Regularity for elliptic equations and systems under either slow or fast growth conditions

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CALCULUS OF VARIATIONS AND APPLICATIONS

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Nonlinear elliptic systems

Let $n \geq 2$, $m \geq 1$, let Ω be an open set of $\mathbb{R}^n$ and let $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a *weak solution* of a *nonlinear elliptic system* of PDE's of the form

$$\operatorname{div} A(Du) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}(Du) = 0, \qquad \alpha = 1, 2 \ldots m,$$

where $Du : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{m \times n}$ denotes the *gradient* of the map u, by components $x = (x_i)_{i=1,2,\ldots,n}$, $u = (u^{\alpha})^{\alpha=1,2,\ldots,m}$ and $Du = (\partial u^{\alpha}/\partial x_i) = (u^{\alpha}_{x_i})^{\alpha=1,2,\ldots,m}_{i=1,2,\ldots,n}$.

Then $A(\xi) = (a_i^{\alpha}(\xi))^{\alpha=1,2,\ldots,m}_{i=1,2,\ldots,n}$ is a given *vector field* $A : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{m \times n}$ of class C^1, satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} \frac{\partial a_i^{\alpha}(\xi)}{\partial \xi_j^{\beta}} \lambda_i^{\alpha} \lambda_j^{\beta} > 0, \qquad \forall\, \lambda, \xi \in \mathbb{R}^{m \times n} : \lambda \neq 0.$$

Nonlinear elliptic systems and Calculus of Variations

In the context of Calculus of Variations we assume the *variational condition* that the vector field $A(\xi)$ is the *gradient* of a function $f(\xi)$; i.e., that there exists a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ of class $C^2\left(\mathbb{R}^{m\times n}\right)$ such that

$$A(\xi) = D_\xi f(\xi)$$

and in terms of components

$$a_i^\alpha = \frac{\partial f}{\partial \xi_i^\alpha} = f_{\xi_i^\alpha}, \quad \forall\, \alpha = 1, 2, \ldots, m;\ \forall\, i = 1, 2, \ldots, n.$$

Under this variational condition, the previous ellipticity condition can be equivalently written in the form

$$\sum_{i,j,\alpha,\beta} \frac{\partial^2 f(\xi)}{\partial \xi_i^\alpha \partial \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta > 0, \quad \forall\, \lambda, \xi \in \mathbb{R}^{m\times n} : \lambda \neq 0.$$

Thus the ellipticity condition of the system is equivalent to the positivity on $\mathbb{R}^{m\times n}$ of the quadratic form $D_\xi^2 f(\xi)$ for every $\xi \in \mathbb{R}^{m\times n}$

$$\left(D_\xi^2 f(\xi)\,\lambda,\lambda\right) > 0, \qquad \forall\, \lambda,\xi \in \mathbb{R}^{m\times n} : \lambda \neq 0$$

which implies the *(strict) convexity* of the function f.

In this case any *weak solution* (in a class of maps u to be defined) to the differential elliptic system is a *minimizer* (also here, we need to define that class of maps which compete with u in the minimization process) of the energy functional

$$F(u) = \int_\Omega f(Du)\,dx$$

(and, in general, the vice-versa does not hold).

The regularity approach to nonlinear elliptic systems

In the general *vectorial setting* of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ which are *weak solution* of *nonlinear elliptic system* of PDE's of the previous form, it is well known that, in general, we can look for the so called partial regularity, since the pioneering work of Morrey and De Giorgi.

If some additional structure conditions are assumed, then some studies can be found in the mathematical literature on the subject for everywhere regularity.

For instance, the celebrated everywhere regularity results, about minimizers of the $p-$*Laplace* energy-integral, obtained by Uhlenbeck in 1977, with $f(\xi) = |\xi|^p$ and $p \geq 2$; that is

$$F(u) = \int_\Omega |Du\,(x)|^p \; dx\,.$$

The *regularity problem* for the previous elliptic system consists in asking if the solution $u = u(x) = (u^\alpha(x))^{\alpha=1,2,\ldots,m}$, *a-priori* only *measurable map* in a given Sobolev class, in fact is *continuos* or more regular; i.e. if u is of class either $C^{0,\alpha}$, C^1, $C^{1,\alpha}$, or C^k for some k, or even C^∞, under suitable assumption of smoothness of the data.

With the aim to explain the situation, we split the regularity process into two main parts (other points of view of smoothness are possible too), both relevant steps by themselves:

$1^{st}-$ from either a *minimizer*, or a *weak solution*, $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ (either in $W^{1,p}$ or in some other Sobolev or Orlicz classes) to $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$;

$2^{nd}-$ from a weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^m) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, under some smoothness of the data, to *more regularity* of the type $C^{1,\alpha}$, or C^k, or C^∞.

The second regularity step (2^{nd})

Let us start to briefly discussing the second regularity step:

from $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$ **to** $C^{1,\alpha}(\Omega, \mathbb{R}^m)$ **and to** $C^{\infty}(\Omega, \mathbb{R}^m)$

I.e., we consider the case when the weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ also belong to $W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$.

First: equations; i.e. the scalar case $m = 1$

Let us first discuss that *scalar case* $m = 1$, that is the case when the nonlinear system reduces to a nonlinear elliptic equation.

