Differential structure associated to axiomatic Sobolev spaces

Original
Differential structure associated to axiomatic Sobolev spaces / Gigli, N.; Pasqualetto, E.. - In: EXPOSITIONES MATHEMATICAE. - ISSN 0723-0869. - 38:4(2019), pp. 480-495. [10.1016/j.exmath.2019.01.002]

Availability:
This version is available at: 20.500.11767/111342 since: 2020-04-30T18:00:07Z

Publisher:

Published
DOI:10.1016/j.exmath.2019.01.002

Differential structure associated to axiomatic Sobolev spaces

Nicola Gigli* Enrico Pasqualetto†

December 20, 2018

Abstract

The aim of this note is to explain in which sense an axiomatic Sobolev space over a general metric measure space (à la Gol'dshtein-Troyanov) induces – under suitable locality assumptions – a first-order differential structure.

MSC2010: primary 46E35, secondary 51Fxx

Keywords: axiomatic Sobolev space, locality of differentials, cotangent module

Contents

Introduction

An axiomatic approach to the theory of Sobolev spaces over abstract metric measure spaces has been proposed by V. Gol'dshtein and M. Troyanov in [5]. Their construction covers many important notions: the weighted Sobolev space on a Riemannian manifold, the Hajłasz Sobolev space [6] and the Sobolev space based on the concept of upper gradient [1, 2, 7, 9].

A key concept in [5] is the so-called *D-structure*: given a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and an exponent $p \in (1, \infty)$, we associate to any function $u \in L^p_{loc}(\mathrm{X})$ a family $D[u]$ of non-negative Borel functions called *pseudo-gradients*, which exert some control from above

*SISSA, Via Bonomea 265, 34136 Trieste. *E-mail address:* ngigli@sissa.it

†SISSA, Via Bonomea 265, 34136 Trieste, and Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä. *E-mail address:* enpasqua@jyu.fi

on the variation of u. The pseudo-gradients are not explicitly specified, but they are rather supposed to fulfil a list of axioms. Then the space $W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$ is defined as the set of all functions in $L^p(\mathfrak{m})$ admitting a pseudo-gradient in $L^p(\mathfrak{m})$. By means of standard functional analytic techniques, it is possible to associate to any Sobolev function $u\in W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$ a uniquely determined minimal object $\underline{D}u\in D[u]\cap L^p(\mathfrak{m})$, called *minimal pseudo-gradient* of the function u. Nevertheless, we point out that the correspondence $u\mapsto \underline{D}u$ is in general not linear, the reason being that $\underline{D}u$ behaves as the 'modulus of the differential of u' rather than the 'differential of u' itself. The purpose of this manuscript is to prove that it is possible to build a linear object $u\mapsto \mathrm{d}u$, called *differential*, which underlies the minimal pseudo-gradient in the sense we are going to describe.

In recent years, the first author of the present paper introduced a differential structure on general metric measure spaces (cf. [3,4]). The key tool in this theory is given by the notion of *L^p-normed L^∞-module*, which constitutes a suitable abstraction of the concept of 'space of p-integrable sections of a Banach bundle'. Shortly said, an L^p-normed L^∞-module is a vector space whose elements v can be multiplied by L^∞-functions and associated with a *pointwise norm* $|v|\in L^p$, which 'fiberwise' behaves like a norm; the reader might think of, for instance, the space of p-integrable vector fields on a given Riemannian manifold endowed with the natural pointwise operations. The fundamental example of normed module over a general metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is the so-called *cotangent module* $L^2(T^*\mathrm{X})$, whose elements play the role of 'square-integrable 1-forms on X' – in some abstract sense.

The main result of this paper – namely Theorem 3.2 – says that any D-structure (satisfying suitable locality properties) gives rise to a natural notion of cotangent module $L^p(T^*\mathrm{X};D)$, whose properties are analogous to the ones of the cotangent module $L^2(T^*\mathrm{X})$ described in [3]. Roughly speaking, the cotangent module allows us to represent minimal pseudo-gradients as pointwise norms of suitable linear objects. More precisely, this theory provides the existence of an abstract differential $\mathrm{d}: W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)\to L^p(T^*\mathrm{X};D)$, which is a linear operator such that the pointwise norm $|\mathrm{d}u|\in L^p(\mathfrak{m})$ of $\mathrm{d}u$ coincides with $\underline{D}u$ in the $\mathfrak{m}$-a.e. sense for any function $u\in W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$. Finally, we prove that the differential d is a closed operator (cf. Theorem 3.4) and satisfies some basic calculus rules (cf. Proposition 2.13).

Let us conclude this introduction recalling that there exists another, substantially different, way of speaking about differential structures on metric measure spaces: it goes back to Cheeger's celebrated paper [2] (see also the more recent developments [**?**], [8]), is related to a metric version of Rademacher's theorem and led to the definition of *Lipschitz differentiability spaces*. Specifically, a Lipschitz differentiability space is a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ which can be $\mathfrak{m}$-almost all covered by Borel sets A_i and so that for every i there is a finite number of Lipschitz functions $f_{i,1},\ldots,f_{i,n_i}:\mathrm{X}\to\mathbb{R}$ with the following property: for any $f:\mathrm{X}\to\mathbb{R}$ Lipschitz there are unique L^∞ functions $c_{i,1},\ldots,c_{i,n_i}$ on A_i such that

$$\mathrm{lip}\big(f-\sum_j c_{i,j}(x)f_{i,j}\big)(x)=0\qquad\text{for }\mathfrak{m}\text{-a.e. }x\in A_i,\tag{0.1}$$

where $\mathrm{lip} f$ denotes the local Lipschitz constant of the function f. In such spaces one can legitimately call the functions $c_{i,1}, \ldots, c_{i,n_i}$ "coefficients of the differential of f on A_i w.r.t. the base of the cotangent bundle given by the differentials of the $f_{i,j}$'s".

It is then natural to try to understand the relation between this, somehow concrete, notion and the abstract approach studied here. In [3, Section 2.5] it has been shown full compatibility between the notion of differential coming from (0.1) and the one based on the theory of L^∞ modules built upon the standard (i.e. non-axiomatic) notion of Sobolev space. It follows that the same compatibility can hold with the 'axiomatic differential', so to say, built here only in presence of strong rigidity of such axiomatic approach to Sobolev spaces. In other words, for such compatibility to hold we should expect the cotangent modules $L^p(T^*\mathrm{X}; D), L^p(T^*\mathrm{X}; D')$ induced by two strongly local D-structures on X to be intimately connected. While we suspect that indeed this is the case, we will not try to address this question on this paper.

Acknowledgements. We would like to thank Vladimir Gol'dshtein and Marc Troyanov for their helpful comments on this paper. We also thank the anonymous referee for her/his many useful suggestions. This research has been supported by the MIUR SIR-grant 'Nonsmooth Differential Geometry' (RBSI147UG4).

1 General notation

For the purpose of the present paper, a *metric measure space* is a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, where

$$\begin{aligned} &(\mathrm{X}, \mathsf{d}) && \text{is a complete and separable metric space,} \\ &\mathfrak{m} \neq 0 && \text{is a non-negative Borel measure on X, finite on balls.} \end{aligned} \tag{1.1}$$

Fix $p \in [1, \infty)$. Several functional spaces over X will be used in the forthcoming discussion:

- $L^0(\mathfrak{m})$: the Borel functions $u : \mathrm{X} \to \mathbb{R}$, considered up to $\mathfrak{m}$-a.e. equality.
- $L^p(\mathfrak{m})$: the functions $u \in L^0(\mathfrak{m})$ for which $|u|^p$ is integrable.
- $L^p_{loc}(\mathfrak{m})$: the functions $u \in L^0(\mathfrak{m})$ with $u_{|_B} \in L^p(\mathfrak{m}_{|_B})$ for any $B \subseteq \mathrm{X}$ bounded Borel.
- $L^\infty(\mathfrak{m})$: the functions $u \in L^0(\mathfrak{m})$ that are essentially bounded.
- $L^0(\mathfrak{m})^+$: the Borel functions $u : \mathrm{X} \to [0, +\infty]$, considered up to $\mathfrak{m}$-a.e. equality.
- $L^p(\mathfrak{m})^+$: the functions $u \in L^0(\mathfrak{m})^+$ for which $|u|^p$ is integrable.
- $L^p_{loc}(\mathfrak{m})^+$: the functions $u \in L^0(\mathfrak{m})^+$ with $u_{|_B} \in L^p(\mathfrak{m}_{|_B})^+$ for any $B \subseteq \mathrm{X}$ bounded Borel.
- $\mathrm{LIP}(\mathrm{X})$: the Lipschitz functions $u : \mathrm{X} \to \mathbb{R}$, with Lipschitz constant denoted by $\mathrm{Lip}(u)$.
- $\mathsf{Sf}(\mathrm{X})$: the functions $u \in L^0(\mathfrak{m})$ that are simple, i.e. with a finite essential image.

