A Note about the Strong Maximum Principle on RCD Spaces

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Abstract

We give a direct proof of the strong maximum principle on finite dimensional RCD spaces based on the Laplacian comparison of the squared distance.

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Contents

1 Introduction **1**

2 Result **2**

1 Introduction

In the context of analysis in metric measure spaces it is by now well understood that a doubling condition and a Poincaré inequality are sufficient to derive the basics of elliptic regularity theory. In particular, one can obtain the Harnack inequality for harmonic functions which in turns implies the strong maximum principle. We refer to [6] for an overview on the topic and detailed bibliography.

$\mathsf{RCD}^*(K,N)$ spaces ([3], [16], see also [4], [11], [5]) are, for finite N, doubling ([27]) and support a Poincaré inequality ([24]) and thus in particular the above applies (see [16] and [17] for the details). Still, given that in fact such spaces are much more regular than general doubling&Poincaré ones, one might wonder whether there is a simpler proof of the strong maximum principle.

Aim of this short note it to show that this is actually the case: out of the several arguments available in the Euclidean space, the one based on the estimates for the Laplacian of the squared distance carries over to such non-smooth context rather easily. We emphasize that such argument is, with only minor variations, the original one of Hopf, which appeared in [20] (the so called 'boundary point lemma' about the sign of external derivative at a maximum point at the boundary, also due to Hopf, appeared later in [21]).

In order to mimic Hopf's proof we need to know that given a closed set C, for 'many' points $x \notin C$ there is a unique $y \in C$ minimizing the distance from x. In the Euclidean setting this is easy to prove thanks to the strict convexity of balls, but in general metric spaces the same

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property can fail, even in presence of a (non-Riemannian) curvature-dimension condition, see Remark 2.7. In our situation this can be proved using the existence of optimal transport maps proved in [18], see Lemma 2.6. After the work on this manuscript finished, we realized that a very similar statement has been obtained, with a similar proof, in the recent paper [12], see Theorem 4.7 there.

We conclude remarking that the present result simplifies the proofs of those properties of $\mathsf{RCD}^*(K, N)$ spaces which depend on the strong maximum principle, like for instance the splitting theorem ([14], [15]).

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2 Result

All metric measure spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ we will consider will be such that (X, d) is complete and separable, $\mathfrak{m}$ is a Radon non-negative measure with $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$.

To keep the presentation short we assume that the reader is familiar with the definition of RCD^* spaces and with calculus on them (see [3] and [16]). Here we only recall those definitions and facts that will be used in the course of the proofs. In particular, we shall take for granted the notion of $W^{1,2}(\mathrm{X})$ space on the metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and, for $f \in W^{1,2}(\mathrm{X})$, of the minimal weak upper gradient $|\mathrm{D}f|$. Recall that the minimal weak upper gradient is a local object, i.e.:

$$|\mathrm{D}f| = |\mathrm{D}g| \qquad \mathfrak{m} - a.e. \text{ on } \{f = g\} \qquad \forall f, g \in W^{1,2}(\mathrm{X}). \tag{2.1}$$

Then the notion of Sobolev space over an open set can be easily given:

Definition 2.1 (Sobolev space on an open subset of X). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and let $\Omega \subset \mathrm{X}$ open. Then we define*

$$W^{1,2}_{\mathsf{loc}}(\Omega) := \{f \in L^2_{\mathsf{loc}}(\Omega) : \textit{for every } x \in \Omega \textit{ there exists } U \subset \Omega \textit{ neighbourhood of } x \\ \textit{and there exists } f_U \in W^{1,2}_{\mathsf{loc}}(\mathrm{X}) \textit{ such that } f_{|_U} = f_U\}.$$

For $f \in W^{1,2}_{\mathsf{loc}}(\Omega)$ the function $|\mathrm{D}f| \in L^2_{\mathsf{loc}}(\Omega)$ is defined as

$$|\mathrm{D}f| := |\mathrm{D}f_U| \quad \mathfrak{m} - a.e. \textit{ on } U,$$

where $|\mathrm{D}f_U|$ is the minimal weak upper gradient of f_U and the locality of this object ensures that $|\mathrm{D}f|$ is well defined.

