# Asymptotic Development by Gamma Convergence

## Gianni Dal Maso's 65th birthday

Giovanni Leoni

Carnegie Mellon University

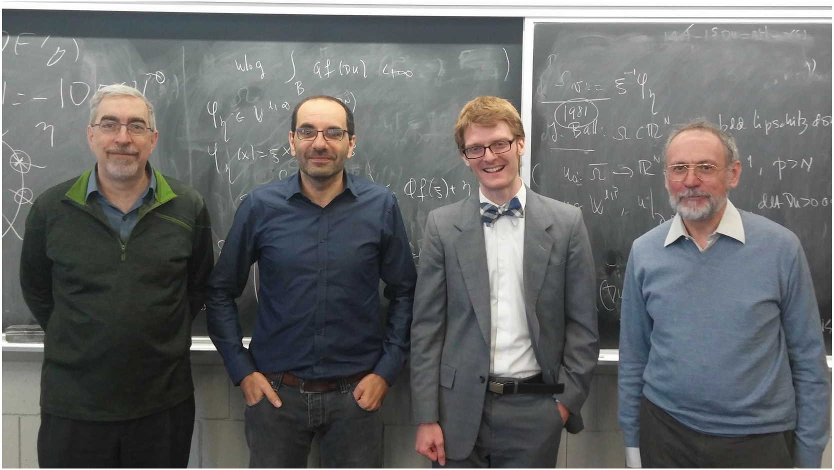
January 27, 2020

# CMU, October 2002

# CMU, Workshop 2003

# Miami, December 2009

# CMU, May 2016



- 2-quasiconvexity. Dal Maso, Fonseca, G. L., & Morini, ARMA, 2004;
- Image restoration. Dal Maso, Fonseca, G. L., & Morini, SIMA, 2009;
- Counterexample integral representation. Dal Maso, Fonseca, & G. L., Adv. Calc. Var., 2010;
- Singularly perturbed problems. Chermisi, Dal Maso, Fonseca, & G. L., Indiana Univ. Math. J, 2011;
- Singular parabolic equations, Hilbert transform. Dal Maso, Fonseca, & G. L., ARMA, 2014;
- Second order Gamma convergence. Dal Maso, Fonseca, & G. L., Calc. Var. Partial Differential Equations, 2015;
- Nonlocal functionals. Dal Maso, Fonseca, & G. L., Trans. Amer. Math. Soc., 2018;
- Minimizing movements, elliptic systems in domains with corners. Dal Maso, Fonseca, & G. L., in preparation.

# Outline of the Talk:

- Gamma-Asymptotic Development of Order k

# Outline of the Talk:

- Gamma-Asymptotic Development of Order k
- Singularly perturbed Neumann-Robin boundary problems

# Outline of the Talk:

- Gamma-Asymptotic Development of Order k
- Singularly perturbed Neumann-Robin boundary problems
  - G.Gravina and G. L., 2019, SIAM J. Math. Anal.,

# Outline of the Talk:

- Gamma-Asymptotic Development of Order k
- Singularly perturbed Neumann-Robin boundary problems
  - G.Gravina and G. L., 2019, SIAM J. Math. Anal.,
- Asymptotic development of order 2 for Cahn–Hilliard functional

# Outline of the Talk:

- Gamma-Asymptotic Development of Order k
- Singularly perturbed Neumann-Robin boundary problems
  - G.Gravina and G. L., 2019, SIAM J. Math. Anal.,
- Asymptotic development of order 2 for Cahn–Hilliard functional
  - G.L. and R. Murray, 2016, ARMA, & 2019, Proc. Amer. Math. Soc.

# Outline of the Talk:

- Gamma-Asymptotic Development of Order k
- Singularly perturbed Neumann-Robin boundary problems
  - G.Gravina and G. L., 2019, SIAM J. Math. Anal.,
- Asymptotic development of order 2 for Cahn–Hilliard functional
  - G.L. and R. Murray, 2016, ARMA, & 2019, Proc. Amer. Math. Soc.
  - G. Dal Maso, I. Fonseca and G.L., 2015, Calc. Var. Partial Differential Equations.

# Gamma-Convergence

- De Giorgi (1975), De Giorgi and Franzoni (1975)

**Definition**

$X$ metric space, $\mathcal{F}_\varepsilon : X \to [-\infty, \infty]$, $\varepsilon > 0$, $\Gamma$-converges if there exists $\mathcal{F}^{(0)} : X \to [-\infty, \infty]$ such that

- for every $x \in X$ and every $x_\varepsilon \to x$,

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}^{(0)}(x),$$

We write $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}^{(0)}$.

# Gamma-Convergence

- De Giorgi (1975), De Giorgi and Franzoni (1975)

**Definition**

$X$ metric space, $\mathcal{F}_\varepsilon : X \to [-\infty, \infty]$, $\varepsilon > 0$, $\Gamma$-converges if there exists $\mathcal{F}^{(0)} : X \to [-\infty, \infty]$ such that

- for every $x \in X$ and every $x_\varepsilon \to x$,

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}^{(0)}(x),$$

- for every $x \in X$ there exists $x_\varepsilon \to x$ such that

$$\lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \leq \mathcal{F}^{(0)}(x)$$

We write $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}^{(0)}$.

# Gamma-Asymptotic Developments

- Anzellotti and Baldo (1993)

**Definition**

$X$ metric space, $\mathcal{F}_\varepsilon : X \to (-\infty, \infty]$ has a $\Gamma$-asymptotic development of order $k$,

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

if there exist $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 1, \ldots, k$, such that

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)}$, where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$,

# Gamma-Asymptotic Developments

- Anzellotti and Baldo (1993)

**Definition**

$X$ metric space, $\mathcal{F}_\varepsilon : X \to (-\infty, \infty]$ has a $\Gamma$-asymptotic development of order $k$,

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

if there exist $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 1, \ldots, k$, such that

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)}$, where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$,
- $\mathcal{F}_\varepsilon^{(i)} := \dfrac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)}$ for $i = 1, \ldots, k$.

$$\mathcal{F}_\varepsilon^{(0)} \overset{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}$

## Gamma-Asymptotic Development of Order k

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}$
- $\mathcal{F}^{(i)} \equiv \infty$ in $X \setminus \mathcal{U}_{i-1}$

## Gamma-Asymptotic Development of Order k

$$\mathcal{F}_\varepsilon^{(0)} \overset{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}$
- $\mathcal{F}^{(i)} \equiv \infty$ in $X \setminus \mathcal{U}_{i-1}$
- let $\varepsilon_m \to 0^+$ and assume that there is a minimizer $x_m \in X$ of $\mathcal{F}_{\varepsilon_m}$ and $x_m \to x$

$$\mathcal{F}^{(0)}_\varepsilon \overset{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon\mathcal{F}^{(1)} + \cdots + \varepsilon^k\mathcal{F}^{(k)} + o(\varepsilon^k),$$

$$\mathcal{F}^{(i)}_\varepsilon := \frac{\mathcal{F}^{(i-1)}_\varepsilon - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}$
- $\mathcal{F}^{(i)} \equiv \infty$ in $X \setminus \mathcal{U}_{i-1}$
- let $\varepsilon_m \to 0^+$ and assume that there is a minimizer $x_m \in X$ of $\mathcal{F}_{\varepsilon_m}$ and $x_m \to x$
- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} \subseteq \mathcal{U}_k \subseteq \cdots \subseteq \mathcal{U}_0,$

