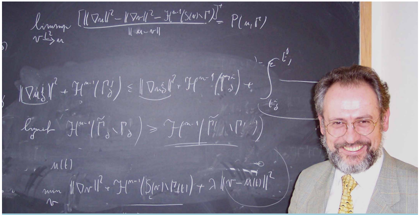
Linear (and nonlinear) Dirichlet problems with singular convection/drift terms

LUCIO BOCCARDO

(Istituto Lombardo)

CALCULUS OF VARIATIONS AND APPLICATIONS



AN INTERNATIONAL CONFERENCE TO CELEBRATE GIANNI DAL MASO'S 65^{th} BIRTHDAY

This workshop will focus on the most recent developments and achievements in a broad range of topics in the Calculus of Variations, with an emphasis on its applications to material science and imaging.
It will provide an exceptional opportunity to present the state of the art of modern methods in the Calculus of Variations and their applications and to stimulate exchange of ideas and knowledge, through a rich selection of talks by leading experts in the field.
The conference is an occasion to celebrate the 65th birthday of Gianni Dal Maso and his deep and vast contribution to the Calculus of Variations, from its most fundamental theoretical foundations to its applications to solid mechanics, homogenization, fracture, plasticity, and imaging.
It will also be an opportunity to bring together a large community of former students, postdocs, collaborators, and friends who have benefited of Gianni's knowledge and mathematical advice throughout the years.

Speakers
G. Alberti, Univ. Pisa
L. Ambrosio, SNS Pisa
L. Boccardo, Sapienza Univ. Roma
G. Bouchitté, Univ. Toulon
A. Braides, Univ. Roma Tor Vergata
G. Buttazzo, Univ. Pisa
A. Chambolle*, École Polytechnique, Paris
S. Conti, Univ. Bonn
G. De Philippis, SISSA and Courant Institute
A. De Simone, SISSA and Scuola Superiore Sant'Anna
G. Dell'Antonio, Sapienza Univ. Roma and SISSA
I. Fonseca, Carnegie Mellon Univ., Pittsburgh
G. Francfort, Univ. Paris 13
H. Frankowska, Sorbonne Univ.
N. Fusco, Univ. Napoli
C. Larsen, Worcester Polytechnic Institute
G. Leoni, Carnegie Mellon Univ., Pittsburgh
P. Marcellini, Univ. Firenze
A. Mielke, WIAS and Humboldt Univ., Berlin
J. Morel, ENS Cachan
F. Murat, Sorbonne Univ.
G. Savare, Univ. Pavia
S. Sbordone, Univ. Napoli
* to be confirmed

Organisers
V. Chiadò-Piat, Politecnico di Torino
A. Garroni, Sapienza Univ. Roma
N. Gigli, SISSA
M.G. Mora, Univ. Pavia
F. Solombrino, Univ. Napoli Federico II
R. Toader, Univ. Udine

Where: SISSA
Scuola Internazionale Superiore di Studi Avanzati
Via Bonomea 265
Trieste, Italy

When: 27 January – 1 February 2020

For registration: → toader@uniud.it

Info: → gianni65.weebly.com

SISSA

Sponsors

*Thanks for the invitation + excellent organization **

Good morning

Buon giorno

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Good morning

Buon giorno

and thanks to the organizers:

V. Chiadò-Piat,
A. Garroni,
N. Gigli,
M.G. Mora,
F. Solombrino,
R. Toader.

ma per seguir virtute e canoscenza
SISSA

Papers concerned with. first part of the talk

L. Boccardo: Some developments on Dirichlet problems with discontinuous coefficients; *Boll. Unione Mat. Ital*, 2 (2009) 285–297.
(invited paper in memory of 30-death Stampacchia)

L. Boccardo: Dirichlet problems with singular convection terms and applications; *J. Differential Equations*, 258 (2015) 2290–2314.

L. Boccardo: Stampacchia-Calderon-Zygmund theory for linear elliptic equations with discontinuous coefficients and singular drift; *ESAIM, Control, Optimization and Calculus of Variations*, 25 (2019), Art. 47, 13 pp.

