= 1/q_1 & q_{2,(11)} &= 1/q_2 \end{aligned}$$

An analysis similar to the one carried out in the previous section enables then us to conclude the following

$$\begin{aligned}
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} 1/N_{\mu_0}^{-y}(q_{1,(00)}, q_{2,(00)}) = y^{|\mu_0|-M_0}, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} 1/N_{\mu_0}^{-y}(q_{1,(00)}, q_{2,(01)}) = y^{|\mu_0|}, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} 1/N_{\mu_0}^{-y}(q_{1,(00)}, q_{2,(10)}) = y^{|\mu_0|}, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} 1/N_{\mu_0}^{-y}(q_{1,(00)}, q_{2,(11)}) = y^{|\mu_0|+M_0}, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} T_{\mu_0,\mu_1}^{-y}(q_{1,(00)}, q_{2,(00)}) = 1, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} T_{\mu_0,\mu_1}^{-y}(q_{1,(00)}, q_{2,(01)}) = 1, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} T_{\mu_0,\mu_1}^{-y}(q_{1,(00)}, q_{2,(10)}) = y^{-|\mu_0\setminus\mu_1|}, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} T_{\mu_0,\mu_1}^{-y}(q_{1,(00)}, q_{2,(11)}) = y^{-|\mu_0\setminus\mu_1|}, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} W_{\mu_0,\mu_1,\dots,\mu_N}^{-y}(q_{1,(00)}, q_{2,(00)}) = 1, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} W_{\mu_0,\mu_1,\dots,\mu_N}^{-y}(q_{1,(00)}, q_{2,(01)}) = 1, \\
&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} W_{\mu_0,\mu_1,\dots,\mu_N}^{-y}(q_{1,(00)}, q_{2,(10)}) = 1, \\
,&\lim_{q_2\to+\infty}\lim_{q_1\to+\infty} W_{\mu_0,\mu_1,\dots,\mu_N}^{-y}(q_{1,(00)}, q_{2,(11)}) = 1,
\end{aligned}$$

so that, by putting everything together, we have

$$\begin{aligned}
\sum_{\mathbf{n}} \chi_{-y}^{\mathrm{vir}}\left(\mathrm{Hilb}^{(\hat{\mathbf{n}})}(\mathbb{P}^1\times\mathbb{P}^1)\right)\mathbf{q}^{\mathbf{n}} = &\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}} y^{|\mu_0|-M_0}\right) \\
&\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}} y^{|\mu_0|}\right) \\
&\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}} y^{|\mu_0|-|\mu_0\setminus\mu_1|}\right) \\
&\left(\sum_{\mathbf{n}}\mathbf{q}^{\mathbf{n}}\sum_{\{\mu_i\}} y^{|\mu_0|-|\mu_0\setminus\mu_1|+M_0}\right).
\end{aligned} \tag{4.4.3}$$

5 HIGHER-RANK DT THEORY OF POINTS OF $\mathbb{A}^3$

In this chapter we turn to the study of the K-theoretic Donaldson–Thomas theory of points of $\mathbb{A}^3$. In [14] it is shown that the main player in the theory, the Quot scheme

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) \tag{5.0.1}$$

of length n quotients of the free sheaf $\mathscr{O}^{\oplus r}$, is a global *critical locus*, i.e. it can be realised as $\{df = 0\}$, for f a regular function on a smooth scheme. This structural result, revealing in bright light the symmetries we were talking about, is used to define the *higher rank K-theoretic DT theory of points*. The rank 1 theory, corresponding to $\mathrm{Hilb}^n(\mathbb{A}^3)$, was already defined, and it was solved by Okounkov [184, §3], proving a conjecture by Nekrasov [171]. Our first main result (Thm. 5.1) can be seen as an upgrade of his calculation, completing the study of the degree 0 K-theoretic DT theory of $\mathbb{A}^3$.

In physics, remarkably, the definition of the K-theoretic DT invariants studied here already existed, and gave rise to a conjecture that we prove mathematically in Thm. 5.1. More precisely, the formula for the K-theoretic DT partition function $\mathrm{DT}_r^{\mathrm{K}}$ of $\mathbb{A}^3$ was first conjectured by Nekrasov [171] for $r = 1$ and by Awata and Kanno [11] for arbitrary r as the partition function of a quiver matrix model describing instantons of a topological $\mathrm{U}(r)$ gauge theory on D6 branes.

We also study higher rank *cohomological DT invariants* of $\mathbb{A}^3$. As we show in Corollary 5.22, these can be obtained as a suitable limit of the K-theoretic invariants. Motivated by [11; 172], a closed formula for their generating function $\mathrm{DT}_r^{\mathrm{coh}}$ was conjectured by Szabo [204, Conj. 4.10] as a generalisation of the $r = 1$ case established by Maulik–Nekrasov–Okounkov–Pandharipande [158, Thm. 1]. We prove this conjecture as our Thm. 5.2. To get there, in §5.3 we develop a *higher rank topological vertex* formalism based on the combinatorics of r*-coloured plane partitions*,[1] generalising the classical vertex formalism of [157; 158].

As we mentioned above, the Quot scheme (5.0.1) is a critical locus, thus it carries a natural symmetric ($\mathbb{T}$-equivariant, as we prove) perfect obstruction theory in the sense of Behrend–Fantechi [17; 18]. As we recall in §5.2.1, there is also an induced *twisted virtual structure sheaf* $\widehat{\mathscr{O}}^{\mathrm{vir}} \in K_0^{\mathbb{T}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n))$, which is a twist, by a square root of the virtual canonical bundle, of the ordinary virtual structure sheaf $\mathscr{O}^{\mathrm{vir}}$. The rank r *K-theoretic DT partition function* of the Quot scheme of points of $\mathbb{A}^3$, encoding the rank r K-theoretic DT invariants of $\mathbb{A}^3$, is defined as

$$\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) = \sum_{n \geqslant 0} q^n \chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \widehat{\mathscr{O}}^{\mathrm{vir}}) \in \mathbb{Z}(\!(t, (t_1t_2t_3)^{\frac{1}{2}}, w)\!)[\![q]\!],$$

1 Here, an r-coloured plane partition is an r-tuple of classical plane partitions, see Def. 5.8.

where the half power is caused by the twist by the chosen square root of the virtual canonical bundle (this choice does not affect the invariants, cf. Remark 5.6).

Granting Thm. 5.18, we shall write $\mathsf{DT}_r^K(\mathbb{A}^3, q, t) = \mathsf{DT}_r^K(\mathbb{A}^3, q, t, w)$, ignoring the framing parameters w. In §5.4.2 we determine a closed formula for this series, proving the conjecture by Awata–Kanno [11]. This conjecture was checked for low instanton number in [11, §4].

To state our first main result, we need to recall the definition of the *plethystic exponential*. Given an arbitrary power series

$$f = f(p_1, \dots, p_e; u_1, \dots, u_\ell) \in \mathbb{Q}(p_1, \dots, p_e)[\![u_1, \dots, u_\ell]\!],$$

one sets

$$\operatorname{Exp}(f) = \exp\left(\sum_{n>0} \frac{1}{n} f(p_1^n, \dots, p_e^n; u_1^n, \dots, u_\ell^n)\right), \tag{5.0.2}$$

viewed as an element of $\mathbb{Q}(p_1, \dots, p_e)[\![u_1, \dots, u_\ell]\!]$. Consider, for a formal variable x, the operator $[x] = x^{1/2} - x^{-1/2}$. In §5.4.1 we consider this operator on $K_0^{\mathbb{T}}(\mathrm{pt})$. See §5.6.4 for its physical interpretation. We are now able to state our first main result.

Theorem 5.1 (Thm. 5.17)**.** *The rank r K-theoretic DT partition function of $\mathbb{A}^3$ is given by*

$$\mathsf{DT}_r^K(\mathbb{A}^3, (-1)^r q, t) = \operatorname{Exp}\left(\mathsf{F}_r(q, t_1, t_2, t_3)\right), \tag{5.0.3}$$

where, setting $t = t_1 t_2 t_3$, one defines

$$\mathsf{F}_r(q, t_1, t_2, t_3) = \frac{[t^r]}{[t][t^{\frac{r}{2}} q][t^{\frac{r}{2}} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}.$$

The case $r = 1$ of Thm. 5.1 was proved by Okounkov in [184]. The general case was proposed conjecturally in [11; 22].

It is interesting to notice that Formula (5.0.3) is equivalent to the product decomposition

$$\mathsf{DT}_r^K(\mathbb{A}^3, (-1)^r q, t) = \prod_{i=1}^r \mathsf{DT}_1^K(\mathbb{A}^3, -q t^{\frac{-r-1}{2}+i}, t), \tag{5.0.4}$$

that we obtain in Thm. 5.21. This is precisely the product formula [177, Formula (3.14)] appearing as a limit of the (conjectural) 4-fold theory developed by Nekrasov and Piazzalunga.

In §5.5 we study the generating function of *cohomological DT invariants*, which is defined as

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \sum_{n \geqslant 0} q^n \int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)]^{\mathrm{vir}}} 1 \in \mathbb{Q}(\!(s, v)\!)[\![q]\!]$$

where $s = (s_1, s_2, s_3)$ and $v = (v_1, \dots, v_r)$, with $s_i = c_1^{\mathbb{T}}(t_i)$ and $v_j = c_1^{\mathbb{T}}(w_j)$ respectively, and the integral is defined in Eq. (5.2.2) via $\mathbb{T}$-equivariant residues. It is a consequence of Thm. 5.18 that $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v)$ does not depend on v, so we will shorten it as $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s)$. In §5.5.1 we explain how to recover the cohomological invariants out of the K-theoretic ones. This is the limit formula (Corollary 5.22)

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s) = \lim_{b\to 0} \mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, e^{bs}),$$

essentially a formal consequence of our explicit expression for the K-theoretic *higher rank vertex* (cf. §5.3) attached to the Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$.

Theorem 5.2 (Thm. 5.23)**.** *The rank r cohomological DT partition function of $\mathbb{A}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s) = \mathsf{M}((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}},$$

where $\mathsf{M}(t) = \prod_{m\geqslant 1}(1-t^m)^{-m}$ is the MacMahon function.

The case $r = 1$ of Thm. 5.2 was proved by Maulik–Nekrasov–Okounkov–Pandharipande [158, Thm. 1]. Thm. 5.2 was conjectured by Szabo in [204] and confirmed for $r \leqslant 8$ and $n \leqslant 8$ in [22]. The specialisation

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s)\big|_{s_1+s_2+s_3=0} = \mathsf{M}((-1)^r q)^r$$

was already computed in physics [73].

In §5.6 we define the *virtual chiral elliptic genus* for any scheme with a perfect obstruction theory, which recovers as a special case the virtual elliptic genus defined in [95]. By means of this new invariant we introduce a refinement $\mathsf{DT}_r^{\mathrm{ell}}$ of the generating series $\mathsf{DT}_r^{\mathrm{K}}$, providing a mathematical definition of the *elliptic DT invariants* studied in [22]. We propose a conjecture (Conjecture 5.32) about the behaviour of $\mathsf{DT}_r^{\mathrm{ell}}$ and, granting this conjecture, we obtain a proof of a conjecture formulated by Benini–Bonelli–Poggi–Tanzini (Thm. 5.33).

5.1 THE LOCAL QUOT SCHEME: CRITICAL AND EQUIVARIANT STRUCTURE

5.1.1 Overview

In this section we start working on the local Calabi–Yau 3-fold $\mathbb{A}^3$. Fix integers $r \geqslant 1$ and $n \geqslant 0$. Our focus will be on the *local Quot scheme*

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n),$$

whose points correspond to short exact sequences

$$0 \to S \to \mathscr{O}^{\oplus r} \to T \to 0$$

where T is a 0-dimensional $\mathscr{O}_{\mathbb{A}^3}$-module with $\chi(T) = n$.

We shall use the following notation throughout.

Notation. *If* F *is a locally free sheaf on a variety* X*, and* $F \twoheadrightarrow T$ *is a surjection onto a* 0*-dimensional sheaf of length* n*, with kernel* $S \subset F$*, we denote by*

$$[S] \in \mathrm{Quot}_X(F, n)$$

the corresponding point in the Quot scheme.

In this section, we will:

- recall from [14] the description of the Quot scheme as a critical locus (§5.1.2),
- describe a $\mathbb{T}$-action (for $\mathbb{T} = (\mathbb{C}^*)^3 \times (\mathbb{C}^*)^r$ a torus of dimension $3+r$) on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, with isolated fixed locus consisting of direct sums of monomial ideals (§5.1.3),
- reinterpret the fixed locus $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$ in terms of coloured partitions (§5.1.4),
- prove that the critical perfect obstruction theory on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ is $\mathbb{T}$-equivariant (Lemma 5.10), and that the induced $\mathbb{T}$-fixed obstruction theory on the fixed locus is trivial (Corollary 5.12).

The content of this section is the starting point for the definition (see §5.2.3) of virtual invariants on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, as well as our construction (see §5.3) of the *higher rank vertex formalism*.

5.1.2 The critical structure on the Quot scheme

Let V be an n-dimensional complex vector space. Consider the space $R_{r,n} = \mathrm{Rep}_{(n,1)}(\widetilde{L}_3)$ of r-framed $(n,1)$-dimensional representations of the 3-loop quiver L_3, depicted in Fig. 5.1.1. The notation "$(n,1)$" means that the main vertex carries a copy of V, whereas the framing vertex carries a copy of $\mathbb{C}$.

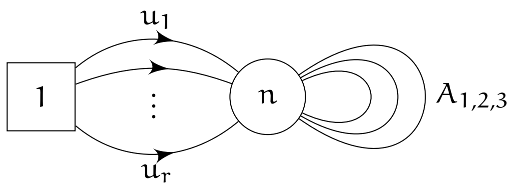


Figure 5.1.1: *The* r*-framed 3-loop quiver* $\widetilde{L}_3$*.*

We have that $R_{r,n}$ is an affine space of dimension $3n^2 + rn$, with an explicit description as

$$\begin{aligned} R_{r,n} &= \left\{ (A_1, A_2, A_3, u_1, \ldots, u_r) | A_j \in \mathrm{End}(V),\ u_i \in V \right\} \\ &= \mathrm{End}(V)^{\oplus 3} \oplus V^{\oplus r}. \end{aligned}$$

By [14, Prop. 2.4], there exists a stability parameter θ on the 3-loop quiver such that θ-stable framed representations $(A_1, A_2, A_3, u_1, \ldots, u_r) \in R_{r,n}$ are precisely those satisfying the condition:

$$\text{the vectors } u_1, \ldots, u_r \in V \text{ jointly generate } (A_1, A_2, A_3) \in \mathrm{Rep}_n(L_3).$$

Imposing this stability condition on $R_{r,n}$ we obtain an open subscheme

$$U_{r,n} \subset R_{r,n}$$

on which $\mathrm{GL}(V)$ acts freely by the rule

$$g \cdot (A_1, A_2, A_3, u_1, \ldots, u_r) = (A_1^g, A_2^g, A_3^g, gu_1, \ldots, gu_r),$$

where $A_i^g = gA_ig^{-1}$ denotes conjugation by $g \in \mathrm{GL}(V)$. The quotient

$$\mathsf{NCQuot}_r^n = U_{r,n}/\mathrm{GL}(V) \tag{5.1.1}$$

is a smooth quasi-projective variety of dimension $2n^2 + rn$. In [14] the scheme NCQuot_r^n is referred to as the *non-commutative Quot scheme*, by analogy with the *non-commutative Hilbert scheme* [182], i.e. the moduli space of left ideals of codimension n in $\mathbb{C}\langle x_1, x_2, x_3\rangle$ (which of course exists for an arbitrary number of free variables).

On $R_{r,n}$ one can define the function

$$h_n : R_{r,n} \to \mathbb{A}^1, \quad (A_1, A_2, A_3, u_1, \ldots, u_r) \mapsto \operatorname{tr} A_1[A_2, A_3],$$

induced by the superpotential $W = A_1[A_2, A_3]$ on the 3-loop quiver. Note that this function

- is symmetric under cyclic permutations of A_1, A_2 and A_3, and
- does not touch the vectors u_i, which are only used to define its domain.

Moreover, $h_n|_{U_{r,n}}$ is $\mathrm{GL}(V)$-invariant, and thus descends to a regular function

$$f_n \colon \mathsf{NCQuot}_r^n \to \mathbb{A}^1. \tag{5.1.2}$$

Proposition 5.3 ([14, Thm. 2.6])**.** *There is an identity of closed subschemes*

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) = \operatorname{crit}(f_n) \subset \mathsf{NCQuot}_r^n.$$

In particular, $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ *carries a symmetric perfect obstruction theory.*

We use the notation $\operatorname{crit}(f)$ for the zero scheme $\{df = 0\}$, for f a function on a smooth scheme. The embedding of the Quot scheme inside a non-commutative quiver model had appeared (conjecturally, and in a slightly different language) in the physics literature [73].

Every critical locus $\operatorname{crit}(f)$ has a canonical symmetric obstruction theory, determined by the Hessian complex attached to the function f. It will be referred to as the *critical obstruction theory* throughout. In the case of $Q = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$, this symmetric obstruction theory is the morphism

$$\begin{tikzcd}
\mathbb{E}_{\mathrm{crit}} = \big[T_{\mathsf{NCQuot}_r^n}|_Q \arrow[r, "\mathsf{Hess}(f_n)"] \arrow[d, "(df_n)^\vee|_Q"'] & \Omega_{\mathsf{NCQuot}_r^n}|_Q \big] \arrow[d, "\mathrm{id}"] \\
\mathbb{L}_Q = \big[\mathscr{I}/\mathscr{I}^2 \arrow[r, "d"] & \Omega_{\mathsf{NCQuot}_r^n}|_Q \big]
\end{tikzcd} \tag{5.1.3}$$

in $D^{[-1,0]}(Q)$, where we represented the truncated cotangent complex by means of the exterior derivative d constructed out of the ideal sheaf $\mathscr{I} \subset \mathscr{O}_{\mathrm{NCQuot}_r^n}$ of the inclusion $Q \hookrightarrow \mathrm{NCQuot}_r^n$.

Remark 5.1. As proved by Cazzaniga-Ricolfi in [64], for any integer $m \geqslant 3$, the Quot scheme $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r}, n)$ is canonically isomorphic to the moduli space of *framed sheaves* on $\mathbb{P}^m$, i.e. the moduli space of pairs (E, φ) where E is a torsion free sheaf on $\mathbb{P}^m$ with Chern character $(r, 0, \ldots, 0, -n)$ and $\varphi \colon E|_D \sim \mathscr{O}_D^{\oplus r}$ is an isomorphism, for $D \subset \mathbb{P}^m$ a fixed hyperplane. ◂

5.1.3 Torus actions on the local Quot scheme

In this section we define a torus action on the Quot scheme. Set

$$\mathbb{T}_1 = (\mathbb{C}^*)^3, \quad \mathbb{T}_2 = (\mathbb{C}^*)^r, \quad \mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2. \tag{5.1.4}$$

The torus $\mathbb{T}_1$ acts on $\mathbb{A}^3$ by the standard action

$$t \cdot x_i = t_i x_i, \tag{5.1.5}$$

and this action lifts to an action on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$. At the same time, the torus $\mathbb{T}_2 = (\mathbb{C}^*)^r$ acts on the Quot scheme by scaling the fibres of $\mathscr{O}^{\oplus r}$. Thus we obtain a $\mathbb{T}$-action on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$.

Remark 5.2. The fixed locus $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}_1}$ is proper. Indeed, a $\mathbb{T}_1$-invariant surjection $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ necessarily has the quotient T entirely supported at the origin $0 \in \mathbb{A}^3$. Hence

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}_1} \hookrightarrow \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)_0$$

sits inside the *punctual Quot scheme* as a closed subscheme. But $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)_0$ is proper, since it is a fibre of the Quot-to-Chow morphism $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) \to \mathrm{Sym}^n \mathbb{A}^3$, which is a proper morphism. ◂

We recall, verbatim from [14, Lemma 2.10], the description of the full $\mathbb{T}$-fixed locus induced by the product action on the local Quot scheme.

Lemma 5.4. *There is an isomorphism of schemes*

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}} = \coprod_{n_1 + \cdots + n_r = n} \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(\mathbb{A}^3)^{\mathbb{T}_1}. \tag{5.1.6}$$

In particular, the $\mathbb{T}$-fixed locus is isolated and compact. Moreover, letting $\mathbb{T}_0 \subset \mathbb{T}$ be the subtorus defined by $t_1 t_2 t_3 = 1$, one has a scheme-theoretic identity

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}_0} = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}. \tag{5.1.7}$$

Proof. The main result proved by Bifet in [32] (in greater generality) implies that

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}_2} = \coprod_{n_1 + \cdots + n_r = n} \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(\mathbb{A}^3). \tag{5.1.8}$$

The isomorphism (5.1.6) follows by taking $\mathbb{T}_1$-invariants. Since $\mathrm{Hilb}^k(\mathbb{A}^3)^{\mathbb{T}_1}$ is isolated (a disjoint union of reduced points, each corresponding to a monomial ideal of colength k), the first claim follows. Let now $\mathbb{T}_0 \subset \mathbb{T}_1$ be the subtorus defined by $t_1t_2t_3 = 1$, so that $\mathbb{T}_0 = \mathbb{T}_0 \times \mathbb{T}_2$. Eq. (5.1.7) follows combining Eq. (5.1.8) and the isomorphism $\mathrm{Hilb}^k(\mathbb{A}^3)^{\mathbb{T}_1} \cong \mathrm{Hilb}^k(\mathbb{A}^3)^{\mathbb{T}_0}$ proved in [18, Lemma 4.1]. ■

Remark 5.3. The $\mathbb{T}$-action on $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ just described can be seen as the restriction of a $\mathbb{T}$-action on the larger space NCQuot_r^n. Indeed, a torus element $\mathbf{t} = (t_1, t_2, t_3, w_1, \ldots, w_r) \in \mathbb{T}$ acts on $(A_1, A_2, A_3, u_1, \ldots, u_r) \in \mathsf{NCQuot}_r^n$ by

$$\mathbf{t} \cdot P = (t_1A_1, t_2A_2, t_3A_3, w_1u_1, \ldots, w_ru_r). \tag{5.1.9}$$

The critical locus $\mathrm{crit}(f_n)$ is $\mathbb{T}$-invariant, and the induced action is precisely the one we described earlier in this section. ◀

5.1.4 Combinatorial description of the $\mathbb{T}$-fixed locus

The $\mathbb{T}$-fixed locus $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$ is described purely in terms of r-*coloured plane partitions* of size n, as we now explain.

We first recall the definition of a partition of arbitrary dimension.

Definition 5.5. *Let $d \geqslant 1$ and $n \geqslant 0$ be integers. A $(d-1)$-dimensional partition of n is a collection of n points $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d_{\geqslant 0}$ with the following property: if $\mathbf{a}_i = (a_{i1}, \ldots, a_{id}) \in \mathcal{A}$, then whenever a point $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{Z}^d_{\geqslant 0}$ satisfies $0 \leqslant y_j \leqslant a_{ij}$ for all $j = 1, \ldots, d$, one has $\mathbf{y} \in \mathcal{A}$. The integer $n = |\mathcal{A}|$ is called the* size *of the partition.*

There is a bijective correspondence between the sets of

- $(d-1)$-dimensional partitions of size n,
- $(\mathbb{C}^*)^d$-fixed points of $\mathrm{Hilb}^n(\mathbb{A}^d)$, and
- monomial ideals $I \subset \mathbb{C}[x_1, \ldots, x_d]$ of colength n.

We will be interested in the case $d = 3$, corresponding by definition to *plane partitions*. These can be visualised (cf. Fig. 5.1.2) as configurations of n boxes stacked in the corner of a room (with gravity pointing in the $(-1,-1,-1)$ direction).

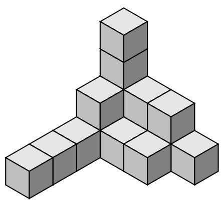


Figure 5.1.2: *A plane partition of size $n = 16$.*

Example 5.6. *If $\mathcal{A}$ is a $(d-1)$-dimensional partition of size n as in Def. 5.5, the associated monomial ideal is*

$$I_{\mathcal{A}} = \left\langle x_1^{i_1} \cdots x_d^{i_d} \,|\, (i_1, \ldots, i_d) \in \mathbb{Z}^d_{\geqslant 0} \setminus \mathcal{A} \right\rangle \subset \mathbb{C}[x_1, \ldots, x_d].$$

For instance, if $d = 3$*, in the case of the plane partition pictured in Fig. 5.1.2, the associated monomial ideal of colength* 16 *is generated by the monomials shaping the staircase of the partition, and is thus equal to*

$$\left\langle x_3^4, x_1x_3^2, x_1^2x_3, x_1^5, x_1^2x_2, x_1x_2x_3, x_2x_3^2, x_1x_2^3, x_2^3x_3, x_2^4 \right\rangle \subset \mathbb{C}[x_1, x_2, x_3].$$

Here is an alternative definition of plane partitions.

Definition 5.7. *A (finite)* plane partition *is a sequence* $\pi = \left\{\pi_{ij} | i, j \geqslant 0\right\} \subset \mathbb{Z}_{\geqslant 0}$ *such that* $\pi_{ij} = 0$ *for* $i, j \gg 0$ *and*

$$\pi_{ij} \geqslant \pi_{i+1,j}, \quad \pi_{ij} \geqslant \pi_{i,j+1} \quad \textit{for all } i, j \geqslant 0.$$

Definition 5.8. *An* r-coloured plane partition *is a tuple* $\overline{\pi} = (\pi_1, \ldots, \pi_r)$*, where each* π_i *is a plane partition.*

Denote by

$$|\pi| = \sum_{i,j \geqslant 0} \pi_{ij}$$

the *size* of a plane partition (i.e. the number n in Def. 5.5) and by $|\overline{\pi}| = \sum_{i=1}^{r} |\pi_i|$ the size of an r-coloured plane partition.

In the light of Def. 5.7, the monomial ideal associated to a plane partition π is

$$I_\pi = \left\langle x_1^i x_2^j x_3^{\pi_{ij}} | i, j \geqslant 0 \right\rangle \subset \mathbb{C}[x_1, x_2, x_3].$$

It is clear that the colength of the ideal I_π is $|\pi|$.

Remark 5.4. A general plane partition may have infinite legs, each shaped by (i.e. asymptotic to) a standard (1-dimensional) partition, or Young diagram. We are not concerned with infinite plane partitions here, since we only deal with quotients $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ with finite support. ◀

Proposition 5.9. *There is a bijection between* $\mathbb{T}$*-fixed points* $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$ *and* r*-coloured plane partitions* $\overline{\pi}$ *of size* n*.*

Proof. For $r = 1$ this is well known: as we recalled above, monomial ideals $I \subset \mathbb{C}[x_1, x_2, x_3]$ are in bijective correspondence with plane partitions. Similarly, to each r-coloured plane partition $\overline{\pi} = (\pi_1, \ldots, \pi_r)$ there corresponds a subsheaf $S_{\overline{\pi}} = \bigoplus_{i=1}^{r} I_{\pi_i} \subset \mathscr{O}^{\oplus r}$. But these are the $\mathbb{T}$-fixed points of the Quot scheme by Lemma 5.4. ∎

Computing the trace of a monomial ideal

Recall the map (1.3.1) sending a torus representation to its weight space decomposition. Consider the 3-dimensional torus $\mathbb{T}_1$ acting on the coordinate ring $R = \mathbb{C}[x_1, x_2, x_3]$ of $\mathbb{A}^3$. Then we have

$$\mathrm{tr}_R = \sum_{\square \in \mathbb{Z}^3_{\geqslant 0}} t^{\square} = \sum_{(i,j,k) \in \mathbb{Z}^3_{\geqslant 0}} t_1^i t_2^j t_3^k = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}.$$

For a cyclic monomial ideal $\mathfrak{m}_{abc} = x_1^a x_2^b x_3^c \cdot R \subset R$, one has

$$\mathrm{tr}_{\mathfrak{m}_{abc}} = \sum_{i\geqslant a}\sum_{j\geqslant b}\sum_{k\geqslant c} t_1^i t_2^j t_3^k = \frac{t_1^a t_2^b t_3^c}{(1-t_1)(1-t_2)(1-t_3)}.$$

More generally, for a monomial ideal $I_\pi \subset \mathbb{C}[x_1,x_2,x_3]$, one has

$$\mathrm{tr}_{I_\pi} = \sum_{(i,j,k)\notin\pi} t_1^i t_2^j t_3^k. \tag{5.1.10}$$

These are the building blocks needed to compute tr_S for an arbitrary sheaf $S = \bigoplus_{i=1}^r I_{\pi_i}$ corresponding to a $\mathbb{T}$-fixed point $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$.

5.1.5 The $\mathbb{T}$-fixed obstruction theory

Recall from [112, Prop. 1] that a torus equivariant obstruction theory on a scheme Y induces a canonical perfect obstruction theory, and hence a virtual fundamental class, on each component of the torus fixed locus. In this subsection we show that the reduced isolated locus

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}} \hookrightarrow \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$$

carries the trivial $\mathbb{T}$-fixed perfect obstruction theory, so the induced virtual fundamental class agrees with the actual (0-dimensional) fundamental class.

We first need to check the equivariance (cf. §1.5) of the critical obstruction theory $\mathbb{E}_{\mathrm{crit}}$ obtained in Prop. 5.3. In fact, this follows from the general fact that the critical obstruction theory on $\mathrm{crit}(f) \subset Y$, for f a function on a smooth scheme Y, acted on by an algebraic torus $\mathbb{T}$, is naturally $\mathbb{T}$-equivariant as soon as f is $\mathbb{T}$-homogeneous. However, for the sake of completeness, we include a direct proof below for the case at hand.

Lemma 5.10. *The critical obstruction theory on $Q = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ is $\mathbb{T}$-equivariant.*

Proof. We start with two observations:

1. The potential $f_n = \mathrm{tr}\, A_1[A_2, A_3]$ recalled in (5.1.2) is homogeneous (of degree 3) in the matrix coordinates of the non-commutative Quot scheme.

2. The potential f_n satisfies the relation

$$f_n(t\cdot P) = t_1t_2t_3 \cdot f_n(P) \tag{5.1.11}$$

for every $t = (t_1,t_2,t_3) \in \mathbb{T}_1$ and $P \in \mathrm{NCQuot}_r^n$.

