A Homogenization Result in the Gradient Theory of Phase Transitions

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It has been 34 years!. . .

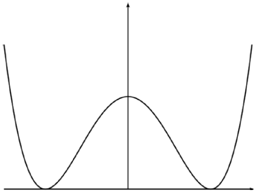
Van Der Waals (1893), Cahn and Hilliard (1958), Gurtin (1987)

...it started in 2003 ...

Equilibrium behavior of a fluid with two stable phases may be described by the Gibbs free energy per unit volume

$$I(u) := \int_\Omega W(u)\, dx$$

$W : \mathbb{R} \to [0, +\infty)$ is a double well potential



$W(p) := (p^2 - 1)^2$, $\{W = 0\} = \{-1, 1\}$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container
- $u : \Omega \to \mathbb{R}$ density of a fluid
- $\int_\Omega u\, dx = m \ldots m$ total mass of the fluid
- W double-well potential energy per unit volume
- $W^{-1}(\{0\}) = \{a, b\} \ldots a < b$ two phases of the fluid

Problem

Minimize total energy

$$I(u) = \int_\Omega W(u)\, dx$$

subject to $\int_\Omega u\, dx = m$

Solution

Assume $|\Omega| = 1$ *and* $a < m < b$. *Then minimizers are of the form*

$$u_E(x) = \begin{cases} a & \text{if } x \in E, \\ b & \text{if } x \in \Omega \setminus E, \end{cases}$$

where $E \subseteq \Omega$ *is any measurable set with* $|E| = \frac{b-m}{b-a}$

NONUNIQUENESS OF SOLUTIONS

Selection via singular perturbations:

$$I_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2 |\nabla u|^2 \right] dx, \quad u \in C^1(\Omega), \varepsilon > 0$$

$\varepsilon^2 \int_\Omega |\nabla u|^2 \, dx$ ... surface energy penalization

Modica–Mortola, 1977

$\{W = 0\} = \{a, b\}$

Gurtin's 1985 conjecture:

Asymptotic behavior of minimizers to E_ε described via Γ-convergence. Scaling by ε^{-1} yields

$$\varepsilon^{-1} I_\varepsilon \xrightarrow{\Gamma} F_0,$$

$$F_0(u) := \begin{cases} c_W \ P(A;\Omega) & u \in BV(\Omega;\{a,b\}), \\ +\infty & u \in L^1(\Omega) \setminus BV(\Omega;\{a,b\}) \end{cases}$$

where

$$A := \{u(x) = a\}, \ c_W := 2\int_a^b \sqrt{W(s)}ds$$

$$I_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2 |\nabla u|^2\right] dx, \quad u \in C^1(\Omega)$$

Gurtin's Conjecture (1987): Minimizers u_ε

$$\min\left\{I_\varepsilon(u) : u \in C^1(\Omega), \quad \int_\Omega u\, dx = m\right\}$$

converge to u_{E_0}, where

$$\mathrm{Per}_\Omega(E_0) \leq \mathrm{Per}_\Omega(E)$$

over all $E \subseteq \Omega$ measurable with $|E| = \frac{b-m}{b-a}$

$$F_\varepsilon(u) := \frac{1}{\varepsilon} I_\varepsilon(u) = \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right] dx$$

F_ε and I_ε have the same minimizers

So ... if we know the Γ-limit of $\{F_\varepsilon\}$ then we know where the minimizers of I_ε converge to ...

$$F_\varepsilon(u) = \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right] dx, \quad u \in C^1(\Omega)$$

Theorem (Modica (1987), Sternberg (1988), F. and Tartar (1989),...)

$F_\varepsilon \xrightarrow{\Gamma} F_0$ *with respect to strong convergence in* $L^1(\Omega)$, *where*

$$F_0(u) := \begin{cases} c_W \operatorname{Per}_\Omega\left(u^{-1}(\{a\})\right) & \text{if } u \in BV(\Omega; \{a, b\}), \int_\Omega u\, dx = m, \\ +\infty & \text{otherwise} \end{cases}$$

$$c_W := 2 \int_a^b \sqrt{W(s)}\, ds$$

What about higher order nonlocal regularizations?

- G. Dal Maso, I.F. and G. Leoni, *Trans. Amer. Math. Soc.* (2018)

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u)\, dx + \mathcal{J}_\varepsilon(u) & \text{if } u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise} \end{cases},$$

where

$$\mathcal{J}_\varepsilon(u) := \varepsilon \int_\Omega \int_\Omega J_\varepsilon(x-y) |\nabla u(x) - \nabla u(y)|^2 dx dy \quad \text{for } u \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

$$J_\varepsilon(x) := \frac{1}{\varepsilon^N} J\left(\frac{x}{\varepsilon}\right)$$

$J : \mathbb{R}^N \to [0, +\infty)$... even measurable function

$$\int_{\mathbb{R}^n} J(x)(|x| \wedge |x|^2)\, dx < +\infty$$

where $a \wedge b := \min\{a, b\}$.

