Homogenization and corrector of the Neumann's brush problem

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Revisiting the work of

Robert Brizzi and Jean-Paul Chabot

- Homogénéisation de frontières
  Ph.D. Thesis, Université de Nice, 1978
- Boundary homogenization and Neumann boundary condition, Ricerche Mat. 46, 1997, pp. 341–387

in a more general geometry,

and in the case where the void can appear in the teeth area.

The problem

$$\begin{cases} -\operatorname{div} A\, Du_\varepsilon + c\, u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ A\, Du_\varepsilon \cdot n_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

(Neumann's boundary condition on $\partial\Omega_\varepsilon$)

with $A \in L^\infty(\Omega)^{N\times N}$, $A \geq \alpha I$, $\alpha > 0$,

$c \in L^\infty(\Omega)$, $c \geq \gamma$, $\gamma > 0$,

$f \in L^2(\Omega)$,

$\Omega_\varepsilon \subset \Omega$ is a comb when $N = 2$,

a brush when $N = 3$.

Homogenization + corrector result

when $\varepsilon \to 0$.

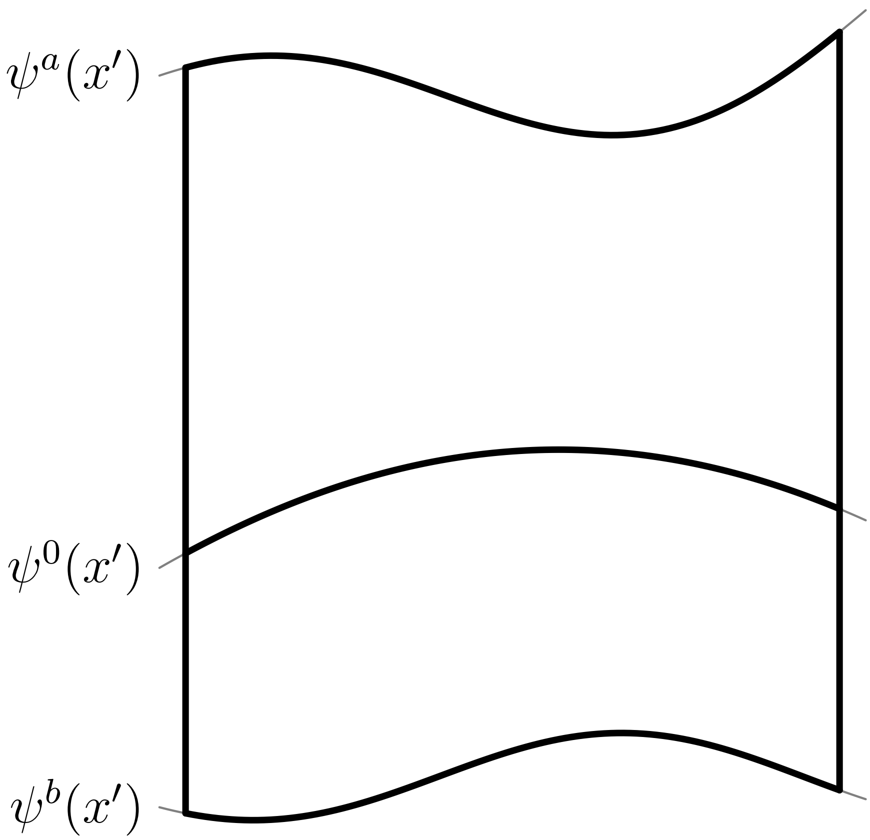


Figure: Ω in 2D

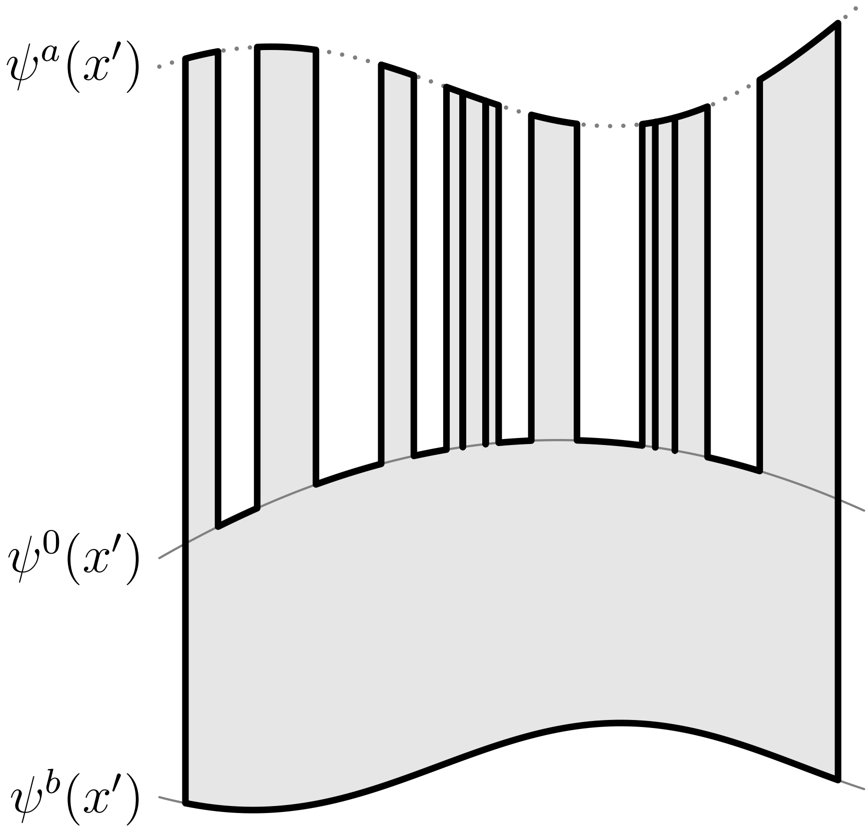


Figure: The comb in 2D

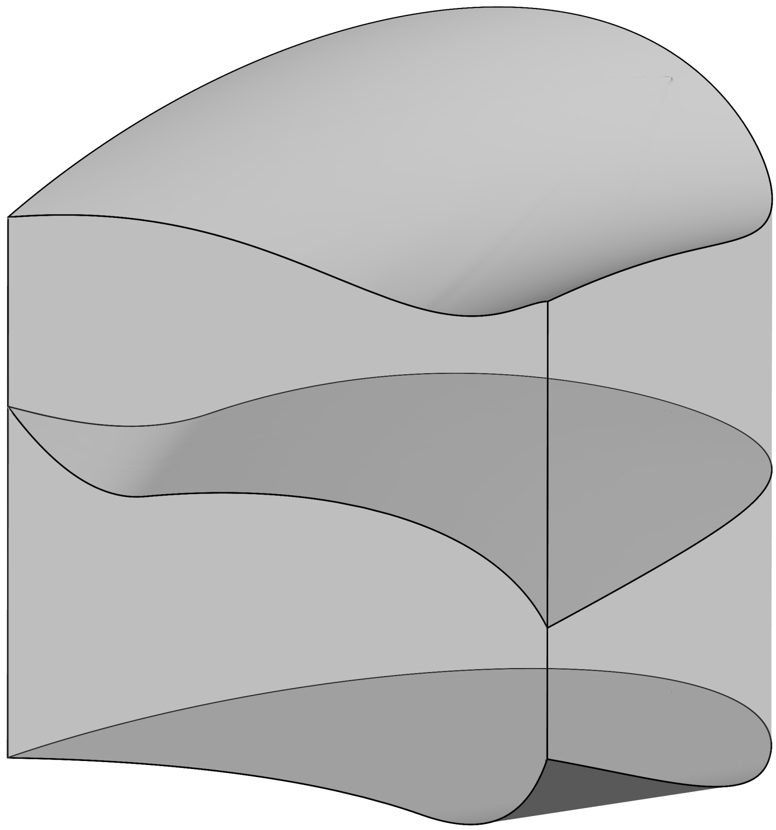


Figure: Ω in 3D

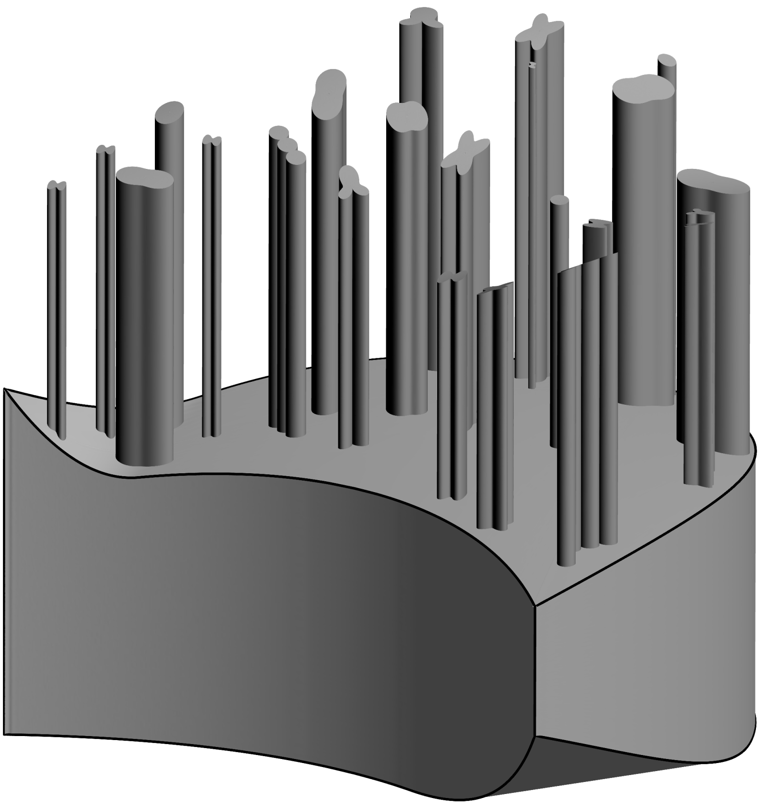
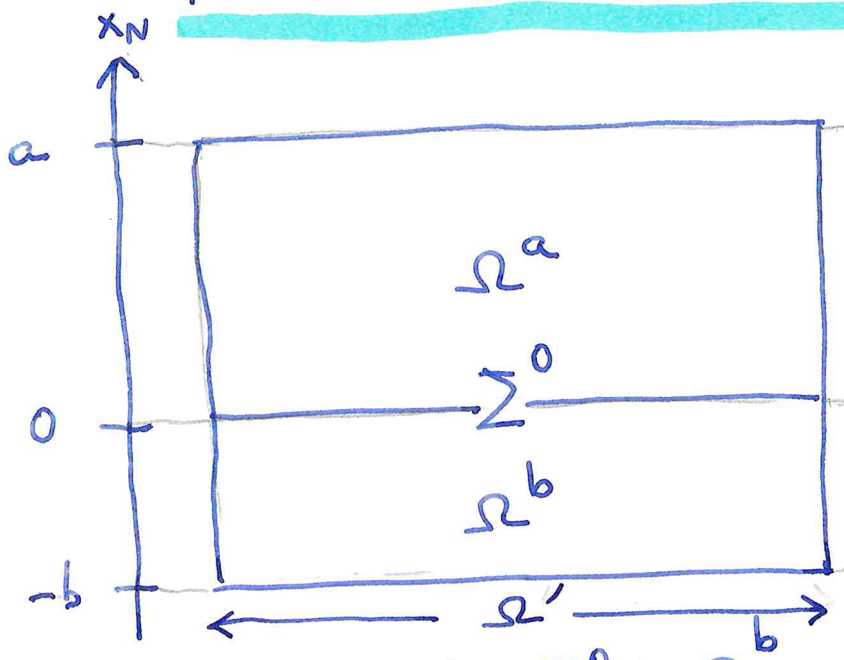
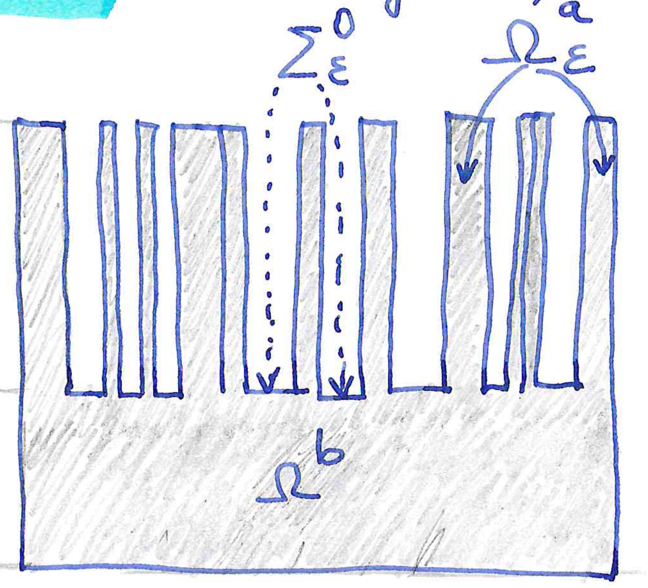


