Chebyshev Polynomials and Best Rank-one Approximation Ratio

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Abstract. We establish a new extremal property of the classical Chebyshev polynomials in the context of best rank-one approximation of tensors. We also give some necessary conditions for a tensor to be a minimizer of the ratio of spectral and Frobenius norms.

Introduction and Outline

The classical Chebyshev polynomials are known to have many extremal properties. The first result was established by Chebyshev himself: he proved [?] that a univariate monic polynomial with real coefficients that least deviates from zero on the interval $[-1,1]$ must be proportional to a Chebyshev polynomial of the first kind. Later, there were further developments highlighting extremal properties of this class of univariate polynomials, see [?] and references therein. In this article we discover a new extremal property of Chebyshev polynomials of the first kind in the context of the theory of rank-one approximations of real tensors.

Let us define *the binary Chebyshev form of degree* d as

$$\text{Ч}_{d,2}(x_1,x_2) = \frac{(x_1+ix_2)^d + (x_1-ix_2)^d}{2} = \sum_{k=0}^{[d/2]} \binom{d}{2k} (-1)^k x_1^{d-2k} x_2^{2k}. \tag{0.1}$$

Note that its restriction to the unit circle $x_1^2+x_2^2=1$ can be identified with the univariate Chebyshev polynomial of the first kind $x \mapsto \text{Ч}_{d,2}(x,\sqrt{1-x^2}) = \cos(d \arccos x)$, $x \in [-1,1]$. In Theorem 1.1 we prove that the binary form (0.1) minimizes the ratio of the uniform norm on the unit circle and the Bombieri norm among all non-zero binary forms of the given degree d.

We then define the family of homogeneous n-ary forms (1.2) that we call *Chebyshev forms* $\text{Ч}_{d,n}$ and conjecture that the Chebyshev form $\text{Ч}_{d,n}$ minimizes the ratio of the uniform norm on the unit sphere and the Bombieri norm among all non-zero forms of a given degree d and number of variables n. Since homogeneous polynomials can be identified with the symmetric tensor of its coefficients, this conjecture can be stated in the language of symmetric tensors as follows: the symmetric tensor n^d tensor associated to the Chebyshev form $\text{Ч}_{d,n}$ minimizes the ratio of spectral norm and Frobenius norm among all real symmetric n^d tensors. Since the spectral norm of a tensor measures its relative distance to the set of rank-one tensors, see Section 2.2, yet another way to state our conjecture is that the symmetric tensor associated to the Chebyshev form achieves the maximum possible relative distance to the set of all rank-one tensors in the space of symmetric tensors.

Besides settling our conjecture for the case of binary forms in Theorem 1.1, we are also able to prove it in case of cubic ternary forms ($d = 3$, $n = 3$) in Theorem 1.2. This latter result in fact follows from a more general result that we obtain in Theorem 1.5: the maximal orthogonal rank of a real $(3, 3, 3)$-tensor is 7. This in particular implies that the minimum value of the ratio of the spectral norm and the Frobenius norm of a non-zero $(3, 3, 3)$-tensor is $1/\sqrt{7}$ and hence gives an affirmative answer to a conjecture in [10]. In Theorem 1.10 we show that if a tensor minimizes the ratio of the spectral and the Frobenius norms, then it lies in the space spanned by its best rank-one approximations. In Theorem 1.11 we prove an analogous result for symmetric tensors or, equivalently, homogeneous forms: if a form minimizes the ratio of the uniform norm on the unit sphere and the Bombieri norm, then it lies in the space spanned by rank-one forms defined by global extrema of the restriction of the form to the unit sphere. These two results imply lower bounds on the number of best rank-one approximations for those tensors (respectively, on the number of global extrema of homogeneous forms) that minimize the ratio of norms, see Corollary 1.12.

In the next section we state all our results in detail. The results are proved in Section 3. Section 2 contains some necessary preliminaries and axillary results.

1 Main results

In this section we state our main results. They are all closely related but can be grouped into somewhat different directions.

1.1 Chebyshev forms and their extremal property

In the following $P_{d,n}$ denotes the space of n-ary real forms of degree d (homogeneous polynomials of degree d in n variables). For a form p, we denote by

$$\|p\|_\infty = \max_{\|x\|_2=1} |p(x)|$$

the *uniform norm* of its restriction to the Euclidean unit sphere in $\mathbb{R}^n$.

Every form $p \in P_{d,n}$ has a standard representation in the basis of monomials: $p(x) = \sum_{|\alpha|=d} c_\alpha x^\alpha \in P_{d,n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, \ldots, d\}^n$ is a multi-index of length $|\alpha| = \alpha_1 + \cdots + \alpha_n = d$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *Bombieri norm* of p is defined as

$$\|p\|_B = \sqrt{\sum_{|\alpha|=d} \binom{d}{\alpha}^{-1} |c_\alpha|^2},$$

where $\binom{d}{\alpha} = \frac{d!}{\alpha_1!\ldots\alpha_n!}$ is the multinomial coefficient.

The *conformal orthogonal group* $CO(n)=\mathbb{R}_+\times O(n)$ acts on the space $P_{d,n}$ of real forms as follows:

$$g=(s,\rho)\in CO(n),\ p\in P_{d,n}\ \mapsto\ g^*p\in P_{d,n},\quad (g^*p)(x)=sp(\rho^{-1}x).$$

Note that both the uniform norm and the Bombieri norm are invariant under the subgroup $O(n)$ of orthogonal transformations and their ratio is invariant under the full group $CO(n)$, see Section 2.2.

In our first result we classify minimizers of the ratio $\|\cdot\|_\infty/\|\cdot\|_B$ of the two norms on the set of all non-zero binary forms.

Theorem 1.1. *For any nonzero $p\in P_{d,2}$ it holds*

$$\frac{\|p\|_\infty}{\|p\|_B}\geq\frac{\|Ч_{d,2}\|_\infty}{\|Ч_{d,2}\|_B}=\frac{1}{\sqrt{2^{d-1}}}. \tag{1.1}$$

When $d=0,1$ one has equality in (1.1) for any $p\in P_{d,2}$, when $d=2$ equality holds if and only if $p=\pm g^*(x_1^2+x_2^2)$ or $p=g^*Ч_2=g^*(x_1^2-x_2^2)$, where $g\in CO(2)$. When $d\geq 3$ equality holds if and only if $p=g^*Ч_{d,2}$, $g\in CO(2)$.

For any $d\geq 0$ and $n\geq 2$ we define the *n-ary Chebyshev form of degree d* as

$$Ч_{d,n}(x_1,\dots,x_n)=\sum_{k=0}^{[d/2]}\binom{d}{2k}(-1)^k x_1^{d-2k}(x_2^2+\cdots+x_n^2)^k. \tag{1.2}$$

Note that the forms $Ч_{d,n}$ are invariant under orthogonal transformations of $\mathbb{R}^n$ that preserve the point $(1,0,\dots,0)$ and for any vector $v=(v_2,\dots,v_n)\in\mathbb{R}^{n-1}$ of unit length one has that $Ч_{d,n}(x_1,v_2x_2,\dots,v_nx_2)=Ч_{d,2}(x_1,x_2)$ is the binary Chebyshev form (0.1). In this work we are particularly concerned with the cubic Chebyshev forms

$$Ч_{3,n}(x_1,\dots,x_n)=x_1^3-3x_1(x_2^2+\cdots+x_n^2). \tag{1.3}$$

It is an easy calculation that

$$\|Ч_{d,n}\|_\infty=1,\quad \|Ч_{d,n}\|_B^2=\sum_{k=0}^{[d/2]}\binom{d}{2k}\sum_{\substack{\beta=(\beta_1,\dots,\beta_{n-1}),\\ |\beta|=k}}\binom{k}{\beta}^2\binom{2k}{2\beta}^{-1} \tag{1.4}$$

and, in particular,

$$\|Ч_{3,n}\|_\infty=3n-2. \tag{1.5}$$

We state the conjecture that the Chebyshev form $Ч_{d,n}$ provides the minimal possible ratio between uniform and Bombieri norm among all real nonzero n-ary forms of degree d.

Conjecture 1. *Let $d\geq 0$, $n\geq 2$. For any nonzero $p\in P_{d,n}$ it holds that*

$$\frac{\|p\|_\infty}{\|p\|_B}\geq\frac{\|Ч_{d,n}\|_\infty}{\|Ч_{d,n}\|_B}.$$

Obviously, Theorem 1.1 settles this conjecture in the binary case $n = 2$. Regarding the case $d = 2$ (quadratic forms) it holds that $\|p\|_\infty/\|p\|_B \geq 1/\sqrt{n}$ for any $p \in P_{2,n}$, as can be seen by passing to the ratio of spectral and Frobenius norm of symmetric matrices. The equality is attained only by quadratic forms $p = g^*(\pm x_1^2 \pm \cdots \pm x_n^2)$, where $g \in CO(n)$. Note that among these extremal quadratic forms there is the Chebyshev quadric $Ч_{2,n}(x) = x_1^2 - x_2^2 - \cdots - x_n^2$ which is classically known as *the Lorentz quadric*. Hence Conjecture 1 is true in the case $d = 2$.

In this work, we settle Conjecture 1 also in the case $d = 3$, $n = 3$ of ternary cubics.

Theorem 1.2. *Let $p \in P_{3,3}$ be a non-zero ternary cubic form. Then*

$$\frac{\|p\|_\infty}{\|p\|_B} \geq \frac{1}{\sqrt{7}}$$

and equality holds if $p = g^* Ч_{3,3}$, where $g \in CO(3)$.

Theorem 1.2 is part of Corollary 1.6 further below.

We believe that for forms of degree $d \geq 3$ the minimal ratio in Conjecture 1 is attained at $p \in P_{d,n}$ if and only if p is in the orbit of the Chebyshev form $Ч_{d,n}$ under the group $CO(n)$. Note that for binary forms this claim is a part of Theorem 1.1, but for ternary cubics the "only if" direction is not asserted in Theorem 1.2.

The following result is a local version of Conjecture 1 for cubic Chebyshev forms.

Theorem 1.3. *Let $n \geq 2$. For all $p \in P_{3,n}$ in a small neighbourhood of $Ч_{3,n}$ we have*

$$\frac{\|p\|_\infty}{\|p\|_B} \geq \frac{\|Ч_{3,n}\|_\infty}{\|Ч_{3,n}\|_B}$$

In other words, Theorem 1.3 asserts that $Ч_{3,n}$ is a local minimum of the ratio of the two norms on the set of non-zero cubic n-ary forms.

1.2 Best rank-one approximation ratio, orthogonal rank and orthogonal tensors

Let $\otimes_{j=1}^d \mathbb{R}^{n_j}$ denote the space of real $(n_1, \ldots, n_d)$-tensors, considered as $n_1 \times \cdots \times n_d$ tables $A = (a_{i_1 \ldots i_d})$ of real numbers. For two $(n_1, \ldots, n_d)$-tensors their *Frobenius inner product* is given by

$$\langle A, A' \rangle_F = \sum_{i_1, \ldots, i_d = 1}^{n_1, \ldots, n_d} a_{i_1 \ldots i_d} a'_{i_1 \ldots i_d}$$

and $\|A\|_F = \sqrt{\langle A, A \rangle_F}$ denotes the induced Frobenius norm .