Observe that for any $u \in L^p_{loc}(\mathfrak{m})^+$ it holds that $u(x) < +\infty$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. We also recall that the space $\mathsf{Sf}(\mathrm{X})$ is strongly dense in $L^p(\mathfrak{m})$ for every $p \in [1, \infty]$.

Remark 1.1 In [5, Section 1.1] a more general notion of $L^p_{loc}(\mathfrak{m})$ is considered, based upon the concept of $\mathcal{K}$*-set*. We chose the present approach for simplicity, but the following discussion would remain unaltered if we replaced our definition of $L^p_{loc}(\mathfrak{m})$ with the one of [5]. ■

2 Axiomatic theory of Sobolev spaces

We begin by briefly recalling the axiomatic notion of Sobolev space that has been introduced by V. Gol'dshtein and M. Troyanov in [5, Section 1.2]:

Definition 2.1 (D-structure) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p\in[1,\infty)$ be fixed. Then a D-*structure *on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is any map D associating to each function $u\in L^p_{loc}(\mathfrak{m})$ a family $D[u]\subseteq L^0(\mathfrak{m})^+$ of* pseudo-gradients *of u, which satisfies the following axioms:*

- **A1 (Non triviality)** *It holds that $\mathrm{Lip}(u)\,\chi_{\{u>0\}}\in D[u]$ for every $u\in L^p_{loc}(\mathfrak{m})^+\cap\mathrm{LIP}(\mathrm{X})$.*
- **A2 (Upper linearity)** *Let $u_1,u_2\in L^p_{loc}(\mathfrak{m})$ be fixed. Consider $g_1\in D[u_1]$ and $g_2\in D[u_2]$. Suppose that the inequality $g\geq|\alpha_1|\,g_1+|\alpha_2|\,g_2$ holds $\mathfrak{m}$-a.e. in X for some $g\in L^0(\mathfrak{m})^+$ and $\alpha_1,\alpha_2\in\mathbb{R}$. Then $g\in D[\alpha_1\,u_1+\alpha_2\,u_2]$.*
- **A3 (Leibniz rule)** *Fix a function $u\in L^p_{loc}(\mathfrak{m})$ and a pseudo-gradient $g\in D[u]$ of u. Then for every $\varphi\in\mathrm{LIP}(\mathrm{X})$ bounded it holds that $g\,\sup_{\mathrm{X}}|\varphi|+\mathrm{Lip}(\varphi)\,|u|\in D[\varphi\,u]$.*
- **A4 (Lattice property)** *Fix $u_1,u_2\in L^p_{loc}(\mathfrak{m})$. Given any $g_1\in D[u_1]$ and $g_2\in D[u_2]$, one has that $\max\{g_1,g_2\}\in D\big[\max\{u_1,u_2\}\big]\cap D\big[\min\{u_1,u_2\}\big]$.*
- **A5 (Completeness)** *Consider two sequences $(u_n)_n\subseteq L^p_{loc}(\mathfrak{m})$ and $(g_n)_n\subseteq L^p(\mathfrak{m})$ that satisfy $g_n\in D[u_n]$ for every $n\in\mathbb{N}$. Suppose that there exist $u\in L^p_{loc}(\mathfrak{m})$ and $g\in L^p(\mathfrak{m})$ such that $u_n\to u$ in $L^p_{loc}(\mathfrak{m})$ and $g_n\to g$ in $L^p(\mathfrak{m})$. Then $g\in D[u]$.*

Remark 2.2 It follows from axioms **A1** and **A2** that $0\in D[c]$ for every constant map $c\in\mathbb{R}$. Moreover, axiom **A2** grants that the set $D[u]\cap L^p(\mathfrak{m})$ is convex and that $D[\alpha\,u]=|\alpha|\,D[u]$ for every $u\in L^p_{loc}(\mathfrak{m})$ and $\alpha\in\mathbb{R}\setminus\{0\}$, while axiom **A5** implies that each set $D[u]\cap L^p(\mathfrak{m})$ is closed in the space $L^p(\mathfrak{m})$. ■

Given any Borel set $B\subseteq\mathrm{X}$, we define the *p-Dirichlet energy* of a map $u\in L^p(\mathfrak{m})$ on B as

$$\mathcal{E}_p(u|B):=\inf\left\{\int_B g^p\,\mathrm{d}\mathfrak{m}\;\middle|\;g\in D[u]\right\}\in[0,+\infty]. \tag{2.1}$$

For the sake of brevity, we shall use the notation $\mathcal{E}_p(u)$ to indicate $\mathcal{E}_p(u|\mathrm{X})$.

Definition 2.3 (Sobolev space) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p\in[1,\infty)$ be fixed. Given any D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we define the* homogeneous Sobolev space *associated to D as*

$$\mathcal{L}^{1,p}(\mathrm{X})=\mathcal{L}^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D):=\big\{u\in L^p_{loc}(\mathfrak{m})\;:\;\mathcal{E}_p(u)<+\infty\big\}. \tag{2.2}$$

Moreover, the Sobolev space *associated to D is defined as*

$$W^{1,p}(\mathrm{X})=W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D):=L^p(\mathfrak{m})\cap\mathcal{L}^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D). \tag{2.3}$$

Theorem 2.4 *The space $W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$ is a Banach space if endowed with the norm*

$$\|u\|_{W^{1,p}(\mathrm{X})} := \Big(\|u\|^p_{L^p(\mathfrak{m})} + \mathcal{E}_p(u)\Big)^{1/p} \qquad \text{for every } u \in W^{1,p}(\mathrm{X}). \tag{2.4}$$

For a proof of the previous result, we refer to [5, Theorem 1.5].

Proposition 2.5 (Minimal pseudo-gradient) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $p \in (1,\infty)$. Consider any D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Let $u \in \mathcal{L}^{1,p}(\mathrm{X})$ be given. Then there exists a unique element $\underline{D}u \in D[u]$, which is called the* minimal pseudo-gradient *of u, such that $\mathcal{E}_p(u) = \|\underline{D}u\|^p_{L^p(\mathfrak{m})}$.*

Both existence and uniqueness of the minimal pseudo-gradient follow from the fact that the set $D[u]\cap L^p(\mathfrak{m})$ is convex and closed by Remark 2.2 and that the space $L^p(\mathfrak{m})$ is uniformly convex; see [5, Proposition 1.22] for the details.

In order to associate a differential structure to an axiomatic Sobolev space, we need to be sure that the pseudo-gradients of a function depend only on the local behaviour of the function itself, in a suitable sense. For this reason, we propose various notions of locality:

Definition 2.6 (Locality) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix $p \in (1,\infty)$. Then we define five notions of locality for D-structures on $(\mathrm{X},\mathsf{d},\mathfrak{m})$:*

L1 *If $B \subseteq \mathrm{X}$ is Borel and $u \in \mathcal{L}^{1,p}(\mathrm{X})$ is $\mathfrak{m}$-a.e. constant in B, then $\mathcal{E}_p(u|B) = 0$.*

L2 *If $B \subseteq \mathrm{X}$ is Borel and $u \in \mathcal{L}^{1,p}(\mathrm{X})$ is $\mathfrak{m}$-a.e. constant in B, then $\underline{D}u = 0$ $\mathfrak{m}$-a.e. in B.*

L3 *If $u \in \mathcal{L}^{1,p}(\mathrm{X})$ and $g \in D[u]$, then $\chi_{\{u>0\}}\, g \in D[u^+]$.*

L4 *If $u \in \mathcal{L}^{1,p}(\mathrm{X})$ and $g_1, g_2 \in D[u]$, then $\min\{g_1, g_2\} \in D[u]$.*

L5 *If $u \in \mathcal{L}^{1,p}(\mathrm{X})$ then $\underline{D}u \leq g$ holds $\mathfrak{m}$-a.e. in X for every $g \in D[u]$.*

Remark 2.7 In the language of [5, Definition 1.11], the properties **L1** and **L3** correspond to *locality* and *strict locality*, respectively. ■

We now discuss the relations among the several notions of locality:

Proposition 2.8 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p \in (1,\infty)$. Fix a D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Then the following implications hold:*

$$\begin{aligned} \mathbf{L3} \quad &\Longrightarrow \quad \mathbf{L2} \quad \Longrightarrow \quad \mathbf{L1}, \\ \mathbf{L4} \quad &\Longleftrightarrow \quad \mathbf{L5} \\ \mathbf{L1} + \mathbf{L5} \quad &\Longrightarrow \quad \mathbf{L2} + \mathbf{L3}. \end{aligned} \tag{2.5}$$

Proof.
L2 $\Longrightarrow$ **L1**. Simply notice that $\mathcal{E}_p(u|B) \leq \int_B (\underline{D}u)^p \,\mathrm{d}\mathfrak{m} = 0$.

