Homogenization of ferromagnetic energies on Poisson random sets in the plane

(work in collaboration with A. Piatnitski)

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Calculus of Variations and Applications

An International Conference to celebrate
Gianni Dal Maso's 65th Birthday

SISSA, January 28, 2020

My first image of Gianni

"…il più veloce ostacolista d'Italia, meglio di Ottoz…" (De Giorgi circa 1982)

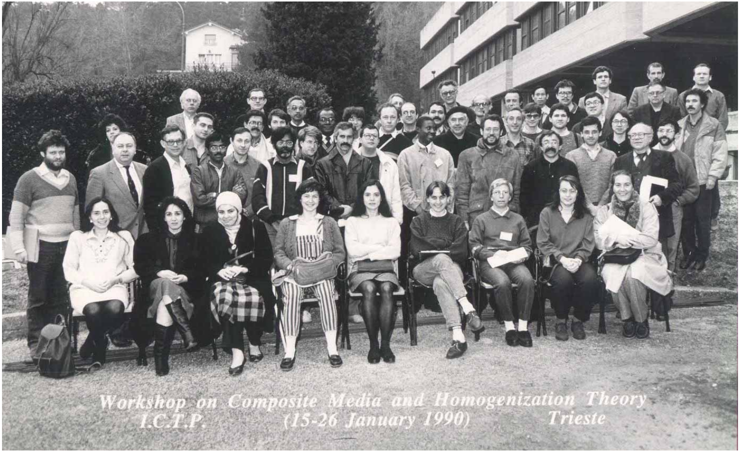
"…the fastest hurdler in Italy, better than Ottoz…"

(Eddy Ottoz)

G. Dal Maso and P. Longo. Γ-limits of obstacle problems

hurdle = ostacolo in Italian

My first International Workshop, organized by Gianni in 1985



The great Workshops and School on Homogenization in 1990 and 1993 (organized by Gianni and Gianfausto). By this time Gianni had already a considerable group of students.

Between 1988 and 1995 Gianni hosted me within his group, but for some reason we never ended up writing a paper together.

In 1995 I moved to SISSA, and we had the occasion to write two papers.

A. Braides and G. Dal Maso. Non-local approximation of the Mumford-Shah functional. *Calc. Var. PDE* **5** (1997), 293–322.

A. Braides, G. Dal Maso and A. Garroni. Variational formulation of softening phenomena in Fracture Mechanics: the one-dimensional case. *Arch. Rational Mech. Anal.* **146** (1999), 23–58.

These papers initiated an interest both towards discrete-to-continuum problems and towards applications to Materials Science.

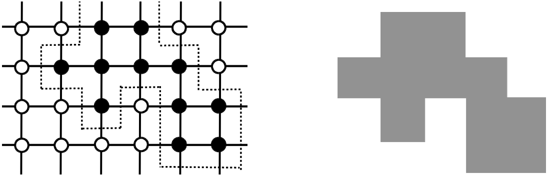
This brings us to the subject of this talk. . .

Surface lattice energies (Ising systems) in the plane

We start with an easy model lattice system
(cf. Caffarelli-De la Llave, Alicandro-B-Cicalese. Earlier work by Chambolle, etc.)

1) Regular lattice: $\mathbb{Z}^2$
Parameter: (scalar) spin function $i \mapsto u_i \in \{0,1\}$
Ferromagnetic energy: $E(u) = \sum_{|i-j|=1} |u_i - u_j|$

Piecewise-constant interpolation and identification with a set:
$u \sim \{u = 1\} =: A(u)$



Energy as a perimeter functional $E(u) = \mathrm{Per}(A(u))$ in a 1-periodic environment (the perimeter is the same as the number of edges in $\partial A(u)$)

Discrete-to-continuum analysis

In this context we may scale E and compute its homogenization within energies on sets of finite perimeter:

Define: $E_\varepsilon(u) = \sum_{|i-j|=1} \varepsilon|u_{\varepsilon i} - u_{\varepsilon j}|$ for $u : \varepsilon\mathbb{Z}^2 \to \{0,1\}$

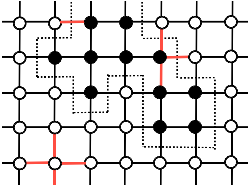
If $A_\varepsilon(u)(= \varepsilon A(u_{\varepsilon I}))$ is the union of the ε-cubes such that $u_{\varepsilon i} = 1$) still $E_\varepsilon(u) = \mathrm{Per}(A_\varepsilon(u))$ in a ε-periodic environment

