Stability of the Riesz potential inequality

Nicola Fusco

Calculus of Variations and Applications

Trieste, January 29, 2020

Riesz inequality

Let $f, g : \mathbb{R}^n \to [0, \infty)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing

$$(*) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, h(|x-y|)\, g(y)\, dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)\, h(|x-y|)\, g^*(y)\, dx\, dy$$

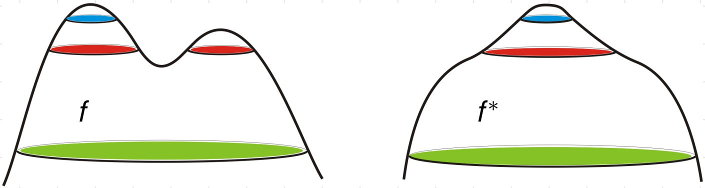
f^*, g^* are the Schwartz symmetrization of f, g

Riesz inequality

Let $f, g : \mathbb{R}^n \to [0, \infty)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing

$$(*) \quad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, h(|x-y|)\, g(y)\, dxdy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)\, h(|x-y|)\, g^*(y)\, dx\, dy$$

f^*, g^* are the Schwartz symmetrization of f, g



Riesz inequality

Let $f, g : \mathbb{R}^n \to [0, \infty)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing

$$(*) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, h(|x-y|)\, g(y)\, dxdy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)\, h(|x-y|)\, g^*(y)\, dxdy$$

f^*, g^* are the Schwartz symmetrization of f, g

Riesz inequality

Let $f, g : \mathbb{R}^n \to [0, \infty)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing

$$(*) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, h(|x-y|)\, g(y)\, dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)\, h(|x-y|)\, g^*(y)\, dxdy$$

f^*, g^* are the Schwartz symmetrization of f, g

If $f = g$ and h is strictly decreasing

Equality holds in $(*)$ $\iff$ $f = f^*$ up to a translation

Now take

- $f = g = \chi_E$ with $|E| < \infty$ $\Longrightarrow$ $f^* = \chi_{B_r(0)}$, $|E| = |B_r|$
- $h(t) = t^{\lambda - n}$ with $0 < \lambda < n$

Riesz potentials

If $f, g = \chi_E$, $h(t) = t^{\lambda - n}$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, h(|x - y|)\, g(y)\, dxdy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)\, h(|x - y|)\, g^*(y)\, dxdy$$

becomes the Riesz potential inequality,

$$(**) \quad \int_E\int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy \le \int_{B_r}\int_{B_r} \frac{1}{|x-y|^{n-\lambda}}\, dxdy, \qquad |B_r| = |E|$$

and $=$ holds iff E is a ball

Riesz potentials

If $f, g = \chi_E$, $h(t) = t^{\lambda - n}$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\, h(|x{-}y|)\, g(y)\, dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)\, h(|x{-}y|)\, g^*(y)\, dxdy$$

becomes the Riesz potential inequality,

$$(**) \quad \int_E\int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy \leq \int_{B_r}\int_{B_r} \frac{1}{|x-y|^{n-\lambda}}\, dxdy, \qquad |B_r| = |E|$$

and $=$ holds iff E is a ball

If $n = 3$, $\lambda = 2$, Riesz potential $\rightsquigarrow$ Coulombic potential

$$\int_E\int_E \frac{1}{|x-y|}\, dxdy$$

Stability for the Riesz potential

$$\mathcal{P}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\lambda}} \, dxdy$$

Stability for the Riesz potential

$$\mathcal{P}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy$$

Stability of

$$\mathcal{P}(E) \leq \mathcal{P}(B_r), \quad |B_r| = |E|$$

Stability for the Riesz potential

$$\mathcal{P}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy$$

Stability of

$$\mathcal{P}(E) \leq \mathcal{P}(B_r), \quad |B_r| = |E|$$

In other words:

if $\mathcal{P}(E) \precapprox \mathcal{P}(B_r)$

can we say that E is close to $B_r(x)$?

$$\mathcal{P}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy$$

Stability of

$$\mathcal{P}(E) \le \mathcal{P}(B_r), \quad |B_r| = |E|$$

$$\mathcal{P}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy$$

Stability of

$$\mathcal{P}(E) \le \mathcal{P}(B_r), \quad |B_r| = |E|$$

From now on $|E| = |B| = \omega_n$, B the unit ball. Set

$$\mathcal{D}(E) := \mathcal{P}(B) - \mathcal{P}(E), \qquad \alpha(E) := \min_{x\in\mathbb{R}^n} |E \Delta B(x)| < 2\omega_n$$

(Potential gap) (Fraenkel asymmetry)

$$\mathcal{P}(E) = \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy$$

Stability of

$$\mathcal{P}(E) \leq \mathcal{P}(B_r), \quad |B_r| = |E|$$

From now on $|E| = |B| = \omega_n$, B the unit ball. Set

$$\mathcal{D}(E) := \mathcal{P}(B) - \mathcal{P}(E), \qquad \alpha(E) := \min_{x \in \mathbb{R}^n} |E \Delta B(x)| < 2\omega_n$$

(Potential gap) (Fraenkel asymmetry)

Theorem (Burchard-Chambers, 2015)

Let $n = 3, \lambda = 2$ *There exists* $C > 0$ *s.t. if* $|E| = \omega_3 = 4\pi/3$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E)$$