Then we set

$$W^{1,2}(\Omega) := \{f \in W^{1,2}_{\mathsf{loc}}(\Omega) : f, |\mathrm{D}f| \in L^2(\Omega)\}.$$

The definition of (sub/super)-harmonic functions can be given in terms of minimizers of the Dirichlet integral (see [6] for a thorough discussion on the topic):

Definition 2.2 (**Subharmonic/Superharmonic/Harmonic functions).** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* Ω *be an open subset in* X*. We say that* f *is subharmonic (resp. superharmonic) in* Ω *if* $f \in W^{1,2}(\Omega)$ *and for any* $g \in W^{1,2}(\Omega)$*,* $g \le 0$ *(resp.* $g \ge 0$*) with* $\operatorname{supp} g \subset\subset \Omega$*, it holds*

$$\frac{1}{2}\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m} \le \frac{1}{2}\int_\Omega |\mathrm{D}(f+g)|^2 \,\mathrm{d}\mathfrak{m}. \tag{2.2}$$

The function f *is harmonic in* Ω *if it is both subharmonic and superharmonic.*

On $\mathsf{RCD}(K,\infty)$ spaces, the weak maximum principle can be deduced directly from the definition of subharmonic function and the following property, proved in [3]:

$$\begin{aligned}&\text{Let } (\mathrm{X}, \mathsf{d}, \mathfrak{m}) \text{ be } \mathsf{RCD}(K,\infty),\ K\in\mathbb{R}, \text{ and } f \in W^{1,2}(\mathrm{X}) \text{ be such that } |\mathrm{D}f| \in L^\infty(\mathrm{X}).\\ &\text{Then there exists } \tilde f = f\ \mathfrak{m}\text{-a.e. such that } \mathrm{Lip}(\tilde f) \le \||\mathrm{D}f|\|_{L^\infty}.\end{aligned} \tag{2.3}$$

We can now easily prove the following:

Theorem 2.3 (**Weak Maximum Principle).** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space,* $\Omega \subset \mathrm{X}$ *open with finite measure and let* $f \in W^{1,2}(\Omega) \cap C(\bar\Omega)$ *be subharmonic. Then*

$$\sup_\Omega f \le \sup_{\partial\Omega} f, \tag{2.4}$$

to be intended as 'f is constant' in the case $\Omega = \mathrm{X}$*.*

proof We argue by contradiction. If (2.4) does not hold, regardless of whether Ω coincides with X or not, we can find $c < \sup_\Omega f$ such that the function

$$\tilde f := \min\{c, f\}$$

agrees with f on $\partial\Omega$. The locality of the differential grants that

$$|\mathrm{D}\tilde f| = \chi_{\{f<c\}}|\mathrm{D}f| \tag{2.5}$$

and from the assumption that f is subharmonic and the fact that $\tilde f \le f$ we deduce that

$$\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m} \le \int_\Omega |\mathrm{D}\tilde f|^2 \,\mathrm{d}\mathfrak{m} \overset{(2.5)}{=} \int_{\{f<c\}\cap\Omega} |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m},$$

which forces

$$|\mathrm{D}f| = 0 \ \mathfrak{m}\text{-a.e. on } \{f \ge c\}. \tag{2.6}$$

Now consider the function $g := \max\{c, \chi_\Omega f\}$, notice that our assumptions grant that $g \in C(\mathrm{X})$ and that the locality of the differential yields

$$|\mathrm{D}g| = \chi_{\Omega\cap\{f>c\}}|\mathrm{D}f| \overset{(2.6)}{=} 0. \tag{2.7}$$

Hence property (2.3) gives that g is constant, i.e. $f \le c$ on Ω. This contradicts our choice of c and gives the conclusion. □

We remark that in the finite-dimensional case one could conclude from (2.7) by using the Poincaré inequality in place of property (2.3).

To prove the strong maximum principle we need to recall few facts. The first is the concept of measure-valued Laplacian (see [16]), for which we restrict the attention to proper (=closed bounded sets are compact) and infinitesimally Hilbertian (=$W^{1,2}(\mathrm{X})$ is an Hilbert space, see [16]) spaces. Recall that on infinitesimally Hilbertian spaces, given two functions $f, g \in W^{1,2}_{loc}$ the quantity $\langle \nabla f, \nabla g\rangle \in L^1_{loc}$ is well defined by polarization of $f \mapsto |\mathrm{D} f|^2$ (see [16]):