The outer product $x^{(1)} \otimes \cdots \otimes x^{(d)}$ of vectors $x^{(j)} \in \mathbb{R}^{n_j}$ is an $(n_1, \ldots, n_d)$-tensor X with entries $(x_{i_1}^{(1)} \cdots x_{i_d}^{(d)})$. Non-zero tensors of this form are said to be of *rank one*, denoted $\operatorname{rank}(X) = 1$. The *spectral norm* on $\otimes_{j=1}^d \mathbb{R}^{n_j}$ is defined as

$$\|A\|_2 = \max_{\|x^{(1)}\| = \cdots = \|x^{(d)}\| = 1} \langle A, x^{(1)} \otimes \cdots \otimes x^{(d)} \rangle_F = \max_{\substack{\|X\|_F = 1 \\ \operatorname{rank}(X) = 1}} \langle A, X \rangle_F, \tag{1.6}$$

where $\|\cdot\|$ denotes the standard Euclidean norm.

The notion of *best rank-one approximation ratio* of a tensor space was introduced by Qi in [12]. For the space of $(n_1,\dots,n_d)$-tensors it is defined as the constant

$$\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j}) = \min_{0\neq A\in\otimes_{j=1}^d \mathbb{R}^{n_j}} \frac{\|A\|_2}{\|A\|_F}. \tag{1.7}$$

It is the largest constant c satisfying $\|A\|_2 \geq c\|A\|_F$ for all $A\in\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})$.

Definition 1.4. A non-zero tensor $A\in\otimes_{j=1}^d \mathbb{R}^{n_j}$ is called *extremal*, if it is a minimizer in (1.7), that is, if it satisfies

$$\frac{\|A\|_2}{\|A\|_F} = \mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j}).$$

The space $\operatorname{Sym}^d(\mathbb{R}^n)$ of symmetric n^d-tensors consists of tensors $A=(a_{i_1\dots i_d})$ in $\otimes_{j=1}^d \mathbb{R}^n$ that satisfy $a_{i_{\sigma_1}\dots i_{\sigma_d}} = a_{i_1\dots i_d}$ for any permutation σ on d elements. This space is isomorphic to the space $P_{d,n}$ of homogeneous forms as explained in Subsection 2.1. *The best rank-one approximation ratio* $\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))$ of the space of symmetric tensors is defined by replacing $\otimes_{j=1}^d \mathbb{R}^{n_j}$ with $\operatorname{Sym}^d(\mathbb{R}^n)$ in (1.7) and it is equal to the minimal possible ratio between uniform and Bombieri norms of a non-zero form in $P_{d,n}$, see Subsection 2.1. In this context it is important to note that the definition of the spectral norm of a symmetric tensor does not change if the maximum in (1.6) is taken over symmetric rank-one tensors only, see Subsection 2.1.

A general formula for $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})$ or $\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))$ is not known except for special cases, see [11]. Determining or estimating these constants is an interesting problem on its own, and may have some useful applications for rank-truncated tensor optimization methods [13]. The present work contains some new contributions with main focus on symmetric tensors.

One always has

$$0<\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})\leq 1 \quad\text{and}\quad 0<\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n})\leq \mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))\leq 1.$$

Lower bounds on the best rank-one approximation ratio can be obtained from decomposition of tensors into pairwise orthogonal tensors of rank one. For a tensor $A\in\otimes_{j=1}^d \mathbb{R}^{n_j}$ let

$$A = Y_1+\dots+Y_r \tag{1.8}$$

be a decomposition into pairwise orthogonal rank-one tensors, i.e., $Y_1,\dots,Y_r$ are rank-one $(n_1,\dots,n_d)$-tensors with $\langle Y_\ell, Y_{\ell'}\rangle_F = 0$ for $\ell\neq\ell'$. The smallest possible number r of that allows such a decomposition (1.8) is called the *orthogonal rank* of the tensor A [7] and will be denoted by $\operatorname{rank}_\perp(A)$. Since at least one of the terms in (1.8) has to satisfy $\langle A, Y_i\rangle_F \geq \|A\|_F^2/r$, it follows that

$$\frac{\|A\|_2}{\|A\|_F} \geq \frac{1}{\sqrt{\operatorname{rank}_\perp(A)}}$$

for all $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$. Thus an upper bound on the maximal orthogonal rank in a given tensor space leads to a lower bound on the best rank-one approximation ratio of that tensor space:

$$\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j}) \geq \frac{1}{\sqrt{\max_{\otimes_{j=1}^d \mathbb{R}^{n_j}} \operatorname{rank}_\perp(A)}}. \tag{1.9}$$

It appears that for all known values of $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})$ this is actually an equality [9, 11].

The values for $\mathscr{A}(\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2} \otimes \mathbb{R}^{n_3})$ have been determined in [10] for all combinations $n_1, n_2, n_3 \leq 4$, except for $(3,3,3)$-tensors. In the present work we are able to settle this remaining case, by determining the maximum possible orthogonal rank of a $(3,3,3)$-tensor.

Theorem 1.5. *The maximal orthogonal rank of a* $(3,3,3)$*-tensor is seven.*

In [10] it has been shown that $1/\sqrt{7}$ is an upper bound for $\mathscr{A}(\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3)$ and conjectured that it actually is the exact value. Due to (1.9), Theorem 1.5 shows that $1/\sqrt{7}$ is also a lower bound and hence proves this conjecture. On the other hand, we see from (1.5) that the minimal ratio $1/\sqrt{7}$ can be achieved by symmetric $(3,3,3)$-tensors, in particular the ones associated with the Chebyshev form $Ч_{3,3}$. Due to the isomorphy of $\operatorname{Sym}^3(\mathbb{R}^3)$ and $P_{3,3}$, Theorem 1.2 is therefore part of the following corollary of Theorem 1.5.

Corollary 1.6. *We have*

$$\mathscr{A}(\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3) = \mathscr{A}(\operatorname{Sym}^3(\mathbb{R}^3)) = \frac{1}{\sqrt{\max_{A \in \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3} \operatorname{rank}_\perp(A)}} = \frac{1}{\sqrt{7}}$$

and the symmetric tensor corresponding to the Chebyshev cubic $Ч_{3,3}$ *is extremal.*

Assume now that $n_1 \leq \cdots \leq n_d$. Then it is not difficult to show that the orthogonal rank of an $(n_1, \ldots, n_d)$ is not larger than $n_1 \cdots n_{d-1}$. It follows from (1.9) that

$$\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j}) \geq \frac{1}{\sqrt{n_1 \cdots n_{d-1}}}, \quad n_1 \leq \cdots \leq n_d. \tag{1.10}$$

In [11] the concept of an *orthogonal tensor* is defined by the property that its contraction with any $d-1$ vectors of unit length along the first $d-1$ modes (assuming n_d is the largest dimension) results in a vector of unit length. It is then shown that equality in (1.10) is attained if and only if the space contains orthogonal tensors and only those are then the extremal ones. Moreover, for n^d-tensors this is the case if and only if $n = 1, 2, 4, 8$. Therefore, Theorem 1.1 in particular shows that

$$\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^2)) = \frac{1}{\sqrt{2^{d-1}}} = \mathscr{A}(\otimes_{j=1}^d \mathbb{R}^2),$$

and since the symmetric tensors associated to Chebyshev forms attain these constants, they are orthogonal in the sense of [11]. In light of Corollary 1.6 one hence may wonder whether $\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))$ equals $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^n)$ in general, or at least in the case $d = 3$. Note that for matrices it is true. The answer to this question, however, is negative. At least, in the cases $n = 4$ and $n = 8$ it would imply the existence of symmetric orthogonal tensors, which we will show is not the case.

Proposition 1.7. *If $A \in \mathrm{Sym}^d(\mathbb{R}^n)$ is an orthogonal symmetric tensor of order $d \geq 3$, then $n = 1$ or $n = 2$. For $n = 2$ the only such tensors are the ones associated to rotated Chebyshev forms $p = Ч_{d,2} \circ \rho^{-1}$, $\rho \in O(2)$, that is, are of the form $(\rho, \cdots, \rho) \cdot A$ (see (2.4)) with A given by (2.7).*

Corollary 1.8. *For $d \geq 3$ we have*

$$\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^4) = \frac{1}{\sqrt{4^{d-1}}} < \mathscr{A}(\mathrm{Sym}^d(\mathbb{R}^4)) \quad \text{and} \quad \mathscr{A}(\otimes_{j=1}^d \mathbb{R}^8) = \frac{1}{\sqrt{8^{d-1}}} < \mathscr{A}(\mathrm{Sym}^d(\mathbb{R}^8)).$$

The cases of 2^d- and $(3,3,3)$-tensors are therefore exceptional in the sense that the "non-symmetric" best rank-one approximation ratio can be achieved by symmetric tensors. Currently, we are not aware of any lower bound on $\mathscr{A}(\mathrm{Sym}^3(\mathbb{R}^n))$ for $n \geq 4$ better than (1.10), but if Conjecture 1 was true it would imply that

$$\mathscr{A}(\mathrm{Sym}^3(\mathbb{R}^n)) = \frac{1}{\sqrt{3n-2}} \tag{1.11}$$

and Corollary 1.6 confirms this in the case $n = 3$ of cubic ternary forms. Stated in this way, Conjecture 1 becomes quite remarkable from another viewpoint: since it is known [5] that

$$\mathscr{A}(\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3) \leq \frac{3\sqrt{\pi/2}}{n}$$

for all n, (1.11) would imply that the best rank-one approximation ratio for symmetric (n,n,n)-tensors deteriorates asymptotically slower with n than for the space of all (n,n,n)-tensors, although the dimension $\dim(\mathrm{Sym}^3(\mathbb{R}^n)) = O(n^3)$ of this subspace grows at the same rate as the dimension $\dim(\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n) = n^3$ of the full space. More generally, it is stated in [5] that $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^n) \leq d\sqrt{\pi/2}/\sqrt{n^{d-1}}$, whereas Conjecture 1 together with (1.4) would yield that

$$\mathscr{A}(\mathrm{Sym}^d(\mathbb{R}^n)) = \frac{\|Ч_{d,n}\|_\infty}{\|Ч_{d,n}\|_B} = \left((2[d/2]-1)!!\binom{d}{2[d/2]}^{-1}\right)^{1/2} \frac{1}{\sqrt{n^{[d/2]}}}\left(1 + O(n^{-1})\right)$$

as $n \to +\infty$.

1.3 Variational characterization and critical tensors

The problem of determining the best rank-one approximation ratio of a tensor space and finding associated extremal tensors can be seen as a constrained optimization problem for a Lipschitz function. The spectral norm $A \mapsto \|A\|_2$ is a Lipschitz function on the normed space $(\otimes_{j=1}^d \mathbb{R}^{n_j}, \|\cdot\|_F)$ (with Lipschitz constant one). The best rank-one approximation ratio $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})$ equals the minimal value of this function on the Euclidean sphere $\{A \in \otimes_{j=1}^d \mathbb{R}^{n_j} : \|A\|_F = 1\}$ defined by the Frobenius norm, and extremal tensors (of unit Frobenius norm) are its global minima. Global as well as local minima of a Lipschitz function on a closed set are among its critical points on that set. The notion of a critical point of a Lipschitz function on a sphere will be explained in section 2.3 and motivates the following terminology.