If $n > 3, \lambda = 2$ *there exists* $C(n)$ *s.t. if* $|E| = \omega_n$

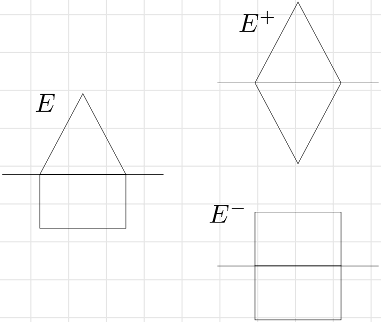
$$\alpha(E)^{n+2} \leq C\,\mathcal{D}(E)$$

Steps of the proof of Burchard and Chambers:

1) Reduce to the case of a bounded set E', with $-E' = E'$

Steps of the proof of Burchard and Chambers:

1) Reduce to the case of a bounded set E', with $-E' = E'$



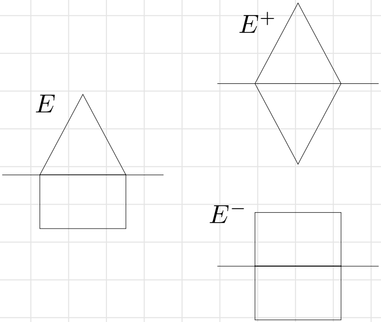
To prove Step 1 they use that for $0 < \lambda \leq 2$, $n \geq 3$ Riesz potential is reflection positive

$$\mathcal{P}(E) \leq \frac{1}{2}\mathcal{P}(E^+) + \frac{1}{2}\mathcal{P}(E^-)$$

(a deep result by Frank-Lieb, 2010)

Steps of the proof of Burchard and Chambers:

1) Reduce to the case of a bounded set E', with $-E' = E'$



To prove Step 1 they use that for $0 < \lambda \leq 2$, $n \geq 3$ Riesz potential is reflection positive

$$\mathcal{P}(E) \leq \frac{1}{2}\mathcal{P}(E^+) + \frac{1}{2}\mathcal{P}(E^-)$$

(a deep result by Frank-Lieb, 2010)

Therefore:

$$\mathcal{D}(E^+) + \mathcal{D}(E^-) \leq 2\mathcal{D}(E)$$

Lemma (F.-Maggi-Pratelli, 2008) Given E, one can always order the orthogonal directions $\{e_1, \ldots, e_n\}$ in such a way that the set E' obtained by subsequent reflections of E in the directions $\{e_{i_1}, \ldots, e_{i_n}\}$ has the property that

$$\alpha(E) \leq 2^n \alpha(E')$$

Lemma (F.-Maggi-Pratelli, 2008) Given E, one can always order the orthogonal directions $\{e_1, \dots, e_n\}$ in such a way that the set E' obtained by subsequent reflections of E in the directions $\{e_{i_1}, \dots, e_{i_n}\}$ has the property that

$$\alpha(E) \leq 2^n \alpha(E')$$

This lemma + Frank-Lieb

$$\Downarrow$$

$$\alpha(E) \leq 2^n \alpha(E') \qquad \mathcal{D}(E') \leq 2^n \mathcal{D}(E)$$

Lemma (F.-Maggi-Pratelli, 2008) Given E, one can always order the orthogonal directions $\{e_1, \dots, e_n\}$ in such a way that the set E' obtained by subsequent reflections of E in the directions $\{e_{i_1}, \dots, e_{i_n}\}$ has the property that

$$\alpha(E) \leq 2^n \alpha(E')$$

This lemma + Frank-Lieb

$$\Downarrow$$

$$\alpha(E) \leq 2^n \alpha(E') \qquad \mathcal{D}(E') \leq 2^n \mathcal{D}(E)$$

1) So, if $0 < \lambda \leq 2$ they may assume that $-E = E$

Lemma (F.-Maggi-Pratelli, 2008) Given E, one can always order the orthogonal directions $\{e_1, \ldots, e_n\}$ in such a way that the set E' obtained by subsequent reflections of E in the directions $\{e_{i_1}, \ldots, e_{i_n}\}$ has the property that

$$\alpha(E) \leq 2^n \alpha(E')$$

This lemma + Frank-Lieb

$$\Downarrow$$

$$\alpha(E) \leq 2^n \alpha(E') \qquad \mathcal{D}(E') \leq 2^n \mathcal{D}(E)$$

1) So, if $0 < \lambda \leq 2$ they may assume that $-E = E$

2) Prove by a direct computation $\mathcal{D}(E') \geq c|E' \Delta B|^2 \geq c\alpha(E')^2$

To prove Step 2 they need $\lambda = 2$

Theorem (F.-Pratelli, ArXiv, September 25, 2019)

Let $n \geq 2$, $1 < \lambda < n$ *There exists* $C(n,\lambda) > 0$ *s.t. if* $|E| = \omega_n = |B|$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

Theorem (F.-Pratelli, ArXiv, September 25, 2019)

Let $n \geq 2, 1 < \lambda < n$ *There exists* $C(n, \lambda) > 0$ *s.t. if* $|E| = \omega_n = |B|$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

- The exponent 2 is optimal

Theorem (F.-Pratelli, ArXiv, September 25, 2019)