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}$
- $\mathcal{F}^{(i)} \equiv \infty$ in $X \setminus \mathcal{U}_{i-1}$
- let $\varepsilon_m \to 0^+$ and assume that there is a minimizer $x_m \in X$ of $\mathcal{F}_{\varepsilon_m}$ and $x_m \to x$
- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} \subseteq \mathcal{U}_k \subseteq \cdots \subseteq \mathcal{U}_0$,
- $\inf \mathcal{F}_{\varepsilon_m} = \inf \mathcal{F}^{(0)} + \varepsilon_m \inf \mathcal{F}^{(1)} + \cdots + \varepsilon_m^k \inf \mathcal{F}^{(k)} + o(\varepsilon_m^k)$,

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k),$$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}$
- $\mathcal{F}^{(i)} \equiv \infty$ in $X \setminus \mathcal{U}_{i-1}$
- let $\varepsilon_m \to 0^+$ and assume that there is a minimizer $x_m \in X$ of $\mathcal{F}_{\varepsilon_m}$ and $x_m \to x$
- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} \subseteq \mathcal{U}_k \subseteq \cdots \subseteq \mathcal{U}_0$,
- $\inf \mathcal{F}_{\varepsilon_m} = \inf \mathcal{F}^{(0)} + \varepsilon_m \inf \mathcal{F}^{(1)} + \cdots + \varepsilon_m^k \inf \mathcal{F}^{(k)} + o(\varepsilon_m^k)$,
- Goal/Hope find $k$ so that $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} = \mathcal{U}_k$.

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

### Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = \varepsilon^k |x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

### Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = \varepsilon^k |x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(i)}(x) = \varepsilon^{k-i}|x|$, $\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i = 1, \dots, k-1$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

## Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = \varepsilon^k |x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(i)}(x) = \varepsilon^{k-i}|x|$, $\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i = 1, \dots, k-1$,
- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\} = \mathbb{R}$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

## Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = \varepsilon^k |x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(i)}(x) = \varepsilon^{k-i}|x|$, $\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i = 1, \dots, k-1$,
- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(k)}(x) = |x|$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

**Example**

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = \varepsilon^k |x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(i)}(x) = \varepsilon^{k-i}|x|$, $\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i = 1, \ldots, k-1$,
- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(k)}(x) = |x|$,
- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} = \mathcal{U}_k \subset \mathcal{U}_{k-1} = \cdots = \mathcal{U}_0$.

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

**Example**

Let $X = [0, 1]$ and

$$\mathcal{F}_\varepsilon(x) = \begin{cases} \varepsilon^n |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = [0, 1]$,

$$\mathcal{F}_\varepsilon^{(k)}(x) = \begin{cases} \varepsilon^{n-k} |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

## Example

Let $X = [0, 1]$ and

$$\mathcal{F}_\varepsilon(x) = \begin{cases} \varepsilon^n |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = [0, 1]$,

$$\mathcal{F}_\varepsilon^{(k)}(x) = \begin{cases} \varepsilon^{n-k} |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

- $\mathcal{U}_k := \{\text{minimizers of } \mathcal{F}^{(k)}\} = [0, 2^{-k}]$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

## Example

Let $X = [0,1]$ and

$$\mathcal{F}_\varepsilon(x) = \begin{cases} \varepsilon^n |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$, $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = [0,1]$,

$$\mathcal{F}_\varepsilon^{(k)}(x) = \begin{cases} \varepsilon^{n-k} |x| & \text{if } x \in (2^{-n}, 2^{-n+1}],\ n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

- $\mathcal{U}_k := \{\text{minimizers of } \mathcal{F}^{(k)}\} = [0, 2^{-k}]$,
- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} = \bigcap_k \mathcal{U}_k$.

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

**Example**

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = e^{-1/\varepsilon}|x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \ldots, k.$$

### Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = e^{-1/\varepsilon}|x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$
- $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R},$

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

## Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = e^{-1/\varepsilon}|x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$
- $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(i)}(x) = \frac{e^{-1/\varepsilon}}{\varepsilon^i}|x|$, $\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

**Example**

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = e^{-1/\varepsilon}|x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$
- $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R}$,
- $\mathcal{F}_\varepsilon^{(i)}(x) = \frac{e^{-1/\varepsilon}}{\varepsilon^i}|x|$, $\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i$,
- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\} = \mathbb{R}$,

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\varepsilon} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \quad \text{for } i = 1, \dots, k.$$

## Example

Let $X = \mathbb{R}$ and

$$\mathcal{F}_\varepsilon(x) = e^{-1/\varepsilon}|x|$$

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)} \equiv 0$
- $\mathcal{U}_0 := \{\text{minimizers of } \mathcal{F}^{(0)}\} = \mathbb{R},$
- $\mathcal{F}_\varepsilon^{(i)}(x) = \dfrac{e^{-1/\varepsilon}}{\varepsilon^i}|x|,\ \mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)} \equiv 0$ for all $i$,
- $\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\} = \mathbb{R},$
- $\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} = \{0\} \subset \mathcal{U}_i = \cdots = \mathcal{U}_0 = \mathbb{R}$ for all $i$

- More generally,

## Definition

$X$ metric space, $\mathcal{F}_\varepsilon : X \to (-\infty, \infty]$ has a $\Gamma$-asymptotic development of order $k$,

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \omega_1(\varepsilon)\mathcal{F}^{(1)} + \cdots + \omega_k(\varepsilon)\mathcal{F}^{(k)} + o(\varepsilon^k),$$

if there exist $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 1, \ldots, k$, such that

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)}$, where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$,

- More generally,

**Definition**

$X$ metric space, $\mathcal{F}_\varepsilon : X \to (-\infty, \infty]$ has a $\Gamma$-asymptotic development of order $k$,

$$\mathcal{F}_\varepsilon^{(0)} \overset{\Gamma}{=} \mathcal{F}^{(0)} + \omega_1(\varepsilon)\mathcal{F}^{(1)} + \cdots + \omega_k(\varepsilon)\mathcal{F}^{(k)} + o(\varepsilon^k),$$

if there exist $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 1, \ldots, k$, such that

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)}$, where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$,
- $\omega_0 \equiv 1$, $\omega_i > 0$, $\omega_i(\varepsilon) \to 0$, $\omega_i(\varepsilon)/\omega_{i-1}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$,

- More generally,

**Definition**

$X$ metric space, $\mathcal{F}_\varepsilon : X \to (-\infty, \infty]$ has a **$\Gamma$-asymptotic development of order $k$**,

$$\mathcal{F}_\varepsilon^{(0)} \stackrel{\Gamma}{=} \mathcal{F}^{(0)} + \omega_1(\varepsilon)\mathcal{F}^{(1)} + \cdots + \omega_k(\varepsilon)\mathcal{F}^{(k)} + o(\varepsilon^k),$$

if there exist $\mathcal{F}^{(i)} : X \to [-\infty, \infty]$, $i = 1, \ldots, k$, such that

- $\mathcal{F}_\varepsilon^{(0)} \xrightarrow{\Gamma} \mathcal{F}^{(0)}$, where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$,
- $\omega_0 \equiv 1$, $\omega_i > 0$, $\omega_i(\varepsilon) \to 0$, $\omega_i(\varepsilon)/\omega_{i-1}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$,
- $\mathcal{F}_\varepsilon^{(i)} := \dfrac{\mathcal{F}_\varepsilon^{(i-1)} - \inf \mathcal{F}^{(i-1)}}{\omega_i(\varepsilon)/\omega_{i-1}(\varepsilon)} \xrightarrow{\Gamma} \mathcal{F}^{(i)}$ for $i = 1, \ldots, k$.