Definition 1.9. A non-zero tensor $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ is *critical* if $A/\|A\|_F$ is a critical point of the spectral norm function on the Frobenius unit sphere in the sense that λA belongs to the subdifferential of the spectral norm at $A/\|A\|_F$ for some $\lambda \in \mathbb{R}$.

We then can give a characterization of critical $(n_1, \dots, n_d)$-tensors in terms of their decomposition into best rank-one approximations, that is, rank-one tensors of closest possible Frobenius distance.

Theorem 1.10. *A non-zero tensor $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ is critical if and only if the rescaled tensor $\|A\|_2^2/\|A\|_F^2\, A$ can be written as a finite convex linear combination of some best rank-one approximations of A.*

More explicitly, the theorem states that tensor A is critical if and only if there exists a decomposition

$$\left(\frac{\|A\|_2}{\|A\|_F}\right)^2 A = \sum_{\ell=1}^r \alpha_\ell Y_\ell, \quad \sum_{\ell=1}^r \alpha_\ell = 1, \ \alpha_1, \dots, \alpha_r > 0, \tag{1.12}$$

where the $Y_1, \dots, Y_r$ are of rank-one and satisfy

$$\|Y_\ell\|_F = \|A\|_2, \quad \text{and} \quad \langle A, Y_\ell \rangle_F = \|A\|_2^2$$

for $\ell = 1, \dots, r$. These two properties (and being rank-one) fully characterize best rank-one approximations of A, see section 2.2.

In particular, if $A^* \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ is an extremal tensor, then

$$\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})^2 A^* = \sum_{\ell=1}^r \alpha_\ell Y_\ell, \quad \sum_{\ell=1}^r \alpha_\ell = 1, \ \alpha_1, \dots, \alpha_r > 0, \tag{1.13}$$

for some best rank-one approximations $Y_1, \dots, Y_r$ of A^*.

Results of Ottaviani?

An analogue of Theorem 1.10 holds for symmetric tensors or, equivalently, homogeneous forms. Considering the spectral norm as a function on the space $\mathrm{Sym}^d(\mathbb{R}^n)$ only, it is again a Lipschitz function, and the best rank-one approximation ratio of $\mathrm{Sym}^d(\mathbb{R}^n)$ equals its minimum value on the Frobenius unit sphere in the space $\mathrm{Sym}^d(\mathbb{R}^n)$ of symmetric tensors. A non-zero symmetric tensor $A \in \mathrm{Sym}^d(\mathbb{R}^n)$ will be called *critical in* $\mathrm{Sym}^d(\mathbb{R}^n)$ if the normalized symmetric tensor $A/\|A\|_F$ is a critical point (section 2.3) of the restricted spectral norm on the Frobenius unit sphere in the space $\mathrm{Sym}^d(\mathbb{R}^n)$. We will sometimes also say that a form $p \in P_{d,n}$ is critical which means that the associated symmetric tensor is critical.

Theorem 1.11. *A non-zero tensor $A \in \operatorname{Sym}^d(\mathbb{R}^n)$ is critical in $\operatorname{Sym}^d(\mathbb{R}^n)$ if and only if the rescaled tensor $\|A\|_2^2/\|A\|_F^2 A$ can be written as a convex linear combination of some symmetric best rank-one approximations of A. Then A is also critical in the space $\otimes_{j=1}^d \mathbb{R}^n$.*

Here the second statement follows immediately from Theorem 1.10 by noting that symmetric best rank-one approximations of a symmetric tensor have the same distance as possibly non-symmetric rank-one approximations of that tensor, due to Banach's result, see section 2.1. However, if $A^* \in \operatorname{Sym}^d(\mathbb{R}^n)$ is an extremal symmetric tensor, then, by the theorem,

$$\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))^2 A^* = \sum_{\ell=1}^r \alpha_\ell Y_\ell, \quad \sum_{\ell=1}^r \alpha_\ell = 1,\ \alpha_1, \dots, \alpha_r > 0 \tag{1.14}$$

for some symmetric best rank-one approximations $Y_1, \dots, Y_r$ of A^*, and A^* is critical in $\otimes_{j=1}^d \mathbb{R}^{n_j}$. But in general A^* is not extremal in $\otimes_{j=1}^d \mathbb{R}^{n_j}$ according to the considerations at the end of the previous subsection. Exceptions are symmetric orthogonal matrices, as well as the Chebyshev forms $\text{Ч}_{d,2}$ (Theorem 1.1) and $\text{Ч}_{3,3}$ (Corollary 1.6).

Theorems 1.10 and 1.11 combined with Proposition 2.2 from section 2.2 imply that extremal tensors must have several best rank-one approximations.

Corollary 1.12. *Let $d \geq 2$. Then any extremal tensor in $\otimes_{j=1}^d \mathbb{R}^n$ has at least n distinct best rank-one approximations. Similarly, any extremal symmetric tensor in $\operatorname{Sym}^d(\mathbb{R}^n)$ has at least n distinct symmetric best rank-one approximations.*

We conjecture that extremal tensors actually have infinitely many best rank-one approximations.

We can also relate critical tensors to their nuclear norm. The nuclear norm of a $(n_1, \dots, n_d)$-tensor $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ is defined by

$$\|A\|_* = \inf \left\{ \sum_{\ell=1}^r \|Y_\ell\|_F :\ A = \sum_{\ell=1}^r Y_\ell,\ r \in \mathbb{N},\ \operatorname{rank}(Y_\ell) = 1, \ell = 1, \dots, r \right\}. \tag{1.15}$$

It is a result of Friedland and Lim [8] that for a symmetric $A \in \operatorname{Sym}^d(\mathbb{R}^n)$ it is enough to take the infimum in (1.15) over symmetric rank-one tensors only. Hence the nuclear norm may be intrinsically defined in the subspace $\operatorname{Sym}^d(\mathbb{R}^n)$ only. In either case, the infimum is attained.

The nuclear and spectral norms are dual to each other (see section 2.1) and for any tensor $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ it holds

$$\|A\|_F^2 \leq \|A\|_2 \|A\|_*. \tag{1.16}$$

Our next result characterizes tensors achieving equality in (1.16).

Theorem 1.13. *The following two properties are equivalent for a non-zero tensor A in $\otimes_{j=1}^d \mathbb{R}^{n_j}$ or $\operatorname{Sym}^d(\mathbb{R}^n)$:*

(i) *A is critical,*

(ii) $\|A\|_2\|A\|_* = \|A\|_F^2$.

We remark that the fact that extremal tensors achieve equality in (1.16) has already been proved in [6, Theorems 2.2 and 3.1].

1.4 Decomposition of Chebyshev forms

For symmetric tensors, the statement of Theorem 1.11 can be rephrased for homogeneous forms. To do this one has to note (see section 2.1 and (2.12)) that a symmetric rank one tensor $Y = \pm y \otimes \cdots \otimes y$ is a best symmetric rank-one approximation of the symmetric tensor associated to a homogenous form p, if and only if it holds

$$\|y\|_2^d = \|p\|_\infty, \quad \text{and} \quad p(y) = \pm\|p\|_\infty^2. \tag{1.17}$$

On the other hand, by (2.10), the homogenous form associated to such a rank-one tensor is a d-th power of a linear form,

$$p_Y(x) = \pm\langle y, x\rangle_2^d = \pm(y_1x_1 + \cdots + y_nx_n)^d.$$

Therefore, in analogy to (1.12), Theorem 1.11 states that a form $p \in P_{d,n}$ is critical for the ratio $\|p\|_\infty/\|p\|_B$ if and only if it can be written as

$$\left(\frac{\|p\|_\infty}{\|p\|_B}\right)^2 p(x) = \sum_{\ell=1}^r \alpha_\ell s_\ell \left(y_1^\ell x_1 + \cdots + y_n^\ell x_n\right)^d, \quad \sum_{\ell=1}^r \alpha_\ell = 1,\ \alpha_1, \ldots, \alpha_r > 0, \tag{1.18}$$

where the points $y^1, \ldots, y^r \in \mathbb{R}^n$ satisfy (1.17), and

$$s_\ell = \operatorname{sgn} p(y^\ell).$$

From Theorem 1.1 we know that the binary Chebyshev forms $Ч_{d,2}$ are extremal in $P_{2,d}$ and therefore they must admit a decomposition like (1.18). In Theorem 1.14 we find such a decomposition with the scaling as in (1.14). It immediately implies a bound on the rank of the associated symmetric tensors. For $k = 0, \ldots, d-1$ denote $\theta_k = \pi k/d$ and $a_k = \cos(\theta_k)$, $b_k = \sin(\theta_k)$. Then $a_k + ib_k = e^{i\theta_k}$ are $2d$-th roots of unity.

Theorem 1.14. *For any $d \geq 1$ we have*

$$\frac{1}{2^{d-1}} Ч_{d,2}(x_1, x_2) = \frac{1}{d}\sum_{k=0}^{d-1}(-1)^k \left(x_1a_k + x_2b_k\right)^d \tag{1.19}$$

or, in polar coordinates,

$$\frac{1}{2^{d-1}} Ч_{d,2}(\cos\theta, \sin\theta) = \frac{1}{2^{d-1}}\cos(d\theta) = \frac{1}{d}\sum_{k=0}^{d-1}(-1)^k \cos(\theta - \theta_k)^d. \tag{1.20}$$

In particular, the rank of the associated symmetric 2^d-tensor (2.7) is at most d.

The second equality in (1.20) constitutes an interesting trigonometric identity, which we were not able to find in the literature,

Regarding cubic forms, we have stated the conjecture that the Chebyshev forms $Ч_{3,n}$ are extremal in $P_{3,n}$ (and the corresponding tensor is extremal in $\mathrm{Sym}^d(\mathbb{R}^n)$). In the following corollary of Theorem 1.14 we provide a decomposition like (1.18) for $Ч_{3,n}$, which at least shows that these forms are critical in $P_{3,n}$.

Corollary 1.15. *For $n \geq 2$ we have*

$$\frac{1}{3n-2} Ч_{3,n}(x) = \left(\frac{n+2}{9n-6}\right) x_1^3 + \frac{4}{9n-6} \sum_{i=2}^{n} \left(\frac{-x_1 + \sqrt{3}x_i}{2}\right)^3 + \left(\frac{-x_1 - \sqrt{3}x_i}{2}\right)^3. \tag{1.21}$$

In particular, $Ч_{3,n}$, $n \geq 2$, is critical for the ratio $\|p\|_\infty / \|p\|_B$, $p \in P_{3,n}$, and the rank of the associated symmetric n^3-tensor (1.22) *is at most $2n-1$.*

In section 3.6 we will use Corollary 1.15 to prove Theorem 1.3.