Fix a point $p \in Q = \mathrm{crit}(f_n) \subset \mathrm{NCQuot}_r^n$. Then, setting $N = 2n^2 + rn = \dim \mathrm{NCQuot}_r^n$, let $x_1,\ldots,x_N$ be local holomorphic coordinates of NCQuot_r^n around p. Let the torus $\mathbb{T}$ act on these coordinates as prescribed by Eq. (5.1.9), i.e. t_1 (resp. t_2 and t_3) rescales each x_k corresponding to the entries of the first (resp. second and third) matrix, and w_l rescales the coordinates of the vector u_l, for $l = 1, \ldots, r$. Formally, for a matrix coordinate x_k, we set

$$(t_1, t_2, t_3, w_1, \ldots, w_r) \cdot x_k = t_{\ell(k)} x_k$$

where $\ell(k) \in \{1, 2, 3\}$ depends on whether x_k comes from an entry of A_1, A_2 or A_3. We also have to prescribe an action on tangent vectors and 1-forms. For a matrix coordinate x_k, we set

$$\begin{aligned} (t_1, t_2, t_3, w_1, \ldots, w_r) \cdot \frac{\partial}{\partial x_k} &= \frac{t_1 t_2 t_3}{t_{\ell(k)}} \frac{\partial}{\partial x_k} \\ (t_1, t_2, t_3, w_1, \ldots, w_r) \cdot dx_k &= t_{\ell(k)} dx_k. \end{aligned} \tag{5.1.12}$$

If x_k comes from a vector component of the l-th vector, we set

$$\begin{aligned} (t_1, t_2, t_3, w_1, \ldots, w_r) \cdot \frac{\partial}{\partial x_k} &= w_l^{-1} \frac{\partial}{\partial x_k} \\ (t_1, t_2, t_3, w_1, \ldots, w_r) \cdot dx_k &= w_l dx_k. \end{aligned} \tag{5.1.13}$$

However, the $\mathbb{T}_2$-action (5.1.13) will be invisible in the Hessian since the function f_n does not touch the vectors.

The Hessian can be seen as a section

$$\mathsf{Hess}(f_n) \in \Gamma\left(Q, T^*_{\mathsf{NCQuot}^n_r}|_Q \otimes T^*_{\mathsf{NCQuot}^n_r}|_Q\right).$$

In checking the equivariance relation

$$\mathbf{t} \cdot \mathsf{Hess}(f_n)(\xi) = \mathsf{Hess}(f_n)(\mathbf{t} \cdot \xi), \quad \mathbf{t} \in \mathbb{T},$$

we may ignore local coordinates x_k corresponding to vector entries, because the Hessian is automatically equivariant in these coordinates (equivariance translates into the identity $0 = 0$).

So, let us fix an x_k coming from one of the matrices. The (i, j)-component of the Hessian applied to $\partial/\partial x_k$ is given by

$$\mathsf{Hess}_{ij}(f_n)\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x_1, \ldots, x_N)\, dx_j.$$

This will vanish unless $k \in \{i, j\}$. Without loss of generality we may assume $k = i$. In this case we obtain, up to a sign convention,

$$\mathsf{Hess}_{ij}(f_n)\left(\frac{t_1 t_2 t_3}{t_{\ell(k)}} \frac{\partial}{\partial x_k}\right) = \frac{t_1 t_2 t_3}{t_{\ell(k)}} \frac{\partial^2 f_n}{\partial x_k \partial x_j}(x_1, \ldots, x_N)\, dx_j. \tag{5.1.14}$$

On the other hand, combining the observations (1) and (2) with (5.1.12), we obtain

$$\begin{aligned} \mathbf{t} \cdot \mathsf{Hess}_{ij}(f_n)\left(\frac{\partial}{\partial x_k}\right) &= \frac{\partial^2 f_n}{(\partial t_{\ell(k)} x_k)(\partial t_{\ell(j)} x_j)}(t_{\ell(1)} x_1, \ldots, t_{\ell(N)} x_N) t_{\ell(j)} dx_j \\ &= \frac{t_1 t_2 t_3}{t_{\ell(k)} t_{\ell(j)}} t_{\ell(j)} \frac{\partial^2 f_n}{\partial x_k \partial x_j}(x_1, \ldots, x_N)\, dx_j, \end{aligned}$$

which agrees with the right hand side of Eq. (5.1.14). Thus we conclude that the Hessian complex is $\mathbb{T}$-equivariant, as well as the morphism (5.1.3) to the cotangent complex. This finishes the proof. ∎

The property (5.1.11) of f_n exhibits the differential df_n as a GL_3-equivariant section

$$df_n \otimes t^{-1} \colon \mathscr{O}_{\mathrm{NCQuot}_r^n} \to \Omega_{\mathrm{NCQuot}_r^n} \otimes t^{-1},$$

where $t^{-1} = (t_1t_2t_3)^{-1}$ is the determinant representation of $\mathbb{C}^3 = \bigoplus_{1 \leqslant i \leqslant 3} t_i^{-1} \cdot \mathbb{C}$. Therefore, explicitly, the morphism in $D^{[-1,0]}(\mathrm{Coh}_Q^{\mathbb{T}})$ lifting the critical obstruction theory (5.1.3) is

$$\begin{tikzcd}
\left[t \otimes T_{\mathrm{NCQuot}_r^n}\big|_Q \arrow[r, "\mathsf{Hess}(f_n)"] \arrow[d, "(df_n)^\vee|_Q"'] & \Omega_{\mathrm{NCQuot}_r^n}\big|_Q\right] \arrow[d, "\mathrm{id}"] \\
\left[\mathscr{I}/\mathscr{I}^2 \arrow[r, "d"] & \Omega_{\mathrm{NCQuot}_r^n}\big|_Q\right]
\end{tikzcd} \tag{5.1.15}$$

so that, in particular, the equivariant K-theory class of the virtual tangent bundle attached to the (equivariant) perfect obstruction theory (5.1.15) is

$$T_Q^{\mathrm{vir}} = T_{\mathrm{NCQuot}_r^n}\big|_Q - \Omega_{\mathrm{NCQuot}_r^n}\big|_Q \otimes t^{-1} \in K_{\mathbb{T}}^0(Q). \tag{5.1.16}$$

This fact will be recalled and used in Prop. 5.15.

Lemma 5.10 implies the existence of a "$\mathbb{T}$-fixed" obstruction theory

$$\mathbb{E}_{\mathrm{crit}}\big|^{\mathrm{fix}}_{\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \to \mathbb{L}_{\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \tag{5.1.17}$$

on the fixed locus of the Quot scheme. We proved in Lemma 5.4 that this fixed locus is 0-dimensional, isolated and reduced. The next result will imply that the virtual fundamental class induced by (5.1.17) on the fixed locus agrees with the actual fundamental class.

Proposition 5.11. *Let* $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}$ *be a torus fixed point. The deformations and obstructions of* $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$ *at* $[S]$ *are entirely* $\mathbb{T}$*-movable. In particular, the virtual tangent space at* $[S]$ *can be written as*

$$T_S^{\mathrm{vir}} = \mathbb{E}_{\mathrm{crit}}^\vee\big|_{[S]}^{\mathrm{mov}} = T_Q\big|_{[S]} - \Omega_Q\big|_{[S]} \otimes t^{-1} \in K_{\mathbb{T}}^0(\mathrm{pt}). \tag{5.1.18}$$

Proof. The perfect obstruction theory $\mathbb{E}_{\mathrm{crit}}$ on $Q = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$, made explicit in Diagram (5.1.15), satisfies $\mathbb{E}_{\mathrm{crit}} \cong \mathbb{E}_{\mathrm{crit}}^\vee[1] \otimes t$. Its equivariant K-theory class is therefore

$$\mathbb{E}_{\mathrm{crit}} = \Omega_Q - T_Q \otimes t \in K_0^{\mathbb{T}}(Q).$$

We know by Eq. (5.1.7) in Lemma 5.4 that no power of t is a weight of $T_Q|_p$ for any fixed point $p \in Q^{\mathbb{T}}$, which implies that

$$\left(T_Q\big|_p \otimes t\right)^{\mathrm{fix}} = 0, \quad \Omega_Q\big|_p^{\mathrm{fix}} = 0. \tag{5.1.19}$$

The claim follows. ∎

Corollary 5.12. *There is an identity*

$$\left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}\right]^{\mathrm{vir}} = \left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}\right] \in A_0^{\mathbb{T}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}).$$

5.2 INVARIANTS ATTACHED TO THE LOCAL QUOT SCHEME

In this section we introduce cohomological and K-theoretic DT invariants of $\mathbb{A}^3$, starting from the Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$ studied in the previous section. We first need to introduce some notation and terminology.

5.2.1 Some notation

Recall the tori $\mathbb{T}_1 = (\mathbb{C}^*)^3$ and $\mathbb{T}_2 = (\mathbb{C}^*)^r$ from (5.1.4). We let t_1, t_2, t_3 and $w_1, \ldots, w_r$ be the generators of the representation rings $K^0_{\mathbb{T}_1}(\mathrm{pt})$ and $K^0_{\mathbb{T}_2}(\mathrm{pt})$, respectively. Then one can write the equivariant cohomology rings of $\mathbb{T}_1$ and $\mathbb{T}_2$ as

$$H^*_{\mathbb{T}_1} = \mathbb{Q}[s_1,s_2,s_3], \quad H^*_{\mathbb{T}_2} = \mathbb{Q}[v_1,\ldots,v_r],$$

where $s_i = c_1^{\mathbb{T}_1}(t_i)$ and $v_j = c_1^{\mathbb{T}_2}(w_j)$. For a virtual $\mathbb{T}$-module $V \in K^0_{\mathbb{T}}(\mathrm{pt})$, we let

$$\mathrm{tr}_V \in \mathbb{Z}(\!(t_1,t_2,t_3,w_1,\ldots,w_r)\!)$$

denote its character, i.e. its decomposition into weight spaces. We denote by $\overline{(\,\cdot\,)}$ the involution defined on $\mathbb{Z}(\!(t_1,t_2,t_3,w_1,\ldots,w_r)\!)$ by

$$\overline{P}(t_1,t_2,t_3,w_1,\ldots,w_r) = P(t_1^{-1},t_2^{-1},t_3^{-1},w_1^{-1},\ldots,w_r^{-1}).$$

Twisted virtual structure sheaf

For any scheme X endowed with a perfect obstruction theory $\mathbb{E} \to \mathbb{L}_X$, define as in [95, Def. 3.12], the *virtual canonical bundle*

$$\mathcal{K}_{X,\mathrm{vir}} = \det \mathbb{E} = \det(T_X^{\mathrm{vir}})^{\vee}.$$

This is just $\det E^0 \otimes (\det E^{-1})^{\vee}$ if $\mathbb{E} = E^0 - E^{-1} \in K^0(X)$. We will simply write $\mathcal{K}_{\mathrm{vir}}$ when X is clear from the context.

Lemma 5.13. *Let A be a smooth variety equipped with a regular function $f\colon A \to \mathbb{A}^1$, and let $X = \mathrm{crit}(f) \subset A$ be the critical locus of f, with its critical (symmetric) perfect obstruction theory $\mathbb{E}_{\mathrm{crit}} \to \mathbb{L}_X$. Then $\mathcal{K}_{X,\mathrm{vir}} \in \mathrm{Pic}(X)$ admits a square root, i.e. there exists a line bundle*

$$\mathcal{K}_{X,\mathrm{vir}}^{\frac{1}{2}} \in \mathrm{Pic}(X)$$

whose second tensor power equals $\det \mathbb{E}_{\mathrm{crit}}$.

Proof. The K-theory class of the critical perfect obstruction theory is

$$\mathbb{E}_{\mathrm{crit}} = \Omega_A\big|_X - T_A\big|_X,$$

and by definition one has

$$\mathcal{K}_{X,\mathrm{vir}} = \frac{\det \Omega_A|_X}{\det T_A|_X} = \frac{\det \Omega_A|_X}{(\det \Omega_A|_X)^{-1}} = K_A\big|_X \otimes K_A\big|_X. \qquad \blacksquare$$

Let X be a scheme endowed with a perfect obstruction theory, and let $\mathcal{O}_X^{\mathrm{vir}} \in K_0(X)$ be the induced virtual structure sheaf. Assume the virtual canonical bundle admits a square root. Following [176], we define the *twisted* (or *symmetrised*) *virtual structure sheaf* as

$$\widehat{\mathcal{O}}_X^{\mathrm{vir}} = \mathcal{O}_X^{\mathrm{vir}} \otimes \mathcal{K}_{X,\mathrm{vir}}^{\frac{1}{2}}.$$

In case X carries a torus action, we will see in Remark 5.5 that $\widehat{\mathcal{O}}_X^{\mathrm{vir}}$ acquires a canonical weight.

5.2.2 Classical enumerative invariants

Degree 0 Donaldson–Thomas invariants of various flavours have been computed in [16; 18; 63; 115; 150; 151; 158; 192]. This work is concerned with the general theory of *Quot schemes*, hence in the (virtual) enumeration of 0-dimensional quotients of locally free sheaves on 3-folds.

The naive (topological) Euler characteristic of the Quot scheme is computed via the Gholampour–Kool formula [107, Prop. 2.3]

$$\sum_{n\geqslant 0} e(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n))q^n = M(q)^r,$$

where $M(q) = \prod_{m\geqslant 1}(1-q^m)^{-m}$ is the MacMahon function, the generating function for the number of plane partitions of non-negative integers. On the other hand, the Behrend weighted Euler characteristic of the Quot scheme can be computed via the formula

$$\sum_{n\geqslant 0} e_{\mathrm{vir}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n))q^n = M((-1)^r q)^r, \tag{5.2.1}$$

proved in [14, Cor. 2.8]. For a complex scheme Z of finite type over $\mathbb{C}$, we have set $e_{\mathrm{vir}}(Z) = e(Z, \nu_Z)$, where ν_Z is Behrend's constructible function [15]. See [14, Thm. A] for a proof of the analogue of (5.2.1) for an arbitrary pair (Y, F) consisting of a smooth 3-fold Y along with a locally free sheaf F on it. It was shown by Toda [15] that, on a projective Calabi–Yau 3-fold Y, the wall-crossing factor in the higher rank DT/PT correspondence is precisely $M((-1)^r q)^{re(Y)}$. The relationship between Toda's wall-crossing formula [211] and the Gholampour–Kool's formula for Euler characteristics of Quot schemes on 3-folds [107] was clarified in [14] via a Hall algebra argument.

5.2.3 Virtual invariants of the Quot scheme

The scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ is not proper, but carries a torus action with proper fixed locus. Thus we may define virtual invariants via equivariant residues, by setting

$$\int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)]^{\mathrm{vir}}} 1 := \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \frac{1}{e^{\mathbb{T}}(T_S^{\mathrm{vir}})} \in \mathbb{Q}(\!(s,v)\!), \tag{5.2.2}$$

where $s = (s_1, s_2, s_3)$ and $v = (v_1, \ldots, v_r)$ are the equivariant parameters of the torus $\mathbb{T}$ and T_S^{vir} is the virtual tangent space (5.1.18). The sum runs over all $\mathbb{T}$-fixed points $[S]$, which are isolated, reduced and with the trivial perfect obstruction theory induced from the critical obstruction theory on the Quot scheme (cf. Corollary 5.12). We refer to these invariants as (degree 0) *cohomological rank* r *DT invariants*, as they take value in (an extension of) the fraction field $\mathbb{Q}(s,v)$ of the $\mathbb{T}$-equivariant cohomology ring $H^*_{\mathbb{T}}$. We will study their generating function

$$\mathrm{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \sum_{n\geqslant 0} q^n \int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)]^{\mathrm{vir}}} 1 \in \mathbb{Q}(\!(s,v)\!)[\![q]\!] \tag{5.2.3}$$

in §5.5. On the other hand, K-theoretic invariants arise as natural refinements of their cohomological counterpart. Naively, one would like to study the virtual holomorphic Euler characteristic

$$\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \mathscr{O}^{\mathrm{vir}}) = \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \mathrm{tr}\left(\frac{1}{\Lambda^\bullet T_S^{\mathrm{vir},\vee}}\right) \in \mathbb{Z}(\!(t,w)\!),$$

where $t = (t_1, t_2, t_3)$, $w = (w_1, \ldots, w_r)$, and via the trace map tr we identify a (possibly infinite-dimensional) virtual $\mathbb{T}$-module with its decomposition into weight spaces. It turns out that guessing a closed formula for these invariants is incredibly difficult and, after all, not what one should look at. Instead, Nekrasov–Okounkov [176] teach us that we should focus our attention on

$$\begin{aligned}\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \widehat{\mathscr{O}}^{\mathrm{vir}}) &= \\ &= \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \mathrm{tr}\left(\frac{\mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}}{\Lambda^\bullet T_S^{\mathrm{vir},\vee}}\right) \in \mathbb{Z}(\!(t, (t_1t_2t_3)^{\frac{1}{2}}, w)\!),^2\end{aligned} \tag{5.2.4}$$

where the *twisted virtual structure sheaf* $\widehat{\mathscr{O}}^{\mathrm{vir}}$ is defined in §5.2.1 — a square root of the virtual canonical bundle exists by Lemma 5.13 and Prop. 5.3. The generating function of rank r K-theoretic DT invariants

$$\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) = \sum_{n\geqslant 0} q^n \chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \widehat{\mathscr{O}}^{\mathrm{vir}}) \in \mathbb{Z}(\!(t, (t_1t_2t_3)^{\frac{1}{2}}, w)\!)[\![q]\!] \tag{5.2.5}$$

will be studied in §5.4.

Remark 5.5. To be precise, we should replace the torus $\mathbb{T}$ with its double cover $\mathbb{T}_t$, the minimal cover of $\mathbb{T}$ where the character $t^{-1/2}$ is defined, as in [176, §7.1.2]. Then $\mathcal{K}_{\mathrm{vir}}^{1/2}$ is a $\mathbb{T}_t$-equivariant sheaf with character $t^{-(\dim \mathrm{NCQuot}_r^n)/2}$. To ease the notation, we keep denoting the torus acting as $\mathbb{T}$. ◀

Remark 5.6. As remarked in [5; 176], choices of square roots of $\mathcal{K}_{\mathrm{vir}}$ differ by a 2-torsion element in the Picard group, which implies that $\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \widehat{\mathscr{O}}^{\mathrm{vir}})$ does not depend on such choices of square roots. Thus there is no ambiguity in (5.2.5). ◀

5.3 HIGHER RANK VERTEX ON THE LOCAL QUOT SCHEME

5.3.1 The virtual tangent space of the local Quot scheme

By Lemma 5.4, we can represent the sheaf corresponding to a $\mathbb{T}$-fixed point

$$[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$$

as a direct sum of ideal sheaves

$$S = \bigoplus_{\alpha=1}^{r} \mathscr{I}_{Z_\alpha} \subset \mathscr{O}^{\oplus r},$$

with $Z_\alpha \subset \mathbb{A}^3$ a finite subscheme of length n_α supported at the origin, and satisfying $n = \sum_{1 \leqslant \alpha \leqslant r} n_\alpha$. In this subsection we derive a formula for the character of

$$\chi(\mathscr{O}^{\oplus r}, \mathscr{O}^{\oplus r}) - \chi(S, S) \in K_{\mathbb{T}}^0(\mathrm{pt}),$$

where for F and G in $K_{\mathbb{T}}^0(\mathrm{pt})$, we set

$$\chi(F, G) = \mathbf{R}\,\mathrm{Hom}(F, G) = \sum_{i \geqslant 0} (-1)^i \,\mathrm{Ext}^i(F, G).$$

Our method follows the approach of [157, §4.7]. Moreover, we show in Prop. 5.15 that such character agrees with the virtual tangent space T_S^{vir} induced by the critical obstruction theory.

Let Q_α be the $\mathbb{T}_1$-character of the α-summand of $\mathscr{O}^{\oplus r}/S$, i.e. (cf. Eq. (5.1.10))

$$Q_\alpha = \mathrm{tr}_{\mathscr{O}_{Z_\alpha}} = \sum_{(i,j,k) \in \pi_\alpha} t_1^i t_2^j t_3^k,$$

where $\pi_\alpha \subset \mathbb{Z}_{\geqslant 0}^3$ is the plane partition corresponding to the monomial ideal $\mathscr{I}_{Z_\alpha} \subset R = \mathbb{C}[x, y, z]$. Let $P_\alpha(t_1, t_2, t_3)$ be the Poincaré polynomial of $\mathscr{I}_{Z_\alpha}$. This can be computed via a $\mathbb{T}_1$-equivariant free resolution

$$0 \to E_{\alpha,s} \to \cdots \to E_{\alpha,1} \to E_{\alpha,0} \to \mathscr{I}_{Z_\alpha} \to 0.$$

Writing

$$E_{\alpha,i} = \bigoplus_j R(d_{\alpha,ij}), \quad d_{\alpha,ij} \in \mathbb{Z}^3,$$

one has, independently of the chosen resolution, the formula

$$P_\alpha(t_1,t_2,t_3)=\sum_{i,j}(-1)^i t^{d_{\alpha,ij}}.$$

By [157, §4.7] we know that there is an identity

$$Q_\alpha=\frac{1+P_\alpha}{(1-t_1)(1-t_2)(1-t_3)}. \tag{5.3.1}$$

For each $1\leqslant\alpha,\beta\leqslant r$, we can compute

$$\begin{aligned}\chi(\mathscr{I}_{Z_\alpha},\mathscr{I}_{Z_\beta})&=\sum_{i,j,k,l}(-1)^{i+k}\operatorname{Hom}_R(R(d_{\alpha,ij}),R(d_{\beta,kl}))\\&=\sum_{i,j,k,l}(-1)^{i+k}R(d_{\beta,kl}-d_{\alpha,ij}),\end{aligned}$$

which immediately yields the identity

$$\operatorname{tr}_{\chi(\mathscr{I}_{Z_\alpha},\mathscr{I}_{Z_\beta})}=\frac{P_\beta\overline{P}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}\in\mathbb{Z}(\!(t_1,t_2,t_3)\!).$$

It follows that, as a $\mathbb{T}$-representation, one has

$$\begin{aligned}\chi(S,S)&=\chi\left(\sum_\alpha w_\alpha\otimes\mathscr{I}_{Z_\alpha},\sum_\beta w_\beta\otimes\mathscr{I}_{Z_\beta}\right)\\&=\sum_{1\leqslant\alpha,\beta\leqslant r}\chi(w_\alpha\otimes\mathscr{I}_{Z_\alpha},w_\beta\otimes\mathscr{I}_{Z_\beta}),\end{aligned}$$

which yields

$$\operatorname{tr}_{\chi(S,S)}=\sum_{1\leqslant\alpha,\beta\leqslant r}\frac{w_\alpha^{-1}w_\beta\cdot P_\beta\overline{P}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}. \tag{5.3.2}$$

On the other hand,

$$\operatorname{tr}_{\chi(\mathscr{O}^{\oplus r},\mathscr{O}^{\oplus r})}=\sum_{1\leqslant\alpha,\beta\leqslant r}\frac{w_\alpha^{-1}w_\beta}{(1-t_1)(1-t_2)(1-t_3)}. \tag{5.3.3}$$

Combining Formulae (5.3.2) and (5.3.3) with Formula (5.3.1) yields the following result.

Proposition 5.14. *Let* $[S]\in\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}$ *be a torus fixed point. There is an identity*

$$\begin{aligned}&\operatorname{tr}_{\chi(\mathscr{O}^{\oplus r},\mathscr{O}^{\oplus r})-\chi(S,S)}=\\&=\sum_{1\leqslant\alpha,\beta\leqslant r}w_\alpha^{-1}w_\beta\left(Q_\beta-\frac{\overline{Q}_\alpha}{t_1t_2t_3}+\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}Q_\beta\overline{Q}_\alpha\right)\end{aligned} \tag{5.3.4}$$

in $\mathbb{Z}(\!(t_1,t_2,t_3,w_1,\dots,w_r)\!)$.

For every $\mathbb{T}$-fixed point $[S]$ we define associated *vertex* terms

$$V_{ij} = w_i^{-1} w_j \left(Q_j - \frac{\overline{Q}_i}{t_1 t_2 t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} Q_j \overline{Q}_i \right) \qquad (5.3.5)$$

for every $i, j = 1, \ldots, r$. It is immediate to see that for $r = 1$ (forcing $i = j$) we recover the vertex formalism developed in [157].

Prop. 5.14 can then be restated as

$$\mathrm{tr}_{\chi(\mathscr{O}^{\oplus r}, \mathscr{O}^{\oplus r})} - \mathrm{tr}_{\chi(S,S)} = \sum_{1 \leqslant i,j \leqslant r} V_{ij}.$$

We now relate this to the virtual tangent space (cf. eq. (1.5.4)) T_S^{vir} of a point $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$.

Proposition 5.15. *Let* $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$ *be a* $\mathbb{T}$*-fixed point. Let* $T_S^{\mathrm{vir}} = \mathbb{E}_{\mathrm{crit}}^{\vee}|_{[S]}$ *be the virtual tangent space induced by the* $\mathbb{T}$*-equivariant critical obstruction theory. Then there are identities*

$$T_S^{\mathrm{vir}} = \chi(\mathscr{O}^{\oplus r}, \mathscr{O}^{\oplus r}) - \chi(S,S) = \sum_{1 \leqslant i,j \leqslant r} V_{ij} \in K_0^{\mathbb{T}}(\mathrm{pt}).$$

Proof. Let NCQuot_r^n be the non-commutative Quot scheme defined in (5.1.1). The superpotential $f_n \colon \mathsf{NCQuot}_r^n \to \mathbb{A}^1$ defined in (5.1.2) is equivariant with respect to the character $(t_1, t_2, t_3) \mapsto t_1 t_2 t_3$, so it gives rise to a GL_3-equivariant section

$$\mathscr{O}_{\mathsf{NCQuot}_r^n} \xrightarrow{df_n \otimes t^{-1}} \Omega_{\mathsf{NCQuot}_r^n} \otimes t^{-1} \qquad (5.3.6)$$

where, starting from the representation $\mathbb{C}^3 = \bigoplus_{1 \leqslant i \leqslant 3} t_i^{-1} \cdot \mathbb{C} \in K_{\mathbb{T}_1}^0(\mathrm{pt})$, we have set

$$t^{-1} = \det \mathbb{C}^3 = (t_1 t_2 t_3)^{-1}.$$

Here, and throughout this proof, we are identifying a representation with its own character via the isomorphism (1.3.1). The zero locus of the section (5.3.6) is our Quot scheme

$$Q = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n),$$

endowed with the $\mathbb{T}$-action described in §5.1.3. According to Eq. (5.1.16), the virtual tangent space computed with respect to the critical $\mathbb{T}$-equivariant obstruction theory on Q is

$$T_S^{\mathrm{vir}} = \left(T_{\mathsf{NCQuot}_r^n} - \Omega_{\mathsf{NCQuot}_r^n} \otimes t^{-1} \right) \big|_{[S]}. \qquad (5.3.7)$$

The tangent space to the smooth scheme NCQuot_r^n can be written, around a point $S \hookrightarrow \mathscr{O}^{\oplus r} \twoheadrightarrow V$, as

$$T_{\mathsf{NCQuot}_r^n}\big|_{[S]} = (\mathbb{C}^3 - 1) \otimes (\overline{V} \otimes V) + \bigoplus_{\alpha=1}^{r} \mathrm{Hom}(w_\alpha \mathbb{C}, V), \qquad (5.3.8)$$

where
$$\bigoplus_{\alpha=1}^{r} \mathrm{Hom}(w_\alpha \mathbb{C}, V) = \bigoplus_{\alpha=1}^{r} w_\alpha^{-1} \otimes V$$
represents the r framings on the 3-loop quiver. Let V be written as a direct sum of structure sheaves
$$V = \bigoplus_{\alpha=1}^{r} \mathscr{O}_{Z_\alpha},$$
where the α-summand has $\mathbb{T}$-character $w_\alpha Q_\alpha$. Substituting
$$\mathbb{C}^3 - 1 = t_1^{-1} + t_2^{-1} + t_3^{-1} - 1 = \frac{t_1t_2 + t_1t_3 + t_2t_3 - t_1t_2t_3}{t_1t_2t_3}$$
$$V = \sum_{\alpha=1}^{r} w_\alpha Q_\alpha$$
into Formula (5.3.8) yields
$$T_{\mathrm{NCQuot}_r^n}\big|_{[S]} = \\ = \frac{t_1t_2 + t_1t_3 + t_2t_3 - t_1t_2t_3}{t_1t_2t_3} \sum_{1 \leqslant \alpha,\beta \leqslant r} w_\alpha^{-1} w_\beta \overline{Q}_\alpha Q_\beta + \sum_{1 \leqslant \alpha,\beta \leqslant r} w_\alpha^{-1} w_\beta Q_\beta,$$
and hence
$$\left(\Omega_{\mathrm{NCQuot}_r^n} \otimes t^{-1}\right)\big|_{[S]} = \\ = \frac{t_1 + t_2 + t_3 - 1}{t_1t_2t_3} \sum_{1 \leqslant \alpha,\beta \leqslant r} w_\alpha^{-1} w_\beta \overline{Q}_\alpha Q_\beta + \sum_{1 \leqslant \alpha,\beta \leqslant r} w_\alpha^{-1} w_\beta \frac{\overline{Q}_\alpha}{t_1t_2t_3},$$
which by Formula (5.3.7) yields
$$T_S^{\mathrm{vir}} = \sum_{1 \leqslant \alpha,\beta \leqslant r} w_\alpha^{-1} w_\beta \left(Q_\beta - \frac{\overline{Q}_\alpha}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} \overline{Q}_\alpha Q_\beta \right).$$
The right hand side is shown to be equal to $\chi(\mathscr{O}^{\oplus r}, \mathscr{O}^{\oplus r}) - \chi(S, S)$ in Prop. 5.14. ■

5.3.2 A small variation of the vertex formalism

All locally free sheaves on $\mathbb{A}^3$ are trivial, but this is not true equivariantly. For example, we have $K_{\mathbb{A}^3} = \mathscr{O}_{\mathbb{A}^3} \otimes t_1t_2t_3 \in K^0_{\mathbb{T}_1}(\mathbb{A}^3)$, even though the ordinary canonical bundle is trivial. Consider
$$F = \bigoplus_{i=1}^{r} \mathscr{O}_{\mathbb{A}^3} \otimes \lambda_i \in K^0_{\mathbb{T}_1}(\mathbb{A}^3) \tag{5.3.9}$$
where $\lambda = (\lambda_i)_i$ are weights of the $\mathbb{T}_1$-action, i.e. monomials in the representation ring of $\mathbb{T}_1$. Let $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(F, n)^{\mathbb{T}}$ be a $\mathbb{T}$-fixed point. It decomposes as $S = \bigoplus_{i=1}^{r} \mathscr{I}_{Z_i} \otimes \lambda_i \in K^0_{\mathbb{T}_1}(\mathbb{A}^3)$, where the weights λ_i are naturally inher-

ited from F. This generalises the discussion in §5.3.1, which can be recovered by setting all weights λ_i to be trivial. Just as in Prop. 5.15, we can compute

$$T^{\mathrm{vir}}_{S,\lambda} = \chi(F,F) - \chi(S,S) \in K^0_{\mathbb{T}}(\mathrm{pt}).$$

We find

$$T^{\mathrm{vir}}_{S,\lambda} = \chi\left(\bigoplus_{i=1}^{r} \mathscr{O}_{\mathbb{A}^3}\otimes\lambda_i w_i, \bigoplus_{j=1}^{r} \mathscr{O}_{\mathbb{A}^3}\otimes\lambda_j w_j\right) + \\ -\chi\left(\bigoplus_{j=1}^{r} \mathscr{I}_{Z_i}\otimes\lambda_i w_i, \bigoplus_{i=1}^{r} \mathscr{I}_{Z_j}\otimes\lambda_j w_j\right).$$

Therefore we derive the same expression for the vertex formalism as before, just substituting w_i with $\lambda_i w_i$.

