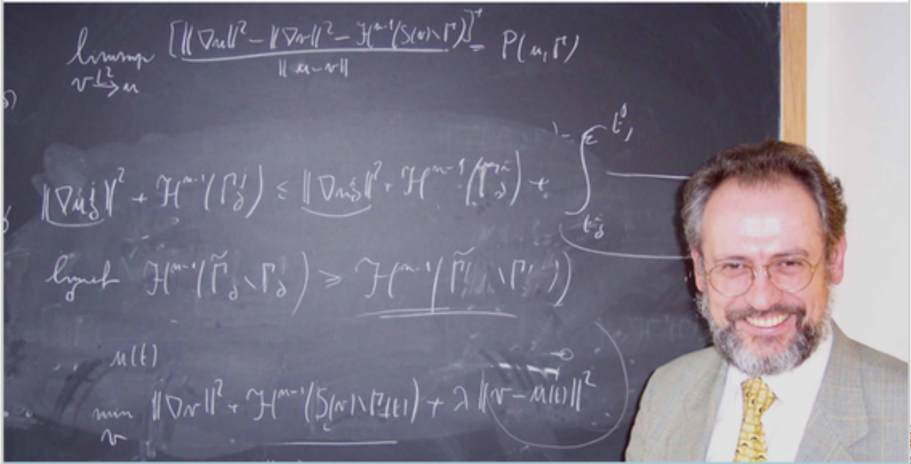
CALCULUS OF VARIATIONS AND APPLICATIONS



My three papers with Gianni

A general chain rule for distributional derivatives. Proc. Amer. Mat. Soc., **108** (1990), 691–702.

On the representation in $BV(\Omega; \mathbf{R}^m)$ *of quasi-convex integrals.* Journal of Functional Analysis, **109** (1992), 76–97.

Fine properties of functions in BD. (also in collaboration with A.Coscia), Arch. Rat. Mech. Anal., **139** (1997), 201–238.

SCUOLA NORMALE SUPERIORE

Semigroups and Geometric Measure Theory

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Overview

I want to illustrate, by a few recent examples, how semigroup techniques can be used to attack problems in Geometric Measure Theory, Real Analysis, more generally “calculus” problems in structures that can be very far from being Euclidean, in some lucky cases overcoming lack of doubling property, infinite dimensionality, lack of local coordinates,....

Plan

1. Lusin-Lipschitz property
2. Regular Lagrangian Flows in $RCD(K,\infty)$ spaces
3. Sets of finite perimeter in Gaussian spaces
4. Sets of finite perimeter in $RCD(K,N)$ spaces

Lusin-type Lipschitz approximation

In a metric measure space $(X, \mathrm{d}, \mathrm{m})$, a function $f : X \to \mathbb{R}$ is said to be approximable in Lusin's sense by Lipschitz functions on $A \in \mathscr{B}(X)$ if for all $\epsilon > 0$ there exists $C \in \mathscr{B}(X)$ such that

$$\mathrm{m}(A \setminus C) < \epsilon \qquad \text{and} \qquad f|_C \text{ is d-Lipschitz.}$$

We are interested in the *quantitative* version of this statement, in particular to

$$|f(x) - f(y)| \le \mathrm{d}(x, y)(g(x) + g(y)) \qquad \forall x,\, y \in X \setminus N$$

with $\mathrm{m}(N) = 0$, for some $g \in L^p(X, \mathrm{m})$. One can then take $C = \{g \le M\} \setminus N$, so that $\mathrm{Lip}(f|_C) \le 2M$ and

$$\mathrm{m}(X \setminus C) \le M^{-1/p} \int_{\{g>M\}} |g|^p \, d\mathrm{m} = o(M^{-1/p})$$

by Markov inequality. In Euclidean and other "nice" spaces, the decay property $\mathrm{m}(X \setminus C) = o\big((\mathrm{Lip}(f|_C))^{-1/p}\big)$ *characterizes* $W^{1,p}$ Sobolev spaces for $p \in [1, \infty)$ (the so-called Hajlasz-Sobolev spaces).