It is interesting to note that a decomposition of $Ч_{3,n}$, more precisely of its representing symmetric tensor, into non-symmetric best rank-one approximations is trivially obtained. By (1.3), the associated symmetric tensor is

$$A_n = e_1 \otimes e_1 \otimes e_1 - \sum_{k=2}^{n} (e_1 \otimes e_k \otimes e_k + e_k \otimes e_1 \otimes e_k + e_k \otimes e_k \otimes e_1), \tag{1.22}$$

with e_k denoting the standard unit vectors in $\mathbb{R}^n$. Since $\|A_n\|_2 = 1$ by (1.5), this "decomposition into entries" is a decomposition into best rank-one approximations with equal weights. Scaling by $\|A_n\|_2^2 / \|A_n\|_F^2 = 1/(3n-2)$ provides a desired convex decomposition (1.12). While this proves (Theorem 1.10) that A_n is critical in $\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, it does not imply by itself that A_n is critical in $\mathrm{Sym}^3(\mathbb{R}^n)$. Thus Corollary 1.15 is a stronger statement. As a side remark, however, observe that (1.22) is a decomposition into pairwise orthogonal rank-one tensors. Due to (1.9) this shows that the tensor A_n associated to the Chebyshev form $Ч_{3,n}$ has orthogonal rank $3n-2$.

2 Preliminaries

In this section we gather some more basic properties upon which our arguments for proving the main results in section 3 will be based.

2.1 Tensors, forms and their norms

The space of $(n_1, \ldots, n_d)$-tensors is isomorphic to the space of multilinear maps on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$. The map associated to tensor A is

$$(x^{(1)}, \ldots, x^{(d)}) \mapsto \langle A, x^{(1)} \otimes \cdots \otimes x^{(d)} \rangle_F = \sum_{i_1, \ldots, i_d = 1}^{n_1, \ldots, n_d} a_{i_1 \ldots i_d} x_{i_1}^{(1)} \ldots x_{i_d}^{(d)}. \tag{2.1}$$

The spectral norm of A as defined in (1.6) can be seen as the natural norm of this associated multilinear map.

As for the nuclear norm defined in (1.15), it can be shown that the infimum is always attained, and a minimizing decomposition $A = \sum_{\ell=1}^{r} X_\ell$ of A into rank-one tensors satisfying $\|A\|_* = \sum_{\ell=1}^{r} \|X_\ell\|_F$ is called *a nuclear decomposition.* We have already stated that the spectral and the nuclear norms are dual to each other, that is,

$$\|A\|_2 = \max_{\|A'\|_* \le 1} \left|\langle A, A'\rangle_F\right|, \quad \|A\|_* = \max_{\|A'\|_2 \le 1} \left|\langle A, A'\rangle_F\right|, \tag{2.2}$$

and the three above introduced norms satisfy

$$\|A\|_2 \le \|A\|_F, \quad \|A\|_F \le \|A\|_* \quad \text{and} \quad \|A\|_F^2 \le \|A\|_2\|A\|_*. \tag{2.3}$$

Moreover, in the first two inequalities in (2.3) equality holds if and only if A is a rank-one tensor. We refer to [8] for these statements.

The product of orthogonal groups $O(n_1, \dots, n_d) = O(n_1) \times \cdots \times O(n_d)$, whose elements are denoted $(\rho^{(1)}, \dots, \rho^{(d)})$ acts on the space $\otimes_{j=1}^{d} \mathbb{R}^{n_j}$ by multilinear multiplication,

$$(\rho^{(1)}, \dots, \rho^{(d)}) \cdot A = \left(\sum_{j_1,\dots,j_d=1}^{n_1,\dots,n_d} \rho^{(1)}_{i_1 j_1} \cdots \rho^{(d)}_{i_d j_d} a_{j_1 \dots j_d} \right), \tag{2.4}$$

preserving the Frobenius inner product, the spectral and the nuclear norms.

We make some additional comments on symmetric tensors. The $\binom{n+d-1}{d}$-dimensional space $\mathrm{Sym}^d(\mathbb{R}^n) \subset \otimes_{j=1}^{d} \mathbb{R}^n$ of symmetric n^d-tensors is isomorphic to the space $P_{d,n}$ of n-ary d-homogeneous real forms. The symmetric tensor A is identified with the form p_A defined as

$$p_A(x) = \langle A, x \otimes \cdots \otimes x\rangle_F = \sum_{i_1,\dots,i_d=1}^{n} a_{i_1 \dots i_d} x_{i_1} \dots x_{i_d}, \quad x \in \mathbb{R}^n, \tag{2.5}$$

which equals the restriction of the multilinear map (2.1) to the "diagonal" in $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$. The representation (2.5) contains many equal terms due to symmetry of A. The compact standard representation of a homogeneous form is obtained by collecting multi-indices according to the number of occurrences of every entry, which is the same as collecting all permutations of multi-indices. This gives

$$p_A(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha,$$

with $\alpha \in \{0, 1, \dots, d\}^n$ and coefficients and

$$a_\alpha = \binom{d}{\alpha} a_{i_1 \dots i_d}, \tag{2.6}$$

where $(i_1, \dots, i_d)$ is any multi-index such that for $i = 1, \dots, n$ the value i occurs α_i times among $i_1, \dots, i_d$.

As an example, the homogeneous Chebyshev forms $Ч_{d,2}$ in (0.1) correspond to symmetric tensors with entries

$$a_{i_1\dots i_d} = \begin{cases} (-1)^k, & \text{if } \#\{i_j = 2\} = 2k, \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

In desymmetrized form (2.1) (and slightly abusive notation) they are hence given as the following multilinear maps:

$$Ч_{d,2}(x^{(1)}, \dots, x^{(d)}) = \langle A, x^{(1)} \otimes \cdots \otimes x^{(d)} \rangle = \sum_{k=0}^{[d/2]} (-1)^k \sum_{\#\{i_j=2\}=2k} x^{(1)}_{i_1} \dots x^{(d)}_{i_d}.$$

Banach proved [1] that for a symmetric coefficient tensor A, the maximum absolute value of the multilinear form (2.1) on a product of spheres can be attained at diagonal inputs, in other words

$$\|A\|_2 = \max_{\|x\|_2=1} |p_A(x)| = \|p_A\|_\infty. \tag{2.8}$$

This is a generalization of the fact that for a symmetric matrix A the maximum absolute value of the bilinear form $x^T Ay$ is, modulo scaling, attained when $x = y$ is an eigenvector for the eigenvalue with largest modulus. Therefore, spectral norm for symmetric tensors may be intrinsically defined in the space $\mathrm{Sym}^d(\mathbb{R}^n)$. Since any symmetric rank-one tensor is necessarily of the form $\pm x \otimes \cdots \otimes x$, Banach's result admits the geometric interpretation that for a symmetric tensor A its projections onto the cones of non-symmetric respectively symmetric rank-one tensors take the same value. In particular, the maximum in the definition (1.6) of the spectral norm can be achieved for a symmetric rank-one tensor.

Next, one can easily check that the Frobenius inner product between two symmetric tensors $A = (a_{i_1\dots i_d})$, $A' = (a'_{i_1\dots i_d}) \in \mathrm{Sym}^d(\mathbb{R}^n)$ equals the *Bombieri product* between the corresponding homogeneous forms $p_A(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ and $p_{A'}(x) = \sum_{|\alpha|=d} a'_\alpha x^\alpha$ with coefficients defined through (2.6):

$$\langle A, A' \rangle_F = \sum_{i_1,\dots,i_d=1}^{n} a_{i_1\dots i_d} a'_{i_1\dots i_d} = \sum_{|\alpha|=d} \binom{d}{\alpha}^{-1} a_\alpha a'_\alpha =: \langle p_A, p_{A'} \rangle_B. \tag{2.9}$$

By (2.8) and (2.9), the isomorphism $A \mapsto p_A$ establishes an isometry between $(\mathrm{Sym}^d(\mathbb{R}^n), \|\cdot\|_2)$ and $(P_{d,n}, \|\cdot\|_\infty)$, as well as between $(\mathrm{Sym}^d(\mathbb{R}^n), \|\cdot\|_F)$ and $(P_{d,n}, \|\cdot\|_B)$.

When $n_1 = \cdots = n_d = n$ the diagonal subaction of the action (2.4) preserves the subspace $\mathrm{Sym}^d(\mathbb{R}^n)$ of symmetric tensors and it corresponds to the action of the orthogonal group on the space $P_{d,n}$ of homogeneous forms by orthogonal change of variables:

$$\rho \in O(n),\ p \in P_{d,n} \ \mapsto\ p \circ \rho^{-1}.$$

Due to (2.9), this shows that the Bombieri inner product is invariant under such a change of variables.

Finally, we have already noted that according to (2.5) a symmetric rank-one tensor $Y = \pm y \otimes \cdots \otimes y$ corresponds to the d-th power of a linear form $L_y(x) = \langle y, x \rangle_2$ as follows:

$$p_Y(x) = \langle \pm y \otimes \cdots \otimes y, x \otimes \cdots \otimes x \rangle_F = \pm \langle y, x \rangle_2^d = \pm (L_y(x))^d. \tag{2.10}$$

Hence a decomposition of a symmetric tensor into symmetric rank-one tensors corresponds to a decomposition of the associated homogeneous form into powers of linear forms. Note that by (2.5) the Bombieri inner product of any homogeneous form $p \in P_{d,n}$ with a d-th power of a linear form L_y equals $\langle p, L_y^d \rangle_B = p(y)$.

2.2 Best rank-one approximation ratio

Given a non-zero tensor $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$, a rank-one $(n_1, \ldots, n_d)$-tensor Y is called a *best rank-one approximation* to A if

$$\|A - Y\|_F = \min_{\operatorname{rank}(X)=1} \|A - X\|_F,$$

that is, X^* is a closest rank-one tensor to A in the Frobenius norm. Note that for a given normalized rank-one tensor X, $\|X\|_F = 1$, the closest one to A on the line λX, $\lambda \in \mathbb{R}$, is given by the orthogonal projection

$$\lambda = \langle A, X \rangle_F \leq \|A\|_2,$$

which gives the squared distance

$$\|A - \lambda X\|_F^2 = \|A\|_F^2 - \langle A, X \rangle_F^2. \tag{2.11}$$

A normalized best rank-one approximation $X = Y / \|Y\|_F$ hence takes the maximal value $\langle A, X \rangle_F = \|A\|_2$. Therefore, Y is a best rank-one approximation if and only if

$$\langle A, Y \rangle_F = \|A\|_2 \|Y\|_F, \quad \text{and} \quad \|Y\|_F = \|A\|_2. \tag{2.12}$$

In light of (2.5), Banach's result (2.8) now implies that a non-zero symmetric tensor A has at least one symmetric best rank-one approximation, and all symmetric best rank-one approximations are given through $Y = \operatorname{sgn}(p_A(x)) \, |p_A(x)| \cdot x \otimes \cdots \otimes x$, where $\|x\|_2 = 1$ and x is a maximizer of $|p_A|$ on the unit sphere.

We mention another property of best rank-one approximations that will be used later. If $\lambda y^{(1)} \otimes \cdots \otimes y^{(d)}$ is a multiple of a best rank-one approximation of A, it follows from the previous considerations that for every $j = 1, \ldots, d$ the vector $y^{(j)} / \|y^{(j)}\|_2$ maximizes the linear form $x^{(j)} \mapsto \langle A, y^{(1)} \otimes \cdot \otimes x^{(j)} \otimes \cdots \otimes y^{(d)} \rangle_F$ subject to the spherical constraint $\|x^{(j)}\|_2 = 1$. Hence the linear form vanishes on the orthogonal complement of $y^{(j)}$, that is,

$$\langle A, y^{(1)} \otimes \cdots \otimes y^{(j-1)} \otimes x^{(j)} \otimes y^{(j+1)} \otimes \cdots \otimes y^{(d)} \rangle_F = 0$$

for all $x^{(j)}$ that are orthogonal to $y^{(j)}$.