# Dirichlet–Neumann Problems

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, $\partial\Omega$ Lipschitz continuous,

# Dirichlet–Neumann Problems

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, $\partial\Omega$ Lipschitz continuous,
- $\nu$ outward unit normal to $\partial\Omega$,

# Dirichlet–Neumann Problems

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, $\partial\Omega$ Lipschitz continuous,
- $\nu$ outward unit normal to $\partial\Omega$,
- $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N$, $\Gamma_D$ relatively open, nonempty

# Dirichlet–Neumann Problems

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, $\partial\Omega$ Lipschitz continuous,
- $\nu$ outward unit normal to $\partial\Omega$,
- $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N$, $\Gamma_D$ relatively open, nonempty
- $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

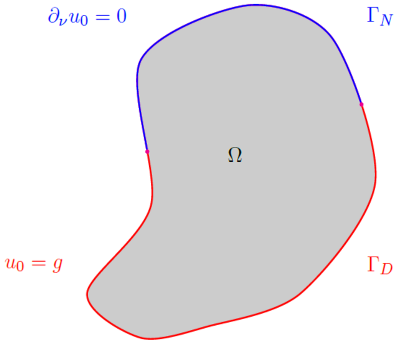



Figure: Courtesy of G. Gravina

# Loss of Regularity

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = 0 & \text{on } \Gamma_D, \end{cases}$$

- $N = 2$, $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$,

# Loss of Regularity

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = 0 & \text{on } \Gamma_D, \end{cases}$$

- $N = 2$, $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$,
- $\Gamma_N = (-\infty, 0) \times \{0\}$, $\Gamma_D = (0, \infty) \times \{0\}$,

# Loss of Regularity

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = 0 & \text{on } \Gamma_D, \end{cases}$$

- $N = 2$, $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$,
- $\Gamma_N = (-\infty, 0) \times \{0\}$, $\Gamma_D = (0, \infty) \times \{0\}$,
- in polar coordinates

$$u_0(r, \theta) = r^{1/2} \sin(\theta/2),$$

# Loss of Regularity

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = 0 & \text{on } \Gamma_D, \end{cases}$$

- $N = 2$, $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$,
- $\Gamma_N = (-\infty, 0) \times \{0\}$, $\Gamma_D = (0, \infty) \times \{0\}$,
- in polar coordinates

$$u_0(r, \theta) = r^{1/2} \sin(\theta/2),$$

- $u_0 \notin H^2(\Omega \cap B((0,0), r))$ for every $r > 0$.

# Main Hypotheses

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ open, bounded, connected $\partial\Omega$ of class $C^{1,1}$,

# Main Hypotheses

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ open, bounded, connected $\partial\Omega$ of class $C^{1,1}$,
- $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\overline{\Gamma_N} \cap \overline{\Gamma_D} = \{P_1, P_2\}$, $\Gamma_N$, $\Gamma_D$ relatively open, nonempty,

# Main Hypotheses

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ open, bounded, connected $\partial\Omega$ of class $C^{1,1}$,
- $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\overline{\Gamma_N} \cap \overline{\Gamma_D} = \{P_1, P_2\}$, $\Gamma_N$, $\Gamma_D$ relatively open, nonempty,
- $\partial\Omega \cap B(P_i, \delta_i)$ is a segment, $i = 1, 2$, $\delta_i > 0$ small,

# Main Hypotheses

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ open, bounded, connected $\partial\Omega$ of class $C^{1,1}$,
- $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\overline{\Gamma_N} \cap \overline{\Gamma_D} = \{P_1, P_2\}$, $\Gamma_N$, $\Gamma_D$ relatively open, nonempty,
- $\partial\Omega \cap B(P_i, \delta_i)$ is a segment, $i = 1, 2$, $\delta_i > 0$ small,
- $(r_i, \theta_i)$ polar coordinates at $P_i$, , $i = 1, 2$,

# Main Hypotheses

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

- $\Omega \subset \mathbb{R}^2$ open, bounded, connected $\partial\Omega$ of class $C^{1,1}$,
- $\partial\Omega = \overline{\Gamma_N} \cup \overline{\Gamma_D}$, $\overline{\Gamma_N} \cap \overline{\Gamma_D} = \{P_1, P_2\}$, $\Gamma_N$, $\Gamma_D$ relatively open, nonempty,
- $\partial\Omega \cap B(P_i, \delta_i)$ is a segment, $i = 1, 2$, $\delta_i > 0$ small,
- $(r_i, \theta_i)$ polar coordinates at $P_i$, , $i = 1, 2$,
- $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$.

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

## Theorem

$$u_0 = u_{\text{reg}} + \sum_{i=1}^{2} c_i \varphi_i(r_i) r_i^{1/2} \sin(\theta_i/2),$$

- *$u_{\text{reg}} \in H^2(\Omega)$, $c_i \in \mathbb{R}$, with*

$$\|u_{\text{reg}}\|_{H^2(\Omega)} + \sum_{i=1}^{2} |c_i| \leq c\|f\|_{L^2(\Omega)} + c\|g\|_{H^{3/2}(\Omega)},$$

Grisvard, Kondratev, Maz'ja, Oleinik, Plamenevskiĭ, etc.

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ \partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D, \end{cases}$$

## Theorem

$$u_0 = u_{\text{reg}} + \sum_{i=1}^{2} c_i \varphi_i(r_i) r_i^{1/2} \sin(\theta_i/2),$$

- *$u_{\text{reg}} \in H^2(\Omega)$, $c_i \in \mathbb{R}$, with*

$$\|u_{\text{reg}}\|_{H^2(\Omega)} + \sum_{i=1}^{2} |c_i| \le c\|f\|_{L^2(\Omega)} + c\|g\|_{H^{3/2}(\Omega)},$$

- *$\varphi_i \in C^\infty([0,\infty))$, $\varphi_i = 1$ in $[0, \delta_i/2]$ and $\varphi_i = 0$ outside $[0, \delta_i]$.*

Grisvard, Kondratev, Maz'ja, Oleinik, Plamenevskiĭ, etc.

# Singularly Perturbed Neumann–Robin

Lions (73), Colli-Franzone (73, 74), Costabel & Dauge (93)

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon\partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases}$$

- What is the advantage?