L3 $\Longrightarrow$ L2. Take a constant $c \in \mathbb{R}$ such that the equality $u = c$ holds $\mathfrak{m}$-a.e. in B. Given that $\underline{D}u \in D[u-c] \cap D[c-u]$ by axiom **A2** and Remark 2.2, we deduce from **L3** that

$$\begin{aligned}\chi_{\{u>c\}}\,\underline{D}u &\in D\big[(u-c)^+\big],\\ \chi_{\{u<c\}}\,\underline{D}u &\in D\big[(c-u)^+\big].\end{aligned}$$

Given that $u - c = (u-c)^+ - (c-u)^+$, by applying again axiom **A2** we see that

$$\chi_{\{u\neq c\}}\,\underline{D}u = \chi_{\{u>c\}}\,\underline{D}u + \chi_{\{u<c\}}\,\underline{D}u \in D[u-c] = D[u].$$

Hence the minimality of $\underline{D}u$ grants that

$$\int_{\mathrm{X}} (\underline{D}u)^p \,\mathrm{d}\mathfrak{m} \leq \int_{\{u\neq c\}} (\underline{D}u)^p \,\mathrm{d}\mathfrak{m},$$

which implies that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $\{u = c\}$, thus also $\mathfrak{m}$-a.e. in B. This means that the D-structure satisfies the property **L2**, as required.
L4 $\Longrightarrow$ L5. We argue by contradiction: suppose the existence of $u \in \mathcal{L}^{1,p}(\mathrm{X})$ and $g \in D[u]$ such that $\mathfrak{m}\big(\{\underline{D}u > g\}\big) > 0$, whence $h := \min\{\underline{D}u, g\} \in L^p(\mathfrak{m})$ satisfies $\int h^p \,\mathrm{d}\mathfrak{m} < \int (\underline{D}u)^p \,\mathrm{d}\mathfrak{m}$. Since $h \in D[u]$ by **L4**, we deduce that $\mathcal{E}_p(u) < \int (\underline{D}u)^p \,\mathrm{d}\mathfrak{m}$, getting a contradiction.
L5 $\Longrightarrow$ L4. Since $\underline{D}u \leq g_1$ and $\underline{D}u \leq g_2$ hold $\mathfrak{m}$-a.e., we see that $\underline{D}u \leq \min\{g_1, g_2\}$ holds $\mathfrak{m}$-a.e. as well. Therefore $\min\{g_1, g_2\} \in D[u]$ by **A2**.
L1+L5 $\Longrightarrow$ L2+L3. Property **L1** grants the existence of $(g_n)_n \subseteq D[u]$ with $\int_B (g_n)^p \,\mathrm{d}\mathfrak{m} \to 0$. Hence **L5** tells us that $\int_B (\underline{D}u)^p \,\mathrm{d}\mathfrak{m} \leq \lim_n \int_B (g_n)^p \,\mathrm{d}\mathfrak{m} = 0$, which implies that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in B, yielding **L2**. We now prove the validity of **L3**: it holds that $D[u] \subseteq D[u^+]$, because we know that $h = \max\{h, 0\} \in D\big[\max\{u, 0\}\big] = D[u^+]$ for every $h \in D[u]$ by **A4** and $0 \in D[0]$, in particular $u^+ \in \mathcal{L}^{1,p}(\mathrm{X})$. Given that $u^+ = 0$ $\mathfrak{m}$-a.e. in the set $\{u \leq 0\}$, one has that $\underline{D}u^+ = 0$ holds $\mathfrak{m}$-a.e. in $\{u \leq 0\}$ by **L2**. Hence for any $g \in D[u]$ we have $\underline{D}u^+ \leq \chi_{\{u>0\}}\, g$ by **L5**, which implies that $\chi_{\{u>0\}}\, g \in D[u^+]$ by **A2**. Therefore **L3** is proved. □

Definition 2.9 (Strong locality) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$*. Then a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be* strongly local *provided it satisfies* **L1** *and* **L5** *(thus also* **L2**, **L3** *and* **L4** *by Proposition 2.8).*

We now recall other two notions of locality for D-structures that appeared in the literature:

Definition 2.10 (Two alternative notions of strong locality) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$*. Consider a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Then we give the following definitions:*

i) *We say that* D *is* strongly local in the sense of Timoshin *provided*

$$\chi_{\{u_1<u_2\}}\, g_1 + \chi_{\{u_2<u_1\}}\, g_2 + \chi_{\{u_1=u_2\}}\, (g_1 \wedge g_2) \in D[u_1 \wedge u_2] \tag{2.6}$$

whenever $u_1, u_2 \in \mathcal{L}^{1,p}(\mathrm{X})$*,* $g_1 \in D[u_1]$ *and* $g_2 \in D[u_2]$*.*

ii) *We say that D is* strongly local in the sense of Shanmugalingam *provided*

$$\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u_2] \quad \textit{for every } g_1 \in D[u_1] \textit{ and } g_2 \in D[u_2] \tag{2.7}$$

whenever $u_1, u_2 \in \mathcal{L}^{1,p}(\mathrm{X})$ *satisfy* $u_1 = u_2$ $\mathfrak{m}$*-a.e. on some Borel set* $B \subseteq \mathrm{X}$.

The above two notions of strong locality have been proposed in [11] and [10], respectively. We now prove that they are actually both equivalent to our strong locality property:

Lemma 2.11 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$. *Fix any D-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. *Then the following are equivalent:*

i) *D is strongly local (in our sense).*

ii) *D is strongly local in the sense of Shanmugalingam.*

iii) *D is strongly local in the sense of Timoshin.*

Proof.
i) $\Longrightarrow$ ii) Fix $u_1, u_2 \in \mathcal{L}^{1,p}(\mathrm{X})$ such that $u_1 = u_2$ $\mathfrak{m}$-a.e. on some $E \subseteq \mathrm{X}$ Borel. Pick $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Observe that $\underline{D}(u_2 - u_1) + g_1 \in D\big[(u_2 - u_1) + u_1\big] = D[u_2]$ by **A2**, so that we have $\big(\underline{D}(u_2 - u_1) + g_1\big) \wedge g_2 \in D[u_2]$ by **L4**. Since $\underline{D}(u_2 - u_1) = 0$ $\mathfrak{m}$-a.e. on B by **L2**, we see that $\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \geq \big(\underline{D}(u_2 - u_1) + g_1\big) \wedge g_2$ holds $\mathfrak{m}$-a.e. in X, whence accordingly we conclude that $\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u_2]$ by **A2**. This shows the validity of ii).
ii) $\Longrightarrow$ i) First of all, let us prove **L1**. Let $u \in \mathcal{L}^{1,p}(\mathrm{X})$ and $c \in \mathbb{R}$ satisfy $u = c$ $\mathfrak{m}$-a.e. on some Borel set $B \subseteq \mathrm{X}$. Given any $g \in D[u]$, we deduce from ii) that $\chi_{\mathrm{X}\setminus B}\, g \in D[u]$, thus accordingly $\mathcal{E}_p(u|B) \leq \int_B (\chi_{\mathrm{X}\setminus B}\, g)^p \,\mathrm{d}\mathfrak{m} = 0$. This proves the property **L1**.

To show property **L4**, fix $u \in \mathcal{L}^{1,p}(\mathrm{X})$ and $g_1, g_2 \in D[u]$. Let us denote $B := \{g_1 \leq g_2\}$. Therefore ii) grants that $g_1 \wedge g_2 = \chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u]$, thus obtaining **L4**. By recalling Proposition 2.8, we conclude that D is strongly local.
i)+ii) $\Longrightarrow$ iii) Fix $u_1, u_2 \in \mathcal{L}^{1,p}(\mathrm{X})$, $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Recall that $g_1 \vee g_2 \in D[u_1 \wedge u_2]$ by axiom **A4**. Hence by using property ii) twice we obtain that

$$\begin{aligned} &\chi_{\{u_1 \leq u_2\}}\, g_1 + \chi_{\{u_1 > u_2\}}\, (g_1 \vee g_2) \in D[u_1 \wedge u_2], \\ &\chi_{\{u_2 \leq u_1\}}\, g_2 + \chi_{\{u_2 > u_1\}}\, (g_1 \vee g_2) \in D[u_1 \wedge u_2]. \end{aligned} \tag{2.8}$$