Define, for $\lambda = (\lambda_1, \dots, \lambda_r)$ as above and F as in (5.3.9), the equivariant integral

$$\int_{[\mathrm{Quot}_{\mathbb{A}^3}(F,n)]^{\mathrm{vir}}} 1 := \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \frac{1}{e^{\mathbb{T}}(T^{\mathrm{vir}}_{S,\lambda})} \in \mathbb{Q}((s,v)), \tag{5.3.10}$$

and let

$$\mathrm{DT}^{\mathrm{coh}}_r(\mathbb{A}^3, q, s, v)_\lambda = \sum_{n\geqslant 0} q^n \int_{[\mathrm{Quot}_{\mathbb{A}^3}(F,n)]^{\mathrm{vir}}} 1 \tag{5.3.11}$$

be the generating function of the invariants (5.3.10). We shall see (cf. Corollary 5.24) that this expression does not depend on the equivariant weights λ_i.

5.4 THE HIGHER RANK K-THEORETIC DT PARTITION FUNCTION

5.4.1 Symmetrised exterior algebras and brackets

We recall some constructions is equivariant K-theory which will be used to prove Thm. 5.1. For a recent and more complete reference, the reader may consult [184, §2].

Let $\mathbb{T}$ be a torus, $V = \sum_\mu t^\mu$ a $\mathbb{T}$-module. Assume that $\det(V)$ is a square in $K^0_{\mathbb{T}}(\mathrm{pt})$. Define the *symmetrised exterior algebra* of V as

$$\widehat{\Lambda}^\bullet V := \frac{\Lambda^\bullet V}{\det(V)^{\frac{1}{2}}}.$$

It satisfies the relation

$$\widehat{\Lambda}^\bullet V^\vee = (-1)^{\operatorname{rk} V}\widehat{\Lambda}^\bullet V.$$

Define the operator $[\cdot]$ by

$$[t^\mu] = t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}}.$$

One can compute

$$\operatorname{tr}(\widehat{\Lambda}^\bullet V^\vee) = \prod_\mu \frac{1-t^{-\mu}}{t^{-\frac{\mu}{2}}} = \prod_\mu (t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}}) = \prod_\mu [t^\mu],$$

For a virtual $\mathbb{T}$-representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu$, where the weight $\nu = 0$ never appears, we extend $\widehat{\Lambda}^\bullet$ and $[\cdot]$ by linearity and find

$$\operatorname{tr}(\widehat{\Lambda}^\bullet V^\vee) = \frac{\prod_\mu [t^\mu]}{\prod_\nu [t^\nu]}.$$

5.4.2 Proof of Thm. 5.1

By the description of the $\mathbb{T}$-fixed locus $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$ given in §5.1.4, every coloured plane partition $\overline{\pi} = (\pi_1, \dots, \pi_r)$ corresponds to a unique $\mathbb{T}$-fixed point $S = \bigoplus_{i=1}^r \mathscr{I}_{Z_i}$, for which we defined in Eq. (5.3.5) the vertex terms V_{ij} by

$$V_{ij} = w_i^{-1} w_j \left(Q_j - \frac{\overline{Q}_i}{t_1 t_2 t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} Q_j \overline{Q}_i \right)$$

with notation as in §5.3.1. The generating function (5.2.3) of higher rank cohomological DT invariants can be rewritten in a purely combinatorial fashion as

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \sum_{\overline{\pi}} q^{|\overline{\pi}|} \prod_{i,j=1}^r e^{\mathbb{T}}(-V_{ij}).$$

Similarly, the generating function (5.2.5) of the K-theoretic invariants can be rewritten as

$$\mathsf{DT}_r^{K}(\mathbb{A}^3, q, t, w) = \sum_{\overline{\pi}} q^{|\overline{\pi}|} \prod_{i,j=1}^r [-V_{ij}].$$

A closed formula for $\mathsf{DT}_1^K(\mathbb{A}^3, q, t, w)$ was conjectured in [171] and has recently been proven by Okounkov.

Theorem 5.16 ([184, Thm. 3.3.6])**.** *The rank* 1 *K-theoretic DT partition function of* $\mathbb{A}^3$ *is given by*

$$\mathsf{DT}_1^K(\mathbb{A}^3, -q, t, w) = \operatorname{Exp}\left(\mathsf{F}_1(q, t_1, t_2, t_3)\right)$$

where, setting $t = t_1 t_2 t_3$, *one defines*

$$\mathsf{F}_1(q, t_1, t_2, t_3) = \frac{1}{[t^{\frac{1}{2}} q][t^{\frac{1}{2}} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}.$$

Remark 5.7. It is clear from the expression of the vertex in rank 1 that there is no dependence on w_1. As pointed out to us by N. Arbesfeld, this can in fact be seen as a shadow of the fact that $\mathbb{T}_2 = \mathbb{C}^*$ acts trivially on $\operatorname{NCQuot}_1^n$

and on df_n. Surprisingly, the same phenomenon occurs in the higher rank case (cf. Thm. 5.18). ◀

We devote the rest of this section to proving a generalisation of Thm. 5.16 to higher rank.

Theorem 5.17. *The rank* r *K-theoretic DT partition function of* $\mathbb{A}^3$ *is given by*

$$\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3,(-1)^r q,t,w) = \operatorname{Exp}\left(\mathsf{F}_r(q,t_1,t_2,t_3)\right),$$

where, setting $t = t_1t_2t_3$, *one defines*

$$\mathsf{F}_r(q,t_1,t_2,t_3) = \frac{[t^r]}{[t][t^{\frac{r}{2}}q][t^{\frac{r}{2}}q^{-1}]}\frac{[t_1t_2][t_1t_3][t_2t_3]}{[t_1][t_2][t_3]}.$$

Remark 5.8. This result was conjectured in [11] by Awata and Kanno, who also proved it $\operatorname{mod} q^4$, i.e. up to 3 instantons. The conjecture was confirmed numerically up to some order by Benini–Bonelli–Poggi–Tanzini [22]. ◀

The proof of Theorem 5.17 will follow essentially by taking suitable limits of the weights w_i. To perform such limits, we prove the following key result.

Theorem 5.18. *The generating function* $\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3,q,t,w)$ *does not depend on the weights* $w_1,\ldots,w_r$.

Proof. The n-th coefficient of $\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3,q,t,w)$ is a sum of contributions

$$[-T_{\overline{\pi}}^{\mathrm{vir}}], \qquad |\overline{\pi}| = n.$$

A simple manipulation shows that

$$\begin{aligned}
[-T_{\overline{\pi}}^{\mathrm{vir}}] &= A(t)\prod_{1\leqslant i<j\leqslant r}\frac{\prod_{\mu_{ij}} w_i - w_j t^{\mu_{ij}}}{\prod_{\nu_{ij}} w_i - w_j t^{\nu_{ij}}}\\
&= A(t)\prod_{1\leqslant i<j\leqslant r}\frac{\prod_{\mu_{ij}} (1 - w_i^{-1}w_j t^{\mu_{ij}})}{\prod_{\nu_{ij}} (1 - w_i^{-1}w_j t^{\nu_{ij}})},
\end{aligned} \tag{5.4.1}$$

where $A(t) \in \mathbb{Q}(\!(t_1,t_2,t_3,(t_1t_2t_3)^{\frac{1}{2}})\!)$ and the number of weights μ_{ij} and ν_{ij} is the same. Thus, $\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3,q,t,w)$ is a homogeneous rational expression of total degree 0 with respect to the variables $w_1,\ldots,w_r$. We aim to show that $\mathrm{DT}_r^{\mathrm{K}}(\mathbb{A}^3,q,t,w)$ has no poles of the form $1-w_i^{-1}w_jt^{\nu_{ij}}$, implying that it is a degree 0 polynomial in the w_i, hence constant in the w_i. This generalises the strategy of [11, §4].

Set $\mathsf{w} = w_i^{-1}w_jt^{\nu}$ for fixed $i<j$ and $\nu \in \widehat{\mathbb{T}}_1$. To see that $1-\mathsf{w}$ is not a pole, we use [5, Prop. 3.2], which asserts the following: if M is a quasiprojective $\mathbb{T}$-scheme with a $\mathbb{T}$-equivariant perfect obstruction theory and proper (nonempty) fixed locus, then for any $V \in K_0^{\mathbb{T}}(M)$, the only poles of the form $(1-\mathsf{w})$ that *may* appear in $\chi(M, V\otimes\mathscr{O}^{\mathrm{vir}})$ arise from *noncompact weights* $\mathsf{w}\in\widehat{\mathbb{T}}$. A weight w is called compact if the fixed locus $M^{\mathbb{T}_{\mathsf{w}}}\subset M$ is proper, where $\mathbb{T}_{\mathsf{w}}$ is the maximal torus in $\ker(\mathsf{w})\subset\mathbb{T}$, and is called noncompact otherwise [5, Def. 3.1].

We of course want to apply [5, Prop. 3.2] to $M = Q = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$, $\mathbb{T} = \mathbb{T}_1\times\mathbb{T}_2$ and $V = \mathcal{K}_{\mathrm{vir}}^{1/2}$. By Equation (5.4.1), our goal is to prove that $\mathsf{w} = w_i^{-1}w_jt^{\nu}$ is a compact weight for all $i<j$ and $\nu\in\widehat{\mathbb{T}}_1$.

First of all, we observe that $\mathbb{T}_w = \ker(w)$. Indeed w is not a product of powers of weights of $\mathbb{T}$, hence

$$\begin{aligned}\mathscr{O}(\ker w) &= \mathscr{O}(\mathbb{T})/(w_i^{-1}w_j t^\nu - 1)\\ &\cong \mathbb{C}\left[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, w_1^{\pm 1}, \ldots, w_{i-1}^{\pm 1}, w_{i+1}^{\pm 1}, \ldots, w_r^{\pm 1}\right],\end{aligned}$$

which shows that $\ker(w)$ is itself a torus (of dimension $3 + r - 1$). Next, consider the automorphism $\tau_\nu \colon \mathbb{T} \sim \mathbb{T}$ defined by

$$(t_1, t_2, t_3, w_1, \ldots, w_r) \mapsto (t_1, t_2, t_3, w_1, \ldots, w_i t^{-\nu}, \ldots, w_j, \ldots, w_r).$$

It maps $\mathbb{T}_w \subset \mathbb{T}$ isomorphically onto the subtorus $\mathbb{T}_1 \times \{w_i = w_j\} \subset \mathbb{T}$. This yields an inclusion of tori

$$\mathbb{T}_1 \sim \mathbb{T}_1 \times \{(1, \ldots, 1)\} \hookrightarrow \tau(\mathbb{T}_w). \tag{5.4.2}$$

We consider the action $\sigma_\nu \colon \mathbb{T} \times Q \to Q$ where $\mathbb{T}_1$ translates the support of the quotient sheaf in the usual way, the i-th summand of $\mathscr{O}^{\oplus r}$ gets scaled by $w_i t^\nu$ and all other summands by w_k for $k \neq i$. In other words, in terms of the matrix-and-vectors description of Q, we set

$$\begin{aligned}&\sigma_\nu(\mathbf{t}, (A_1, A_2, A_3, u_1, \ldots, u_r)) =\\ &\qquad = (t_1 A_1, t_2 A_2, t_3 A_3, w_1 u_1, \ldots, w_i t^\nu u_i, \ldots, w_r u_r),\end{aligned}$$

just a variation of Equation (5.1.9) in the i-th vector component. Then, upon restricting this action to $\mathbb{T}_w$, we have a commutative diagram

$$\begin{tikzcd} \mathbb{T}_w \times Q \arrow[r, "\sigma"] \arrow[d, "\tau_\nu \times \mathrm{id}"'] & Q \arrow[d, equal] \\ \tau_\nu(\mathbb{T}_w) \times Q \arrow[r, "\sigma_\nu"] & Q \end{tikzcd}$$

where σ is the restriction of the usual action (5.1.9). This diagram induces a natural isomorphism $Q^{\mathbb{T}_w} \sim Q^{\tau_\nu(\mathbb{T}_w)}$, which combined with (5.4.2) yields an inclusion

$$Q^{\mathbb{T}_w} \sim Q^{\tau_\nu(\mathbb{T}_w)} \hookrightarrow Q^{\mathbb{T}_1},$$

where $Q^{\mathbb{T}_1}$ is the fixed locus with respect to the action σ_ν. But by the same reasoning as in Remark 5.2, this fixed locus is proper (because, again, a $\mathbb{T}_1$-fixed surjection $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ necessarily has the quotient T entirely supported at the origin $0 \in \mathbb{A}^3$). Thus w is a compact weight, and the result follows. ∎

Remark 5.9. After a first draft of this work was already finished, we were informed of an alternative way to prove Theorem 5.18, which, in a nutshell, goes as follows: one exploits the (proper) Quot-to-Chow morphism $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) \to \mathrm{Sym}^n \mathbb{A}^3$ to express the K-theoretic DT invariants as equivariant holomorphic Euler characteristics on $\mathrm{Sym}^n \mathbb{A}^3$, where the framing torus $\mathbb{T}_2$ is acting trivially on the symmetric product. One concludes by an application of Okounkov's *rigidity principle* [184, § 2.4.1]. This strategy will be carried out in [6]. ◀

Thanks to Theorem 5.18, we may now specialise $w_1, \dots, w_r$ to arbitrary values and take arbitrary limits. We set $w_i = L^i$ for $i = 1, \dots, r$ and compute the limit for $L \to \infty$.

Lemma 5.19. *Let $i < j$. Then we have*

$$\lim_{L\to\infty} [-V_{ij}][-V_{ji}]\big|_{w_i=L^i} = (-t^{\frac{1}{2}})^{|\pi_j|-|\pi_i|}.$$

Proof. Notice that all monomials in V_{ij} are of the form $w_i^{-1}w_j\lambda$ for λ a monomial in t_1, t_2, t_3. Then

$$[w_i^{-1}w_j\lambda]\big|_{w_i=L^i} = (L^{j-i}\lambda)^{\frac{1}{2}}(1-L^{i-j}\lambda^{-1}).$$

Write $Q_i = \sum_\mu t^\mu$ and $Q_j = \sum_\nu t^\nu$. Taking limits, we obtain

$$\begin{aligned}\lim_{L\to\infty} [-V_{ij}]\big|_{w_i=L^i} &= \lim_{L\to\infty} \Big[-w_i^{-1}w_j(Q_j - \overline{Q}_i t^{-1} \\ &\qquad + \overline{Q}_i Q_j t^{-1}(1-t_1)(1-t_2)(1-t_3))\Big]\Big|_{w_i=L^i} \\ &= \lim_{L\to\infty} L^{\frac{j-i}{2}(|\pi_i|-|\pi_j|)} \frac{\prod_\mu (t^{-\frac{\mu}{2}}t^{-\frac{1}{2}})}{\prod_\nu t^{\frac{\nu}{2}}}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\lim_{L\to\infty} [-V_{ji}]\big|_{w_i=L^i} &= (-1)^{\mathrm{rk}(-V_{ji})} \lim_{L\to\infty} [-\overline{V}_{ji}]\Big|_{w_i=L^i} \\ &= (-1)^{|\pi_i|-|\pi_j|} \lim_{L\to\infty} \Big[-w_i^{-1}w_j(\overline{Q}_i - Q_j t \\ &\qquad - \overline{Q}_i Q_j(1-t_1)(1-t_2)(1-t_3))\Big]\Big|_{w_i=L^i} \\ &= (-1)^{|\pi_i|-|\pi_j|} \lim_{L\to\infty} L^{\frac{j-i}{2}(|\pi_j|-|\pi_i|)} \frac{\prod_\nu (t^{\frac{\nu}{2}}t^{\frac{1}{2}})}{\prod_\mu t^{-\frac{\mu}{2}}}.\end{aligned}$$

We conclude, as required, that

$$\lim_{L\to\infty} [-V_{ij}][-V_{ji}]\big|_{w_i=L^i} = (-t^{\frac{1}{2}})^{|\pi_j|-|\pi_i|}. \qquad \blacksquare$$

Lemma 5.20. *Let x be a variable and $c_i \in \mathbb{Z}$, for $i = 1, \dots, r$. Then we have*

$$\prod_{1\leqslant i<j\leqslant r} x^{c_j - c_i} = \prod_{i=1}^{r} x^{(-r-1+2i)c_i}.$$

Proof. The assertion holds for $r = 1$ as the productory on the left hand side is empty. Assume it holds for $r-1$. Then we have:

$$\begin{aligned}\prod_{1\leqslant i<j\leqslant r} x^{c_j-c_i} &= \prod_{1\leqslant i<j\leqslant r-1} x^{c_j-c_i} \prod_{i=1}^{r-1} x^{c_r - c_i} \\ &= x^{(r-1)c_r} \prod_{i=1}^{r-1} x^{(-r-1+2i)c_i}\end{aligned}$$

$$= \prod_{i=1}^{r} x^{(-r-1+2i)c_i}. \qquad \blacksquare$$

Combining Lemma 5.19 with Lemma 5.20 we can express the rank r K-theoretic DT theory of $\mathbb{A}^3$ as a product of r copies of the rank 1 K-theoretic DT theory. This product formula already appeared as a limit of the (conjectural) 4-fold theory developed by Nekrasov and Piazzalunga [177, Formula (3.14)].[3]

Theorem 5.21. *There is an identity*

$$\mathrm{DT}_r^K(\mathbb{A}^3, (-1)^r q, t, w) = \prod_{i=1}^{r} \mathrm{DT}_1^K(\mathbb{A}^3, -qt^{\frac{-r-1}{2}+i}, t).$$

Proof. Set $w_i = L^i$. The generating series $\mathrm{DT}_r^K(\mathbb{A}^3, q, t, w)$ can be computed in the limit $L \to \infty$:

$$\begin{aligned}
\lim_{L\to\infty} \mathrm{DT}_r^K(\mathbb{A}^3, q, t, w) &= \lim_{L\to\infty} \sum_{\overline{\pi}} q^{|\overline{\pi}|} \prod_{i,j=1}^{r} [-V_{ij}] \\
&= \lim_{L\to\infty} \sum_{\overline{\pi}} \prod_{i=1}^{r} q^{|\pi_i|}[-V_{ii}] \prod_{1\leqslant i<j\leqslant r} [-V_{ij}][-V_{ji}] \\
&= \sum_{\overline{\pi}} \prod_{i=1}^{r} q^{|\pi_i|}[-V_{ij}] \prod_{1\leqslant i<j\leqslant r} (-t^{\frac{1}{2}})^{|\pi_j|-|\pi_i|} \\
&= \sum_{\overline{\pi}} \prod_{i=1}^{r} q^{|\pi_i|}[-V_{ii}] \prod_{i=1}^{r} (-t^{\frac{1}{2}})^{(-r-1+2i)|\pi_i|} \\
&= \sum_{\overline{\pi}} \prod_{i=1}^{r} [-V_{ii}] q^{|\pi_i|} (-1)^{(r+1)|\pi_i|} t^{(\frac{-r-1}{2}+i)|\pi_i|} \\
&= \sum_{\overline{\pi}} \prod_{i=1}^{r} [-V_{ii}] \left((-1)^{(r+1)} qt^{\frac{-r-1}{2}+i}\right)^{|\pi_i|} \\
&= \prod_{i=1}^{r} \mathrm{DT}_1^K(\mathbb{A}^3, (-1)^{(r+1)} qt^{\frac{-r-1}{2}+i}, t). \qquad \blacksquare
\end{aligned}$$

We can now prove Thm. 5.17 (i.e. Thm. 5.1 from the Introduction).

Proof of Thm. 5.17. Define

$$G_{r,i}(q, t_1, t_2, t_3) = F_1(qt^{\frac{-r-1}{2}+i}, t_1, t_2, t_3).$$

We have

$$\mathrm{DT}_1^K(\mathbb{A}^3, -qt^{\frac{-r-1}{2}+i}, t) = \exp\left(\sum_{n\geqslant 1} \frac{1}{n} \frac{1}{[t^{\frac{n}{2}} q^n t^{n(\frac{-r-1}{2}+i)}][t^{\frac{n}{2}} q^{-n} t^{n(\frac{r+1}{2}-i)}]} \frac{[t_1^n t_2^n][t_1^n t_3^n][t_2^n t_3^n]}{[t_1^n][t_2^n][t_3^n]}\right)$$

3 Typo warning: N. Piazzalunga kindly pointed out to us that in [177, Formula (3.14)] one should read '$\frac{N+1}{2} - l$' instead of '$N + 1 - 2l$'.

$$= \mathrm{Exp}\left(G_{r,i}(q,t_1,t_2,t_3)\right).$$

By Thm. 5.21 and Thm. 5.16 it is enough to show that $F_r = \sum_{i=1}^r G_{r,i}$, or equivalently

$$\sum_{i=1}^{r} \frac{1}{[t^{\frac{1}{2}}qt^{\frac{-r-1}{2}+i}][t^{\frac{1}{2}}q^{-1}t^{\frac{r+1}{2}-i}]} = \frac{[t^r]}{[t][t^{\frac{r}{2}}q][t^{\frac{r}{2}}q^{-1}]}.$$

It is easy to check this is true for $r = 1, 2$. Let now $r \geqslant 3$: we perform induction separately on even and odd cases. Assume the claimed identity holds for $r-2$. In both cases we have

$$\begin{aligned}
&\sum_{i=1}^{r} \frac{1}{[t^{\frac{1}{2}}qt^{\frac{-r-1}{2}+i}][t^{\frac{1}{2}}q^{-1}t^{\frac{r+1}{2}-i}]}\\
&= \sum_{i=1}^{r-2} \frac{1}{[t^{\frac{1}{2}}qt^{\frac{-(r-2)-1}{2}+i}][t^{\frac{1}{2}}q^{-1}t^{\frac{(r-2)+1}{2}-i}]} + \frac{1}{[qt^{-\frac{r}{2}+1}][q^{-1}t^{\frac{r}{2}}]} + \frac{1}{[qt^{\frac{r}{2}}][q^{-1}t^{-\frac{r}{2}+1}]}\\
&= \frac{[t^{r-2}]}{[t][t^{\frac{r-2}{2}}q][t^{\frac{r-2}{2}}q^{-1}]} - \frac{1}{[t^{\frac{r-2}{2}}q^{-1}][t^{\frac{r}{2}}q^{-1}]} - \frac{1}{[t^{\frac{r}{2}}q][t^{\frac{r-2}{2}}q]}\\
&= \frac{1}{[t][t^{\frac{r}{2}}q][t^{\frac{r}{2}}q^{-1}]} \cdot \frac{[t^{r-2}][t^{\frac{r}{2}}q][t^{\frac{r}{2}}q^{-1}] - [t][t^{\frac{r}{2}}q][t^{\frac{r-2}{2}}q] - [t][t^{\frac{r}{2}}q^{-1}][t^{\frac{r-2}{2}}q^{-1}]}{[t^{\frac{r-2}{2}}q][t^{\frac{r-2}{2}}q^{-1}]}\\
&= \frac{[t^r]}{[t][t^{\frac{r}{2}}q][t^{\frac{r}{2}}q^{-1}]}
\end{aligned}$$

by which we conclude the proof. ∎

5.4.3 Comparison with motivic DT invariants

Let $f\colon U \to \mathbb{A}^1$ be a regular function on a smooth scheme U, and let $\hat{\mu}$ be the group of all roots of unity. The critical locus $Z = \mathrm{crit}(f) \subset U$ inherits a canonical *virtual motive* [16], i.e. a $\hat{\mu}$-equivariant motivic class

$$[Z]_{\mathrm{vir}} = -\mathbb{L}^{-\frac{\dim U}{2}}[\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})\left[\mathbb{L}^{-\frac{1}{2}}\right]$$

such that $e[Z]_{\mathrm{vir}} = e_{\mathrm{vir}}(Z)$, where $e_{\mathrm{vir}}(-)$ is Behrend weighted Euler characteristic and the Euler number specialisation prescribes $e(\mathbb{L}^{-1/2}) = -1$. The motivic class $[\phi_f]$ is the (absolute) motivic vanishing cycle class introduced by Denef and Loeser [81].

The virtual motive of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) = \mathrm{crit}(f_n)$, with respect to the critical structure of Prop. 5.3, was computed in [190, Prop. 2.3.6]. The result is as follows. Let $\mathrm{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, q) \in \mathcal{M}_{\mathbb{C}}[\![q]\!]$ be the generating function of the virtual motives $[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)]_{\mathrm{vir}}$. Then one has

$$\mathrm{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, q) = \prod_{m\geqslant 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{2+k-\frac{rm}{2}}q^m\right)^{-1}. \tag{5.4.3}$$

The case $r = 1$ was computed in [16]. The general proof of Formula (5.4.3) is obtained in a similar fashion in [62; 190], and via a wall-crossing technique in [63]. Moreover, it is immediate to verify that $\mathrm{DT}_r^{\mathrm{mot}}$ satisfies a product formula analogous to the one proved in Thm. 5.21 for the K-theoretic invariants: we have

$$\mathsf{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, q) = \prod_{i=1}^{r} \mathsf{DT}_1^{\mathrm{mot}}\left(\mathbb{A}^3, q\mathbb{L}^{\frac{-r-1}{2}+i}\right). \tag{5.4.4}$$

In particular, up to the substitution $t^{\frac{1}{2}} \to -\mathbb{L}^{\frac{1}{2}}$, the factorisation (5.4.4) is equivalent to the K-theoretic one (Thm. 5.21). As observed in [192, §4], the (signed) motivic partition function admits an expression in terms of the motivic exponential, namely

$$\mathsf{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, (-1)^r q) = \operatorname{Exp}\left(\frac{(-1)^r q\mathbb{L}^{\frac{3}{2}}}{(1-(-1)^r q\mathbb{L}^{\frac{r}{2}})(1-(-1)^r q\mathbb{L}^{-\frac{r}{2}})} \frac{\mathbb{L}^{\frac{r}{2}} - \mathbb{L}^{-\frac{r}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right). \tag{5.4.5}$$

Given their structural similarities, we believe it is an interesting problem to compare the K-theoretic partition function with the motivic one.

It is worth noticing that Formula (5.4.5) can be recovered from the factorisation (5.4.4), just as we discovered in the K-theoretic case during the proof of Thm. 5.17. This fact follows immediately from the properties of the plethystic exponential.

Remark 5.10. A virtual motive for $\mathrm{Quot}_Y(F, n)$ was defined in [192, §4] for every locally free sheaf F on a 3-fold Y. Just as in the case of the naive motives of the Quot scheme [191], the resulting partition function $\mathsf{DT}_r^{\mathrm{mot}}(Y, q)$ only depends on the motivic class $[Y] \in K_0(\mathrm{Var}_{\mathbb{C}})$ and on $r = \operatorname{rk} F$. See also [65] for calculations of motivic higher rank DT and PT invariants in the presence of nonzero curve classes: the generating function $\mathsf{DT}_r^{\mathrm{mot}}(Y, q)$, computed easily starting with Formula (5.4.3), is precisely the DT/PT wall-crossing factor. ◀

5.5 THE HIGHER RANK COHOMOLOGICAL DT PARTITION FUNCTION

5.5.1 Cohomological reduction

One should think of K-theoretic invariants as refinements of the cohomological ones, as by taking suitable limits one fully recovers $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s)$ from $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t)$. We make this precise in the remainder of this section.

Let $\mathbb{T} \cong \mathbb{C}^g$ be an algebraic torus and let $t_1, \ldots, t_g$ be its coordinates. Recall that the Chern character gives a natural transformation from (equivariant) K-theory to the (equivariant) Chow group with rational coefficients by sending $t_i \mapsto e^{s_i}$, where $s_i = c_1^{\mathbb{T}}(t_i)$. We can formally extend it to

$$\begin{tikzcd} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_g^{\pm 1}] \arrow[r, "\mathrm{ch}"] \arrow[d] & \mathbb{Q}[\![s_1, \ldots, s_g]\!] \arrow[d] \\ \mathbb{Z}[t_1^{\pm b}, \ldots, t_g^{\pm b} \mid b \in \mathbb{C}] \arrow[r, "\mathrm{ch}"] & \mathbb{C}[\![s_1, \ldots, s_g]\!] \end{tikzcd}$$

by sending $t_i^b \mapsto e^{bs_i}$, where $b \in \mathbb{C}$.

In §5.4.1 we defined the symmetrised transformation $[t^\mu] = t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}}$. We set $[\mathrm{ch}(t^{b\mu})] = e^{\frac{b\mu\cdot s}{2}} - e^{-\frac{b\mu\cdot s}{2}}$ as an expression in rational cohomology, which enjoys the following *linearisation* property:

$$[\mathrm{ch}(t^{b\mu})] = e^{\frac{b\mu\cdot s}{2}}(1 - e^{-b\mu\cdot s}) = be^{\mathbb{T}}(t^\mu) + o(b^2).$$

In other words, $e^{\mathbb{T}}(\,\cdot\,)$ is the first-order approximation of $[\,\cdot\,]$ in $\mathbb{T}$-equivariant Chow groups. For a virtual representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu \in K_0^{\mathbb{T}}(\mathrm{pt})$, denote by $V^b = \sum_\mu t^{b\mu} - \sum_\nu t^{b\nu}$ the virtual representation where we formally substitute each weight t^μ with $t^{b\mu}$. We have the identity

$$[\mathrm{ch}(V^b)] = \frac{\prod_\mu [\mathrm{ch}(t^{b\mu})]}{\prod_\nu [\mathrm{ch}(t^{b\nu})]} = b^{\mathrm{rk}\, V}\frac{\prod_\mu (e^{\mathbb{T}}(t^\mu) + o(b))}{\prod_\nu (e^{\mathbb{T}}(t^\nu) + o(b))}.$$

If $\mathrm{rk}\, V = 0$, by taking the limit for $b \to 0$ we conclude

$$\lim_{b\to 0} [\mathrm{ch}(V^b)] = e^{\mathbb{T}}(V). \tag{5.5.1}$$

It is clear from the definition of $\mathrm{ch}(\,\cdot\,)$ and $[\,\cdot\,]$ that these two transformations commute with each other. This proves the following relation between K-theoretic invariants and cohomological invariants of the local model.

Corollary 5.22. *There is an identity*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \lim_{b\to 0} \mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, e^{bs}, e^{bv}).$$

Proof. Follows from the description of the generating series of K-theoretic invariants as

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) = \sum_{n\geqslant 0} q^n \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}} [-T_S^{\mathrm{vir}}]$$

and by noticing that $\mathrm{rk}\, T_S^{\mathrm{vir}} = 0$. ∎

Thanks to the v-independence, we can now rename

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s) = \mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v).$$

We are ready to prove Thm. 5.2 from the Introduction.