We continue with some further remarks on extremal tensors and best rank-one approximation ratio. By (2.11) and (2.12), the squared Frobenius distance of a tensor A to any of its best rank-one approximations Y equals

$$\|A - Y\|_F^2 = \|A\|_F^2 - \|A\|_2^2.$$

Recalling the definition (1.7) of the best rank-one approximation ratio $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})$, the maximum relative distance of a tensor to the set of rank-one tensors in the tensor space hence is given as

$$\max_{0 \neq A \in \otimes_{j=1}^d \mathbb{R}^{n_j}} \min_{\operatorname{rank}(X)=1} \frac{\|A - X\|_F}{\|A\|_F} = \sqrt{1 - \mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})^2}, \tag{2.13}$$

and is achieved for extremal tensors. This relation explains the name 'best rank-one approximation ratio' for the constant $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n_j})$. When restricting to symmetric tensors, (2.13) holds with $\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))$ instead.

In the following lemma we show that the best rank-one approximation ratio strictly decreases with the dimension.

Lemma 2.1. *Let $\mathscr{A}_{d,n}$ denote either the constants $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^n)$ or the constants $\mathscr{A}(\operatorname{Sym}^d(\mathbb{R}^n))$. Then for any $d \geq 1$ and $n \geq 1$ we have*

$$\mathscr{A}_{d,n+1} \leq \frac{\mathscr{A}_{d,n}}{\sqrt{1 + \mathscr{A}_{d,n}^2}}.$$

Proof. Let $A \in \otimes_{j=1}^d \mathbb{R}^n$ be an n^d-tensor of of unit Frobenius norm, $\|A\|_F = 1$. For $\varepsilon \in [0,1]$, let $A^\varepsilon \in \otimes_{j=1}^d \mathbb{R}^{n+1}$ be the $(n+1)^d$-tensor with entries

$$a^\varepsilon_{i_1 \dots i_d} = \begin{cases} \sqrt{1 - \varepsilon^2 \|A\|_2^2}\, a_{i_1 \dots i_d}, & \text{if } i_1, \dots, i_d \leq n, \\ \varepsilon \|A\|_2, & \text{if } i_1 = \dots = i_d = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\|A^\varepsilon\|_F = 1$, and A^ε is symmetric if A is. Let $\xi^{(1)}, \dots, \xi^{(d)}$ be unit norm vectors in $\mathbb{R}^{n+1}$ partitioned as $\xi^{(j)} = (x^{(j)}, z^{(j)})$ with $x^{(j)} \in \mathbb{R}^n$ and $z^{(j)} \in \mathbb{R}$. Then from the 'block diagonal' structure of A^ε it is follows that

$$\begin{aligned} \langle A^\varepsilon, \xi^{(1)} \otimes \dots \otimes \xi^{(d)} \rangle_F &= \sqrt{1 - \varepsilon^2 \|A\|_2^2} \langle A, x^{(1)} \otimes \dots \otimes x^{(d)} \rangle_F + \varepsilon \|A\|_2 z^{(1)} \cdots z^{(d)} \\ &\leq \max\left(\sqrt{1 - \varepsilon^2 \|A\|_2^2}, \varepsilon \right) \|A\|_2 (\|x^{(1)}\|_2 \cdots \|x^{(d)}\|_2 + z^{(1)} \cdots z^{(d)}). \end{aligned}$$

By a generalized Hölder inequality, the term in the right brackets is bounded by one. The maximum on the left, on the other hand, takes its minimal value for $\varepsilon = 1/\sqrt{1 + \|A\|_2^2}$. Since $\xi^{(1)}, \cdots, \xi^{(d)}$ were arbitrary, this shows

$$\|A^\varepsilon\|_2 \leq \frac{\|A\|_2}{\sqrt{1 + \|A\|_2^2}}.$$

The assertions follow by choosing as A extremal tensors in the spaces $\otimes_{j=1}^d \mathbb{R}^n$ or $\mathrm{Sym}^d(\mathbb{R}^n)$, respectively. $\square$

The previous lemma provides a lower bound on the rank of extremal tensors. Recall that the *(real) rank* of a tensor $A \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ is the smallest number r that is needed to represent A as the linear combination

$$A = X_1 + \dots + X_r \tag{2.14}$$

of rank-one tensors $X_1, \dots, X_r$. The *(real) symmetric rank* of a symmetric tensor A is the smallest number of symmetric rank-one tensors that one can have in (2.14).

Proposition 2.2. *If $A \in \otimes_{j=1}^d \mathbb{R}^n$ is an extremal tensor, its rank must be at least n. If $A \in \mathrm{Sym}^d(\mathbb{R}^n)$ is an extremal symmetric tensor, its symmetric rank must be at least n.*

Proof. Let $A \in \otimes_{j=1}^d \mathbb{R}^n$ be a tensor of rank at most $n-1$, that is,

$$A = v_1^{(1)} \otimes \dots \otimes v_1^{(d)} + \dots + v_{n-1}^{(1)} \otimes \dots \otimes v_{n-1}^{(d)}.$$

For $j = 1, \dots, d$ let $V^{(j)} \simeq \mathbb{R}^{n-1}$ be any $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that contains vectors $v_1^{(j)}, \dots, v_{n-1}^{(j)}$. Since $A \in V^{(1)} \otimes \dots \otimes V^{(d)} \simeq \otimes_{j=1}^d \mathbb{R}^{n-1}$ we have

$$\frac{\|A\|_2}{\|A\|_F} \geq \mathscr{A}(V^{(1)} \otimes \dots \otimes V^{(d)}) = \mathscr{A}(\otimes_{j=1}^d \mathbb{R}^{n-1}).$$

Thus, by Lemma 2.1, A cannot be extremal in $\mathrm{Sym}^d(\mathbb{R}^n)$.

When A is symmetric and of symmetric rank at most $n-1$ we can choose $V^{(1)} = \dots = V^{(d)} = V$ so that $A \in \mathrm{Sym}^d(V) \simeq \mathrm{Sym}^d(\mathbb{R}^{n-1})$, leading to the analogous conclusion. $\square$

2.3 Generalized gradients and local optimality of Lipschitz functions

The problem of determining the best rank-one approximation ratio of a tensor space and finding extremal tensors is a constrained optimization problem for a Lipschitz function. The theory of generalized gradients developed by Clarke [3] provides necessary optimality conditions. We provide here only the most necessary facts of this theory needed for our results. A comprehensive introduction is given, e.g., in [4].

A function $f : \mathbb{R}^m \to \mathbb{R}$ is called *Lipschitz*, if there exist a constant L such that $|f(p) - f(q)| \leq L\|p - q\|$ for all pairs $p, q \in \mathbb{R}^m$. In finite dimension, the property of being Lipschitz does not depend on the chosen norm in this definition, but the constant L of course does. By the classical Rademacher's theorem, a Lipschitz function f is differentiable at almost all (in the sense of Lebesgue measure) points $p \in \mathbb{R}^m$. Denote by $\nabla f(p)$ the gradient of f at such a point. The *generalized gradient* or *subdifferential* of f at any $p \in \mathbb{R}^m$, denoted as $\partial f(p)$, is then defined as the convex hull of the set of all limits $\nabla f(p_i)$, where p_i is a sequence in the set of differentiable points converging to p. It turns out that $\partial f(p)$ is a non-empty convex compact subset of $\mathbb{R}^m$. Moreover $\partial f(p)$ is a singleton if and only if f is differentiable in p, in which case $\partial f(p) = \{\nabla f(p)\}$.

Let S be differentiable submanifold in $S \subseteq \mathbb{R}^m$. Then a necessary condition for the Lipschitz function f to attain a local minimum relative to S at $x \in S$ is that

$$\partial f(p) \cap N_S(p) \neq \emptyset, \tag{2.15}$$

where $N_S(p)$ denotes the normal space, that is, the orthogonal complement of the tangent space, of S at p. Note that this is a "Lipschitz" analogue of the classical Lagrange multiplier rule for continuously differentiable functions. We refer to [4, Sec. 2.4]. Every point $p \in S$ that satisfies (2.15) is called a *critical point* of f on S. Hence local minima of f on S are among the critical points.

The proofs of Theorems 1.10 and 1.11 in section 3.4 basically consist in applying the necessary optimality condition (2.15) to the spectral norm function on the Euclidean sphere. Here two things are of relevance. First, for an Euclidean sphere S we have $N_s(p) = \mu p$, $\mu \in \mathbb{R}$. Hence the condition (2.15) becomes

$$\mu p \in \partial f(p) \tag{2.16}$$

for some $\mu \in \mathbb{R}$. Second, by (1.6), the spectral norm is an example of a so called *max function*, that is, a function of the type

$$f(p) = \max_{u \in C} g(p, u) \tag{2.17}$$

where C is compact. Under certain smoothness conditions on the function g, which are satisfied for spectral norm (1.6), Clarke [3, Thm. 2.1] has determined the following characterization of the generalized derivative:

$$\partial f(p) = \operatorname{conv}\{\nabla_p g(p, u) : u \in M(p)\}, \tag{2.18}$$

where conv denotes the convex hull and $M(p)$ is the set of all maximizers u in (2.17) for fixed p. For the spectral norm (1.6), this set consists of all normalized best rank-one approximations of given tensor, see (3.4).

3 Proof of main results

Our main results are proved in this section. We are going to repeatedly use the equivalence (2.5) between symmetric tensors and homogeneous forms and the corresponding relations (2.8), (2.9) for the different norms.

3.1 Binary forms

This subsection is devoted to the proof of Theorem 1.1. While the given proof is self-contained, some arguments could be omitted with reference to results in [11].

Proof of Theorem 1.1. By (1.4),

$$\frac{\|Ч_{d,2}\|_2}{\|Ч_{d,2}\|_B} = \frac{1}{\sqrt{2^{d-1}}}.$$

It then follows from (1.10) that this value equals $\mathscr{A}(\otimes_{j=1}^d \mathbb{R}^n)$, so the symmetric tensor associated to the Chebyshev form must be extremal both in $\otimes_{j=1}^d \mathbb{R}^n$ and in $\mathrm{Sym}^d(\mathbb{R}^n)$.

We now consider the uniqueness statements. When $d = 1$, the space $P_{1,n}$ consists of linear forms $p(x) = \langle a, x\rangle$, for any of which it holds $\|p\|_\infty/\|p\|_B = 1$. In the case $d = 2$ of quadratic forms, the minimal ratio between spectral and Frobenius norm of a symmetric $n \times n$ matrix is attained for multiples of symmetric orthogonal matrices only and takes the value $1/\sqrt{n}$. When $n = 2$, all such matrices can be obtained by orthogonal transformation and scaling from the two diagonal matrices with diagonal entries $(1, 1)$ and $(1, -1)$, respectively. This corresponds to the asserted quadratic forms $p \in P_{d,2}$.