Good smile

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We discuss the existence of distributional solutions for the boundary value problems (the first with a convection term, the second with a drift term)

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$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(u\,E(x)) + f(x) & \textbf{in } \Omega, \\ u = 0 & \textbf{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\mathrm{div}(M(x)\nabla \psi) = E(x)\,\nabla\psi + g(x) & \textbf{in } \Omega, \\ \psi = 0 & \textbf{on } \partial\Omega, \end{cases}$$

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We note that

- **at least formally, if $M(x)$ is symmetric, the two above linear problems are in duality.**

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We note that

- **the differential operators may be not coercive, unless one assumes that either the norm of $|E|$ in $L^N(\Omega)$ is small, or that $\mathrm{div}(\|E\|_N) = 0$: ...**

Coercivity of $\quad -\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(u\,E(x)) + f(x)$

Assumptions

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = -\mathrm{div}(u\,E(x)) + f(x) & : \Omega, \\ u = 0 & : \partial\Omega. \end{cases}$$

- Ω **bounded subset of** $\mathbb{R}^N$,

[1] **1: dependence w.r.t. x / 2: nonsmooth dependence / Mingione**

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Boundary value problem and Lax-Milgram

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u weak solution of the boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u)) = -\mathrm{div}(u\,E(x)) + f(x) & : \Omega, \\ u = 0 & : \partial\Omega. \end{cases}$$

means

$$u \in W_0^{1,2}(\Omega) : \int_\Omega M(x)\nabla u \nabla v = \int_\Omega u\,E(x)\nabla v + \int_\Omega f(x)v, \ \forall\ v \in W_0^{1,2}(\Omega).$$

Coercivity of $-\mathrm{div}(M(x)\nabla u) + \mathrm{div}(u\, E(x))$

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$$\geq \alpha \int_\Omega |\nabla v|^2 - \Big[\int_\Omega |v|^{2^*}\Big]^{\frac{1}{2^*}} \Big[\int_\Omega |E(x)|^N\Big]^{\frac{1}{N}} \Big[\int_\Omega |\nabla v|^2\Big]^{\frac{1}{2}}$$

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- $E \in L^N$
- $\|E\|_{L^N}$ **not too large**

Our approach hinges on test function method

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The proofs of all the results are very easy

if we assume $\mathrm{div}(E) = 0$

Stampacchia-Calderon-Zygmund for the two problems

[2] paper invitation U.M.I. in memory of 30-Stampacchia

[3] ESAIM-COCV 2019

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and we prove for both the b.v.p. the same Stampacchia-Calderon-Zygmund results of the case

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and we prove for both the b.v.p. the same Stampacchia-Calderon-Zygmund results of the case $E = 0$.

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[3] ESAIM-COCV 2019

$|E| \in L^N(\Omega)$

as $E = 0$

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$$u : \int_\Omega M\nabla u \nabla\phi = \int_\Omega u\,E\nabla\phi + \int_\Omega f\phi, \;\forall\,\phi \in \mathcal{D}.$$

as $E = 0$

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Theorem (70-Brezis)

$E = 0$, $m > \frac{N}{2}$, it is false that $u \in W_0^{1,m^*}(\Omega)$

$|E| \in L^N(\Omega)$

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Remark

$E = 0,\ \frac{2N}{N+2} + \delta_{Meyers} < m < \frac{N}{2},\ u \in ?$

Same results for the drift problem

[4] **ESAIM-COCV 2019**

Same results for the drift problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla\psi) = E(x)\,\nabla\psi + g(x) & \textbf{in } \Omega, \\ \psi = 0 & \textbf{on } \partial\Omega, \end{cases}$$ [4]

[4] **ESAIM-COCV 2019**

Other recent papers

L. Boccardo, S. Buccheri, G.R. Cirmi: Two linear noncoercive Dirichlet problems in duality; Milan J. Math. 86 (2018), 97–104.

L. Boccardo, S. Buccheri, R.G. Cirmi: Calderon-Zygmund theory for infinite energy solutions of nonlinear elliptic equations with singular drift; preprint.