The pointwise minimum between the two functions that are written in (2.8) – namely given by $\chi_{\{u_1 < u_2\}}\, g_1 + \chi_{\{u_2 < u_1\}}\, g_2 + \chi_{\{u_1 = u_2\}}\, (g_1 \wedge g_2)$ – belongs to the class $D[u_1 \wedge u_2]$ as well by property **L4**, thus showing iii).
iii) $\Longrightarrow$ i) First of all, let us prove **L1**. Fix a function $u \in \mathcal{L}^{1,p}(\mathrm{X})$ that is $\mathfrak{m}$-a.e. equal to some constant $c \in \mathbb{R}$ on a Borel set $B \subseteq \mathrm{X}$. By using iii) and the fact that $0 \in D[0]$, we have that

$$\begin{aligned} &\chi_{\{u < c\}}\, g \in D\big[(u - c) \wedge 0\big] = D\big[-(u - c)^+\big] = D\big[(u - c)^+\big], \\ &\chi_{\{u > c\}}\, g \in D\big[(c - u) \wedge 0\big] = D\big[-(c - u)^+\big] = D\big[(c - u)^+\big]. \end{aligned} \tag{2.9}$$

Since $u - c = (u-c)^+ - (c-u)^+$, we know from **A2** and (2.9) that

$$\chi_{\{u\neq c\}}\, g = \chi_{\{u<c\}}\, g + \chi_{\{u>c\}}\, g \in D[u-c] = D[u],$$

whence $\mathcal{E}_p(u|B) \leq \int_B (\chi_{\{u\neq c\}}\, g)^p \, \mathrm{d}\mathfrak{m} = 0$. This proves the property **L1**.

To show property **L4**, fix $u \in \mathcal{L}^{1,p}(\mathrm{X})$ and $g_1, g_2 \in D[u]$. Hence (2.6) with $u_1 = u_2 := u$ simply reads as $g_1 \wedge g_2 \in D[u]$, which gives **L4**. This proves that D is strongly local. □

Remark 2.12 (L1 does not imply L2) In general, as we are going to show in the following example, it can happen that a D-structure satisfies **L1** but not **L2**.

Let $G = (V, E)$ be a locally finite connected graph. The distance $\mathsf{d}(x, y)$ between two vertices $x, y \in V$ is defined as the minimum length of a path joining x to y, while as a reference measure $\mathfrak{m}$ on V we choose the counting measure. Notice that any function $u : V \to \mathbb{R}$ is locally Lipschitz and that any bounded subset of V is finite. We define a D-structure on the metric measure space $(V, \mathsf{d}, \mathfrak{m})$ in the following way:

$$D[u] := \Big\{ g : V \to [0, +\infty] \;\Big|\; \big|u(x) - u(y)\big| \leq g(x) + g(y) \text{ for any } x, y \in V \text{ with } x \sim y \Big\} \tag{2.10}$$

for every $u : V \to \mathbb{R}$, where the notation $x \sim y$ indicates that x and y are adjacent vertices, i.e. that there exists an edge in E joining x to y.

We claim that D fulfills **L1**. To prove it, suppose that some function $u : \mathrm{X} \to \mathbb{R}$ is constant on some set $B \subseteq V$, say $u(x) = c$ for every $x \in B$. Define the function $g : V \to [0, +\infty)$ as

$$g(x) := \begin{cases} 0 & \text{if } x \in B, \\ |c| + \big|u(x)\big| & \text{if } x \in V \setminus B. \end{cases}$$

Hence $g \in D[u]$ and $\int_B g^p \, \mathrm{d}\mathfrak{m} = 0$, so that $\mathcal{E}_p(u|B) = 0$. This proves the validity of **L1**.

On the other hand, if V contains more than one vertex, then **L2** is not satisfied. Indeed, consider any non-constant function $u : V \to \mathbb{R}$. Clearly any pseudo-gradient $g \in D[u]$ of u is not identically zero, thus there exists $x \in V$ such that $\underline{D}u(x) > 0$. Since u is trivially constant on the set $\{x\}$, we then conclude that property **L2** does not hold. ■

Hereafter, we shall focus our attention on the strongly local D-structures. Under these locality assumptions, one can show the following calculus rules for minimal pseudo-gradients, whose proof is suitably adapted from analogous results that have been proved in [1].

Proposition 2.13 (Calculus rules for $\underline{D}u$) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $p \in (1, \infty)$. *Consider a strongly local* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. *Then the following hold:*

i) *Let* $u \in \mathcal{L}^{1,p}(\mathrm{X})$ *and let* $N \subseteq \mathbb{R}$ *be a Borel set with* $\mathcal{L}^1(N) = 0$. *Then the equality* $\underline{D}u = 0$ *holds* $\mathfrak{m}$*-a.e. in* $u^{-1}(N)$.

ii) Chain rule. *Let* $u \in \mathcal{L}^{1,p}(\mathrm{X})$ *and* $\varphi \in \mathrm{LIP}(\mathbb{R})$. *Then* $|\varphi'| \circ u\, \underline{D}u \in D[\varphi \circ u]$. *More precisely,* $\varphi \circ u \in \mathcal{L}^{1,p}(\mathrm{X})$ *and* $\underline{D}(\varphi \circ u) = |\varphi'| \circ u\, \underline{D}u$ *holds* $\mathfrak{m}$*-a.e. in* X.

iii) Leibniz rule. *Let $u, v \in \mathcal{L}^{1,p}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Then $|u|\, \underline{D}v + |v|\, \underline{D}u \in D[uv]$. In other words, $uv \in \mathcal{L}^{1,p}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and $\underline{D}(uv) \leq |u|\, \underline{D}v + |v|\, \underline{D}u$ holds $\mathfrak{m}$-a.e. in* X.

Proof.
Step 1. First, consider φ affine, say $\varphi(t) = \alpha\, t + \beta$. Then $|\varphi'| \circ u\, \underline{D}u = |\alpha|\, \underline{D}u \in D[\varphi \circ u]$ by Remark 2.2 and **A2**. Now suppose that the function φ is piecewise affine, i.e. there exists a sequence $(a_k)_{k \in \mathbb{Z}} \subseteq \mathbb{R}$, with $a_k < a_{k+1}$ for all $k \in \mathbb{Z}$ and $a_0 = 0$, such that each $\varphi|_{[a_k, a_{k+1}]}$ is an affine function. Let us denote $A_k := u^{-1}\big([a_k, a_{k+1})\big)$ and $u_k := (u \vee a_k) \wedge a_{k+1}$ for every index $k \in \mathbb{Z}$. By combining **L3** with the axioms **A2** and **A5**, we can see that $\chi_{A_k}\, \underline{D}u \in D[u_k]$ for every $k \in \mathbb{Z}$. Called $\varphi_k : \mathbb{R} \to \mathbb{R}$ that affine function coinciding with φ on $[a_k, a_{k+1})$, we deduce from the previous case that $|\varphi_k'| \circ u_k\, \underline{D}u_k \in D[\varphi_k \circ u_k] = D[\varphi \circ u_k]$, whence we have that $|\varphi'| \circ u_k\, \chi_{A_k}\, \underline{D}u \in D[\varphi \circ u_k]$ by **L5**, **A2** and **L2**. Let us define $(v_n)_n \subseteq \mathcal{L}^{1,p}(\mathrm{X})$ as

$$v_n := \varphi(0) + \sum_{k=0}^{n} \big(\varphi \circ u_k - \varphi(a_k)\big) + \sum_{k=-n}^{-1} \big(\varphi \circ u_k - \varphi(a_{k+1})\big) \qquad \text{for every } n \in \mathbb{N}.$$

Hence $g_n := \sum_{k=-n}^{n} |\varphi'| \circ u_k\, \chi_{A_k}\, \underline{D}u \in D[v_n]$ for all $n \in \mathbb{N}$ by **A2** and Remark 2.2. Given that one has $v_n \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$ and $g_n \to |\varphi'| \circ u\, \underline{D}u$ in $L^p(\mathfrak{m})$ as $n \to \infty$, we finally conclude that $|\varphi'| \circ u\, \underline{D}u \in D[\varphi \circ u]$, as required.
Step 2. We aim to prove the chain rule for $\varphi \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R})$. For any $n \in \mathbb{N}$, let us denote by φ_n the piecewise affine function interpolating the points $\big(k/2^n, \varphi(k/2^n)\big)$ with $k \in \mathbb{Z}$. We call $D \subseteq \mathbb{R}$ the countable set $\big\{k/2^n : k \in \mathbb{Z},\, n \in \mathbb{N}\big\}$. Therefore φ_n uniformly converges to φ and $\varphi_n'(t) \to \varphi'(t)$ for all $t \in \mathbb{R} \setminus D$. In particular, the functions $g_n := |\varphi_n'| \circ u\, \underline{D}u$ converge $\mathfrak{m}$-a.e. to $|\varphi'| \circ u\, \underline{D}u$ by **L2**. Moreover, $\mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\varphi)$ for every $n \in \mathbb{N}$ by construction, so that $(g_n)_n$ is a bounded sequence in $L^p(\mathfrak{m})$. This implies that (up to a not relabeled subsequence) $g_n \rightharpoonup |\varphi'| \circ u\, \underline{D}u$ weakly in $L^p(\mathfrak{m})$. Now apply Mazur lemma: for any $n \in \mathbb{N}$, there exists $(\alpha_i^n)_{i=n}^{N_n} \subseteq [0,1]$ such that $\sum_{i=n}^{N_n} \alpha_i^n = 1$ and $h_n := \sum_{i=n}^{N_n} \alpha_i^n\, g_i \overset{n}{\to} |\varphi'| \circ u\, \underline{D}u$ strongly in $L^p(\mathfrak{m})$. Given that $g_n \in D[\varphi_n \circ u]$ for every $n \in \mathbb{N}$ by Step 1, we deduce from axiom **A2** that $h_n \in D[\psi_n \circ u]$ for every $n \in \mathbb{N}$, where $\psi_n := \sum_{i=n}^{N_n} \alpha_i^n\, \varphi_i$. Finally, it clearly holds that $\psi_n \circ u \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$, whence $|\varphi'| \circ u\, \underline{D}u \in D[\varphi \circ u]$ by **A5**.
Step 3. We claim that