Some classical applications

The Lusin-Lipschitz property has a variety of applications:

• Lower semicontinuity in vectorial Calculus of Variations (Acerbi-Fusco);

• Regularity in Elliptic PDE's (Diening, Duzaar, Mingione, Stroffolini, Verde,....);

• Theory of currents (A-Kirchheim, De Lellis-Spadaro);

• Flow of nonsmooth vector fields (Crippa-De Lellis);

• Rate of convergence in the matching problem (A-Stra-Trevisan).

Strategies of proof of Lusin-Lip: Euclidean case

For $f \in W^{1,p}(X, d, \mathrm{m})$, $p > 1$, we want to find $g \in L^p(X, \mathrm{m})$ and a m-negligible set N such that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \qquad \forall x,\, y \in X \setminus N.$$

In Euclidean spaces the proof of can be achieved, for instance, writing f as

$$f(x) = -\int \langle \nabla f(y), \nabla_x G(x, y) \rangle \, dy$$

with G fundamental solution of Laplace's operator Δ, in the end a suitable g is proportional to $M(|\nabla f|)$ (the Hardy-Littlewood's maximal function), namely

$$M(|\nabla f|)(x) := \sup_{r>0} \fint_{B_r(x)} |\nabla f| \, dy.$$

Strategies of proof of Lusin-Lip: PI metric measure spaces

In more general metric measure structures, we can compare f with a regularization f_r, for instance $f_r(x) = \fint_{B_r(x)} f\, d\mathrm{m}$. Choosing $r \sim d(x, y)$, $f_r(x)$ is comparable to $f_r(y)$ (with an estimate $\sim r \fint_{B_{Cr}(x)} |\nabla f|\, d\mathrm{m}$) and the problem reduces to the *pointwise* estimate of $f(x) - f_r(x)$.

This estimate involves once more $M(|\nabla f|)(x)$.

However, these strategies seem to fail when either m is not doubling or the local Poincaré inequality fails. This happens for instance for Gaussian spaces (even when they are topologically finite-dimensional), for the Wiener space and for $\mathrm{RCD}(K, \infty)$ spaces.

Our method covers all these important cases, and builds upon another powerful maximal theorem.

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New strategy of proof

Theorem. (Rota) *For $p \in (1,\infty]$ and for the m-a.e. continuous version of a Markov semigroup R_t one has*

$$\|\sup_{t>0} R_t f\|_p \leq C_p \|f\|_p \qquad \forall f \in L^p(X,\mathrm{m}).$$

In addition, for all $f \in L^p(X,\mathrm{m})$, one has $R_t f \to f$ m-a.e. as $t \to 0^+$.

Then, our method uses the semigroup R_t associated to the Sobolev class $W^{1,2}$ instead of the inversion of Laplace's operator: the regularization is $f_t = R_t f$, now with $t \sim d^2(x,y)$.

It follows that we need to estimate

$$|f(x) - f(y)| \leq |f(x) - R_t f(x)| + |R_t f(x) - R_t f(y)| + |R_t f(y) - f(y)|.$$

Roughly speaking the estimates of all terms involve $|\nabla f|$, but while the estimate of the oscillation $|R_t(x) - R_t(y)|$ involves mostly the curvature properties of the metric measure space, the estimate of $f - R_t f$ is more related to the regularity of the transition probabilities $p_t(x,y)$ of R_t.

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Lip-Lusin property in $\mathrm{RCD}(K,\infty)$ spaces

Recall that a metric measure space (X,d,m) is said to be $\mathrm{RCD}(K,\infty)$ if it is $\mathrm{CD}(K,\infty)$ according to Lott-Villani and Sturm, i.e.

$$\mathrm{Ent}_{\mathrm{m}}(\varrho\mathrm{m}) = \int_X \varrho \log \varrho \, d\mathrm{m} \text{ is } K\text{-convex along geodesics of } (\mathscr{P}_2(X), W_2)$$

and Cheeger's energy

$$\mathrm{Ch}(f) := \inf\left\{ \liminf_{h\to\infty} \int_X |\nabla f_h|^2 \, d\mathrm{m} : \ f_h \in \mathrm{Lip}(X,\mathrm{d}), \ \|f_h - f\|_2 \to 0 \right\}$$

is a quadratic form in $L^2(X,\mathrm{m})$.