Let $n \geq 2$, $1 < \lambda < n$ *There exists* $C(n, \lambda) > 0$ *s.t. if* $|E| = \omega_n = |B|$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

Consider

$$\mathcal{F}(E) := P(E) + \varepsilon \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\, dxdy, \qquad 0 < \lambda < n$$

Theorem (F.-Pratelli, ArXiv, September 25, 2019)

Let $n \geq 2$, $1 < \lambda < n$ *There exists* $C(n,\lambda) > 0$ *s.t. if* $|E| = \omega_n = |B|$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

Consider

$$\mathcal{F}(E) := P(E) + \varepsilon \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\,dxdy, \qquad 0 < \lambda < n$$

If $\varepsilon \leq \varepsilon_0(n,\lambda)$, then
Knüpfer-Muratov, Bonacini-Cristoferi,
Figalli-F.-Maggi-Millot-Morini, ...

$$\mathcal{F}(B) \leq \mathcal{F}(E), \qquad |E| = |B|$$

and equality holds iff E is a ball

Theorem (F.-Pratelli, ArXiv, September 25, 2019)

Let $n \geq 2$, $1 < \lambda < n$ *There exists* $C(n, \lambda) > 0$ *s.t. if* $|E| = \omega_n = |B|$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

Consider

$$\mathcal{F}(E) := P(E) + \varepsilon \int_E \int_E \frac{1}{|x-y|^{n-\lambda}}\,dxdy, \qquad 0 < \lambda < n$$

If $\varepsilon \leq \varepsilon_0(n, \lambda)$, then
Knüpfer-Muratov, Bonacini-Cristoferi,
Figalli-F.-Maggi-Millot-Morini, ...

$$\mathcal{F}(B) \leq \mathcal{F}(E), \qquad |E| = |B|$$

and equality holds iff E is a ball
There exists $C = C(n, \lambda)$ s.t. if $|E| = |B|$

$$\alpha(E)^2 \leq C\left[\mathcal{F}(E) - \mathcal{F}(B)\right]$$

and the proof is easier

September 26, 2019: A message from R. Frank

Theorem (Frank-Lieb, ArXiv, September 10, 2019)

Let $n \geq 2$, $0 < \lambda < n$ *There exists* $C(n, \lambda) > 0$ *s.t. if* $|E| = \omega_n$

$$\alpha(E)^2 \leq C\, \mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

Theorem (Frank-Lieb, ArXiv, September 10, 2019)

Let $n \geq 2$, $0 < \lambda < n$ *There exists* $C(n, \lambda) > 0$ *s.t. if* $|E| = \omega_n$

$$\alpha(E)^2 \leq C\,\mathcal{D}(E) = C\left[\mathcal{P}(B) - \mathcal{P}(E)\right]$$

The proof is based on a deep stability result by M. Christ

Theorem (Christ, ArXiv, June 6, 2017)

Let $n \geq 2$ *There exists* $C(n) > 0$ *s.t. if* $f : \mathbb{R}^n \to [0,1]$, $\|f\|_{L^1} = \omega_n$, *then*

$$\int_B \int_B \chi_B(x-y)\,dxdy - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)\chi_B(x-y)f(y)\,dxdy \geq CA(f)^2$$

where

$$A(f) = \min_{x \in \mathbb{R}^n} \left\| f - \chi_{B(x)} \right\|_{L^1}$$

Nearly spherical sets

Theorem $(0 < \lambda < n)$

There exist $\varepsilon_1 \in (0,1), C_0 > 0$ *s.t. if* $|E| = \omega_n$, $\mathrm{bar}(E) = 0$ *and*

$$E = \{t\,z : z \in \mathbb{S}^{n-1},\ t \in [0, 1 + u(z)]\}$$

with $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq \varepsilon_1$, *then*

$$\mathcal{P}(B) - \mathcal{P}(E) \geq C_0 |E \Delta B|^2$$

Nearly spherical sets

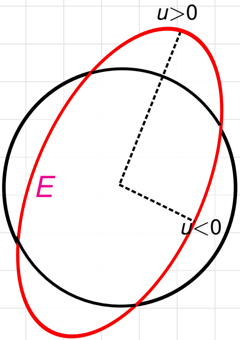
Theorem ($0 < \lambda < n$)

There exist $\varepsilon_1 \in (0,1), C_0 > 0$ *s.t. if* $|E| = \omega_n$, $\mathrm{bar}(E) = 0$ *and*

$$E = \{t\, z : z \in \mathbb{S}^{n-1},\, t \in [0, 1 + u(z)]\}$$

with $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \le \varepsilon_1$, *then*

$$\mathcal{P}(B) - \mathcal{P}(E) \ge C_0 |E \Delta B|^2$$



Proof by a second variation argument (Fuglede's style)

Nearly spherical sets

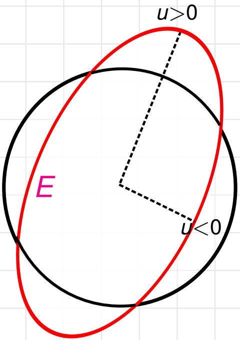
Theorem ($0 < \lambda < n$)

There exist $\varepsilon_1 \in (0,1), C_0 > 0$ *s.t. if* $|E| = \omega_n$, $\mathrm{bar}(E) = 0$ *and*

$$E = \{t\,z : \, z \in \mathbb{S}^{n-1},\, t \in [0, 1 + u(z)]\}$$

with $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq \varepsilon_1$, *then*

$$\mathcal{P}(B) - \mathcal{P}(E) \geq C_0 |E \Delta B|^2$$



Proof by a second variation argument

In [FFMMM] it is proved that

for a nearly spherical set

$$\mathcal{P}(B) - \mathcal{P}(E)$$

$$\leq C_1 \left(|E \Delta B|^2 + \int_{\partial B} \int_{\partial B} \frac{|u(x) - u(y)|^2}{|x - y|^{n-\lambda}} \right)$$

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0$

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \dfrac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

vanishing: $\forall R > 0$ one has $\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} |E_h \cap B_R(x)| = 0$ NO!