# Singularly Perturbed Neumann–Robin

Lions (73), Colli-Franzone (73, 74), Costabel & Dauge (93)

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases}$$

- What is the advantage?
- $u_0 \sim r^{1/2}$

# Singularly Perturbed Neumann–Robin

Lions (73), Colli-Franzone (73, 74), Costabel & Dauge (93)

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases}$$

- What is the advantage?
- $u_0 \sim r^{1/2}$
- $u_\varepsilon \sim r \log r$

# Asymptotics

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases}$$

**Theorem (Costabel & Dauge)**

*Under the main hypotheses, with* $f = 0$,

- $\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon \log \varepsilon)$,

# Asymptotics

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases}$$

**Theorem (Costabel & Dauge)**

*Under the main hypotheses, with* $f = 0$,

- $\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon \log \varepsilon)$,
- $\|u_\varepsilon - u_0\|_{H^{1+s}(\Omega)} = \mathcal{O}(\varepsilon^{1/2-s})$, $s \in (-\frac{1}{2}, \frac{1}{2})$,

# Asymptotics

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases}$$

**Theorem (Costabel & Dauge)**

*Under the main hypotheses, with* $f = 0$,

- $\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon \log \varepsilon)$,
- $\|u_\varepsilon - u_0\|_{H^{1+s}(\Omega)} = \mathcal{O}(\varepsilon^{1/2-s})$, $s \in (-\frac{1}{2}, \frac{1}{2})$,
- $\|u_\varepsilon - u_0\|_{L^2(\Gamma_D)} = \mathcal{O}(\varepsilon |\log \varepsilon|^{1/2})$.

# Gamma Asymptotic Development

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases} \tag{N-R}$$

- Solutions $u_\varepsilon$ of (N-R) are critical points of

$$F_\varepsilon(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

# Gamma Asymptotic Development

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases} \tag{N-R}$$

- Solutions $u_\varepsilon$ of (N-R) are critical points of

$$F_\varepsilon(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\,dx + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2 ds, \quad v \in H^1(\Omega).$$

- Study compactness of bounded sequences.

# Gamma Asymptotic Development

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases} \tag{N-R}$$

- Solutions $u_\varepsilon$ of (N-R) are critical points of

$$F_\varepsilon(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right) dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- Study compactness of bounded sequences.
- Study Gamma asymptotic development.

# Gamma Asymptotic Development

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D. \end{cases} \tag{N-R}$$

- Solutions $u_\varepsilon$ of (N-R) are critical points of

$$F_\varepsilon(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right) dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- Study compactness of bounded sequences.
- Study Gamma asymptotic development.
- Extend $F_\varepsilon$ to $L^2(\Omega)$ via $F_\varepsilon(v) := \infty$ for $v \in L^2(\Omega) \setminus H^1(\Omega)$.

# Gamma Asymptotic Development of Order 0

$$F_\varepsilon(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,* $F_\varepsilon \xrightarrow{\Gamma} F_0 \quad$ *in* $L^2(\Omega)$*, where*

$$F_0(v) = \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx$$

*if* $v \in H^1(\Omega)$ *and* $v = g$ *on* $\Gamma_D$*,* $F_0(v) = \infty$ *otherwise in* $L^2(\Omega)$*.*

- $\min F_0 = F_0(u_0)$, $u_0$ solution to Dirichlet–Neumann problem.

# Compactness (of Order 1)

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2 ds, \quad v \in H^1(\Omega).$$

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses, let $\varepsilon_n \to 0^+$ and $v_n \in H^1(\Omega)$ be such that*

$$\sup_n F_{\varepsilon_n}(v_n) \le F_0(u_0) + c\varepsilon_n|\log\varepsilon_n|.$$

*Then (up to a subsequence)*

$$\frac{v_n - u_0}{\varepsilon_n^{1/2}|\log\varepsilon_n|^{1/2}} \rightharpoonup w_0 \quad \text{in } H^1(\Omega),$$

$$\frac{v_n - u_0}{\varepsilon_n|\log\varepsilon_n|^{1/2}} \rightharpoonup v_0 \quad \text{in } L^2(\Gamma_D).$$

# Hardy's Inequality p=N

Compactness relies on the Hardy-type inequality

**Theorem (Machihara, Ozawa, & Wadade, '12)**

*If $N = 2$ and $u \in H^1(B(0,r))$, $r > 0$, then*

$$\left(\int_{B(0,r)} \frac{u^2(x)}{|x|^2(1+\log(r/|x|))^2}dx\right)^{1/2} \leq \frac{\sqrt{2}}{r}\left(\int_{B(0,r)} u^2(x)\,dx\right)^{1/2} + 2(1+\sqrt{2})\left(\int_{B(0,r)} \left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2 dx\right)^{1/2}.$$

# Hardy's Inequality p=N

- $\dot{W}^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}$

# Hardy's Inequality p=N

- $\dot{W}^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}$
- $\dot{H}^{1,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N) : u_B = 0\}}^{\dot{W}^{1,p}(\mathbb{R}^N)}$,
  $u_B := \frac{1}{\mathcal{L}^N(B(0,1))} \int_{B(0,1)} u \, dx$,

# Hardy's Inequality p=N

- $\dot{W}^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N;\mathbb{R}^N)\}$
- $\dot{H}^{1,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N) : u_B = 0\}}^{\dot{W}^{1,p}(\mathbb{R}^N)}$,
  $u_B := \frac{1}{\mathcal{L}^N(B(0,1))} \int_{B(0,1)} u\, dx$,
- Problem: $\dot{W}^{1,p}(\mathbb{R}^N) = \dot{H}^{1,p}(\mathbb{R}^N)$?

# Hardy's Inequality p=N

- $\dot{W}^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N;\mathbb{R}^N)\}$
- $\dot{H}^{1,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N) : u_B = 0\}}^{\dot{W}^{1,p}(\mathbb{R}^N)}$,
  $u_B := \frac{1}{\mathcal{L}^N(B(0,1))} \int_{B(0,1)} u \, dx$,
- Problem: $\dot{W}^{1,p}(\mathbb{R}^N) = \dot{H}^{1,p}(\mathbb{R}^N)$?
- Ladyzhenskaya & Solonnikov, '78

# Hardy's Inequality p=N

- $\dot{W}^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N;\mathbb{R}^N)\}$
- $\dot{H}^{1,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N) : u_B = 0\}}^{\dot{W}^{1,p}(\mathbb{R}^N)}$,
  $u_B := \frac{1}{\mathcal{L}^N(B(0,1))} \int_{B(0,1)} u \, dx$,
- Problem: $\dot{W}^{1,p}(\mathbb{R}^N) = \dot{H}^{1,p}(\mathbb{R}^N)$?
- Ladyzhenskaya & Solonnikov, '78

**Theorem**

*Let $N \geq 2$ and let $u \in \dot{H}^{1,N}(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} \frac{|u(x) - u_B|^N}{(1 + |x|^2 \log^2 |x|)^{N/2}} dx \leq c \int_{\mathbb{R}^N} |\nabla u(x)|^N dx$$

*for some constant $c = c(N) > 0$.*

# Gamma Asymptotic Development of Order 1

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2 ds, \quad v \in H^1(\Omega).$$

- Recall $u_0 = u_{\text{reg}} + \sum_{i=1}^2 c_i \varphi_i(r_i) r_i^{1/2}\sin(\theta_i/2)$, $u_0$ solution to Dirichlet-Neumann problem.

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,*

$$F_\varepsilon^{(1)} := \frac{F_\varepsilon - \min F_0}{\varepsilon|\log\varepsilon|} \xrightarrow{\Gamma} F_1 \quad \text{in } L^2(\Omega),$$

*where* $F_1(v) = -\frac{1}{8}\sum_{i=1}^2 c_i^2$ *if* $v = u_0$, $F_1(v) = \infty$ *otherwise in* $L^2(\Omega)$.

