Calculus of Variations and Applications
for Gianni Dal Maso's birthday

Minimal planar N-partitions for large N

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joint work with
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My connection with Gianni

- G. Buttazzo, G. Dal Maso:
  On Nemyckii operators and integral representation of local functionals.
  Rend. Mat. (7), 3 (1983), 491–509.

$$|F_\lambda(u,B) - F_\lambda(v,B)| \le \lambda\, c_p \Big\{ \|u-v\|_p^p + \big(\|u\|_p^{p-1} \vee \|u\|_p^{p-1}\big)\|u-v\|_p \Big\}$$

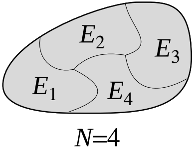
- G. Alberti, G. Bouchitté, G. Dal Maso:
  The calibration method for the Mumford-Shah functional.
  C. R. Acad. Sci. Paris Sér. I Math., 329 (1999), 249–254.
- G. Alberti, G. Bouchitté, G. Dal Maso:
  The calibration method for the Mumford-Shah functional and free-discontinuity problems.
  Calc. Var. Partial Differential Equations, 16 (2003), 299–333.

Summary

- ~~Asymptotic shape of minimal clusters in the plane~~
- Minimal partitions and Hales's Honeycomb Theorem
- Uniform energy distribution for minimal partitions
- Towards a description of the structure of minimal partitions

Partitions

Let Ω be a two-dimensional domain with finite area.



An *N-partition* of Ω is a collection $\mathcal{E} = \{E_1, \ldots, E_N\}$ of closed sets in Ω (called *cells* of the partition)

- with pairwise disjoint interiors and union Ω;
- with equal area $|E_i| = |\Omega|/N$;
- with sufficiently regular boundaries...

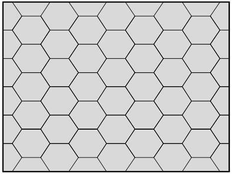
The *perimeter* of a partition $\mathcal{E}$ is

$$
\begin{aligned}
\operatorname{Per}(\mathcal{E}) &:= \operatorname{length}\big(\partial E_1 \cup \cdots \cup \partial E_n\big) \\
&= \frac{1}{2}\sum_{i=1}^{N} \operatorname{length}(\partial E_i) + \frac{1}{2}\operatorname{length}(\partial \Omega) \\
&= \frac{1}{2}\sum_{i=1}^{N} \operatorname{Per}(E_i) + \frac{1}{2}\operatorname{Per}(\Omega)
\end{aligned}
$$

- Ω admits a minimal N-partition for every integer N;
- the local structure of minimal N-partitions is simple;
- computing minimal N-partitions is complicated.

Hales's Honeycomb Theorem (T.C. Hales, 2001)

Let Ω be a flat torus which admits a regular hexagonal N-partition $\mathcal{E}_{\text{hex}}$.



N=48

Then $\mathcal{E}_{\text{hex}}$ is the unique minimal N-partition of Ω.

- Not all flat tori admit a regular hexagonal partition.
- No counterpart in higher dimension!

Key tool: Hales's isoperimetric inequality

Simplified version (polygons only):

- let E be an n-polygon with area 1,
- let R_n be the regular n-polygon with area 1,
- let $H = R_6$ be the regular hexagon with area 1.

Then:

$$\mathrm{Per}(E) \geq \mathrm{Per}(R_n) \geq \mathrm{Per}(H) - c(n-6)$$

where we use that $n \mapsto \mathrm{Per}(R_n)$ is a convex function (!)

We can do better:

$$\mathrm{Per}(E) \geq \mathrm{Per}(H) - c(n-6) + \delta\, \mathrm{dist}(E, H)^2$$

where $\mathrm{dist}(E, H)$ is *for example* Hausdorff distance.

Sketch of proof of Hales's theorem

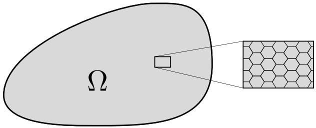
Let $\mathcal{E}$ be an N-partition of Ω with E_i an n_i-polygon:

$$\begin{aligned}
\mathrm{Per}(\mathcal{E}) &= \frac{1}{2}\sum_i \mathrm{Per}(E_i) \\
&\geq N \cdot \frac{1}{2}\mathrm{Per}(H) - \frac{c}{2}\sum_i (n_i - 6) + \frac{\delta}{2}\sum_i \mathrm{dist}(E_i, H)^2 \\
&= N \cdot \frac{1}{2}\mathrm{Per}(H) + \frac{\delta}{2}\sum_i \mathrm{dist}(E_i, H)^2 \\
&\geq N \cdot \frac{1}{2}\mathrm{Per}(H)\,.
\end{aligned}$$

For the following we set $\sigma := \frac{1}{2}\mathrm{Per}(H) = \sqrt[4]{12}$.