Under (so called *"natural"*) *ellipticity* and $p-$*growth conditions* ($p \geq 2$) on the function $f \in C^2\left(\mathbb{R}^{m\times n}\right)$, of the type

$$\begin{cases} \left(D_\xi^2 f(\xi)\lambda,\lambda\right) \geq m\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2 \\ \left|D_\xi^2 f(\xi)\right| \leq M\left(1+|\xi|^2\right)^{\frac{p-2}{2}} \end{cases}, \quad \forall\,\lambda,\xi \in \mathbb{R}^n,$$

it is possible to see that u admits second derivatives in weak form, i.e., $u \in W^{2,2}_{\text{loc}}(\Omega)$. Then, fixed $k \in \{1,2,\ldots,n\}$, we can take the $k-$derivative in both sides to the equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(Du) = 0\,,$$

and we obtain

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n \frac{\partial a_i(Du(x))}{\partial \xi_j}\left(u_{x_j}\right)_{x_k}\right) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du(x))}{\partial \xi_j}\left(u_{x_k}\right)_{x_j}\right) = 0\,.$$

Therefore the partial derivative u_{x_k} satisfies an elliptic differential equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial a_i\left(Du\left(x\right)\right)}{\partial \xi_j} \left(u_{x_k}\right)_{x_j} \right) = 0\,.$$

Recall that the map u is given (u is fixed); then we can *"forget"* the explicit dependence of $\partial a_i / \partial \xi_j$ on $Du\left(x\right)$. We define the element a_{ij} of an $n \times n$ matrix

$$a_{ij}\left(x\right) = \frac{\partial a_i\left(Du\left(x\right)\right)}{\partial \xi_j} = f_{\xi_i \xi_j}\left(Du\left(x\right)\right), \qquad i, j = 1, 2, \ldots, n.$$

Recalling that

$$m\left(1 + \left|Du\right|^2\right)^{\frac{p-2}{2}} \left|\lambda\right|^2 \leq \left(D_\xi^2 f\left(Du\right) \lambda, \lambda\right) \leq M\left(1 + \left|Du\right|^2\right)^{\frac{p-2}{2}}$$

and since *the gradient Du is locally bounded* in Ω, then the $n \times n$ square matrix $\left(a_{ij}\left(x\right)\right)_{n \times n}$ is uniformly elliptic, with *measurable locally bounded coefficients*.

Thus - as well known - we can apply the celebrated

De Giorgi's Hölder continuity result, 1957,

for the linear elliptic equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u_{x_k}}{\partial x_j} \right) = 0$$

with measurable coefficients:

for every $k \in \{1, 2, \ldots, n\}$ the partial derivative u_{x_k} is Hölder continuous for some exponent $\alpha \in (0, 1)$.

Thus $u \in C^{1,\alpha}(\Omega, \mathbb{R}^m)$.

The vector-valued case $m \geq 1$

In the vector-valued case $m \geq 1$ we need to assume a structure condition, of the type

$$f(\xi) = g(|\xi|) \,, \qquad \forall\, \xi \in \mathbb{R}^{m \times n}.$$

Then, again it is possible to show (in some cases) that

$$u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)\,,\ A \in C^{1,\gamma} \text{ for some } \gamma \in (0,1)$$

$$\Downarrow$$

$$u \in C^{1,\alpha} \text{ for some } \gamma \in (0,1)\,.$$

See for instance the $p-$*Laplace* energy-integral, studied by Uhlenbeck in 1977, with $f(\xi) = |\xi|^p$ and $p \geq 2$.

Moreover more regularity applies; in fact, if the function f is smooth, say $f \in C^{2,\gamma}\left(\mathbb{R}^{m\times n}\right)$, similarly to the scalar case, u admits second derivatives in weak form and, fixed $k \in \{1,2,\dots,n\}$, we can take the $k-$derivative in the system

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}(Du) = 0, \qquad \alpha = 1,2\dots m.$$

Thus the partial derivative $u_{x_k} = \left(u_{x_k}^{\beta}\right)^{\beta=1,2,\dots,m}$ satisfies
(u_{x_k} is a vector-valued map, a vector-valued partial derivative)

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\sum_{\beta=1}^{m} \frac{\partial a_i^{\alpha}(Du(x))}{\partial \xi_j^{\beta}} \left(u_{x_j}^{\beta}\right)_{x_k}\right)$$

$$= \sum_{i,j,\beta} \frac{\partial}{\partial x_i}\left(\frac{\partial a_i^{\alpha}(Du(x))}{\partial \xi_j^{\beta}} \left(u_{x_k}^{\beta}\right)_{x_j}\right) = 0\,, \quad \alpha = 1,2\dots m.$$

That is, for every $k \in \{1, 2, \dots, n\}$, the (vector-valued) map $u_{x_k} = \left(u^{\beta}_{x_k}\right)^{\beta=1,2,\dots,m}$ is a weak solution to the elliptic differential system

$$\sum_{i,j,\beta} \frac{\partial}{\partial x_i} \left(a^{\alpha\beta}_{ij}(x) \, \frac{\partial u^{\beta}_{x_k}}{\partial x_j} \right) = 0 \, , \qquad \alpha = 1, 2 \dots m.$$

where $a^{\alpha\beta}_{ij}(x) \stackrel{\text{def}}{:=} \partial a^{\alpha}_{i} / \partial \xi^{\beta}_{j} (Du(x)) = f_{\xi^{\alpha}_{i} \xi^{\beta}_{j}} (Du(x))$

are now Hölder continuous coefficients, since $u \in C^{1,\alpha}$.