Nonlocal higher order singular perturbations

$J : \mathbb{R}^N \to [0, +\infty)\ldots$ even measurable function

$$\int_{\mathbb{R}^N} J(x)(|x| \wedge |x|^2)\, dx < +\infty$$

For example

$$J(x) := |x|^{-N-2s}, \quad \frac{1}{2} < s < 1$$

leads to Gagliardo's seminorm for the fractional Sobolev space $H^s(\mathbb{R})$
In this case

$$J_\varepsilon(x) = \varepsilon^{2s}|x|^{-N-2s}$$

- G. Alberti and G. Belletini, *Math. Ann.* (1998)

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \frac{1}{\varepsilon} W(u)\, dx + \tilde{\mathcal{J}}_\varepsilon(u) & \text{if } u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega) \\ +\infty & \text{otherwise,} \end{cases},$$

$$\tilde{\mathcal{J}}_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega\int_\Omega J_\varepsilon(x-y)(u(x)-u(y))^2 dx dy \quad \text{for } u \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

$$J_\varepsilon(x) := \frac{1}{\varepsilon^N} J\left(\frac{x}{\varepsilon}\right)$$

(statistical mechanics) free energies of continuum limits of Ising spin systems on lattices

u . . . macroscopic magnetization

J . . . ferromagnetic Kac potential

but dependence on ∇u in place of u adds **remarkable** difficulties!

Relevant Spaces:

$\nu \in \mathbb{S}^{N-1} := \partial B_1(0)$

$\nu_1, \ldots, \nu_N$ ... orthonormal basis in $\mathbb{R}^N$ with $\nu_N = \nu$

$$V^\nu := \{x \in \mathbb{R}^N : \ |x \cdot \nu_i| < 1/2 \text{ for } i = 1, \ldots, N-1\}$$

$$Q^\nu := \{x \in \mathbb{R}^N : \ |x \cdot \nu_i| < 1/2 \text{ for } i = 1, \ldots, N\}$$

$W^{1,2}_{\nu_1,\ldots,\nu_{N-1}} := \{v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N) : v(x+\nu_i) = v(x) \text{ for a.e. } \in \mathbb{R}^N, i = 1, \ldots, N-1\}$

$X^\nu := \{v \in W^{1,2}_{\nu_1,\ldots,\nu_{N-1}} : \ v(x) = \pm 1 \text{ for a.e. } x \in \mathbb{R}^N \text{ with } \pm x \cdot \nu \geq 1/2\}$

When $N = 1$ take $\nu = \pm 1$, $V^\nu := \mathbb{R}$, $Q^\nu := (-1/2, 1/2)$

$X^\nu := \{v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}) : v(x) = \pm 1 \text{for a.e.} x \in \mathbb{R} \text{ with } \pm x \geq 1/2\}$

Surface Energy

$$\psi(\nu) := \inf_{0<\varepsilon<1} \inf_{v \in X^\nu} \mathcal{F}_\varepsilon^\nu(v)$$

where

$$\mathcal{F}_\varepsilon^\nu(u) := \frac{1}{\varepsilon}\int_{Q^\nu} W(u(x))\,dx + \varepsilon \int_{V^\nu}\int_{\mathbb{R}^N} J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy$$

Define $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ by

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{S_u} \psi(\nu_u)\ d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega; \{-1, 1\})\ , \\ +\infty & \text{otherwise in } L^2(\Omega) \end{cases}$$

Compactness in L^2 of energy bounded sequences

$\{\mathcal{F}_\varepsilon\}$ Γ-converges to $\mathcal{F}$ in $L^2(\Omega)$

nonlocality $\to$ remarkable technical difficulties!

Localized energies:

$$\mathcal{W}_\varepsilon(u, A) := \frac{1}{\varepsilon} \int_A W(u(x))\ dx$$

$$\mathcal{J}_\varepsilon(u, A, B) := \varepsilon \int_A \int_B J_\varepsilon(x - y)|\nabla u(x) - \nabla u(y)|^2 dxdy$$

When $A = B$ we set

$$\mathcal{F}_\varepsilon(u, A) := \mathcal{W}_\varepsilon(u, A) + \mathcal{J}_\varepsilon(u, A, A) \quad \text{and} \quad \mathcal{J}_\varepsilon(u, A) := \mathcal{J}_\varepsilon(u, A, A)$$

Theorem (Interpolation Inequality)

$$\varepsilon \int_A |\nabla u(x)|^2 dx \leq C\mathcal{F}_\varepsilon(u, (A)^{2\varepsilon\gamma_J})$$

for every $\varepsilon > 0$, for every open set $A \subset \mathbb{R}^N$, and for every $u \in W^{1,2}_{\mathrm{loc}}((A)^{2\varepsilon\gamma_J})$

$(A)^\eta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, A) < \eta\}$

$$\varepsilon \int_A |\nabla u(x)|^2 dx \leq C \mathcal{F}_\varepsilon(u, (A)^{2\varepsilon\gamma_J})$$

$(A)^\eta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, A) < \eta\}$

γ_J: For all $\xi \in \mathbb{S}^{N-1}$ there exist $-\gamma_J < \alpha(\xi) < \beta(\xi) < \gamma_J$ s.t.

$$\int_{\alpha(\xi)}^{\beta(\xi)} \frac{1}{J(t\xi)|t|^{N-1}}\, dt \leq C_J$$

Next ... "modification lemma" ... proof 11 pages long ...