L. Boccardo, S. Buccheri: A nonlinear homotopy between two linear Dirichlet problems; preprint.

L. Boccardo: Two semilinear Dirichlet problems "almost" in duality; Boll. Unione Mat. Ital. 12 (2019), 349–356.

L. Boccardo, L. Orsina, A. Porretta: Some noncoercive parabolic equations with lower order terms in divergence form. Dedicated to Philippe Bénilan. J. Evol. Equ. 3 (2003), 407–418.

~~$E \in (L^N(\Omega))^N$~~

If $E \notin (L^N(\Omega))^N$, even for nothing, as in

$\boxed{|E| \le \frac{|A|}{|x|}, \quad A \in \mathbb{R}}$**, $0 \in \Omega$,**

[5] **JDE 2015, Nonlin.Anal. 2019**

If $E \notin (L^N(\Omega))^N$, even for nothing, as in

$$\boxed{|E| \le \frac{|A|}{|x|}, \quad A \in \mathbb{R}}, \ 0 \in \Omega,$$

the framework changes completely:
$u \in W_0^{1,2}(\Omega)$ **or** $u \in W_0^{1,q}(\Omega)$ **depends on the size of A.** [5]

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1) **if** $|A| < \frac{\alpha(N-2m)}{m}$**, and** $\frac{2N}{N+2} \le m < \frac{N}{2}$**, then** $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$**;**
2) **if** $|A| < \frac{\alpha(N-2m)}{m}$**, and** $1 < m < \frac{2N}{N+2}$**, then** $u \in W_0^{1,m^*}(\Omega)$**;**
3) **if** $|A| < \alpha(N-2)$**, and** $m = 1$**, then** $\nabla u \in (M^{\frac{N}{N-1}}(\Omega))^N$ **and** $u \in W_0^{1,q}(\Omega)$**, for every** $q < \frac{N}{N-1}$**;**
4 **if** $\alpha(N-2) \le |A| < \alpha(N-1)$**, then** $u \in W_0^{1,q}(\Omega)$**, for every** $q < \frac{N\alpha}{|A|+\alpha}$

[5] **JDE 2015, Nonlin.Anal. 2019**

Radial ex.

~~$E \in (L^2(\Omega))^N$~~

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[6] **JDE 2015**

$E \in (L^2(\Omega))^N$

$$\begin{cases} \textit{definition of solution;} \\ \textit{existence of solution.} \end{cases}$$ [6]

[6] **JDE 2015**

If we add the zero order term "$+u$", the framework changes completely.

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A, $u_n \in W_0^{1,2}(\Omega)$:

$$-\mathrm{div}(M(x)\nabla u_n) + \mathbf{A}\, u_n = -\mathrm{div}\Big(\frac{u_n}{1+\frac{1}{n}|u_n|}E(x)\Big) + f(x)$$

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$$\mathbf{A} \int_\Omega |u_n| \le \int_\Omega |f|$$

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! only (again)

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Third estimate

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! only (again) $E \in L^2$ **is needed.**

By duality: problems with very singular drifts

[7] **DIE 2019**
[8] **only L^2**

By duality: problems with very singular drifts

$$\begin{cases} -\mathrm{div}(M(x)\nabla\psi) + \psi = E(x)\,\nabla\psi + g(x) & \textbf{in } \Omega, \\ \psi = 0 & \textbf{on } \partial\Omega, \end{cases}$$ [7]

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By duality: problems with very singular drifts

$$\begin{cases} -\mathrm{div}(M(x)\nabla\psi) + \psi = E(x)\,\nabla\psi + g(x) & \textbf{in } \Omega, \\ \psi = 0 & \textbf{on } \partial\Omega, \end{cases}$$ [7]

- $E \in L^2$[8], f **bounded** $\Rightarrow u \in W_0^{1,2}(\Omega)$, **bounded.**

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[7] **DIE 2019**
[8] **only** L^2

An elliptic system connected with the mathematical study of PDE models for chemotaxis

[9] **JDE 2015**
[10] **Comm.PDE + L. Orsina**

An elliptic system connected with the mathematical study of PDE models for chemotaxis

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + u = -\mathrm{div}(u\, M(x)\nabla \psi) + f(x)\,, \\ -\mathrm{div}(M(x)\nabla \psi) = u^{\theta}\,. \end{cases}$$ [9][10]