# Gamma Asymptotic Development of Order 1

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0$ solution to Dirichlet–Neumann problem.

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,*

$$F_\varepsilon(u_\varepsilon) = F_0(u_0) + \varepsilon|\log\varepsilon| F_1(u_0) + o(\varepsilon\log\varepsilon),$$
$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon\log\varepsilon),$$
$$\|u_\varepsilon - u_0\|_{H^1(\Omega)} = \mathcal{O}(\varepsilon^{1/2}).$$

# Gamma Asymptotic Development of Order 1

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right) \, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0$ solution to Dirichlet–Neumann problem.
- $u_\varepsilon$ solution to Robin–Neumann problem.

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,*

$$F_\varepsilon(u_\varepsilon) = F_0(u_0) + \varepsilon|\log\varepsilon| F_1(u_0) + o(\varepsilon\log\varepsilon),$$
$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon\log\varepsilon),$$
$$\|u_\varepsilon - u_0\|_{H^1(\Omega)} = \mathcal{O}(\varepsilon^{1/2}).$$

# Compactness (of Order 2)

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses, let $\varepsilon_n \to 0^+$ and $w_n \in H^1(\Omega)$ be such that*

$$\sup_n F_{\varepsilon_n}(w_n) \le F_0(u_0) + \varepsilon_n |\log \varepsilon_n| F_1(u_0) + c\varepsilon_n.$$

*Then (up to a subsequence)*

$$\frac{w_n - u_0 - \varepsilon_n u_1}{\varepsilon_n^{1/2}} \rightharpoonup p_0 \quad \text{in } H^1(\Omega),$$

$$\frac{w_n - u_0}{\varepsilon_n} - u_1 - \sum_{i=1}^{2} c_i \psi_i [1 - \chi_{B(P_i,\varepsilon_n)}] \rightharpoonup q_0 - \sum_{i=1}^{2} c_i \psi_i \quad \text{in } L^2(\Gamma_D).$$

# Compactness (of Order 2)

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0 = u_{\text{reg}} + \sum_{i=1}^2 c_i \varphi_i(r_i) r_i^{1/2} \sin(\theta_i/2)$, $u_0$ solution to Dirichlet-Neumann problem.

# Compactness (of Order 2)

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0 = u_{\text{reg}} + \sum_{i=1}^{2} c_i \varphi_i(r_i) r_i^{1/2} \sin(\theta_i/2)$, $u_0$ solution to Dirichlet-Neumann problem.
- $u_1$ solution to

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega, \\ \partial_\nu u_1 = 0 & \text{on } \Gamma_N, \\ u_1 = -\partial_\nu u_{\text{reg}} & \text{on } \Gamma_D. \end{cases}$$

# Compactness (of Order 2)

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\,dx + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0 = u_{\text{reg}} + \sum_{i=1}^{2} c_i \varphi_i(r_i) r_i^{1/2} \sin(\theta_i/2)$, $u_0$ solution to Dirichlet-Neumann problem.
- $u_1$ solution to

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega, \\ \partial_\nu u_1 = 0 & \text{on } \Gamma_N, \\ u_1 = -\partial_\nu u_{\text{reg}} & \text{on } \Gamma_D. \end{cases}$$

- $\psi_i(r_i) = \frac{1}{2}\varphi_i(r_i) r_i^{-1/2}$.

# Gamma Asymptotic Development of Order 2

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0 = u_{\rm reg} + \sum_{i=1}^2 c_i \varphi_i(r_i) r_i^{1/2} \sin(\theta_i/2)$, $u_0$ solution to Dirichlet-Neumann problem.

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,*

$$F_\varepsilon^{(2)} := \frac{\frac{F_\varepsilon - \min F_0}{\varepsilon|\log\varepsilon|} - \min F_1}{1/|\log\varepsilon|} \xrightarrow{\Gamma} F_2 \quad \text{in } L^2(\Omega),$$

*where* $F_1(v) = A - \frac{1}{2}\int_{\Gamma_D}(\partial_\nu u_{\rm reg})^2\, ds$ *if* $v = u_0$, $F_2(v) = \infty$ *otherwise in* $L^2(\Omega)$.

# Gamma Asymptotic Development of Order 2

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\,dx + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0$ solution to Dirichlet–Neumann problem.

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,*

$$F_\varepsilon(u_\varepsilon) = F_0(u_0) + \varepsilon|\log\varepsilon| F_1(u_0) + \varepsilon F_2(u_0) + o(\varepsilon),$$
$$\|u_\varepsilon - u_0\|_{L^2(\Gamma_D)} = \mathcal{O}(\varepsilon|\log\varepsilon|^{1/2}).$$

# Gamma Asymptotic Development of Order 2

$$F_\varepsilon(v) := \int_\Omega \left(\tfrac{1}{2}|\nabla v|^2 + fv\right)\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_D} (v-g)^2 ds, \quad v \in H^1(\Omega).$$

- $u_0$ solution to Dirichlet–Neumann problem.
- $u_\varepsilon$ solution to Robin–Neumann problem.

**Theorem (G. Gravina & G.L.)**

*Under the main hypotheses,*

$$F_\varepsilon(u_\varepsilon) = F_0(u_0) + \varepsilon|\log\varepsilon|F_1(u_0) + \varepsilon F_2(u_0) + o(\varepsilon),$$
$$\|u_\varepsilon - u_0\|_{L^2(\Gamma_D)} = \mathcal{O}(\varepsilon|\log\varepsilon|^{1/2}).$$

# Future Work/Open Problems

- Asymptotic development of order 3.

# Future Work/Open Problems

- Asymptotic development of order 3.
- $N \geq 3$.

# Future Work/Open Problems

- Asymptotic development of order 3.
- $N \geq 3$.
- $p$-Laplacian

$$\begin{cases} \operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) = f & \text{in } \Omega, \\ |\nabla u_0|^{p-2}\nabla u_0 \cdot \nu = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D. \end{cases}$$

# Van Der Waals– Cahn–Hilliard Theory for Phase Transitions

- Free energy

$$F_\varepsilon(u) = \int_\Omega \left( W(u) + \varepsilon^2 |\nabla u|^2 \right) \, dx.$$

# Van Der Waals– Cahn–Hilliard Theory for Phase Transitions

- Free energy

$$F_\varepsilon(u) = \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right)\, dx.$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container

# Van Der Waals– Cahn–Hilliard Theory for Phase Transitions

- Free energy

$$F_\varepsilon(u) = \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right)\, dx.$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container
- $u$ density of a fluid,

# Van Der Waals– Cahn–Hilliard Theory for Phase Transitions

- Free energy

$$F_\varepsilon(u) = \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right) \, dx.$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container
- $u$ density of a fluid,
- $\int_\Omega u \, dx = m$, where $m$ total mass of the fluid,

# Van Der Waals– Cahn–Hilliard Theory for Phase Transitions

- Free energy

$$F_\varepsilon(u) = \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right)\, dx.$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container
- $u$ density of a fluid,
- $\int_\Omega u\, dx = m$, where $m$ total mass of the fluid,
- $W$ double-well potential, $W \geq 0$, $W^{-1}(\{0\}) = \{a, b\}$, $a < b$,

# Van Der Waals– Cahn–Hilliard Theory for Phase Transitions

- Free energy

$$F_\varepsilon(u) = \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right)\, dx.$$

- $\Omega \subset \mathbb{R}^N$ open, bounded, container
- $u$ density of a fluid,
- $\int_\Omega u\, dx = m$, where $m$ total mass of the fluid,
- $W$ double-well potential, $W \geq 0$, $W^{-1}(\{0\}) = \{a, b\}$, $a < b$,
- Van Der Waals (1893), Cahn and Hilliard (1958), Gurtin (1987)

## Zero Order Gamma Limit

Take $X = L^1(\Omega)$, $W^{-1}(\{0\}) = \{\pm 1\}$, and

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right) dx$$

if $u \in H^1(\Omega)$, $\int_\Omega u\,dx = m$, and $\mathcal{F}_\varepsilon(u) := \infty$ otherwise in $L^1(\Omega)$.