Thus we can apply the classical regularity results in the literature for linear elliptic systems with smooth coefficients (see for instance Section 3 of Chapter 3 of the book by Giaquinta, 1983) to infer

$$u \in C^{1,\alpha}, \;\; A \in A \in C^{k,\gamma} \qquad \Longrightarrow \qquad u \in C^{k,\alpha}, \quad \forall \, k = 2, 3, \dots$$

In particular, if $A \in C^{\infty}$ (or equivalently $f \in C^{\infty}$) then $u \in C^{\infty}(\Omega, \mathbb{R}^m)$.

The first regularity step (1^{st})

from $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ **to** $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$

Therefore the problem which remains to be considered is: *under which conditions on the vector field* $A : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$, $A(\xi) = \left(a_i^\alpha(\xi)\right)_{i=1,2,\dots,n}^{\alpha=1,2,\dots,m}$ *the gradient* Du *is in fact locally bounded? I.e.: we look for sufficient conditions for* $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$.

Why the local boundedness of the gradient Du is a so relevant condition for regularity? Because the differential system heavily depends on Du in a nonlinear way, in particular trough $a_i^\alpha(Du)$ and, if $Du(x)$ is bounded, then $a_i^\alpha(Du(x))$ (here "$p \geq 2$") is bounded too and far away from zero. Thus the behavior of $A(\xi) = \left(a_i^\alpha(\xi)\right)$ for $|\xi| \to +\infty$ becomes irrelevant.

On the contrary, the local boundedness of the gradient is a property related to the behavior of $A(\xi)$ as $|\xi| \to +\infty$.

For $W^{1,\infty}_{\text{loc}}$ estimates, growth conditions play a relevant role.

We summarize with a scheme:

$1^{st}-$ from $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ to $W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$; growth conditions, either of the vector field $A(\xi) = (a_i^\alpha(\xi))$ or of the integrand $f(\xi)$, play a central role;

$2^{nd}-$ from $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$ to $C^{1,\alpha}$ or C^∞; essentially growth conditions are not considered (however, of course, some uniform ellipticity must be considered, in this step too).

VECTOR-VALUED MAPS - some former results

First, let us mention again the celebrated example by De Giorgi 1968, who considered an integral of the type

$$F(u) = \int_{\Omega} f\left(x, Du\left(x\right)\right)\, dx$$

and he proved that his techniques valid for the scalar case $m = 1$ cannot be extended to the vector-valued case $m > 2$ with systems. Later (but published in the same year) Giusti and M. Miranda 1968 proposed a similar example for a minimizer of the energy-integral

$$F(u) = \int_{\Omega} f\left(u, Du\left(x\right)\right)\, dx\,.$$

The first example of a singular minimizer of an energy-integral without x and u explicit dependence is due to Nečas 1977, for

$$F(u) = \int_{\Omega} f\left(Du\left(x\right)\right)\, dx\,,$$

The minimizer found by Nečas 1977 is a map $u : \mathbb{R}^n \longrightarrow \mathbb{R}^{n^2}$ with n large.

Later Šverák & Yan 2000 found an example of a singular minimizer in 3 dimensions; precisely for a map $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^5$.

More recently Connor Mooney & Ovidiu Savin, ARMA 2016, constructed a singular minimizing map $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ of a smooth uniformly convex energy-integral.

There exists a study by Mooney alone, on Arxiv March 2019, for a singular minimizer map u defined in $\Omega \subset \mathbb{R}^4$.

As already said, for regularity in the vector-valued case $m > 1$ we need to assume a structure condition. Usually the condition is

$$f(\xi) = g(|\xi|) \,, \qquad \forall\, \xi \in \mathbb{R}^{m \times n}.$$

If this additional structure condition is assumed, then several results can be found in the mathematical literature for everywhere regularity. As already said, a main reference is the $p-$*Laplace* energy-integral, studied by Uhlenbeck in 1977, with $f(\xi) = |\xi|^p$ and $p \geq 2$. The energy-integral has the form

$$F(u) = \int_\Omega |Du(x)|^p \, dx$$

and the $p-$*Laplace* operator - as well known - either in the scalar context $m = 1$ for equations, or for vector-valued maps and systems $m > 1$, is

$$\operatorname{div} A(Du) = \operatorname{div} \, |Du|^{p-2} \, Du$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|Du|^{p-2} \, u^\alpha_{x_i} \right) , \qquad \alpha = 1, 2 \ldots m.$$

We can also consider the non-degenerate $p-$*Laplace* energy-integral and the corresponding $p-$*Laplace* operator with exponent $p > 1$, respectively given by

$$F(u) = \int_\Omega \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} dx\,;$$

$$\text{div}\, A(Du) = \text{div}\left(1 + |Du(x)|^2\right)^{\frac{p-2}{2}} Du\,.$$

We have a natural generalization of these examples with the integrand of the form $f(\xi) = g(|\xi|)$, where $g : [0, +\infty) \to [0, +\infty)$ is an increasing convex function. The energy-integral F takes the form

$$F(u) = \int_\Omega g(|Du|)\, dx\,.$$

A relevant difference with the $p-$Laplacian relies on the growth assumptions that we assume on $g = g(t)$; precisely, on the behavior of $g(t)$ as $t \to +\infty$.