[9] JDE 2015

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~~$\|E\|_{L^N}$ *not too large*~~

Nonlinear approach to linear problems

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$$u_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla u_n) = -\mathrm{div}\Big(\frac{u_n}{1+\frac{1}{n}|u_n|}\,E(x)\Big) + f(x)$$

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Existence u_n: Schauder

Log-estimate (first estimate)

$$u_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla u_n) = -\mathrm{div}\Big(\frac{u_n}{1+\frac{1}{n}|u_n|}\,E(x)\Big)+f(x)$$

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Truncation-estimate (second estimate)

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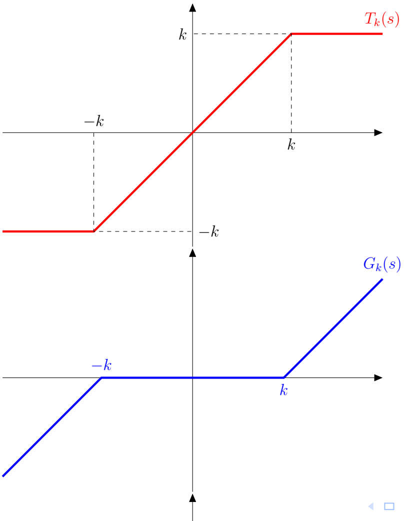

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$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \le \int_\Omega |u_n|\,|E||\nabla T_k(u_n)| + \int_\Omega f T_k(u_n)$$
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Two important estimates

$$u_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla u_n) = -\mathrm{div}\Big(\frac{u_n}{1+\frac{1}{n}|u_n|}\,E(x)\Big) + f(x)$$

$$\Big[\int_\Omega |\log(1+|u_n|)|^{2^*}\Big]^{\frac{2}{2^*}} \leq \frac{1}{S^2\alpha^2}\int_\Omega |E|^2 + \frac{2}{S^2\alpha}\int_\Omega |f|^1$$

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^2 \leq \frac{2}{\alpha}k^2\int_\Omega |E|^2 + k\int_\Omega |f|^1$$

Proof of

$$\|G_k(u_n)\|_{W_0^{1,2}(\Omega)} \leq C(E, f, \alpha), \quad k > k_E(\|E\|_N)$$

Let $\delta > 0$. Use $G_k(u_n)$ as test f.; Young, Hölder, Sob.$\Rightarrow$

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega |G_k(u_n)||E||\nabla G_k(u_n)| + k\int_\Omega |E||\nabla G_k(u_n)| + \int_\Omega |G_k(u_n)||f|$$

$$\leq \frac{1}{\mathcal{S}}\Big(\int_{k<|u_n|} |E|^N\Big)^{\frac{1}{N}} \int_\Omega |\nabla G_k(u_n)|^2 + \delta\int_\Omega |\nabla G_k(u_n)|^2 + \frac{k^2}{4\delta}\int_{k<|u_n|} |E|^2$$

$$+\delta\int_\Omega |\nabla G_k(u_n)|^2 + \frac{\mathcal{S}^2}{4\delta}\left[\int_{k<|u_n|} |f|^{\frac{2N}{N+2}}\right]^{\frac{N+2}{N}},$$

$$\left[\alpha - \frac{1}{\mathcal{S}}\left(\int_{k<|u_n|} |E|^N\right)^{\frac{1}{N}} - 2\delta\right]\int_\Omega |\nabla G_k(u_n)|^2$$

$$\leq \frac{k^2}{4\delta}\int_{k<|u_n|} |E|^2 + \frac{\mathcal{S}^2}{4\delta}\left[\int_{k<|u_n|} |f|^{\frac{2N}{N+2}}\right]^{\frac{N+2}{N}}.$$

$$\left[\alpha - \frac{1}{\mathcal{S}}\left(\int_{k<|u_n|}|E|^N\right)^{\frac{1}{N}} - 2\delta\right]\int_\Omega |\nabla G_k(u_n)|^2$$

$$\leq \frac{k^2}{4\delta}\int_{k<|u_n|}|E|^2 + \frac{\mathcal{S}^2}{4\delta}\left[\int_{k<|u_n|}|f|^{\frac{2N}{N+2}}\right]^{\frac{N+2}{N}}.$$