In the case $d \geq 3$ we have to show that the only symmetric 2^d-tensors A satisfying

$$\|A\|_2 = 1, \quad \|A\|_F = \sqrt{2^{d-1}} \tag{3.1}$$

are obtained from orthogonal transformations of the Chebyshev form $Ч_{d,2}$. To this end, we show that under the additional condition

$$p_A(e_1) = \langle A, e_1 \otimes \cdots \otimes e_1\rangle_F = 1 = \|A\|_2, \tag{3.2}$$

the form p_A equals $Ч_{d,2}$ (where we use the notation $e_1 = (1, 0)$ and $e_2 = (0, 1)$). The prove is given by induction over $d \geq 3$. Before giving this prove we note that for a 2^d-tensor A satisfying (3.1), its two *slices* $A_1 = (a_{i_1 \dots i_{d-2} 1})$ and $A_2 = (a_{i_1 \dots i_{d-1} 2})$ necessarily have the same Frobenius norm $\|A_1\|_F = \|A_2\|_F = \sqrt{2^{d-2}}$. In fact, $\|A\|_2 = 1$ implies $\|A_1\|_2 \leq 1$ and hence, by (1.10), $\|A_1\|_F \leq \sqrt{2^{d-2}}$. Since the same holds for A_2 and $\|A\|_F^2 = \|A_1\|_F^2 + \|A_2\|_F^2$ the claim follows. Even more $\|A_1\|_2 = \|A_2\|_2 = 1$, again by (1.10), so that both slices are necessarily extremal. Note that by the same argument, every $2^{d'}$-subtensor of A with $d' < d$ must be extremal.

We begin the induction with $d = 3$. Assume $A \in \mathrm{Sym}^3(\mathbb{R}^2)$ satisfies (3.1) and (3.2). Then we have seen that both, say, frontal slices of A are themselves extremal symmetric 2×2 matrices. By (3.2), $a_{111} = 1$ and the tensor $e_1 \otimes e_1 \otimes e_1$ is a best rank-one approximation. From Lemma 2.2 we then deduce that entries $a_{112} = a_{121} = a_{211} = 0$. The only two remaining options for the slices of A are

$$A = \left(\begin{matrix} 1 & 0 \\ 0 & \pm 1 \end{matrix} \,\middle|\, \begin{matrix} 0 & \pm 1 \\ \pm 1 & 0 \end{matrix}\right).$$

But the case $a_{122} = a_{221} = a_{212} = +1$ is also not possible, since it corresponds to the form $p_A(x) = x_1^3 + 3x_1x_2^2$ whose maximum on the sphere is $\|p_A\|_\infty = \sqrt{2} > 1$. Therefore, $a_{122} = a_{221} = a_{212} = -1$ and $p_A = x_1^3 - 3x_1x_2^2$ is the cubic Chebyshev form.

We proceed with the induction step. If $A \in \mathrm{Sym}^{d+1}(\mathbb{R}^2)$ satisfies (3.1) and (3.2), then its two slices $A_1 = (a_{i_1 \dots i_d 1})$ and $A_2 = (a_{i_1 \dots i_d 2})$ are extremal 2^d tensors. Since $p_{A_1}(e_1) = p_A(e_1) = 1$, it follows from the induction hypothesis that $A_1 = Ч_{d,2}$. So its entries are given by (2.7). Let $a_{i_1 \dots i_d 2}$ be an entry of the second slice. Due to symmetry of A, every entry in the second slice, except for the entry $a_{2 \dots 2}$, equals an entry in the first slice after a permutation of the multi-index. Since this permutation does not affect the number of occurrences of the value 2, the definition (2.7) applies to all these entries as well. It remains to show that the entry $a_{2\dots 2}$ satisfies (2.7), that is, equals zero in case $d+1$ is odd, and equals $(-1)^m$ in case $d+1=2m$ is even. This entry is part of the symmetric subtensor $A'=(a_{i_1 i_2 i_3 2\dots 2})$, which as we have noted above must be extremal as well. Since we know the entries of the first slice A_1 through (2.7), we find that

$$p_{A'}(x) = (-1)^{m-1}(x_1^3 - 3x_1x_2^2) + a_{2\cdots 2}x_2^3$$

if $d+1=2m+1$ is odd. Since A' is extremal, it then follows from the base case $d=3$ that $a_{2\cdots 2}=0$. In case $d+1=2m$ is even, we get

$$p_{A'}(x_1,x_2) = (-1)^{m-1}3x_1^2x_2 + a_{2\cdots 2}x_2^3,$$

which by a small consideration implies $a_{2\cdots 2}=(-1)^m$. This concludes the proof. $\square$

3.2 Ternary cubic tensors

In this section we prove Theorem 1.5. It has been mentioned in section 1.2 how Corollary 1.6 follows from it, and that the statement of Theorem 1.2 is included in the latter.

The proof of Theorem 1.5 requires a fact from [9]. Since it is not explicitly formulated there, we state it here as a lemma and include the proof.

Lemma 3.1. *For odd n let $A_1, A_2 \in \mathbb{R}^n \otimes \mathbb{R}^n$ be two $n\times n$ matrices. If at least one of them is invertible, then there exist orthogonal matrices $\rho, \rho' \in O(n)$ such that*

$$\rho A_1 \rho' = \begin{pmatrix} B_1 & c_1 \\ 0 & d_1 \end{pmatrix}, \quad \rho A_2 \rho' = \begin{pmatrix} B_2 & c_2 \\ 0 & d_2 \end{pmatrix},$$

where B_1, B_2 are matrices of size $(n-1)\times(n-1)$, c_1, c_2 are $(n-1)$-dimensional vectors and d_1, d_2 are real numbers.

Proof. We can assume A_1 is invertible. Since n is odd, the matrix $A_1^{-1}A_2$ has at least one real eigenvalue d. Then there exists an invertible matrix P such that

$$P^{-1}A_1^{-1}A_2P = \begin{pmatrix} B & c \\ 0 & d \end{pmatrix},$$

where B is a matrix of size $(n-1)\times(n-1)$ and c is a $(n-1)$-dimensional vector. Consider QR decompositions of A_1P and P, that is,

$$A_1P = Q_1R_1, \quad P = Q_2R_2,$$

where Q_1, Q_2 are orthogonal, and R_1, R_2 are upper triangular and invertible. We set $\rho = Q_1^{-1}$ and $\rho' = Q_2$. Then

$$\rho A_1 \rho' = R_1 P^{-1} A_1^{-1} A_1 P R_2^{-1} = R_1 R_2^{-1}$$

is the product of two upper block triangular matrices, hence upper block triangular. Similarly,

$$\rho A_2 \rho' = R_1 P^{-1} A_1^{-1} A_2 P R_2^{-1} = R_1 \begin{pmatrix} B & c \\ 0 & d \end{pmatrix} R_2^{-1}$$

has the asserted upper block triangular structure. $\square$

In [9] the previous lemma is used to show that for odd n the maximum possible orthogonal rank of an $(n, n, 2)$-tensor is $2n - 1$. We will only need that the orthogonal rank of a $(3, 3, 2)$-tensor is not larger than 5, which actually follows quite easily from the lemma by applying it to the slices.

Proof of Theorem 1.5. For $A \in \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$, it is convenient to write $A = (A_1|A_2|A_3)$, where A_1, A_2, A_3 are the 3×3 slices along the third dimension. If none of the matrices A_1, A_2, A_3 is invertible, each of them can be decomposed into a sum of two rank-one matrices that are orthogonal in Frobenius inner product: $A_i = u_i^{(1)} \otimes u_i^{(2)} + v_i^{(1)} \otimes v_i^{(2)}$, $i = 1, 2, 3$. This leads to a decomposition of A into at most six pairwise orthogonal rank-one tensors:

$$A = \sum_{i=1}^{3} u_i^{(1)} \otimes u_i^{(2)} \otimes e_i + v_i^{(1)} \otimes v_i^{(2)} \otimes e_i.$$

Assume without loss of generality that the first slice A_1 is invertible. Lemma 3.1 together with the invariance of orthogonal rank under orthogonal transformations (2.4) allows to assume that A has the form

$$\begin{aligned} A &= \left(\begin{array}{ccc|ccc|ccc} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & 0 & 0 & * & * & * & * \end{array}\right) \\ &= \left(\begin{array}{ccc|ccc|ccc} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * & * & * & * \end{array}\right) \end{aligned}$$

The first term is essentially a $(2, 3, 3)$-tensor, so its orthogonal rank is at most five by the result of [9]. In particular, it has a decomposition into at most five pairwise orthogonal rank-one tensors with zero bottom rows. Since the bottom row of the second term is a rank-two matrix the orthogonal rank of A is at most seven. $\square$

3.3 On symmetric orthogonal tensors

We prove Proposition 1.7 below. For the general definition of orthogonal tensors of arbitrary size we refer to [11]. For n^d-tensors we can use the recursive definition that $A \in \otimes_{j=1}^{d} \mathbb{R}^n$ is orthogonal if $A \times_j u$ is orthogonal for every $j = 1, \ldots, d$ and every unit norm vector $u \in \mathbb{R}^n$, where for $d = 2$ we agree to the standard definition of an orthogonal matrix. Here and in the proof below we use standard notation $A \times_j u = \left(\sum_{i_j=1}^{n_j} a_{i_1 \ldots i_j \ldots i_d} u_{i_j} \right)$ for partial contraction of a tensor A with a vector u along mode j, resulting in a tensor of order $d - 1$. Note that the above definition implies that every $n^{d'}$-subtensor, $d' < d$, of A is itself orthogonal.

Proof of Proposition 1.7 and Corollary 1.8. It has been shown in [11] that an n^d-tensor A is orthogonal if and only if it satisfies $\|A\|_2 = 1$ and $\|A\|_F = \sqrt{n^{d-1}}$, and such tensors only exist when $n = 1, 2, 4, 8$. Therefore, the statement that for $n = 2$ the only symmetric orthogonal tensors are the ones obtained from the Chebyshev form $Ч_{d,2}$ is hence equivalent to Theorem 1.1. Also, Corollary 1.8 is immediate from Proposition 1.7.