Minimal N-partitions

We fix an arbitrary planar domain Ω with finite area.



We consider minimal N-partitions of Ω for large N.

- Hales's theorem suggests that most cells are close to regular hexagons $\longrightarrow$ local hexagonal patterns.
- We expect some "disturbance" close to the boundary of Ω. Does such disturbance decay away from the boundary?
- Is the orientation of the local hexagonal pattern constant? If not, is it piecewise constant $\longrightarrow$ emergence of "grains"?

Uniform energy distribution

- We want to prove uniform distribution of energy (that is, perimeter) in the spirit of A. + Choksi + Otto (2009).
- Purpose: prove that "most" cells are close to be regular hexagons (in a quantified way).
- From now on we replace N with the length parameter

$$\varepsilon := \sqrt{|\Omega|/N}\,.$$

  Thus ε^2 is the area of the cells of N-partitions, which we now call ε-partitions.

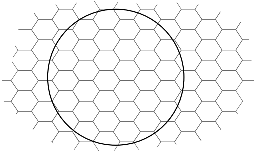
Average energy distribution of the hexagonal partition.

Let $\mathcal{E}_{\text{hex}}$ be the regular hexagonal partition with cells of area 1.

The average energy density of $\mathcal{E}_{\text{hex}}$ is $\sigma := \sqrt[4]{12}$.

That is, for every ball $B(x, r)$ with radius $r \gg 1$ there holds

$$\mathrm{Per}_{B(x,r)}(\mathcal{E}_{\text{hex}}) = \sigma \,\mathrm{area}(B(x,r)) + O(r^{2/3})$$



- Statement similar to Gauss's Circle Theorem.
- Proof by Fourier transform.

Let $\mathcal{E}^\varepsilon_{\text{hex}}$ be the regular hexagonal partition with cells of area ε^2. Then

$$\mathrm{Per}_{B(x,r)}(\mathcal{E}^\varepsilon_{\text{hex}}) = \frac{\sigma}{\varepsilon}\text{area}(B(x,r)) + O(\varepsilon^{1/3}r^{2/3})$$

Uniform distribution of energy

Let $\mathcal{E}$ be a minimal ε-partition of Ω. Let $B_\varepsilon = B(x_\varepsilon, r_\varepsilon)$ be a disc in Ω with $r_\varepsilon \gg \varepsilon$ and $\mathrm{dist}(B_\varepsilon, \partial\Omega) \gg \varepsilon$. Then

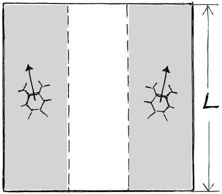
$$\mathrm{Per}_{B_\varepsilon}(\mathcal{E}) = \frac{\sigma}{\varepsilon}|B_\varepsilon| + O(r_\varepsilon)\,.$$

- Proof of lower bound is based Hales inequality.
- Proof of upper bound is based on “cut and paste” technique.
- The actual statement depends on the variant of the problem considered.

Towards a precise description of minimal ε-partitions

- Recall the questions: *Is the orientation of the local hexagonal pattern constant?* (we think NO)
  Is it piecewise constant? (we think YES)
- From now on we consider the "excess energy" of an ε-partition $\mathcal{E}$:
  $$F_\varepsilon(\mathcal{E}) := \varepsilon \operatorname{Per}(\mathcal{E}) - \sigma|\Omega|$$
- Ideally, we would like to write a Γ-limit of F_ε as $\varepsilon \to 0$.
  But what the variable of the Γ-limit should be?
  Claim: the "limit" of the orientation of the local hexagonal patterns…
- We did not write the Γ-limit, but we did identify and partially address some key questions ("cell problems").

Excess energy due to change of orientation



- Consider a square of side-length $L \gg 1$;
- consider all 1-partitions $\mathcal{E}$ which are prescribed in the grey zone (and satisfy suitable boundary periodic conditions);
- $\theta :=$ angle between the imposed orientations.