Fix δ so that $2\delta = \frac{\alpha}{4}$. Then log-estimate implies that there exists k_E, such that

$$\frac{1}{\mathcal{S}}\left[\int_{k<|u_n|}|E|^N\right]^{\frac{1}{N}} \leq \frac{\alpha}{4}, \quad k \geq k_E.$$

Thus, for some $C_1 > 0$, we have, if $k \geq k_E$,

$$C(\|E\|_{L^N})\int_\Omega |\nabla G_k(u_n)|^2 \leq C_1 k^2 \int_{k<|u_n|}|E|^2 + C_2\left[\int_{k<|u_n|}|f|^{\frac{2N}{N+2}}\right]^{\frac{N+2}{N}}$$

3 estimates

$$\left[\int_\Omega |\log(1+|u_n|)|^{2^*}\right]^{\frac{2}{2^*}} \le \frac{1}{S^2\alpha^2}\int_\Omega |E|^2 + \frac{2}{S^2\alpha}\int_\Omega |f|^1$$

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^2 \le \frac{2}{\alpha}k^2\int_\Omega |E|^2 + k\int_\Omega |f|^1, \text{ forall } k > 0$$

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$k = k_E$

$$C_0 \int_\Omega |\nabla T_k(u_n)|^2 \leq k^2 \int_\Omega |E|^2 + k \int_\Omega |f|^1, \; forall \; k > 0$$

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Theorem

Let $E \in (L^N(\Omega))^N$, $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$. *Then there exists* $u \in W_0^{1,2}(\Omega)$ *weak solution, that is*

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$$\sqrt{\left[C_0 + C(\|E\|_{L^N})\right]}\left[\int_\Omega |\nabla u|^2\right]^{\frac{1}{2}} \leq \|f\|_{L^{\frac{2N}{N+2}}}$$

Last part of the talk: new presentation of some results

Written for Gianni

Estimate on the sequence $\{u_n\}$

$$\sqrt{\left[C_0 + C(\|E\|_{L^N})\right]}\left[\int_\Omega |\nabla u_n|^2\right]^{\frac{1}{2}} \leq \|f\|_{L^{\frac{2N}{N+2}}}$$

$$u_n \in W_0^{1,2}(\Omega):$$

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$$\psi_n \in W_0^{1,2}(\Omega): -\mathrm{div}(M^*(x)\nabla\psi_n) = \frac{E(x)\cdot\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|}\frac{1}{1+\frac{1}{n}|u_n|} + g(x)$$

Estimate on the sequence $\{u_n\}$

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Estimate on the sequence $\{u_n\}$

$$\sqrt{\left[C_0 + C(\|E\|_{L^N})\right]}\left[\int_\Omega |\nabla u_n|^2\right]^{\frac{1}{2}} \leq \|f\|_{L^{\frac{2N}{N+2}}}$$

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Theorem

Let $E \in (L^N(\Omega))^N$, $g \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$. Then there exists $\psi \in W_0^{1,2}(\Omega)$ weak solution, that is

$$\int_\Omega M^*(x)\nabla\psi\nabla v = \int_\Omega E(x)\cdot\nabla\psi\, v + \int_\Omega f\, v, \quad \forall\, v \in W_0^{1,2}(\Omega).$$

Existence th. $\Rightarrow$ the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,2}(\Omega)$: $u_\varepsilon \rightharpoonup u^*$

$E \in (L^N(\Omega))^N$, $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, $u_\varepsilon \in W_0^{1,2}(\Omega)$ **sol. of**

$$\int_\Omega M_\varepsilon(x)\nabla u_\varepsilon \nabla v = \int_\Omega [\nabla u_\varepsilon\, E(x)]v + \int_\Omega f\, v, \qquad \forall\ v \in W_0^{1,2}(\Omega).$$

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Let $\varphi_0 \in W_0^{1,2}(\Omega)$. **Consider** $\varphi_\varepsilon \in W_0^{1,2}(\Omega)$ **solution of**

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$E_n \in (L^N(\Omega))^N$, $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$

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Work in progress

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Work in progress

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