We thus only have to show that for $n = 4, 8$ an orthogonal n^d-tensor cannot be symmetric. We only consider the case $n = 4$, the arguments for $n = 8$ are analogous. Since $n^{d'}$-subtensors of an orthogonal tensor are necessarily orthogonal, it is enough to show that orthogonal $4 \times 4 \times 4$ tensors cannot be symmetric. Assume to the opposite that such a tensor A exists. Then $\|A\|_2 = 1$ and A admits a symmetric best rank-one approximation of Frobenius norm one. By e_i we denote the standard unit vectors in $\mathbb{R}^n$. Since orthogonality and symmetry are preserved under the multilinear action of $O(4)$ we can assume that $e_1 \otimes e_1 \otimes e_1$ is the best rank-one approximation of A, that is, $a_{111} = \langle A, e_1 \otimes e_1 \otimes e_1 \rangle_F = \|A\|_2 = 1$. On the other hand, the first frontal slice $A \times_3 e_1$ must be a symmetric orthogonal matrix, so it is of the form

$$A \times_3 e_1 = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

where B is a symmetric orthgonal 3×3 matrix. By applying further orthogonal transformation that fix the vector e_1, we can assume that B is a diagonal matrix with diagonal entries $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{+1, -1\}$. Since A is symmetric and in fact every slice has to be an orthogonal matrix, we find that $A = (A \times_3 e_1 | A \times_3 e_2 | A \times_3 e_3 | A \times_3 e_4)$ must be of the form

$$A = \left(\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 0 & 0 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & \varepsilon_3 & 0 & 0 & 0 & 0 & \varepsilon_4 \\ 0 & \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_0 & 0 & 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 & 0 & 0 & 0 & \varepsilon_0 & \varepsilon_3 & 0 & 0 & 0 & 0 & \varepsilon_0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_3 & 0 & 0 & \varepsilon_0 & 0 & 0 & \varepsilon_0 & 0 & 0 & \varepsilon_4 & 0 & 0 & 0 \end{array}\right),$$

where also $\varepsilon_0 \in \{+1, -1\}$. For $i = 2, 3, 4$ the matrices $A \times_3 (e_1 + e_i)/\sqrt{2}$ must be orthogonal as well, which yields the values $\varepsilon_0 = 1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = -1$. But then the matrix

$$A \times_3 \left(\frac{e_1 - e_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

is not orthogonal, which contradicts the assumption that A is an orthogonal tensor. $\square$

3.4 Variational characterization

In Theorems 1.10 and 1.11 we characterize critical tensors in $\otimes_{j=1}^d \mathbb{R}^{n_j}$ and $\mathrm{Sym}^d(\mathbb{R}^n)$ in terms of decompositions into best rank-one approximations. We now prove these results and then derive Corollary 1.12. Afterwards we prove Theorem 1.13.

Proof of Theorems 1.10 and (1.11). From section 2.3, specifically (2.16), it follows that a non-zero tensor $A' \in \otimes_{j=1}^d \mathbb{R}^{n_j}$ is critical in the sense of Definition 1.9, if the tensor $A = A'/\|A'\|_F$ of Frobenius norm one satisfies

$$\mu A \in \partial \|A\|_2 \tag{3.3}$$

for some $\mu \in \mathbb{R}$. By (1.6), the spectral norm is a max function of the type (2.17) which is easily shown to satisfy the conditions of [3, Thm. 2.1]. Therefore, its generalized derivative is given by the formula (2.18), which in the case of the max function (1.6) reads

$$\partial\|A\|_2 = \operatorname{conv}\{X : \|X\|_F = 1, \operatorname{rank}(X) = 1, \langle A, X\rangle_F = \|A\|_2\}, \tag{3.4}$$

where conv denotes the convex hull. This lets us write (3.3) as

$$\mu A = \sum_{\ell=1}^{r} \alpha_\ell X_\ell, \tag{3.5}$$

where $r > 0$ is a natural number[1], $\alpha_1, \dots, \alpha_r > 0$ are such that $\alpha_1 + \dots + \alpha_r = 1$, and X_ℓ are rank-one tensors of unit Frobenius norm satisfying $\langle A, X_\ell\rangle_F = \|A\|_2$. By taking the Frobenius inner product with A itself in (3.5), we find that

$$\mu = \frac{\|A\|_2}{\|A\|_F^2}.$$

Therefore, after multiplying the resulting equation (3.5) by $\|A\|_2$ we obtain the asserted statement of Theorem 1.10, since, by (2.12), the rank-one tensors $Y_\ell = \|A\|_2 X_\ell$ are best rank-one approximations of A.

Considering symmetric tensors instead of general ones in the previous arguments yields a proof of Theorem 1.11. Here it is crucial that in the definition (1.6) of spectral norm for symmetric tensors one can restrict to take the maximum over symmetric rank-one tensors of unit Frobenius norm thanks to Banach's theorem, cf. (2.8). □

Proof of Corollary 1.12. By Proposition 2.2 any extremal tensor in $\otimes_{j=1}^{d}\mathbb{R}^n$ or $\operatorname{Sym}^d(\mathbb{R}^n)$ must be of rank (respectively, symmetric rank) at least n. In particular, there cannot be less than n best rank-one approximations in the expansions (1.13) and (1.14). □

We now give a proof of Theorem 1.13.

Proof of Theorem 1.13. Let a tensor A (either in $\otimes_{j=1}^{d}\mathbb{R}^{n_j}$ or in $\operatorname{Sym}^d(\mathbb{R}^n)$) be critical, that is, by Theorems 1.10 resp. 1.11,

$$A = \left(\frac{\|A\|_F}{\|A\|_2}\right)^2 \sum_{\ell=1}^{r} \alpha_\ell Y_\ell \tag{3.6}$$

for some (symmetric, if A is symmetric) best rank-one approximations $Y_1, \dots, Y_r$ to A, and coefficients $\alpha_1, \dots, \alpha_r > 0$ that sum up to one. Recall from section 2.1 that the nuclear norm is dual to the spectral norm. By (2.2), this in particular means there exists a tensor A^* satisfying $\|A^*\|_2 \le 1$ and $\|A\|_* = \langle A, A^*\rangle_F$. Note that we then have

[1]By the classical Carathéodory theorem one can take $r \le \dim \otimes_{j=1}^{d}\mathbb{R}^{n_j} + 1 = n_1 \cdots n_d + 1$.

$\langle X, A^* \rangle_F \leq \|X\|_F \|A^*\|_2 \leq \|X\|_F$ for every rank-one tensor X. Since $\|Y_\ell\|_F = \|A\|_2$, it hence follows from (3.6) that

$$\|A\|_* = \langle A, A^* \rangle_F = \left(\frac{\|A\|_F}{\|A\|_2} \right)^2 \sum_{\ell=1}^{r} \alpha_\ell \langle Y_\ell, A^* \rangle_F \leq \frac{\|A\|_F^2}{\|A\|_2},$$

which is the converse inequality to (1.16). This shows that (i) implies (ii).

Assume now that (ii) holds for a non-zero tensor A, that is, $\|A\|_2 \|A\|_* = \|A\|_F^2$. By the definition of the nuclear norm there exist $r \in \mathbb{N}$, positive numbers $\beta_1, \dots, \beta_r > 0$ and rank-one tensors $X_1, \dots, X_r$ of unit Frobenius norm such that

$$A = \sum_{\ell=1}^{r} \beta_\ell X_\ell \quad \text{and} \quad \|A\|_* = \sum_{\ell=1}^{r} \beta_\ell. \tag{3.7}$$

If A is symmetric, the X_ℓ can be taken symmetric [8]. Taking the Frobenius inner product with A in the first of these equations gives

$$\|A\|_2 \|A\|_* = \langle A, A \rangle_F = \sum_{\ell=1}^{r} \beta_\ell \langle A, X_\ell \rangle_F.$$

Since $\langle A, X_\ell \rangle_F \leq \|A\|_2$ for every ℓ and the β_ℓ sum up to $\|A\|_*$, this equality can only hold $\langle A, X_\ell \rangle_F = \|A\|_2$ for all ℓ. Since, by (2.12), the rank-one tensors $Y_\ell = \|A\|_2 X_\ell$ are then best rank-one approximations of A, we see that (3.7) is equivalent to (3.6), which by Theorems 1.10 resp. 1.11 means that A is critical. □

Remark 3.2. Observe from the proof that decomposition (3.6) of a critical tensor into its best rank-one approximations is also its nuclear decomposition. Vice versa, any nuclear decomposition of a tensor A satisfying $\|A\|_2 \|A\|_* = \|A\|_F^2$ can be turned into a convex linear combination of best rank-one approximations of the rescaled tensor $\|A\|_2^2 / \|A\|_F^2 \, A$.

3.5 Decomposition of Chebyshev forms

In this section we give the proof of Proposition 1.14, that realizes the decomposition of critical tensors into symmetric best rank-one approximations, that is, corresponding powers of linear forms, for the Chebyshev forms $\text{Ч}_{d,2}$.

Proof of Proposition 1.14. Recall that for any $k = 0, \dots, d-1$ we denote $\theta_k = \pi k/d$ and $a_k = \cos(\theta_k), b_k = \sin(\theta_k)$. Let us observe that for any such k we can write

$$\cos(d\theta) = \mathrm{Re}((-1)^k e^{id(\theta - \theta_k)}) = (-1)^k \sum_{\ell=0}^{[d/2]} \binom{d}{2\ell} (-1)^\ell \cos(\theta - \theta_k)^{d-2\ell} \sin(\theta - \theta_k)^{2\ell}$$

and therefore

$$\cos(d\theta) = \frac{1}{d} \sum_{\ell=0}^{[d/2]} \binom{d}{2\ell} (-1)^\ell \sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^{d-2\ell} \sin(\theta - \theta_k)^{2\ell}.$$

Below we show that for any $\ell = 0, \dots, [d/2]$ it holds

$$(-1)^\ell \sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^{d-2\ell} \sin(\theta - \theta_k)^{2\ell} = \sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^d. \tag{3.8}$$

This together with the identity $\sum_{\ell=0}^{[d/2]} \binom{d}{2\ell} = 2^{d-1}$ implies (1.20) (and hence also (1.19)).

To derive (3.8) we write

$$\begin{aligned}
&(-1)^\ell \sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^{d-2\ell} \sin(\theta - \theta_k)^{2\ell} \\
&= \sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^{d-2\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} \cos(\theta - \theta_k)^{2j} (-1)^{\ell - j} \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell - j} \sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^{d-2(\ell - j)}
\end{aligned}$$

and claim that for $j = 0, \dots, \ell - 1$ the inner sum in the last formula is zero. In fact, we will show that for $s = 1, \dots, [d/2]$

$$\sum_{k=0}^{d-1} (-1)^k \cos(\theta - \theta_k)^{d-2s} = 0. \tag{3.9}$$

For this let us observe first that Chebyshev polynomials of the first kind $T_{d-2j}(\cos\theta) = \cos((d-2j)\theta)$, $j = 1, \dots, [d/2]$ form a basis in the space spanned by univariate real polynomials of degrees $d-2, d-4, \dots, d-2[d/2]$. As a consequence one can express $\cos(\theta - \theta_k)^{d-2s}$ in terms of $T_{d-2j}(\cos(\theta - \theta_k))$ for $j = s, \dots, [d/2]$, and thus in order to prove (3.9), it is enough to show that for $s = 1, \dots, [d/2]$

$$\sum_{k=0}^{d-1} (-1)^k \cos((d-2s)(\theta - \theta_k)) = 0.$$

But this follows more details? from the identity $\sum_{k=0}^{d-1} \left(e^{i2\pi s/d}\right)^k = 0$. □

We now derive Corollary 1.15 which, in particular, implies that the cubic Chebyshev forms Ч$_{3,n}$ are critical for the ratio $\|p\|_\infty / \|p\|_B, p \in P_{3,n}$.

Proof of Corollary 1.15. From Proposition 1.14 we get

$$\text{Ч}_{3,2}(x_1, x_2) = x_1^3 - 3x_1 x_2^2 = \frac{4}{3}\left(x_1^3 - \left(\frac{x_1 - \sqrt{3}x_2}{2}\right)^3 + \left(\frac{-x_1 + \sqrt{3}x_2}{2}\right)^3\right). \tag{3.10}$$

We then write

$$
\begin{aligned}
Ч_{3,n}(x) &= x_1^3 - 3x_1(x_2^2 + \cdots + x_n^2) = -(n-2)x_1^3 + \sum_{i=2}^{n} \left(x_1^3 - 3x_1 x_i^2\right) \\
&= -(n-2)x_1^3 + \frac{4}{3}\sum_{i=2}^{n} x_1^3 - \left(\frac{x_1 - \sqrt{3}x_i}{2}\right)^3 + \left(\frac{-x_1 + \sqrt{3}x_i}{2}\right)^3,
\end{aligned}
$$

where we applied (3.10) to each binary Chebyshev form $Ч_{3,2}(x_1, x_i) = x_1^3 - 3x_1x_i^2$. The obtained formula is equivalent to the asserted one (1.21). □

3.6 Local minimality of cubic Chebyshev forms

This subsection is devoted to the proof of Theorem 1.3 for which we will need the following lemma.