$$\underline{D}u = 0 \quad \mathfrak{m}\text{-a.e. in } u^{-1}(K), \qquad \text{for every } K \subseteq \mathbb{R} \text{ compact with } \mathcal{L}^1(K) = 0. \tag{2.11}$$

For any $n \in \mathbb{N} \setminus \{0\}$, define $\psi_n := n\, \mathsf{d}(\cdot, K) \wedge 1$ and denote by φ_n the primitive of ψ_n such that $\varphi_n(0) = 0$. Since each ψ_n is continuous and bounded, any function φ_n is of class C^1 and Lipschitz. By applying the dominated convergence theorem we see that the $\mathcal{L}^1$-measure of the ε-neighbourhood of K converges to 0 as $\varepsilon \searrow 0$, thus accordingly φ_n uniformly converges to $\mathrm{id}_{\mathbb{R}}$ as $n \to \infty$. This implies that $\varphi_n \circ u \to u$ in $L^p_{loc}(\mathfrak{m})$. Moreover, we know from Step 2 that $|\psi_n| \circ u\, \underline{D}u \in D[\varphi_n \circ u]$, thus also $\chi_{\mathrm{X} \setminus u^{-1}(K)}\, \underline{D}u \in D[\varphi_n \circ u]$. Hence $\chi_{\mathrm{X} \setminus u^{-1}(K)}\, \underline{D}u \in D[u]$ by **A5**, which forces the equality $\underline{D}u = 0$ to hold $\mathfrak{m}$-a.e. in $u^{-1}(K)$, proving (2.11).
Step 4. We are in a position to prove i). Choose any $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ and call $\mu := u_* \mathfrak{m}'$. Then μ is a Radon measure on $\mathbb{R}$, in particular it is inner regular. We can thus

find an increasing sequence of compact sets $K_n \subseteq N$ such that $\mu\big(N \setminus \bigcup_n K_n\big) = 0$. We already know from Step 3 that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $\bigcup_n u^{-1}(K_n)$. Since $u^{-1}(N) \setminus \bigcup_n u^{-1}(K_n)$ is $\mathfrak{m}$-negligible by definition of μ, we conclude that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $u^{-1}(N)$. This shows the validity of property i).

Step 5. We now prove ii). Let us fix $\varphi \in \mathrm{LIP}(\mathbb{R})$. Choose some convolution kernels $(\rho_n)_n$ and define $\varphi_n := \varphi * \rho_n$ for all $n \in \mathbb{N}$. Then $\varphi_n \to \varphi$ uniformly and $\varphi_n' \to \varphi'$ pointwise $\mathcal{L}^1$-a.e., whence accordingly $\varphi_n \circ u \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$ and $|\varphi_n'| \circ u \, \underline{D}u \to |\varphi'| \circ u \, \underline{D}u$ pointwise $\mathfrak{m}$-a.e. in X. Since $|\varphi_n'| \circ u \, \underline{D}u \leq \mathrm{Lip}(\varphi) \, \underline{D}u$ for all $n \in \mathbb{N}$, there exists a (not relabeled) subsequence such that $|\varphi_n'| \circ u \, \underline{D}u \rightharpoonup |\varphi'| \circ u \, \underline{D}u$ weakly in $L^p(\mathfrak{m})$. We know that $|\varphi_n'| \circ u \, \underline{D}u \in D[\varphi_n \circ u]$ for all $n \in \mathbb{N}$ because the chain rule holds for all $\varphi_n \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R})$, hence by combining Mazur lemma and **A5** as in Step 2 we obtain that $|\varphi'| \circ u \, \underline{D}u \in D[\varphi \circ u]$, so that $\varphi \circ u \in \mathcal{L}^{1,p}(\mathrm{X})$ and the inequality $\underline{D}(\varphi \circ u) \leq |\varphi'| \circ u \, \underline{D}u$ holds $\mathfrak{m}$-a.e. in X.

Step 6. We conclude the proof of ii) by showing that one actually has $\underline{D}(\varphi \circ u) = |\varphi'| \circ u \, \underline{D}u$. We can suppose without loss of generality that $\mathrm{Lip}(\varphi) = 1$. Let us define the functions $\psi_\pm$ as $\psi_\pm(t) := \pm t - \varphi(t)$ for all $t \in \mathbb{R}$. Then it holds $\mathfrak{m}$-a.e. in $u^{-1}\big(\{\pm\varphi' \geq 0\}\big)$ that

$$\underline{D}u = \underline{D}(\pm u) \leq \underline{D}(\varphi \circ u) + \underline{D}(\psi_\pm \circ u) \leq \big(|\varphi'| \circ u + |\psi_\pm'| \circ u\big) \, \underline{D}u = \underline{D}u,$$

which forces the equality $\underline{D}(\varphi \circ u) = \pm\varphi' \circ u \, \underline{D}u$ to hold $\mathfrak{m}$-a.e. in the set $u^{-1}\big(\{\pm\varphi' \geq 0\}\big)$. This grants the validity of $\underline{D}(\varphi \circ u) = |\varphi'| \circ u \, \underline{D}u$, thus completing the proof of item ii).

Step 7. We show iii) for the case in which $u, v \geq c$ is satisfied $\mathfrak{m}$-a.e. in X, for some $c > 0$. Call $\varepsilon := \min\{c, c^2\}$ and note that the function log is Lipschitz on the interval $[\varepsilon, +\infty)$, then choose any Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ that coincides with log on $[\varepsilon, +\infty)$. Now call C the constant $\log\big(\|uv\|_{L^\infty(\mathfrak{m})}\big)$ and choose a Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \exp$ on the interval $[\log \varepsilon, C]$. By applying twice the chain rule ii), we thus deduce that $uv \in \mathcal{L}^{1,p}(\mathrm{X})$ and the $\mathfrak{m}$-a.e. inequalities

$$\begin{aligned} \underline{D}(uv) &\leq |\psi'| \circ \varphi \circ (uv) \, \underline{D}\big(\varphi \circ (uv)\big) \leq |uv| \, \big(\underline{D} \log u + \underline{D} \log v\big) \\ &= |uv| \left(\frac{\underline{D}u}{|u|} + \frac{\underline{D}v}{|v|} \right) = |u| \, \underline{D}v + |v| \, \underline{D}u. \end{aligned}$$

Therefore the Leibniz rule iii) is verified under the additional assumption that $u, v \geq c > 0$.