- Zero Order $\Gamma$-limit

$$\mathcal{F}^{(0)}(u) = \int_\Omega W(u)\,dx$$

if $u \in L^1(\Omega)$, $\int_\Omega u\,dx = m$, and $\mathcal{F}^{(0)}(u) = \infty$ otherwise in $L^1(\Omega)$,

## Zero Order Gamma Limit

Take $X = L^1(\Omega)$, $W^{-1}(\{0\}) = \{\pm 1\}$, and

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right) dx$$

if $u \in H^1(\Omega)$, $\int_\Omega u\, dx = m$, and $\mathcal{F}_\varepsilon(u) := \infty$ otherwise in $L^1(\Omega)$.

- Zero Order $\Gamma$-limit

$$\mathcal{F}^{(0)}(u) = \int_\Omega W(u)\, dx$$

  if $u \in L^1(\Omega)$, $\int_\Omega u\, dx = m$, and $\mathcal{F}^{(0)}(u) = \infty$ otherwise in $L^1(\Omega)$,

- minimizers of $\mathcal{F}^{(0)}$:

$$\mathcal{U}_0 = \left\{1\chi_E - 1\chi_{\Omega\setminus E} :\ E \text{ Borel},\ \text{meas}\, E = \frac{\text{meas}\,\Omega + m}{2}\right\}$$

## Zero Order Gamma Limit

Take $X = L^1(\Omega)$, $W^{-1}(\{0\}) = \{\pm 1\}$, and

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right) dx$$

if $u \in H^1(\Omega)$, $\int_\Omega u\,dx = m$, and $\mathcal{F}_\varepsilon(u) := \infty$ otherwise in $L^1(\Omega)$.

- Zero Order $\Gamma$-limit

$$\mathcal{F}^{(0)}(u) = \int_\Omega W(u)\,dx$$

  if $u \in L^1(\Omega)$, $\int_\Omega u\,dx = m$, and $\mathcal{F}^{(0)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- minimizers of $\mathcal{F}^{(0)}$:

$$\mathcal{U}_0 = \left\{1\chi_E - 1\chi_{\Omega\setminus E} : \ E \text{ Borel}, \ \text{meas}\, E = \tfrac{\text{meas}\,\Omega + m}{2}\right\}$$

- $\inf \mathcal{F}^{(0)} = 0$,

## Zero Order Gamma Limit

Take $X = L^1(\Omega)$, $W^{-1}(\{0\}) = \{\pm 1\}$, and

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(W(u) + \varepsilon^2 |\nabla u|^2\right) dx$$

if $u \in H^1(\Omega)$, $\int_\Omega u\,dx = m$, and $\mathcal{F}_\varepsilon(u) := \infty$ otherwise in $L^1(\Omega)$.

- Zero Order $\Gamma$-limit

$$\mathcal{F}^{(0)}(u) = \int_\Omega W(u)\,dx$$

  if $u \in L^1(\Omega)$, $\int_\Omega u\,dx = m$, and $\mathcal{F}^{(0)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- minimizers of $\mathcal{F}^{(0)}$:

$$\mathcal{U}_0 = \left\{1\chi_E - 1\chi_{\Omega\setminus E} : \ E \text{ Borel}, \ \text{meas}\, E = \tfrac{\text{meas}\,\Omega + m}{2}\right\}$$

- $\inf \mathcal{F}^{(0)} = 0$,
- Non-physical solutions.

## First Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(1)} := \frac{\mathcal{F}_\varepsilon^{(0)} - \inf \mathcal{F}^{(0)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(0)} = 0$,

  $\mathcal{F}_\varepsilon^{(1)}(u) = \int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx$ if $u \in H^1(\Omega)$, $\int_\Omega u\, dx = m$

  and $\mathcal{F}_\varepsilon^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$,

## First Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(1)} := \frac{\mathcal{F}_\varepsilon^{(0)} - \inf \mathcal{F}^{(0)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(0)} = 0$,

  $$\mathcal{F}_\varepsilon^{(1)}(u) = \int_\Omega \left(\tfrac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx \text{ if } u \in H^1(\Omega), \int_\Omega u\, dx = m$$

  and $\mathcal{F}_\varepsilon^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- First Order $\Gamma$-limit:

  $$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \int_\Omega u\, dx = m$$

  and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$

## First Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(1)} := \frac{\mathcal{F}_\varepsilon^{(0)} - \inf \mathcal{F}^{(0)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(0)} = 0$,

  $$\mathcal{F}_\varepsilon^{(1)}(u) = \int_\Omega \left(\tfrac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx \text{ if } u \in H^1(\Omega), \int_\Omega u\, dx = m$$

  and $\mathcal{F}_\varepsilon^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- First Order $\Gamma$-limit:

  $$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \int_\Omega u\, dx = m$$

  and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$
- $\operatorname{Per}_\Omega \{u = 1\}$ surface area of $\partial \{u = 1\}$ in $\Omega$,
  $c_0 := 2 \int_{-1}^{1} \sqrt{W(s)}\, ds$,

## First Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(1)} := \frac{\mathcal{F}_\varepsilon^{(0)} - \inf \mathcal{F}^{(0)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(0)} = 0$,

  $$\mathcal{F}_\varepsilon^{(1)}(u) = \int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx \text{ if } u \in H^1(\Omega), \ \int_\Omega u \, dx = m$$

  and $\mathcal{F}_\varepsilon^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- First Order $\Gamma$-limit:

  $$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \ \int_\Omega u \, dx = m$$

  and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$
- $\operatorname{Per}_\Omega \{u = 1\}$ surface area of $\partial \{u = 1\}$ in $\Omega$, $c_0 := 2 \int_{-1}^{1} \sqrt{W(s)} \, ds$,
- Carr, Gurtin, and Slemrod (1984) for $N = 1$, Modica and Mortola (1979) $W(s) = \sin^2(\pi s)$, Modica (1987), Sternberg (1988) for $N \geq 1$.