Interaction Phase Transition/Homogenization

Consider fluids which exhibit periodic heterogeneity at small scales, i.e.

$$F_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\delta(\varepsilon)}, u\right) + \varepsilon |\nabla u|^2\right] dx$$

where

- $W(x,p) = 0$ iff $p \in \{a, b\}$
- $W(\cdot, p)$ is Q-periodic for every p, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$

Example: $W(x,p) = \chi_E(x) W_1(p) + \chi_{Q \setminus E} W_2(p)$

Goal: Identify Γ-limit of F_ε

Ansini, Braides, Chiadò-Piat (2003): W homogeneous, regularization $f\left(\frac{x}{\delta(\varepsilon)}, \nabla u\right)$

Braides, Zeppieri (2009): $\int_0^1 \left[W^{(k)}\left(\frac{x}{\delta(\varepsilon)}, u\right) + \varepsilon^2 |u'|^2\right] dx$

Scaling regime $\delta(\varepsilon) = \varepsilon$

Theorem (Cristoferi, F., Hagerty, Popovici. *Interfaces Free Bound.*(2019))

Let $\delta(\varepsilon) = \varepsilon$. *Then* $F_\varepsilon \xrightarrow{\Gamma} F_0$,

$$F_0(u) := \begin{cases} \int_{\partial^* A} \sigma(\nu) d\mathcal{H}^{N-1} & u \in BV(\Omega; \{a, b\}), \\ +\infty & \textit{otherwise} \end{cases}$$

where

$$A := \{u(x) = a\}, \ \nu \textit{ is the outward normal to } A,$$

and

$$\sigma(\nu) := \lim_{T \to \infty} \inf_{u \in \mathcal{A}_{\nu,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

Cell Problem

$$\sigma(\nu) = \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{\nu,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where

$$\mathcal{A}_{\nu,T} := \left\{ u \in H^1(TQ_\nu; \mathbb{R}^d) : u(x) = (\rho_T * u_0)(x\cdot\nu) \text{ on } \partial TQ_\nu \right\}$$

$$u_0(t) := \begin{cases} b & \text{if } t > 0, \\ a & \text{if } t < 0 \end{cases}$$

$$\rho_T(x) := T^N \rho(Tx),\ \ \rho \in C_c^\infty(\mathbb{R}) \text{ with } \int_{\mathbb{R}} \rho = 1$$

R. Choksi, I. F., J. Lang and R. Venkatraman (to appear): isotropy
σ is constant!

Outline of Proof

- **Compactness: Bounded energy $\rightarrow BV$ structure**
  - Reduction to classical Modica-Mortola technique
  - $W(x,p) = 0$ iff $p \in \{a,b\}$
  - $(x,p) \rightarrow W(x,p)$ Carathéodory, only measurability in x
  - $W(x,p) \geq \tilde{W}(p)$, $\tilde{W}(p) = 0$ iff $p \in \{a,b\}$, $\tilde{W}(p) \geq C|p|$ for $|p| >> 1$
- **Γ-liminf: "Lower-semicontinuity" result using blow-up techniques**
- $\frac{|p|^q}{C} - C \leq W(x,p) \leq C(1+|p|^q)$, some $q \geq 2$
  - "Blow up" at points in jump set
  - De Giorgi's slicing method $\rightarrow$ prescribe boundary conditions from σ
  - Compare with optimal profiles given by σ
- **Γ-limsup: Recovery sequences**
  - Blow-Up Method
  - Recovery sequences for polyhedral sets with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$
  - Density result and upper semicontinuity of σ

Compactness

Reduce to

$$E_\varepsilon(u) := \int_\Omega \left[\tilde{W}(u) + \varepsilon^2 |\nabla u|^2 \right] dx$$

Use F. and Tartar (1989)

$u \in BV(\Omega; \{a, b\})$, $A := \{u = a\}$, $\varepsilon_n \to 0^+$

Claim: there exists $\{u_n\} \subset H^1(\Omega; \mathbb{R}^d)$ s.t. $u_n \to u$ in L^1 and

$$\limsup F_{\varepsilon_n}(u_n) \leq F(u) = \int_{\partial^* A} \sigma(\nu) d\mathcal{H}^{N-1}$$

Localization: for $U \subset \Omega$ open

$$\mathcal{F}_{\{\varepsilon_n\}}(u; U) := \inf\{\liminf F_{\varepsilon_n}(u_n, U) : u_n \to u \text{ in } L^1(U; \mathbb{R}^d)\}$$

