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Periodic solutions of nearly integrable Hamiltonian systems bifurcating from infinite-dimensional tori

Alessandro Fonda, Giuliano Klun and Andrea Sfecci

Dedicated to Shair Ahmad, on the occasion of his 85th birthday

Abstract

We prove the existence of periodic solutions of some infinite-dimensional nearly integrable Hamiltonian systems, bifurcating from infinite-dimensional tori, by the use of a generalization of the Poincaré–Birkhoff Theorem.

1 Introduction

The aim of this paper is to provide the existence of periodic solutions bifurcating from an infinite-dimensional invariant torus for a nearly integrable Hamiltonian system.

The finite-dimensional case was treated in [1, 2, 4, 5, 6] by assuming the existence of an invariant torus made of periodic solutions all sharing the same period, under some non-degeneracy conditions. Let us briefly describe the main result in this setting. Denoting by $H(I,\varphi) = \mathcal{K}(I)$ the Hamiltonian of a completely integrable system in $\mathbb{R}^{2N}$ (as usual, we denote by φ and I the angle and the action variables, respectively), we can write the corresponding system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) \\ -\dot I = 0\,. \end{cases}$$

Assume that there is a $I^0 \in \mathbb{R}^N$ such that

$$\det \mathcal{K}''(I^0) \neq 0\,. \tag{1.1}$$

Consider now the perturbed system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t,\varphi,I) \\ -\dot I = \varepsilon\nabla_\varphi P(t,\varphi,I)\,, \end{cases}$$

where $P(\cdot,\varphi,I)$ is T-periodic, and $P(t,\cdot,I)$ is τ_k-periodic in φ_k, for every $k = 1,\dots,N$. Assume that there exist some integers $m_1,\dots,m_N$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1,\dots,m_N\tau_N)\,. \tag{1.2}$$

Then, for $|\varepsilon|$ small enough, there are at least $N+1$ solutions $(\varphi(t), I(t))$ satisfying

$$\varphi(t+T) = \varphi(t) + T\,\nabla\mathcal{K}(I^0)\,, \quad I(t+T) = I(t)\,, \quad \text{for every } t\in\mathbb{R}\,, \tag{1.3}$$

and these solutions are near to some solutions of the unperturbed problem, i.e., briefly,

$$\varphi(t) \approx \varphi(0) + t\nabla\mathcal{K}(I^0)\,, \quad I(t) \approx I^0.$$

Notice that, by (1.2) and (1.3), $\varphi_k(t+T) = \varphi_k(t) + m_k\tau_k$, for every $k = 1, \dots, N$. Since usually φ_k is interpreted as an angle, with $\tau_k = 2\pi$, we may consider these as "periodic solutions" having period T. However, in the following, it will be better to keep more freedom in the choice of the periods τ_k.

Clearly enough, being $P(\cdot, \varphi, I)$ also mT-periodic for every positive integer m, one could search "periodic solutions" having period mT, as well (the so-called "subharmonic solutions"). We refer to [6] for a complete description of the problem, and for a more general statement, obtained by the use of the Poincaré–Birkhoff theorem.

The above result was recently extended in [7] for systems of the type

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t,\varphi,I,z) \\ -\dot I = \varepsilon\nabla_\varphi P(t,\varphi,I,z) \\ J\dot z = \mathcal{A}z + \varepsilon\nabla_z P(t,\varphi,I,z)\,, \end{cases} \tag{1.4}$$

where $J = \begin{pmatrix} 0 & -I_M \\ I_M & 0 \end{pmatrix}$ denotes the standard $2M \times 2M$ symplectic matrix and $\mathcal{A}$ is a symmetric non-resonant matrix, meaning that the only T-periodic solution of the unperturbed equation $J\dot z = \mathcal{A}z$ is the constant $z = 0$. Assuming (1.1), (1.2) and that ∇P, the gradient of P with respect to (φ, I, z), is uniformly bounded, the existence of at least $N+1$ solutions $(\varphi(t), I(t), z(t))$ satisfying (1.3) and $z(t+T) = z(t)$ was proved, when $|\varepsilon|$ is small enough.

The aim of this paper is to extend the above results to an infinite-dimensional setting. Let X and Z be the separable Hilbert spaces which will replace $\mathbb{R}^N$ and $\mathbb{R}^{2M}$, respectively. So, when looking at system (1.4), the functions $\varphi(t)$ and $I(t)$ will vary in X, while $z(t)$ will belong to Z. The spaces X and Z may be infinite-dimensional, finite-dimensional, or even reduced to $\{0\}$. If X is finite-dimensional, the cases $Z = \{0\}$ and Z finite-dimensional correspond to the settings in [6] and [7], respectively. However, if X or Z are infinite-dimensional, we will be able to prove the bifurcation of *at least one* periodic orbit from an invariant torus, which can also be infinite-dimensional. The multiplicity problem remains open.

In order to obtain our existence result in infinite-dimensions, we ask all the functions to be Lipschitz continuous on bounded sets, and the perturbing term ∇P to be uniformly bounded. Moreover, we need a special structure of the autonomous Hamiltonian function: in our assumptions **(Dec1)** and **(Dec2)** below, roughly speaking, the functions involved must be decomposable in a sequence of finite-dimensional blocks. This allows us to tackle the problem by a finite-dimensional approximating process, applying a version of the Poincaré–Birkhoff theorem for the reduced systems, and carefully estimating the so-found periodic solutions in order to guarantee their convergence to a periodic solution of the infinite-dimensional system.