Definition 2.4 (Measure-valued Laplacian). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be proper and infinitesimally Hilbertian,* $\Omega \subset \mathrm{X}$ *open and* $f \in W^{1,2}(\Omega)$*. We say that* f *has a measure-valued Laplacian in* Ω*, and write* $f \in D(\mathbf{\Delta}, \Omega)$*, provided there exists a Radon measure, that we denote by* $\mathbf{\Delta} f_{|\Omega}$*, such that for every* $g \colon \mathrm{X} \to \mathbb{R}$ *Lipschitz with compact support contained in* Ω *it holds*

$$\int g \, \mathrm{d}\mathbf{\Delta} f_{|\Omega} = -\int \langle \nabla f, \nabla g\rangle \, \mathrm{d}\mathfrak{m}. \tag{2.8}$$

If $\Omega = \mathrm{X}$ *we write* $f \in D(\mathbf{\Delta})$ *and* $\mathbf{\Delta} f$*.*

Much like in the smooth case, it turns out that being subharmonic is equivalent to having non-negative Laplacian. This topic has been investigated in [16] and [17], here we report the proof of this fact because in [17] it has been assumed the presence of a Poincaré inequality, while working on proper infinitesimally Hilbertian spaces allows to easily remove such assumption.

Theorem 2.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a proper infinitesimally Hilbertian space,* $\Omega \subset \mathrm{X}$ *open and* $f \in W^{1,2}(\Omega)$*.*

Then f *is subharmonic (resp. superharmonic, resp. harmonic) if and only if* $f \in D(\mathbf{\Delta}, \Omega)$ *with* $\mathbf{\Delta} f_{|\Omega} \geq 0$ *(resp.* $\mathbf{\Delta} f_{|\Omega} \leq 0$*, resp.* $\mathbf{\Delta} f_{|\Omega} = 0$*).*

proof

Only if Let $\mathrm{LIP}_c(\Omega) \subset W^{1,2}(\Omega)$ be the space of Lipschitz functions with support compact and contained in Ω. For $g \in \mathrm{LIP}_c(\Omega)$ non-positive and $\varepsilon > 0$ apply (2.2) with εg in place of g to deduce

$$\int_\Omega |\mathrm{D}(f + \varepsilon g)|^2 - |\mathrm{D} f|^2 \, \mathrm{d}\mathfrak{m} \geq 0$$

and dividing by ε and letting $\varepsilon \downarrow 0$ we conclude

$$\int_\Omega \langle \nabla f, \nabla g\rangle \, \mathrm{d}\mathfrak{m} \geq 0.$$

In other words, the linear functional $\mathrm{LIP}_c(\Omega) \ni g \mapsto -\int_\Omega \langle \nabla f, \nabla g\rangle \, \mathrm{d}\mathfrak{m}$ is positive. It is then well known, see e.g. [7, Theorem 7.11.3], that the monotone extension of such functional to the space of continuous and compactly supported functions on Ω is uniquely represented by integration w.r.t. a non-negative measure, which is the claim.

If Recall from [2] that on general metric measure spaces Lipschitz functions are dense in energy in $W^{1,2}$; since infinitesimally Hilbertianity implies uniform convexity of $W^{1,2}$, we see that in our case they are dense in the $W^{1,2}-$norm. Then by truncation and cut-off argument we easily see that

$$\big\{g \in \mathrm{LIP}_c(\Omega) : g \leq 0\big\} \quad \text{is } W^{1,2}\text{−dense in} \quad \big\{g \in W^{1,2}(\Omega) : \ g \leq 0 \ \ \mathrm{supp}(g) \subset\subset \Omega\big\}. \tag{2.9}$$

Now notice that the convexity of $g \mapsto \frac{1}{2}\int_\Omega |\mathrm{D}g|^2\,\mathrm{d}\mathfrak{m}$ grants that for any $g \in W^{1,2}(\Omega)$ it holds

$$|\mathrm{D}(f+g)|^2 - |\mathrm{D}f|^2 \geq \lim_{\varepsilon\downarrow 0}\frac{|\mathrm{D}(f+\varepsilon g)|^2 - |\mathrm{D}f|^2}{\varepsilon} = 2\,\langle \nabla f, \nabla g\rangle$$

and thus from the assumption $\mathbf{\Delta} f_{|\Omega} \geq 0$ we deduce that

$$\int_\Omega |\mathrm{D}(f+g)|^2 - |\mathrm{D}f|^2\,\mathrm{d}\mathfrak{m} \geq 0 \tag{2.10}$$

for every $g \in \mathrm{LIP}_c(\Omega)$ non-positive. Taking (2.9) into account we see that (2.10) also holds for any $g \in W^{1,2}(\Omega)$ non-negative with $\mathrm{supp}(g) \subset \Omega$, which is the thesis. □