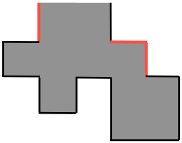
Convergence: $u^\varepsilon \to A$ if $A_\varepsilon(u^\varepsilon)$ tends to A locally in $\mathbb{R}^2$

Γ-limit: $F(A) = \int_{\partial^* A} (|\nu_1| + |\nu_2|) d\mathcal{H}^1$ for A set of finite perimeter

2) Randomly distributed weak inclusions in a regular lattice (“dilute spins”) (cf. B-Piatnitski)

We randomly “remove bonds” (**Bernoulli bond percolation**):





$E(u) = \sum_{|i-j|=1} a^\omega_{ij} |u_i - u_j|$ with $a^\omega_{ij} \in \{0, 1\}$,

$a^\omega_{ij} = 1$ with probability $p < 1/2$ (the case $p \geq 1/2$ being trivial)
according to an i.i.d. random variable
(ω = realization of the random variable)

Analogy with the stochastic Γ-convergence of Dal Maso and Modica (with “random perforated domains”) (but large perforations are almost surely very very ‘far away’)

Fundamental issue: to understand the geometry of the graph of the ‘active’ bonds

Discrete-to-continuum analysis

We may scale $E = E^\omega$ and compute its homogenization:

Define: $E^\omega_\varepsilon(u) = \sum_{|i-j|=1} \varepsilon\, a^\omega_{ij} |u_{\varepsilon i} - u_{\varepsilon j}|$ for $u : \varepsilon\mathbb{Z}^2 \to \{0,1\}$

Percolation techniques allow to prove that functions may be extended 'inside' the perforated domain so that $E_\varepsilon(u) \sim \mathrm{Per}(A_\varepsilon(Tu))$, where T is the extension operator

Convergence: $u^\varepsilon \to A$ if $A_\varepsilon(Tu^\varepsilon)$ tends to A locally in $\mathbb{R}^2$

Γ**-limit**: Almost surely $F(A) = \int_{\partial^* A} \varphi_p(\nu)\, d\mathcal{H}^1$ for A set of finite perimeter

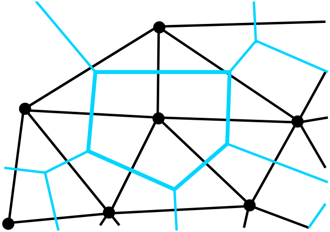
φ_p a.s. given by a "first-passage percolation formula" (asymptotic metric on the connected graph of active bonds)

3) Stochastic lattice:
(cf. Blanc-Le Bris-Lions, Alicandro-Cicalese-Gloria, Ruf)

The set $\mathcal{L}$ is a "perturbation of a regular lattice"; the location of points is random but the arrangement is *regular*:

- the distance between any two points in $\mathcal{L}$ is at least $\alpha > 0$
- any ball of diameter $1/\alpha$ contains at least a point of $\mathcal{L}$

Nearest neighbors are defined via Voronoi cells
$C_i = \{x \in \mathbb{R}^2 : |x - i| \leq |x - j|$ for all $j \in \mathcal{L}\}$ with a common edge

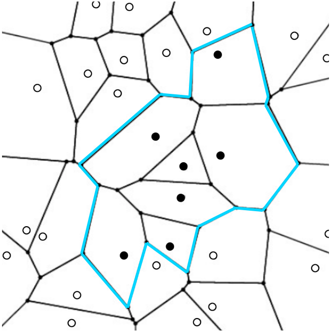


Notation: $\langle i, j\rangle$ means "i and j are nearest neighbors"; bonds between nearest neighbors give the related Delaunay triangulation

Parameter: (scalar) spin function $\mathcal{L} \ni i \mapsto u_i \in \{0,1\}$

Ferromagnetic energy: $E(u) = \sum_{\langle i,j \rangle} |u_i - u_j|$

Piecewise-constant interpolation on Voronoi cells, and identification with a set: $u \sim \{u = 1\} =: V(u)$ (union of the Voronoi cells where $u_i = 1$; if $\mathcal{L} = \mathbb{Z}^2$ then $V(u) = A(u)$).