By now this class of spaces, and the smaller class $\mathrm{RCD}(K,N)$ is well understood and characterized in many ways, after the work of many authors (A., Bolley, Cavalletti, Gentil, Gigli, Guillin, Kuwada, Milman, Mondino, Savaré, Sturm,..), via properties of H_t, contractivity properties, Bochner inequalities (Bakry-Émery).

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Lip-Lusin in $\mathrm{RCD}(K,\infty)$ spaces

Theorem. (A-Brué-Trevisan) *For all $f \in W^{1,2}(X,d,\mathrm{m})$ and all $\alpha \in (1,2)$ there exists a m-negligible set $N \subset X$ such that*

$$|f(x)-f(y)| \le C_{\alpha,K}\mathsf{d}(x,y)(g(x)+g(y)) \qquad \forall x,\, y \in X \setminus N,$$

with

$$g := \left(\sup_{t>0} H_t|\nabla f|^{\alpha}\right)^{1/\alpha} + \sup_{s>0} |H_s\sqrt{-\Delta} f| \in L^2(X,\mathrm{m}).$$

In addition, this property characterizes the space $W^{1,2}(X,d,\mathrm{m})$.

The $H_t\sqrt{-\Delta}f$ term in the definition of g is due to the fractional representation

$$H_t f(x) - f(x) = \int_0^{\infty} \mathscr{K}(s,t) H_s\sqrt{-\Delta} f(x)\, ds.$$

Lip-Lusin in Gaussian spaces

The result covers also Sobolev spaces $W^{1,2}_{\mathcal{E}}(H,\mathrm{m})$, with H separable Hilbert and m Gaussian and non-degenerate, induced by the Dirichlet form

$$\mathcal{E}(f) := \int_H |\nabla f|^2\,d\mathrm{m}.$$

They are indeed particular cases of $\mathrm{RCD}(K,\infty)$ spaces, but in this case g takes the simpler form

$$g := \sup_{t>0} P_t|\nabla f| + \sup_{t>0} |P_t\sqrt{-\Delta_{\mathcal{E}}}f| \in L^2(H,\mathrm{m})$$

for all $f \in W^{1,2}_{\mathcal{E}}(H,\mathrm{m})$, where P_t is the standard Markov semigroup associated to $\mathcal{E}$ and $\Delta_{\mathcal{E}}$ the infinitesimal generator.

Also a more extreme case (the Wiener space, an *extended* metric measure space) can be covered, as well as the cases $p \neq 2$ and f Hilbert-valued.

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Regular Lagrangian Flows

The DiPerna-Lions theory provides existence, uniqueness and stability to the flow $\boldsymbol{X}(t,x)$ of a large class of vector fields $\boldsymbol{b}_t(x)$, $t \in (0,T)$, including Sobolev vector fields. We follow the axiomatization of A. '04, based on the concept of *Regular Lagrangian Flow*.

Definition. We say that $\boldsymbol{X}(t,x)$ is a Regular Lagrangian Flow associated to $\boldsymbol{b}_t$ if:

(i) $\boldsymbol{X}(\cdot,x)$ is an absolutely continuous solution in $[0,T]$ to the ODE $\dot\gamma = \boldsymbol{b}_t(\gamma)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$;

(ii) for some constant $L \geq 0$, called compression constant, one has $\boldsymbol{X}(t,\cdot)_\#\mathscr{L}^n \leq L\mathscr{L}^n$ for all $t \in [0,T]$.

Recently, using notions from Γ-calculus, the DiPerna-Lions theory has been extended in A.-Trevisan to a large class of metric measure structures, including all $\mathrm{RCD}(K,\infty)$ metric measure spaces.

Regular Lagrangian Flows

Existence and uniqueness of the RLF are equivalent to the well-posedness, in the class of nonnegative functions ϱ in $L^\infty_t(L^1_x \cap L^\infty_x)$, of

$$\frac{d}{dt}\varrho_t + \mathrm{div}\left(\boldsymbol{b}_t\varrho_t\right) = 0.$$

The DiPerna-Lions strategy relies on a regularization $\varrho_t \mapsto \varrho^\varepsilon_t := R_\epsilon\varrho_t$ and on the fine analysis of the commutator

$$\frac{d}{dt}\varrho^\varepsilon_t + \mathrm{div}\left(\boldsymbol{b}_t\varrho^\varepsilon_t\right) = \mathscr{C}^\epsilon(\boldsymbol{b}_t, \varrho_t),$$

i.e. $\mathscr{C}^\epsilon(\boldsymbol{b}_t, \varrho_t) := \mathrm{div}\left(\boldsymbol{b}_t R_\varepsilon\varrho_t\right) - R_\varepsilon\left(\mathrm{div}\left(\boldsymbol{b}_t\varrho_t\right)\right)$.