## First Order Gamma Limit

$$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \ \int_\Omega u \, dx = m$$

and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$

- minimizers of $\mathcal{F}^{(1)}$: $\mathcal{U}_1 = \{1\chi_{E_0} - 1\chi_{\Omega \setminus E_0}\}$, where $E_0 \subset \Omega$ minimizes

$$\min \left\{ \operatorname{Per}_\Omega E : \ E \subset \Omega, \ E \text{ Borel}, \ \operatorname{meas} E = \tfrac{\operatorname{meas}\Omega + m}{2} \right\},$$

## First Order Gamma Limit

$$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \; \int_\Omega u \, dx = m$$

and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$

- minimizers of $\mathcal{F}^{(1)}$: $\mathcal{U}_1 = \{1\chi_{E_0} - 1\chi_{\Omega \setminus E_0}\}$, where $E_0 \subset \Omega$ minimizes

$$\min \left\{ \operatorname{Per}_\Omega E : \; E \subset \Omega, \; E \text{ Borel}, \; \operatorname{meas} E = \frac{\operatorname{meas} \Omega + m}{2} \right\},$$

- $E_0$ is regular for $N \leq 7$, has constant mean curvature $\kappa$,

## First Order Gamma Limit

$$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \int_\Omega u \, dx = m$$

and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$

- minimizers of $\mathcal{F}^{(1)}$: $\mathcal{U}_1 = \{1\chi_{E_0} - 1\chi_{\Omega \setminus E_0}\}$, where $E_0 \subset \Omega$ minimizes

$$\min \left\{ \operatorname{Per}_\Omega E : \; E \subset \Omega, \; E \text{ Borel}, \; \operatorname{meas} E = \frac{\operatorname{meas}\Omega + m}{2} \right\},$$

- $E_0$ is regular for $N \leq 7$, has constant mean curvature $\kappa$,
- if $\Omega$ is regular, then $E_0$ meets $\partial\Omega$ transversally,

## First Order Gamma Limit

$$\mathcal{F}^{(1)}(u) = c_0 \operatorname{Per}_\Omega \{u = 1\} \text{ if } u \in BV(\Omega; \{\pm 1\}), \int_\Omega u \, dx = m$$

and $\mathcal{F}^{(1)}(u) = \infty$ otherwise in $L^1(\Omega)$

- minimizers of $\mathcal{F}^{(1)}$: $\mathcal{U}_1 = \{1\chi_{E_0} - 1\chi_{\Omega \setminus E_0}\}$, where $E_0 \subset \Omega$ minimizes

$$\min \left\{ \operatorname{Per}_\Omega E : \; E \subset \Omega, \; E \text{ Borel}, \; \operatorname{meas} E = \tfrac{\operatorname{meas}\Omega + m}{2} \right\},$$

- $E_0$ is regular for $N \leq 7$, has constant mean curvature $\kappa$,
- if $\Omega$ is regular, then $E_0$ meets $\partial\Omega$ transversally,
- Gonzalez, Massari and Tamanini (1983), Grüter (1987)

## Second Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(2)} := \frac{\mathcal{F}_\varepsilon^{(1)} - \inf \mathcal{F}^{(1)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(1)} = c_0 \operatorname{Per}_\Omega E_0$, $E_0 \subset \Omega$ minimizer,

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}$$

if $u \in H^1(\Omega)$, $\int_\Omega u \, dx = m$ and $\mathcal{F}_\varepsilon^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$,

## Second Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(2)} := \frac{\mathcal{F}_\varepsilon^{(1)} - \inf \mathcal{F}^{(1)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(1)} = c_0 \operatorname{Per}_\Omega E_0$, $E_0 \subset \Omega$ minimizer,

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}$$

  if $u \in H^1(\Omega)$, $\int_\Omega u \, dx = m$ and $\mathcal{F}_\varepsilon^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- $N = 1$, $\Omega = (-L, L)$, Carr, Gurtin, and Slemrod (1984): If $u_\varepsilon$ mininimizer of $\mathcal{F}_\varepsilon$:

$$\int_{-L}^{L} \left(W(u_\varepsilon) + \varepsilon^2 |u_\varepsilon'|^2\right) dx = c_0 \varepsilon + O(e^{-c\varepsilon}).$$

## Second Order Gamma Limit

$$\mathcal{F}_\varepsilon^{(2)} := \frac{\mathcal{F}_\varepsilon^{(1)} - \inf \mathcal{F}^{(1)}}{\varepsilon}$$

- $\inf \mathcal{F}^{(1)} = c_0 \operatorname{Per}_\Omega E_0$, $E_0 \subset \Omega$ minimizer,

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left(\frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2\right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}$$

  if $u \in H^1(\Omega)$, $\int_\Omega u \, dx = m$ and $\mathcal{F}_\varepsilon^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$,
- $N = 1$, $\Omega = (-L, L)$, Carr, Gurtin, and Slemrod (1984): If $u_\varepsilon$ mininimizer of $\mathcal{F}_\varepsilon$:

$$\int_{-L}^{L} \left(W(u_\varepsilon) + \varepsilon^2 |u_\varepsilon'|^2\right) dx = c_0 \varepsilon + O(e^{-c\varepsilon}).$$

- $\mathcal{F}^{(k)} = 0$ for all $k \geq 2$.

## Second Order Gamma Limit, $N = 1$

- Anzellotti and Baldo (1993)
  $W^{-1}(\{0\}) = [-1-\delta, -1+\delta] \cup [1-\delta, 1+\delta]$, where $0 < \delta < 1$,

$$\int_{-L}^{L} u \, dx = m \rightsquigarrow \quad u(-L) = \alpha, \quad u(L) = \beta.$$

## Second Order Gamma Limit, $N = 1$

- Anzellotti and Baldo (1993)
  $W^{-1}(\{0\}) = [-1-\delta, -1+\delta] \cup [1-\delta, 1+\delta]$, where $0 < \delta < 1$,

$$\int_{-L}^{L} u \, dx = m \rightsquigarrow \quad u(-L) = \alpha, \quad u(L) = \beta.$$

- $\Omega = \mathbb{T}$ torus, Bellettini, Nayam, Novaga (2013): If $\{u_\varepsilon\} \subset H^1(\mathbb{T})$ with $u_\varepsilon \to u_0$ in $L^1(\mathbb{T})$, $u_0 \in BV(\mathbb{T}; \{-1, 1\})$, there exist $d_{n,\varepsilon} \to x_{n+1} - x_n$, $n = 1, \ldots, \ell$, where $x_i$'s are the jumps of $u_0$ such that

$$\frac{\int_{\mathbb{T}} \left(\frac{1}{4\varepsilon}(u_\varepsilon^2 - 1)^2 + \varepsilon |u_\varepsilon'|^2\right) dx - c_0 \ell}{\varepsilon}$$

$$\geq -16\sqrt{2} \sum_{n=1}^{\ell} e^{-\sqrt{2} d_{n,\varepsilon}/\varepsilon} + o\left(e^{-3\sqrt{2} d_{n,\varepsilon}/(2\varepsilon)}\right),$$

## Second Order Gamma Limit, $N = 1$

- Anzellotti and Baldo (1993)
  $W^{-1}(\{0\}) = [-1-\delta, -1+\delta] \cup [1-\delta, 1+\delta]$, where $0 < \delta < 1$,
  $$\int_{-L}^{L} u \, dx = m \rightsquigarrow \quad u(-L) = \alpha, \quad u(L) = \beta.$$
- $\Omega = \mathbb{T}$ torus, Bellettini, Nayam, Novaga (2013): If $\{u_\varepsilon\} \subset H^1(\mathbb{T})$ with $u_\varepsilon \to u_0$ in $L^1(\mathbb{T})$, $u_0 \in BV(\mathbb{T}; \{-1, 1\})$, there exist $d_{n,\varepsilon} \to x_{n+1} - x_n$, $n = 1, \ldots, \ell$, where $x_i$'s are the jumps of $u_0$ such that
  $$\frac{\int_{\mathbb{T}} \left(\frac{1}{4\varepsilon}(u_\varepsilon^2 - 1)^2 + \varepsilon |u_\varepsilon'|^2\right) \, dx - c_0 \ell}{\varepsilon}$$
  $$\geq -16\sqrt{2} \sum_{n=1}^{\ell} e^{-\sqrt{2} d_{n,\varepsilon}/\varepsilon} + o\left(e^{-3\sqrt{2} d_{n,\varepsilon}/(2\varepsilon)}\right),$$
- This inequality is sharp