For $x \in \mathrm{X}$ we write d_x for the function $y \mapsto \mathsf{d}(x,y)$. We shall need the following two properties of the squared distance function valid on $\mathsf{RCD}^*(K,N)$ spaces, $N<\infty$:

$$\mathsf{d}^2_{x_0} \in W^{1,2}_{\mathsf{loc}}(\mathrm{X}) \quad \text{and} \quad |\mathrm{D}(\mathsf{d}^2_{x_0})|^2 = 2\mathsf{d}^2_{x_0} \quad \mathfrak{m}\text{-a.e.}, \tag{2.11}$$

$$\mathsf{d}^2_{x_0} \in D(\mathbf{\Delta}) \quad \text{and} \quad \mathbf{\Delta}\mathsf{d}^2_{x_0}(x) \leq \ell_{K,N}(\mathsf{d}_{x_0})\mathfrak{m}, \tag{2.12}$$

where $\ell_{K,N} : [0,+\infty) \to [0,+\infty)$ is some continuous function depending only on K,N. Property (2.11) can be seen as a consequence of Cheeger's work [10]: recall that $\mathsf{CD}(K,N)$ spaces are doubling ([27]) and support a 1-2 weak Poincaré inequality ([24]) and notice that, since X is geodesic, the local Lipschitz constant of d_x is identically 1. An alternative proof, more tailored to the RCD setting, passes through the fact that $\mathsf{d}^2_{x_0}/2$ is c-concave and uses the regularity of W_2-geodesics, see for instance [19] for the details of the argument.

The Laplacian comparison estimate (2.12) is one of the main results in [16]. Notice that in [16] such inequality has been obtained in its sharp form, but for our purposes the above formulation is sufficient.

Beside these facts, we shall need the following geometric property of RCD spaces, which we believe is interesting on its own. For the notions of c-transform and c-superdifferential see for instance [28], [1], [26].

Lemma 2.6 (a.e. unique projection)**.** *Let $K \in \mathbb{R}$, $N \in [1,\infty)$, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ an $\mathsf{RCD}^*(K,N)$ space and $C \subset \mathrm{X}$ a closed set. Then for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ there exists a unique $y \in C$ such that*

$$\mathsf{d}(x,y) = \min_{z\in C}\mathsf{d}(x,z). \tag{2.13}$$

proof Existence follows trivially from the fact that X is proper. For uniqueness define

$$\varphi(x) := \inf_{z\in C}\frac{\mathsf{d}^2(x,z)}{2} = \psi^c(x) \qquad \text{where} \qquad \psi(y) := \begin{cases} 0, & \text{if } y \in C,\\ -\infty, & \text{if } y \in \mathrm{X}\setminus C.\end{cases}$$

Since $\varphi^c = \psi^{cc} \geq \psi$, if $x \in \mathrm{X}$ and $y \in C$ are such that (2.13) holds, we have

$$\varphi(x) + \varphi^c(y) \geq \varphi(x) + \psi(y) \overset{(2.13)}{=} \frac{\mathsf{d}^2(x,y)}{2},$$

i.e. $y \in \partial^c\varphi(x)$. Conclude recalling that since φ is c-concave and real valued, Theorem 3.4 in [18] grants that for $\mathfrak{m}$-a.e. x there exists a unique $y \in \partial^c\varphi(x)$. □

Remark 2.7. The simple proof of this lemma relies on quite delicate properties of RCD spaces, notice indeed that the conclusion can fail on the more general $\mathsf{CD}(K,N)$ spaces. Consider for instance $\mathbb{R}^2$ equipped with the distance coming from the L^∞ norm and the Lebesgue measure $\mathcal{L}^2$. This is a $\mathsf{CD}(0,2)$ space, as shown in the last theorem in [28]. Then pick $C := \{(z_1,z_2) : z_1 \geq 0\}$ and notice that for every $(x_1,x_2) \in \mathbb{R}^2$ with $x_1 < 0$ there are uncountably many minimizers in (2.13).