We discovered that the use of P_ϵ as regularizing operator R_ϵ leads to a coordinate-free formula for the commutator, namely (if $\mathrm{div}\,\boldsymbol{b}_t = 0$)

$$\langle\mathscr{C}^\epsilon(\boldsymbol{b}_t, \varrho_t), \phi\rangle = \int_0^\epsilon\left(\int_X \langle\nabla\boldsymbol{b}_t^{\mathrm{sym}}\nabla P_{\epsilon-s}\varrho_t, \nabla P_s\phi\rangle\,d\mathrm{m}\right)ds$$

and this is starting point for the "synthetic approach" to the theory of RLF's in $\mathrm{RCD}(K,\infty)$ spaces (including the Gaussian case of A-Figalli).

Sets of finite perimeter in $\mathbb{R}^m$

$E \in \mathscr{B}(\mathbb{R}^m)$ has finite perimeter if $D\chi_E \in \left[\mathscr{M}(\mathbb{R}^m)\right]^m$, namely

$$\int_E \frac{\partial\phi}{\partial x_i}\,dx = -\int_{\mathbb{R}^m}\phi\,dD_i\chi_E \qquad \phi \in C_c^\infty(\mathbb{R}^m),\ i = 1,\dots,m.$$

Caccioppoli's original definition:

$$P(E) := \inf\left\{\liminf_{n\to\infty} \mathrm{Area}(\partial E_n) :\ E_n \text{ polyhedra},\ E_n \to E \text{ in } L^1_{\mathrm{loc}}\right\}.$$

The two definitions are equivalent, and $P(E) = |D\chi_E|(\mathbb{R}^m)$.

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Two more definitions

De Giorgi (1951): E has finite perimeter if

$$I(E) := \lim_{t\downarrow 0} \int_{\mathbb{R}^m} |\nabla T_t \chi_E| \, dx < \infty.$$

Here $T_t\chi_E = G_t * \chi_E$ is the solution to the heat equation with χ_E as initial datum. Again, $I(E) = P(E) = |D\chi_E|(\mathbb{R}^m)$ and the existence of the limit is a simple consequence of the properties

$$\nabla T_{s+t}\chi_E = \nabla T_s T_t \chi_E = T_s \nabla T_t \chi_E.$$

Ledoux (1994):

$$P(E) = \lim_{t\downarrow 0} \sqrt{\frac{\pi}{t}} K_t(E, \mathbb{R}^n \setminus E),$$

where

$$K_t(E,F) := \int_{\mathbb{R}^n} T_{t/2}\chi_E T_{t/2}\chi_F \, dx.$$

In addition, always $P(E) \geq \sqrt{\frac{\pi}{t}} K_t(E, \mathbb{R}^n \setminus E)$ holds.

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Gauss-Green formulas

Writing the polar decomposition $D\chi_E = \nu_E |D\chi_E|$, with ν_E provided by the Radon-Nikodym theorem, we have

$$\text{(weak)} \qquad \int_E \frac{\partial \phi}{\partial x_i}\, dx = -\int_{\mathbb{R}^m} \phi \nu_{E,i}\, d|D\chi_E|$$

On the other hand, under reasonable smoothness assumptions,

$$\text{(classical)} \qquad \int_E \frac{\partial \phi}{\partial x_i}\, dx = -\int_{\partial E} n_{E,i} \phi\, d\mathscr{H}^{m-1}$$

where $n_E = (n_{E,1}, \ldots, n_{E,m})$ is the inner normal.

How far from each other are the two formulas?

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Measure-theoretic boundaries

Federer's **essential boundary.**

$$\partial^* E := \left\{ x \in \mathbb{R}^m : \limsup_{r \downarrow 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} > 0 \text{ and } \limsup_{r \downarrow 0} \frac{|B_r(x) \setminus E|}{|B_r(x)|} > 0 \right\}.$$

By Lebesgue's theorem, $\partial^* E$ is $\mathscr{L}^m$-negligible.