Figure: The brush in 3D

The model : $N=2$: a comb (rectangular)





$$\Omega = \Omega^a \cup \Sigma^0 \cup \Omega^b$$

$\Omega' \subset \mathbb{R}^{N-1}$
$$\begin{cases} \Omega^a = \Omega' \times \,]0,a[\\ \Sigma^0 = \Omega' \times \{x_N = 0\} \\ \Omega^b = \Omega' \times \,]-b,0[\end{cases}$$

$\forall x \in \mathbb{R}^N$, $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$

$$\chi_{\omega_\varepsilon} \longrightarrow \theta(x') \; L^\infty(\mathbb{R}^{N-1}) \text{ weak} *$$

$$0 \le \theta(x') \le 1$$

$$\Omega_\varepsilon = \Omega_\varepsilon^a \cup \Sigma_\varepsilon^0 \cup \Omega^b$$

$$\Omega_\varepsilon^a = \omega_\varepsilon \times \,]0,a[$$

$$\Sigma_\varepsilon^0 = \omega_\varepsilon^c \times \{x_N = 0\}$$

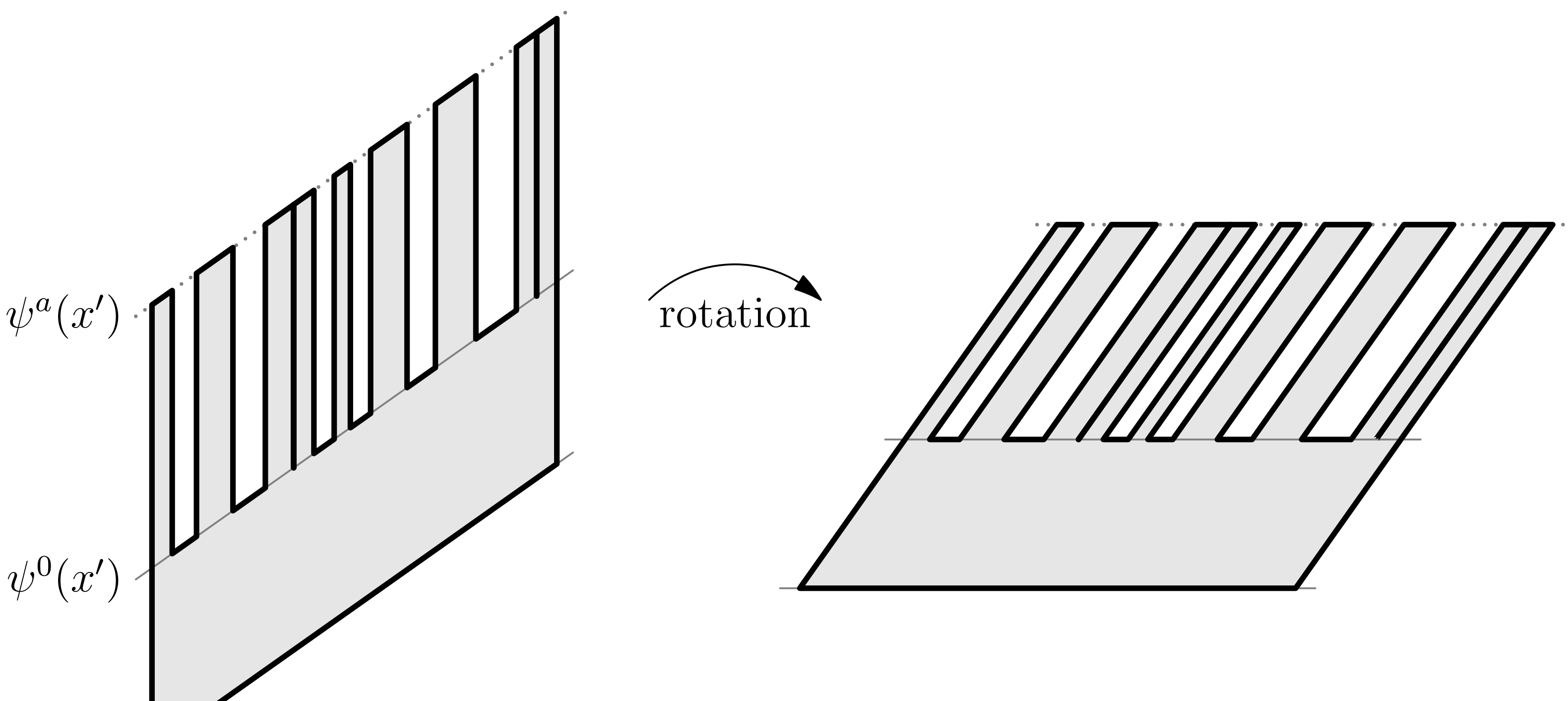
$$\omega_\varepsilon = \bigcup_j \omega_\varepsilon^j \qquad \omega_\varepsilon^j \subset \mathbb{R}^{N-1}$$

ω_ε^j open, disjoint, diam $\omega_\varepsilon^j \le \varepsilon$

no regularity on $\partial\omega_\varepsilon^j$

$\omega_\varepsilon^j \cap \omega_\varepsilon^k = \emptyset$ if $j \neq k$

but $\overline{\omega_\varepsilon^j} \cap \overline{\omega_\varepsilon^k}$ can be non empty



$\psi^a(x')$
$\psi^0(x')$
$\psi^b(x')$
rotation

The teeth are cylindrical: $\omega_\varepsilon^j \times]0,a[$

$\theta(x) = \theta(x')$ is their limit density in Ω^a

with $0 \leq \theta(x') \leq 1$ in Ω^a

The void can appear in the teeth area $(\theta(x') = 0)$

and then the problem is degenerated

Variational formulation of the ε-problem

$$\begin{cases} u_\varepsilon \in H^1(\Omega_\varepsilon), \\ \displaystyle\int_{\Omega_\varepsilon} A\, Du_\varepsilon\, Dv_\varepsilon + \int_{\Omega_\varepsilon} c\, u_\varepsilon v_\varepsilon = \int_{\Omega_\varepsilon} f\, v_\varepsilon, \quad \forall v_\varepsilon \in H^1(\Omega_\varepsilon). \end{cases}$$

No need of any regularity on Ω_ε, except Ω_ε open.