2 The main result

We want to treat a system of the type (1.4) in an infinite-dimensional setting. To this aim, let X and E be two separable Hilbert spaces, and set $\mathcal{X} = X^2 \times E^2$. We will use the notation $\omega = (\varphi, I, z)$ for the elements of $\mathcal{X}$, with $\varphi, I \in X$ and $z = (x, y) \in E^2$. For simplicity, we will write $Z = E^2$, and we define $J : Z \to Z$ as $J(x,y) = (-y, x)$. (The same notation J will also be used with the same meaning in similar settings.) Let us introduce all the assumptions we need.

The continuous functions $\mathcal{K} : X \to \mathbb{R}$ and $P : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ are assumed to be continuously differentiable with respect to I and ω, respectively. The function $t \mapsto P(t,\omega)$ is T-periodic, for some $T > 0$. Moreover, we assume the following Lipschitz condition on bounded sets.

(L) For every $R > 0$ there exist two positive constants $L_R, \mathcal{L}_R$ such that

$$\|\nabla\mathcal{K}(I') - \nabla\mathcal{K}(I'')\| \leq L_R\|I' - I''\| ,$$

for every $I', I'' \in X$ with $\|I'\| < R$, $\|I''\| < R$, and

$$\|\nabla_\omega P(t,\omega') - \nabla_\omega P(t,\omega'')\| \leq \mathcal{L}_R\|\omega' - \omega''\| ,$$

for every $t \in [0,T]$ and $\omega', \omega'' \in \mathcal{X}$ with $\|\omega'\| < R$ and $\|\omega''\| < R$.

Introducing some Hilbert bases of X and E, we can identify these spaces either with some $\mathbb{R}^n$, if they are finite-dimensional, or with ℓ^2, the space of real sequences $(\alpha_k)_k$ which satisfy $\sum_{k=1}^\infty \alpha_k^2 < \infty$. Each of the vectors φ, I in X and z in Z will then be written in their coordinates, e.g., $\varphi = (\varphi_1, \varphi_2, \dots)$, or $\varphi = (\varphi_k)_k$, with $\varphi_k \in \mathbb{R}$, while $I = (I_k)_k$ and $z = (z_l)_l$, with $z_l = (x_l, y_l) \in \mathbb{R}^2$. Notice that these sequences may be finite.

We also ask P to be periodic in the φ-variables, as follows.

(P$_\tau$) The function $P(t,\varphi,I,z)$ is τ_k-periodic in each φ_k, i.e., for $k = 1,2,\dots$,

$$P(t,\dots,\varphi_k+\tau_k,\dots,I,z) = P(t,\dots,\varphi_k,\dots,I,z) , \qquad \text{for every } (t,\varphi,I,z) \in [0,T] \times \mathcal{X} ;$$

moreover, if $\dim X = \infty$, then the sequence $(\tau_k)_k$ belongs to ℓ^2.

Concerning $\nabla_\omega P$, we assume it to be bounded and precompact, in the following sense.

(P$_{bd}$) There exist $(\alpha_k^\star)_k$ and $(\alpha_l^\sharp)_l$ such that, for every $k,l = 1,2,\dots$,

$$\left|\frac{\partial P}{\partial\varphi_k}(t,\omega)\right| + \left|\frac{\partial P}{\partial I_k}(t,\omega)\right| \leq \alpha_k^\star , \qquad \left|\frac{\partial P}{\partial x_l}(t,\omega)\right| + \left|\frac{\partial P}{\partial y_l}(t,\omega)\right| \leq \alpha_l^\sharp ,$$

for every $(t,\omega) \in [0,T] \times \mathcal{X}$. If $\dim X = \infty$ or $\dim Z = \infty$, then $(\alpha_k^\star)_k$ or $(\alpha_l^\sharp)_l$ belong to ℓ^2, respectively.

Notice that the sets $\prod_{k=1}^\infty [-\alpha_k^\star, \alpha_k^\star]$ and $\prod_{l=1}^\infty [-\alpha_l^\sharp, \alpha_l^\sharp]$ are Hilbert cubes, hence compact sets in ℓ^2.

Let $\mathcal{A} : Z \to Z$ be a linear *bounded selfadjoint* operator. We need the following non-resonance assumption.

(NR) Denoting by

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2([0,T],Z) \to L^2([0,T],Z) , \quad \mathcal{L}z = J\dot{z} ,$$

the unbounded selfadjoint operator with domain

$$\mathcal{D}(\mathcal{L}) = \{z \in H^1([0,T],Z) : z(0) = z(T)\} ,$$

we assume that $0 \notin \sigma(\mathcal{L} - \mathcal{A})$.

In the case when Z is infinite-dimensional, we need to assume a particular structure for the function $\mathcal{A}$.

(Dec1) If $\dim Z = \infty$, there exists a sequence of positive integers $(N_m^\sharp)_{m\geq 1}$ and functions $\mathcal{A}_m : \mathbb{R}^{2N_m^\sharp} \to \mathbb{R}^{2N_m^\sharp}$ such that, writing any vector $z \in Z$ as $z = (\vec{z}_1, \dots, \vec{z}_m, \dots)$, with $\vec{z}_m = (\vec{x}_m, \vec{y}_m) \in \mathbb{R}^{2N_m^\sharp}$, we have that

$$\mathcal{A}z = (\mathcal{A}_1\vec{z}_1, \dots, \mathcal{A}_m\vec{z}_m, \dots)\,.$$

Concerning the function $\mathcal{K}$, its gradient will be "guided" by some linear *bounded selfadjoint invertible* operator $\mathcal{B} : X \to X$, with bounded inverse, as we now specify. First of all, similarly as before, in the case when X is infinite-dimensional, we need to assume a particular structure for the functions $\mathcal{B}$ and $\mathcal{K}$.