For instance, the local Lipschitz regularity result by Marcellini (1996) can be applied to the *exponential growth*, and also to any finite composition of exponential, such as (with $p_i \geq 1, \forall i = 1, 2 \dots k$)

$$g(|\xi|) = (\exp(\dots(\exp(\exp|\xi|^2)^{p_1})^{p_2})\dots)^{p_k}.$$

However, some other restrictions ware imposed, such as, for instance, the fact that $t \in (0, +\infty) \to \frac{g'(t)}{t}$ is assumed to be an increasing function. To exemplify, the model case $g(t) = t^p$ gives the restriction $p \geq 2$.

Also $p, q-$growth can be considered, with the energy-integrand $f(\xi) = g(|\xi|)$ which do not behave like a power when $|\xi| \to \infty$. For instance, for $|\xi|$ large (i.e., $|\xi| \geq e$), the integrand could be of the type

$$f(\xi) = g(|\xi|) = |\xi|^{a+b \sin\log\log|\xi|}, \qquad g(t) = t^{a+b\sin\log\log t}.$$

In fact a computation shows that such an integrand is a convex function for $|\xi| \geq e$ $(t \geq e)$.

Therefore the function

$$g(|\xi|) = |\xi|^{a+b \sin \log \log |\xi|}$$

a-priori defined for $|\xi| \geq e$, can be extended to all $\xi \in \mathbb{R}^{m \times n}$ as a convex function on $\mathbb{R}^{m \times n}$ if a, b are positive real numbers such that $a > 1 + b\sqrt{2}$. In this case our integrand satisfies the $p, q-$growth conditions, with $p < q$, where $p = a - b$ and $q = a + b$,

$$|\xi|^p \leq f(\xi) \leq 1 + |\xi|^q, \quad \forall\, \xi \in \mathbb{R}^{m \times n}.$$

A remark: the "Δ_2-condition" (which can be found in the literature) is considered to be the "generalized uniformly elliptic case". The function $f(\xi)$ above satisfies the Δ_2-condition. While we can construct (details by Bögelein-Duzaar-Marcellini-Scheven, 2018) some convex functions $f(\xi) = g(|\xi|)$, satisfying $p, q-$growth conditions, with $q > p$ and q arbitrarily close to p, which do not satisfy the Δ_2-condition and which enter in the regularity theory presented here.

Some other references:

Fuchs-Mingione (2000) concentrate on the case of nearly-linear growth. Typical examples are the logarithmic case

$$f(\xi) = |\xi| \log(1 + |\xi|)$$

and its iterated version, for $k \in$ arbitrary,

$$\begin{cases} f_k(\xi) & = & |\xi| L_k(|\xi|), \\ L_{s+1}(t) & = & \log(1 + L_s(t)), \ L_1(t) = \log(1 + t) \end{cases} .$$

Leonetti-Mascolo-Siepe (2003) consider the case of subquadratic $p, q-$growth, i.e. they assume $1 < p < q < 2$;

their result includes energy densities f of the type (here $p < 2$)

$$f(\xi) = |\xi|^p \log^\alpha(1 + |\xi|) .$$

Also Bildhauer (2003) considers nearly-linear growth; he gives conditions that can keep "γ-elliptic linear growth" with $\gamma < 1 + \frac{2}{n}$.

Examples of γ-elliptic linear integrands are given by

$$g_\gamma(t) = \int_0^t \int_0^s (1 + z^2)^{-\frac{\gamma}{2}} dz ds, \ \forall t \geq 0.$$

For $\gamma = 1$, $g_\gamma(t)$ behaves like $t \log(1 + t)$

and in the (not included) limit case $\gamma = 3$, $g_\gamma(t)$ becomes $(1 + t^2)^{1/2}$.

Note that the *minimal surface integrand* $g(t) = \sqrt{1 + t^2}$ does not enter in the assumptions of these quoted regularity results.

Marcellini and Gloria Papi (2006) gave conditions which include different kind of growths: *more general conditions on the function g embracing growths moving between linear and exponential functions.*

The conditions are the following (we consider explicitly the case $n \geq 3$, while if $n = 2$ then the exponent $\frac{n-2}{n}$ can be replaced by any real number):
Let $t_0, H > 0$ and let $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$. For every $\alpha \in \left(1, \frac{n}{n-1}\right]$ there exist $K = K(\alpha)$ such that, for all $t \geq t_0$,

$$Ht^{-2\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \leq \; g''(t) \; \leq K\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right].$$

The exponent α in the right hand side is a parameter to be used to test more functions g. The condition in the left-hand side allows us to achieve functions - for instance - with second derivative going to zero as a power $t^{-\gamma}$, with γ small, i.e. $\gamma < 1 + \frac{2}{n}$.

In the paper by Marcellini and Papi in 2006 the following two results are proved, the first one valid under *general growth conditions*, the second one specific for the *linear case*.