The homogenized problem

Space $\quad \mathcal{V}_\theta(\Omega^a, \Omega^b) = \{ v = (v^a, v^b) :$

$$v^a = 0 \text{ on } \{\theta = 0\}^a = \{x \in \Omega^a : \theta(x') = 0\},$$

$$\sqrt{\theta}\, v^a \in L^2(\Omega^a),\ D_{x_N}(\sqrt{\theta}\, v^a) \in L^2(\Omega^a),$$

$$v^b \in H^1(\Omega^b),$$

$$\hat{\gamma}^a(\sqrt{\theta}\, v^a) = \sqrt{\theta}\, \gamma^b(v^b) \text{ on } \Sigma^0 \quad \},$$

with $\displaystyle \|v\|^2_{\mathcal{V}_\theta} = \int_{\Omega^a} |\sqrt{\theta} v^a|^2 + \int_{\Omega^a} |D_{x_N}(\sqrt{\theta} v^a)|^2 + \int_{\Omega^b} |v^b|^2 + \int_{\Omega^b} |D v^b|^2.$

Equation

$$\begin{cases} u = (u^a, u^b) \in \mathcal{V}_\theta(\Omega^a, \Omega^b), \\ \displaystyle\int_{\Omega^a} a_0(x)\, D_{x_N}(\sqrt{\theta}\, u^a)\, D_{x_N}(\sqrt{\theta}\, v^a) + \int_{\Omega^a} c \sqrt{\theta}\, u^a \sqrt{\theta}\, v^a + \\ \displaystyle + \int_{\Omega^b} A(x)\, D u^b\, D v^b + \int_{\Omega^b} c\, u^b v^b = \int_{\Omega^a} \sqrt{\theta} f\, \sqrt{\theta}\, v^a + \int_{\Omega^b} f v^b, \\ \qquad \forall v = (v^a, v^b) \in \mathcal{V}_\theta(\Omega^a, \Omega^b). \end{cases}$$

Remarks on the space $V_\theta(\Omega^a, \Omega^b)$

- v^a appears only through $\sqrt{\theta}\, v^a$...
  Therefore v^a is not really defined on $\{\theta = 0\}^a$, where
  $$\{\theta = 0\}^a = \{x \in \Omega^a : \theta(x') = 0\}$$
  By convenience we set $v^a = 0$ on $\{\theta = 0\}^a$.

- $\sqrt{\theta}\, v^a$ belongs to $L^2(\Omega^a)$
  but $v^a = \frac{1}{\sqrt{\theta}} (\sqrt{\theta}\, v^a)$ does not belong to $L^2(\Omega^a)$ in general

- γ^b is the trace operator $\gamma^b \in \mathcal{L}(H^1(\Omega^b); H^{1/2}(\Sigma^0))$
  $\hat{\gamma}^a$ is the trace operator $\hat{\gamma}^a \in \mathcal{L}(H(\Omega^a; D_{x_N}); L^2(\Sigma^0))$
  where $H(\Omega^a, D_{x_N}) = \{w^a : w^a \in L^2(\Omega^a),\ D_{x_N} w^a \in L^2(\Omega^a)\}$

- $V_\theta(\Omega^a, \Omega^b)$ is an Hilbert space for the norm
  $\|v\|_{V_\theta}$

Remarks on the homogenized equation

- The coefficient $a_0 = a_0(x)$ will be defined later. It depends only on the matrix $A(x)$ and satisfies

$$a_0 \in L^\infty(\Omega), \qquad a_0(x) \geqslant \alpha \quad \text{a.e. } x \in \Omega$$

(same α as the matrix $A \geqslant \alpha I$).

- The equation has a unique solution (apply Lax-Milgram lemma)

The homogenization result

For any function $y^a_\varepsilon \in L^2(\Omega^a_\varepsilon)$ (Ω^a_ε = the teeth)

we define
$$\widetilde{y^a_\varepsilon}(x) = \begin{cases} y^a_\varepsilon(x) & \text{if } x \in \Omega^a_\varepsilon, \\ 0 & \text{if } x \in \Omega^a \setminus \Omega^a_\varepsilon. \end{cases}$$

Theorem (Homogenization)

Let $u_\varepsilon \in H^1(\Omega^\varepsilon)$ be the sol. of the ε-problem, and let $u = (u^a, u^b) \in V_\theta(\Omega^a, \Omega^b)$ be the solution of the homogenized problem. Then, as $\varepsilon \to 0$,

$$\begin{cases} \widetilde{u^a_\varepsilon} \longrightarrow \sqrt{\theta}\,.\sqrt{\theta}\, u^a \text{ in } L^2(\Omega^a) \text{ weakly}, \\ \widetilde{D.u^a_\varepsilon} \longrightarrow \sqrt{\theta}\; z(x)\, D_{x_N}(\sqrt{\theta}\, u^a) \text{ in } (L^2(\Omega^a))^N \text{ weakly}, \\ u^b_\varepsilon \longrightarrow u^b \text{ in } H^1(\Omega^b) \text{ weakly}, \end{cases}$$

where $z = z(x) \in L^\infty(\Omega)^N$ will be defined later and depends only on the matrix $A(x)$ on Ω^a.