Up to a subsequence

$$\lambda : \mathcal{A}(\Omega) \to [0, +\infty), \qquad \lambda(B) := \mathcal{F}_{\{\varepsilon_n\}}(u; B), \quad B \text{ Borel set}$$

is a positive finite measure, and

$$\lambda << \mu := \mathcal{H}^{N-1} \lfloor \partial^* A$$

Done if

$$\frac{d\lambda}{d\mu}(x_0) \le \sigma(\nu(x_0))$$

To prove it:

$$\frac{d\lambda}{d\mu}(x_0) = \lim \frac{\lambda(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}$$

$$\frac{\lambda(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \le \liminf_{n\to\infty} \frac{1}{\varepsilon^{N-1}} F_{\varepsilon_n}(u_{n,\varepsilon}, Q_\nu(x_0,\varepsilon))$$

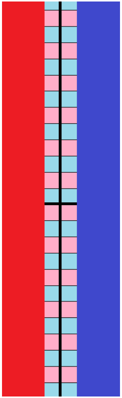
with $u_{n,\varepsilon} \to u$, $n \to \infty$, in $L^1(Q_\nu(x_0,\varepsilon))$

How do we construct these approximating sequences?

Easy Case: Transition Layer Aligned with Principal Axes

If $\nu \in \{e_1, \ldots, e_N\}$, create recovery sequence by tiling optimal profiles from definition of σ.

Say $\nu = e_N$



Pick $T_k \subset \mathbb{N}$ and u_k s.t.

$$\sigma(e_N) = \lim_{k\to\infty} \frac{1}{T_k^{N-1}} \int_{T_k Q} \left[W(y, u_k(y)) + |\nabla u_k(y)|^2\right] dy$$

$$v_k(x) := u_k(T_k x), \text{ extended by } Q'\text{-periodicity}$$

$$v_{k,\varepsilon,r}(x) := \begin{cases} u_0(x) & |x_N| \geq \frac{\varepsilon T_k}{2r} \\ v_k\left(\frac{rx}{\varepsilon T_k}\right) & |x_N| < \frac{\varepsilon T_k}{2r} \end{cases}$$

$$u_{k,\varepsilon,r}(x) := v_{k,\varepsilon,r}\left(\frac{x}{r}\right) \to u \text{ in } L^1(rQ)$$

Transition Layer Aligned with Principal Axes, cont.

Blow up:

$$
\begin{aligned}
\lim_{r\to 0}\frac{F(u;rQ)}{r^{N-1}} &\le \lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{1}{r^{N-1}}\int_{rQ}\left[\frac{1}{\varepsilon}W(x,u_{k,\varepsilon,r})+\varepsilon|\nabla u_{k,\varepsilon,r}|^2\right]dx \\
&= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-\varepsilon T_k/2r}^{\varepsilon T_k/2r}\left[\frac{r}{\varepsilon}W\left(\frac{r}{\varepsilon}y, v_k\left(\frac{ry}{\varepsilon T_k}\right)\right)\right. \\
&\qquad\qquad \left. + \frac{r}{\varepsilon T_k^2}\left|\nabla v_k\left(\frac{ry}{\varepsilon T_k}\right)\right|^2\right]dy \\
&= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N, v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right)\right)\right. \\
&\qquad\qquad \left. + \frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\right]dz
\end{aligned}
$$

$$\lim_{r\to 0}\frac{F(u;rQ)}{r^{N-1}} \le \lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{1}{r^{N-1}}\int_{rQ}\left[\frac{1}{\varepsilon}W(x,u_{k,\varepsilon,r})+\varepsilon|\nabla u_{k,\varepsilon,r}|^2\right]dx$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-\varepsilon T_k/2r}^{\varepsilon T_k/2r}\left[\frac{r}{\varepsilon}W\left(\frac{r}{\varepsilon}y, v_k\left(\frac{ry}{\varepsilon T_k}\right)\right)\right.$$

$$\left. + \frac{r}{\varepsilon T_k^2}\left|\nabla v_k\left(\frac{ry}{\varepsilon T_k}\right)\right|^2\right]dy$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\left[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N, v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right)\right.\right.$$

$$\left. + \frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\right]dz$$

Transition Layer aligned with Principal Axes, cont.

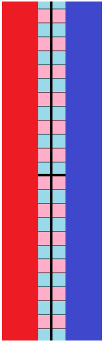
Since W and v_k are **BOTH** Q'-periodic and $T_k \in \mathbb{N}$, we can use the Riemann Lebesgue Lemma:

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{Q'}\int_{-1/2}^{1/2}\Bigg[T_k W\left(\left(T_k\frac{rz'}{\varepsilon T_k}, T_k z_N\right), v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right) + \frac{1}{T_k}\left|\nabla v_k\left(\frac{rz'}{\varepsilon T_k}, z_N\right)\right|^2\Bigg]dz$$

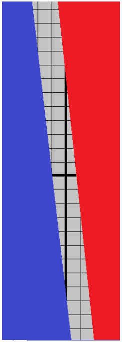
$$= \lim_{r\to 0}\int_{Q'}\int_{-1/2}^{1/2}\Bigg[T_k W((T_k y', T_k z_N), v_k(y', z_N)) + \frac{1}{T_k}|\nabla v_k(y', z_N)|^2 dz_N\Bigg]dy'$$

$$= \frac{1}{T_k^{N-1}}\int_{T_k Q}\left[W(x, u_k(x)) + |\nabla u_k(x)|^2\right]dx$$

Other Transition Directions?