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

vanishing: $\forall R > 0$ one has $\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} |E_h \cap B_R(x)| = 0$ NO!

$$\alpha(E_h) = \inf_x |E \Delta B(x)| = 2 \inf_x |B(x) \setminus E_h| = 2\omega_n - 2 \sup_x |B(x) \cap E_h| \to 2\omega_n$$

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

vanishing: $\forall R > 0$ one has $\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} |E_h \cap B_R(x)| = 0$ NO!

$$\alpha(E_h) = \inf_x |E \Delta B(x)| = 2 \inf_x |B(x) \setminus E_h| = 2\omega_n - 2 \sup_x |B(x) \cap E_h| \to 2\omega_n$$

Lemma

There exist $\xi(n), c(n) > 0$ such that if $\alpha(E) \geq 2\omega_n - \xi(n)$ then

$$D(E) \geq c(n)$$

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

vanishing: $\forall R > 0$ one has $\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} |E_h \cap B_R(x)| = 0$ NO!

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

vanishing: $\forall R > 0$ one has $\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} |E_h \cap B_R(x)| = 0$ NO!

dichotomy: $\exists\, 0 < m < \omega_n$ s.t. $\forall \varepsilon > 0 \ \exists R_\varepsilon, \quad E_h^1, E_h^2 \subset E_h$

$$\limsup_{h\to\infty} \big||E_h^1| - m\big| < \varepsilon, \quad \limsup_{h\to\infty} \big||E_h^2| - (\omega_n - m)\big| < \varepsilon, \quad \mathrm{dist}(E_h^1, E_h^2) \to \infty$$

NO!

The Strategy

To show that for $\delta > 0$ small

(1) $\alpha(E) \geq \delta \implies \mathcal{D}(E) \geq c_\delta > 0 \implies \alpha(E)^2 \leq \frac{4\omega_n^2}{c_\delta}\mathcal{D}(E)$

(2) $\alpha(E) < \delta \implies$ we may reduce to the nearly spherical case

(1) $\mathcal{D}(E_h) \to 0 \implies \alpha(E_h) \to 0$

vanishing: $\forall R > 0$ one has $\lim_{h\to\infty} \sup_{x\in\mathbb{R}^n} |E_h \cap B_R(x)| = 0$ NO!

dichotomy: $\exists\, 0 < m < \omega_n$ s.t. $\forall \varepsilon > 0\ \exists R_\varepsilon,\ E_h^1, E_h^2 \subset E_h$

$$\limsup_{h\to\infty} \big||E_h^1| - m\big| < \varepsilon, \quad \limsup_{h\to\infty} \big||E_h^2| - (\omega_n - m)\big| < \varepsilon, \quad \mathrm{dist}(E_h^1, E_h^2) \to \infty$$

NO!

compactness: $\forall \varepsilon > 0\ \exists R_\varepsilon$ s.t. $\limsup_{h\to\infty} |E_h \setminus B_{R_\varepsilon}| < \varepsilon$

compactness: $\forall \varepsilon > 0 \ \exists R_\varepsilon$ s.t. $\limsup_{h\to\infty} |E_h \setminus B_{R_\varepsilon}| < \varepsilon$

compactness: $\forall \varepsilon > 0 \ \exists R_\varepsilon$ s.t. $\limsup_{h\to\infty} |E_h \setminus B_{R_\varepsilon}| < \varepsilon$

This is the difficult case!

compactness: $\forall \varepsilon > 0 \ \exists R_\varepsilon$ s.t. $\limsup_{h\to\infty} |E_h \setminus B_{R_\varepsilon}| < \varepsilon$

This is the difficult case!

$$\chi_{E_h} \rightharpoonup f \qquad \text{weakly* in } L^\infty \qquad \int_{\mathbb{R}^n} f\,dx = |B|$$

compactness: $\forall \varepsilon > 0 \ \exists R_\varepsilon$ s.t. $\limsup_{h\to\infty} |E_h \setminus B_{R_\varepsilon}| < \varepsilon$

This is the difficult case!