De Giorgi's **reduced boundary.**

$$\mathscr{F} E := \left\{ x \in \mathrm{supp}\, |D\chi_E| : \exists \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ and } |\mathrm{limit}| = 1 \right\}.$$

At points $x \in \mathscr{F} E$ the (weak) inner normal ν_E is defined by the value of the limit. Furthermore, by Besicovitch theorem,

$$|D\chi_E| \text{ is concentrated on } \mathscr{F} E \text{ and } D\chi_E = \nu_E |D\chi_E|.$$

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De Giorgi and Federer results

Theorem. *For any set of finite perimeter E we have*

(a) $|D\chi_E|(B) = \mathscr{H}^{m-1}(B \cap \mathscr{F}E)$ *for all* $B \in \mathscr{B}(\mathbb{R}^m)$;

(b) $\mathscr{F}E \subset \partial^* E$, *but* $\mathscr{H}^{m-1}(\partial^* E \setminus \mathscr{F}E) = 0$;

(c) $\mathscr{F}E$ *is contained in the union of countably many Lipschitz hypersurfaces.*

These results, of central importance for the development of modern GMT, reduce somehow the gap between the weak and the classical Gauss-Green formulas.

The proof of these statements is mostly based on a blow-up analysis, and in particular in the proof of the convergence

$$\frac{1}{r}(E - x) \to \text{halfspace} \perp \text{to } \nu_E(x) \text{ as } r \downarrow 0 \text{ for all } x \in \mathscr{F}E.$$

This works because points in $\mathscr{F}E$ are Lebesgue points of ν_E, relative to $|D\chi_E|$.

Gaussian theory: finite dimensions

$X = \mathbb{R}^m$, $G_m(x) = (2\pi)^{-m/2} e^{-|x|^2/2}$, $\gamma_m = G_m \mathscr{L}^m$ standard Gaussian.
Since $\partial_h \gamma_m = -\langle x, h\rangle \gamma_m$ we have the integration by parts formula

$$\int_X f \partial_h \phi \, d\gamma_m = -\int_X \phi \partial_h f \, d\gamma_m + \int_X \langle x, h\rangle f \phi \, d\gamma_m \qquad h \in X$$

It can be used, with $f = \chi_E$, to define a weak derivative $D_{\gamma_m}\chi_E$. Obviously $D_\gamma \chi_E = G_m D\chi_E$ and all "local" regularity properties remain true. For instance

$$D_{\gamma_m}\chi_E = G_m D\chi_E = G_m \mathscr{H}^{m-1} \llcorner \partial^* E$$

leads to a "more explicit" GG formula even in Gaussian spaces.

The Ornstein-Uhlenbeck semigroup

$$\partial_t f = \mathrm{div}\big(\nabla f_t + x f_t\big)$$

If we write $f_t \mathscr{L}^m = \varrho_t \gamma_m$ (i.e. $f_t = \varrho_t G_m$), a fundamental and dimension-free representation formula for ρ_t is

(Mehler) $$\varrho_t(x) = \int_X \varrho_0\big(e^{-t}x + \sqrt{1 - e^{-2t}}y\big)\, d\gamma_m(y).$$

We still have

$$|D_{\gamma_m}\chi_E| = \lim_{t\downarrow 0} \int_X |\nabla \varrho_t|\, d\gamma_m \qquad\qquad \varrho_0 = \chi_E$$

and many properties are known in finite dimension, for instance the only isoperimetric sets are the halfspaces (Sudakov).

What happens in infinite dimensions?

X separable Banach space, $\gamma \in \mathscr{P}(X)$ with $\int_X x\,d\gamma(x) = 0$, not supported in a proper subspace of X.
We say that γ is Gaussian if $x \mapsto \langle x^*, x\rangle$ has a Gaussian law (in $\mathbb{R}$) for all $x \in X^* \setminus \{0\}$.

The Cameron-Martin subspace $H \subset X$ is defined by

$$H := \{h \in X : \ \tau_h \gamma \ll \gamma\}$$

It turns out that H is dense in X, but $\gamma(H) = 0$!