(Dec2) If $\dim X = \infty$, there exists a sequence of positive integers $(N_j^\star)_{j\geq 1}$ and functions $\mathcal{B}_j : \mathbb{R}^{N_j^\star} \to \mathbb{R}^{N_j^\star}$, $\mathcal{K}_j : \mathbb{R}^{N_j^\star} \to \mathbb{R}$ such that, writing any vector $I \in X$ as $I = (\vec{I}_1, \dots, \vec{I}_j, \dots)$, with $\vec{I}_j \in \mathbb{R}^{N_j^\star}$, we have that

$$\mathcal{B}\,I = (\mathcal{B}_1\vec{I}_1, \dots, \mathcal{B}_j\vec{I}_j, \dots)\,, \qquad \mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j)\,.$$

We now fix $I^0 \in X$, and introduce our *twist condition*.

(Tw) There exist two positive constants $\bar{c}, \bar{\rho}$ such that, for every $j = 1, 2, \dots,$

$$\|\vec{I}_j - \vec{I}_j^0\| \leq \bar{\rho} \quad \Rightarrow \quad \Big\langle \nabla\mathcal{K}_j(\vec{I}_j) - \nabla\mathcal{K}_j(\vec{I}_j^0)\,,\, \mathcal{B}_j(\vec{I}_j - \vec{I}_j^0) \Big\rangle \geq \bar{c}\,\|\vec{I}_j - \vec{I}_j^0\|^2\,.$$

Finally, we assume a compatibility condition between T and the periods introduced in **(P$_\tau$)**.

(C$_\tau$) There exist some integers $m_1, m_2, \dots$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1, m_2\tau_2, \dots)\,.$$

We are now ready to state our main result.

Theorem 2.1. *Let the above assumptions hold. Then, for every $\sigma > 0$ there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, there is a solution of system*

$$\begin{cases} \dot{\varphi} = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t,\varphi,I,z) \\ -\dot{I} = \varepsilon\nabla_\varphi P(t,\varphi,I,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z P(t,\varphi,I,z)\,, \end{cases} \tag{2.1}$$

satisfying

$$\varphi(t+T) = \varphi(t) + T\nabla\mathcal{K}(I^0)\,, \quad I(t+T) = I(t)\,, \quad z(t+T) = z(t)\,, \tag{2.2}$$

and such that

$$\|\varphi(t) - \varphi(0) - t\nabla\mathcal{K}(I^0)\| + \|I(t) - I^0\| + \|z(t)\| < \sigma\,, \quad \textit{for every } t \in \mathbb{R}\,. \tag{2.3}$$

Remark 2.2. When X is finite-dimensional, we will see that condition **(Tw)** can be generalized to

(Tw') There exists a positive constant $\bar{\rho}$ such that

$$\|I - I^0\| \leq \bar{\rho} \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \big\rangle > 0\,;$$

a still more general condition, adopted in [6], is the following:

$$0 \in \mathrm{cl}\Big\{\rho \in \,]0, +\infty[\,: \min_{\|I-I^0\|=\rho} \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \big\rangle > 0\Big\}\,,$$

where $\mathrm{cl}\,\mathcal{S}$ denotes the closure of a set $\mathcal{S}$.

3 Preliminaries for the proof

We will carry out the proof of Theorem 2.1 in the case $\dim X = \infty$ and $\dim Z = \infty$, with some specific remarks on the finite-dimensional cases. By the change of variables

$$(\xi(t), I(t), z(t)) = (\varphi(t) - t\nabla\mathcal{K}(I^0), I(t), z(t)), \tag{3.1}$$

system (2.1) becomes

$$\begin{cases} \dot\xi = \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) + \varepsilon\nabla_I \widehat{P}(t,\xi,I,z) \\ -\dot I = \varepsilon\nabla_\xi \widehat{P}(t,\xi,I,z) \\ J\dot z = \mathcal{A}z + \varepsilon\nabla_z \widehat{P}(t,\xi,I,z), \end{cases} \tag{3.2}$$

where

$$\widehat{P}(t,\xi,I,z) = P(t,\xi + t\nabla\mathcal{K}(I^0), I, z).$$

We use the notation $\zeta = (\xi, I, z)$; the Hamiltonian function is thus

$$\widehat{H}(t,\zeta) = \mathcal{K}(I) - \left\langle \nabla\mathcal{K}(I^0), I\right\rangle + \tfrac12 \left\langle \mathcal{A}z, z\right\rangle + \varepsilon\widehat{P}(t,\zeta).$$

Combining **($\mathbf{P}_\tau$)** with **($\mathbf{C}_\tau$)**, we see that the function $\widehat{P}(\cdot,\xi,I,z)$ is T-periodic, and $\widehat{P}(t,\cdot,I,z)$ is τ_k-periodic in ξ_k, for every $k = 1, 2, \dots$

Some additional notations are now necessary. By assumption **(Dec2)**, the vectors $\xi, I \in X$ decompose in vectors $\vec\xi_j, \vec I_j \in \mathbb{R}^{N_j^\star}$. Setting

$$S_0^\star = 0, \quad S_j^\star = \sum_{i=1}^{j} N_i^\star \quad \text{for } j \geq 1,$$

we can explicitly write the components of $\vec\xi_j, \vec I_j$ as

$$\vec\xi_j = (\xi_{S_{j-1}^\star+1}, \xi_{S_{j-1}^\star+2}, \dots, \xi_{S_j^\star}), \qquad \vec I_j = (I_{S_{j-1}^\star+1}, I_{S_{j-1}^\star+2}, \dots, I_{S_j^\star}).$$

Similarly, by assumption **(Dec1)**, the vector $z \in Z$ decomposes in vectors $\vec z_m \in \mathbb{R}^{2N_m^\sharp}$. Setting

$$S_0^\sharp = 0, \quad S_m^\sharp = \sum_{i=1}^{m} N_i^\sharp \quad \text{for } m \geq 1,$$

we can explicitly write the components of $\vec z_m$ as

$$\vec z_m = (z_{S_{m-1}^\sharp+1}, z_{S_{m-1}^\sharp+2}, \dots, z_{S_m^\sharp}).$$