$\chi_{E_h} \rightharpoonup f$ weakly* in L^∞ $\quad \int_{\mathbb{R}^n} f\,dx = |B|$

Then one has to prove that

$$\int_{E_h}\int_{E_h} \frac{1}{|x-y|^{n-\lambda}} \to \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{|x-y|^{n-\lambda}} = \int_B\int_B \frac{1}{|x-y|^{n-\lambda}}$$

$$\Longrightarrow \qquad f = \chi_B \qquad \Longrightarrow \qquad \alpha(E_h) \to 0$$

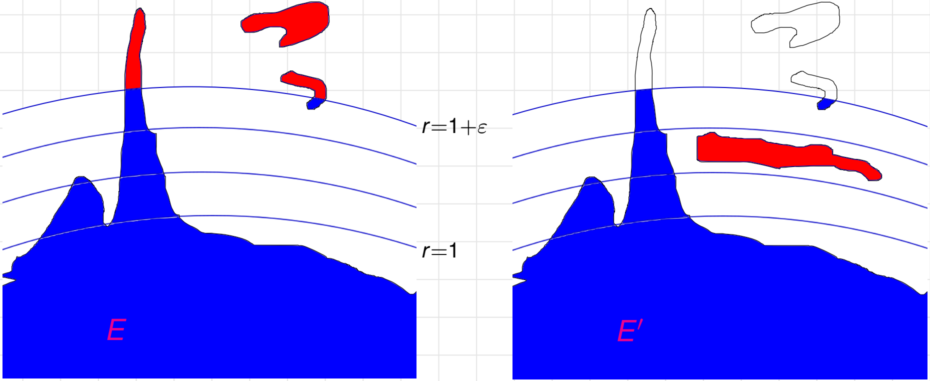
Lemma

Given $\varepsilon \in (0,1)$, *there exists* $\delta > 0$ *such that if* $\alpha(E) < \delta$ *then one can find a set* E' *with* $|E'| = \omega_n$ *and*

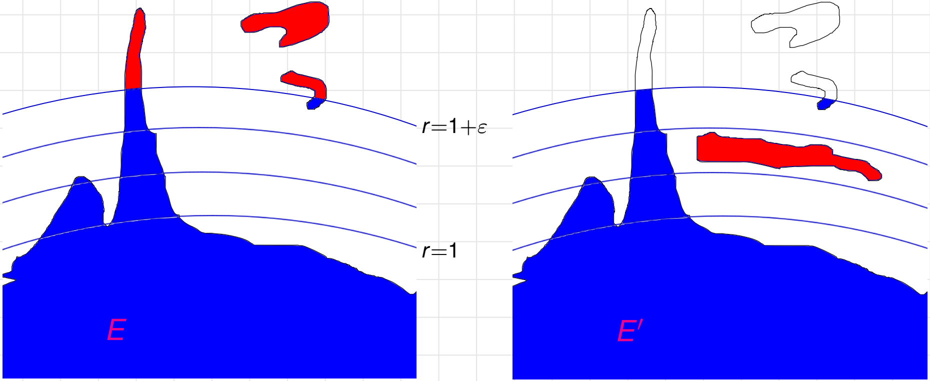
$$B_{1-\varepsilon}(0) \subset E' \subset B_{1+\varepsilon}(0)$$

$$\alpha(E) = \alpha(E') \qquad \mathcal{D}(E') \leq \mathcal{D}(E)$$

$$\text{if } \alpha(E) < \delta << \varepsilon \quad \Longrightarrow \quad |E \backslash B_{1+\varepsilon}(0)| \leq |E \backslash B_1(0)| = \frac{1}{2}\alpha(E) << \varepsilon$$

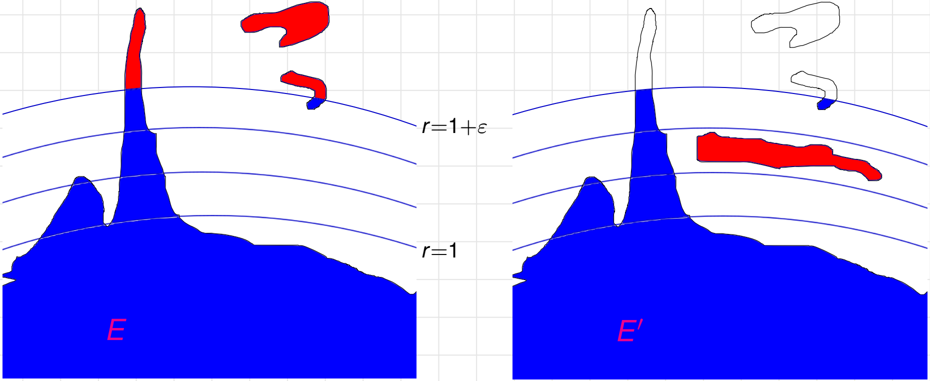


if $\alpha(E) < \delta << \varepsilon \quad \Longrightarrow \quad |E \backslash B_{1+\varepsilon}(0)| \leq |E \backslash B_1(0)| = \frac{1}{2}\alpha(E) << \varepsilon$



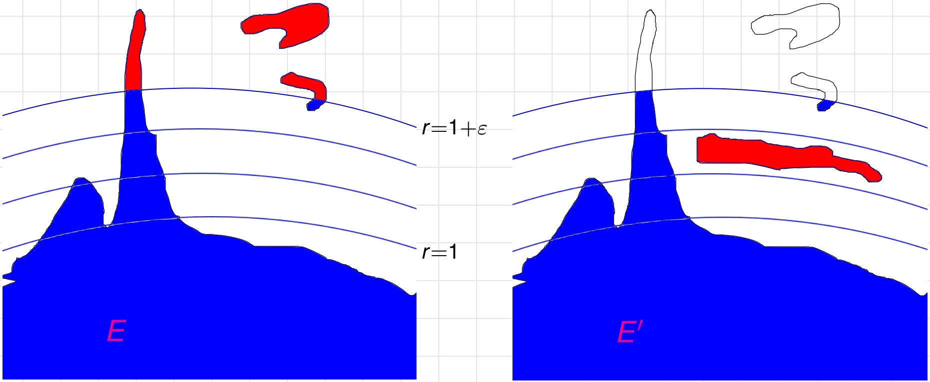
The proof that $\mathcal{P}(E') \geq \mathcal{P}(E)$ is easy

if $\alpha(E) < \delta << \varepsilon \quad \Longrightarrow \quad |E \backslash B_{1+\varepsilon}(0)| \leq |E \backslash B_1(0)| = \frac{1}{2}\alpha(E) << \varepsilon$