Further to the above weak convergences, we have:

Theorem (Corrector) $\forall \varphi^\delta \in L^2(\Omega^a)$, $\forall \psi^\delta \in L^2(\Omega^a)$,

$$\begin{cases} u_\varepsilon^b \longrightarrow u^b \text{ in } H^1(\Omega^b) \text{ strongly}, \\ \widetilde{u_\varepsilon^a} = \chi_{\Omega_\varepsilon^a} \varphi^\delta + \rho_\varepsilon^\delta \quad (\text{def. of } \rho_\varepsilon^\delta \in L^2(\Omega^a)), \\ \widetilde{Du_\varepsilon^a} = \chi_{\Omega_\varepsilon^a} z(x) \psi^\delta + \sigma_\varepsilon^\delta \ (\text{def. of } \sigma_\varepsilon^\delta \in (L^2(\Omega^a))^N), \end{cases}$$

with

$$\begin{cases} \limsup_\varepsilon \left[\gamma \|\rho_\varepsilon^\delta\|^2_{L^2(\Omega^a)} + \alpha \|\sigma_\varepsilon^\delta\|^2_{(L^2(\Omega^a))^N} \right] \leq \\ \leq \left[\|c\|_{L^\infty(\Omega^a)} \|\sqrt{\theta}\, u^a - \sqrt{\theta}\, \varphi^\delta\|^2_{L^2(\Omega^a)} + \|a_0\|_{L^\infty(\Omega^a)} \|D_{x_N}(\sqrt{\theta}\, u^a) - \sqrt{\theta}\, \psi^\delta\|^2_{L^2(\Omega^a)} \right] \end{cases}$$

Difficult to understand, but the idea is:

Take $\varphi^\delta = u^a$, $\psi_\delta = D_{x_N} u^a$ (impossible in general....)

$$\text{then} \begin{cases} \widetilde{u_\varepsilon^a} - \chi_{\Omega_\varepsilon^a} u^a \longrightarrow 0 \text{ in } L^2(\Omega^a) \text{ strongly}, \\ \widetilde{Du_\varepsilon^a} - \chi_{\Omega_\varepsilon^a} z(x) D_{x_N} u^a \longrightarrow 0 \text{ in } (L^2(\Omega^a))^N \text{ strongly}. \end{cases}$$

Remark : It is impossible to take $\varphi^\delta = u^a$, $\psi^\delta = D_{x_N} u^a$ since they do not belong to $L^2(\Omega^a)$, but one can take

$$\forall \delta > 0, \quad \hat{\varphi}^\delta = \underbrace{\chi_{\{\theta \geq \delta\}^a} \frac{1}{\sqrt{\theta}}}_{\in L^\infty(\Omega^a)} \underbrace{\sqrt{\theta} u^a}_{\in L^2(\Omega^a)},$$

$$\hat{\psi}^\delta = \underbrace{\chi_{\{\theta \geq \delta\}^a} \frac{1}{\sqrt{\theta}}}_{\in L^\infty(\Omega^a)} \underbrace{D_{x_N}(\sqrt{\theta} u^a)}_{\in L^2(\Omega^a)},$$

and then : $\sqrt{\theta}\, \hat{\varphi}^\delta = \chi_{\{\theta \geq \delta\}^a} \sqrt{\theta} u^a \in L^2(\Omega^a)$

$$\sqrt{\theta} u^a - \sqrt{\theta}\, \hat{\varphi}^\delta = \chi_{\{\theta < \delta\}^a} \sqrt{\theta} u^a \xrightarrow{\delta} \chi_{\{\theta = 0\}^a} \sqrt{\theta} u^a \equiv 0$$

in $L^2(\Omega^a)$ strongly as $\delta \to 0$

and similarly for $\sqrt{\theta}\, \hat{\psi}^\delta$ ...

so that

$$\limsup_{\varepsilon} \left[\gamma \| \hat{\rho}^\delta_\varepsilon \|^2_{L^2(\Omega^a)} + \alpha \| \hat{\sigma}^\delta_\varepsilon \|^2_{(L^2(\Omega^a))^N} \right] \longrightarrow 0 \text{ as } \delta \to 0.$$

Sketch of the proof

① **First test function** : $v_\varepsilon = u_\varepsilon$ in the ε-problem

$$\int_{\Omega_\varepsilon} A\, Du_\varepsilon\, Du_\varepsilon + \int_{\Omega_\varepsilon} c\, u_\varepsilon^2 = \int_{\Omega_\varepsilon} f u_\varepsilon$$

$$\alpha \int_{\Omega_\varepsilon^a} |Du_\varepsilon^a|^2 + \gamma \int_{\Omega_\varepsilon^a} |u_\varepsilon^a|^2 + \alpha \int_{\Omega^b} |Du_\varepsilon^b|^2 + \gamma \int_{\Omega^b} |u_\varepsilon^b|^2 \leq \int_{\Omega_\varepsilon} f u_\varepsilon \leq$$

$$\leq \frac{\gamma}{2} \int_{\Omega_\varepsilon^a} |u_\varepsilon^a|^2 + \frac{\gamma}{2} \int_{\Omega^b} |u_\varepsilon^b|^2 + \underbrace{\frac{1}{2\gamma} \int_{\Omega_\varepsilon^a} |f|^2 + \frac{1}{2\gamma} \int_{\Omega^b} |f|^2}_{\leq C_0}$$

$\forall y_\varepsilon^a \in L^2(\Omega_\varepsilon^a)$: **Notation**: $\quad \widetilde{y_\varepsilon^a} = \begin{cases} y_\varepsilon^a & \text{in } \Omega_\varepsilon^a, \\ 0 & \text{in } \Omega^a \setminus \Omega_\varepsilon^a, \end{cases}$

$$\alpha \int_{\Omega^a} |\widetilde{Du_\varepsilon^a}|^2 + \frac{\gamma}{2} \int_{\Omega^a} |\widetilde{u_\varepsilon^a}|^2 + \alpha \int_{\Omega^b} |Du_\varepsilon^b|^2 + \frac{\gamma}{2} \int_{\Omega^b} |u_\varepsilon^b|^2 \leq C_0$$