Theorem 5.23. *The rank r cohomological DT partition function of $\mathbb{A}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s) = \mathsf{M}((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}.$$

Proof. By Corollary 5.22 and Thm. 5.17, we just need to compute the limit

$$\lim_{b\to 0} \mathsf{DT}_r^{\mathrm{K}}\left(\mathbb{A}^3, (-1)^r q, e^{bs}\right) = \lim_{b\to 0} \mathrm{Exp}\left(\mathsf{F}_r(q, t_1^b, t_2^b, t_3^b)\right).$$

Denote for ease of notation $\mathfrak{s} = c_1^{\mathbb{T}}(t) = s_1 + s_2 + s_3$. By the definition of plethystic exponential, recalled in (5.0.2), we have

$$\lim_{b\to 0} \operatorname{Exp}\left(\mathsf{F}_r(q,t_1^b,t_2^b,t_3^b)\right) = \exp \sum_{k\geqslant 1} \frac{1}{k}\left(\lim_{b\to 0} \frac{[e^{bkr\mathfrak{s}}]}{[e^{bk\mathfrak{s}}][e^{\frac{bkr}{2}\mathfrak{s}}q^k][e^{\frac{bkr}{2}\mathfrak{s}}q^{-k}]} \right.$$
$$\left. \frac{[e^{bk(s_1+s_2)}][e^{bk(s_1+s_3)}][e^{bk(s_2+s_3)}]}{[e^{bks_1}][e^{bks_2}][e^{bks_3}]}\right).$$

We have

$$\lim_{b\to 0} \frac{[e^{bk(s_1+s_2)}][e^{bk(s_1+s_3)}][e^{bk(s_2+s_3)}]}{[e^{bks_1}][e^{bks_2}][e^{bks_3}]} = \frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3},$$

and

$$\lim_{b\to 0} \frac{[e^{bkr\mathfrak{s}}]}{[e^{bk\mathfrak{s}}][e^{\frac{bkr}{2}\mathfrak{s}}q^k][e^{\frac{bkr}{2}\mathfrak{s}}q^{-k}]} = \frac{r}{[q^k][q^{-k}]} = -r\cdot \frac{q^k}{(1-q^k)^2}.$$

Recall the plethystic exponential form of the MacMahon function

$$\mathsf{M}(q) = \prod_{n\geqslant 1}(1-q^n)^{-n} = \operatorname{Exp}\left(\frac{q}{(1-q)^2}\right).$$

We conclude

$$\lim_{b\to 0} \mathsf{DT}_r^K(\mathbb{A}^3,(-1)^r q, e^{bs}) = \exp\left(-r\cdot \frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3} \right.$$
$$\left. \sum_{k\geqslant 1}\frac{1}{k}\frac{q^k}{(1-q^k)^2}\right)$$
$$= \mathsf{M}(q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}}. \qquad \blacksquare$$

Thus we proved Szabo's conjecture [204, Conj. 4.10].

Remark 5.11. The specialisation

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathsf{A}^3,q,s)\big|_{s_1+s_2+s_3=0} = \mathsf{M}((-1)^r q)^r,$$

recovering Formula (5.2.1), was already known in physics, see e.g. [73]. ◂

We end this section with a small variation of Thm. 5.2.

Corollary 5.24. *Fix an r-tuple $\lambda = (\lambda_1,\dots,\lambda_r)$ of $\mathbb{T}_1$-equivariant line bundles on $\mathbb{A}^3$. Then there is an identity*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s) = \mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s,v)_\lambda,$$

where the right hand side was defined in (5.3.11).

Proof. We have

$$\mathsf{T}^{\mathrm{vir}}_{S,\lambda} = \sum_{i,j}\lambda_i^{-1}\lambda_j \mathsf{V}_{ij}.$$

Let $\mathsf{V}_{ij} = \sum_\mu w_i^{-1}w_j t^\mu$ be the decomposition into weight spaces. A monomial in $\mathsf{T}^{\mathrm{vir}}_{S,\lambda}$ is of the form $\lambda_i^{-1}\lambda_j w_i^{-1}w_j t^\mu$ and its Euler class is

$$e^{\mathbb{T}}(\lambda_i^{-1}\lambda_j w_i^{-1}w_j t^\mu) = \mu\cdot s + v_j + c_1^{\mathbb{T}}(\lambda_j) - v_i - c_1^{\mathbb{T}}(\lambda_i)$$

$$= \mu \cdot s + \bar{\nu}_j - \bar{\nu}_i$$

where we define $\bar{\nu}_i = \nu_i + c_1^{\mathbb{T}}(\lambda_i)$. We conclude that

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, \nu)_\lambda = \mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, \bar{\nu}),$$

which does not depend on $\bar{\nu}$ by Thm. **??**. ■

Example 5.25. *Set $r = 2$, $n = 1$, so that the only $\mathbb{T}$-fixed points in $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus 2}, 1)$ are the direct sums of ideal sheaves*

$$S_1 = \mathscr{I}_{\mathrm{pt}} \oplus \mathscr{O} \subset \mathscr{O}^{\oplus 2}, \quad S_2 = \mathscr{O} \oplus \mathscr{I}_{\mathrm{pt}} \subset \mathscr{O}^{\oplus 2},$$

where $\mathrm{pt} = (0,0,0) \in \mathbb{A}^3$ is the origin. One computes

$$\begin{aligned} T_{S_1}^{\mathrm{vir}} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_1^{-1}w_2\frac{1}{t_1t_2t_3} + w_2^{-1}w_1 \\ T_{S_2}^{\mathrm{vir}} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_2^{-1}w_1\frac{1}{t_1t_2t_3} + w_1^{-1}w_2. \end{aligned}$$

Therefore, the cohomological DT invariant is

$$\int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus 2},1)]^{\mathrm{vir}}} 1 = e^{\mathbb{T}}(-T_{S_1}^{\mathrm{vir}}) + e^{\mathbb{T}}(-T_{S_2}^{\mathrm{vir}}).$$

The part that could possibly depend on the framing parameters ν_1 and ν_2 is, in fact, constant:

$$\begin{aligned} e^{\mathbb{T}}\left(w_1^{-1}w_2\frac{1}{t_1t_2t_3} - w_2^{-1}w_1\right) + e^{\mathbb{T}}\left(w_2^{-1}w_1\frac{1}{t_1t_2t_3} - w_1^{-1}w_2\right) &= \\ = \frac{-\nu_1+\nu_2-\mathfrak{s}}{\nu_1-\nu_2} + \frac{\nu_1-\nu_2-\mathfrak{s}}{\nu_2-\nu_1} &= -2. \end{aligned}$$

Let λ_1 and λ_2 be two $\mathbb{T}_1$-equivariant line bundles. After the substitutions $w_i \to w_i\lambda_i$, and setting $\bar{\nu}_i = \nu_i + c_1(\lambda_i)$, the final sum of Euler classes depending on $\bar{\nu}$ becomes

$$2\frac{\bar{\nu}_2 - \bar{\nu}_1}{\bar{\nu}_1 - \bar{\nu}_2} = -2.$$

5.6 ELLIPTIC DONALDSON–THOMAS INVARIANTS

5.6.1 Chiral elliptic genus

In [22] an elliptic generalisation of DT invariants is given. In physics the invariants computed in loc. cit. are obtained as the superconformal index of a D1-D7 brane system on a type IIB $\mathcal{N} = 1$ supersymmetric background, where r D7-branes wrap the product of a 3-fold by a torus, i.e. $X_3 \times T^2$, while n D1-branes wrap T^2. The connection with enumerative geometry is then given via BPS-bound states counting, as brane systems often provide interesting constructions of relevant moduli spaces. In the case at hand, when $X_3 = \mathbb{A}^3$, the D1-D7 brane system considered has a BPS moduli space which can be naturally identified with the moduli space parametrising quotients of length

n of $\mathscr{O}_{\mathbb{A}^3}^{\oplus r}$. The superconformal index is usually identified in the physics literature with the elliptic genus of such a moduli space. This however does not coincide with the usual notion of (virtual) elliptic genus, as the coupling to the D7-branes breaks half of the chiral supersymmetry, thus leading to an effective 2d $\mathcal{N} = (0, 2)$ GLSM on T^2 for the D1-brane dynamics, whose Witten index generalises the K-theoretic DT invariants of $\mathbb{A}^3$ and provide a sort of chiral (or 1/2 BPS) version of the elliptic genus of the Quot scheme. In this section we give a mathematical definition of the elliptic invariants computed in [22], and show that our definition leads precisely to the computation of the same quantities found in [22, §3].

Let X be a scheme carrying a perfect obstruction theory $\mathbb{E} \to \mathbb{L}_X$ of virtual dimension $\mathrm{vd} = \mathrm{rk}\,\mathbb{E}$.

Definition 5.26. *If* F *is a rank* r *vector bundle on* X *we define*

$$\mathcal{E}_{1/2}(F) = \bigotimes_{n\geqslant 1} \mathrm{Sym}^{\bullet}_{p^n}\left(F \oplus F^{\vee}\right) \in 1 + p \cdot K^0(X)[\![p]\!] \tag{5.6.1}$$

where the total symmetric algebra $\mathrm{Sym}^{\bullet}_p(F) = \sum_{i\geqslant 0} p^i[S^iF] \in K^0(X)[\![p]\!]$ *satisfies* $\mathrm{Sym}^{\bullet}_p(F) = 1/\Lambda^{\bullet}_{-p}(F)$. *Note that* $\mathcal{E}_{1/2}$ *defines a homomorphism from the additive group* $K^0(X)$ *to the multiplicative group* $1 + p \cdot K^0(X)[\![p]\!]$. *Set*

$$\mathcal{E}\ell\ell_{1/2}(F; p) = (-p^{-\frac{1}{12}})^{\mathrm{rk}\,F}\,\mathrm{ch}\left(\mathcal{E}_{1/2}(F)\right) \cdot \mathrm{td}(F) \in A^*(X)[\![p]\!][p^{\pm\frac{1}{12}}], \tag{5.6.2}$$

where $\mathrm{td}(-)$ *is the Todd class, so that* $\mathcal{E}\ell\ell_{1/2}(-;p)$ *extends to a group homomorphism from* $K^0(X)$ *to the multiplicative group of units in* $A^*(X)[\![p]\!][p^{\pm\frac{1}{12}}]$.

We can then define the virtual chiral elliptic genus as follows.

Definition 5.27. *Let* X *be a proper scheme with a perfect obstruction theory and* $V \in K^0(X)$. *The* virtual chiral elliptic genus *is defined as*

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(X, V; p) = (-p^{-\frac{1}{12}})^{\mathrm{vd}}\chi^{\mathrm{vir}}\left(X, \mathcal{E}_{1/2}(T^{\mathrm{vir}}_X) \otimes V\right) \in \mathbb{Z}[\![p]\!][p^{\pm\frac{1}{12}}].$$

By the virtual Riemann–Roch theorem of [95] we can also compute the virtual chiral elliptic genus as

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(X, V; p) = \int_{[X]^{\mathrm{vir}}} \mathcal{E}\ell\ell_{1/2}(T^{\mathrm{vir}}_X; p) \cdot \mathrm{ch}(V).$$

Remark 5.12. One may give a more general definition by adding a "mass deformation" and defining $\mathcal{E}^{(y)}_{1/2}(F)$ for $F \in K^0(X)$ as

$$\mathcal{E}^{(y)}_{1/2}(F;p) = \bigotimes_{n\geqslant 1} \mathrm{Sym}^{\bullet}_{y^{-1}p^n}(F) \otimes \mathrm{Sym}^{\bullet}_{yp^n}\left(F^{\vee}\right) \in 1 + p \cdot K^0(X)[y, y^{-1}][\![p]\!],$$

so we recover the standard definition of virtual elliptic genus by taking $\mathcal{E}(F) = \mathcal{E}^{(1)}_{1/2}(F;p) \otimes \mathcal{E}^{(y)}_{1/2}(-F;p)$, cf. [95, §6]. ◀

Proposition 5.28. *Let* X *be a proper scheme with a perfect obstruction theory and let* $V \in K^0(X)$. *Then the virtual chiral elliptic genus* $\mathrm{Ell}^{\mathrm{vir}}_{1/2}(X, V; p)$ *is deformation invariant.*

Proof. The statement follows directly from Def. 5.27 and [95, Thm. 3.15]. ∎

Let now $V = \sum_\mu t^\mu$ be a $\mathbb{T}$-module as in §5.4.1. The trace of its symmetric algebra is given by

$$\operatorname{tr}\left(\mathrm{Sym}^\bullet_p(V)\right) = \operatorname{tr}\left(\frac{1}{\Lambda^\bullet_{-p}(V)}\right) = \prod_\mu \frac{1}{1-pt^\mu}.$$

Let us now assume as in §5.4.1 that $\det V$ is a square in $K^0_{\mathbb{T}}(\mathrm{pt})$ and $\mu = 0$ is not a weight of V. We can then compute the trace of the symmetric product in (5.6.1) as

$$\operatorname{tr}\left(\bigotimes_{n\geqslant 1} \mathrm{Sym}^\bullet_{p^n}\left(V\oplus V^\vee\right)\right) = \prod_\mu \prod_{n\geqslant 1} \frac{1}{(1-p^n t^\mu)(1-p^n t^{-\mu})},$$

$$\begin{aligned}\operatorname{tr}\left(\frac{\bigotimes_{n\geqslant 1} \mathrm{Sym}^\bullet_{p^n}\left(V\oplus V^\vee\right)}{\Lambda^\bullet V^\vee}\right) &= \prod_\mu \frac{1}{1-t^{-\mu}} \prod_{n\geqslant 1} \frac{1}{(1-p^n t^\mu)(1-p^n t^{-\mu})} \\ &= \left(-ip^{1/8}\phi(p)\right)^{\operatorname{rk} V} \prod_\mu \frac{t^{\mu/2}}{\theta(p;t^\mu)},\end{aligned}$$

where $\phi(p)$ is the Euler function, i.e. $\phi(p) = \prod_n (1-p^n)$, and $\theta(p;y)$ denotes the Jacobi theta function

$$\theta(p;y) = -ip^{1/8}(y^{1/2}-y^{-1/2}) \prod_{n=1}^\infty (1-p^n)(1-yp^n)(1-y^{-1}p^n).$$

Combining everything together we are able to compute the trace

$$A = (-p^{-\frac{1}{12}})^{\operatorname{rk} V} \operatorname{tr}\left(\frac{\bigotimes_{n\geqslant 1} \mathrm{Sym}^\bullet_{p^n}\left(V\oplus V^\vee\right)\otimes \det(V^\vee)^{1/2}}{\Lambda^\bullet V^\vee}\right)$$

to get the identity

$$\begin{aligned}A &= (-p^{-\frac{1}{12}})^{\operatorname{rk} V}\left(-ip^{1/8}\phi(p)\right)^{\operatorname{rk} V} \prod_\mu \frac{1}{\theta(p;t^\mu)} \\ &= \prod_\mu i\frac{\eta(p)}{\theta(p;t^\mu)},\end{aligned}$$

where $\eta(p)$ is the Dedekind eta function

$$\eta(p) = p^{\frac{1}{24}} \prod_{n\geqslant 1} (1-p^n).$$

For a virtual $\mathbb{T}$-representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu \in K^0_{\mathbb{T}}(\mathrm{pt})$ where $\mu = 0$ is not a weight of V, we compute

$$(-p^{-\frac{1}{12}})^{\operatorname{rk} V} \operatorname{tr}\left(\frac{\bigotimes_{n\geqslant 1} \mathrm{Sym}^\bullet_{p^n}\left(V\oplus V^\vee\right)\otimes \det(V^\vee)^{1/2}}{\Lambda^\bullet V^\vee}\right) =$$

$$= (i \cdot \eta(p))^{\operatorname{rk} V} \frac{\prod_\nu \theta(p; t^\nu)}{\prod_\mu \theta(p; t^\mu)}.$$

For the remainder of the section we set $p = e^{2\pi i\tau}$, with $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$. Denoting $\theta(\tau|z) = \theta(e^{2\pi i\tau}; e^{2\pi i z})$, θ enjoys the modular behaviour

$$\theta(\tau|z + a + b\tau) = (-1)^{a+b} e^{-2\pi i b z} e^{-i\pi b^2\tau} \theta(\tau|z), \quad a, b \in \mathbb{Z}.$$

Analogously to the measure $[\cdot]$ for K-theoretic invariants, we define the *elliptic measure*

$$\theta[V] = (i \cdot \eta(p))^{-\operatorname{rk} V} \frac{\prod_\mu \theta(p; t^\mu)}{\prod_\nu \theta(p; t^\nu)},$$

which satisfies $\theta[\overline{V}] = (-1)^{\operatorname{rk} V} \theta[V]$. Notice that, if $\operatorname{rk} V = 0$, the elliptic measure refines both $[\cdot]$ and $e^{\mathbb{T}}(\cdot)$

$$\theta[V] \xrightarrow{p \to 0} [V] \xrightarrow{b \to 0} e^{\mathbb{T}}(V)$$

where the second limit was dicussed in §5.5.1.

Remark 5.13. The definition we gave for virtual chiral elliptic genus is reminiscent of what is commonly known in physics as the elliptic genus (or superconformal index) of a 2d $\mathcal{N} = (0, 2)$ sigma model. Our definition actually matches the one in [138] for $\mathcal{N} = (0, 2)$ Landau–Ginzburg models. Indeed, in this case we are given an n-dimensional (compact) Kähler manifold X together with a holomorphic vector bundle $E \to X$ such that $c_1(E) - c_1(T_X) = 0$ mod 2. If we then consider the K-theory class $[V] = [T_X] - [E]$, the superconformal index of [138] can be written as

$$\mathcal{I}(X, E; p) = (-ip^{-\frac{1}{12}})^{\operatorname{rk} V} \chi\left(X, \mathcal{E}_{1/2}(V) \otimes \det(V^\vee)^{\frac{1}{2}}\right)$$

and in terms of the Chern roots ν_i, w_j of T_X and E, respectively, we also have (cf. [98])

$$\mathcal{I}(X, E; p) = \int_X \prod_{i=1}^{r} \frac{\theta(\tau|\frac{w_i}{2\pi i})}{\eta(p)} \prod_{j=1}^{n} \frac{\nu_j \eta(p)}{\theta(\tau|\frac{-\nu_j}{2\pi i})}.$$ ◂

5.6.2 Elliptic DT invariants

Definition 5.29. *The generating series of elliptic DT invariants* $\mathrm{DT}_r^{\mathrm{ell}}(\mathbb{A}^3, q, t, w; p)$ *is defined as*

$$\mathrm{DT}_r^{\mathrm{ell}}(\mathbb{A}^3, q, t, w; p) = \sum_{n \geqslant 0} q^n \operatorname{Ell}^{\mathrm{vir}}_{1/2}(\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}; p) \in \mathbb{Z}(\!(t, t^{\frac{1}{2}}, w)\!)[\![p, q]\!].$$

Being that $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ *is not projective, but nevertheless carries the action of an algebraic torus* $\mathbb{T}$ *with proper* $\mathbb{T}$*-fixed locus, we define the invariants by means of virtual localisation, as we explained in §1.5.*

At each $\mathbb{T}$-fixed point $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$, the localised contribution is

$$\operatorname{tr}\left(\frac{\bigotimes_{n\geqslant 1}\mathrm{Sym}^{\bullet}_{p^n}\left(\mathrm{T}^{\mathrm{vir}}_S \oplus \mathrm{T}^{\mathrm{vir},\vee}_S\right)}{\widehat{\Lambda}^{\bullet}\mathrm{T}^{\mathrm{vir},\vee}_S}\right)$$

from which we deduce that we can recover the K-theoretic invariants in the limit $p \to 0$. As for K-theoretic invariants, we have

$$\begin{aligned}\mathrm{DT}^{\mathrm{ell}}_r(\mathbb{A}^3, q, t, w; p) &= \sum_{n\geqslant 0} q^n \sum_{[S]\in\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbb{T}}} \theta[-\mathrm{T}^{\mathrm{vir}}_S] \\ &= \sum_{\overline{\pi}} q^{|\overline{\pi}|}\prod_{i,j=1}^{r}\theta[-\mathsf{V}_{ij}],\end{aligned}$$

where $\overline{\pi}$ runs over all r-coloured plane partitions.

Contrary to the case of K-theoretic and cohomological invariants, there exists no conjectural closed formula for elliptic DT invariants yet, even for the rank 1 case. Moreover, the generating series depends on the equivariant parameters of the framing torus, as shown in the following example.

Example 5.30. *Consider* $Q^3_1 = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus 3}, 1)$*, whose only* $\mathbb{T}$*-fixed points are*

$$\begin{aligned}S_1 &= \mathscr{I}_{\mathrm{pt}} \oplus \mathscr{O}^{\oplus 2} \subset \mathscr{O}^{\oplus 3},\\ S_2 &= \mathscr{O} \oplus \mathscr{I}_{\mathrm{pt}} \oplus \mathscr{O} \subset \mathscr{O}^{\oplus 3},\\ S_3 &= \mathscr{O}^{\oplus 2} \oplus \mathscr{I}_{\mathrm{pt}} \subset \mathscr{O}^{\oplus 3},\end{aligned}$$

with $\mathrm{pt} = (0,0,0) \in \mathbb{A}^3$ *as in Example 5.25. We have*

$$\begin{aligned}\mathrm{T}^{\mathrm{vir}}_{S_1} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_1^{-1}w_2\frac{1}{t_1t_2t_3}\\ &\quad + w_2^{-1}w_1 - w_1^{-1}w_3\frac{1}{t_1t_2t_3} + w_3^{-1}w_1,\\ \mathrm{T}^{\mathrm{vir}}_{S_2} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_2^{-1}w_1\frac{1}{t_1t_2t_3}\\ &\quad + w_1^{-1}w_2 - w_2^{-1}w_3\frac{1}{t_1t_2t_3} + w_3^{-1}w_2,\\ \mathrm{T}^{\mathrm{vir}}_{S_3} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_3^{-1}w_1\frac{1}{t_1t_2t_3}\\ &\quad + w_1^{-1}w_3 - w_3^{-1}w_2\frac{1}{t_1t_2t_3} + w_2^{-1}w_3,\end{aligned}$$

by which we may compute the corresponding elliptic invariant. Set $w_j = e^{2\pi i v_j}$ *and* $t_\ell = e^{2\pi i s_\ell}$*, so that*

$$\begin{aligned}\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{\frac{1}{2}}_{\mathrm{vir}}, t, w; p) &= \diamondsuit \cdot \left(\frac{\theta(\tau|v_2-v_1-s)\theta(\tau|v_3-v_1-s)}{\theta(\tau|v_1-v_2)\theta(\tau|v_1-v_3)}\right.\\ &\left. + \frac{\theta(\tau|v_1-v_2-s)\theta(\tau|v_3-v_2-s)}{\theta(\tau|v_2-v_1)\theta(\tau|v_2-v_3)} + \frac{\theta(\tau|v_1-v_3-s)\theta(\tau|v_2-v_3-s)}{\theta(\tau|v_3-v_1)\theta(\tau|v_3-v_2)}\right),\end{aligned}$$

where $s = s_1 + s_2 + s_3$, *with the overall factor*

$$\diamondsuit = \frac{\theta(\tau|s_1+s_2)\theta(\tau|s_1+s_3)\theta(\tau|s_2+s_3)}{\theta(\tau|s_1)\theta(\tau|s_2)\theta(\tau|s_3)}.$$

Moreover, by evaluating residues in $\nu_i - \nu_j = 0$ *one can realise that* $\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{1/2}_{\mathrm{vir}}, t, w; p)$ *has no poles in* ν_i. *Indeed*

$$\begin{aligned}\mathrm{Res}_{\nu_1-\nu_2=0}\, &\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{\frac{1}{2}}_{\mathrm{vir}}; p) = \\ &= \diamondsuit \cdot \left(\frac{\theta(\tau|-s)\theta(\tau|\nu_3-\nu_2-s)}{\theta(\tau|\nu_2-\nu_3)} - \frac{\theta(\tau|-s)\theta(\tau|\nu_3-\nu_2-s)}{\theta(\tau|\nu_2-\nu_3)} \right) = 0,\end{aligned}$$

and the same occurs for any other pole involving the ν_i*'s. However, this does not imply the independence of the elliptic invariants from* ν, *as we now suggest.*

Set $\bar{\nu}_i = \nu_i + a_i + b_i\tau$, *with* $a_i, b_i \in \mathbb{Z}$, *for* $i = 1,2,3$. *Applying the quasi-periodicity of theta functions, we get*

$$\begin{aligned}\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{\frac{1}{2}}_{\mathrm{vir}}, t, \overline{w}; p) = &\frac{\diamondsuit}{\theta(\tau|\bar{\nu}_1-\bar{\nu}_2)\theta(\tau|\bar{\nu}_1-\bar{\nu}_3)\theta(\tau|\bar{\nu}_2-\bar{\nu}_3)} \cdot \\ &(\theta(\tau|\bar{\nu}_2-\bar{\nu}_1-s)\theta(\tau|\bar{\nu}_3-\bar{\nu}_1-s)\theta(\tau|\bar{\nu}_2-\bar{\nu}_3) \\ &-\theta(\tau|\bar{\nu}_1-\bar{\nu}_2-s)\theta(\tau|\bar{\nu}_3-\bar{\nu}_2-s)\theta(\tau|\bar{\nu}_1-\bar{\nu}_3) \\ &+\theta(\tau|\bar{\nu}_1-\bar{\nu}_3-s)\theta(\tau|\bar{\nu}_2-\bar{\nu}_3-s)\theta(\tau|\bar{\nu}_1-\bar{\nu}_2)) \\ = &\frac{\diamondsuit}{\theta(\tau|\nu_1-\nu_2)\theta(\tau|\nu_1-\nu_3)\theta(\tau|\nu_2-\nu_3)} \cdot \\ &\Big(e^{2\pi i s(b_2+b_3-2b_1)}\theta(\tau|\nu_2-\nu_1-s)\theta(\tau|\nu_3-\nu_1-s)\theta(\tau|\nu_2-\nu_3) \\ &-e^{2\pi i s(b_1+b_3-2b_2)}\theta(\tau|\nu_1-\nu_2-s)\theta(\tau|\nu_3-\nu_2-s)\theta(\tau|\nu_1-\nu_3) \\ &+e^{2\pi i s(b_1+b_2-2b_3)}\theta(\tau|\nu_1-\nu_3-s)\theta(\tau|\nu_2-\nu_3-s)\theta(\tau|\nu_1-\nu_2)\Big).\end{aligned}$$

Notice that for general values of s *we have*

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{1/2}_{\mathrm{vir}}, t, \overline{w}; p) \neq \mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{1/2}_{\mathrm{vir}}, t, w; p).$$

However, if we specialise $s \in \frac{1}{3}\mathbb{Z}$, *we see that in the previous example the chiral elliptic genus becomes constant and periodic with respect to* ν *on the lattice* $\mathbb{Z} + 3\tau\mathbb{Z}$ *and is holomorphic in* ν, *from which we conclude that it is constant on* ν *under this specialisation. Therefore, by choosing* $w_j = e^{2\pi i \frac{j}{3}}$ *to be third roots of unity, one can show*

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q^3_1, \mathcal{K}^{\frac{1}{2}}_{\mathrm{vir}}, t, w; p)\Big|_{t=e^{2\pi i \frac{k}{3}}} = \begin{cases} (-1)^{m+1}3, & \textit{if } k = 3m, \quad m \in \mathbb{Z} \\ 0, & \textit{if } k \notin 3\mathbb{Z}. \end{cases}$$

5.6.3 Limits of elliptic DT invariants

Even if a closed formula for the higher rank generating series of elliptic DT invariants is not available, we can still study its behaviour by looking at some particular limits of the variables p, t_i, w_j.

It is easy to see that, under the Calabi–Yau restriction $t = 1$, the generating series of elliptic DT invariants does not carry any more refined information than the cohomological one; in particular, we have no more dependence on the framing parameters w_j and the elliptic parameter p. We generalise this phenomenon in the following setting. Denote by $\mathbb{T}_k \subset \mathbb{T}_1$ the subtorus where $t^{\frac{1}{2}} = e^{\pi i k/r}$, $k \in \mathbb{Z}$. Define by

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{DT}^{\mathrm{ell}}_{r}(\mathbb{A}^3, q, t, w; p)\Big|_{\mathbb{T}_k}$$

the restriction of the generating series to the subtorus $\mathbb{T}_k \subset \mathbb{T}_1$, which is well-defined as no powers of the Calabi–Yau weight appear in the vertex terms (5.3.5) by Lemma 5.4.

Proposition 5.31. *If $k = rm \in r\mathbb{Z}$, then*

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{M}((-1)^{r(m+1)}q)^r.$$

In particular, the dependence on t_i, w_j and p drops.

Proof. Let $S \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}}$. Denote $T^{\mathrm{vir}}_S = T^{nc}_S - \overline{T^{nc}_S}t^{-1}$ as in Eq. (5.1.16), where T^{nc}_S is the tangent space of NCQuot^n_r at S. Denote by $T^{nc}_{S,l}$ the sub-representation of T^{nc}_S corresponding to t^l, with $l \in \mathbb{Z}$. As there are no powers of the Calabi–Yau weight in T^{vir}_S, we have an identity $T^{nc}_{S,l} = \overline{T^{nc}_{S,-l-1}}t^{-1}$. Set

$$W = T^{nc}_S - \sum_{n \in \mathbb{Z}} T^{nc}_{S,n}.$$

We have that

$$T^{nc}_S - \overline{T^{nc}_S}t^{-1} = W - \overline{W}t^{-1}$$

and, in particular, neither W nor $\overline{W}t^{-1}$ contain constant terms and powers of the Calabi-Yau weight. Using the quasi-periodicity of the theta function $\theta(\tau|z)$, we have

$$\theta[-T^{\mathrm{vir}}_S] = \frac{\theta[\overline{W}t^{-1}]}{\theta[W]} = (-1)^{m \operatorname{rk} W}\frac{\theta[\overline{W}]}{\theta[W]} = (-1)^{\operatorname{rk} W(m+1)}.$$

We conclude by noticing that

$$\operatorname{rk} W = \operatorname{rk} T^{nc}_S = rn \mod 2. \qquad \blacksquare$$

Motivated by Example 5.30 and Prop. 5.31, we propose the following conjecture.

Conjecture 5.32. *The series $\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p)$ does not depend on the elliptic parameter p.*

Remark 5.14. Notice that the independence from the elliptic parameter p implies that we can reduce our invariants to the K-theoretic ones by setting $p = 0$, i.e.

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{DT}^{K}_{r}(\mathbb{A}^3, q, t)\big|_{\mathbb{T}_k},$$

which in particular do not depend on the framing parameters. ◀

Assuming Conjecture 5.32, we derive a closed expression for $\mathrm{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p)$, which was conjectured in [22, Eq. (3.20)], motivated by string-theoretic phenomena.