The proof that $\mathcal{P}(E') \geq \mathcal{P}(E)$ is easy $\quad \Longrightarrow \quad \mathcal{D}(E') \leq \mathcal{D}(E)$

if $\alpha(E) < \delta << \varepsilon \quad \Longrightarrow \quad |E \backslash B_{1+\varepsilon}(0)| \leq |E \backslash B_1(0)| = \frac{1}{2}\alpha(E) << \varepsilon$



The proof that $\mathcal{P}(E') \geq \mathcal{P}(E)$ is easy $\quad \Longrightarrow \quad \mathcal{D}(E') \leq \mathcal{D}(E)$

The proof that $\alpha(E') = \alpha(E)$ is trickier

Thus from now on we may suppose that $|E| = |B|$ and

$$B_{1-\varepsilon}(0) \subset E \subset B_{1+\varepsilon}(0) \qquad \text{for some small } \varepsilon > 0$$

Theorem

There exists C_1 *s.t. if* $B_{1-\varepsilon}(0) \subset E \subset B_{1+\varepsilon}(0)$, $|E| = |B|$,

either $\alpha(E)^2 \leq C_1 \mathcal{D}(E)$

or $\exists\ E' = \left\{ t\,z :\ z \in \mathbb{S}^{n-1},\ t \in [0, 1 + u(z)],\ \|u\|_{L^\infty} \leq \varepsilon \right\}$

s.t. $\alpha(E) \leq 6|E' \Delta B|, \qquad \mathcal{D}(E') \leq 2\mathcal{D}(E)$

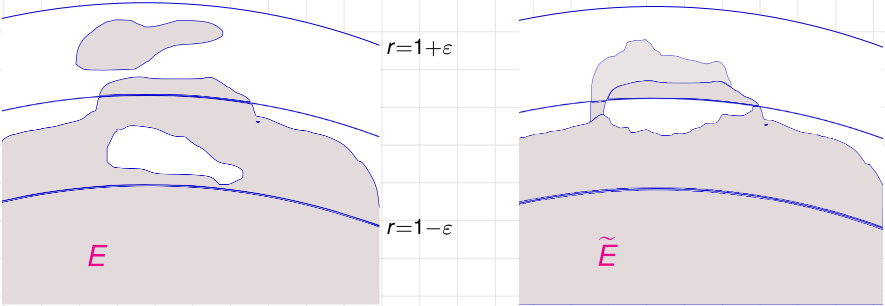
Theorem

There exists C_1 *s.t. if* $B_{1-\varepsilon}(0) \subset E \subset B_{1+\varepsilon}(0)$, $|E| = |B|$,

either $\alpha(E)^2 \leq C_1 \mathcal{D}(E)$

or $\exists\ E' = \{t\,z :\ z \in \mathbb{S}^{n-1},\ t \in [0, 1 + u(z)],\ \|u\|_{L^\infty} \leq \varepsilon\}$

s.t. $\alpha(E) \leq 6|E' \Delta B|, \qquad \mathcal{D}(E') \leq 2\mathcal{D}(E)$



$$\widetilde{E} = \{t\,z :\ z \in \mathbb{S}^{n-1},\ t \in [0, 1 - u_1(z)] \cup [0, 1 + u_2(z)],\ \|u\|_{L^\infty} \leq \varepsilon\}$$

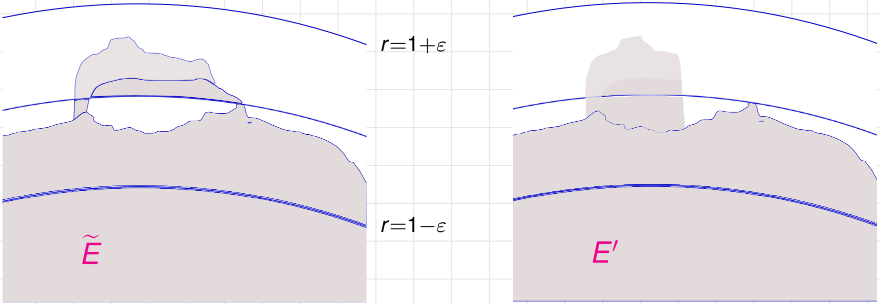
Theorem

There exists C_1 s.t. if $B_{1-\varepsilon}(0) \subset E \subset B_{1+\varepsilon}(0)$, $|E| = |B|$,

either $\alpha(E)^2 \le C_1 \mathcal{D}(E)$

or $\exists\ E' = \{t\,z :\ z \in \mathbb{S}^{n-1},\ t \in [0, 1 + u(z)],\ \|u\|_{L^\infty} \le \varepsilon\}$

s.t. $\alpha(E) \le 6|E'\Delta B|$, $\mathcal{D}(E') \le 2\mathcal{D}(E)$



E' is nearly spherical. Can we say that $\mathrm{bar}(E') = 0$?