2 We extract a subsequence ε such that

$$\begin{cases} \exists\, U^a \in L^2(\Omega^a), \\ \exists\, \Xi^a \in (L^2(\Omega^a))^N, \\ \exists\, u^b \in H^1(\Omega^b), \end{cases} \quad \text{s.t} \quad \begin{cases} \widetilde{u^a_\varepsilon} \longrightarrow U^a \text{ in } L^2(\Omega^a) \text{ weak}, \\ \widetilde{Du^\varepsilon_a} \longrightarrow \Xi^a \text{ in } (L^2(\Omega^a))^N \text{ weak}, \\ u^b_\varepsilon \longrightarrow u^b \text{ in } H^1(\Omega^b) \text{ weak}. \end{cases}$$

We define u^a by $\quad u^a = \begin{cases} \dfrac{U^a}{\theta} & \text{on } \{\theta > 0\}^a, \\ 0 & \text{on } \{\theta = 0\}^a = \{x \in \Omega^a : \theta(x) = 0\}. \end{cases}$

Then $U^a = \theta u^a$

since $U^a = 0$ on $\{\theta = 0\}^a$:

indeed: $\underset{\substack{\downarrow \\ \text{weakly } U^a}}{\widetilde{u^a_\varepsilon}} = \underset{\substack{\downarrow \\ U^a}}{\widetilde{u^a_\varepsilon}} \cdot \underset{\substack{\downarrow \\ \theta}}{\chi_{\Omega^a_\varepsilon}}$ but $\chi^a_\varepsilon \to 0$ strongly on $\{\theta = 0\}^a$
$L^2(\Omega^a)$

but $u^a \notin L^2(\Omega^a)$ since $u^a = \dfrac{U^a}{\sqrt{\theta}}$ on $\{\theta > 0\}^a$

Similarly we define ξ^a such that $\xi^a = 0$ on $\{\theta = 0\}^a$
$\Xi^a = \theta \xi^a$ in Ω^a.

3 Actually $\sqrt{\theta}\, u^a \in L^2(\Omega^a)$ and $\sqrt{\theta}\, \zeta^a \in (L^2(\Omega^a))^N$

Lemma let $y^a_\varepsilon \in L^2(\Omega^a_\varepsilon)$ with $\|\widetilde{y^\varepsilon_a}\|^2_{L^2(\Omega^a)} = \|y^\varepsilon_a\|^2_{L^2(\Omega^a_\varepsilon)} \leq C$

Assume that

$$y^a_\varepsilon \xrightarrow{\varepsilon} Y^a = \theta y^a \text{ in } L^2(\Omega^a) \text{ weakly,}$$

where as before $y^a = 0$ on $\{\theta = 0\}^a$. We have $\theta y^a \in L^2(\Omega^a)$

But $\sqrt{\theta}\, y^a \in L^2(\Omega^a)$ and

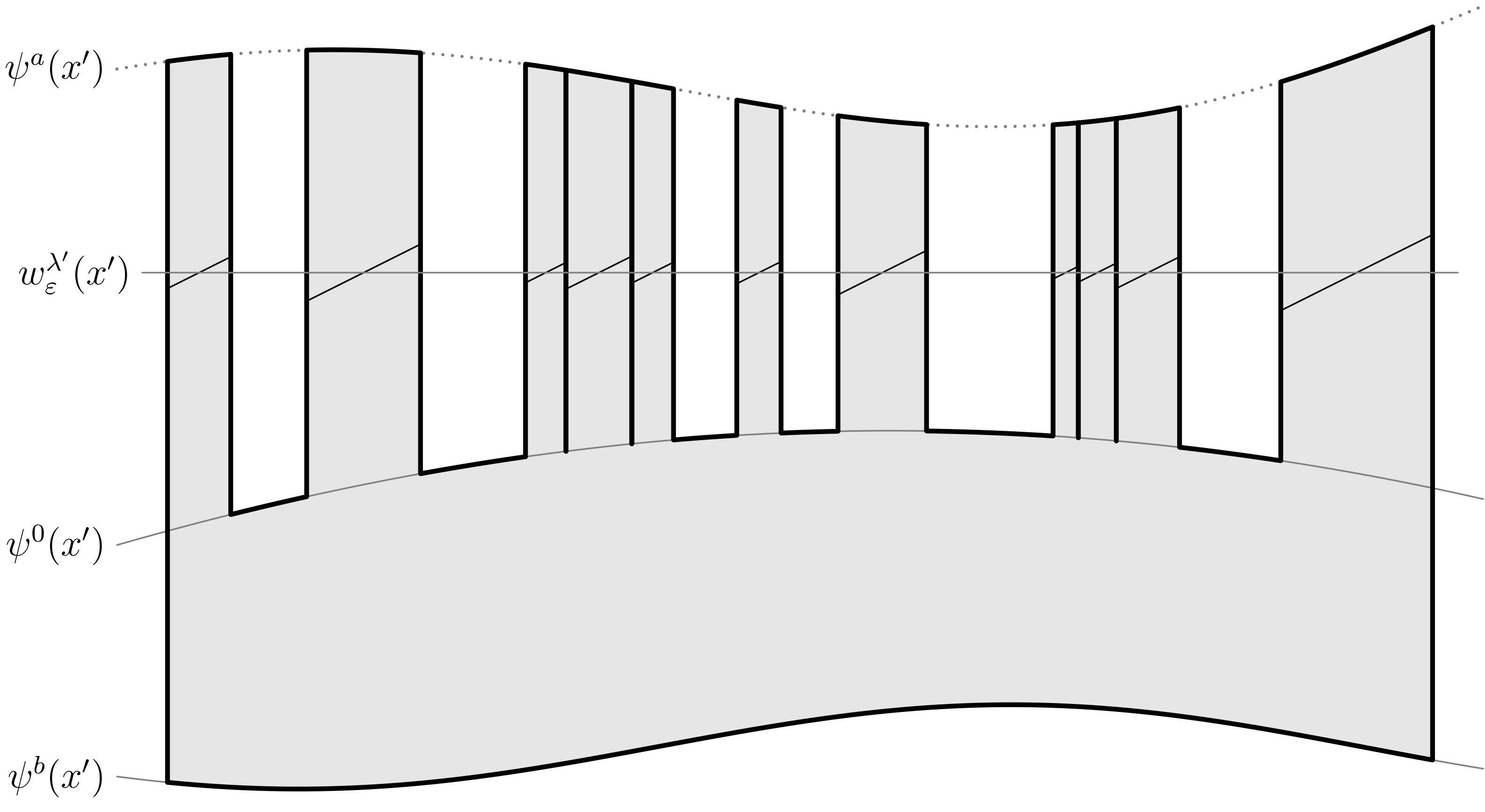
$$\liminf_\varepsilon \int_{\Omega^a} |\widetilde{y^a_\varepsilon}|^2 \geq \int_{\Omega^a} \theta |y^a|^2 .$$ lower semi continuity