Step 8. We conclude by proving item iii) for general $u, v \in \mathcal{L}^{1,p}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Given any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let us denote $I_{n,k} := \big[k/n, (k+1)/n\big)$. Call $\varphi_{n,k} : \mathbb{R} \to \mathbb{R}$ the continuous function that is the identity on $I_{n,k}$ and constant elsewhere. For any $n \in \mathbb{N}$, let us define

$$\begin{aligned} u_{n,k} &:= u - \frac{k-1}{n}, & \tilde{u}_{n,k} &:= \varphi_{n,k} \circ u - \frac{k-1}{n} & &\text{for all } k \in \mathbb{Z}, \\ v_{n,\ell} &:= v - \frac{\ell-1}{n}, & \tilde{v}_{n,\ell} &:= \varphi_{n,\ell} \circ v - \frac{\ell-1}{n} & &\text{for all } \ell \in \mathbb{Z}. \end{aligned}$$

Notice that the equalities $u_{n,k} = \tilde{u}_{n,k}$ and $v_{n,\ell} = \tilde{v}_{n,\ell}$ hold $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively. Hence $\underline{D}u_{n,k} = \underline{D}\tilde{u}_{n,k} = \underline{D}u$ and $\underline{D}v_{n,\ell} = \underline{D}\tilde{v}_{n,\ell} = \underline{D}v$ hold $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively, but we also have that

$$\underline{D}(u_{n,k} \, v_{n,\ell}) = \underline{D}(\tilde{u}_{n,k} \, \tilde{v}_{n,\ell}) \quad \text{is verified } \mathfrak{m}\text{-a.e. in } u^{-1}(I_{n,k}) \cap v^{-1}(I_{n,\ell}).$$

Moreover, we have the $\mathfrak{m}$-a.e. inequalities $1/n \leq \tilde{u}_{n,k}, \tilde{v}_{n,\ell} \leq 2/n$ by construction. Therefore for any $k, \ell \in \mathbb{Z}$ it holds $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k}) \cap v^{-1}(I_{n,\ell})$ that

$$\begin{aligned}
\underline{D}(uv) &\leq \underline{D}(\tilde{u}_{n,k}\,\tilde{v}_{n,\ell}) + \frac{|k-1|}{n}\,\underline{D}v_{n,\ell} + \frac{|\ell-1|}{n}\,\underline{D}u_{n,k} \\
&\leq |\tilde{v}_{n,\ell}|\,\underline{D}\tilde{u}_{n,k} + |\tilde{u}_{n,k}|\,\underline{D}\tilde{v}_{n,\ell} + \frac{|k-1|}{n}\,\underline{D}v_{n,\ell} + \frac{|\ell-1|}{n}\,\underline{D}u_{n,k} \\
&\leq \left(|v| + \frac{4}{n}\right)\underline{D}u + \left(|u| + \frac{4}{n}\right)\underline{D}v,
\end{aligned}$$

where the second inequality follows from the case $u, v \geq c > 0$, treated in Step 7. This implies that the inequality $\underline{D}(uv) \leq |u|\,\underline{D}v + |v|\,\underline{D}u + 4\,(\underline{D}u + \underline{D}v)/n$ holds $\mathfrak{m}$-a.e. in X. Given that $n \in \mathbb{N}$ is arbitrary, the Leibniz rule iii) follows. □

3 Cotangent module associated to a D-structure

It is shown in [3] that any metric measure space possesses a first-order differential structure, whose construction relies upon the notion of $L^p(\mathfrak{m})$-*normed* $L^\infty(\mathfrak{m})$-*module*. For completeness, we briefly recall its definition and we refer to [3,4] for a comprehensive exposition of this topic.

Definition 3.1 (Normed module) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in [1, \infty)$. *Then an* $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module *is any quadruplet* $\big(\mathscr{M}, \|\cdot\|_{\mathscr{M}}, \cdot\,, |\cdot|\big)$ *such that*

i) $\big(\mathscr{M}, \|\cdot\|_{\mathscr{M}}\big)$ *is a Banach space,*

ii) $(\mathscr{M}, \cdot)$ *is an algebraic module over the commutative ring* $L^\infty(\mathfrak{m})$,

iii) $|\cdot| : \mathscr{M} \to L^p(\mathfrak{m})^+$ *is an operator, called* pointwise norm, *which satisfies*

$$\begin{aligned}
|f\cdot v| &= |f||v| \quad \mathfrak{m}\text{-}a.e. &&\text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathscr{M}, \\
\|v\|_{\mathscr{M}} &= \big\||v|\big\|_{L^p(\mathfrak{m})} &&\text{for every } v \in \mathscr{M}.
\end{aligned} \tag{3.1}$$

A key role in [3] is played by the *cotangent module* $L^2(T^*\mathrm{X})$, which has a structure of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module; see [4, Theorem/Definition 2.8] for its characterisation. The following result shows that a generalised version of such object can be actually associated to any D-structure, provided the latter is assumed to be strongly local.

Theorem 3.2 (Cotangent module associated to a D-structure) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be any metric measure space and let* $p \in (1, \infty)$. *Consider a strongly local D-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. *Then there exists a unique couple* $\big(L^p(T^*\mathrm{X}; D), \mathrm{d}\big)$, *where* $L^p(T^*\mathrm{X}; D)$ *is an* $L^p(\mathfrak{m})$-*normed* $L^\infty(\mathfrak{m})$-*module and* $\mathrm{d} : \mathcal{L}^{1,p}(\mathrm{X}) \to L^p(T^*\mathrm{X}; D)$ *is a linear map, such that the following hold:*

i) *the equality* $|\mathrm{d}u| = \underline{D}u$ *is satisfied* $\mathfrak{m}$-*a.e. in* X *for every* $u \in \mathcal{L}^{1,p}(\mathrm{X})$,

ii) *the vector space* $\mathcal{V}$ *of all elements of the form* $\sum_{i=1}^n \chi_{B_i}\,\mathrm{d}u_i$, *where* $(B_i)_i$ *is a Borel partition of* X *and* $(u_i)_i \subseteq \mathcal{L}^{1,p}(\mathrm{X})$, *is dense in the space* $L^p(T^*\mathrm{X}; D)$.

Uniqueness has to be intended up to unique isomorphism: given another such couple $(\mathscr{M}, \mathrm{d}')$, there exists a unique isomorphism $\Phi : L^p(T^*\mathrm{X}; D) \to \mathscr{M}$ such that $\Phi(\mathrm{d}u) = \mathrm{d}'u$ for all $u \in \mathcal{L}^{1,p}(\mathrm{X})$.

The space $L^p(T^*\mathrm{X}; D)$ is called cotangent module*, while the map* d *is called* differential*.*

Proof.
UNIQUENESS. Consider any element $\omega \in \mathcal{V}$ written as $\omega = \sum_{i=1}^n \chi_{B_i} \, \mathrm{d}u_i$, with $(B_i)_i$ Borel partition of X and $u_1, \ldots, u_n \in \mathcal{L}^{1,p}(\mathrm{X})$. Notice that the requirements that Φ is $L^\infty(\mathfrak{m})$-linear and $\Phi \circ \mathrm{d} = \mathrm{d}'$ force the definition $\Phi(\omega) := \sum_{i=1}^n \chi_{B_i} \, \mathrm{d}'u_i$. The $\mathfrak{m}$-a.e. equality

$$\big|\Phi(\omega)\big| = \sum_{i=1}^{n} \chi_{B_i} \, |\mathrm{d}'u_i| = \sum_{i=1}^{n} \chi_{B_i} \, \underline{D}u_i = \sum_{i=1}^{n} \chi_{B_i} \, |\mathrm{d}u_i| = |\omega|$$

grants that $\Phi(\omega)$ is well-defined, in the sense that it does not depend on the particular way of representing ω, and that $\Phi : \mathcal{V} \to \mathscr{M}$ preserves the pointwise norm. In particular, one has that the map $\Phi : \mathcal{V} \to \mathscr{M}$ is (linear and) continuous. Since $\mathcal{V}$ is dense in $L^p(T^*\mathrm{X}; D)$, we can uniquely extend Φ to a linear and continuous map $\Phi : L^p(T^*\mathrm{X}; D) \to \mathscr{M}$, which also preserves the pointwise norm. Moreover, we deduce from the very definition of Φ that the identity $\Phi(h \, \omega) = h \, \Phi(\omega)$ holds for every $\omega \in \mathcal{V}$ and $h \in \mathsf{Sf}(\mathrm{X})$, whence the $L^\infty(\mathfrak{m})$-linearity of Φ follows by an approximation argument. Finally, the image $\Phi(\mathcal{V})$ is dense in $\mathscr{M}$, which implies that Φ is surjective. Therefore Φ is the unique isomorphism satisfying $\Phi \circ \mathrm{d} = \mathrm{d}'$.
EXISTENCE. First of all, let us define the *pre-cotangent module* as

$$\mathsf{Pcm} := \left\{ \big\{(B_i, u_i)\big\}_{i=1}^n \;\middle|\; \begin{array}{c} n \in \mathbb{N}, \; u_1, \ldots, u_n \in \mathcal{L}^{1,p}(\mathrm{X}), \\ (B_i)_{i=1}^n \text{ Borel partition of X} \end{array} \right\}.$$