Theorem A (General growth). *Let $g : [0, +\infty) \to [0, +\infty)$ be a convex function of class $W^{2,\infty}_{\mathrm{loc}}$ with $g(0) = g'(0) = 0$, satisfying the general growth conditions stated above. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega; R^m)$ be a minimizer. Then*

$$u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; R^m)\,.$$

Moreover, for every $\epsilon > 0$ and $R > \rho > 0$ there exists a constant $C = C(\epsilon, n, \rho, R)$ such that

$$\|Du\|^{2-\beta n}_{L^\infty(B\rho;\mathbb{R}^{m\times n})} \leq C \left\{ \int_{B_R} (1 + g(|Du|))\, dx \right\}^{\frac{1}{1-\beta}+\epsilon}.$$

Theorem B (Linear growth). *Let* $g : [0,+\infty) \to [0,+\infty)$ *be a convex function of class* $W^{2,\infty}_{\mathrm{loc}}$ *with* $g(0) = g'(0) = 0$. *If* g *has the linear behavior at infinity*

$$\lim_{t \to +\infty} \frac{g(t)}{t} = l \in (0,+\infty)$$

please note: linear behavior as $t \to +\infty$

and if its second derivative satisfies the inequalities

$$H \frac{1}{t^{\gamma}} \le g''(t) \le K \frac{1}{t}\,, \qquad \forall t \ge t_0\,,$$

for some positive constants H, K, t_0 *and for some* $\gamma \in \left[1, 1 + \frac{2}{n}\right)$, *then* $u \in W^{1,\infty}_{loc}(\Omega; R^m)$ *and, for every* $R > \rho > 0$, *there exists a constant* $C = C(n, \rho, R, l, H, K)$ *such that*

$$\|Du\|^{\frac{2-n(\gamma-1)}{2}}_{L^{\infty}(B\rho;\mathbb{R}^{m\times n})} \le C \int_{B_R} (1 + g(|Du|))\, dx\,.$$

Mascolo-Migliorini (2003) studied some cases of integrands $f(x,\xi) = g(x, |\xi|)$ with general growth conditions, which however ruled out the slow growth and power growth with exponents $p \in (1,2)$.

Recently Beck-Mingione (2018) introduced in the integrand some $x-$dependence in lower order terms, of the form

$$F(u) = \int_\Omega \{g(|Du|) + h(x) \cdot u\} \, dx \, .$$

They obtained the local boundedness of the gradient Du of the local minimizer under some general growth assumptions on the principal part $g(|\xi|)$, which is assumed to be independent of x. They considered some sharp assumptions on the function $h(x)$, of the type $h \in L(n,1)(\Omega;\mathbb{R}^m)$ in dimension $n > 2$:

$$\int_0^{+\infty} \text{meas}\,\{x \in \Omega : |h(x)| > \lambda\}^{1/n} \, d\lambda < +\infty \, .$$

For the Lorentz space $L(n,1)$ note that $L^{n+\varepsilon} \subset L(n,1) \subset L^n$; otherwise $h \in L^2 (\log L)^\alpha (\Omega;\mathbb{R}^m)$ for some $\alpha > 2$ when $n = 2$.

VECTOR-VALUED MAPS - $x-$dependence

This is a recent joint research - submitted, under referee - joint with Tommaso Di Marco , from the University of Firenze.

We are concerned with the regularity of *local minimizers* of energy-integrals of the calculus of variations explicitly depending on x, with energy-integrals and differential systems respectively of the form

$$F(u) = \int_\Omega f(x, Du)\, dx = \int_\Omega g(x, |Du|)\, dx\,,$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{g_t(x, |Du|)}{|Du|} u_{x_i}^\alpha \right) = 0, \qquad \alpha = 1, 2, \ldots, m,$$

where $g = g(x, t)$ and $t = |Du|$ is the gradient variable.

We consider a general integrand of the form $g = g(x,t)$, with $g : \Omega \times [0,\infty) \to [0,\infty)$ Carathéodory function, convex and increasing with respect to $t \in [0,\infty)$. Our assumptions allow us to consider *both fast and slow growth* on the integrand $g(x,|Du|)$. Model energy-integrals that we have in mind are, for instance, *exponential growth* with local Lipschitz continuous coefficients a, b $(a(x), b(x) \geq c > 0)$

$$\int_\Omega e^{a(x)|Du|^2}\, dx \quad or \quad \int_\Omega b(x) \exp\left(\ldots \exp\left(a(x)|Du|^2\right)\right)\, dx\,;$$

variable exponents $(a, p \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, $a(x) \geq c > 0$ and $p(x) \geq p > 1)$

$$\int_\Omega a(x)|Du|^{p(x)}\, dx \quad \text{or} \quad \int_\Omega a(x)\left(1+|Du|^2\right)^{p(x)/2}\, dx\,;$$

of course the classical $p-$*Laplacian* energy-integral, with a constant p strictly greater than 1 and integrand $f(x,Du) = a(x)|Du|^p$, can be allowed: the theory considered here applies to the $p-$Laplacian.

Also Orlicz-type energy-integrals (see the recent papers by Chlebicka *et al.* 2018-2019), again with local Lipschitz continuous exponent $p(x) \geq p > 1$, of the type

$$\int_{\Omega} a(x) |Du|^{p(x)} \log(1 + |Du|) \, dx \, .$$

Note that we can also consider some cases with *slow growth*, for instance when $p(x)$ is identically equal to 1. Again, here the $x-$dependence makes delicate the case $p(x) \geq 1$.

The $p, q-$growth case with $x-$dependence, of the type

$$\int_{\Omega} \alpha(x) |Du|^{a(x)+b(x) \sin \log \log |Du|} \, dx \, .$$

Also some $g(x,|\xi|)$ with *slow growth*, precisely *linear growth* as $t=|Du|\to+\infty$. Let us consider in low dimension $n=2,3$

$$g(x,|\xi|)=|\xi|-a(x)\sqrt{|\xi|}\,,\quad \forall\, x\in\Omega,\ \ \forall\,\xi\in\mathbb{R}^{m\times n},\ |\xi|\geq 1\,;$$

i.e. we consider

$$t\to g(x,t)=t-a(x)\sqrt{t}$$

for $t\geq 1$ and we extend it to $[0,+\infty)$ as a smooth convex increasing function in $[0,+\infty)$, with derivative equal to zero at $t=0$. With abuse of notation, we will continue to denote by $g(x,t)=t-a(x)\sqrt{t}$ this extended function. The regularity result that we will state below applies also to this energy-integral (which we will continue to denote with the same expression)

$$\int_\Omega\left\{|Du|-a(x)\sqrt{|Du|}\right\}dx\,,\quad \boxed{\text{please note: linear behavior as } t\to+\infty}$$

with $a\in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, $a(x)\geq c>0$.