Note: $E(u)$ = **number** of edges of $V(u)$, but still $E(u) \sim \mathrm{Per}(V(u))$ (more precisely, $\alpha \,\mathrm{Per}(V(u)) \leq E(u) \leq \frac{1}{\alpha}\mathrm{Per}(V(u))$)

Discrete-to-continuum analysis

Define: $E_\varepsilon(u) = \sum_{\langle i,j\rangle} \varepsilon|u_{\varepsilon i} - u_{\varepsilon j}|$ for $u : \varepsilon\mathcal{L} \to \{0,1\}$

Convergence: $u^\varepsilon \to A$ if $V_\varepsilon(u^\varepsilon)$ tends to A locally in $\mathbb{R}^2$
($V_\varepsilon(u)$ is the union of the ε-Voronoi cells with $u_{\varepsilon i} = 1$)

Under hypotheses of *stationarity and ergodicity* we have almost sure Γ-convergence

Γ-limit: $F(A) = \int_{\partial^* A} \varphi(\nu)\, d\mathcal{H}^1$ for A set of finite perimeter

using subadditive theorems to prove the existence of a deterministic φ

(cf. also B-Cicalese-Ruf, Cagnetti-Dal Maso-Scardia-Zeppieri)

Note: φ is isotropic only for very special choices of $\mathcal{L}$ (cf. Ruf)

Homogenization on Poisson random sets

A Poisson random set $\mathcal{L}$ with intensity λ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is characterized by

• for any bounded Borel set $B \subset \mathbb{R}^2$ the number of points in $B \cap \mathcal{L}$ has a Poisson law with parameter $\lambda|B|$

$$\mathbf{P}(\{\#(B \cap \mathcal{L}) = n\}) = e^{-\lambda|B|}\frac{(\lambda|B|)^n}{n!};$$

• for any collection of bounded disjoint Borel subsets in $\mathbb{R}^2$ the random variables defined as the number of points of $\mathcal{L}$ in these subsets are independent.

Technical detail: The probability space is equipped with a dynamical system $T_x : \Omega \mapsto \Omega$, $x \in \mathbb{R}^2$, Such that for any bounded Borel set B and any $x \in \mathbb{R}^2$ we have $\#\big((B + x) \cap \mathcal{L}\big)(\omega) = \#\big(B \cap \mathcal{L}\big)(T_x\omega)$. We suppose that T_x is a group of measurable measure preserving transformations in Ω and is ergodic.

Note: contrary to a stochastic lattice

- $\mathcal{L}$ is not regular: we have pairs of points of $\mathcal{L}$ arbitrarily close, and balls of arbitrary size not containing points of $\mathcal{L}$
- $\mathcal{L}$ is isotropic since the properties of Poisson random sets are invariant under (translations and) rotations

In the same way as for the stochastic lattice, we define the energy in terms of nearest neighbors for the Delaunay triangulation:

Ferromagnetic energy: $E(u) = \sum_{\langle i,j \rangle} |u_i - u_j|$ for $u : \mathcal{L} \to \{0, 1\}$

and define $V(u)$ as the union of the Voronoi cells C_i where $u_i = 1$

Issue: we cannot estimate Per$V(u)$ in terms of $E(u)$.
(Even for a single Voronoi cell, we have large C_i with few edges or small C_i with many edges).

However, we may estimate sets $V(u)$ containing ‘many’ cells thanks to a Percolation lemma.

Pimentel's "polyomino" lemma

Π denotes the set of finite connected unions of Voronoi cells. If $P \in \Pi$ we set

$$\mathbf{A}(P) = \{z \in \mathbb{Z}^2 : (z + (0,1)^2) \cap P \neq \emptyset\}.$$

Lemma

Let $R > 0$ and $\gamma > 0$. Then there exists a deterministic constant C such that for almost all ω there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that if $P \in \Pi$ and $\varepsilon < \varepsilon_0$ satisfy

$$P \cap \frac{R}{\varepsilon}(0,1)^2 \neq \emptyset, \qquad \max\Big\{\#\{i : C_i \subset P\}, \#\mathbf{A}(P)\Big\} \geq \varepsilon^{-\gamma}$$

then we have

$$\frac{1}{C}\#\{i : C_i \subset P\} \leq \#\mathbf{A}(P) \leq C\,\#\{i : C_i \subset P\}.$$