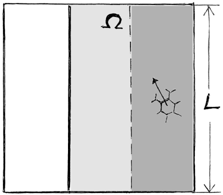
Define

$$\Phi(\theta) := \liminf_{L \to +\infty} \frac{1}{L} \left\{ \inf_{\mathcal{E}} F_1(\mathcal{E}) \right\}.$$

- Explicit construction gives $\Phi(\theta) = O\big(\theta |\log \theta|\big)$;
- Is $\Phi(\theta) > 0$? Yes, but proved under undesired assumptions.
- Is Φ superlinear in 0, i.e., $\liminf_{\theta \to 0^+} \dfrac{\Phi(\theta)}{\theta} = +\infty$? Yes, but ...

Excess energy due to shift? Fortunately not!

Excess energy due to boundary



- Consider a square of side-length $L \gg 1$ and take Ω as in the picture;
- consider all 1-partitions of Ω which are prescribed in the grey zone (and ...);
- $\theta :=$ angle between the imposed orientation and the vertical direction.

Define

$$\Phi_b(\theta) := \liminf_{L \to +\infty} \frac{1}{L} \left\{ \inf_{\mathcal{E}} F_1(\mathcal{E}) \right\}.$$

- Hales (isoperimetric inequality) gives $C \leq \Phi_b \leq C'$.
- Does $\Phi_b(\theta) > 0$ depends on θ?
  We think so, but we have no clue about a proof.
  Indeed we cannot prove even the most basic conjecture...

Conclusions

- If Φ_b does NOT depend on θ, then minimal ε-partitions of Ω have constant orientation (in the regime $\varepsilon \ll 1$).
- If Φ_b depends on θ, and Φ is strictly positive then minimal ε-partitions of Ω may not have constant orientation.
- If in addition Φ is super-linear at 0 then the orientation of minimal ε-partitions is piecewise constant (and in $SBV(\Omega)$) $\longrightarrow$ emergence of "grains".

From partitions to maps

- We use rigidity estimates a la Friesecke+James+Müller (2002), and precisely Müller+Scardia+Zeppieri (2015).
- We pass from a partition $\mathcal{E}$ of Ω to a map $u : \Omega \to \mathbb{R}^2$ in several steps.
- We fix $\varepsilon = 1$ and consider for simplicity a polygonal partition. First we construct the dual network N, connecting the barycenters of neighbouring cells.
- If the partitions contains only hexagons and has only triple points, then the the dual network N is made of triangles and contains only 6-nodes. Then we construct in a natural way a map u from N to the regular triangular network, which we extend to Ω by linearity.
- We use Hales's inequality $\mathrm{Per}(E) \geq \mathrm{Per}(H) + \delta\, \mathrm{dist}(E, H)^2$ to get:

$$F_1(\mathcal{E}) \gtrsim \int_\Omega \mathrm{dist}\big(\nabla u, SO(2)\big)^2\, dx$$

From partitions to maps, continued

- If the partitions contains non hexagonal cells (pentagons, heptagons,...) and has only triple points, then the the dual network N is still made of triangles but contains also n-nodes with $n \neq 6$ (*defects*).
- In this case we *cannot* construct a global map u from N to the regular triangular network (there is a topological issue).
  Indeed we can properly define only a matrix-field $\beta : \Omega \to \mathbb{R}^{2\times 2}/R_6$ where $R_6 \subset SO(2)$ is the subgroup generated by a rotation by $60°$.
- For a suitable $\Gamma \subset N$ we construct a lift $\beta : \Omega \setminus \Gamma \to \mathbb{R}^{2\times 2}$.
  We use Hales's inequality $\mathrm{Per}(E) \geq \mathrm{Per}(H) - c(n-6) + \delta\, \mathrm{dist}(E, H)^2$ to get:
  $$F_1(\mathcal{E}) \gtrsim \int_\Omega \mathrm{dist}\big(\beta, SO(2)\big)^2\, dx + \#\{\text{defects}\}$$
- Here is the difficulty! If $\#\{\text{defects}\} \gtrsim \text{length}(\Gamma)$ we are in game: we can use Müller+Scardia+Zeppieri and Lauteri+Luckhaus (2017).
  This estimate looks quite plausible but has eluded us (so far).

Thanks for the attention!

E ancora tanti auguri, Gianni!