Lemma 3.3. *The tangent space to the $O(n)$-orbit of $Ч_{3,n}(x) = x_1^3 - 3x_1(x_2^2 + \cdots + x_n^2)$ has dimension $n-1$ and consists of reducible cubics of the form $\ell \cdot q$, where ℓ is a linear form that vanishes at $(1, 0, \ldots, 0)$ and $q = 3x_1^2 - x_2^2 - \cdots - x_n^2$.*

Proof. Recall that $Ч_{3,n}$ is stabilized by the subgroup $G \simeq O(n-1) \subset O(n)$ of orthogonal transformations that preserve $(1, 0, \ldots, 0)$. Note that G is of codimension $n-1$ and, as a consequence, the $O(n)$-orbit of $Ч_{3,n}$ is at most $(n-1)$-dimensional. For $j = 2, \ldots, n$ let us consider the elementary rotation $R_j(\varphi) \in O(n)$ in the $(1, j)$-plane, that is, $R_j(\varphi)$ is given by the $(n-1)\times(n-1)$ matrix whose only non-zero entries are $(R_j(\varphi))_{11} = (R_j(\varphi))_{jj} = \cos(\varphi)$, $(R_j(\varphi))_{1j} = -(R_j(\varphi))_{j1} = \sin\varphi$ and $(R_j(\varphi))_{ii} = 1$ for $i \neq 1, j$. It is straightforward to check that the tangent vector to the curve $R_j(\varphi)^* Ч_{3,n}$ at $\varphi = 0$ is a non-zero cubic proportional to $x_j q$. Since the $O(n)$-orbit of $Ч_{3,n}$ is at most $(n-1)$-dimensional and since the tangent vectors to $n-1$ curves $R_j(\varphi)$, $j = 2, \ldots, n$, are independent, the claim follows. □

Proof of Theorem 1.3. The subgroup $G \subset O(n)$ of orthogonal transformations that preserve the point $(1, 0, \ldots, 0)$ is naturally identified with orthogonal transformations $O(n-1)$ in the variables $x_2, \ldots, x_n$. As a consequence, together with the decomposition (1.21) we have the whole family of decompositions

$$
\begin{aligned}
\frac{1}{3n-2} Ч_{3,n}(x) = \frac{n+2}{9n-6}x_1^3 + \frac{4}{9n-6}\sum_{i=2}^{n} &- \left(\frac{x_1 + \sqrt{3}(\rho_{i2}x_2 + \cdots + \rho_{in}x_n)}{2}\right)^3 \\
&+ \left(\frac{-x_1 + \sqrt{3}(\rho_{i2}x_2 + \cdots + \rho_{in}x_n)}{2}\right)^3,
\end{aligned}
\tag{3.11}
$$

where $\rho = (\rho_{ij})_{i,j=2}^n \in O(n-1)$ is an orthogonal matrix. Using the G-invariance of the cubic Chebyshev form one can check that rank-one forms in (3.11) with different

$\rho \in O(n-1)$ exhaust all symmetric best rank-one approximations of $Ч_{3,n}, n \geq 2$. Let us consider the set

$$C_n = \{(\pm 1, 0, \ldots, 0)\} \cup \left\{ \left(\pm\frac{1}{2}, \frac{\sqrt{3}}{2}\rho_2, \ldots, \frac{\sqrt{3}}{2}\rho_n \right) : \rho_2^2 + \cdots + \rho_n^2 = 1 \right\}$$
$$= \{(\pm 1, 0, \ldots, 0)\} \cup \{x \in S^{n-1} : 3x_1^2 - x_2^2 - \cdots - x_n^2 = 0\}$$

of unit vectors corresponding to symmetric best rank-one approximations of $Ч_{3,n}$, it consists of global maxima of the restriction to S^{n-1} of the absolute value of $Ч_{3,n}$ (see Subsection 2.2). Observe that for any unit vector $(\rho_2, \ldots, \rho_n)$ we have

$$Ч_{3,n}(\pm 1, 0, \ldots, 0) = \pm 1 \quad \text{and} \quad Ч_{3,n}\left(\pm\frac{1}{2}, \frac{\sqrt{3}}{2}\rho_2, \ldots, \frac{\sqrt{3}}{2}\rho_n \right) = \mp 1. \tag{3.12}$$

Let us denote by $\tilde{Ч}_{3,n} = Ч_{3,n}/\sqrt{3n-2} \in S_B$ the normalized cubic Chebyshev form, where $S_B = \{p \in P_{3,n} : \|p\|_B = 1\}$ is the sphere in $P_{3,n}$ defined by the Bombieri norm, and let $M = \{p \in S_B : \|p\|_\infty < \|\tilde{Ч}_{3,n}\|_\infty\}$ be the semialgebraic subset of the sphere S_B that consists of forms having smaller uniform norm than $\tilde{Ч}_{3,n}$. Note that if Conjecture holds, the set M is empty. To prove the theorem we need to show that $\tilde{Ч}_{3,n}$ does not belong to the closure $\overline{M}$ of M. Assume by contradiction that $\tilde{Ч}_{3,n} \in \overline{M}$. Then by the Curve Selection Lemma [2, Prop. 8.1.13] there exists an analytic curve $p : [0, t_0] \to S_B$ such that $p(0) = \tilde{Ч}_{3,n}$ and $p(t) \in M$ for $t \in (0, t_0]$. Up to a reparametrization, this curve has the form

$$p(t) = \tilde{Ч}_{3,n} + tv + o(t^\alpha), \tag{3.13}$$

where $\alpha > 1$ is a rational number and v is some non-zero cubic form orthogonal to $\tilde{Ч}_{3,n}$, that is $\langle \tilde{Ч}_{3,n}, v \rangle_B = 0$. There are now two cases:

i) the cubic form v does not vanish on the set C_n,

ii) v vanishes on C_n.

In the former case there exists $x \in C_n$ such that $\tilde{Ч}_{3,n}(x)v(x) > 0$. Indeed, as v does not vanish on C_n we have that $v(x) \neq 0$ for some $x \in C_n$. If $\tilde{Ч}_{3,n}(x)v(x) < 0$, we write the decomposition (3.11) for an appropriate ρ and take its Bombieri product with v:

$$0 = \frac{1}{3n-2}\langle v, Ч_{3,n} \rangle_B = \frac{n+2}{9n-6} v(1, 0, \ldots, 0) + \frac{4}{9n-6} \sum_{i=2}^{n} - v\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\rho_{i2}, \ldots, \frac{\sqrt{3}}{2}\rho_{in} \right)$$
$$+ v\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\rho_{i2}, \ldots, \frac{\sqrt{3}}{2}\rho_{in} \right). \tag{3.14}$$

Since $\tilde{Ч}_{3,n} = Ч_{3,n}/\sqrt{3n-2}$, (3.12) and (3.14) imply that for some $x' \in C_n$ (corresponding to the same ρ as x) $\tilde{Ч}_{3,n}(x')v(x') > 0$. Without loss of generality we can assume that

$\tilde{\text{Ч}}_{3,n}(x')$ and $v(x')$ are positive. Then for a sufficiently small time $t > 0$

$$\|p(t)\|_\infty = \max_{x \in S^{n-1}} |\tilde{\text{Ч}}_{3,n}(x) + tv(x) + o(t^\alpha)| \geq \tilde{\text{Ч}}_{3,n}(x') + t\,v(x') + o(t^\alpha)$$
$$> \tilde{\text{Ч}}_{3,n}(x') = \frac{1}{\sqrt{3n-2}} = \|\tilde{\text{Ч}}_{3,n}\|_\infty,$$

but this is impossible as $p(t) \in M$ for $t > 0$. Thus we obtain a contradiction with the assumption we made that $\tilde{\text{Ч}}_{3,n} \in \overline{M}$.

Let us now treat the case *ii*). For this let us observe first that a cubic form v vanishes on C_n if and only if it is a product of the quadric $q = 3x_1^2 - x_2^2 - \cdots - x_n^2$ and a linear form vanishing at $(1, 0, \ldots, 0)$. By Lemma 3.3 the space of such cubics coincides with the $(n-1)$-dimensional tangent space to the $O(n)$-orbit of $\tilde{\text{Ч}}_{3,n}$.

Denote by W any $(n-1)$-dimensional analytic submanifold of $O(n)$ that passes through the identity $\mathrm{id} \in O(n)$ and intersects G transversally at $\mathrm{id} \in W \cap G$. Denote also by H any analytic submanifold of S_B that has codimension $\operatorname{codim} H = n - 1$, passes through $\tilde{\text{Ч}}_{3,n} \in S_B$ and intersects the $O(n)$-orbit of $\tilde{\text{Ч}}_{3,n}$ transversally at $\tilde{\text{Ч}}_{3,n}$. Consider now the analytic map $f : W \times H \to S_B$, $(w, h) \mapsto w^*h$ and note that by construction the differential of f at $(\mathrm{id}, \tilde{\text{Ч}}_{3,n})$ is surjective. Therefore, by the analytic inverse function theorem [] there exist a neighbourhood $\tilde{W} \times \tilde{H} \subset W \times H$ of $(\mathrm{id}, \tilde{\text{Ч}}_{3,n})$, a neighbourhood $\tilde{S} \subset S_B$ of $\tilde{\text{Ч}}_{3,n}$ and a local analytic inverse $f^{-1} : \tilde{S} \to \tilde{W} \times \tilde{H}$ to f. Let us now consider the analytic curve $f^{-1}(p)$, where p is defined above, and let us denote by $\tilde{p} \subset \tilde{H}$ the projection of $f^{-1}(p)$ onto $\tilde{H}$. The curve $\tilde{p}$ is analytic, $\tilde{p}(0) = \tilde{\text{Ч}}_{3,n}$ and, by the $O(n)$-invariance of the set M, we have $\tilde{p}(t) \in M$ for $t > 0$. Up to a reparametrization $\tilde{p}$ has the form (3.13), where v is a non-zero cubic form tangent to H at $\tilde{\text{Ч}}_{3,n}$. By construction $T_{\tilde{\text{Ч}}_{3,n}} H$ intersects trivially the tangent space to the $O(n)$-orbit of $\tilde{\text{Ч}}_{3,n}$ or, equivalently (by Lemma 3.3 and the above reasonings), the space of cubic forms vanishing on C_n. As a consequence, v is of type i) and hence we again obtain a contradiction with the assumption that $\tilde{\text{Ч}}_{3,n} \in \overline{M}$. Thus $\tilde{\text{Ч}}_{3,n} \notin \overline{M}$ or, in other words, it is a local minimum of $\|\cdot\|_\infty : S_B \to \mathbb{R}$.

$\square$

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