Theorem 5.33. *Assume Conjecture 5.32 holds and let* $k \in \mathbb{Z}$. *Then there is an identity*

$$\mathrm{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{M}\left((-1)^{kr}((-1)^r q)^{\frac{r}{\gcd(k,r)}}\right)^{\gcd(k,r)}.$$

Proof. Assuming Conjecture 5.32, by Remark 5.14 we just have to prove the result for K-theoretic invariants. By Thm. 5.17,

$$\begin{aligned}\mathrm{DT}^{K}_{r}(\mathbb{A}^3, (-1)^r q, t) = \exp \sum_{n\geqslant 1} \frac{1}{n} \frac{(1-t_1^{-n}t_2^{-n})(1-t_1^{-n}t_3^{-n})(1-t_2^{-n}t_3^{-n})}{(1-t_1^{-n})(1-t_2^{-n})(1-t_3^{-n})} \\ \frac{1-t^{-rn}}{1-t^{-n}} \frac{1}{(1-t^{-\frac{rn}{2}}q^{-n})(1-t^{-\frac{rn}{2}}q^{n})}.\end{aligned}$$

Assume now that $t^{\frac{1}{2}} = e^{\pi i \frac{k}{r}}$, with $k \in \mathbb{Z}$; we have clearly that $t^{-\frac{rn}{2}} = (-1)^{kn}$. Moreover, we have

$$\frac{1-t^{-rn}}{1-t^{-n}} = \begin{cases} r, & \text{if } n \in \frac{r}{\gcd(r,k)}\mathbb{Z} \\ 0, & \text{if } n \notin \frac{r}{\gcd(r,k)}\mathbb{Z} \end{cases}$$

In particular, if $n \in \frac{r}{\gcd(r,k)}\mathbb{Z}$, we have

$$\frac{(1-t_1^{-n}t_2^{-n})(1-t_1^{-n}t_3^{-n})(1-t_2^{-n}t_3^{-n})}{(1-t_1^{-n})(1-t_2^{-n})(1-t_3^{-n})} = -1$$

Setting $n = \frac{r}{\gcd(r,k)}m$, with $m \in \mathbb{Z}$, we have

$$\mathrm{DT}^{K}_{r}(\mathbb{A}^3, (-1)^r q, t) = \exp \sum_{m\geqslant 1} \frac{1}{m} \gcd(r,k) \cdot \frac{-1}{(1-\overline{q}^{-m})(1-\overline{q}^{m})}$$

where to ease notation we have set $\overline{q} = ((-1)^k q)^{\frac{r}{\gcd(r,k)}}$. We conclude by using the description of the MacMahon function as a plethystic exponential

$$\mathrm{DT}^{K}_{r}(\mathbb{A}^3, (-1)^r q, t) = \mathsf{M}\left((-1)^{kr} q^{\frac{r}{\gcd(r,k)}}\right)^{\gcd(r,k)}. \qquad \blacksquare$$

Remark 5.15. A key technical point in the proof of the conjecture proposed in [22, Eq. (3.20)] was the assumption of the independence of $\mathrm{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3)$ on p, as in Conjecture 5.32. We strongly believe it should be possible to prove this assumption by exploiting modular properties of the generating series of elliptic DT invariants. One could proceed by considering the integral representation for the generating series, given in [22, Eq. (3.1)]. The analysis of the K-thoretic case, which we carried out in the proof of Thm. 5.33, shows that no dependence whatsoever is present in the limit $t^{1/2} = e^{\pi i k/r}$. As elliptic DT invariants take the form of meromorphic Jacobi forms, given by quotients of theta functions, poles in the equivariant parameters are only given by shifts along the lattice $\mathbb{Z} + \tau\mathbb{Z}$ of the poles found in K-theoretic

DT invariants. Then $\mathrm{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3)$, as a function of each of the equivariant parameters v_i, $i = 1, \dots, r$, and s_j, $j = 1, 2, 3$, is holomorphic on the torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, so it also carries no dependence on them. This observation may be not very surprising, if one considers the striking resemblance of the chiral virtual elliptic genus to the usual level$-N$ elliptic genus of almost complex manifolds, which is known to be often rigid. In our case, each q-term in the elliptic generating series restricted to $\mathbb{T}_k$ would be now invariant under modular transformations on τ, hence a constant in $p = e^{2\pi i \tau}$. ◀

5.6.4 Relation to string theory

The definition for the elliptic version of Donaldson–Thomas invariants is motivated by an argument due to Witten [219], which goes as follows: let M be a $2k$-dimensional spin manifold, and take $\mathscr{L}M = C^0(S^1, M)$ to be the free loop space on M. Then $\mathscr{L}M$ always carries a natural S^1-action, given by the rotation of loops, so that fixed points under this action of S^1 will only be constant maps $S^1 \to M$, and $(\mathscr{L}M)^{S^1} \cong M \hookrightarrow \mathscr{L}M$. One can then study the Dirac operator on $\mathscr{L}M$ by formally computing its index using fixed point formulae. In particular, if $D\colon \Gamma(S_+) \to \Gamma(S_-)$ is the Dirac operator on M then

$$\mathrm{Ind}(D) = \dim \ker D - \dim \operatorname{coker} D,$$

and whenever M admits the action of a compact Lie group G one can define the G-equivariant index of D as the virtual character

$$\mathrm{Ind}_G(D)(g) = \mathrm{tr}_{\ker D}\, g - \mathrm{tr}_{\operatorname{coker} D}\, g, \qquad g \in G,$$

which only depends on the conjugacy class of g in G. In the case of Dirac operators on loop spaces over spin manifolds, a formal computation yields

$$\mathrm{Ind}_{S^1}(D)(q) = q^{-\frac{d}{24}} \hat{A}(M) \cdot \mathrm{ch}\left(\bigotimes_{n \geqslant 1} \mathrm{Sym}^{\bullet}_{q^n} T_M \right) \cap [M],$$

where q denotes a topological generator of S^1. The $\hat{A}$-genus is the characteristic class which computes the index of the Dirac complex on a spin manifold. In general, if E is any rank r complex vector bundle on M, one can define $\hat{A}(E)$ in terms of the Chern roots x_i of E as

$$\hat{A}(E) = \prod_{i=1}^{r} \frac{x_i/2}{\sinh x_i/2},$$

and $\hat{A}(M) = \hat{A}(TM)$, which is completely analogous to the K–theoretic invariants we have been studying so far. The previous formula can also be classically interpreted as the index of a twisted Dirac operator over the spin structure of M. It is worth noticing that the DT partition functions coming from physics are indeed interpreted as being computing indices of twisted Dirac operators, where the twist by a vector bundle $V \to M$ makes sense only if $w_2(T_M) = w_2(V)$ so as to extend D to an operator $D\colon \Gamma(S_+ \otimes V) \to \Gamma(S_- \otimes V)$. In this same spirit one might also justify the definition of the half-BPS elliptic genus in terms of computations of Euler characteristics of loop spaces over (compact) almost complex manifolds. Let then X be a d-dimensional almost complex manifold, with holomorphic tangent bundle T_X, and whose corresponding free loop space will be denoted by $\mathscr{L}X$. As it was the case also in the previous situation, $\mathscr{L}X$ is naturally equipped with an S^1 action, whose fixed point will be $(\mathscr{L}X)^{S^1} \cong X \hookrightarrow \mathscr{L}X$. By formally applying the virtual localisation formula to the computation of the Euler characteristic of $\mathscr{L}X$ one gets

$$\chi_{S^1}(\mathscr{L}X) = q^{-\frac{d}{12}} \operatorname{td}(X) \cdot \operatorname{ch}\left(\bigotimes_{n\geqslant 1} \operatorname{Sym}^{\bullet}_{q^n}(T_X \oplus \Omega_X)\right) \cap [X],$$

which can also be seen as the index of a twisted Spin^c-Dirac operator $\overline{\partial} + \overline{\partial}^*$. Moreover, if $c_1(T_X) = 0 \mod 2$, X is also spin, and it is possible to compute the index of the Dirac operator on $\mathscr{L}X$ as before.

6 EIGHT-DIMENSIONAL ADHM CONSTRUCTION

The aim of this chapter is to study the moduli space of solutions of an eight dimensional analog of the celebrated self-duality equation $F = \star F$ for the gauge theory curvature in four dimensions [21]. The equation in eight dimensions reads

$$F \wedge T = \star F \tag{6.0.1}$$

where $F = dA + A \wedge A$ is the curvature of the gauge bundle, T is an invariant closed four-form and $\star$ is the Hodge star operator with respect to a given Riemannian structure on the eight dimensional manifold. Eq. (6.0.1) was introduced in [75] in 1982. The very existence of the invariant four-form T restricts the holonomy group of the eight dimensional manifold X to be contained in $\mathrm{Spin}(7)$ [136].

We provide the eight dimensional analog of the ADHM [9] construction for (6.0.1) with $U(N)$ gauge group for $X = \mathbb{C}^4$ and its discrete Calabi-Yau quotients. As we will discuss, differently from the real four dimensional case, genuine solutions to the above equation exist only on eight dimensional spaces whose local model is a non-commutative deformation of $\mathbb{C}^4$. The latter is obtained by deforming one of the moment maps à la Nekrasov-Schwarz [178].

This further breaks the holonomy group $\mathrm{Spin}(7) \to \mathrm{Spin}(6) \simeq SU(4)$, implying that X is a Calabi-Yau fourfold. The corresponding gauge theory can be engineered as the low energy limit of a $D(-1)/D7$ system in a stabilising non trivial B-field background [225], aligned along the invariant 2-form associated to the deformed moment map. The more general configuration that we will study includes also a set of $\overline{D7}$s which act as a source of matter field-/observables, as in [173; 177]. By resorting to our higher dimensional ADHM construction, we provide explicit solutions of Eq. (6.0.1) in the abelian case. Let us notice that $\mathbb{C}^4$ admits a $(\mathbb{C}^*)^3$ toric action compatible with its (trivial) Calabi-Yau structure, which naturally lifts to the moduli space of solutions to (6.0.1). We describe the invariant solutions supported at the fixed points of this toric action, by making crucial use of the non-commutative deformation, and find that these are classified by solid partitions. These are a four dimensional analog of Young diagrams built with hypercubes accumulating on the corner of $\mathbb{R}^4_+$. This provides a lift to four complex dimensions of the statistical crystal melting model based on plane partitions, see [129] for the $U(1)$ case and [74; 96] for $U(N)$. All this construction has a natural extension on discrete quotients of $\mathbb{C}^4$, the fixed points being described in this case by coloured solid partitions.[1]

As it is well known, the ADHM construction for four-dimensional instantons is at the root of an isomorphism with the moduli space of framed torsion free coherent sheaves on $\mathbb{P}^2$ [86]. We provide here an analog isomor-

1 Let us remark that on (partial) resolutions of $\mathbb{C}^4$ orbifolds one can also construct abelian instantons whose gauge flux is along the non-trivial two cycles $H_2^-(X, \mathbb{Z})$.

phism between the moduli space of solution of (6.0.1) on non-commutative $\mathbb{C}^4$ and the moduli space of framed torsion free coherent sheaves on $\mathbb{P}^4$, extending it also to the orbifold case by adapting the Kronheimer-Nakajima construction [147]. In this context the fixed points are described by ideal sheaves on $\mathbb{C}^4$ and its quotients.

Building on Eq. (6.0.1), one can construct [13] a (semi)Topological Field Theory[2], which is indeed a topological twist of the eight dimensional gauge theory describing the $D(-1)/D7$ system at low energy [1; 33]. This provides the setting for BPS-bound states counting whose mathematical counterpart is given by Donaldson-Thomas theory on four-folds [88]. Let us remark that major progresses have been recently obtained on the compactification of the moduli space of solutions of (6.0.1) and a rigorous definition of the associated enumerative invariants [49; 58–61; 183]. A natural extension is to consider the theory on $S^1 \times X$ computing the Witten index of $D0/D8/\overline{D8}$ bound states whose mathematical counterpart is the lift to K-theory. On toric manifolds one can study the equivariant extension of the sTQFT so providing a geometrically motivated statistical weight for counting solid partitions which describe the fixed points of the gauge theory moduli space [177]. Chiral ring observables can be introduced via descent equations as in the four dimensional case [13]. Their explicit evaluation on $\mathbb{C}^4$ via equivariant localisation recently appeared in [58; 177].

Let us remark that M-theory on local four-folds provides a geometric engineering description of supersymmetric gauge theories in three dimensions [128], analogously to the much better known case of local three-folds [125] which instead describes five dimensional supersymmetric gauge theories. Moreover, interesting classes of $(0,2)$ supersymmetric models in two-dimensions arise from D1-branes probing toric Calabi-Yau four-fold singularities [99]. It is thus interesting to study the eight dimensional BPS counting problem on some examples of local CY four-fold geometries. To this end, in this chapter we also provide the generalisation of the eight-dimensional ADHM-like quiver to orbifolds $\mathbb{C}^4/G$, where G is a discrete subgroup of $SU(4)$. The fixed points in this case are classified by coloured solid partitions whose statistical weight depends on the representation of G. This boils down to count G-coloured hypercubes configurations whose colouring rules are dictated by the specific action of G on $\mathbb{C}^4$. This provides an eight dimensional analog of instanton counting on four-dimensional ALE spaces [19; 20; 40–42; 53; 55; 100]. As an example, we explicitly address the associated K-theoretic counting problem on $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2$.

There are a number of interesting problems to be addressed. Supersymmetric gauged linear sigma models in two dimensions modelled on the ADHM-like quivers presented in this chapter can be studied via localisation technique. In the sphere case this could possibly shed light on the associated quantum cohomology and its relation with quantum integrable hydrodynamics analogously to the four dimensional ADHM quiver studied in [43–45]. The torus case would allow to compute the elliptic genus of the eight dimensional ADHM moduli space and as such to provide an elliptic lift of Donaldson-Thomas invariants on four-folds analogous to the one stud-

2 The "semi" refers to the dependence on the four-form T, which calibrates the volume of the four-cycles in the eight dimensional manifold.

ied in [22] on three-folds. Finally defects operators can be investigated by a generalisation of the ADHM-like quivers, as for example the orbifold version of [134] and the eight dimensional generalisation of nested instantons we studied in chapter 3.

6.1 TOPOLOGICAL GAUGE THEORY IN EIGHT DIMENSIONS

Let X be a real Riemannian eight dimensional differentiable manifold with a torsion free Spin(7) structure [136]. This is determined by a real covariantly constant spinor ψ_+

$$T = \psi_+^T \Gamma^{[\wedge 4]} \psi_+ \in \Lambda^4(X)$$

where $\Gamma \equiv \Gamma_\mu dx^\mu$ and $\{\Gamma_\mu\}_{\mu=1,\dots,8}$ the SO(8) Γ-matrices. We assume ψ_+ to be of positive chirality and normalised as $\psi_+^T \psi_+ = 1$.

On X the vector spaces of p-forms $\Lambda^p(X)$ split in irreducible representations of the holonomy group. In particular one has $\Lambda^2(X) = \Lambda^2_7(X) \oplus \Lambda^2_{21}(X)$. This split corresponds to the projections on $\omega \in \Lambda^2(X)$ given by $T \wedge \omega = -3 \star \omega$ and $T \wedge \omega = \star\omega$ respectively.

The Spin-bundles on X are isomorphic to $S^+(X) \sim \Lambda^1(X)$ and $S^-(X) \sim \Lambda^0(X) \oplus \Lambda^2_7(X)$. A supersymmetric gauge theory can be formulated on a Spin(7) manifold via a topological twist which uses these isomorphisms [1; 13; 33]. The corresponding twisted supersymmetry transformations of the gauge theory can be made equivariant with respect to an isometry V and read

$$\begin{aligned} &QA = \Psi\,, \quad Q\Psi = \iota_V F - iD\Phi\,, \quad Q\Phi = \iota_V \Psi \\ &Q\chi_7 = B_7\,, \quad QB_7 = \mathcal{L}_V \chi_7 + i[\Phi, \chi_7]\,, \quad Q\eta = \bar{\Phi} \\ &Q\bar{\Phi} = \iota_V D\eta + i[\Phi, \eta]\,, \end{aligned} \tag{6.1.1}$$

where V is any isometry of the Spin(7) structure, that is $\star L_V = L_V \star$ and $L_V T = 0$, where $L_V \equiv \iota_V d + d\iota_V$ is the Lie derivative. In (6.1.1), $\mathcal{L}_V \equiv \iota_V D + D\iota_V$ is the covariant Lie derivative. Notice that in (6.1.1), $\Psi \in S^+(X) \sim \Lambda^1(X)$ and in the second line (η, χ_7) and $(\bar{\Phi}, B_7) \in S^-(X) \sim \Lambda^0(X) \oplus \Lambda^2_7(X)$.

The supersymmetric action after the topological twist can be written as a topological term plus a Q-exact one as

$$S = \int_X T \wedge \mathrm{Tr}\,(F \wedge F) + Q\nu\,,$$

where

$$\nu = \int_X \mathrm{Tr}\left[i \star \chi_7 \wedge F + \Psi \wedge \star (Q\Psi)^\dagger + g^2 \chi_7 \wedge \star (Q\chi_7)^\dagger + \eta \wedge \star (Q\eta)^\dagger \right]\,,$$

where g is the Yang-Mills coupling constant which in the topological theory is a gauge fixing parameter. In the path-integral, in the δ-gauge $g = 0$, the B_7 field appears as the Lagrange multiplier for the Spin(7)-instanton equation

$$F_7 = 0\,,$$

which is nothing but a rewriting of eq (6.0.1). In the following Section we will provide an ADHM-like description of the solutions to the above equation and their moduli space. This will turn out to have positive (virtual) dimension, inducing a $U(1)_R$-anomaly due to the presence of chiral fermionic zero-modes. In order to have a non-vanishing result one has thus to insert non-trivial observables in analogy with the well known case of Donaldson theory in four real dimensions [219]. The observables are given by non-trivial cohomology classes of the twisted supersymmetry (6.1.1), and can be obtained from an equivariant version of the usual descent equations, see [30] for the four-dimensional case. Indeed, the supersymmetry transformations (6.1.1) can be rewritten as the equivariant Bianchi identity for the curvature $\mathcal{F} = F + \Psi + \Phi$ of the universal bundle as [12]

$$\mathcal{D}\mathcal{F} \equiv (-Q + D + \iota_V)(F + \Psi + \Phi) = 0\,,$$

and expanding in the de Rham form degree. Picking an ad-invariant polynomial $\mathcal{P}$ on the Lie algebra of the gauge group, we have

$$Q\mathcal{P}(\mathcal{F}) = (d + \iota_V)\,\mathcal{P}(\mathcal{F})\,,$$

so that one can build the equivariant observables as intersection of the above with elements of the equivariant cohomology of the manifold, $\boldsymbol{\Omega} \in H^\bullet_V(X)$ as

$$\mathcal{O}(\boldsymbol{\Omega}, \mathcal{P}) \equiv \int_X \boldsymbol{\Omega} \wedge \mathcal{P}(\mathcal{F}).$$

In the path integral formulation of the gauge theory, we will actually consider the generating function of the equivariant observables through the determinant bundle

$$\mathcal{O}_{\det}(\boldsymbol{\Omega}) \equiv \int_X \boldsymbol{\Omega} \wedge \det(m\mathbf{1} + \mathcal{F})\,,$$

where m is a generating parameter of the observables. In the calculations of the following Sections, we will consider the K-theoretic uplift of the above, or in other terms the index of the equivariant theory on $X \times S^1$.

6.2 ADHM CONSTRUCTION IN EIGHT DIMENSIONS

In this section we describe an eight dimensional generalisation of the classical ADHM construction in four dimensions and show that it describes the moduli space of solutions to (6.0.1). For the sake of completeness we recall in Appendix 6.A the four-dimensional ADHM construction and highlight few aspects of the latter which the reader could find useful to follow the eight dimensional generalisation.

Let us start by fixing the spinorial notation to write equation(6.0.1) and the ADHM representation of its solutions. The Cliff(8) gamma matrices can be chosen as 16×16 real matrices of the form

$$\Gamma^\mu = \begin{pmatrix} 0 & \Sigma^\mu \\ \bar{\Sigma}^\mu & 0 \end{pmatrix}, \tag{6.2.1}$$

where $\Sigma^0 = \bar{\Sigma}^0 = 1_{8\times 8}$ and $\Sigma^i = -\bar{\Sigma}^i$ for $i = 1, \dots, 7$. The latter are real antisymmetric matrices (they are in fact $\sqrt{-1}$ times purely imaginary Cliff(7) gamma-matrices). Let $S_\pm$ denote eight-dimensional real irreducible Majorana-Weyl spinor representations of Spin(8) of positive and negative chirality respectively. Since the representations $S_\pm$ are real, the matrices of Spin(8) generators

$$\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu] = \frac{1}{2}\begin{pmatrix} \Sigma^{[\mu}\bar{\Sigma}^{\nu]} & 0 \\ 0 & \bar{\Sigma}^{[\mu}\Sigma^{\nu]} \end{pmatrix} \tag{6.2.2}$$

are real and antisymmetric and so are the 8×8 blocks $\Sigma^{\mu\nu} = \frac{1}{2}\Sigma^{[\mu}\bar{\Sigma}^{\nu]}$ and $\bar{\Sigma}^{\mu\nu} = \frac{1}{2}\bar{\Sigma}^{[\mu}\Sigma^{\nu]}$. Formulated differently, we have (cf. (6.A.16)) an isomorphism of three Spin(8) representations spaces, each represented by real and antisymmetric 8×8 matrices:

$$\Lambda^2 S_+ = \Lambda^2 S_- = \Lambda^2 \mathbb{R}^8 = \mathrm{adj}_{SO(8)}. \tag{6.2.3}$$

The triality of Spin(8) permutes S_+, S_- and $\mathbb{R}^8$, which is the defining representation of SO(8). Notice that each of the two sets of 28 matrices $(\Sigma^{\mu\nu})_\alpha{}^\beta$ – or $(\bar{\Sigma}^{\mu\nu})_\alpha{}^\beta$ – form a basis in the space of real antisymmetric matrices[3]. Due to this fact, the following Fierz identities hold:

$$(\Sigma^{\mu\nu})_{\alpha\beta}(\Sigma^{\mu\nu})_{\gamma\delta} = (\bar{\Sigma}^{\mu\nu})_{\alpha\beta}(\bar{\Sigma}^{\mu\nu})_{\gamma\delta} = -8(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\beta\delta}\delta_{\alpha\gamma}). \tag{6.2.4}$$

The coefficient in the l.h.s. of Eq. (6.2.4) can be obtained by contracting β and γ indices and using the relations of the Clifford algebra to get $\Gamma^{\mu\nu}\Gamma^{\mu\nu} = -56 \cdot 1_{16\times 16}$.

6.2.1 Eight-dimensional equations

There is no way to formulate a first order equation for $F_{\mu\nu}$ in 8d in an SO(8)-invariant manner. Another way to formulate this is to say that there is no SO(8)-invariant four-index tensor $T^{\mu\nu\lambda\rho}$, which can be used to write

$$\lambda F_{\mu\nu} = \frac{1}{2} T^{\mu\nu\lambda\rho} F_{\lambda\rho}. \tag{6.2.5}$$

with λ being some eigenvalue. However, if we make a choice of a constant spinor on $\mathbb{R}^8$ we can build from it a tensor $T^{\mu\nu\lambda\rho}$ invariant under $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$. This is the largest possible symmetry subgroup which can be preserved by equations of the form (6.2.5) in eight space-time dimensions. For this construction let us fix $\psi_+ \in S_+$ such that $\psi_+^T\psi_+ = 1$ and write

$$T^{\mu\nu\lambda\rho} = \psi_+^T \Gamma^{\mu\nu\lambda\rho}\psi_+, \tag{6.2.6}$$

where

$$\Gamma^{\mu\nu\lambda\rho} = \frac{1}{4!}\Gamma^{[\mu}\Gamma^{\nu}\Gamma^{\lambda}\Gamma^{\rho]} \tag{6.2.7}$$

---

3 A third basis of matrices corresponding to the representation $\Lambda^2\mathbb{R}^8$ is given by $(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)$.

The tensor $T^{\mu\nu\lambda\rho}$ then satisfies the 8d self-duality equation[4]

$$T^{\mu\nu\lambda\rho} = \epsilon^{\mu\nu\lambda\rho\alpha\beta\gamma\delta} T^{\alpha\beta\gamma\delta}, \tag{6.2.8}$$

since

$$\epsilon^{\mu\nu\lambda\rho\alpha\beta\gamma\delta} \Gamma^{\alpha\beta\gamma\delta} = \Gamma^{\mu\nu\lambda\rho}\Gamma^9. \tag{6.2.9}$$

For definiteness we take[5] $\psi_+^\alpha = \delta_0^\alpha$. The choice of the spinor ψ_+ allows us to split S_+ into the one-dimensional subspace proportional to ψ_+ and the seven-dimensional orthogonal complement which we call $\tilde{S}_+$. In a group-theoretical language this corresponds to the splitting of the representation S_+ into $\mathbf{1} \oplus \mathbf{7}$ under the subgroup $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$.

The irreducible two-form representation $\Lambda^2\mathbb{R}^8 = \mathbf{28}$ of Spin(8) splits into a sum of two irreps $\mathbf{7} \oplus \mathbf{21}$ of Spin(7). These two irreps correspond to two different eigenvalues $\lambda = 1$ and $\lambda = -3$ respectively in the first order field equations (6.2.5). In this way the splitting allows us to write two different Spin(7)-invariant conditions[6] on the field strength $F_{\mu\nu}$. The conditions correspond to the vanishing of the component of $F_{\mu\nu}$ lying in one of the two irreps of Spin(7), or equivalently to the eigenspaces corresponding to two different eigenvalues in Eq. (6.2.5). The choice $\lambda = 1$ gives Eq. (6.0.1).

As discussed in [75] (6.0.1) reads then in spinorial form as

$$(\Sigma^{\mu\nu})_{\alpha\beta}\psi_+^\beta F_{\mu\nu} = (\Sigma^{\mu\nu})_{\alpha 0} F_{\mu\nu} = 0, \tag{6.2.10}$$

which imposes $7N^2$ equations, so that – together with N^2 gauge fixing conditions for the gauge group $U(N)$ – eliminate all functional degrees of freedom from $A_\mu(x)$ and therefore a finite dimensional moduli space $\mathcal{M}_{k,N}$ of solutions remains. Similarly to the 4d case, one can view $\mathcal{M}_{k,N}$ as an *octonionic* quotient of the space of connections $\mathcal{A}$ by the gauge group $\mathcal{G}$. Indeed we can introduce seven natural symplectic forms $\omega_{\mu\nu}^{(A)} = (\Sigma^{\mu\nu})_{A0}$ on $\mathbb{R}^8$ and use them to write seven symplectic structures on $\mathcal{A}$:

$$\Omega_A[\delta_1 A_\mu(x), \delta_2 A_\mu(y)] = \int_{\mathbb{R}^8} T \wedge \omega^{(A)} \wedge \delta_1 A(x) \wedge \delta_2 A(x), \tag{6.2.11}$$

where T denotes the four-form with components $T^{\mu\nu\lambda\rho}$. Then the $7N^2$ conditions (6.2.10) correspond to the vanishing of the seven moment maps

$$\mu^A[\phi(x)] = \int_{\mathbb{R}^8} T \wedge \omega^{(A)} \wedge \mathrm{tr}\,\phi(x) F, \tag{6.2.12}$$

and we have the "octonionic" quotient

$$\mathcal{M}_{k,N} = \mathcal{A}/\!/\!/\!/\!/\!/\!/\mathcal{G}. \tag{6.2.13}$$

---

4 We could have started with a negative chirality spinor ψ_- corresponding to anti-self-dual $T^{\mu\nu\lambda\rho}$. The resulting construction is isomorphic due to the triality of SO(8).

5 In our conventions the spinor indices run from 0 to 7 similarly to the indices of the $\mathbb{R}^8$ vectors.

6 These two conditions may be viewed as analogues of the self-duality and anti-self-duality conditions in 4d. However, the latter are more similar to choosing the opposite chirality spinor ψ_- instead of ψ_+.

6.2.2 Derrick's theorem and noncommutativity

Any solution of the first order equations (6.2.5) is automatically a solution of the 8d Yang-Mills equations. Indeed,

$$D_\mu F_{\mu\nu} = \frac{1}{2} T_{\mu\nu\lambda\rho} D_\mu F_{\lambda\rho} = \frac{1}{2} T_{\mu\nu\lambda\rho} D_{[\mu} F_{\lambda\rho]} = 0 \tag{6.2.14}$$

where we have used the Bianchi identity for $F_{\mu\nu}$ and the fact that $T_{\mu\nu\lambda\rho}$ is totally antisymmetric. We are looking for localized solutions, *i.e.* for those sufficiently rapidly decaying at infinity in order to have finite action. Then, the solutions of (6.2.5), if any exist, should be true extrema of the Yang-Mills action. The well-known Derrick's theorem states that in dimensions greater than four no such localized solution are possible. The idea of the proof is to provide for any non-singular field configuration a continuous family of configurations with lower action, so that no true minimum can exist. The family of configurations is obtained by scaling the initial configuration into smaller and smaller volume. A simple power counting then shows that the action on the scaled down configuration is lower.

Thus, classically, the moduli space $\mathscr{M}_{k,N}$ of solutions is empty. However, there is a natural way to deform the problem to get a non-empty space of solutions by introducing noncommutativity. The commutation relations for the coordinates are similar to those of the 4d case (cf. (6.A.32)):

$$[x^\mu, x^\nu] = i\zeta(\omega^{-1})^{\mu\nu}, \tag{6.2.15}$$

where ζ is a real parameter and $\omega_{\mu\nu}$ is a non-degenerate constant 2-form on $\mathbb{R}^8$. In this case Derrick's theorem doesn't apply, since the coordinates cannot be rescaled without affecting their commutation relations (6.2.15). To put it another way, the non-commutativity introduces an additional fundamental scale $\sqrt{\zeta}$ into the problem, which puts a limit on how much one can scale down localized field configurations. So, even when no classical non-singular solutions to field equations exist, additional solutions to the non-commutative version of the problem having typical size $\sqrt{\zeta}$ may appear. This is exactly the situation we have with Eq. (6.2.5), where there is a finite-dimensional moduli space of solutions after the non-commutative deformation. This will be our definition of $\mathscr{M}_{k,N}$. In the next section we will describe this moduli space using an analogue of the ADHM construction.

6.2.3 8d ADHM construction

The ADHM equations for the 8d instantons were written in [173; 174]. They correspond to bound states of k D0 and N D8 branes in a suitable B-field background. As in sec. 6.A.5, the B-field introduces non-commutativity into the 8d gauge theory. Also, as we have explained in sec. 6.2.2 it allows for the very existence of solutions to the first order equations (6.2.5), corresponding to stable low-energy bound states of D-branes.

To introduce the non-commutativity we pick one of the seven complex structures on $\mathbb{R}^8$, or correspondingly one of the symplectic forms $\omega^{(A)}_{\mu\nu}$ to represent the Kähler form. For definiteness we choose $\omega^{(1)}_{\mu\nu}$ and denote the projection of the two-form B on $\omega^{(1)}_{\mu\nu}$ by ζ.

The choice of the complex structure breaks down the Spin(7) symmetry of the seven first order equations (6.2.10) further to $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$. Equivalently we can say that by choosing a complex structure we introduce one more fixed chiral spinor $\chi^\alpha_+ = \delta^\alpha_1$ into our theory (which corresponds to the index 1 in $\omega^{(1)}_{\mu\nu}$). The seven equations (6.2.10), transforming as a Spin(7) spinor, split into an SU(4) singlet (corresponding to the component in the direction of χ_+) and further six equations lying in the representation

$$\mathbf{6} = \mathbb{R}^6_{SO(6)} = (\Omega^{(2,0)} \oplus \Omega^{(0,2)}\mathbb{C}^4)_+, \tag{6.2.16}$$

of complex two-forms obeying $\alpha_{z_i z_j} = \epsilon_{z_i z_j z_k z_l}\alpha_{\bar{z}_k \bar{z}_l}$.

Unlike in the 4d case the B-field not only adds a constant to the value of one of the seven moment maps μ^A, but introduces new degrees of freedom corresponding to the rectangular matrix I, which only appears in one of the moment map equations.

Somewhat similarly to the 4d case we introduce a $(8k+N)\times 8k$ matrix $\Delta(x)$, which can be written as[7]

$$\Delta(x) = \begin{pmatrix} (B_\mu - x_\mu 1_{k\times k}) \otimes \bar{\Sigma}^\mu \\ I^\dagger \otimes (\psi^\dagger_+ + i\chi^\dagger_+) \end{pmatrix}, \tag{6.2.17}$$

where B_μ are eight Hermitian $k\times k$ matrices, Σ^μ are defined in Eq. (6.2.1), $I^\dagger$ is an $N \times k$ matrix.