We define the sequences $(a_j^\star)_j$, $(a_m^\sharp)_m$ in ℓ^2 by

$$a_j^\star = \Big(\sum_{i=1}^{N_j^\star} (\alpha^\star_{S_{j-1}^\star+i})^2\Big)^{1/2}, \quad a_m^\sharp = \Big(\sum_{i=1}^{N_m^\sharp} (\alpha^\sharp_{S_{m-1}^\sharp+i})^2\Big)^{1/2}.$$

Notice that $\|a^\star\|_{\ell^2} = \|\alpha^\star\|_{\ell^2}$ and $\|a^\sharp\|_{\ell^2} = \|\alpha^\sharp\|_{\ell^2}$.

Remark 3.1. When X has a finite dimension d_X, we can define the sequence $(N_j^\star)_j$ taking $N_1^\star = d_X$ and $N_j^\star = 0$ for $j \geq 2$. Similarly when Z is finite-dimensional.

Without loss of generality, from now on we will assume that $I^0 = 0$, a situation which can be recovered by a simple translation. The strategy of the proof of Theorem 2.1 will be to construct a finite-dimensional approximation of system (3.2), and then pass to the limit on the dimension. Precisely, we define the projections $\Pi_{S^\star_{\mathcal{J}}} : X \to X$ and $\Pi_{S^\sharp_{\mathcal{J}}} : Z \to Z$ as

$$\Pi_{S^\star_{\mathcal{J}}} \upsilon = (\vec{\upsilon}_1, \dots, \vec{\upsilon}_{\mathcal{J}}, 0, 0, \dots)\,, \qquad \Pi_{S^\sharp_{\mathcal{J}}} z = (\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}, 0, 0, \dots)\,,$$

and consider the truncated system

$$\begin{cases} \dot{\xi} = \Pi_{S^\star_{\mathcal{J}}} [\nabla \mathcal{K}(I) - \nabla \mathcal{K}(0) + \varepsilon \nabla_I \widehat{P}(t, \xi, I, z)] \\ -\dot{I} = \Pi_{S^\star_{\mathcal{J}}} [\varepsilon \nabla_\xi \widehat{P}(t, \xi, I, z)] \\ J\dot{z} = \Pi_{S^\sharp_{\mathcal{J}}} [\mathcal{A} z + \varepsilon \nabla_z \widehat{P}(t, \xi, I, z)]\,. \end{cases} \tag{3.3}$$

We thus have the Hamiltonian function

$$\widehat{H}_{\mathcal{J}}(t, \zeta) = \mathcal{K}(\Pi_{S^\star_{\mathcal{J}}} I) - \big\langle \nabla \mathcal{K}(0)\,,\, \Pi_{S^\star_{\mathcal{J}}} I \big\rangle + \tfrac{1}{2} \big\langle \mathcal{A} \Pi_{S^\sharp_{\mathcal{J}}} z\,,\, \Pi_{S^\sharp_{\mathcal{J}}} z \big\rangle + \varepsilon \widehat{P}(t, \Pi_{S^\star_{\mathcal{J}}} \xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)\,.$$

Notice that the function

$$\widehat{P}_{\mathcal{J}}(t, \xi, I, z) = \widehat{P}(t, \Pi_{S^\star_{\mathcal{J}}} \xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)$$

satisfies both **(L)** and **($\mathbf{P}_\tau$)** with the same constants, for every index $\mathcal{J} \geq 1$, and observe that system (3.3) is equivalent to

$$\begin{cases} \dot{\vec{\xi}}_j = \nabla \mathcal{K}_j(\vec{I}_j) - \nabla \mathcal{K}_j(0) + \varepsilon \nabla_{\vec{I}_j} \widehat{P}_{\mathcal{J}}(t, \xi, I, z) & \\ -\dot{\vec{I}}_j = \varepsilon \nabla_{\vec{\xi}_j} \widehat{P}_{\mathcal{J}}(t, \xi, I, z) & j \leq \mathcal{J}\,, \\ J\dot{\vec{z}}_j = \mathcal{A}_j \vec{z}_j + \varepsilon \nabla_{\vec{z}_j} \widehat{P}_{\mathcal{J}}(t, \xi, I, z) & \\ & \\ \dot{\vec{\xi}}_i = 0 & \\ -\dot{\vec{I}}_i = 0 & i > \mathcal{J}\,. \\ J\dot{\vec{z}}_i = 0 & \end{cases} \tag{3.4}$$

It can be viewed as two uncoupled systems, the first one in a finite-dimensional space (the "approximating system"), and the second one, infinite-dimensional, having only constant solutions. From now on, we will take $\vec{\xi}_i(t), \vec{I}_i(t), \vec{z}_i(t)$ identically equal to zero when $i \geq \mathcal{J}$.

Concerning the "approximating system", we will need the following slight modification of [7, Corollary 2.3]. Let us consider the finite-dimensional Hamiltonian system

$$J\dot{\zeta} = \nabla_\zeta H(t, \zeta)\,, \tag{3.5}$$

with $\zeta = (\xi, I, z) \in \mathbb{R}^{N+N+2M}$, where the Hamiltonian function is T-periodic in t. Here we use the notation $\xi = (\vec{\xi}_1, \dots, \vec{\xi}_{\mathcal{J}})$, $I = (\vec{I}_1, \dots, \vec{I}_{\mathcal{J}})$.