(a)
Aligned



(b)
Misaligned

Figure : Since W is Q-periodic, can tile along principal axes. What if the transition layer is not aligned?

Q-periodic implies $\lambda_\nu Q_\nu$-periodic

Key observation: Periodic microstructure in principal directions $\rightarrow$ periodicity in other directions.

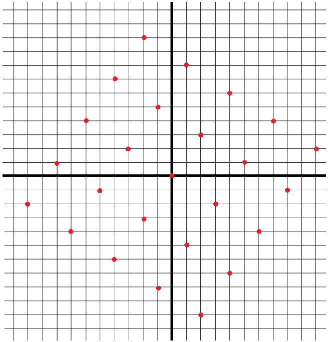


Figure : Integer lattice contains copies of itself, rotated and scaled

$\triangleright$ W is $\lambda_\nu Q_\nu$-periodic for some $\lambda_\nu \in \mathbb{N}$, and for $\nu \in \Lambda := \mathbb{Q}^N \cap \mathbb{S}^{N-1}$: Dense!

Orthonormal Bases in $\mathbb{Q}^N$

Important: Every face of Q_ν has rational normal.

Need an orthonormal basis using rational vectors:

Theorem (Witt, '37)

Any isometry between two subspaces F_1 and F_2 of a finite-dimensional vector space V defined over a field $\mathbb{K}$ of characteristic different from 2 and provided with a metric structure induced from a nondegenerate symmetric or skew-symmetric bilinear form $B[\cdot,\cdot]$ may be extended to a metric automorphism of the entire space V.

In particular:

$$V = \mathbb{Q}^N,\ F_1 := \operatorname{span}_{\mathbb{Q}}(e_N),\ F_2 := \operatorname{span}_{\mathbb{Q}}(\nu),\ B[x,y] := x \cdot y$$

Then, the mapping $e_N \mapsto \nu$ extends to an isometry!

Theorem (Cristoferi, F., Hagerty, Popovici, *Interfaces Free Bound.*(2019))

Let $\nu_N \in \Lambda = \mathbb{Q}^N \cap \mathbb{S}^{N-1}$. *There exist* $\nu_1, \ldots, \nu_{N-1} \in \Lambda$, $\lambda_\nu \in \mathbb{N}$, *s.t.*

$$\nu_1, \ldots, \nu_{N-1}, \nu_N$$

o.n. basis of $\mathbb{R}^N$ *and*

$$W(x + n\lambda_\nu \nu_i, p) = W(x, p)$$

a.e. $x \in Q$, *all* $n \in \mathbb{N}$, $p \in \mathbb{R}^d$.

Also use:

$\varepsilon > 0$, $\nu \in \Lambda$, $S : \mathbb{R}^N \to \mathbb{R}^N$ rotation, $Se_N = \nu$.

Then there is a rotation $R : \mathbb{R}^N \to \mathbb{R}^N$ s.t. $Re_N = \nu$, $Re_i \in \Lambda$ all $i = 1, \ldots, N-1$, $||R - S|| < \varepsilon$

Properties of σ are important

$$\sigma(\nu) = \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{Q_\nu,T}, Q_\nu\in\mathcal{Q}_\nu} \left\{ \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[W(y,u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where

$$\mathcal{A}_{Q_\nu,T} := \left\{ u \in H^1(TQ_\nu;\mathbb{R}^d) : u(x) = (\rho_T * u_0)(x\cdot\nu) \text{ on } \partial TQ_\nu \right\}$$

$$u_0(t) := \begin{cases} b & \text{if } t > 0 \\ a & \text{if } t < 0 \end{cases}$$

$$\rho_T(x) := T^N \rho(Tx),\ \rho \in C_c^\infty(\mathbb{R}) \text{ with } \int_{\mathbb{R}} \rho = 1$$

$\mathcal{Q}_\nu$ . . . unit cubes centered at the origin with two faces orthogonal to ν

Properties of σ (before knowing it is constant!):

- σ is well defined and finite

- the definition of σ does not depend on the choice of the mollifier

- $\sigma : \mathbb{S}^{N-1} \to [0, +\infty)$ is upper semicontinuous

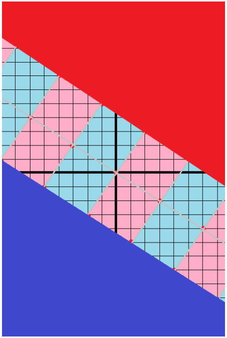
- if $\nu \in \Lambda$ then

$$\sigma(\nu) = \lim_{n\to\infty} \lim_{T\to\infty} \inf_{u\in\mathcal{A}_{Q_n,T}} \left\{ \frac{1}{T^{N-1}} \int_{TQ_n} \left[W(y, u(y)) + |\nabla u(y)|^2 \right] dy \right\}$$

where the normals to all faces of Q_n belong to Λ

Transition Layer aligned with $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$

Same periodic tiling technique: Use $T_k \in \lambda_\nu \mathbb{N}$.