We will be looking for solutions of

$$\Delta^\dagger(x) U(x) = 0, \tag{6.2.18}$$

with Δ satisfying certain moment map conditions. However, differently from the 4d case, these conditions do not imply that $\Delta^\dagger\Delta = 1_{8\times 8} \otimes f^{-1}_{k\times k}$ [8]. Indeed, to solve Eq.(6.2.10), it is enough to impose

$$\Delta^\dagger \Delta \psi_+ = \psi_+ \otimes f^{-1}_{k\times k}. \tag{6.2.20}$$

In our convention for ψ_+ and χ_+ Eq. (6.2.20) has explicit components

$$(\Delta^\dagger\Delta)_{A0} = 0, \qquad A = 1,\dots,7, \tag{6.2.21}$$

7 Notice that the combination $\psi_+ + i\chi_+$ transforms in the complex one-dimensional Weyl spinor representation of the $\mathrm{Spin}(2) = U(1)$ part of $\mathrm{Spin}(6)\times\mathrm{Spin}(2) \subset \mathrm{Spin}(8)$.

8 Indeed, since $(\Sigma^{\mu\nu})_{\alpha\beta}$ is a complete basis of real antisymmetric 8×8 matrices,

$$\Delta^\dagger(x)\Delta(x) = II^\dagger \otimes (\psi_+ + i\chi_+)(\psi^T_+ - i\chi^T_+) +$$
$$+\frac{1}{2}([B_\mu, B_\nu] + i\zeta\omega^{(1)}_{\mu\nu} 1_{k\times k}) \otimes \Sigma^{\mu\nu} + (B_\mu - x_\mu 1_{k\times k})(B_\mu - x_\mu 1_{k\times k}) \otimes 1_{8\times 8} = 1_{8\times 8} \otimes f^{-1}_{k\times k}$$

implies (recall that $\omega^{(1)}_{\mu\nu} = (\Sigma^{\mu\nu})_{01}$)

$$[B_\mu, B_\nu] + i\zeta\omega^{(1)}_{\mu\nu} \otimes 1_{k\times k} + i\omega^{(1)}_{\mu\nu} II^\dagger = 0, \tag{6.2.19}$$

which gives 28 matrix equations, instead of just seven.

or more explicitly

$$i(\omega^{(A)-1})_{\mu\nu}[B_\mu, B_\nu] + \delta^{A,1} II^\dagger = \zeta\delta^{A,1}, \qquad A = 1, \dots, 7. \tag{6.2.22}$$

6.2.4 Formulation in complex coordinates

By explicitely using the complex structure, we can rewrite Eq.(6.2.22) in the form given in [173]. The 8d ADHM data contains four complex $k \times k$ matrices B_a and a complex $k \times N$ matrix I and the equations read

$$\sum_{a=1}^{4} [B_a, B_a^\dagger] + II^\dagger = \zeta 1_{k\times k}, \tag{6.2.23}$$

$$[B_a, B_b] - \frac{1}{2}\epsilon_{abcd}[B_c^\dagger, B_d^\dagger] = 0. \tag{6.2.24}$$

The matrix $\Delta^\dagger$ defined in Eq. (6.2.17) can be written quite explicitly in complex coordinates. Indeed, the main ingredient of $\Delta^\dagger$ is the matrix $x_\mu \Sigma^\mu : S_- \to S_+$, which acts from one Majorana-Weyl representation of Spin(8) to another. Under the SU(4) subgroup the spinor representations $S_\pm$ split into sums of even and odd parts of the exterior algebra:

$$S_+ = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{1} = \Lambda^0 \mathbb{C}^4 \oplus \Lambda^2 \mathbb{C}^4 \oplus \Lambda^4 \mathbb{C}^4, \tag{6.2.25}$$

$$S_- = \mathbf{4} \oplus \bar{\mathbf{4}} = \Lambda^1 \mathbb{C}^4 \oplus \Lambda^3 \mathbb{C}^4. \tag{6.2.26}$$

The matrix $x_\mu \Sigma^\mu$ then acts on the exterior powers as an operator

$$\iota_{z_a \partial_{z_a}} + \bar{z}_a dz_a, \tag{6.2.27}$$

where $\iota_{z_a \partial_{z_a}}$ is the substitution of the vector field $z_a \partial_{z_a}$. In this way we get:

$$\Delta^\dagger = \left(\begin{array}{cccc|cccc|c}
b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & I \\
\hline
b_2^\dagger & -b_1^\dagger & 0 & 0 & 0 & 0 & b_4 & -b_3 & 0 \\
b_3^\dagger & 0 & -b_1^\dagger & 0 & 0 & -b_4 & 0 & b_2 & 0 \\
b_4^\dagger & 0 & 0 & -b_1^\dagger & 0 & b_3 & -b_2 & 0 & 0 \\
0 & b_3^\dagger & -b_2^\dagger & 0 & b_4 & 0 & 0 & -b_1 & 0 \\
0 & -b_4^\dagger & 0 & -b_2^\dagger & b_3 & 0 & -b_1 & 0 & 0 \\
0 & 0 & b_4^\dagger & -b_3^\dagger & b_2 & -b_1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & b_1^\dagger & b_2^\dagger & b_3^\dagger & b_4^\dagger & I^*
\end{array}\right), \tag{6.2.28}$$

where we have abbreviated $b_a = B_a - z_a$. Notice that unlike in the four-dimensional case both representations $S_\pm$ are real, i.e. they admit outer automorphisms which square to one. Explicitly these automorphisms are given by the matrices

$$\tau_+ = \left(\begin{array}{c|cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad \tau_- = \left(\begin{array}{c|c} 0 & 1_{4\times 4} \\ \hline 1_{4\times 4} & 0 \end{array}\right). \tag{6.2.29}$$

The matrix Δ commutes with the automorphisms in the sense that

$$\tau_+ \Delta^\dagger \tau_- = \Delta^*. \tag{6.2.30}$$

6.2.5 Matrix formulation

After choosing the non-commutative deformation (6.2.15), the original instanton equations (6.2.10) become:

$$F_{ab} = \frac{1}{2}\epsilon_{abcd} F_{\bar{c}\bar{d}}, \tag{6.2.31}$$

$$\sum_{a=1}^{4} F_{a\bar{a}} = 0. \tag{6.2.32}$$

Let us rewrite these equations in terms of the matrix variables analogously to the noncommutative 4d case recalled in sec. 6.A.6. Plugging the Z_a variables in the instanton equations (6.2.31), (6.2.32) we get:

$$[Z_a, Z_b] = \frac{1}{2}\epsilon_{abcd}[Z^\dagger_{\bar{c}}, Z^\dagger_{\bar{d}}], \tag{6.2.33}$$

$$\sum_{a=1}^{4} [Z^\dagger_{\bar{a}}, Z_a] = 2\zeta. \tag{6.2.34}$$

Again the nontrivial r.h.s. in Eq. (6.2.34) arises because of the noncommutativity of z_a and $z^\dagger_{\bar{a}}$. The vacuum solution of Eqs. (6.2.33), (6.2.34) is given by

$$Z^a = z^a, \tag{6.2.35}$$

and corresponds to vanishing gauge potential A_a. Let us now discuss the simplest non trivial solutions.

6.2.6 $U(1)$ one-instanton

Nontrivial solutions to Eqs. (6.2.33), (6.2.34) correspond to nontrivial ideals in the ring of polynomials in four variables. Let us consider the simplest solution corresponding to a single abelian instanton sitting at the origin of $\mathbb{C}^4$. In this case

$$Z_a = S_{[[[1]]]} z_a f_{[[[1]]]}(N) S^\dagger_{[[[1]]]}, \tag{6.2.36}$$

where $N = \sum_{a=1}^{4} a_{\bar{a}}^{\dagger} a_a$ and

$$f_{[[[1]]]}(N) = \left(1 - \frac{24}{N(N+1)(N+2)(N+3)}\right)^{\frac{1}{2}}, \tag{6.2.37}$$

and $S_{[[[1]]]}$ is the partial isometry of the Hilbert space $\mathcal{H} = \mathbb{C}[z_1, z_2, z_3, z_4]$, satisfying

$$\begin{aligned} S_{[[[1]]]} S_{[[[1]]]}^{\dagger} &= 1, \\ S_{[[[1]]]}^{\dagger} S_{[[[1]]]} &= 1 - |0,0,0,0\rangle\langle 0,0,0,0| = P_{\mathcal{H}\setminus\{|0,0,0,0\rangle\}}. \end{aligned} \tag{6.2.38}$$

Notice that N in the denominator of $f_{[[[1]]]}(N)$ is never zero, because the state $|0,0,0,0\rangle$ is projected out by the partial isometry .

6.2.7 $U(1)$ multi-instanton

By taking the square of equations (6.2.33) we deduce that the operators Z_a commute with each other:

$$[Z_a, Z_b] = 0. \tag{6.2.39}$$

As we have noticed above the multi-instanton solutions correspond to ideals in the ring $\mathcal{H} = \mathbb{C}[z_1, z_2, z_3, z_4]$. Having such an ideal $\mathcal{I}$, we define a partial isometry $S_{\mathcal{I}}$, which satisfies

$$S_{\mathcal{I}} S_{\mathcal{I}}^{\dagger} = 1, \qquad S_{\mathcal{I}}^{\dagger} S_{\mathcal{I}} = P_{\mathcal{I}}, \tag{6.2.40}$$

where $P_{\mathcal{I}}$ is the projection operator on the ideal $\mathcal{I}$. The matrix $\Delta^{\dagger}$ contains the information about the resolution of the ideal corresponding to the solution of the ADHM equations. Consider for example the 1-instanton solution (6.2.36). It corresponds to the ideal $\mathcal{I}_{[[[1]]]}$ of polynomials without constant terms. The resolution of this ideal is written as the following exact sequence:

$$0 \to \mathcal{O} \xrightarrow{\mu_4} \mathcal{O}^{\oplus 4} \xrightarrow{\mu_3} \mathcal{O}^{\oplus 6} \xrightarrow{\mu_2} \mathcal{O}^{\oplus 4} \xrightarrow{\mu_1} \mathcal{O} \xrightarrow{p} \mathcal{I}_{[[[1]]]} \to 0, \tag{6.2.41}$$

where $\mathcal{O} = \mathbb{C}[z_1, z_2, z_3, z_4]$, p is the projection and the linear operators μ_i are

$$\mu_1 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix}, \tag{6.2.42}$$

$$\mu_2 = \begin{pmatrix} z_2 & z_3 & z_4 & 0 & 0 & 0 \\ -z_1 & 0 & 0 & z_3 & -z_4 & 0 \\ 0 & -z_1 & 0 & -z_2 & 0 & z_4 \\ 0 & 0 & -z_1 & 0 & z_2 & -z_3 \end{pmatrix}, \tag{6.2.43}$$

$$\mu_3 = \begin{pmatrix} 0 & 0 & z_4 & z_3 \\ 0 & -z_4 & 0 & z_2 \\ 0 & z_3 & -z_2 & 0 \\ z_4 & 0 & 0 & -z_1 \\ z_3 & 0 & -z_1 & 0 \\ z_2 & -z_1 & 0 & 0 \end{pmatrix}, \qquad \mu_4 = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}. \tag{6.2.44}$$

Notice that these operators are very similar to those featuring in $\Delta^\dagger$ in Eq. (6.2.28). This similarity seems to be a generic property of any ADHM-like construction.

We are interested in the solutions which are fixed points of the $U(1)^3 \subset SU(4)$ action on $\mathbb{C}^4$. Those are enumerated by solid partitions and correspond to monomial ideals in the ring $\mathbb{C}[z_1, z_2, z_3, z_4]$. We denote the ideal corresponding to a solid partition σ by $\mathcal{I}_\sigma$. The fixed point multi-instanton solutions can be obtained with an ansatz similar to Eq. (6.2.36), but now the function $f(r)$ does not need to be symmetric in $N_a = a_a^\dagger a_a$ (no summation over a), so that

$$Z_a = U_\sigma z_a f_a^{(\sigma)}(N_1, N_2, N_3, N_4) U_\sigma^\dagger \tag{6.2.45}$$

We thus have to determine four functions $f_a^{(\sigma)}(N_1, N_2, N_3, N_4)$ of four variables. Eqs. (6.2.39), (6.2.34) imply the following recurrence relations[9] for f_a

$$f_a(N_b+1) f_b(N) = f_b(N_a+1) f_a(N), \tag{6.2.46}$$

$$\sum_{a=1}^{4} \left\{ (f_a(N))^2 (N_a+1) - (f_a(N_a-1))^2 N_a \right\} = 4. \tag{6.2.47}$$

Eqs. (6.2.46) are "flatness" conditions for f_a and can be solved explicitly. Indeed, one can see that

$$f_a(N) = \frac{h(N)}{h(N_a+1)} \tag{6.2.48}$$

solves Eqs. (6.2.46) for any function $h(N)$. There remains a single recurrence relation (6.2.47) for $h(r)$, which reads

$$\sum_{a=1}^{4} \left\{ \frac{h(N)^2}{h(N_a+1)^2}(N_a+1) - \frac{h(N_a-1)^2}{h(N)^2} N_a \right\} = 4. \tag{6.2.49}$$

An example of instanton solution with second Chern class equal to $k(k+1)(k+2)(k+3)/24$ is given by

$$Z_a = S_k z_a f^{(k)}(N) S_k^\dagger, \tag{6.2.50}$$

where S_k now avoids all the states corresponding to monomials of degree at most k and

$$f^{(k)}(N) = \left(1 - \frac{k(k+1)(k+2)(k+3)}{N(N+1)(N+2)(N+3)}\right)^{\frac{1}{2}}. \tag{6.2.51}$$

Notice that all four functions $f_a(N)$ are in this case equal to each other. The solutions (6.2.51) correspond to "pentachoron" solid partitions (decreasing sequences of tetrahedral plane partitions), e.g. [[[1]]] for $k = 1$ or [[[2, 1], [1]], [[1]]] for $k = 2$.

9 We denote by $f_a(N_b+1)$ the function $f_a(N)$ with $N_b \to N_b + 1$.

6.3 COUNTING SOLID PARTITIONS ON ORBIFOLDS

6.3.1 Orbifolding the quiver

As we showed in section 6.2, eight-dimensional instanton dynamics is encoded in the representation theory of the quiver depicted in Fig.6.3.1, with relations

$$[B_a, B_b] = 0, \qquad 1 \leqslant a < b \leqslant 4 \tag{6.3.1}$$

and a stability condition.[10] The moduli space of its stable representations

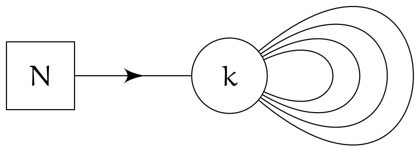


Figure 6.3.1: *Local model for the Quot scheme of points.*

$\mathscr{M}_{k,N}$ is isomorphic to the quot scheme of points $\mathrm{Quot}_{\mathbb{A}^4}(\mathscr{O}^{\oplus N}, k)$ or, equivalently, to the moduli space of framed torsion free sheaves on $\mathbb{P}^4$, as we briefly show in appendix 6.B.1. This isomorphism follows from an application of Beilinson's theorem or, equivalently, from an infinitesimal argument due to [64].

The next natural step is then to study eight-dimensional instanton dynamics on orbifolds of $\mathbb{C}^4$. Then we let G be a finite subgroup $G \subset SU(4)$, and study instantons on $\mathbb{C}^4/G$. From the open string theory perspective this amounts to consider twisted representations of the Chan-Paton factors under the discrete group G, which manifest in the low energy quiver dynamics of fractional and regular branes [91]. The mathematical counterpart of the quiver ADHM-like description for orbifold instantons on $\mathbb{C}^4$ can be obtained as an application of Beilinson's theorem which is outlined in appendix 6.B.2.

The useful thing to point out here is that the monad description for the moduli space $\mathscr{M}^G_{k,N}$ of orbifold instantons, which can be obtained by means of homological algebra, is then given in terms of a sequence of maps between vector spaces. These maps can be easily understood as an equivariant decomposition (in terms of the G-action) of the maps and vector spaces arising in the quiver description of $\mathscr{M}_{k,N}$. If we introduce the action of G on the coordinates z_α in $\mathbb{C}^4$ by $r_\alpha z_\alpha$, we have a decomposition of the fundamental representation $Q = \rho_{r_1} \oplus \cdots \oplus \rho_{r_4}$, where ρ_{r_α} denotes the irreducible representation of G with weight r_α. This decomposition also defines a colouring $\mathbb{N}^{\oplus 4} \to G$ by

$$(n_1, n_2, n_3, n_4) \mapsto \rho_{r_1}^{\otimes n_1} \otimes \rho_{r_2}^{\otimes n_2} \otimes \rho_{r_3}^{\otimes n_3} \otimes \rho_{r_4}^{\otimes n_4}.$$

10 Strictly speaking we showed that the moduli space of SU(4) instantons can be identified with a space of matrices cut by equations (6.3.1) plus an additional real constraint, modulo gauge symmetry. Though it have not rigorously been proved as of this writing, it is believed that the last real condition can be traded for a stability condition, so that the moduli space of instantons may be identified with stable representations of a quiver with relations.

Correspondingly we also have decompositions of the vector spaces

$$W = \bigoplus_r W_r \otimes \rho_r^\vee, \qquad V = \bigoplus_r V_r \otimes \rho_r^\vee,$$

where all of the W_r, V_r are finite dimensional vector spaces carrying a trivial G-action. The corresponding decomposition of the dimensions $k = \dim_{\mathbb{C}} V$, $N = \dim_{\mathbb{C}} W$ is then induced as

$$k = \sum_r k_r = \sum_r \dim_{\mathbb{C}} V_r, \qquad N = \sum_r N_r = \sum_r \dim_{\mathbb{C}} W_r.$$

Here $k_r = \dim_{\mathbb{C}} V_r$ represents the fractional instanton charge in the $\rho_r^\vee$ representation of G, which, from an equivariant localisation point of view, will specify the number of boxes of r-th type in a G-coloured solid partition. On the other hand, the gauge sheaf at infinity transforms in a given representation ρ of G, and the N_r dimensions determine the multiplicities of the decomposition of ρ in irreducible representations. If the theory is abelian, *i.e.* $N = 1$, only one of the N_r is not zero, and equal to one, while in the case of a non-abelian theory one is given with a plethora of different possibilities. We will restrict our attention to the abelian case for the moment. The decomposition of V then induces a decomposition of the linear maps $B_\alpha \in \mathrm{Hom}_G(V, Q \otimes V)$ as

$$B = \bigoplus_r (B_1^r, B_2^r, B_3^r, B_4^r),$$

so that $B_\alpha^r : V_r \to V_{r+r_\alpha}$. This decomposition immediately gives us the orbifold generalisation of the 8d ADHM equations as

$$B_\beta^{r+r_\alpha} B_\alpha^r = B_\alpha^{r+r_\beta} B_\beta^r. \tag{6.3.2}$$

In general, even for abelian theories, one should consider different cases (corresponding in the abelian case to which one of the W_r vector spaces has non-vanishing dimension). These different choices correspond to instanton configurations with different asymptotics at infinity. One can however argue along the lines of [74] that moduli spaces corresponding to different asymptotics are isomorphic, so that when computing partition functions we will just consider the distinguished boundary condition $\overline{N} = (1, 0, \dots, 0)$.

As in the lower dimensional cases (see [74; 147; 164]) all the information about the moduli space of orbifold instantons can be encoded in the datum of a quiver generalising the McKay quiver, which is determined by the representation theory data of the G-action. This quiver will moreover encode the decomposition of the usual ADHM data according to the G-action and it will have the orbifold ADHM equations as relations. One then starts considering all the irreducible representations $\widehat{G}$ of $G \subset SU(4)$. To each representation in $\widehat{G}$, including the trivial one, we associate a node in the quiver, while a node r is connected to a node s by a number of arrows a_{rs} which is determined by the decomposition

$$Q \otimes \rho_r = \bigoplus_s a_{rs} \rho_s, \qquad a_{rs} = \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_s, Q \otimes \rho_r).$$

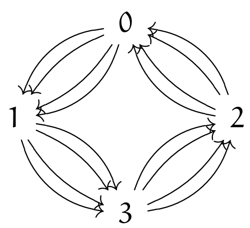
To the resulting quiver Q we also associate the framed quiver Q^f, its path algebra $\mathbb{C}Q^f$, and the bounded quiver (Q^f, R) determined by an ideal $\langle R \rangle$ of relations in $\mathbb{C}Q^f$. Its representations form the category $\mathrm{Rep}(Q^f, R)$, which is equivalent to the A-mod category of left A-modules for the factor path algebra $A = \mathbb{C}Q^f/\langle R \rangle$. Moreover, to each vertex v of Q one can associate a simple module D_v, defined to be the representation $V_v \cong \mathbb{C}$ and $V_w = 0$, for $v \neq w$. Projective resolutions of the simple modules D_v can be constructed by means of the submodule P_v of A generated by paths beginning on vertex v

$$\cdots \longrightarrow \bigoplus_w d^p_{w,v} P_w \longrightarrow \cdots \longrightarrow \bigoplus_w d^1_{w,v} P_w \longrightarrow P_v \longrightarrow D_v \longrightarrow 0,$$

where

$$d^p_{v,w} = \dim_{\mathbb{C}} \mathrm{Ext}^p_A(D_v, D_w).$$

In the lower dimensional case a special role is played by representations $\mathrm{Rep}(Q, R)$ of the bounded quiver with dimensions $k_r = 1$ and $\dim_{\mathbb{C}} W = 1$, as it turns out they correspond to smooth crepant resolutions of toric singularities. In some cases also the path algebra A is a different desingularisation by itself, known as the noncommutative crepant resolution of the toric singularity, which contains the coordinate ring of the singularity as its center. In four complex dimensions, however, a crepant resolution of the orbifold singularity is not even granted to exist, though in some simple classes of examples this is known to be the case [194; 195]. Take as an example the case of $\mathbb{C}^4/\mathbb{Z}_4$ with the diagonal action $(z_1, z_2, z_3, z_4) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta z_4)$, where $\zeta = e^{2\pi i/4} = i$. As $Q = \rho_1 \oplus \rho_1 \oplus \rho_1 \oplus \rho_1$, $r_\alpha = 1$ for each $\alpha = 1, \dots, 4$ and the relevant associated quiver Q is then



and the maps $B^r_\alpha : D_r \to D_{r + r_\alpha \mod 4}$ are of the form

$$\begin{aligned} B^0_\alpha &: D_0 \to D_1, \\ B^1_\alpha &: D_1 \to D_2, \\ B^2_\alpha &: D_2 \to D_3, \\ B^3_\alpha &: D_3 \to D_0. \end{aligned}$$

The relevant relations for the unframed quiver are obtained by decomposing accordingly the ADHM equations, thus obtaining

$$\begin{array}{llll} B^1_2 B^0_1 = B^1_1 B^0_2, & B^1_3 B^0_1 = B^1_1 B^0_3, & B^1_4 B^0_1 = B^1_1 B^0_4, & B^1_3 B^0_2 = B^1_2 B^0_3, \\ B^1_4 B^0_2 = B^1_2 B^0_4, & B^1_4 B^0_3 = B^1_3 B^0_4, & B^2_2 B^1_1 = B^2_1 B^1_2, & B^2_3 B^1_1 = B^2_1 B^1_3, \\ B^2_4 B^1_1 = B^2_1 B^1_4, & B^2_3 B^1_2 = B^2_2 B^1_3, & B^2_4 B^1_2 = B^2_2 B^1_4, & B^2_4 B^1_3 = B^2_3 B^1_4, \end{array}$$

$$
\begin{array}{llll}
B^3_2 B^2_1 = B^3_1 B^2_2, & B^3_3 B^2_1 = B^3_1 B^2_3, & B^3_4 B^2_1 = B^3_1 B^2_4, & B^3_3 B^2_2 = B^3_2 B^2_3, \\
B^3_4 B^2_2 = B^3_2 B^2_4, & B^3_4 B^2_3 = B^3_3 B^2_4, & B^0_2 B^3_1 = B^0_1 B^3_2, & B^0_3 B^3_1 = B^0_1 B^3_3, \\
B^0_4 B^3_1 = B^0_1 B^3_4, & B^0_3 B^3_2 = B^0_2 B^3_3, & B^0_4 B^3_2 = B^0_2 B^3_4, & B^0_4 B^3_3 = B^0_3 B^3_4.
\end{array}
$$

The center $Z(A)$ of the path algebra A associated to the bounded quiver is generated as a ring by elements

$$x_{\alpha\beta\gamma\delta} = B^3_\alpha B^2_\beta B^1_\gamma B^0_\delta, \qquad \alpha \leqslant \beta \leqslant \gamma \leqslant \delta.$$

As the G-action is chosen to be diagonal one can identify the generators $x_{\alpha\beta\gamma\delta}$ with the invariant elements in $\mathbb{C}[z_1, z_2, z_3, z_4]$ by $x_{\alpha\beta\gamma\delta} \rightsquigarrow z_\alpha z_\beta z_\gamma z_\delta$, so that

$$\operatorname{Spec} Z(A) \cong \mathbb{C}^4/\mathbb{Z}_4$$

and the factor path algebra A is a resolution of the orbifold singularity $\mathbb{C}^4/\mathbb{Z}_4$.

6.3.2 Orbifold partition function

The K-theoretic instanton partition function in eight dimensions has been studied in [173] by means of supersymmetric localisation in terms of the quantum mechanics of a D0-D8 system. In the abelian case the moduli space of BPS vacua is identified with the Hilbert scheme of points of $\mathbb{C}^4$. In the general case of a proper Calabi-Yau fourfold X, the Hilbert scheme of points $X^{[n]}$ is known to carry an obstruction theory, though not perfect. The mathematical definition of the DT-like invariants corresponding to the instanton partition function is made difficult precisely by the latter fact. It is known that they depend on the choice of an orientation of the virtual tangent space and that they need insertions in order to be defined properly. Indeed, if $\mathcal{Z} \subseteq X^{[n]} \times X$ denotes the universal object, then the virtual tangent space to $X^{[n]}$ can be written as

$$T^{\mathrm{vir}}_{X^{[n]}} = \mathbf{R}\mathrm{Hom}_{\pi_{X^{[n]}}}(I_\mathcal{Z}, I_\mathcal{Z})_0[1] = \mathbf{R}\pi_{X^{[n]},*} \circ \mathbf{R}\mathrm{Hom}(I_\mathcal{Z}, I_\mathcal{Z})_0[1],$$

and this obstruction theory is not perfect. However, the machinery put forward by the work of Borisov-Joyce, [49], and more recently by Oh-Thomas [183], one can still construct a virtual fundamental class $[X^{[n]}]^{\mathrm{vir}}_{o(\mathcal{L})}$ depending on the choice of a square root of the isomorphism $Q : \mathcal{L} \otimes \mathcal{L} \to \mathscr{O}$, where $\mathcal{L} = \det \mathbf{R}\mathrm{Hom}_{\pi_{X^{[n]}}}(I_\mathcal{Z}, I_\mathcal{Z})$. As, however, we are interested in the case of a quasi-projective variety, namely $(\mathbb{C}^4)^{[n]}$, the previous observations don't provide direct access to definitions of relevant invariants. Equivariant localisation (with respect to the action of $\mathbb{T} = \{(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4 : t_1 t_2 t_3 t_4 = 1\} \subset (\mathbb{C}^*)^4$), however, does provide an easy way out. For a thorough description of this procedure, see [58], while, for the existence of globally defined orientations on CY four-folds, see [34]. For us, let it suffice to say that, given any $\mathbb{T}$-equivariant line bundle L on X, one can define the following K-theoretic invariant, with a slight abuse of notation

$$
\begin{aligned}
Z_X^K(L,y) &= \chi\left(X^{[n]}, \widehat{\mathscr{O}}^{\mathrm{vir}} \otimes \frac{\Lambda^\bullet(L^{[n]}\otimes y^{-1})}{\det^{1/2}(L^{[n]}\otimes y^{-1})}\right) \\
&= \sum_{S\in (X^{[n]})^{\mathbb{T}}} (-1)^{o(\mathcal{L})|_S}\, e\left(\sqrt{\mathrm{Ob}_{X^{[n]}}|_S}^{\mathrm{fix}}\right) \frac{\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}_{X^{[n]}}|S}^{1/2}\right)}{\mathrm{ch}\left(\Lambda^\bullet \sqrt{N^{\mathrm{vir}}|_S}^\vee\right)} \cdot \\
&\quad \cdot \frac{\mathrm{ch}\left(\Lambda^\bullet(L^{[n]}\otimes y^{-1})\right)}{\mathrm{ch}\left(\det^{1/2}(L^{[n]}\otimes y^{-1})\right)}\, \mathrm{td}\left(\sqrt{T^{\mathrm{vir}}_{X^{[n]}}|_S}^{\mathrm{fix}}\right),
\end{aligned}
$$

where $L^{[n]} = \mathbf{R}\pi_{X^{[n]},*}(\mathbf{R}\pi_X^* L \otimes \mathscr{O}_{\mathcal{Z}})$ and the choice of the square root of the virtual tangent space at a fixed point S induces those of $\mathrm{Ob}_{X^{[n]}}|_S = h^1(T^{\mathrm{vir}}_{X^{[n]}})$, $K^{\mathrm{vir}}_{X^{[n]}}|_S = \det(T^{\mathrm{vir}}_{X^{[n]}}|_S^\vee)$ and $N^{\mathrm{vir}}|_S = (T^{\mathrm{vir}}_{X^{[n]}}|_S)^{\mathrm{mov}}$. Before moving on, let us also notice that the choice that square roots is not unique, so that the invariants at hand are defined only up to a sign. Precisely this definition of $Z_X^K(L,y)$ is what the partition function of the D0-D8 system computes, with a given prescription for the orientation choice.