Theorem 3.2. *Assume that $H(t, \zeta) = \frac{1}{2} \langle \mathbb{A} z, z \rangle + G(t, \zeta)$, where $\mathbb{A}$ is a symmetric $2M \times 2M$ matrix such that $z \equiv 0$ is the unique T-periodic solution of equation $J\dot{z} = \mathbb{A} z$, and there exists a constant c_1 such that*

$$|\nabla_\zeta G(t, \zeta)| \leq c_1\,, \qquad \textit{for every } (t, \zeta) \in \mathbb{R} \times \mathbb{R}^{2(M+N)}\,.$$

Let $G(t,\xi,I,z)$ be periodic in the variables $\xi_1,\dots,\xi_N$. Assume moreover the existence of some positive constants $r'_j < r''_j$ and symmetric invertible matrices $\mathcal{B}_j$, with $j=1,\dots,\mathcal{J}$, such that, for any solution $\zeta(t)=(\xi(t),I(t),z(t))$ of (3.5)*, if*

$$r'_j \le \|\vec{I}_j(0)-\vec{I}^0_j\| \le r''_j \quad \textit{and} \quad \|\vec{I}_i(0)-\vec{I}^0_i\| \le r''_i \ \textit{ for every } i\neq j\,,$$

then

$$\Big\langle \vec{\xi}_j(T)-\vec{\xi}_j(0)\,,\,\mathcal{B}_j(\vec{I}_j(0)-\vec{I}^0_j)\Big\rangle > 0\,.$$

Then, there are at least $N+1$ geometrically distinct T-periodic solutions $\zeta(t)=(\xi(t),I(t),z(t))$ of (3.5)*, such that*

$$\|\vec{I}_j(0)-\vec{I}^0_j\| < r'_j\,, \quad \textit{for every } j=1,\dots,\mathcal{J}\,.$$

4 Proof of Theorem 2.1

In what follows, we always assume that $|\varepsilon|\le 1$, and we denote by $\bar\rho$ the constant introduced in assumption **(Tw)**. Moreover, as in the previous section, we assume $I^0=0$.

Lemma 4.1. *There is a constant $C>0$ with the following property: if $\zeta(t)=(\xi(t),I(t),z(t))$ is a solution of* (3.2) *with $\|\vec{I}_j(0)\|\le\bar\rho$, for some $j\ge 1$, then*

$$\|\vec{\xi}_j(t)-\vec{\xi}_j(0)-t[\nabla\mathcal{K}_j(\vec{I}_j(0))-\nabla\mathcal{K}_j(0)]\| + \|\vec{I}_j(t)-\vec{I}_j(0)\| \le C|\varepsilon|a^\star_j\,, \ \textit{ for every } t\in[0,T]\,.$$

The same property holds for the solutions of (3.4)*, when $j=1,\dots,\mathcal{J}$.*

Proof. Let us start computing, for every $t\in[0,T]$ and every $k\in\{S^\star_{j-1}+1,\dots,S^\star_{j-1}+N^\star_j=S^\star_j\}$,

$$|I_k(t)-I_k(0)| \le \int_0^t |\dot I_k(s)|\,ds \le |\varepsilon|\int_0^T \Big|\frac{\partial\widehat{P}}{\partial\xi_k}(s,\zeta(s))\Big|\,ds \le |\varepsilon|\,T\alpha^\star_k\,.$$

Then we easily get

$$\|\vec{I}_j(t)-\vec{I}_j(0)\| \le |\varepsilon|\,T\Big(\sum_{i=1}^{N^\star_j}(\alpha^\star_{S^\star_{j-1}+i})^2\Big)^{1/2} = |\varepsilon|\,Ta^\star_j\,.$$

Moreover,

$$\begin{aligned}\|\vec{\xi}_j(t)-\vec{\xi}_j(0)-t[\nabla\mathcal{K}_j(\vec{I}_j(0))-\nabla\mathcal{K}_j(0)]\| &\le \int_0^t \|\dot{\vec{\xi}}_j(s)-[\nabla\mathcal{K}_j(\vec{I}_j(0))-\nabla\mathcal{K}_j(0)]\|\,ds\\
&\le \int_0^T \|\nabla\mathcal{K}_j(\vec{I}_j(s))-\nabla\mathcal{K}_j(\vec{I}_j(0))\|\,ds + |\varepsilon|\int_0^T\|\nabla_{\vec{I}_j}\widehat{P}(s,\zeta(s))\|\,ds\\
&\le \int_0^T L\|\vec{I}_j(s)-\vec{I}_j(0)\|\,ds + |\varepsilon|\,Ta^\star_j\\
&\le |\varepsilon|\,T(1+LT)a^\star_j\,,\end{aligned}$$

where L is a suitable Lipschitz constant provided by **(L)**. The proof is thus completed. □

Lemma 4.2. *There exist $\bar\varepsilon>0$ and a sequence $(\delta_j)_j$ in ℓ^2, with $\delta_j\in]0,\bar\rho]$, satisfying the following property: if $\zeta(t)=(\xi(t),I(t),z(t))$ is a solution of* (3.2)*, with $|\varepsilon|<\bar\varepsilon$ and $\delta_j\le\|\vec{I}_j(0)\|\le\bar\rho$, for some $j\ge 1$, then*

$$\Big\langle \vec{\xi}_j(T)-\vec{\xi}_j(0)\,,\,\mathcal{B}_j\,\vec{I}_j(0)\Big\rangle > 0\,.$$

The same property holds for the solutions of (3.4)*, when $j=1,\dots,\mathcal{J}$.*