▷ Blow up method → Recovery sequences for polyhedral sets A with normals to its facets in Λ

Blow up method $\to$ Recovery sequences for polyhedral sets A with normals to its facets in Λ

$x_0 \in \Omega \cap \partial^* A$, $\nu := \nu_A(x_0)$.

Find rotation R_ν, $\lambda_\nu \in \mathbb{N}$, s.t. with $Q_\nu := R_\nu(x_0 + Q)$

$$W(x + n\lambda_\nu v, p) = W(x, p)$$

a.e. $x \in \Omega$, every $n \in \mathbb{N}$, every $p \in \mathbb{R}^d$, every v orthogonal to one face of Q_ν

As before, done if

$$\frac{d\lambda}{d\mu}(x_0) \leq \sigma(\nu(x_0))$$

To prove it:

$$\frac{d\lambda}{d\mu}(x_0) = \lim \frac{\lambda(Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}}$$

...and work a little harder ...

Recovery sequences for arbitrary $u \in BV(\Omega; \{a, b\})$

- For $u \in BV(\Omega; \{a, b\})$, we can find $u^{(n)} \in BV(\Omega; \{a, b\})$ such that $A^{(n)}$ are polyhedral,

$$u^{(n)} \to u \text{ in } L^1$$

$$|Du^{(n)}|(\Omega) \to |Du|(\Omega).$$

  Since $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$ dense, can require $\nu^{(n)} \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$.

- Since σ upper-semicontinuous, by Reshetnyak's,

$$\int_{\partial^* A} \sigma(\nu) d\mathcal{H}^{n-1} \leq \limsup_{n\to\infty} \int_{\partial^* A_0^{(n)}} \sigma\left(\nu^{(n)}\right) d\mathcal{H}^{n-1}$$

- Find recovery sequences $u_\varepsilon^{(n)}$ for the $u^{(n)}$ so that

$$\int_{\partial^* A^{(n)}} \sigma\left(\nu^{(n)}\right) d\mathcal{H}^{n-1} \leq \limsup_{\varepsilon\to 0^+} F_\varepsilon\left(u_\varepsilon^{(n)}\right)$$

- Diagonalize!

Other scaling regimes

Recently considered the case where the scale of homogenization is much **smaller** than the scale of the phase transition

$$F_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\delta(\varepsilon)}, u\right) + \varepsilon |\nabla u|^2\right] dx.$$

If $\delta(\varepsilon)$ is **sufficiently small** compared to ε, the homogenization effects are effectively **instantaneous**, and we can pass to a homogenized system

$$F_\varepsilon^H(u) = \int_\Omega \left[\frac{1}{\varepsilon} W_H(u) + \varepsilon |\nabla u|^2\right] dx$$

where

$$W_H(p) := \int_Q W(y,p) dy$$

Scaling regime $\delta(\varepsilon) << \varepsilon$

Theorem (Cristoferi, F., Hagerty (2019))

Let $\delta(\varepsilon)$ *be such that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{\frac{3}{2}}}{\delta(\varepsilon)} = +\infty.$$

Then, $F_\varepsilon \xrightarrow{\Gamma} F_0^H$*, where*

$$F_0^H(u) := \begin{cases} K_H \ \mathrm{Per}_\Omega(A) & u \in BV(\Omega; \{a, b\}), \\ +\infty & u \in L^1(\Omega) \setminus BV(\Omega; \{a, b\}) \end{cases}$$

$W_H(p) := \int_Q W(y, p)\, dy$, $A := \{u(x) = a\}$

$$K_H := 2 \inf \left\{ \int_0^1 \sqrt{W_H(g(s))} |g'(s)| ds : g \text{ piecewise } C^1, g(0) = a, g(1) = b \right\}$$

Outline of Proof

- Homogenization Lemma
  - Compare the bulk energy to a homogenized bulk energy
  - Requires quantitative control on δ vs ε
- Use the result of F. and Tartar to identify Γ-limit of homogenized energy
  - Comparison with homogenized energy yields information about minimizing sequences $\to$ relaxed growth assumptions for W

Theorem (F., Tartar (1989))

Functionals of the form

$$G_\varepsilon(u) = \int_\Omega \left[\frac{1}{\varepsilon} \widetilde{W}(u) + \varepsilon |\nabla u|^2 \right] dx, \ u \in H^1(\Omega; \mathbb{R}^d)$$

have a Γ-limit

$$G_0(u) := K_G \, P(A_0; \Omega), \ u \in BV(\Omega; \{a, b\})$$

Homogenization Lemma

The key tool in comparing F_ε and F_ε^H is a Riemann-Lebesgue type result for all W uniformly bounded.