Assume $B_{1-\varepsilon^2}(0) \subset E \subset B_{1+\varepsilon^2}(0) \quad \rightsquigarrow \quad E'$,

E' is nearly spherical. Can we say that $\mathrm{bar}(E') = 0$?

Assume $B_{1-\varepsilon^2}(0) \subset E \subset B_{1+\varepsilon^2}(0) \quad \rightsquigarrow \quad E'$, but $\mathrm{bar}(E') \neq 0$

Moreover $B_{1-\varepsilon^2}(0) \subset E' \subset B_{1+\varepsilon^2}(0)$

E' is nearly spherical. Can we say that $\mathrm{bar}(E') = 0$?

Assume $B_{1-\varepsilon^2}(0) \subset E \subset B_{1+\varepsilon^2}(0) \quad \rightsquigarrow \quad E'$, but $\mathrm{bar}(E') \neq 0$

Moreover $B_{1-\varepsilon^2}(0) \subset E' \subset B_{1+\varepsilon^2}(0)$

Let's try with another ball.

E' is nearly spherical. Can we say that $\mathrm{bar}(E') = 0$?

Assume $B_{1-\varepsilon^2}(0) \subset E \subset B_{1+\varepsilon^2}(0)$ $\rightsquigarrow$ E', but $\mathrm{bar}(E') \neq 0$

Moreover $B_{1-\varepsilon^2}(0) \subset E' \subset B_{1+\varepsilon^2}(0)$

Let's try with another ball. Let's take $|z| \leq \varepsilon$. Then

$B_{1-2\varepsilon}(z) \subset E \subset B_{1+2\varepsilon}(z)$ $\rightsquigarrow$ a nearly spherical set E_z

E' is nearly spherical. Can we say that $\mathrm{bar}(E') = 0$?

Assume $B_{1-\varepsilon^2}(0) \subset E \subset B_{1+\varepsilon^2}(0)$ $\rightsquigarrow$ E', but $\mathrm{bar}(E') \neq 0$

Moreover $B_{1-\varepsilon^2}(0) \subset E' \subset B_{1+\varepsilon^2}(0)$

Let's try with another ball. Let's take $|z| \leq \varepsilon$. Then

$B_{1-2\varepsilon}(z) \subset E \subset B_{1+2\varepsilon}(z)$ $\rightsquigarrow$ a nearly spherical set E_z

If we are lucky ...... $\mathrm{bar}(E_z) = z$ $\rightsquigarrow$ and we are done!

E' is nearly spherical. Can we say that $\mathrm{bar}(E') = 0$?

Assume $B_{1-\varepsilon^2}(0) \subset E \subset B_{1+\varepsilon^2}(0) \quad \rightsquigarrow \quad E'$, but $\mathrm{bar}(E') \neq 0$

Moreover $B_{1-\varepsilon^2}(0) \subset E' \subset B_{1+\varepsilon^2}(0)$

Let's try with another ball. Let's take $|z| \leq \varepsilon$. Then

$B_{1-2\varepsilon}(z) \subset E \subset B_{1+2\varepsilon}(z) \quad \rightsquigarrow \quad$ a nearly spherical set E_z

If we are lucky . . . . . . $\mathrm{bar}(E_z) = z \quad \rightsquigarrow \quad$ and we are done!

- $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous
- if $0 < |z| \leq \varepsilon$ then $B_{1-\varepsilon^2}(0) \subset E_z \subset B_{1+\varepsilon^2}(0)$
- $\implies$ if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \ \forall z \in \overline{B}_\varepsilon(0)$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \ \forall z \in \overline{B}_\varepsilon(0)$

$$\text{Set} \qquad R = \min_{|z|=\varepsilon} |z - \mathrm{bar}(E_z)| > 0$$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \ \forall z \in \overline{B}_\varepsilon(0)$

$$\text{Set} \qquad R = \min_{|z|=\varepsilon} |z - \mathrm{bar}(E_z)| > 0$$

$$F(w) = \frac{1}{R} \Pi_{\overline{B}_R(0)}(\varepsilon w - \mathrm{bar}(E_{\varepsilon w})) : \overline{B} \mapsto \overline{B} \setminus \{0\}$$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \ \forall z \in \overline{B}_\varepsilon(0)$

$$\text{Set} \qquad R = \min_{|z|=\varepsilon} |z - \mathrm{bar}(E_z)| > 0$$

$$F(w) = \frac{1}{R}\Pi_{\overline{B}_R(0)}(\varepsilon w - \mathrm{bar}(E_{\varepsilon w})) : \overline{B} \mapsto \overline{B} \setminus \{0\}$$

Note that $F_{|\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \;\; \forall z \in \overline{B}_\varepsilon(0)$

$$\text{Set} \qquad R = \min_{|z|=\varepsilon} |z - \mathrm{bar}(E_z)| > 0$$

$$F(w) = \frac{1}{R}\Pi_{\overline{B}_R(0)}(\varepsilon w - \mathrm{bar}(E_{\varepsilon w})) : \overline{B} \mapsto \overline{B} \setminus \{0\}$$