Proof. If $\|\vec{I}_j(0)\| \leq \bar{\rho}$ for some $j \geq 1$, then, by Lemma 4.1 and **(Tw)**,

$$\begin{aligned}\left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0)\,,\, \mathcal{B}_j\, \vec{I}_j(0) \right\rangle &= \left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0) - T[\nabla\mathcal{K}_j(\vec{I}_j(0)) - \nabla\mathcal{K}_j(0)]\,,\, \mathcal{B}_j\, \vec{I}_j(0) \right\rangle + \\ &\quad + T\left\langle \nabla\mathcal{K}_j(\vec{I}_j(0)) - \nabla\mathcal{K}_j(0)\,,\, \mathcal{B}_j\, \vec{I}_j(0) \right\rangle \\ &\geq -C|\varepsilon| a_j^\star\, \|\mathcal{B}_j\|\, \|\vec{I}_j(0)\| + T\bar{c}\|\vec{I}_j(0)\|^2 \\ &= \left(-C|\varepsilon| a_j^\star\, \|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\| \right) \|\vec{I}_j(0)\|\,.\end{aligned}$$

Setting

$$\delta_j := \min\left\{ \bar{\rho}\,,\, \frac{2C}{\bar{c}T}\, a_j^\star\, \|\mathcal{B}_j\| \right\}\,,$$

we easily verify that $(\delta_j)_j \in \ell^2$, since $(\|\mathcal{B}_j\|)_j$ is bounded by $\|\mathcal{B}\|$ and $(a_j^\star)_j \in \ell^2$; in particular, there exists an integer j_0 such that

$$\delta_j = \frac{2C}{\bar{c}T}\, a_j^\star\, \|\mathcal{B}_j\|\,, \qquad \text{for every } j \geq j_0\,.$$

So, we see that, since $|\varepsilon| \leq 1$ and $\|\vec{I}_j(0)\| \geq \delta_j$,

$$-C|\varepsilon| a_j^\star\, \|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\| > 0\,,$$

for every $j \geq j_0$. For the remaining finite number of integers $j \in \{1, \ldots, j_0 - 1\}$ we simply need to choose $|\varepsilon|$ sufficiently small, thus finishing the proof. □

Remark 4.3. When X is finite-dimensional, the above estimate simplifies, in view of the compactness of the closed balls centered at the origin, so the first condition in **(Tw')** is sufficient in this case. Concerning the second condition in **(Tw')**, we see that it guarantees the existence of a sequence of balls, with smaller and smaller radii, over which the twist condition still holds.

Notice that the set

$$\Xi_I = \prod_{j=1}^{\infty} B^{N_j^\star}[0, \delta_j + C a_j^\star]\,,$$

where $B^n[0,R]$ denotes the closed ball $\{v \in \mathbb{R}^n : \|v\| \leq R\}$, is compact, being homeomorphic to a Hilbert cube. We now modify the function $\mathcal{K}$ outside Ξ_I, in order that the gradient of the modified function be bounded. Let $R_I > 0$ be such that $\Xi_I \subseteq \{v \in X : \|v\| \leq R_I\}$, and $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that

$$\psi(s) = 1 \;\text{ if } s \leq R_I\,, \qquad \psi(s) = 0 \;\text{ if } s \geq 2R_I\,.$$

Define $\widetilde{\mathcal{K}} : X \to \mathbb{R}$ as $\widetilde{\mathcal{K}}(I) = \psi(\|I\|)\mathcal{K}(I)$. Then, when $I \neq 0$,

$$\|\nabla\widetilde{\mathcal{K}}(I)\| = \left\| \psi'(\|I\|)\mathcal{K}(I)\frac{I}{\|I\|} + \psi(\|I\|)\nabla\mathcal{K}(I) \right\| \leq c_1|K(I)| + \|\nabla\mathcal{K}(I)\|\,,$$

for some $c_1 > 0$. By assumption **(L)**, we can find a Lipschitz constant L such that, for every $s \in [0,1]$, if $\|I\| \leq 2R_I$,

$$\|\nabla\mathcal{K}(sI)\| \leq \|\nabla\mathcal{K}(sI) - \nabla\mathcal{K}(0)\| + \|\nabla\mathcal{K}(0)\| \leq L\|I\| + \|\nabla\mathcal{K}(0)\|\,.$$

Moreover,

$$|K(I)| = \left| K(0) + \int_0^1 \langle \nabla \mathcal{K}(sI)\,,\, I \rangle\; ds \right| \le |K(0)| + \sup_{s\in[0,1]} \|\nabla \mathcal{K}(sI)\|\, \|I\|$$
$$\le |K(0)| + \big(L\|I\| + \|\nabla \mathcal{K}(0)\| \big)\, \|I\|\,.$$

Hence,

$$\|\nabla \widetilde{\mathcal{K}}(I)\| \le c_1 |K(0)| + (2R_I c_1 + 1)(2R_I L + \|\nabla \mathcal{K}(0)\|)\,, \qquad \text{for every } I \in X.$$

We define $\mathbb{A} = \mathrm{diag}(\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{J}})$ as a *block-diagonal* matrix having a diagonal formed by the matrices $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{J}}$ introduced in **(Dec1)**, i.e. such that

$$\mathbb{A}(\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}) = (\mathcal{A}_1 \vec{z}_1, \dots, \mathcal{A}_{\mathcal{J}} \vec{z}_{\mathcal{J}})\,.$$

It is easy to verify, using **(NR)**, that $z \equiv 0$ is the unique T-periodic solution of equation $J\dot{z} = \mathbb{A}z$. Then, by Theorem 3.2, for every $\mathcal{J}$ there is a T-periodic solution

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t))$$

of (3.4), with

$$\|\vec{I}_{\mathcal{J}_j}(0)\| < \delta_j\,, \quad \text{ for every } j \ge 1\,. \tag{4.1}$$