We emphasize that $a(x) \geq c > 0$.

We cannot consider $g(x, |\xi|) = |\xi|$ alone, with $a(x) = c = 0$.

Moreover, it is relevant to notice that our proof does not allow to us to consider the $x-$dependence in the leading term $|Du|$.

More precisely, we are not able to establish the local Lipschitz continuity of local minimizers to the energy-integral

$$\int_{\Omega} \left\{ m(x)\,|Du| - a(x)\sqrt{|Du|} \right\} dx\,,$$

unless $m(x)$ is a positive constant. This seems strongly due to the exponent $p = 1$ in the leading term $|Du|^p$. On the contrary, when $p > 1$ the gradient bound applies to this case too.

With respect to the previous references, related to these researches, we mention the *double phase problems* recently intensively studied by Colombo-Mingione 2015 and Baroni-Colombo-Mingione 2016-2018; the *double phase* with *variable exponents* by Eleuteri-Marcellini-Mascolo 2016-2018. See also Esposito-Leonetti-Mingione 2004, Rădulescu-Zhang 2018, Cencelja-Rădulescu-Repovš 2018 and De Filippis 2018. For related recent references we quote Bousquet-Brasco 2019, Carozza-Giannetti-Leonetti-Passarelli 2018, Cupini-Giannetti-Giova-Passarelli 2018, Cupini-Marcellini-Mascolo 2009-2012-2018, De Filippis-Mingione 2019, Harjulehto-Hästö-Toivanen 2017, Hästö-Ok 2019, Mingione-Palatucci 2019.

Without loss of generality, by changing $g\left(x,t\right)$ with $g\left(x,t\right)-g\left(x,0\right)$ if necessary, we can reduce ourselves to the case $g(x,0)=0$ for almost every $x\in\Omega$. We assume that the partial derivatives g_t, g_{tt}, g_{tx_k} exist (for every $k=1,2,\ldots n$) and that they are Carathéodory functions too, with $g_t\left(x,0\right)=0$.

The following assumptions cover the previous model examples. Precisely, we require that the growth conditions hold:

Let $t_0 > 0$ be fixed; for every open subset Ω' compactly contained in Ω, there exist $\vartheta \geq 1$ and positive constants m and M_ϑ such that

$$\begin{cases} mh'(t) \leq g_t(x,t) \leq M_\vartheta \left[h'(t)\right]^\vartheta t^{1-\vartheta} \\ mh''(t) \leq g_{tt}(x,t) \leq M_\vartheta \left[h''(t)\right]^\vartheta \\ \left|g_{tx_k}(x,t)\right| \leq M_\vartheta \min\left\{g_t(x,t), t\, g_{tt}(x,t)\right\}^\vartheta \end{cases}$$

for every $t \geq t_0$ and for $x \in \Omega'$, where $h : [0,+\infty) \to [0,+\infty)$ is an increasing convex function as in Marcellini-Papi 2006.

The role of the parameter ϑ can be easily understood if apply these growth conditions with the above model examples. For instance:

$$g(x,Du) = |Du|^{p(x)}, \text{ i.e. } \; g(x,t) = t^{p(x)} \text{ with } t \geq 0,$$

then

$$g_t(x,t) = p(x)\ t^{p(x)-1}, \quad t\, g_{tt}(x,t) = p(x)(p(x)-1)\ t^{p(x)-1}$$

and

$$g_{tx_k}(x,t) = p_{x_k}(x)\, t^{p(x)-1} + p(x) \frac{\partial}{\partial x_k}\left[e^{(p(x)-1)\log t}\right]$$

$$= p_{x_k}(x)\, t^{p(x)-1}\left[1 + p(x)\log t\right].$$

We cannot compare (we cannot estimate) $\left|g_{tx_k}(x,t)\right|$ in terms neither of $g_t(x,t)$ nor of $t g_{tt}(x,t)$.

On the contrary, if we denote by L the Lipschitz constant of $p(x)$ on a fixed open subset Ω' whose closure is contained in Ω, then

$$\frac{\left|g_{tx_k}(x,t)\right|}{(t\, g_{tt}(x,t))^{\vartheta}} \leq L \frac{1 + p(x)\log t}{p^{\vartheta}(x)(p(x)-1)^{\vartheta}\, t^{(\vartheta-1)(p(x)-1)}}$$

and thus the quotient is bounded for $t \in [1, +\infty)$ and $x \in \Omega'$ if $p(x) > 1$ is locally Lipschitz continuous in Ω (i.e., also being $p(x) \geq c > 1$ for some constant $c = c(\Omega')$) and $\vartheta > 1$.

The role of the parameter ϑ, strictly greater than 1, is relevant.

The special case $\vartheta = 1$ corresponds to the so called *natural growth conditions*.