Compactness of Voronoi sets

We still use the notation $E_\varepsilon(u) = \sum_{\langle i,j\rangle} \varepsilon|u_{i\varepsilon} - u_{j\varepsilon}|$ for $u : \varepsilon\mathcal{L} \to \{0,1\}$
$V_\varepsilon(u)$ as the union of the ε-Voronoi cells εC_i such that $u_{\varepsilon i} = 1$

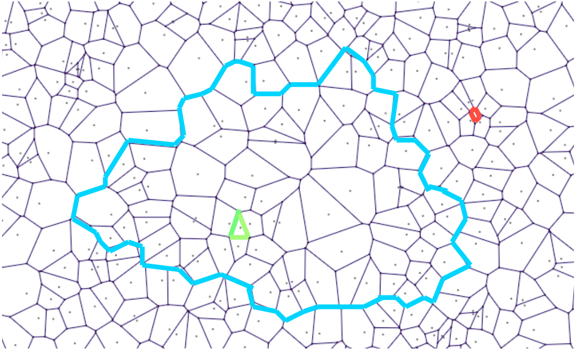
Lemma

Let u^ε be such that $\sup_\varepsilon E_\varepsilon(u^\varepsilon) < +\infty$. Then we can write

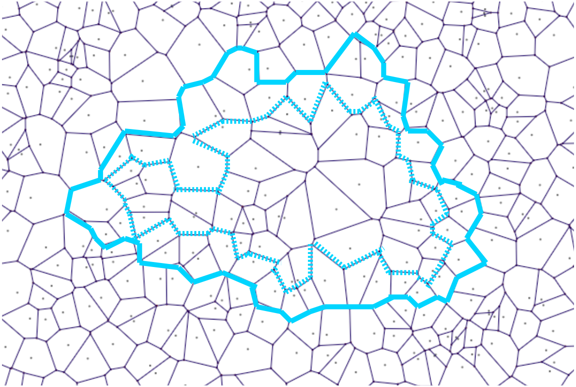
$$V_\varepsilon(u^\varepsilon) = (A_\varepsilon \cup B'_\varepsilon) \setminus B''_\varepsilon,$$

where $|B'_\varepsilon| + |B''_\varepsilon| \to 0$, the family χ_{A_ε} is precompact in $L^1_{\rm loc}(\mathbb{R}^2)$ and each its limit point is the characteristic function of a set of finite perimeter A, so that the same holds for $\chi_{V_\varepsilon(u^\varepsilon)}$.

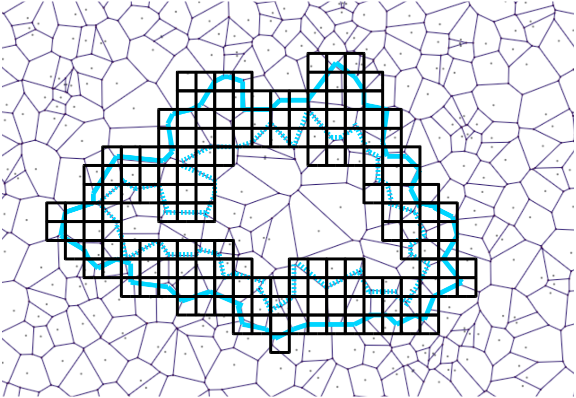
Proof. Subdivide the boundary Voronoi cells of $V_\varepsilon(u^\varepsilon)$ into connected components P^ε_k:
• large components (such that $\frac{1}{\varepsilon}P^\varepsilon_k$ satisfies Pimentel's lemma with e.g. $\gamma = 1/4$).
The ε-cubes covering P^ε_k are asymptotically negligible, and Pimentel's lemma ensures that the 'interior' of such components union such cubes has equibounded perimeter
• small components.
Using the isoperimetric inequality we show that the interior (giving B''_ε) or exterior (giving B'_ε) of these components is asymptotically negligible,



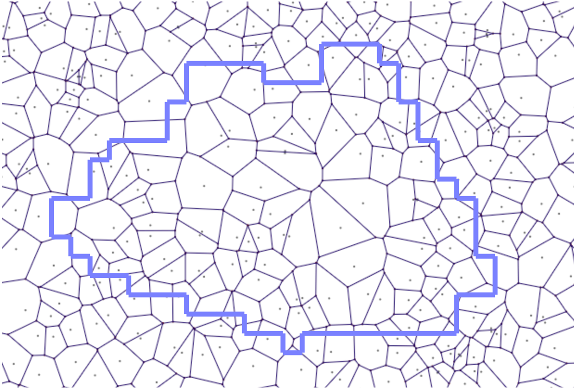
$$V_\varepsilon(u^\varepsilon) = (A_\varepsilon \cup B'_\varepsilon) \setminus B''_\varepsilon,$$