(Recall that we have chosen the last constant components of the solutions of (3.4) to be equal to zero.) By Lemma 4.1, these solutions satisfy

$$\|\vec{I}_{\mathcal{J}_j}(t)\| \le \delta_j + C a_j^\star\,, \quad \text{ for every } t \in [0,T]\,,$$

i.e.,

$$I_{\mathcal{J}}(t) \in \Xi_I\,, \quad \text{ for every } t \in [0,T]\,. \tag{4.2}$$

Let us now consider the component $\xi_{\mathcal{J}}(t)$ of the solution. By the periodicity assumption **($\mathbf{P}_\tau$)**, we can assume without loss of generality that $\xi_k(0) \in [0, \tau_k]$, for every $k \ge 1$. From Lemma 4.1, property **(L)** and (4.1), we have

$$|\xi_k(t) - \xi_k(0)| \le \|\vec{\xi}_j(t) - \vec{\xi}_j(0)\| \le C a_j^\star + TL\delta_j\,, \qquad \text{for every } t \in [0,T]\,,$$

for a suitable Lipschitz constant L. Setting $b_k := C a_j^\star + TL\delta_j$, where j is the index such that $S_{j-1}^\star < k \le S_j^\star$, and defining

$$\Xi_\xi = \prod_{k=1}^{\infty} [-b_k, \tau_k + b_k]\,,$$

we have that

$$\xi_{\mathcal{J}}(t) \in \Xi_\xi\,, \quad \text{ for every } t \in [0,T]\,. \tag{4.3}$$

We now need an a priori estimate on $z_{\mathcal{J}}(t)$.

Lemma 4.4. *There exists a sequence $(R_j)_j \in \ell^2$ of positive constants such that, for every T-periodic solution $\zeta(t) = (\xi(t), I(t), z(t))$ of* (3.2), *we have*

$$\|\vec{z}_j\|_{\mathcal{C}([0,T],\mathbb{R}^{2N_j^\sharp})} \le |\varepsilon| R_j\,,$$

for every $j \ge 1$. The same property holds for every T-periodic solution of (3.4), *when $j = 1, \dots, \mathcal{J}$.*

Proof. Fix $j \geq 1$ and consider the j-th block of the third equation in (3.2), i.e.

$$\mathcal{L}_j \vec{z}_j = \mathcal{A}_j \vec{z}_j + \varepsilon \nabla_{\vec{z}_j} \widehat{P}(t, \zeta) \,, \tag{4.4}$$

where $\mathcal{L}_j$ denotes the j-th block of the linear operator $\mathcal{L}$ introduced in **(NR)**, i.e.

$$\mathcal{L}_j \vec{z}_j = \mathcal{L}_j (z_{S^\sharp_{j-1}+1}, \ldots, z_{S^\sharp_j}) = (J\dot{z}_{S^\sharp_{j-1}+1}, \ldots, J\dot{z}_{S^\sharp_j}) \,. \tag{4.5}$$

From hypothesis **(Dec1)**, we have $\sigma(\mathcal{L}_j - \mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \mathcal{A})$. Hence, using **(NR)**, $0 \notin \sigma(\mathcal{L}_j - \mathcal{A}_j)$ and (4.4) is equivalent to

$$\vec{z}_j = \varepsilon (\mathcal{L}_j - \mathcal{A}_j)^{-1} \nabla_{\vec{z}_j} \widehat{P}(t, \zeta) \,.$$

Moreover,

$$\|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| = \frac{1}{\mathrm{dist}(0, \sigma(\mathcal{L}_j - \mathcal{A}_j))} \leq \frac{1}{\mathrm{dist}(0, \sigma(\mathcal{L} - \mathcal{A}))} = \|(\mathcal{L} - \mathcal{A})^{-1}\| \,,$$

and consequently, setting $r_j := \sqrt{T} a^\sharp_j \|(\mathcal{L} - \mathcal{A})^{-1}\|$, we have that

$$\|\vec{z}_j\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| \, \|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\vec{z}_j} \widehat{P}\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| r_j \,.$$

Since $\vec{z}_j$ solves (4.4), we have that $\dot{\vec{z}}_j \in L^2([0,T], \mathbb{R}^{2N^\sharp_j})$, and

$$\|\dot{\vec{z}}_j\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} \leq \|\mathcal{A}_j\| \|\vec{z}_j\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} + |\varepsilon| \sqrt{T} a^\sharp_j \leq |\varepsilon| \Big(\|\mathcal{A}_j\| r_j + \sqrt{T} a^\sharp_j \Big) \,.$$

So, setting $C_j = (1 + \|\mathcal{A}_j\|) r_j + \sqrt{T} a^\sharp_j$,

$$\|\vec{z}_j\|_{H^1([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| C_j \,. \tag{4.6}$$

By the continuous immersion of $H^1([0,T], Z)$ in $\mathcal{C}([0,T], Z)$, cf. [14, §23.6], we can find a constant $\chi > 0$ such that

$$\|z\|_{\mathcal{C}([0,T],Z)} \leq \chi \|z\|_{H^1([0,T],Z)} \,,$$

for every $z \in H^1([0,T], Z)$. Since $\mathcal{C}([0,T], \mathbb{R}^{2N^\sharp_j})$ and $H^1([0,T], \mathbb{R}^{2N^\sharp_j})$ can be seen as a subsets of $\mathcal{C}([0,T], Z)$ and $H^1([0,T], Z)$, respectively, simply adding an infinite number of null components, we obtain

$$\|\vec{z}_j\|_{\mathcal{C}([0,T], \mathbb{R}^{2N^\sharp_j})} \leq \chi \|\vec{z}_j\|_{H^1([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| \chi C_j \,.$$

The proof is thus completed, taking $R_j = \chi C_j$. □

Defining

$$\Xi_z = \prod_{j=1}^{\infty} B^{2N^\sharp_j}[0, R_j] \,,$$

we have thus proved that

$$z_{\mathcal{J}}(t) \in \Xi_z \,, \qquad \text{for every } t \in [0,T] \,. \tag{4.7}$$

Summing up, by (4.2), (4.3), (4.7), we have that, setting $\Xi = \Xi_\xi \times \Xi_I \times \Xi_z$, the T-periodic solutions we found satisfy

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t)) \in \Xi \,, \qquad \text{for every } t \in [0,T] \,.$$

Notice that Ξ is compact, being the product of three compact sets. We will now prove that there is a subsequence of $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ which uniformly converges to a solution of (3.2).