All these considerations translate into the physical treatment of the problem, where the supersymmetric measure corresponding to the bulk contribution to the Witten index manifest ghost number anomaly, reminiscent of the positive virtual dimension of the underlying moduli space. Then, in the absence of the Ω deformation, the Witten index is vanishing unless observables matching the ghost number anomaly are inserted. This can be neatly done by adding auxiliary hypermultiplets representing the matter deformation necessary in order to cure the anomaly. A similar story goes for the non-abelian case, which generalises the moduli space to the Quot scheme of points of $\mathbb{C}^4$, and was studied in [174]. In general the partition function takes the form

$$
Z_N^{D8} = \sum_k Z_{N,k}^{D8} q^k,
$$

where $Z_{N,k}^{D8}$ is computed by the JK integration

$$
Z_{N,k}^{D8} = \int_{\mathrm{JK}} Z_{N,k}^{1-\mathrm{loop}} d^k u = \frac{1}{k!} \sum_{u_*\in\mathfrak{M}_{\mathrm{sing}}} \text{JK-Res}_{u_*,\zeta}\, \chi_k d^k u
$$

and the instanton measure χ_k is defined by

$$
\chi_k \propto \prod_{i>j} \frac{\sin^2(u_i-u_j)\prod_{a<b}^3 \sin(u_i-u_j-\epsilon_a-\epsilon_b)}{\prod_{a=1}^3 \sin(u_i-u_j-\epsilon_a)\sin(u_j-u_i-\epsilon_a)} \prod_{i=1}^k \prod_{\alpha=1}^N \frac{\sin(u_i-m_\alpha)}{\sin(u_i-a_\alpha)}.
$$

It is then known that poles contributing to the JK integration are only those corresponding to N-tuples of solid partitions $\overline{\pi} = (\pi_1,\dots,\pi_N)$, such that $|\overline{\pi}| = |\pi_1|+\cdots+|\pi_N| = k$. It turns out it is more convenient to work with exponential variables $t_a = e^{2i\epsilon_a}$, $x_i = e^{2iu_i}$, $\nu_\alpha = e^{2ia_\alpha}$ and $\mu_\alpha = e^{2im_\alpha}$, in which case we have $Z^{D8}_{N,k} = \sum_{|\overline{\pi}|=k} M_{\mathbf{z}}(\overline{\pi})$, with

$$M_{\mathbf{z}}(\overline{\pi}) = \mathrm{Res}_{x=x_{\overline{\pi}}}\, \chi_k \prod_i \frac{dx_i}{x_i}$$

and

$$\chi_k = \prod_{i\neq j} \frac{(x_j - x_i)\prod_{a<b}^{3}(x_j - x_i t_a t_b)}{\prod_{a=1}^{4}(x_j - x_i t_a)} \prod_{i=1}^{k}\prod_{\alpha=1}^{N} \frac{\mu_\alpha - x_i}{\nu_\alpha - x_i}$$

up to a normalisation constant. It also turns out that a more geometric interpretation for the index computation is available. Indeed, in the rank one case, if Q denotes the character of a solid partition and $\sqrt{V} = \sum_\mu t^\mu - \sum_\nu t^\nu \in K_0^{\mathbb{T}}(\mathrm{pt})$ is the character of the square root of the virtual tangent space to the BPS moduli space at a fixed point, one has

$$\sqrt{V} = Q - \overline{P}_{123}\overline{Q}Q,$$

where $P_{123} = (1-t_1)(1-t_2)(1-t_3)$ and the involution acts on the generators of $K_0^{\mathbb{T}}(\mathrm{pt})$ as $\bar{t}_i = t_i^{-1}$. Then

$$M_{\mathbf{z}}(\pi) = (-1)^{h(\pi)}\left[-\sqrt{\widetilde{V}}\right],$$

with

$$h(\pi) = |\pi| + \#\{(a,d) : (a,a,a,d) \in \pi \text{ and } a < d\},$$

$\sqrt{\widetilde{V}} = \sqrt{V} - y\overline{Q}$, while the action of the brackets operator $[-]$ on $V = \sum_\mu t^\mu - \sum_\nu t^\nu \in K_0^{T}(\mathrm{pt})$ is defined as

$$[V] = \frac{\prod_\mu [t^\mu]}{\prod_\nu [t^\nu]} = \frac{\prod_\mu (t^{\mu/2} - t^{-\mu/2})}{\prod_\nu (t^{\nu/2} - t^{-\nu/2})}.$$

The same procedure might be followed in order to compute partition functions for orbifold instantons. In this case, however, the bosonic field content is the one associated to the morphisms of the quiver specifying the natural crepant resolution of the orbifold singularity, provided it exists. Let us notice here that as $G \subset SU(4)$ is contained in the localising torus $\mathbb{T}$ the locus on which the computation localises can be identified with the G-invariant part of the $\mathbb{T}$-fixed one. Moreover, as the geometrical interpretation is that of an equivariant count of G-equivariant zero-dimensional schemes, in order to perform computation one can proceed by simply extracting the G-invariant part $\sqrt{\widetilde{V}}^G$ of $\sqrt{\widetilde{V}}$.

In the case of an orbifold theory the JK residue form gets easily generalised. Let us consider for the sake of simplicity the case of $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}^2$. The bosonic field content is now encoded in the relevant quiver describing the resolution of the orbifold. Let then Q_0 and Q_1 be the node set and the edge set of the quiver. We have

$$Z^{\mathrm{orb}}_{\overline{N}} = \sum_{\overline{k}} \underline{q}^{\overline{k}} Z^{\mathrm{orb}}_{\overline{N},\overline{k}},$$

with $\underline{q}^{\overline{k}} = \prod_{a \in Q_0} q_a^{k_a}$ and

$$Z^{\mathrm{orb}}_{\overline{N},\overline{k}} = \int_{\mathrm{JK}} Z^{1-\mathrm{loop}}_{\overline{N},\overline{k}} \mathrm{d}^{\overline{k}}\overline{x} = \sum_{u_* \in \mathfrak{M}_{\mathrm{sing}}} \text{JK-Res}_{u_*,\zeta}\, \chi^{\mathrm{orb}}_{\overline{k}} \mathrm{d}^{\overline{k}}\overline{x},$$

where we denote by $\overline{x}$ the collection of coordinates associated to the gauge nodes in Q^f, *i.e.* $\overline{x} = (x_1^{(0)}, \dots, x_{k_0}^{(0)}, \dots)$ and

$$\mathrm{d}^{\overline{k}}\overline{x} = \prod_{a \in Q_0} \prod_{i=1}^{k_a} \frac{\mathrm{d}x_i^{(a)}}{x_i^{(a)}}.$$

The orbifold instanton measure $\chi^{\mathrm{orb}}_{\overline{k}}$ can be easily read off the edge set Q_1 of the quiver, and we have:

$$\chi^{\mathrm{orb}}_{\overline{k}} \propto \prod_{a \in Q_0} \frac{1}{k_a!} Z^{(a)}_{\mathrm{f/af}} Z^{(a)}_{\mathrm{adj}} Z^{(a)}_{\mathrm{bif}},$$

where $Z^{(a)}_{\mathrm{f/af}}, Z^{(a)}_{\mathrm{adj}}, Z^{(a)}_{\mathrm{bif}}$ encode the bosonic field content of the theory in the fundamental/antifundamental, adjoint and bifundamental representation of the a-th node respectively. In particular we have

$$\begin{aligned}
Z^{(a)}_{\mathrm{f/af}} &= \prod_{i=1}^{k_a} \prod_{\alpha=1}^{N_a} \frac{\mu_\alpha^{(a)} - x_i^{(a)}}{\nu_\alpha^{(a)} - x_i^{(a)}}, \\
Z^{(a)}_{\mathrm{adj}} &= \prod_{i \neq j}^{k_a} \frac{(x_j^{(a)} - x_i^{(a)})(x_j^{(a)} - x_i^{(a)} t_1 t_2)}{(x_j^{(a)} - x_i^{(a)} t_3)(x_j^{(a)} - x_i^{(a)} t_4)} \\
Z^{(a)}_{\mathrm{bif}} &= \prod_{i=1}^{k_a} \prod_{j=1}^{k_{a+1}} \frac{(x_i^{(a)} - x_j^{(a+1)} t_1 t_3)(x_i^{(a)} - x_j^{(a-1)} t_2 t_3)}{(x_i^{(a)} - x_j^{(a+1)} t_1)(x_i^{(a)} - x_j^{(a-1)} t_2)},
\end{aligned}$$

with the node indices being understood to be $a \pmod{n}$.

Remark 6.1. The JK integral formula for the 4-fold orbifold partition function immediately reduces to the integral formula for orbifold counting on 3-folds after the specialisation $\mu \rightsquigarrow t_4$, as is to be expected. ◀

6.3.3 An example: $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$

As an example let us consider a local $\mathbb{P}^1$ realised as $\mathrm{Tot}_{\mathbb{P}^1}\left(\mathscr{O}(-2) \oplus \mathscr{O}^{\oplus 2}\right)$. This can be understood as the canonical crepant resolution of the orbifold singularity $\mathbb{C}^4/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$, with the $\mathbb{Z}_2$-action defined as

$$(z_1, z_2, z_3, z_4) \mapsto (\zeta z_1, \zeta^{-1} z_2, z_3, z_4).$$

The ADHM data associated to the orbifolded $\mathbb{C}^2$ directions can be decomposed according to the $\mathbb{Z}_2$-action in irreducible $\mathbb{Z}_2$ representations as we described in section 6.3.1. The relevant quiver is depicted in Fig. 6.3.2.

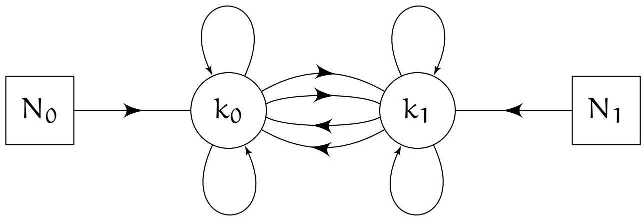


Figure 6.3.2: *Orbifolded* $\mathbb{C}^4/\mathbb{Z}_2^{(1,-1,0,0)}$ *quiver*

For the sake of simplicity we will restrict to the case $N = 1$, which enforces $\overline{N} = (1, 0)$ by the construction in [147] and the observations in 6.3.1. Supersymmetric localisation can be exploited in order to compute partition functions. As $\mathbb{Z}_2$ is a subgroup of the localising torus $\mathbb{T} \cong (\mathbb{C}^*)^3$, the relevant fixed points will simply be identified to be the $\mathbb{Z}_2$–invariant locus of the $\mathbb{T}$–fixed one. In particular, as the $\mathbb{T}$ and the $\mathbb{Z}_2$-actions commute and as the $\mathbb{T}$–fixed locus of the theory on $\mathbb{C}^4$ is into bijective correspondence with solid partitions, if we take the framing to be in the trivial representation of $\mathbb{Z}_2$ their $\mathbb{Z}_2$–analogue will be identified with solid partitions $\pi_{\mathbb{Z}_2}$ decorated by a $\mathbb{Z}_2$–colouring, which must be compatible with the action of $\mathbb{Z}_2$ on $\mathbb{C}^4$. By identifying solid partitions themselves with $\mathbb{Z}_2$ representations, this colouring is in fact induced by the colouring $(n_1, n_2, n_3, n_4) \mapsto \rho_{r_1}^{\otimes n_1} \otimes \rho_{r_2}^{\otimes n_2} \otimes \rho_{r_3}^{\otimes n_3} \otimes \rho_{r_4}^{\otimes n_4}$ we described in section 6.3.1. One way we can construct these $\mathbb{Z}_2$–coloured solid partitions goes as follows: let $|\pi| = k$ and fix an ordinary partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, $|\mu| \leqslant k$, coloured according to the $\mathbb{Z}_2$–action, then associate to each one of the boxes of the Young diagram of μ another partition λ_s, so that $\sum_{s\in\mu} |\lambda_s| = n$, and the partitions λ_s must satisfy a nesting relation in either direction of μ. Graphically we have

$$p \in \text{fixed locus} \quad \longleftrightarrow \quad \begin{array}{|c|c|cc|}
\hline
\lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,\mu_1} \\
\hline
\lambda_{1,1} & \lambda_{1,2} & \cdots & \\
\cline{1-2}
\vdots & \vdots & \vdots & \\
\lambda_{\ell,1} & \cdots & \lambda_{\ell,\mu_\ell} & \\
\hline
\end{array} \quad ,$$

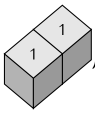
with $\lambda_{i,j} \supset \lambda_{i,j+1}$ and $\lambda_{i,j} \supset \lambda_{i+1,j}$, for $(i,j) \in \mu$. The colouring of the resulting solid partition is then induced by a colouring of the Young diagram μ, where each λ_s acquires the same colour as the underlying box $s \in \mu$. The main difference from the standard instanton counting consists then in the fact that only $\mathbb{Z}_2$-invariant boxes in $\pi_{\mathbb{Z}_2}$ are now going to contribute to the computation of the partition functions.

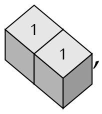
Example 6.1. *Consider the case* $N = 1$, $k = 2$ *of the cohomological limit of the K-theoretic partition function we discussed in the previous sections. This cohomological limit can be interpreted geometrically as follows (cf. §5.5.1): the Chern character provides a natural transformation from the* $\mathbb{T}$*-equivariant K-theory to the* $\mathbb{T}$*-equivariant Chow group with rational coefficient by* $t_i \mapsto e^{s_i}$*, with* $s_i = c_1^{\mathbb{T}}(t_i)$*. This natural map can be extended to complex coefficients as* $t_i^b \mapsto e^{bs_i}$, $b \in \mathbb{C}$*, and it gives a simple linearisation of the K-theoretic brackets* $[-] = (-)^{1/2} - (-)^{-1/2}$ *operator* $\mathrm{ch}[t^{b\mu}] = b e_{\mathbb{T}}(t^\mu) + O(b^2)$*. This linearisation property can be employed* *to define a map* $Z^{D8}_{\overline{N},\overline{k}} \mapsto Z^{D8,\mathrm{coh}}_{\overline{N},\overline{k}}$*, which can in turn be identified with the rational limit of the trigonometric partition function, from a physical standpoint. The rational partition function is also given an integral representation in terms of JK residues, which also depends on all possible decompositions of* (k, N) *in* $(\overline{k}, \overline{N})$*.*

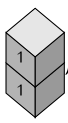
Coming back to the particular case of $(k, N) = (2, 1)$*, one can see that the only possible solid partitions with two boxes are*

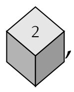
$$\pi_1 = \begin{array}{|c|c|}\hline [1^1] & [1^1] \\ \hline \end{array}, \qquad \pi_2 = \begin{array}{|c|}\hline [1^1] \\ \hline [1^1] \\ \hline \end{array}, \qquad \pi_3 = \begin{array}{|c|}\hline [2^1] \\ \hline \end{array}, \qquad \pi_4 = \begin{array}{|c|}\hline [1^2] \\ \hline \end{array}. \tag{6.3.3}$$

The same solid partitions may also be visualised in the following way:

$\pi_1 =$ 

, $\pi_2 =$ 

, $\pi_3 =$ 

, $\pi_4 =$ 

, (6.3.4)

where the number on the (i, j, k) *box of each plane partition denotes the height of a pile of boxes stacked on* (i, j, k) *along the* ϵ_4 *direction. The corresponding fixed points labelled with the colouring corresponding to the orbifold action will then be*

$$\pi_1 = \begin{array}{|c|c|}\hline [1^1] & [1^1] \\ \hline \end{array}, \qquad \pi_2 = \begin{array}{|c|}\hline [1^1] \\ \hline [1^1] \\ \hline \end{array}, \qquad \pi_3 = \begin{array}{|c|}\hline [2^1] \\ \hline \end{array}, \qquad \pi_4 = \begin{array}{|c|}\hline [1^2] \\ \hline \end{array}. \tag{6.3.5}$$

The partition function (equivariant under $\mathbb{T}_0 = (\mathbb{C}^*)^4|_{s_1+s_2+s_3+s_4=0}$*) for the non orbifolded theory then reads*

$$\begin{aligned} Z^{D8}_{1,2}(\epsilon; q) = \frac{q^2}{2}\Bigg(& \frac{m^2(\epsilon_1+\epsilon_2)^2(\epsilon_1+\epsilon_3)^2(\epsilon_2+\epsilon_3)^2}{\epsilon_1^2\epsilon_2^2\epsilon_3^2\epsilon_4^2} \\ & +5\frac{m(\epsilon_1+\epsilon_2)(\epsilon_1+\epsilon_3)(\epsilon_2+\epsilon_3)}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}\Bigg), \end{aligned} \tag{6.3.6}$$

while the contribution from the regular instanton sector to the orbifold partition function will be

$$\begin{aligned} Z^{D8,\mathrm{orb}}_{\mathrm{reg}}(\epsilon; q_0, q_1) =& Z^{D8,\mathrm{orb}}_{(1,0),(2,0)}(\epsilon; q_0, q_1) + Z^{D8,\mathrm{orb}}_{(1,0),(1,1)}(\epsilon; q_0, q_1) \\ =& \frac{q_0^2 m\epsilon_{12}}{2\epsilon_3\epsilon_{123}(\epsilon_{12}+2\epsilon_3)}\Bigg(\frac{(\epsilon_3-m)(\epsilon_{12}-\epsilon_3)}{\epsilon_3} \\ & +\frac{(\epsilon_{123}+m)(2\epsilon_{12}+\epsilon_3)}{\epsilon_{123}}\Bigg) \\ & + \frac{q_0 q_1 m\epsilon_{12}}{2\epsilon_3\epsilon_{123}(\epsilon_2-\epsilon_1)}\Bigg(\frac{(2\epsilon_2+\epsilon_3)(\epsilon_{13}-\epsilon_2)}{\epsilon_2} \\ & -\frac{(2\epsilon_1+\epsilon_3)(\epsilon_{23}-\epsilon_1)}{\epsilon_1}\Bigg), \end{aligned}$$

with $\epsilon_i = c_1^{\mathbb{T}}(t_i)$, $\epsilon_{ij} = \epsilon_i + \epsilon_j$ *and similarly* $\epsilon_{123} = \epsilon_1 + \epsilon_2 + \epsilon_3$. *The previous formula can be either obtained through an integral formula and iterated residues, or by generalising the geometric correspondence of [174] to the orbifold case*

$$Z_{\text{reg}}^{\text{D8,orb}}(\epsilon; q_0, q_1) = \sum_{|\pi|=2} (-1)^{h(\pi)} e_{\mathbb{T}}\left[-\left(\sqrt{\widetilde{V}_\pi}\right)^{\mathbb{Z}_2} \right] q_0^{|\pi|_0} q_1^{|\pi|_1},$$

where $\sqrt{\widetilde{V}_\pi}$ *is a square root of the virtual tangent space* $T^{\text{vir}}_{(\mathbb{C}^4)^{[k]}}|_\pi$ *twisted by the observable insertion as in §6.3.2 and* $|\pi|_i$ *denotes the number of boxes in* π, *seen as* $\mathbb{Z}_2$*-modules, transforming in the* ρ_i*-th representation of* $\mathbb{Z}_2$.

6.A REVIEW OF THE $4d$ ADHM CONSTRUCTION

6.A.1 Self-dual connections in $4d$

In this appendix we recall some basic facts about the standard ADHM construction [9] of the solutions to the self-duality equation on $\mathbb{R}^4$:

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}, \tag{6.A.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the curvature of a $U(N)$ connection. The instanton number is defined as

$$k = \frac{1}{16\pi^2}\int d^4x\, \epsilon_{\mu\nu\lambda\rho}\, \text{tr}(F_{\mu\nu}F_{\lambda\rho}) \tag{6.A.2}$$

We denote the moduli space of self-dual $U(N)$ connections of instanton number k by $\mathscr{M}_{k,N}$. The (virtual) dimension of the moduli space of instantons is by definition the number of independent deformations

$$(\delta_\mu^\alpha\delta_\nu^\beta - \frac{1}{2}\epsilon^{\alpha\beta\mu\nu})D_\mu\delta A_\nu = 0, \tag{6.A.3}$$

minus the number of gauge degrees of freedom:

$$\delta A_\mu \sim D_\mu\phi, \tag{6.A.4}$$

where D_μ is the covariant derivative in the background A_μ. This difference can be understood as the index of the Atiyah-Singer complex $\mathcal{C}$ (this requires certain vanishing theorem for the cohomologies in degrees 0 and 2):

$$0 \to \Omega^0(\mathbb{R}^4, \mathfrak{su}(N)) \xrightarrow{D_A} \Omega^1(\mathbb{R}^4, \mathfrak{su}(N)) \xrightarrow{(1-*)D_A} \Omega^2_+(\mathbb{R}^4, \mathfrak{su}(N)) \to 0, \tag{6.A.5}$$

which by the index theorem is given by

$$\dim \mathscr{M}_{k,N} = \text{ind}\, \mathcal{C} = 4Nk. \tag{6.A.6}$$

6.A.2 ADHM data and equations

In the ADHM construction we consider a complex $(2k+N)\times N$ matrix U satisfying the equation

$$\Delta^\dagger(x)U(x)=0, \tag{6.A.7}$$

where $\Delta(x)$ is a $(2k+N)\times 2k$ complex matrix

$$\Delta(x)=\begin{pmatrix} B_2^\dagger - z_2^* & -B_1+z_1 \\ B_1^\dagger - z_1^* & B_2-z_2 \\ I^\dagger & J \end{pmatrix}, \tag{6.A.8}$$

and $z_1=x_2+ix_1$, $z_2=x_4+ix_3$. The auxiliary matrix $\Delta(x)$ is required to satisfy the moment map equation

$$\Delta^\dagger(x)\Delta(x)=1_{2\times 2}\otimes f^{-1}_{k\times k}(x), \tag{6.A.9}$$

where $f(x)$ is a $k\times k$ invertible matrix. Eq. (6.A.9) leads to the well-known conditions on $B_{1,2}$, I and J:

$$\mu_{\mathbb{C}}=[B_1,B_2]+IJ=0, \qquad \mu_{\mathbb{R}}=[B_1,B_1^\dagger]+[B_2,B_2^\dagger]-II^\dagger+J^\dagger J=0. \tag{6.A.10}$$

One has to normalize U so that

$$U^\dagger U=1_{N\times N}. \tag{6.A.11}$$

In other words, the matrix U defines a basis of N orthonormal vectors inside $\mathbb{C}^{2k+N}$. The columns of the matrix $\Delta(x)$ form a set of k linearly independent vectors inside $\mathbb{C}^{2k+N}$ (the linear independence is guaranteed by Eq. (6.A.9)). Eq. (6.A.7) means that the basis defined by U is orthogonal to the set of $2k$ vectors inside $\mathbb{C}^{2k+N}$ defined by $\Delta(x)$. Together the columns of U and Δ form a complete basis in $\mathbb{C}^{2k+N}$, and therefore:

$$1_{(2k+N)\times(2k+N)}-U(x)U^\dagger(x)=\Delta(x)(1_{2\times 2}\otimes f_{k\times k}(x))\Delta^\dagger(x). \tag{6.A.12}$$

The self-dual connection $A_\mu(x)$, corresponding to the matrix $U(x)$ is given by

$$A_\mu(x)=U^\dagger\partial_\mu U(x). \tag{6.A.13}$$

It is easy to show that the connection (6.A.13) is indeed self-dual:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu U^\dagger \partial_\nu U(x)-\partial_\nu U^\dagger\partial_\mu U(x)+[U^\dagger\partial_\mu U(x),U^\dagger\partial_\nu U(x)] \\ &= (\partial_{[\mu}U^\dagger)(1-UU^\dagger)(\partial_{\nu]}U^\dagger) \\ &= U^\dagger(\partial_{[\mu}\Delta(x))(1_{2\times 2}\otimes f_{k\times k}(x))(\partial_{\nu]}\Delta^\dagger(x))U \\ &= U^\dagger(\sigma_{\mu\nu}\otimes f_{k\times k}(x))U, \end{aligned} \tag{6.A.14}$$

where

$$\sigma_{\mu\nu}=\sigma_\mu\bar\sigma_\nu-\sigma_\nu\bar\sigma_\mu=i\eta^a_{\mu\nu}\sigma_a, \tag{6.A.15}$$

and $\sigma_\mu=(1,i\vec\sigma)$, $\bar\sigma_\mu=\sigma_\mu^\dagger=(1,-i\vec\sigma)$, $\eta^a_{\mu\nu}$ is the 't Hooft symbol. The matrices $(\sigma_{\mu\nu})_\alpha{}^\beta$ when seen as matrices with indices $\mu\nu$ are known to be self-dual. In fact they give an intertwining operator between the three-dimensional representation $\Lambda^2_+\mathbb{R}^4$ of $SO(4)$ and the adjoint representation of $SU(2)_+$, which is part of $\mathrm{Spin}(4) = SU(2)_+ \times SU(2)_-$, so that

$$\Lambda^2_\pm\mathbb{R}^4 = \mathrm{adj}_{SU(2)_\pm}. \tag{6.A.16}$$

The moduli space $\mathscr{M}_{k,N}$ of solutions to the ADHM equations is the space of matrices (B_1, B_2, I, J) obeying $\mu_\mathbb{C} = 0$ and $\mu_\mathbb{R} = 0$ quotiented by the action of $U(k)$ group (indeed, constant $U(k)$ transformations don't change the connection (6.A.13)). The resulting dimension of the moduli space is

$$\dim_\mathbb{R} \mathscr{M}_{k,N} = \underbrace{2k^2}_{B_1} + \underbrace{2k^2}_{B_2} + \underbrace{2Nk}_{I} + \underbrace{2Nk}_{J} - \underbrace{2k^2}_{\mu_\mathbb{C}} - \underbrace{\mu_\mathbb{R}}_{k^2} - \underbrace{k^2}_{/U(k)} = 4Nk, \tag{6.A.17}$$

is equal to the dimension of the tangent (6.A.6), as it should.

6.A.3 Spinor formalism

Let us see how the ADHM equations can be formulated using spinors. Let γ^μ, $\mu = 0, \ldots, 4$ be four-dimensional gamma-matrices generating the algebra $\mathrm{Cliff}(4)$. Then we have

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = \epsilon^{\mu\nu\lambda\rho}\gamma_5, \tag{6.A.18}$$

where γ_5 is the chirality operator. Let $\psi_\pm$ be a positive (resp. negative) chirality spinor of $\mathrm{Spin}(4)$

$$\gamma_5\psi_\pm = \pm\psi_\pm, \tag{6.A.19}$$

normalized so that $\psi^\dagger_\pm\psi_\pm = 1$. We can then trivially write that

$$\psi^\dagger_\pm\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\psi_\pm = \pm\epsilon^{\mu\nu\lambda\rho}. \tag{6.A.20}$$

Notice that this identity is independent of the concrete value of $\psi_\pm$. We can thus write the self-duality condition for $F_{\mu\nu}$ as

$$F_{\mu\nu} = \frac{1}{2}\psi^\dagger_+\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\psi_+F_{\lambda\rho}. \tag{6.A.21}$$

Alternatively, one can notice that

$$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu\gamma^\nu] = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar\sigma_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \eta^a_{\mu\nu}i\sigma^a & 0 \\ 0 & \bar\eta^a_{\mu\nu}i\sigma^a \end{pmatrix}, \tag{6.A.22}$$

where $\bar\sigma_{\mu\nu} = \bar\sigma_{[\mu}\sigma_{\nu]}$, so that the projection on the self-dual part of $F_{\mu\nu}$ can be written as

$$\gamma^\lambda\gamma^\rho\psi_-F_{\lambda\rho} = 0, \qquad \text{or} \qquad \bar\sigma_{\lambda\rho}F_{\lambda\rho} = 0. \tag{6.A.23}$$

Eq. (6.A.23) contains three independent equations on the components of $A_\mu(x)$. The forth equation is provided by the gauge fixing condition, e.g. $\partial_\mu A_\mu = 0$, so that the total number of components is equal to the total number of (first order) equations. After imposing the equations no *functional* degrees of freedom in $A_\mu(x)$ remain, and the moduli space of solutions $\mathscr{M}_{k,N}$ is finite-dimensional.

6.A.4 Hyperkähler reduction and complex structure(s)

We have considered self-duality equations in $\mathbb{R}^4$ without any reference to a specific choice of the complex structure. However, we can incorporate complex structure in our construction. $\mathbb{R}^4$ is a hypekähler manifold with three basis complex structures I, J and K[11] satisfying the usual quaternionic relations. Correspondingly, there are three Kähler forms ω^I, ω^J and ω^K.

These forms can be used to give the structure of the hyperkähler manifold to the space $\mathcal{A}$ of $U(N)$ connections on $\mathbb{R}^4$. Indeed, the metric on the space of connections is induced from the flat metric on $\mathbb{R}^4$

$$ds^2[\delta_1 A_\mu(x), \delta_2 A_\mu(y)] = \int_{\mathbb{R}^4} \operatorname{tr} \delta_1 A_\mu(x) \delta_2 A_\mu(x), \tag{6.A.24}$$

and we can write the symplectic forms on $\mathcal{A}$ as follows

$$\Omega^a[\delta_1 A_\mu(x), \delta_2 A_\mu(y)] = \int_{\mathbb{R}^4} \omega^a \wedge \operatorname{tr} \delta_1 A(x) \wedge \delta_2 A(x), \tag{6.A.25}$$

where $a = I, J, K$. All three ω^a are actually *self-dual* two-forms. One can then verify that the action of $\mathcal{G}$ on $\mathcal{A}$ is Hamiltonian for all three symplectic forms Ω^a, and the corresponding three moment maps are

$$\mu^a[\phi(x)] = \int_{\mathbb{R}^4} \omega^a \wedge \operatorname{tr}(\phi(x) F(x)), \tag{6.A.26}$$

where $F = dA + [A, A]$ is the field strength. Requiring moment maps to vanish we get precisely the self-duality equations:

$$\omega^a \wedge F = 0. \tag{6.A.27}$$

Indeed, Eq. (6.A.27) mean that the *anti-self-dual* part F vanishes. The space of self-dual connections can be thought of as the hyperkähler reduction of $\mathcal{A}$ by the group $\mathcal{G}$ of all gauge transformations. If we introduce $\mathcal{A}_k$ as the space of connections with instanton number k then we can write

$$\mathscr{M}_{k,N} = \mathcal{A}_k /// \mathcal{G}. \tag{6.A.28}$$

Let us choose a complex structure I, such that z_1 and z_2 defined as in Eq. (6.A.8) are the holomorphic coordinates. Only a subgroup $U(2) \subset SO(4)$ of rotations preserves this choice. The choice of the complex structure I singles out one of the Kähler forms ω^I, which can be written as

$$\omega^I = \omega_{\mathbb{R}} = dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2. \tag{6.A.29}$$

11 There is in fact a S^2 worth of complex structures. Indeed, $aI + bJ + cK$ is a complex structure as long as $a^2 + b^2 + c^2 = 1$.

The other two forms ω^J and ω^K can be recast into the holomorphic symplectic form (and its conjugate)

$$\omega^J + i\omega^K = \omega_{\mathbb{C}} = dz^1 \wedge dz^2. \tag{6.A.30}$$

The subgroup of $SO(4)$, which preserves both the complex structure I and $\omega_{\mathbb{C}}$ is $SU(2) \subset U(2)$ (the remaining $U(1) \subset U(2)$ rotates the phase of $\omega_{\mathbb{C}}$). The field strength $F_{\mu\nu}$ breaks into the following irreducible representations of $SU(2)$ (we also list the number of components):

$$F_{\mu\nu} = \begin{cases} F^{(2,0)}_{z_1 z_2} = F_{\mu\nu}(\omega_{\mathbb{C}}^{-1})^{\mu\nu}, & \dim = 1, \\ F^{(0,2)}_{\bar{z}_1 \bar{z}_2} = F_{\mu\nu}(\bar{\omega}_{\mathbb{C}}^{-1})^{\mu\nu}, & \dim = 1, \\ F^{(1,1)}_{\omega} = F_{\mu\nu}(\omega_{\mathbb{R}}^{-1})^{\mu\nu}, & \dim = 1, \\ (F^{(1,1)}_0)_{z_i \bar{z}_j} = F^{(1,1)}_{z_i \bar{z}_j} - \frac{1}{2} F^{(1,1)}_{z_k \bar{z}_l}(\omega_{\mathbb{R}}^{-1})^{z_k \bar{z}_l}(\omega_{\mathbb{R}})_{z_i \bar{z}_j}, & \dim = 3. \end{cases} \tag{6.A.31}$$

The first three one-dimensional pieces turn out to be self-dual (they are projections on ω^a), while the three-dimensional representation is anti-self-dual. Having these identifications it is easy to see that the moment map equation (6.A.9) is just the requirement that the $\Delta^\dagger\Delta$ lies in the self-dual part $\mathbf{1}$ of the tensor product of $SU(2)$ representations $\bar{\mathbf{2}} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$.