What makes the proof work in the RCD case is the validity of the result in [18] which uses some forms of non-branching and lower Ricci bounds to deduce existence of optimal maps. This kind of argument appeared first in [13] (albeit the main idea of the proof was independently discovered by various authors, see for instance [8] and the references therein) and since then the topic has been pushed quite far: to date, the most general results are those recently obtained by Kell in [22], which in particular cover the case of essential non-branching (see [25]) MCP spaces (see [27], [23]) previously considered by Cavalletti-Mondino in [9]. ■

We can now prove the main result of this note:

Theorem 2.8 (Strong Maximum Principle). *Let $K \in \mathbb{R}$, $N \in [1,\infty)$ and $(\mathrm{X},\mathsf{d},\mathfrak{m})$ an $\mathsf{RCD}^*(K,N)$ space. Let $\Omega \subset \mathrm{X}$ be open and connected and let $f \in W^{1,2}(\Omega) \cap C(\bar\Omega)$ be subharmonic and such that for some $\bar x \in \Omega$ it holds $f(\bar x) = \max_{\bar\Omega} f$. Then f is constant.*

proof Put $m := \sup_\Omega f$, $C := \{x \in \bar\Omega : f(x) = m\}$ and define

$$\Omega' := \big\{x \in \Omega \setminus C \ : \ \mathsf{d}(x,C) < \mathsf{d}(x,\partial\Omega)\big\}.$$

By assumption we know that $C \cap \Omega \neq \emptyset$ and that Ω is connected, thus since C is closed, either $C \supset \Omega$, in which case we are done, or $\partial C \cap \Omega \neq \emptyset$, in which case $\Omega' \neq \emptyset$. We now show that such second case cannot occur, thus concluding the proof.

Assume by contradiction that $\Omega' \neq \emptyset$, notice that Ω' is open and thus $\mathfrak{m}(\Omega') > 0$. Hence by Lemma 2.6 we can find $x \in \Omega'$ and $y \in C$ such that (2.13) holds. Notice that the definition of Ω' grants that $y \in \Omega$, put $r := \mathsf{d}(x,y)$ and define

$$h(z) := e^{-A\mathsf{d}^2(z,x)} - e^{-Ar^2},$$

where $A \gg 1$ will be fixed later. By the chain rule for the measure-valued Laplacian (see [16]) we have that $h \in D(\mathbf{\Delta})$ with

$$\mathbf{\Delta} h = A^2 e^{-A\mathsf{d}_x^2}|\mathrm{D}\mathsf{d}_x^2|^2\,\mathfrak{m} - Ae^{-A\mathsf{d}_x^2}\mathbf{\Delta}\mathsf{d}_x^2 \overset{(2.11),(2.12)}{\geq} 2e^{-A\mathsf{d}_x^2}\big(A^2\mathsf{d}_x^2 - A\ell_{K,N}(\mathsf{d}_x)\big)\mathfrak{m}.$$

Hence we can, and will, choose A so big that $\mathbf{\Delta} h|_{B_{r/2}(y)} \geq 0$. Now let $r' < r/2$ be such that $B_{r'}(y) \subset \Omega$ and notice that for every $\varepsilon > 0$ the function $f_\varepsilon := f + \varepsilon h$ is subharmonic in $B_{r'}(y)$ and thus according to Theorem 2.3 we have

$$f_\varepsilon(y) \leq \sup_{\partial B_{r'}(y)} f_\varepsilon, \qquad \forall \varepsilon > 0. \tag{2.14}$$

Since $\{h < 0\} = \mathrm{X} \setminus \bar B_r(x)$ and $h(y) = 0$ we have

$$f_\varepsilon(y) > f_\varepsilon(z) \qquad \forall z \in \partial B_{r'}(y) \setminus \bar B_r(x),\ \forall \varepsilon > 0. \tag{2.15}$$

On the other hand, $\partial B_{r'}(y) \cap \bar{B}_r(x)$ is a compact set contained in $\Omega \setminus C$, hence by continuity and the definition of C we have

$$f(y) > \sup_{\partial B_{r'}(y) \cap \bar{B}_r(x)} f$$

and thus for $\varepsilon > 0$ sufficiently small we also have

$$f_\varepsilon(y) > \sup_{\partial B_{r'}(y) \cap \bar{B}_r(x)} f_\varepsilon.$$

This inequality, (2.15) and the continuity of f_ε contradict (2.14); the thesis follows. □

Remark 2.9. The proof uses the weak maximum principle, the Laplacian comparison of the distance, its linearity and Lemma 2.6 only. Since the Laplacian comparison for the distance holds in the more general class of infinitesimally strictly convex MCP spaces (see [16]), taking Remark 2.7 into account we see that the strong maximum principle holds in the class of essentially non-branching and infinitesimally Hilbertian MCP spaces. ■

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