Moreover if there is $=$ here, then $\forall \varphi^\delta \in L^2(\Omega^a)$,

$$\widetilde{y^a_\varepsilon} = \chi_{\Omega^a_\varepsilon} \varphi^\delta + \rho^\delta_\varepsilon \quad (\text{def of } \rho^\delta_\varepsilon \in L^2(\Omega^a))$$

with $\limsup_\varepsilon \|\rho^\delta_\varepsilon\|^2_{L^2(\Omega^a)} \leq \|\sqrt{\theta}\, y^a - \sqrt{\theta}\, \varphi^\delta\|^2_{L^2(\Omega^a)}$

Corrector result when $\hat{\varphi}^\delta = \chi_{\{\theta \geq \delta\}^a} \frac{1}{\sqrt{\theta}} \sqrt{\theta}\, y^a \quad \forall \delta > 0.$


$\psi^a(x')$
$w_\varepsilon^{\lambda'}(x')$
$\psi^0(x')$
$\psi^b(x')$

4 What is Ξ^a ? A new test function

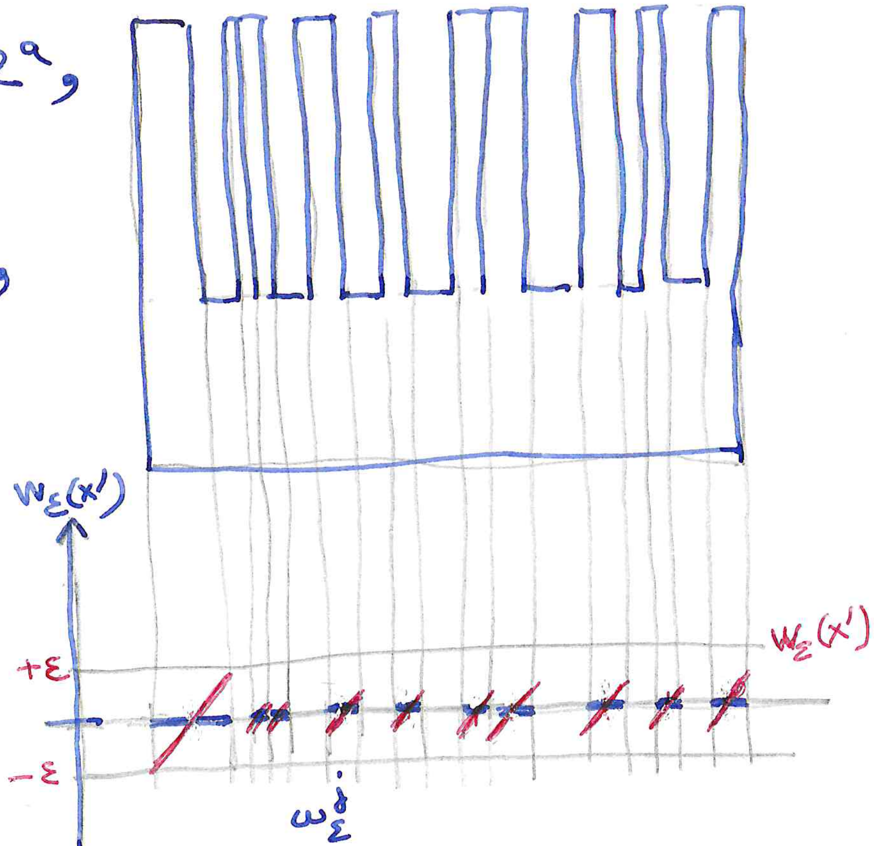
An idea of Robert BRIZZI and Jean-Paul CHALOT

Ph.D. Thesis, 1978

$\forall \lambda' \in \mathbb{R}^{N-1}$ define $w_\varepsilon(x) = w_\varepsilon(x') = \lambda'.(x' - c'^j_\varepsilon)$ in ω^j_ε

for some $c'^j_\varepsilon \in \omega^j_\varepsilon$

Then $\begin{cases} w_\varepsilon \in H^1(\Omega^a_\varepsilon), \\ |w_\varepsilon(x)| \leq |\lambda'|\,\varepsilon \quad \forall x \in \Omega^a, \\ (\text{since } \operatorname{diam} \omega^j_\varepsilon \leq \varepsilon) \\ D w_\varepsilon(x) = \begin{pmatrix} \lambda' \\ 0 \end{pmatrix} \ \forall x \in \Omega^a_\varepsilon, \end{cases}$



$\forall \varphi \in C^\infty_c(\Omega^a) \quad \varphi\, w^\varepsilon \in H^1(\Omega^\varepsilon)$

$$\Longrightarrow \begin{cases} \displaystyle\int_{\Omega^a} A\, \Xi^a \begin{pmatrix} \lambda' \\ 0 \end{pmatrix} \varphi = 0 \\ \forall \lambda' \in \mathbb{R}^{N-1}, \ \forall \varphi \in C^\infty_c(\Omega^a). \end{cases}$$

Hence, writing by blocks

$$\forall \lambda' \in \mathbb{R}^{N-1}, \quad 0 = A\,\Xi^a \begin{pmatrix} \lambda' \\ 0 \end{pmatrix} = \begin{pmatrix} A' & V' \\ H' & A_{NN} \end{pmatrix} \begin{pmatrix} \Xi^{a\prime} \\ \Xi^a_N \end{pmatrix} \begin{pmatrix} \lambda' \\ 0 \end{pmatrix} = \begin{pmatrix} A'\,\Xi^{a\prime} + V'\,\Xi^a_N \\ H'\,\Xi^{a\prime} + A_{NN}\,\Xi^a_N \end{pmatrix} \begin{pmatrix} \lambda' \\ 0 \end{pmatrix}$$

Define $z' \in L^\infty(\Omega^a)^{N-1}$ by $z' = -(A')^{-1} V'$ ie $A' z' + V' = 0$

Then $\Xi^{a\prime} = z'\,\Xi^a_N$

Define $a_0 \in L^\infty(\Omega^a)$ by $a_0 = H' z' + A_{NN} = -H'(A')^{-1}V' + A_{NN}$

and $a_0 \geq \alpha$ a.e.