There is a natural way to extract from the density β_h of $\tau_h\gamma$ w.r.t. γ an Hilbert norm which makes the inclusion of H in X compact. In finite dimensions, with the standard Gaussian,

$$\beta_h(x) = e^{-|h|^2/2 + \langle x, h\rangle}.$$

To what extent can we reproduce the DeGiorgi-Federer theory?

Good news

• Still the integration by parts formula along directions in H makes sense, and this leads to a Sobolev (and BV) theory (Gross, Malliavin, Fukushima).

• Mehler's formula

$$\varrho_t(x) = \int_X \varrho_0\big(e^{-t}x + \sqrt{1-e^{-2t}}y\big)\,d\gamma(y)$$

still makes sense and provides a nice contraction semigroup $P_t\varrho := \varrho_t$ in all $L^p(\gamma)$ spaces. Furthermore $P_t\varrho$, understood in the pointwise sense of Mehler's formula, is smooth for $t > 0$ (along directions of H).

Bad news

• Preiss-Tiser showed that Lebesgue theorem holds if the covariance operator of γ decays sufficiently fact (quite fast, indeed). Preiss provided also an example of a Gaussian measure γ in a Hilbert space X and $f \in L^\infty(X,\gamma)$ such that

$$\limsup_{r\downarrow 0} \frac{1}{\gamma(B_r(x))}\int_{B_r(x)} f\,d\gamma > f(x) \quad \text{in a set of } \gamma\text{-positive measure.}$$

So, no Lebesgue theorem can be expected in general and Federer's definition of essential boundary, based on volume fractions, becomes really problematic.

• Of course also no Besicovitch theorem can be expected, so there is no hope to define the reduced boundary in the traditional way.

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Blow-up analysis via OU semigroup

Theorem. (A-Figalli, A-Figalli-Runa) *For any set E of finite perimeter one has*

$$\lim_{t\downarrow 0}\int_X |P_t\chi_E - \frac{1}{2}|^2\, d|D_\gamma\chi_E| = 0,$$

more precisely (so that $r=\sqrt{1-e^{-2t}}\sim\sqrt{2t}$ and $e^{-t}x - x = o(r)$)

$$\lim_{t\downarrow 0}\int_X\int_X \left|\chi_E(e^{-t}x+\sqrt{1-e^{-2t}}y)-\chi_{S(x)}(y)\right|\, d\gamma(y)\, d|D_\gamma\chi_E|(x)=0$$

where S_x is the halfspace determined by $\nu_E(x)$.

Eventually this can be used to give coordinate-free definitions of $\partial^* E$ and $\mathscr{F}E$, proving the representation formula (Hino)

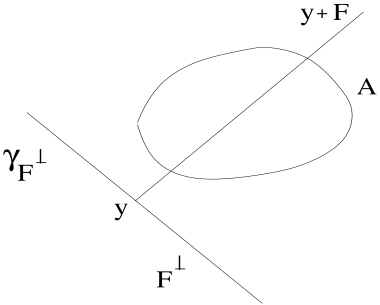
$$|D_\gamma\chi_E| = \mathscr{H}^{\infty-1}\llcorner\partial^* E = \mathscr{H}^{\infty-1}\llcorner\mathscr{F}E,$$

where $\mathscr{H}^{\infty-1}$ is the Feyel-De la Pradelle codimension 1 (Gaussian) Hausdorff measure.

Finite-codimension Hausdorff measures

As illustrated in the picture, $\gamma_F^\perp \times \gamma_F$ is a factorization of γ induced by a m-dimensional subspace F of H (γ_F is the standard Gaussian in F, with the metric induced by H) and the sets A_y are the m-dimensional sections of A, keeping $y \in (I - \pi_F)(X)$ fixed.

$$\mathscr{H}_F^{\infty-k}(A) := \frac{1}{\sqrt{2\pi}^m} \int_{F^\perp} \int_{A_y} e^{-|x|^2/2} \, d\mathscr{H}_F^{m-k}(x) \, d\gamma_{F^\perp}(y).$$



Sets of finite perimeter in metric measure spaces

In a metric measure space (X, d, m), we say that a Borel set E has finite perimeter if there exist Lipschitz functions f_n convergent to χ_E in $L^1(X, \mathrm{m})$ with

$$\limsup_{n\to\infty} \int_X |\nabla f_n| \, d\mathrm{m} < \infty.$$

Optimization of the sequence (f_n) and localization of this construction to open subsets then give a finite Borel measure $|D\chi_E| = \mathrm{Per}(E, \cdot)$, playing the role of "surface measure". The "distributional" point of view is still possible, but requires the language of derivations (Gigli, Di Marino).