We consider boundary Voronoi cells
(for the components of the boundary with many edges)



We use Pimentel's Lemma to prove that unions of 'boundary cubes' have finite perimeter



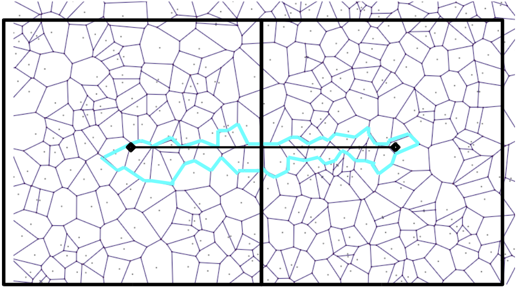
We prove the pre-compactness of sets with boundary with many edges (plus boundary squares, which are asymptotically negligible)

Geometry of regular Voronoi cells

Given $\alpha > 0$ we define the set of α-regular points of $\mathcal{L}$ as

$$\{i \in \mathcal{L} : C_i \text{ contains a ball of radius } \alpha, \operatorname{diam} C_i \leq \tfrac{1}{\alpha}, \#\text{edges} \leq \tfrac{1}{\alpha}\}$$

If α is small enough, there exists $L > 0$ such that the event that $k, k' \in \mathbb{Z}^2$ with $|k - k'| = 1$ are such that the segment $[Lk, Lk']$ intersects only α-regular sets has probability $p > 1/2$.



We can then apply Bernoulli bond-percolation theory to the bonds $[k, k']$ of nearest neighbors in $\mathbb{Z}^2$

Consequence: the subset of the Delaunay triangulation of $\mathcal{L}$ with α-regular endpoints contains an infinite connected component $\mathcal{D}_\alpha$ whose complement is composed of isolated bounded sets.

Furthermore $\mathcal{D}_\alpha$ is 'regular', in the sense that there exists τ_α such that each two points $x, y \in \mathcal{D}_\alpha$ are connected by a path with length not more than $\tau_\alpha |x - y|$.

As a consequence, we immediately have the finiteness of the Γ-limsup on sets of finite perimeter; more precisely,

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} E_\varepsilon(A) \leq \tau_\alpha \mathcal{H}^1(\partial^* A)$$

Another consequence (with more refined properties of 'uniform regularity' of $\mathcal{D}_\alpha$): we may use paths in $\mathcal{D}_\alpha$ in "discrete area-formula-type arguments" (in particular to match boundary conditions).

The possibility to match boundary conditions, and the invariance by rotations, gives a candidate formula for the surface tension

$$\tau = \lim_{t\to+\infty} \frac{1}{t} \min\Big\{\#\Big(\text{segments of paths in } \mathcal{D}\text{‘almost’ joining } (0,0) \text{ and } (t,0)\Big)\Big\}$$

A subadditive argument allows to show that this limit exists a.s. and is deterministic.

This formula allows to construct a matching upper bound. Finally, after a scaling argument which shows that $\tau = \tau_0\sqrt{\lambda}$ (λ the intensity of the Poisson random set) we have

Theorem (B-Piatnitski 2020)

Almost surely the functionals E_ε Γ-converge to $\tau_0\sqrt{\lambda}\,\mathcal{H}^1(\partial^* A)$ with respect to the L^1-loc convergence of $V_\varepsilon(u^\varepsilon)$ to A.

Some final remarks

1. Poisson random sets are a relatively simple environment to obtain **isotropic surface energies**. Their treatment is based on the analysis of geometric properties of clusters of Voronoi cells.

2. The result can be extended to the original energies of Blake-Zisserman type (truncated quadratic potentials) considered by Chambolle obtaining **isotropic Mumford-Shah functionals** (work in progress with Marco Caroccia)

3. The **extension to higher dimension** seems feasible but requires (a) finer properties of α-regular clusters; (b) different formulas for τ_0 (work in progress with A. Piatnitski). Note that Pimentel's Lemma and compactness hold in any dimension.

Happy birthday, Gianni!