Lemma

Let ε_n, δ_n *and* $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ *be such that*

$$\sup_{n\in\mathbb{N}} \int_\Omega \varepsilon_n |\nabla u_n|^2 dx < \infty \text{ and } \lim_{n\to\infty} \varepsilon_n^{-\frac{3}{2}} \delta_n = 0.$$

Then,

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n} \int_\Omega \left[W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_H(u_n(x)) \right] dx = 0$$

- Uniform boundedness: NOT required for the main theorem- will be discussed later
- Scaling: More on this...

Scaling

- The homogenization lemma requires a particular exponent $\varepsilon^{\frac{3}{2}}$
  - If the regularization is of the form $|\nabla u|^p$, the exponent would be $\varepsilon^{1+\frac{1}{p}}$.
- This same exponent is necessary Ansini, Braides, Chiadò-Piat (2003) who homogenized the regularization term
- Unclear if this is purely technical or if truly different behavior is possible in the intermediate regime

Homogenization Lemma - Outline of Proof

At scale δ_n, decompose Ω into δ_n-cubes and a remainder R_n

$$\Omega = \bigcup_{i=1}^{M_n} Q(p_i, \delta_n) \cup R_n,$$

where p_i are on the lattice $\delta_n \mathbb{Z}^N$
R_n . . . collection of cubes $Q(z, \delta_n)$, $z \in \delta_n \mathbb{Z}^N$, intersecting $\partial\Omega$

$$|R_n| \leq C\delta_n$$

Uniform boundedness:

$$\frac{1}{\varepsilon_n} \int_{R_n} W\left(\frac{x}{\delta_n}, u_n(x)\right) dx \leq C \frac{\delta_n}{\varepsilon_n} \to 0$$

Homogenization Lemma - Outline of Proof, cont.

Sufficient to control

$$\frac{1}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_{Q(p_i,\delta_n)} W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_H(u_n(x))dx\right|$$

Apply the substitution $x = p_i + \delta_n y$ and periodicity:

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_Q W(y, u_n(p_i+\delta_n y)) - W_H(u_n(p_i+\delta_n y))dy\right|$$

Recast as the double integral

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_Q\int_Q W(y, u_n(p_i+\delta_n y)) - W(z, u_n(p_i+\delta_n y))dzdy\right|$$

Homogenization Lemma - Outline of Proof, cont.

After another change of variables, this is

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\left|\int_Q\int_Q W(y,u_n(p_i+\delta_n y))-W(y,u_n(p_i+\delta_n z))dzdy\right|$$

and by Lipschitz behavior of W, enough to control

$$\frac{\delta_n^N}{\varepsilon_n}\sum_{i=1}^{M_n}\int_Q\int_Q |u_n(p_i+\delta_n y)-u_n(p_i+\delta_n z)|\,dzdy$$

By Poincaré, we can estimate via

$$\begin{aligned}\delta_n\frac{\delta^N}{\varepsilon_n}\sum_{i=1}^{M_n}\int_Q|\nabla u_n(p_i+\delta_n y)|dy &\le \frac{\delta_n}{\varepsilon_n}\int_\Omega|\nabla u_n|dx\\ &\le \frac{\delta_n}{\varepsilon_n}\varepsilon_n^{-1/2}\left(\varepsilon_n\int_\Omega|\nabla u_n|^2dx\right)^{1/2}\end{aligned}$$

Uniform Boundedness

To apply the homogenization lemma to potentials which may be unbounded, we use a cut-off trick- possible because by F.-Tartar, the homogenized problem is based on the 1-dimensional optimization

$$K_H = 2\inf\left\{\int_0^1 \sqrt{W_H(g(s))}|g'(s)|ds\right\}$$

where the g are pointwise C^1 so that $g(0) = a,\ g(1) = b$. Pick $R > 0$ so that for optimal curves g, $|g(t)| \leq R$. Let

$$M = \operatorname*{ess\,sup}_{x\in\Omega} \max_{|p|\leq R} W(x,p)$$

and define the truncated potential

$$\widetilde{W}(x,p) := \min\{W(x,p), M\}$$

Gradient Flow: Current work with Rustum Choksi, Jessica Lin and Raghavendra (Raghav) Venkatraman

L^2-gradient flow of F_ε:

$$F_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} a\left(\frac{x}{\varepsilon}\right) \overline{W}(u^\varepsilon) + \varepsilon \left|\nabla u^\varepsilon\right|^2 \right] dx.$$