We define an equivalence relation on Pcm as follows: we declare that $\big\{(B_i, u_i)\big\}_i \sim \big\{(C_j, v_j)\big\}_j$ provided $\underline{D}(u_i - v_j) = 0$ holds $\mathfrak{m}$-a.e. on $B_i \cap C_j$ for every i, j. The equivalence class of an element $\big\{(B_i, u_i)\big\}_i$ of Pcm will be denoted by $[B_i, u_i]_i$. We can endow the quotient $\mathsf{Pcm}/\sim$ with a vector space structure:

$$\begin{aligned} [B_i, u_i]_i + [C_j, v_j]_j &:= [B_i \cap C_j, u_i + v_j]_{i,j}, \\ \lambda \, [B_i, u_i]_i &:= [B_i, \lambda \, u_i]_i, \end{aligned} \tag{3.2}$$

for every $[B_i, u_i]_i, [C_j, v_j]_j \in \mathsf{Pcm}/\sim$ and $\lambda \in \mathbb{R}$. We only check that the sum operator is well-defined; the proof of the well-posedness of the multiplication by scalars follows along the same lines. Suppose that $\big\{(B_i, u_i)\big\}_i \sim \big\{(B'_k, u'_k)\big\}_k$ and $\big\{(C_j, v_j)\big\}_j \sim \big\{(C'_\ell, v'_\ell)\big\}_\ell$, in other words $\underline{D}(u_i - u'_k) = 0$ $\mathfrak{m}$-a.e. on $B_i \cap B'_k$ and $\underline{D}(v_j - v'_\ell) = 0$ $\mathfrak{m}$-a.e. on $C_j \cap C'_\ell$ for every i, j, k, ℓ, whence accordingly

$$\underline{D}\big((u_i + v_j) - (u'_k + v'_\ell)\big) \overset{\textbf{L5}}{\leq} \underline{D}(u_i - u'_k) + \underline{D}(v_j - v'_\ell) = 0 \quad \text{holds } \mathfrak{m}\text{-a.e. on } (B_i \cap C_j) \cap (B'_k \cap C'_\ell).$$

This shows that $\big\{(B_i \cap C_j, u_i + v_j)\big\}_{i,j} \sim \big\{(B'_k \cap C'_\ell, u'_k + v'_\ell)\big\}_{k,\ell}$, thus proving that the sum operator defined in (3.2) is well-posed. Now let us define

$$\big\|[B_i, u_i]_i\big\|_{L^p(T^*\mathrm{X}; D)} := \sum_{i=1}^{n} \left(\int_{B_i} (\underline{D}u_i)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} \quad \text{for every } [B_i, u_i]_i \in \mathsf{Pcm}/\sim. \tag{3.3}$$

Such definition is well-posed: if $\{(B_i,u_i)\}_i \sim \{(C_j,v_j)\}_j$ then for all i,j it holds that

$$|\underline{D}u_i - \underline{D}v_j| \overset{\textbf{L5}}{\leq} \underline{D}(u_i - v_j) = 0 \quad \mathfrak{m}\text{-a.e. on } B_i \cap C_j,$$

i.e. that the equality $\underline{D}u_i = \underline{D}v_j$ is satisfied $\mathfrak{m}$-a.e. on $B_i \cap C_j$. Therefore one has that

$$\begin{aligned}\sum_i \bigg(\int_{B_i} (\underline{D}u_i)^p \,\mathrm{d}\mathfrak{m} \bigg)^{1/p} &= \sum_{i,j} \bigg(\int_{B_i\cap C_j} (\underline{D}u_i)^p \,\mathrm{d}\mathfrak{m} \bigg)^{1/p} = \sum_{i,j} \bigg(\int_{B_i\cap C_j} (\underline{D}v_j)^p \,\mathrm{d}\mathfrak{m} \bigg)^{1/p} \\ &= \sum_j \bigg(\int_{C_j} (\underline{D}v_j)^p \,\mathrm{d}\mathfrak{m} \bigg)^{1/p},\end{aligned}$$

which grants that $\|\cdot\|_{L^p(T^*\mathrm{X};D)}$ in (3.3) is well-defined. The fact that it is a norm on $\mathsf{Pcm}/\sim$ easily follows from standard verifications. Hence let us define

$$\begin{aligned}&L^p(T^*\mathrm{X};D) := \text{completion of } \big(\mathsf{Pcm}/\sim, \|\cdot\|_{L^p(T^*\mathrm{X};D)}\big),\\ &\mathrm{d} : \mathcal{L}^{1,p}(\mathrm{X}) \to L^p(T^*\mathrm{X};D), \quad \mathrm{d}u := [\mathrm{X},u] \text{ for every } u \in \mathcal{L}^{1,p}(\mathrm{X}).\end{aligned}$$

Observe that $L^p(T^*\mathrm{X};D)$ is a Banach space and that d is a linear operator. Furthermore, given any $[B_i,u_i]_i \in \mathsf{Pcm}/\sim$ and $h = \sum_j \lambda_j \,\chi_{C_j} \in \mathsf{Sf}(\mathrm{X})$, where $(\lambda_j)_j \subseteq \mathbb{R}$ and $(C_j)_j$ is a Borel partition of X, we set

$$\begin{aligned}\big|[B_i,u_i]_i\big| &:= \sum_i \chi_{B_i}\,\underline{D}u_i,\\ h\,[B_i,u_i]_i &:= [B_i \cap C_j, \lambda_j\,u_i]_{i,j}.\end{aligned}$$

One can readily prove that such operations – which are well-posed again by the strong locality of D – can be uniquely extended to a pointwise norm $|\cdot| : L^p(T^*\mathrm{X};D) \to L^p(\mathfrak{m})^+$ and to a multiplication by L^∞-functions $L^\infty(\mathfrak{m}) \times L^p(T^*\mathrm{X};D) \to L^p(T^*\mathrm{X};D)$, respectively. Therefore the space $L^p(T^*\mathrm{X};D)$ turns out to be an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module when equipped with the operations described so far. In order to conclude, it suffices to notice that

$$|\mathrm{d}u| = \big|[\mathrm{X},u]\big| = \underline{D}u \quad \text{holds } \mathfrak{m}\text{-a.e.} \qquad \text{for every } u \in \mathcal{L}^{1,p}(\mathrm{X})$$

and that $[B_i,u_i]_i = \sum_i \chi_{B_i}\,\mathrm{d}u_i$ for all $[B_i,u_i]_i \in \mathsf{Pcm}/\sim$, giving i) and ii), respectively. □

Remark 3.3 At this level of generality, the cotangent module $L^p(T^*\mathrm{X};D)$ cannot be viewed (to the best of our knowledge) as the space of p-integrable sections of some notion of 'measurable cotangent bundle' of X. In particular, the differential $\mathrm{d}u$ of a Sobolev function $u \in \mathcal{L}^{1,p}(\mathrm{X})$ is a rather abstract object, which does not admit any sort of '$\mathfrak{m}$-a.e. representative'. ■

In full analogy with the properties of the cotangent module that is studied in [3], we can show that the differential d introduced in Theorem 3.2 is a closed operator, which satisfies both the chain rule and the Leibniz rule.

Theorem 3.4 (Closure of the differential) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and let* $p\in(1,\infty)$. *Consider a strongly local* D*-structure on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$. *Then the differential operator* d *is* closed, *i.e. if a sequence* $(u_n)_n\subseteq\mathcal{L}^{1,p}(\mathrm{X})$ *converges in* $L^p_{loc}(\mathfrak{m})$ *to some* $u\in L^p_{loc}(\mathfrak{m})$ *and* $\mathrm{d}u_n\rightharpoonup\omega$ *weakly in* $L^p(T^*\mathrm{X};D)$ *for some* $\omega\in L^p(T^*\mathrm{X};D)$, *then* $u\in\mathcal{L}^{1,p}(\mathrm{X})$ *and* $\mathrm{d}u=\omega$.

Proof. Since d is linear, we can assume with no loss of generality that $\mathrm{d}u_n\to\omega$ in $L^p(T^*\mathrm{X};D)$ by Mazur lemma, so that $\mathrm{d}(u_n-u_m)\to\omega-\mathrm{d}u_m$ in $L^p(T^*\mathrm{X};D)$ for any $m\in\mathbb{N}$. In particular, one has $u_n-u_m\to u-u_m$ in $L^p_{loc}(\mathfrak{m})$ and $\underline{D}(u_n-u_m)=\big|\mathrm{d}(u_n-u_m)\big|\to|\omega-\mathrm{d}u_m|$ in $L^p(\mathfrak{m})$ as $n\to\infty$ for all $m\in\mathbb{N}$, whence $u-u_m\in\mathcal{L}^{1,p}(\mathrm{X})$ and $\underline{D}(u-u_m)\le|\omega-\mathrm{d}u_m|$ holds $\mathfrak{m}$-a.e. for all $m\in\mathbb{N}$ by **A5** and **L5**. Therefore $u=(u-u_0)+u_0\in\mathcal{L}^{1,p}(\mathrm{X})$ and

$$\begin{aligned}\varlimsup_{m\to\infty}\|\mathrm{d}u-\mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)}&=\varlimsup_{m\to\infty}\big\|\underline{D}(u-u_m)\big\|_{L^p(\mathfrak{m})}\le\varlimsup_{m\to\infty}\|\omega-\mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)}\\&=\varlimsup_{m\to\infty}\lim_{n\to\infty}\|\mathrm{d}u_n-\mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)}=0,\end{aligned}$$

which grants that $\mathrm{d}u_m\to\mathrm{d}u$ in $L^p(T^*\mathrm{X};D)$ as $m\to\infty$ and accordingly that $\mathrm{d}u=\omega$. □