## Second Order Gamma Limit, $N = 1$

- Anzellotti and Baldo (1993)
  $W^{-1}(\{0\}) = [-1-\delta, -1+\delta] \cup [1-\delta, 1+\delta]$, where $0 < \delta < 1$,
  $$\int_{-L}^{L} u\, dx = m \rightsquigarrow \quad u(-L) = \alpha, \quad u(L) = \beta.$$
- $\Omega = \mathbb{T}$ torus, Bellettini, Nayam, Novaga (2013): If $\{u_\varepsilon\} \subset H^1(\mathbb{T})$ with $u_\varepsilon \to u_0$ in $L^1(\mathbb{T})$, $u_0 \in BV(\mathbb{T}; \{-1, 1\})$, there exist $d_{n,\varepsilon} \to x_{n+1} - x_n$, $n = 1, \ldots, \ell$, where $x_i$'s are the jumps of $u_0$ such that
  $$\frac{\int_{\mathbb{T}} \left(\frac{1}{4\varepsilon}(u_\varepsilon^2 - 1)^2 + \varepsilon |u_\varepsilon'|^2\right)\, dx - c_0 \ell}{\varepsilon}$$
  $$\geq -16\sqrt{2} \sum_{n=1}^{\ell} e^{-\sqrt{2} d_{n,\varepsilon}/\varepsilon} + o\left(e^{-3\sqrt{2} d_{n,\varepsilon}/(2\varepsilon)}\right),$$
- This inequality is sharp
- Similar result for $W$ of class $C^2$ quadratic at $\pm 1$.

## Second Order Gamma Limit, $N \geq 2$

- Anzellotti, Baldo, and Orlandi (1996) $W(s) = s^2$,

$$\int_\Omega u \, dx = m \rightsquigarrow \quad u = g > 0 \quad \text{on } \partial\Omega.$$

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left(\frac{1}{4\varepsilon}(u^2-1)^2 + \varepsilon|\nabla u|^2\right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}$$

if $u \in H^1(\Omega)$, $\int_\Omega u\, dx = m$ and $\mathcal{F}_\varepsilon^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$, where $E_0$ minimizes

$$\min\left\{\operatorname{Per}_\Omega E : \ E \subset \Omega,\ E \text{ Borel},\ \operatorname{meas} E = \tfrac{\operatorname{meas}\Omega + m}{2}\right\}.$$

**Theorem (G.L. and Murray)**

*Let $\Omega \subset \mathbb{R}^N$, $2 \leq N \leq 7$, be open, bounded, of class $C^{2,\alpha}$, $\alpha > 0$. Then*

$$\mathcal{F}^{(2)}(u) = -\frac{(N-1)^2}{9}\kappa^2$$

*if $u = 1\chi_{E_0} - 1\chi_{\Omega\setminus E_0}$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.*

- Similar result for $W$ of class $C^2$ quadratic at $\pm 1$.

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left( \frac{1}{2\beta\varepsilon} |u^2 - 1|^\beta + \varepsilon |\nabla u|^2 \right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}, \quad 1 < \beta < 2.$$

if $u \in H^1(\Omega)$, $\int_\Omega u \, dx = m$ and $\mathcal{F}_\varepsilon^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$, where $E_0$ minimizer

**Theorem (G.L. and Murray)**

*Let $\Omega \subset \mathbb{R}^N$, $2 \leq N \leq 7$, be open, bounded, of class $C^{2,\alpha}$, $\alpha > 0$. Then*

$$\mathcal{F}^{(2)}(u) = 0$$

*if $u = 1\chi_{E_0} - 1\chi_{\Omega \setminus E_0}$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.*

- Dal Maso, Fonseca and G.L. (2015) under the additional hypothesis $u_\varepsilon = 1$ on $\partial\Omega$.

$$\mathcal{F}_\varepsilon^{(2)}(u) = \frac{\int_\Omega \left( \frac{1}{2\beta\varepsilon} |u^2 - 1|^\beta + \varepsilon |\nabla u|^2 \right) dx - c_0 \operatorname{Per}_\Omega E_0}{\varepsilon}, \quad 1 < \beta < 2.$$

if $u \in H^1(\Omega)$, $\int_\Omega u \, dx = m$ and $\mathcal{F}_\varepsilon^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$, where $E_0$ minimizer

**Theorem (G.L. and Murray)**

*Let $\Omega \subset \mathbb{R}^N$, $2 \leq N \leq 7$, be open, bounded, of class $C^{2,\alpha}$, $\alpha > 0$. Then*

$$\mathcal{F}^{(2)}(u) = 0$$

*if $u = 1\chi_{E_0} - 1\chi_{\Omega \setminus E_0}$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.*

- Dal Maso, Fonseca and G.L. (2015) under the additional hypothesis $u_\varepsilon = 1$ on $\partial\Omega$.
- $E_0 \Rightarrow$ Ball compactly contained in $\Omega$.

# Main Tools

- Asymptotic development by Gamma convergence. Anzellotti and Baldo (1993)

# Main Tools

- Asymptotic development by Gamma convergence. Anzellotti and Baldo (1993)
- A type of rearrangement for Neumann problems Cianchi, Edmunds, and Gurka (1996).

# Main Tools

- Asymptotic development by Gamma convergence. Anzellotti and Baldo (1993)
- A type of rearrangement for Neumann problems Cianchi, Edmunds, and Gurka (1996).
- Gamma convergence for weighted 1-dimensional functional

$$H_\varepsilon(u) = \int_{-T}^{T} \left[\frac{1}{4}(u^2-1)^2 + \varepsilon^2 |u'|^2\right] \omega(t)\, dt.$$

# Main Tools

- Asymptotic development by Gamma convergence. Anzellotti and Baldo (1993)
- A type of rearrangement for Neumann problems Cianchi, Edmunds, and Gurka (1996).
- Gamma convergence for weighted 1-dimensional functional

$$H_\varepsilon(u) = \int_{-T}^{T} \left[\frac{1}{4}(u^2-1)^2 + \varepsilon^2|u'|^2\right]\omega(t)\, dt.$$

- In the radial case $\omega(t) = t^{N-1}$ Niethammer (1995)

# Main Tools

- Asymptotic development by Gamma convergence. Anzellotti and Baldo (1993)
- A type of rearrangement for Neumann problems Cianchi, Edmunds, and Gurka (1996).
- Gamma convergence for weighted 1-dimensional functional

$$H_\varepsilon(u) = \int_{-T}^{T} \left[ \frac{1}{4}(u^2 - 1)^2 + \varepsilon^2 |u'|^2 \right] \omega(t)\, dt.$$

- In the radial case $\omega(t) = t^{N-1}$ Niethammer (1995)
- Sternberg and Zumbrun (1998, 1999)

Thank you!
Happy Birthday, Gianni!