$W(y,u) := a(y)\overline{W}(u)$
$a : \mathbb{R}^N \to [\lambda, \Lambda]$, $0 < \lambda < \Lambda$, C^2 and periodic
$\{\overline{W} = 0\} = \{-1, 1\}$ C^2 double-well potential

$$\begin{cases} u_t^\varepsilon - 2\Delta u^\varepsilon = -\frac{1}{\varepsilon^2} a\left(\frac{x}{\varepsilon}\right) \overline{W}'(u^\varepsilon) & \text{in } (0,\infty) \times \Omega, \\ u^\varepsilon(0,x) \approx \chi_A - \chi_{\overline{A}^c} & \text{in } \Omega, \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } (0,\infty) \times \partial\Omega, \end{cases}$$

∂A ... interface

To show: u^ε converge to a $1, -1$ sharp interface limit which is governed by the mean curvature equation

$$\begin{cases} \overline{u}_t - \sigma \text{div}\left(\frac{D\overline{u}}{|D\overline{u}|}\right)|D\overline{u}| = 0 & \text{in } (0,\infty)\times\Omega, \\ \overline{u}(0,x) = \chi_A - \chi_{\overline{A}^c} & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,\infty)\times\partial\Omega \end{cases}$$

Recall: $F_\varepsilon \xrightarrow{\Gamma - L^1} F_0$ where

$$F_0(u) = \begin{cases} \int_{\partial^* A} \sigma(\nu_A(x))\, d\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

for $\sigma : \mathbb{S}^{N-1} \to [0, +\infty)$ given by the cell formula (AND constant!)

$$\sigma(\nu) := \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \left[a(y)\overline{W}(u(y)) + |\nabla u(y)|^2\right] dy : u \in \mathcal{A}(\nu, T)\right\}$$

The PDE now becomes:

$$u_t^\varepsilon = -\nabla_{X_\varepsilon} F_\varepsilon(u).$$

with

$$\nabla_{X_\varepsilon} F_\varepsilon(u) = -2\Delta u + \frac{1}{\varepsilon^2} a\left(\frac{x}{\varepsilon}\right) \overline{W}'(u),$$

and $\| \cdot \|_{X_\varepsilon}^2 := \varepsilon \| \cdot \|_{L^2(\Omega)}^2$

Ideas from : Sandier-Serfaty, Mugnai-Röger, Röger- Schäzle

Many references when $a = 1$, including:
Alikakos-Bates-Chen, Xinfu Chen, Bronsard- Kohn, Rubinstein-Sternberg-Keller, Ilmanen, Tonegawa-Hutchinson, Tonegawa, Tim Laux and Thilo Simon, Evans-Soner-Souganidis, Lions-Souganidis

Future problems

- Moving wells
- Scaling regime $\varepsilon << \delta(\varepsilon)$ ... homogenization of the "surface Cahn-Hilliard limiting energy". Forthcoming
- multiple wells
- More general regularization terms, i.e. $|\nabla u|^2 \to f(x, u, \nabla u)$
- Nonlocal stochastic homogenization
- Solid-solid phase transitions: $W\left(\frac{x}{\delta(\varepsilon)}, \nabla u(x)\right)$

Solid-sold phase transitions without homogenization:

$$W(F) \approx |F|^p, \text{ Conti, Fonseca, Leoni, '02.}$$

$$W(F) \approx \mathrm{dist}^p(F, SO(N)A \cup SO(N)B)$$

only studied for N=2 (Conti–Schweizer, '06) ... and in arbitrary dimensions under a suitable anisotropic penalization of second variations **Elisa Davoli and Manuel Friedrich, 2018**

Something funny about moving wells . . .

$$W(x, p) = 0 \text{ iff } p \in \{a(x), b(x)\}$$

$\{u_\varepsilon\}$ with bounded energy, so that

$$\frac{1}{\varepsilon} \int_\Omega W\left(\frac{x}{\varepsilon}, u_\varepsilon(x)\right)\, dx < +\infty$$

Now, if $\{u_\varepsilon\}$ has a L^1 limit, then its **2-scale limit** $u(x, y)$ is actually just $u(x)$, and so

$$\int_\Omega W\left(\frac{x}{\varepsilon}, u_\varepsilon(x)\right)\, dx \to \int_Q \int_\Omega W(y, u(x))\, dx\, dy = 0$$

But then

$$W(y, u(x)) = 0 \text{ for almost every } (x, y) \in \Omega \times Q$$

Something funny about moving wells ...

$$W(y, u(x)) = 0 \text{ for almost every } (x, y) \in \Omega \times Q$$

and so

$$u(x) \in \{a(y), b(y)\} \text{ for almost every } (x, y) \in \Omega \times Q$$

... basically $\{a(y), b(y)\} = \{a(y'), b(y')\}$ a.e. ... **NOT** moving wells ...

wrong scaling?

(without homogenization) sharp interface limit $W(x, p) = 0$ iff $p \in \{z_1(x), z_2(x), \ldots, z_k(x)\}$ by **Riccardo Cristoferi and Giovanni Gravina, 2020**

HAPPY BIRTHDAY GIANNI!

A good place to stop . . .