Question. Can we prove that at $|D\chi_E|$-a.e. point the set E has a "good" behaviour on sufficiently small scales? For instance, under fairly general assumptions, one can prove that *any* tangent set F in *any* tangent metric measure structure (Y, δ, ν) is an entire minimizer of the perimeter.

Sets of finite perimeter in $\mathrm{RCD}(K,N)$ spaces

In a recent work with Brué and Semola, we studied the fine structure of sets of finite perimeter in $\mathrm{RCD}(K,N)$ spaces.

Definition. A $\mathrm{RCD}(K,\infty)$ space (X,d,m) is $\mathrm{RCD}(K,N)$ if Bochner's inequality holds

$$\Delta\frac{1}{2}|\nabla f|^2 \geq \langle \nabla f, \nabla \Delta f\rangle + K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2.$$

• Any $\mathrm{RCD}(K,N)$ space has an "essential dimension" (Brué-Semola), i.e. there exists an integer $k \in [1,N]$ with $\mathrm{m}(X \setminus \mathcal{R}_k) = 0$. Here $\mathcal{R}_k$ is the k-dimensional set, i.e. the set of points x where the limit

$$\lim_{r\to 0^+}\big(X, \frac{1}{r}d, \frac{1}{Z(x,r)}\mathrm{m}\big) = \big(\mathbb{R}^k, d_{eu}, \mathscr{L}^k\big)$$

occurs in the pointed, measured Gromov-Hausdorff sense.

• We call the space $\mathrm{RCD}(K,N)$ "non-collapsed" (De Philippis-Gigli) if there is no dimension gap, i.e. $k = N$.

Sets of finite perimeter in $\mathrm{RCD}(K,N)$ spaces

Difficulty: Even though $\mathrm{RCD}(K,N)$ spaces are doubling and their tangent bundle (according to Cheeger's theory) is finite-dimensional, it is hard to identify $D\chi_E$ and its "components", because the bundle is defined only up to m-negligible sets (but see Debin-Gigli-Pasqualetto).

Theorem. *If (X,d,m) is non-collapsed $\mathrm{RCD}(K,N)$, $|D\chi_E|$ is concentrated on $\mathcal{R}_N$ and, at $|D\chi_E|$-a.e. point $x\in X$, the tangent set to E at x is the half-Euclidean N-space.*

The first ingredient of the proof is the fact that blow-up provides the equality case in the Bakry-Émery inequality

$$(BE)_1 \qquad |\nabla P_t f| \le P_t|\nabla f|.$$

Theorem A. *If (X,d,m) is $\mathrm{RCD}(K,N)$, at $|D\chi_E|$-a.e. point $x\in X$, any tangent F to E at x satisfies*

$$|\nabla P_{t+s}\chi_F| = P_t|\nabla P_s\chi_F| \qquad \nu\text{-a.e. in } Y,$$

so that $f_s = P_s\chi_F$ satisfy $(BE)_1$ with equality.

Sets of finite perimeter in $\mathrm{RCD}(K, N)$ spaces

The second ingredient is a "dual" splitting theorem, where instead of starting from a line (Cheeger-Gromoll, Gigli), we start from a function satisfying $(BE)_1$ with equality.

Theorem B. (Rigidity of $(BE)_1$ and splitting) *In a* $\mathrm{RCD}(0, N)$ *space* (Y, δ, ν)*, the condition*

$$|\nabla P_t f| = P_t |\nabla f| \qquad \nu\text{-a.e. in } Y$$

with $f \not\equiv 0$*, implies the splitting*

$$(Y, \delta, \nu) \sim (Y', \delta', \nu') \times (\mathbb{R}, d_{eu}, \mathscr{L}^1)$$

and, with coordinates $y = (y', s)$*,* f *is a monotone function of* s*.*

Building also on Theorems A and B, very recently Brué-Pasqualetto-Semola obtained the rectifiability, for all sets of finite perimeter, of a suitable measure-theoretic boundary.

Thank you for the attention!

Slides available upon request