Proposition 3.5 (Calculus rules for $\mathrm{d}u$**)** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be any metric measure space and let* $p\in(1,\infty)$. *Consider a strongly local* D*-structure on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$. *Then the following hold:*

i) *Let* $u\in\mathcal{L}^{1,p}(\mathrm{X})$ *and let* $N\subseteq\mathbb{R}$ *be a Borel set with* $\mathcal{L}^1(N)=0$. *Then* $\chi_{u^{-1}(N)}\,\mathrm{d}u=0$.

ii) Chain rule. *Let* $u\in\mathcal{L}^{1,p}(\mathrm{X})$ *and* $\varphi\in\mathrm{LIP}(\mathbb{R})$ *be given. Recall that* $\varphi\circ u\in\mathcal{L}^{1,p}(\mathrm{X})$ *by Proposition 2.13. Then* $\mathrm{d}(\varphi\circ u)=\varphi'\circ u\,\mathrm{d}u$.

iii) Leibniz rule. *Let* $u,v\in\mathcal{L}^{1,p}(\mathrm{X})\cap L^\infty(\mathfrak{m})$ *be given. Recall that* $uv\in\mathcal{L}^{1,p}(\mathrm{X})\cap L^\infty(\mathfrak{m})$ *by Proposition 2.13. Then* $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$.

Proof.
i) We have that $|\mathrm{d}u|=\underline{D}u=0$ holds $\mathfrak{m}$-a.e. on $u^{-1}(N)$ by item i) of Proposition 2.13, thus accordingly $\chi_{u^{-1}(N)}\,\mathrm{d}u=0$, as required.
ii) If φ is an affine function, say $\varphi(t)=\alpha\,t+\beta$, then $\mathrm{d}(\varphi\circ u)=\mathrm{d}(\alpha\,u+\beta)=\alpha\,\mathrm{d}u=\varphi'\circ u\,\mathrm{d}u$. Now suppose that φ is a piecewise affine function. Say that $(I_n)_n$ is a sequence of intervals whose union covers the whole real line $\mathbb{R}$ and that $(\psi_n)_n$ is a sequence of affine functions such that $\varphi|_{I_n}=\psi_n$ holds for every $n\in\mathbb{N}$. Since φ' and ψ_n' coincide $\mathcal{L}^1$-a.e. in the interior of I_n, we have that $\mathrm{d}(\varphi\circ f)=\mathrm{d}(\psi_n\circ f)=\psi_n'\circ f\,\mathrm{d}f=\varphi'\circ f\,\mathrm{d}f$ holds $\mathfrak{m}$-a.e. on $f^{-1}(I_n)$ for all n, so that $\mathrm{d}(\varphi\circ u)=\varphi'\circ u\,\mathrm{d}u$ is verified $\mathfrak{m}$-a.e. on $\bigcup_n u^{-1}(I_n)=\mathrm{X}$.

To prove the case of a general Lipschitz function $\varphi:\mathbb{R}\to\mathbb{R}$, we want to approximate φ with a sequence of piecewise affine functions: for any $n\in\mathbb{N}$, let us denote by φ_n the function that coincides with φ at $\{k/2^n\,:\,k\in\mathbb{Z}\}$ and that is affine on the interval $\big[k/2^n,(k+1)/2^n\big]$ for every $k\in\mathbb{Z}$. It is clear that $\mathrm{Lip}(\varphi_n)\le\mathrm{Lip}(\varphi)$ for all $n\in\mathbb{N}$. Moreover, one can readily check that, up to a not relabeled subsequence, $\varphi_n\to\varphi$ uniformly on $\mathbb{R}$ and $\varphi_n'\to\varphi'$ pointwise $\mathcal{L}^1$-almost everywhere. The former grants that $\varphi_n\circ u\to\varphi\circ u$ in $L^p_{loc}(\mathfrak{m})$. Given that $|\varphi_n'-\varphi'|^p\circ u\,(\underline{D}u)^p\le 2^p\,\mathrm{Lip}(\varphi)^p\,(\underline{D}u)^p\in L^1(\mathfrak{m})$ for all $n\in\mathbb{N}$ and $|\varphi_n'-\varphi'|^p\circ u\,(\underline{D}u)^p\to 0$

pointwise $\mathfrak{m}$-a.e. by the latter above together with i), we obtain $\int |\varphi_n' - \varphi'|^p \circ u \, (\underline{D}u)^p \, \mathrm{d}\mathfrak{m} \to 0$ as $n \to \infty$ by the dominated convergence theorem. In other words, $\varphi_n' \circ u \, \mathrm{d}u \to \varphi' \circ u \, \mathrm{d}u$ in the strong topology of $L^p(T^*\mathrm{X}; D)$. Hence Theorem 3.4 ensures that $\mathrm{d}(\varphi \circ u) = \varphi' \circ u \, \mathrm{d}u$, thus proving the chain rule ii) for any $\varphi \in \mathrm{LIP}(\mathbb{R})$.

iii) In the case $u, v \geq 1$, we argue as in the proof of Proposition 2.13 to deduce from ii) that

$$\frac{\mathrm{d}(uv)}{uv} = \mathrm{d}\log(uv) = \mathrm{d}\big(\log(u) + \log(v)\big) = \mathrm{d}\log(u) + \mathrm{d}\log(v) = \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}v}{v},$$

whence we get $\mathrm{d}(uv) = u \, \mathrm{d}v + v \, \mathrm{d}u$ by multiplying both sides by uv.

In the general case $u, v \in L^\infty(\mathfrak{m})$, choose a constant $C > 0$ so big that $u + C, v + C \geq 1$. By the case treated above, we know that

$$\begin{aligned} \mathrm{d}\big((u+C)(v+C)\big) &= (u+C)\,\mathrm{d}(v+C) + (v+C)\,\mathrm{d}(u+C) \\ &= (u+C)\,\mathrm{d}v + (v+C)\,\mathrm{d}u \\ &= u\,\mathrm{d}v + v\,\mathrm{d}u + C\,\mathrm{d}(u+v), \end{aligned} \tag{3.4}$$

while a direct computation yields

$$\mathrm{d}\big((u+C)(v+C)\big) = \mathrm{d}\big(uv + C(u+v) + C^2\big) = \mathrm{d}(uv) + C\,\mathrm{d}(u+v). \tag{3.5}$$

By subtracting (3.5) from (3.4), we finally obtain that $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$, as required. This completes the proof of the Lebniz rule iii). □

Remark 3.6 (Locality of the differential) It also holds that

$$\chi_{\{u=v\}}\,\mathrm{d}u = \chi_{\{u=v\}}\,\mathrm{d}v \qquad \text{for every } u, v \in \mathcal{L}^{1,p}(\mathrm{X}). \tag{3.6}$$

Indeed, given that $\{u = v\} = (u - v)^{-1}\big(\{0\}\big)$ we know from item i) of Proposition 2.13 that $\chi_{\{u=v\}}\,\mathrm{d}(u - v) = 0$, whence (3.6) follows from the linearity of d. ■

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195 (2014), pp. 289–391.

[2] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), pp. 428–517.

[3] N. Gigli, *Nonsmooth differential geometry - an approach tailored for spaces with Ricci curvature bounded from below.* Mem. Amer. Math. Soc. 1196, 2014.

[4] ———, *Lecture notes on differential calculus on* RCD *spaces.* Publ. RIMS Kyoto Univ. 54, 2018.

[5] V. Gol'dshtein and M. Troyanov, *Axiomatic theory of Sobolev Spaces*, Expositiones Mathematicae, 19 (2001), pp. 289–336.

[6] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Analysis, 5 (1996), pp. 403–415.

[7] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math., 181 (1998), pp. 1–61.

[8] A. S. Jeff Cheeger, Bruce Kleiner, *Infinitesimal structure of differentiability spaces, and metric differentiation*, Analysis and Geometry in Metric Spaces, 4 (2016), pp. 104–159, electronic only.

[9] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana, 16 (2000), pp. 243–279.

[10] ———, *A Universality Property of Sobolev Spaces in Metric Measure Spaces*, Springer New York, New York, NY, 2009, pp. 345–359.

[11] S. Timoshin, *Regularity in metric spaces*, (2006). PhD thesis, École polytechnique fédérale de Lausanne, available at: https://infoscience.epfl.ch/record/85799/files/EPFL_TH3571.pdf.