6.A.5 Non-commutative deformation.

The instanton moduli space $\mathscr{M}_{k,N}$ contains singularities, which correspond to instantons of zero size. A clever way to regularize the moduli space is to consider instantons on the non-commutative spacetime $\mathbb{R}^4$ with coordinates satisfying:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{6.A.32}$$

where $\theta^{\mu\nu}$ is a constant 2-form. We further assume that $\theta^{\mu\nu}$ is anti-self-dual, then with an $SO(4)$ rotation it can be aligned with one of the three basis Kähler structures, e.g.

$$\theta^{\mu\nu} = \frac{\zeta}{4}\bar{\eta}^3_{\mu\nu}, \tag{6.A.33}$$

where ζ is a real non-commutativity parameter which we take to be positive.

As shown in [178], the ADHM construction for the non-commutative case requires only a minor update. The expression (6.A.8) for $\Delta(x)$ is still valid and the conditions (6.A.9) still holds. However, since the coordinates no longer commute, one of the moment maps is modified:

$$\mu_{\mathbb{R}} = [B_1^\dagger, B_1] + [B_2^\dagger, B_2] + II^\dagger - J^\dagger J = \zeta 1_{k\times k}. \tag{6.A.34}$$

Notice that after the non-commutative deformation, the instanton solutions also appear in the $U(1)$ gauge theory.

In string theory language, the self-dual connections we are studying correspond to bound states of D0 and D4 branes (or, more generally Dp and D$(p+4)$ branes). Non-commutativity arises if we turn on the nonzero B-field background along the D4 brane.

6.A.6 Matrix form of the non-commutative self-duality equations.

In the noncommutative setting it will be convenient for us to recast the self-duality equations in matrix form. To this end we introduce the operator analogues of covariant derivatives

$$X^{\mu} = x^{\mu} + i\theta^{\mu\nu} A_{\nu}, \tag{6.A.35}$$

The commutator of covariant derivatives gives the field strength:

$$[X^{\mu}, X^{\nu}] = i\theta^{\mu\nu} + \theta^{\mu\lambda}\theta^{\nu\rho} F_{\lambda\rho}, \tag{6.A.36}$$

where the second term in the r.h.s. is due to Eq. (6.A.32).

The self-duality equations can be rewritten as an equation for the operators X^{μ}

$$[X^{\mu}, X^{\nu}] - i\theta^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}([X^{\lambda}, X^{\rho}] - i\theta^{\lambda\rho}). \tag{6.A.37}$$

Let us choose a Kähler structure on $\mathbb{R}^4$ proportional to the B-field. In the complex structure compatible with B the noncommutativity reads

$$[z_a, \bar{z}_b] = -\frac{\zeta}{2}\delta_{ab}, \qquad [z_a, z_b] = [\bar{z}_a, \bar{z}_b] = 0. \tag{6.A.38}$$

We introduce complex covariant derivatives Z_a corresponding to complex coordinates z_a. Eqs. (6.A.37) are then written as one real and one complex equation for Z_a operators:

$$[Z_1, Z_2] = 0, \qquad [Z_1^{\dagger}, Z_2^{\dagger}] = 0, \tag{6.A.39}$$

$$[Z_1^{\dagger}, Z_1] + [Z_2^{\dagger}, Z_2] = \zeta. \tag{6.A.40}$$

The commutation relations (6.A.38) are of course nothing but a pair of standard Heisenberg algebras of creation and annihilation operators, so that

$$z_a = \sqrt{\frac{2}{\zeta}}a_a^{\dagger}, \qquad \bar{z}_a = \sqrt{\frac{2}{\zeta}}a_a, \tag{6.A.41}$$

and

$$[a_a, a_b^{\dagger}] = \delta_{ab}, \qquad [a_a, a_b] = [a_a^{\dagger}, a_b^{\dagger}] = 0. \tag{6.A.42}$$

The operators a_a, $a_a^{\dagger}$ act on the Hilbert space $\mathcal{H}$, which is spanned by the eigenstates $|n, m\rangle$ of the number operators:

$$N_a = a_a^{\dagger}a_a, \qquad N_1|n, m\rangle = n|n, m\rangle, \qquad N_2|n, m\rangle = m|n, m\rangle. \tag{6.A.43}$$

$\mathcal{H}$ can be identified with the ring of polynomials in $a_a^{\dagger}$ (or in z_a) acting on the vacuum $|0, 0\rangle$.

The simplest solution to Eqs. (6.A.39) is the vacuum solution

$$Z_a = z_a, \tag{6.A.44}$$

which corresponds to zero A_{μ} and vanishing instanton charge.

Another example of a solution is the non-commutative $U(1)$ instanton sitting at the origin of $\mathbf{C}^2$:

$$Z_a = \sqrt{\frac{2}{\zeta}} S_{[1]} a_a^\dagger \sqrt{\frac{N(N+3)}{(N+1)(N+2)}} S^\dagger = \sqrt{\frac{2}{\zeta}} S \sqrt{\frac{N+2}{N}} a_a^\dagger \sqrt{\frac{N}{N+2}} S_{[1]}^\dagger, \tag{6.A.45}$$

where $N = a_1^\dagger a_1 + a_2^\dagger a_2$ and $S_{[1]}^\dagger$ acts on $\mathcal{H}$ by relabelling the (infinite number of) basis vectors, so that the state $|0,0\rangle$ does not belong to its image. An example of such a transformation is

$$S_{[1]}^\dagger |n,m\rangle = \begin{cases} |n,m\rangle & m \neq 0, \\ |n+1,m\rangle & m = 0. \end{cases} \tag{6.A.46}$$

The solution (6.A.45) is non-singular (all of its matrix elements are well-defined) and invariant under $U(2)$ rotations of the space-time $\mathbf{C}^2$ (the operator $S_{[1]}$ is invariant only up to a unitary transformation, which can be viewed as a gauge transformation). We give some more examples of instanton solutions invariant under $U(1)^2 \subset U(2)$ in 6.A.8.

6.A.7 $U(1)$ instantons and ideals.

Let us recall the correspondence between $U(1)$ noncommutative instantons and ideals in the ring of polynomials $\mathbb{C}[z_1, z_2]$. The matrix U for gauge group $U(1)$ is a $(2k+1)$-dimensional column vector, which we write as

$$U(z) = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \\ \xi(z) \end{pmatrix}, \tag{6.A.47}$$

where $\psi_{1,2}(z)$ are two k-dimensional column vectors of polynomials and $\xi(z)$ is a polynomial. Eq. (6.A.7) for U is then equivalent to two equations

$$I\xi(z) = (z_2 - B_2)\psi_1(z) + (z_1 - B_1)\psi_2(z), \tag{6.A.48}$$

$$0 = (B_1^\dagger - \bar{z}_1)\psi_1(z) + (\bar{z}_2 - B_2^\dagger)\psi_2(z). \tag{6.A.49}$$

Eq. (6.A.48) implies (see e.g.[103] for details) that $\xi(z)$ belongs to an ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ of polynomials which vanish when one subsitutes in them matrices $B_{1,2}$ instead of the variables $z_{1,2}$. Indeed, acting with Eq. (6.A.48) on the states $|n,m\rangle \in \mathcal{H}$ we get

$$\xi(z)|n,m\rangle I = (z_2 - B_2)\psi_1(z)|n,m\rangle + (z_1 - B_1)\psi_2(z)|n,m\rangle. \tag{6.A.50}$$

After the substitution $z_{1,2} \to B_{1,2}$ we get

$$\xi(B_1, B_2)|n,m\rangle I = 0. \tag{6.A.51}$$

Acting with $B_1^k B_2^l$ on Eq. (6.A.51) and using the commutativity of B_1 and B_2 we see that $\xi(B_1, B_2)|n,m\rangle$ is zero on the whole $\mathbb{C}^k$, so it is a zero matrix and thus $\xi(B_1, B_2) = 0$ as a matrix equation. This determines the ideal completely. Vice versa, each element $\xi(z)$ of $\mathcal{I}$ by definition can be written as a linear combination from Eq. (6.A.48), but not uniquely: one can shift

$$\psi_1(z) \to \psi_1(z) + (z_1 - B_1)\chi(z), \tag{6.A.52}$$

$$\psi_2(z) \to \psi_2(z) + (z_2 - B_2)\chi(z), \tag{6.A.53}$$

for any column-vector $\chi(z)$. To fix a unique decomposition we have to gauge fix the symmetry (6.A.52), (6.A.53), which is done by requiring Eq. (6.A.49) (note that $\bar{z}_a$ act as the derivatives in z_a).

For the single $U(1)$ instanton solution described in the previous section we have $B_1 = B_2 = 0$, $I = \sqrt{\zeta}$ and

$$U(z) = \begin{pmatrix} \bar{z}_2 \\ \bar{z}_1 \\ \frac{1}{\sqrt{\zeta}}(z_1\bar{z}_1 + z_2\bar{z}_2) \end{pmatrix} \lambda(z). \tag{6.A.54}$$

For any $\lambda(z)$ the vector (6.A.54) solves (6.A.7). However to satisfy the normalization condition (6.A.11), we have to put the following normalization factor instead of $\lambda(z)$:

$$\mathcal{N} = \frac{1}{\sqrt{(z_1\bar{z}_1 + z_2\bar{z}_2)(z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta)}} S^\dagger. \tag{6.A.55}$$

The solution (6.A.45) is obtained from the normalized $U(z)$ as

$$Z_a = U^\dagger z_a U, \qquad Z_a^\dagger = U^\dagger \bar{z}_a U. \tag{6.A.56}$$

6.A.8 4d fixed point instantons

We give here some examples of noncommutative instanton solutions on $\mathbb{R}^4$ which are equivariant under the $U(1)^2$ subgroup of $SO(4)$. The ideals in $\mathbb{C}[z_1, z_2]$ corresponding to these solutions are monomial ideals enumerated by Young diagrams.

All the solutions have the form

$$Z_a = \sqrt{\frac{\zeta}{2}} S_Y h_Y(N_1, N_2)^{-1} a_a^\dagger h_Y(N_1, N_2) S_Y^\dagger, \tag{6.A.57}$$

where S_Y is the partial isometry associated with the monomial ideal labelled by the Young diagram Y, $N_a = a_a^\dagger a_a$ and

$$h_Y(N_1, N_2) = \sqrt{\frac{g_Y(N_1, N_2)}{g_Y(N_1 + 1, N_2 + 1)}}. \tag{6.A.58}$$

Let us display the functions $g_Y(N_1, N_2)$ for some elementary Young diagrams

$$g_{[1]} = N, \tag{6.A.59}$$

$$g_{[2]} = N^2 - N_1 + N_2, \tag{6.A.60}$$

$$g_{[1,1]} = N^2 + N_1 - N_2, \tag{6.A.61}$$

$$g_{[3]} = N(N^2 - 3(N_1 - N_2) + 2, \tag{6.A.62}$$
$$g_{[1,1,1]} = N(N^2 + 3(N_1 - N_2) + 2, \tag{6.A.63}$$
$$g_{[2,1]} = (N-1)N(N+1), \tag{6.A.64}$$
$$g_{[3,2,1]} = (N-2)(N-1)N^2(N+1)(N+2). \tag{6.A.65}$$

where $N = N_1 + N_2$. The first solution (6.A.59) is the one-instanton solution discussed in sec. 6.A.6. One can check that for any function g_Y given above the matrix equations (6.A.39), (6.A.40) are indeed satisfied. In fact Eq. (6.A.39) is satisfied automatically by the ansatz (6.A.57), and it is only necessary to check a single recurrence relation for the function $h(N_1, N_2)$:

$$\sum_{a=1}^{2} \left\{ \frac{h(N)^2}{h(N_a+1)^2}(N_a+1) - \frac{h(N_a-1)^2}{h(N)^2} N_a \right\} = 2, \tag{6.A.66}$$

which turns out to be true. Of the solutions given above only (6.A.59) and (6.A.65) are invariant under the full $SU(2)$ rotation symmetry. There exists an infinite family of such fully symmetric solutions corresponding to triangular Young diagrams with

$$h_{[k,k-1,\ldots,1]}(N_1, N_2) = \left(1 - \frac{k(k+1)}{N(N+1)}\right)^{\frac{1}{2}} = \left(\frac{(N-k)(N+k+1)}{N(N+1)}\right)^{\frac{1}{2}}. \tag{6.A.67}$$

Notice that $g_Y(i,j)$ vanishes if the box (i,j) lies on the border of the Young diagram Y. This guarantees that the action of Z_a and $Z_a^\dagger$ from Eq. (6.A.57) is well-defined.

6.B 8D INSTANTONS AND SHEAF COHOMOLOGY

6.B.1 Moduli spaces of 8d instantons via Beilinson's Theorem

Here we will briefly study moduli spaces of framed torsion-free sheaves on $\mathbb{P}^4$ and their relation to spaces of $SU(4)$ instantons (and their generalisations) on $\mathbb{C}^4$. Let us first notice that, in general, if $\mathscr{E}$ is a torsion-free sheaf of rank r on $\mathbb{P}^4$ with $\mathrm{ch}(\mathscr{E}) = (N, 0, 0, 0, -k)$, framed along a divisor D, including $\mathscr{E}$ into its double dual we get the exact sequence (for the proof of this result, see [64])

$$0 \longrightarrow \mathscr{E} \hookrightarrow \mathscr{O}_{\mathbb{P}^d}^{\oplus r} \twoheadrightarrow \mathscr{Q} \longrightarrow 0,$$

and $\mathscr{Q}$ has finite support in $\mathbb{A}^4 \cong \mathbb{P}^4/D$. Consider then the moduli space

$$\left\{ \begin{array}{l} \mathscr{E} \in \mathsf{Coh}(\mathbb{P}^4),\ \mathscr{E}\ \text{torsion}-\text{free},\ \mathrm{rk}(\mathscr{E}) = N, \\ \mathscr{E} \hookrightarrow \mathscr{O}_{\mathbb{P}^4}^{\oplus r} \twoheadrightarrow \mathscr{Q} \quad \text{s.t.} \\ \mathscr{Q}\ \text{of finite support in}\ \mathbb{A}^4 \cong \mathbb{P}^4/D \end{array} \right\} \Big/ \text{iso}, \tag{6.B.1}$$

parametrising torsion–free sheaves on $\mathbb{P}^4$ fitting in a given short exact sequence. Of course the previous moduli space is identified with the quot scheme of points in $\mathbb{A}^4$, as the Grothendieck moduli functor

$$\mathrm{Quot}_{\mathbb{P}^4}(\mathscr{O}^{\oplus N}, k) : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Sets}$$

contains an open subfunctor $\mathrm{Quot}_{\mathbb{A}^4}(\mathscr{O}^{\oplus N}, k) \hookrightarrow \mathrm{Quot}_{\mathbb{P}^4}(\mathscr{O}^{\oplus N}, k)$ parametrising precisely the quotients above. What one might want to do is then to study a model for deformations of the moduli space of sheaves

$$\mathscr{M}_{k,N}(\mathbb{P}^4) = \left\{ \begin{array}{l} \mathscr{E} \in \mathrm{Coh}(\mathbb{P}^4),\ \mathscr{E} \text{ torsion} - \text{free},\ \mathrm{rk}(\mathscr{E}) = N, \\ c_1(\mathscr{E}) = c_2(\mathscr{E}) = c_3(\mathscr{E}) = 0,\ c_4(\mathscr{E}) = k,\ \mathscr{E}|_{\wp^i_\infty} \cong \mathscr{O}^{\oplus N}_{\mathbb{P}^4} \\ H^3(\mathbb{P}^4, \mathscr{E}(-3)) = 0,\ H^2(\mathbb{P}^4, \mathscr{E}(-\ell)) = 0, \forall \ell \end{array} \right\} \Big/ \text{iso},$$

where $\wp^i_\infty$, $i = 1, \ldots, 4$, are hyperplanes at infinity in $\mathbb{P}^4$ defined by $z_i = 0$ in homogeneous coordinates. The aim is to generalize the monad construction for the usual $SU(N)$ instantons to the case of $\mathbb{C}^4$. Let then $\mathscr{E} \in \mathrm{Coh}(\mathbb{P}^4)$ a sheaf on $\mathbb{P}^4$ as in the definition and consider $\mathbb{P}^4 \times \mathbb{P}^4$ with the projections on the two factors

$$\begin{tikzcd} & \mathbb{P}^4 \times \mathbb{P}^4 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^4 & & \mathbb{P}^4 \end{tikzcd}$$

One then has the Koszul resolution of $\mathscr{O}_\Delta$, Δ being the diagonal $\Delta \cong \mathbb{P}^4 \hookrightarrow \mathbb{P}^4 \times \mathbb{P}^4$:

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^4 \widehat{\mathscr{O}} \arrow[r] & \bigwedge^3 \widehat{\mathscr{O}} \arrow[r] & \bigwedge^2 \widehat{\mathscr{O}} \arrow[dlll, rounded corners] \\ \widehat{\mathscr{O}} \arrow[r] & \mathscr{O}_{\mathbb{P}^4 \times \mathbb{P}^4} \arrow[r] & \mathscr{O}_\Delta \arrow[r] & 0, \end{tikzcd}$$

where $\widehat{\mathscr{O}} = \mathscr{O}_{\mathbb{P}^4}(1) \boxtimes \mathscr{Q}^\vee$, and $\mathscr{Q} \cong T_{\mathbb{P}^4}(-1)$. The previous sequence tells us that $[\mathscr{O}_\Delta] \cong [\bigwedge^\bullet \left(\mathscr{O}_{\mathbb{P}^4}(-1) \boxtimes \mathscr{Q}^\vee\right)]$ in the derived category. We then define

$$C^p = \bigwedge^{-i} \left(\mathscr{O}_{\mathbb{P}^4}(-1) \boxtimes \mathscr{Q}^\vee\right),$$

by means of which we will define the Beilinson spectral sequence in this case. As $\mathscr{E} \in \mathrm{Coh}(\mathbb{P}^4)$, we have the trivial identity $p_{1*}\left(p_2^* \mathscr{E} \otimes \mathscr{O}_\Delta\right) = \mathscr{E}$, and if we replace $\mathscr{O}_\Delta$ by its Koszul resolution we get the double complex for the hyperdirect image, which can be expressed in terms of the Fourier-Mukai transform

$$\mathbf{R}^\bullet p_{1,*}\left(p_2^* \mathscr{E} \otimes C^\bullet\right).$$

There are then two different spectral sequences that can be taken for the Fourier-Mukai transform. One of them gives back the trivial identity we started with, while the other one has E_1–term given by

$$E_1^{p,q} = R^q p_{1,*}\left(p_2^* \mathscr{E} \otimes C^p\right),$$

and as $C^p = \mathscr{F}_1^p \boxtimes \mathscr{F}_2^p$, the E_1–term can be written as

$$E_1^{p,q} = \mathscr{F}_1^p \otimes H^q(\mathbb{P}^4, \mathscr{E} \otimes \mathscr{F}_2^p).$$

This spectral sequence converges to

$$E_\infty^{q,p} = \begin{cases} \mathscr{E}(-\ell), & \text{if } q+p=0 \\ 0, & \text{otherwise} \end{cases}$$

for each $\ell \geqslant 0$. We can actually make the first term in the sequence explicit:

$$E_1^{p,q} = \mathscr{O}_{\mathbb{P}^4}(p) \otimes H^q\left(\mathbb{P}^4, \mathscr{E}(-\ell) \otimes \Omega_{\mathbb{P}^4}^{-p}(-p)\right),$$

and we can visualise the E_1-page of the spectral sequence as

$$\begin{array}{ccccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & E_1^{-4,4} & E_1^{-3,4} & E_1^{-2,4} & E_1^{-1,4} & E_1^{0,4} & 0 & \cdots \\ \cdots & 0 & E_1^{-4,3} & E_1^{-3,3} & E_1^{-2,3} & E_1^{-1,3} & E_1^{0,3} & 0 & \cdots \\ \cdots & 0 & E_1^{-4,2} & E_1^{-3,2} & E_1^{-2,2} & E_1^{-1,2} & E_1^{0,2} & 0 & \cdots \\ \cdots & 0 & E_1^{-4,1} & E_1^{-3,1} & E_1^{-2,1} & E_1^{-1,1} & E_1^{0,1} & 0 & \cdots \\ \cdots & 0 & E_1^{-4,0} & E_1^{-3,0} & E_1^{-2,0} & E_1^{-1,0} & E_1^{0,0} & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \tag{6.B.2}$$

Though the spectral sequence is naturally a doubly graded object, fixing certain conditions on the sheaf cohomology groups makes its E_1-page to collapse to an ordinary exact sequence. Indeed it is easy, albeit tedious, to show by means of homological algebra that the appropriate conditions are exactly those in the definition of $\mathscr{M}_{N,k}(\mathbb{P}^4)$. By doing so, the spectral sequence displayed in (6.B.2) converges to $E_\infty^{-1,1} \cong \mathscr{E}(-2)$, all the other terms being identically vanishing. Moreover the $E_1^{p,q}$ term of the Beilinson spectral sequence is reduced to the following (6.B.3)

$$\begin{array}{l} 0 \longrightarrow A \otimes \mathscr{O}_{\mathbb{P}^4}(-4) \longrightarrow B \otimes \mathscr{O}_{\mathbb{P}^4}(-3) \longrightarrow C \otimes \mathscr{O}_{\mathbb{P}^4}(-2) \longrightarrow \\ \qquad \longrightarrow D \otimes \mathscr{O}_{\mathbb{P}^4}(-1) \longrightarrow E \otimes \mathscr{O}_{\mathbb{P}^4} \longrightarrow 0, \end{array} \tag{6.B.3}$$

where

$$\begin{aligned} A &= H^1(\mathbb{P}^4, \mathscr{E}(-3)), & B &= H^1(\mathbb{P}^4, \mathscr{E}(-2) \otimes \Omega_{\mathbb{P}^4}^3(3)), \\ C &= H^1(\mathbb{P}^4, \mathscr{E}(-2) \otimes \Omega_{\mathbb{P}^4}^2(2)), & D &= H^1(\mathbb{P}^4, \mathscr{E}(-2) \otimes \Omega_{\mathbb{P}^4}^1(1)), \\ E &= H^1(\mathbb{P}^4, \mathscr{E}(-2)). \end{aligned}$$

We then see that (6.B.3) is a perfect extended monad in the sense of [120, Definition 3.1/3.2], and it provides a model for the deformation of the Quot scheme of points on $\mathbb{A}^4$. Analogously one can lift the set theoretic isomorphism between moduli spaces of framed sheaves and quot schemes of points to a scheme theoretic isomorphism by virtue of a version in families of Beilinson's theorem. Indeed, let S be a scheme over $\mathbb{C}$. Given any coherent sheaf $\mathscr{E}$ on $\mathbb{P}^m \times S$ there exists a spectral sequence $E_i^{p,q}$ whose $E_1^{p,q}$ term is

$$E_1^{p,q} = \mathscr{O}_{\mathbb{P}^m}(p) \boxtimes R^q p_{2,*}\left(\mathscr{E} \otimes \Omega^{-p}_{\mathbb{P}^4 \times S/S}(-p)\right),$$

where p_2 is the projection of $\mathbb{P}^m \times S$ on the second factor. The spectral sequence $E_i^{p,q}$ converges to

$$E_\infty^{q,p} = \begin{cases} \mathscr{E}, & \text{if } q+p=0 \\ 0, & \text{otherwise.} \end{cases}$$

This approach was studied in much greater generality in [120], and more details about the proof can be found also in [121; 122]. Moreover, the scheme-theoretic isomorphism between moduli spaces of framed torsion-free sheaves on $\mathbb{P}^m$ and quot schemes of points on $\mathbb{A}^m$ was recently proved by means of an infinitesimal argument in [64].

Remark 6.2. The choice of conditions constraining the sheaves $\mathscr{E} \in \mathrm{Coh}_{\mathbb{P}^4}$ are different from the instanton sheaf conditions found in [130] as the latter would exclude certain sheaf configurations, such as ideal sheaves of points, which are instead known to be interesting to the problem at hand. ◀

6.B.2 Orbifold instantons and ADHM data decomposition

It is interesting to notice here that in general it is not known whether a Gorenstein orbifold in dimension 4 admits a crepant resolution, and many singularities are terminal. Nonetheless some classification results, though a bit scattered, are available. It is known, for example, that orbifolds of the form $\mathbb{C}^r/G$ admit crepant resolutions in the case of finite *abelian* subgroups $G \subset SL(r, \mathbb{C})$ for which $\mathbb{C}^r/G$ is a complete intersection, [77; 78], while some arithmetic condition have been derived in the case of some series of cyclic quotient singularities [79; 80]. Moreover, contrary to the lower dimensional cases, here the Hilbert-Chow morphism

$$\mathrm{Hilb}^G(\mathbb{C}^r) \xrightarrow{\pi} \mathbb{C}^r/G$$

does not necessarily provide a crepant resolution, as there are even cases in which $\mathrm{Hilb}^G(\mathbb{C}^r)$ is singular despite $\mathbb{C}^r/G$ being known to have projective crepant resolutions. Here $\mathrm{Hilb}^G(\mathbb{C}^r)$ denotes the G-Hilbert scheme of zero-dimensional G-invariant subschemes $Z \subset \mathbb{C}^r$ of length $|G|$ such that $H^0(\mathscr{O}_Z)$ is the regular representation of G. It turns out, however, that in some cases which will be of interest to us a canonical crepant (projective) resolution of the orbifold singularity is indeed provided by the G-Hilbert scheme, and toric geometry techniques are well suited to check whether this is the case or not, [194; 195]. Let's assume for the moment that $X = \mathrm{Hilb}^G(\mathbb{C}^4)$ is a crepant resolution of $\mathbb{C}^4/G$: this will enable us to justify the ADHM data decomposition which we will use in the following in order to study the orbifold instantons in eight dimensions. Take then the universal object $\mathcal{Z} \subset X \times \mathbb{C}^4$ with the projections π_i on the i-th factor.

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[d, dash] & \\
 & X \times \mathbb{C}^4 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & \mathbb{C}^4
\end{tikzcd}$$

We can then introducte the tautological bundle $\mathscr{T}$ on X by pushing forward

$$\mathscr{T} = \pi_{1*}\mathscr{O}_{\mathcal{Z}}.$$

Under the G-action, $\mathscr{T}$ transforms under the regular representation, and it is easy to show that its fibers are of the form $\mathbb{C}[z_1, z_2, z_3, z_4]/I \cong H^0(\mathscr{O}_Y)$, where I is a G-invariant ideal corresponding to the zero-dimensional subscheme $Y \subseteq \mathbb{C}^4$ of length $|G|$. Multiplication by the coordinates along the fibers of $\mathscr{T}$ induces a G-invariant homomorphism $B \in Q \otimes \mathrm{End}(\mathscr{T})$ (where Q denotes the regular representation of G), which is representable by a quadruple of endomorphisms $(B_1, \ldots, B_4)$, such that $B \wedge B = \sum_{a<b}[B_a, B_b] = 0 \in \mathrm{Hom}_G(\mathscr{T}, \bigwedge^2 Q \otimes \mathscr{T})$. As we noticed in section 6.3.1 the regular representation Q may be decomposed in irreducible representations of G. This induces a decomposition of the tautological bundle as

$$\mathscr{T} = \bigoplus_r \mathscr{T}_r \otimes \rho_r.$$

The monad construction for the ADHM representation of orbifold instantons then follows from a generalisation of [74; 147; 164]. One starts from the resolution of the diagonal in $X \times X$, which is

$$\begin{aligned}
0 \longrightarrow \left(\widehat{\mathscr{T}} \otimes \Lambda^4 \mathscr{Q}^\vee\right)^G \longrightarrow \left(\widehat{\mathscr{T}} \otimes \Lambda^3 \mathscr{Q}^\vee\right)^G &\longrightarrow \\
\left(\widehat{\mathscr{T}} \otimes \Lambda^2 \mathscr{Q}^\vee\right)^G \longrightarrow \left(\widehat{\mathscr{T}} \otimes \mathscr{Q}^\vee\right)^G \longrightarrow \widehat{\mathscr{T}}^G &\longrightarrow \\
\mathscr{O}_\Delta \longrightarrow 0&,
\end{aligned}$$

where $\widehat{\mathscr{T}} = \mathscr{T} \boxtimes \mathscr{T}^\vee$. We then compactify X to $\overline{X}$ by compactifying $\mathbb{C}^4/G$ to $\mathbb{P}^4/G$ and resolving the singularity at the origin. We can do this by defining $\overline{X} = X \sqcup \mathscr{P}_\infty$, where $\mathscr{P}_\infty \cong \mathbb{P}^3/G$. This is useful as we can then glue objects on X and G-invariant objects on $\mathscr{P}_\infty$ so as to get globally defined objects on

$\overline{X}$. By defining $\widehat{\mathscr{T}}(-i) = \mathscr{T}(-i\mathscr{P}_\infty) \boxtimes \mathscr{T}^\vee$, a globally defined resolution of the diagonal sheaf in $\overline{X} \times \overline{X}$ is

$$
\begin{aligned}
0 &\longrightarrow \left(\widehat{\mathscr{T}}(-4) \otimes \textstyle\bigwedge^4 \mathscr{Q}^\vee\right)^G \longrightarrow \left(\widehat{\mathscr{T}}(-3) \otimes \textstyle\bigwedge^3 \mathscr{Q}^\vee\right)^G \\
&\longrightarrow \left(\widehat{\mathscr{T}}(-2) \otimes \textstyle\bigwedge^2 \mathscr{Q}^\vee\right)^G \longrightarrow \left(\widehat{\mathscr{T}}(-1) \otimes \mathscr{Q}^\vee\right)^G \\
&\longrightarrow \widehat{\mathscr{T}}^G \longrightarrow \mathscr{O}_\Delta \longrightarrow 0.
\end{aligned}
$$

The construction then follows the same strategy adopted in the previous section. In particular, if $\mathscr{E}$ is a coherent sheaf on $\overline{X}$, one can take its Fourier-Mukai transform, where now $C^\bullet$ is the resolution of the diagonal $\mathscr{O}_\Delta$ in $\overline{X}$. Then, associated to the sheaf $\mathscr{E}(-\ell) = \mathscr{E} \otimes \mathscr{O}_{\overline{X}}(-\ell\mathscr{P}_\infty)$ there is a spectral sequence $E_i^{p,q}$, whose $E_1^{p,q}$-term is given by

$$
E_1^{p,q} = \left(\mathscr{T}(p) \otimes H^q(\overline{X}, \mathscr{E}(-\ell) \times \mathscr{T}^\vee \otimes \bigwedge^{-p} \mathscr{Q}^\vee)\right)^G .
$$

The tensor product decomposition of the tautological bundle $\mathscr{T}$ as $\mathscr{T} = \bigoplus \mathscr{T}_r \otimes \rho_r$ makes it possible to reduce the homological algebra coming from the orbifold spectral sequence to the easier $\mathbb{P}^4$ case. Indeed, cohomology groups relevant to the computations all take the form $H^q(\overline{X}, \mathscr{E}(-\ell) \otimes \mathscr{T}_r)$, which can in turn be interpreted as the equivariant decomposition of the ADHM datum.

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