Then $H'\,\Xi^{a\prime} + A_{NN}\,\Xi^a_N = a_0\,\Xi^a_N$

Define $z \in L^\infty(\Omega)^N$ by $z = \begin{pmatrix} z' \\ 1 \end{pmatrix}$

Then $A\,\Xi^a = \begin{pmatrix} 0 \\ a_0 \end{pmatrix} \Xi^a_N$

Note that z and a_0 only depend on the matrix A (more precisely on $A(x)$ for $x \in \Omega^a$)

5 Now we use the fact that the teeth are vertical cylinders (never used before). Therefore

$$\widehat{D_{x_N} u_\varepsilon^a} = D_{x_N} \widehat{u_\varepsilon^a}$$

$$\Rightarrow \quad \Xi_N^a = D_{x_N} U^a \quad \Rightarrow \quad \theta\, \xi_N^a = D_{x_N}(\theta u^a)$$

$$\sqrt{\theta}\sqrt{\theta}\, \xi_N^a = D_{x_N}(\sqrt{\theta}\sqrt{\theta}\, u^a)$$

$$= \sqrt{\theta}(x')\, D_{x_N}(\sqrt{\theta}\, u^a)$$

Therefore $\sqrt{\theta}\, u^a \in L^2(\Omega^a)$

with $D_{x_N}(\sqrt{\theta}\, u^a) \in L^2(\Omega^a)$

6 Moreover $\left\{ \begin{array}{l} u_\varepsilon \in H^1(\Omega_\varepsilon) \\ \widehat{u_\varepsilon^a} \in H(\Omega^a, D_{x_N}) \end{array} \right\} \Rightarrow \hat{\gamma}^a(\widehat{u_\varepsilon^a}) = \chi_{\Sigma_\varepsilon^0}\, \gamma^b(u_\varepsilon^b)$ on Σ^0

$$\hat{\gamma}^a(U^a) = \underbrace{\theta}_{\text{weak}} \cdot \underbrace{\gamma^b(u^b)}_{\text{strong}}$$

Therefore $\hat{\gamma}^a(\sqrt{\theta}\, u^a) = \sqrt{\theta}\, \gamma^b(u^b) \quad \Leftarrow \quad \hat{\gamma}^a(\sqrt{\theta}\sqrt{\theta}\, u^a) = \theta\, \gamma^b(u^b)$

$$\hat{\gamma}^a(\sqrt{\theta}\sqrt{\theta}\, u^a) = \sqrt{\theta}\, \hat{\gamma}^a(\sqrt{\theta}\, u^a)$$

7 In conclusion we have proved that

$$u = (u^a, u^b) \in V_\theta(\Omega^a, \Omega^b)$$

and passing to the limit in the ε-equation

$$\int_{\Omega^a} A \widetilde{Du^a_\varepsilon}\, Dv^a + \int_{\Omega^a} c\, \widetilde{u^a_\varepsilon}\, v^a_a + \int_{\Omega^b} A Du^b_\varepsilon\, Dv^b + \int_{\Omega^b} c\, u^b_\varepsilon v^b =$$

$$\widetilde{Du^a_\varepsilon} \downarrow \qquad A\, \Xi^a\, Dv^a = \begin{pmatrix} 0 \\ a_0 \end{pmatrix} \Xi^a_N\, Dv^a = a_0\, \theta\, \xi^a_N\, D_{x_N} v^a = a_0 \cancel{\sqrt{\theta}}\, D_{x_N}(\sqrt{\theta} u^a)\, D_{x_N}(\sqrt{\theta} v^a)$$

$$\widetilde{u^a_\varepsilon} \downarrow \qquad U^a = \theta u^a, \qquad = c\sqrt{\theta} u^a \sqrt{\theta} v^a$$

$$= \int_{\Omega^a} \chi_{\Omega^a_\varepsilon} f v^a + \int_{\Omega^b} f v^b$$

$$\forall v = (v^a, v^b) \in H^1(\Omega).$$

At the end we have

$$\begin{cases} \displaystyle\int_{\Omega^a} a_0 D_{x_N}(\sqrt{\theta} u^a)\, D_{x_N}(\sqrt{\theta} v^a) + \int_{\Omega^a} c \sqrt{\theta} u^a \sqrt{\theta} v^a + \\ \displaystyle\quad + \int_{\Omega^b} A Du^b Dv^b + \int_{\Omega^b} c u^b v^b = \int_{\Omega^a} \sqrt{\theta} f \sqrt{\theta} v^a + \int_{\Omega^b} f v^b \\ \qquad \forall v = (v^a, v^b) \in H^1(\Omega). \end{cases}$$

8 It remains to prove that one can replace $\forall v = (v^a, v^b) \in H^1(\Omega)$ by $\forall v = (v^a, v^b) \in \mathcal{V}_\theta(\Omega^a, \Omega^b)$

i.e. that one can approximate every $v \in \mathcal{V}_\theta(\Omega^a, \Omega^b)$ (in the norm of $\mathcal{V}_\theta$) by a sequence of $v_n \in H^1(\Omega)$

This proves that $u = (u^a, u^b)$ is the solution of the homogenized pb. The proof of the Homogenization Theorem is complete.

9 The convergence of the right-hand sides of the ε-problem with $v_\varepsilon = u_\varepsilon$ to the right-hand side of the homogenized problem proves the convergence of the energies, therefore that in the above lemma lim inf $\geq$ is actually lim inf $=$; this implies the corrector result and completes the proof of the Corrector Theorem.

HAPPY BIRTHDAY GIANNI !