Note that $F_{|\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$

F is homotopic to the constant $\dfrac{F(0)}{|F(0)|}$: $\quad \dfrac{F((1-t)w)}{|F((1-t)w)|} \quad t \in [0,1]$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \ \forall z \in \overline{B}_\varepsilon(0)$

$$\text{Set} \qquad R = \min_{|z|=\varepsilon} |z - \mathrm{bar}(E_z)| > 0$$

$$F(w) = \frac{1}{R}\Pi_{\overline{B}_R(0)}(\varepsilon w - \mathrm{bar}(E_{\varepsilon w})) : \overline{B} \mapsto \overline{B} \setminus \{0\}$$

Note that $F_{|\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$

F is homotopic to the constant $\dfrac{F(0)}{|F(0)|}$: $\dfrac{F((1-t)w)}{|F((1-t)w)|}$ $\quad t \in [0,1]$

But $F_{|\mathbb{S}^{n-1}}$ is also homotopic to the identity thanks to 2)

$$\frac{(1-t)F(\omega) + t\omega}{|(1-t)F(\omega) + t\omega|} \qquad t \in [0,1]$$

1) $z \in \overline{B}_\varepsilon(0) \to \mathrm{bar}(E_z)$ is continuous

2) if $|z| = \varepsilon$ then $(z - \mathrm{bar}(E_z)) \cdot z > 0$

Assume by contradiction that $\mathrm{bar}(E_z) \neq z \;\; \forall z \in \overline{B}_\varepsilon(0)$

$$\text{Set} \qquad R = \min_{|z|=\varepsilon} |z - \mathrm{bar}(E_z)| > 0$$

$$F(w) = \frac{1}{R} \Pi_{\overline{B}_R(0)}(\varepsilon w - \mathrm{bar}(E_{\varepsilon w})) : \overline{B} \mapsto \overline{B} \setminus \{0\}$$

Note that $F_{|\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \mapsto \mathbb{S}^{n-1}$

F is homotopic to the constant $\dfrac{F(0)}{|F(0)|}$: $\dfrac{F((1-t)w)}{|F((1-t)w)|}$ $\quad t \in [0,1]$

But $F_{|\mathbb{S}^{n-1}}$ is also homotopic to the identity thanks to 2)

$$\frac{(1-t)F(\omega) + t\omega}{|(1-t)F(\omega) + t\omega|} \qquad t \in [0,1]$$

Impossible!

A related result (work in progress with A. Pratelli)

Let $g : (0, \infty) \to [0, \infty)$ be a continuous, decreasing function, such that

$$\int_0^1 t^{n-1} g(t)\, dt < \infty$$

(this includes in particular the case

$$g(t) = \frac{1}{t^{n-\lambda}} \qquad 0 < \lambda < n)$$

A related result (work in progress with A. Pratelli)

Let $g : (0, \infty) \to [0, \infty)$ be a continuous, decreasing function, such that

$$\int_0^1 t^{n-1} g(t)\, dt < \infty$$

Set for $\varepsilon > 0$

$$\mathcal{F}(E) := P(E) + \varepsilon \int_E \int_E g(|x - y|)\, dxdy$$

A related result (work in progress with A. Pratelli)

Let $g : (0, \infty) \to [0, \infty)$ be a continuous, decreasing function, such that

$$\int_0^1 t^{n-1} g(t)\, dt < \infty$$

Set for $\varepsilon > 0$

$$\mathcal{F}(E) := P(E) + \varepsilon \int_E \int_E g(|x-y|)\, dxdy$$

There exists ε_0 such that if $\varepsilon \leq \varepsilon_0$, then

$$\mathcal{F}(B) \leq \mathcal{F}(E), \qquad |E| = |B|$$

and equality holds iff E is a ball

$$\min \left\{ \mathcal{F}(E) = P(E) + \varepsilon \int_E \int_E g(|x-y|)\, dxdy : \ |E| = |B| \right\}$$

$$\min\left\{\mathcal{F}(E) = P(E) + \varepsilon \int_E\int_E g(|x-y|)\,dxdy \;:\; |E| = |B|\right\}$$

Again,

1) for ε small any minimizer is $L^1 \Longrightarrow C^{2,\alpha}$ close to B

$$\min\left\{\mathcal{F}(E) = P(E) + \varepsilon \int_E\int_E g(|x-y|)\,dxdy :\ |E| = |B|\right\}$$

Again,

1) for ε small any minimizer is $L^1 \Longrightarrow C^{2,\alpha}$ close to B

2) if $E = \{t\,z :\ z \in \mathbb{S}^{n-1},\ t \in [0, 1+u(z)]\}$ is C^1 close to B

$$P(E) - P(B) \geq C_0\left(|E\Delta B|^2 + \|\nabla u\|^2_{L^2(\partial B)}\right)$$

3) But one can prove that

$$\int_B\int_B g(|x-y|)\,dxdy - \int_E\int_E g(|x-y|)\,dxdy$$
$$\leq C\left(|E\Delta B|^2 + \int_{\partial B}\int_{\partial B} |u(x)-u(y)|^2 g(|x-y|)\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\right)$$
$$\leq C_1\left(|E\Delta B|^2 + \|\nabla u\|^2_{L^2(\partial B)}\right)$$

Happy birthday to Gianni!