As already said, here we follow a similar condition by Marcellini-Papi 2006 on the function $h : [0, +\infty) \to [0, +\infty)$, which is a convex increasing function of class $W^{2,\infty}_{\mathrm{loc}}$ satisfying the following property: for some $\beta > \frac{1}{n}$ such that $(2\vartheta - 1)\vartheta < (1 - \beta)\frac{2^*}{2}$, and for every α such that $1 < \alpha \leq \frac{n}{n-1}$, there exist constants m_β and M_α such that, for every $t \geq t_0$,

$$\frac{m_\beta}{t^{2\beta}}\left[\left(\frac{h'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{h'(t)}{t}\right] \leq h''(t) \leq M_\alpha\left[\left(\frac{h'(t)}{t}\right)^{\alpha} + \frac{h'(t)}{t}\right]$$

The following a-priori gradient estimate holds.

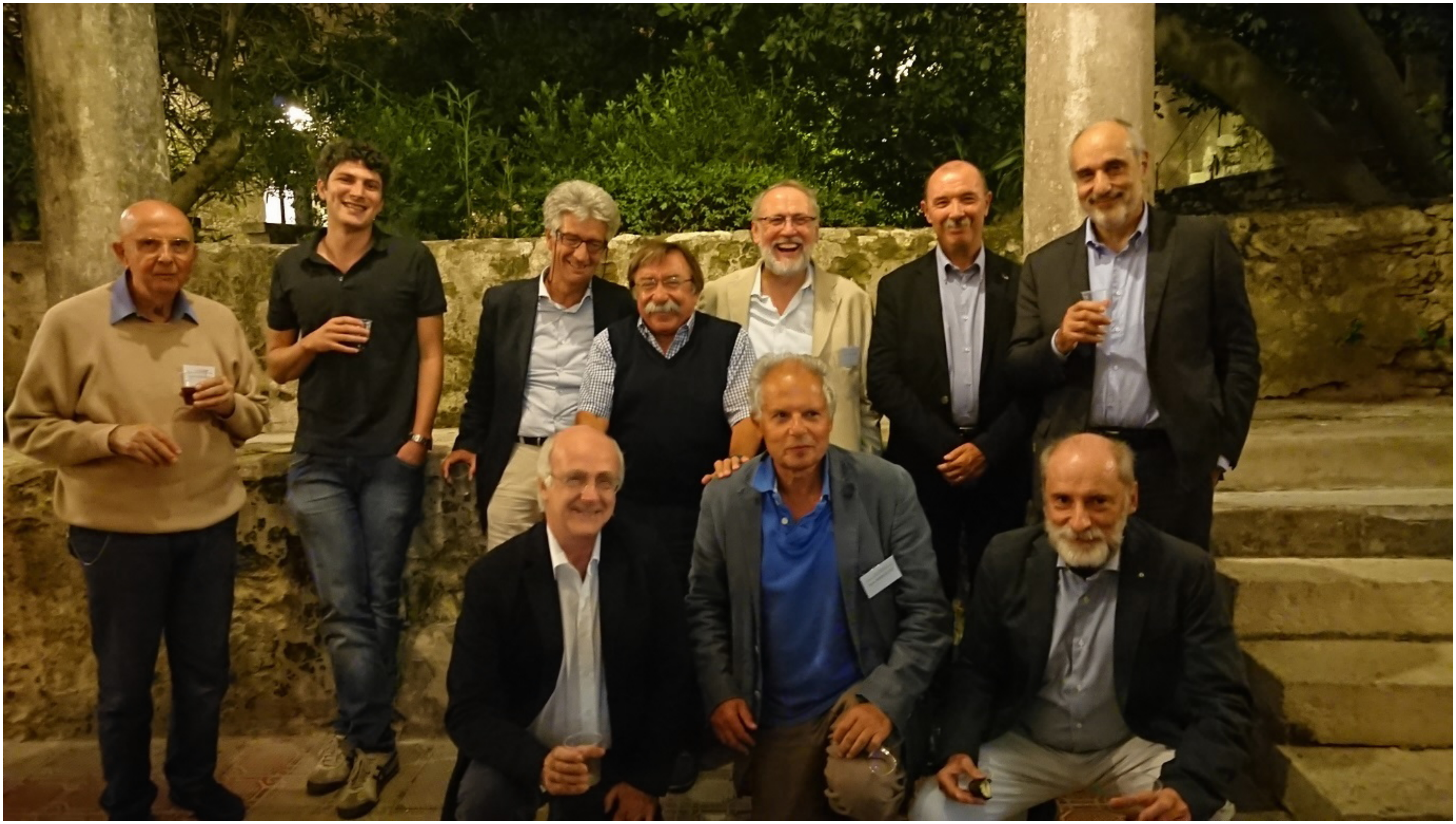
Theorem (joint work with Tommaso Di Marco, University of Firenze) Submitted - under referee.

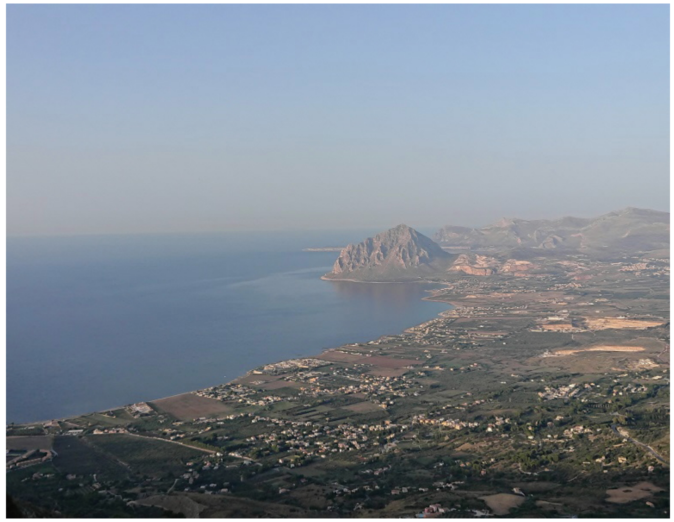
Under the previous assumptions (satisfied by the given examples) the gradient of any smooth local minimizer is uniformly locally bounded in Ω.

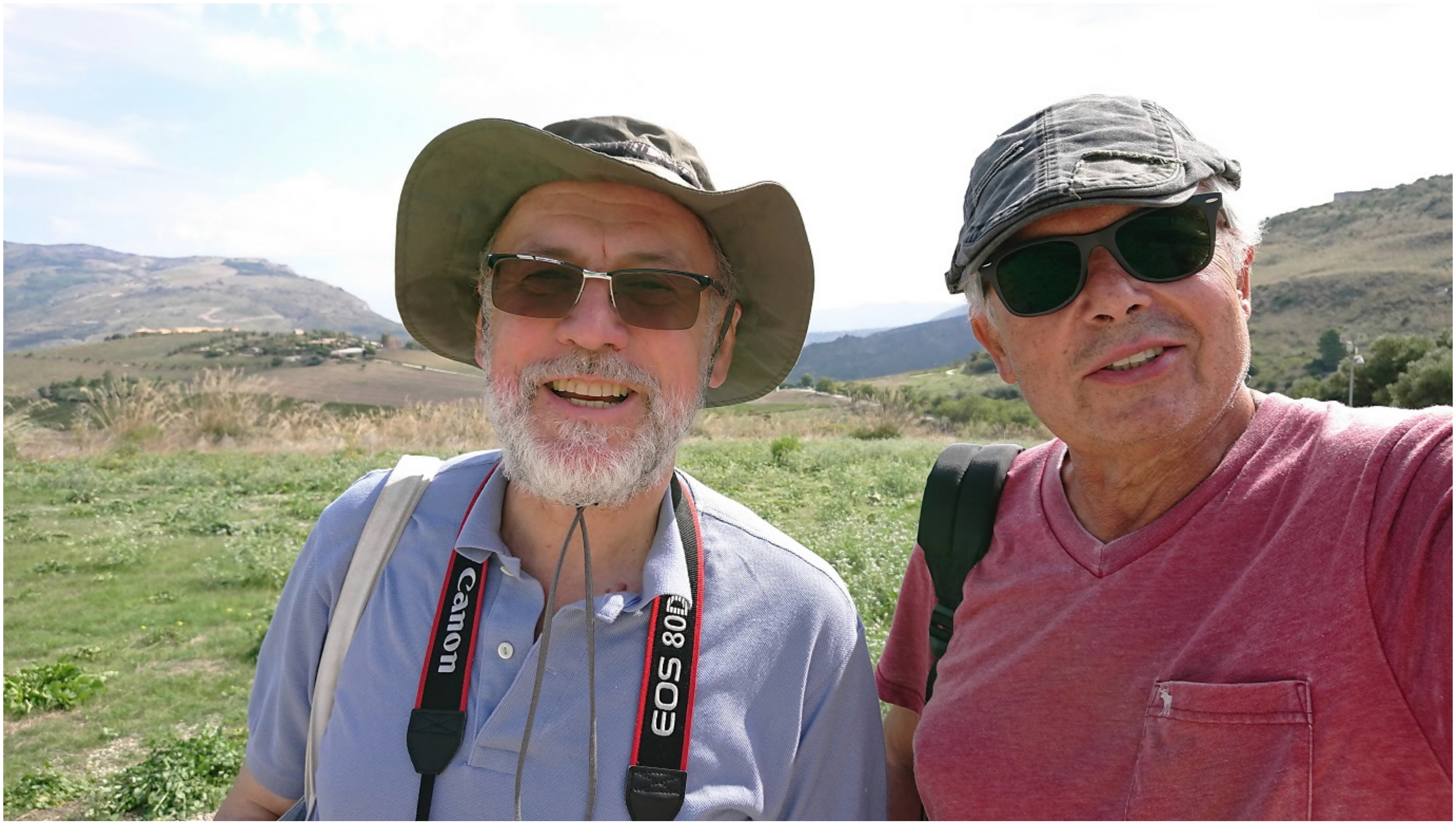
Precisely, there exist an exponent $\omega > 1$ and, for every ρ, R, $0 < \rho < R$, there exist a positive constant C such that

$$\|Du\|_{L^\infty(B_\rho,\mathbb{R}^{m\times n})} \le C\left\{\int_{B_R} (1 + g(x, |Du|))\, dx\right\}^{\omega}.$$

The exponent ω depends on ϑ, β, n, while the constant C depends on $\rho, R, n, \alpha, \beta, \vartheta, t_0$ and $\sup\{h''(t) :\ t \in [0, t_0]\}$.



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CALCULUS OF VARIATIONS AND APPLICATIONS

January 27^{th}- February 1^{st}, 2020 — Trieste, SISSA

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Thanks to the Organizers:

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Happy Birthday Gianni!!

and thanks to all the audience