From (4.6), recalling that $|\varepsilon| \le 1$, we have

$$\|z_{\mathcal{J}}(t_1) - z_{\mathcal{J}}(t_2)\| \le |t_1 - t_2|^{1/2} \left(\int_0^T \|\dot z_{\mathcal{J}}(s)\|^2\, ds \right)^{1/2} \le |t_1 - t_2|^{1/2} \Big(\sum_{j=1}^{\infty} C_j^2 \Big)^{1/2}.$$

Looking at the variables $I_{\mathcal{J}}(t)$, by **(P$_{bd}$)** we have that

$$\|I_{\mathcal{J}}(t_1) - I_{\mathcal{J}}(t_2)\| \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\dot I_{\mathcal{J}}(s)\|^2\, ds \right)^{1/2} \le |t_2 - t_1|^{1/2} \sqrt{T}\, \|a^\star\|_{\ell^2}.$$

Concerning the variables $\xi_{\mathcal{J}}(t)$, we first observe that

$$\begin{aligned} \|\dot\xi_{\mathcal{J}}(s)\| &\le \|\nabla\mathcal{K}(I_{\mathcal{J}}(s)) - \nabla\mathcal{K}(0)\| + \|a^\star\|_{\ell^2} \\ &\le L\|I_{\mathcal{J}}(s)\| + \|a^\star\|_{\ell^2} \le L\Big(\sum_{j=1}^{\infty}(\delta_j + Ca_j^\star)^2\Big)^{1/2} + \|a^\star\|_{\ell^2} := \widehat{C}, \end{aligned}$$

where L is a suitable Lipschitz constant provided by **(L)**. Then,

$$\|\xi_{\mathcal{J}}(t_1) - \xi_{\mathcal{J}}(t_2)\| \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\dot\xi_{\mathcal{J}}(s)\|^2\, ds \right)^{1/2} \le |t_2 - t_1|^{1/2}\sqrt{T}\, \widehat{C}.$$

Hence, the sequence $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ is equi-uniformly continuous on $[0,T]$ and takes its values in a compact subset of $\mathcal{X}$. By the Ascoli–Arzelà Theorem, we find a subsequence, still denoted by $(\zeta_{\mathcal{J}})_{\mathcal{J}}$, which uniformly converges to a certain continuous function $\zeta^\natural : [0,T] \to \mathcal{X}$, such that $\zeta^\natural(t) \in \Xi$ for every $t \in [0,T]$, and $\zeta^\natural(0) = \zeta^\natural(T)$. We are going to prove that $\zeta^\natural$ solves (3.2), following the lines of the proof of [3, Theorem 3].

Let us consider the solution ζ_∞ of system (3.2) such that $\zeta_\infty(0) = \zeta^\natural(0)$ which, by the boundedness of $\nabla\mathcal{K}$ and $\nabla_\zeta \widehat{P}$, is certainly defined on $[0,T]$. We will prove that the sequence $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ converges uniformly to ζ_∞, thus obtaining that $\zeta_\infty = \zeta^\natural$. To this aim, we write the integral formulation of systems (3.2) and (3.3), for $\mathcal{J} \ge 1$:

$$\zeta_\infty(t) = \zeta_\infty(0) - \int_0^t J\nabla_\zeta \widehat{H}(s, \zeta_\infty(s))\, ds, \tag{4.8}$$

$$\zeta_{\mathcal{J}}(t) = \zeta_{\mathcal{J}}(0) - \int_0^t J\nabla_\zeta \widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s))\, ds. \tag{4.9}$$

In order to simplify the notations, we introduce the projection

$$\mathscr{P}_{\mathcal{J}}(\zeta) = \mathscr{P}_{\mathcal{J}}(\xi, I, z) = (\Pi_{S^\star_{\mathcal{J}}}\xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z).$$

Let us write

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le \|\zeta_{\mathcal{J}}(t) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(t)\| + \|\mathscr{P}_{\mathcal{J}}\zeta_\infty(t) - \zeta_\infty(t)\|.$$

By an elementary argument,

$$\|\mathscr{P}_{\mathcal{J}}\zeta_\infty(t) - \zeta_\infty(t)\| \to 0, \quad \text{as } \mathcal{J} \to \infty, \tag{4.10}$$

uniformly with respect to $t \in [0,T]$. From (4.8) and (4.9), since $\mathscr{P}_{\mathcal{J}} J = J \mathscr{P}_{\mathcal{J}}$, we have

$$\|\zeta_{\mathcal{J}}(t) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(t)\| \le \|\zeta_{\mathcal{J}}(0) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(0)\| + \\ + \int_0^t \|J\nabla_\zeta \widehat{H}_{\mathcal{J}}(s,\zeta_{\mathcal{J}}(s)) - J\mathscr{P}_{\mathcal{J}}\nabla_\zeta \widehat{H}(s,\zeta_\infty(s))\|\, ds\,. \tag{4.11}$$

Notice that

$$\|\zeta_{\mathcal{J}}(0) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(0)\| \le \|\zeta_{\mathcal{J}}(0) - \zeta_\infty(0)\| = \|\zeta_{\mathcal{J}}(0) - \zeta^\natural(0)\| \to 0\,, \quad \text{as } \mathcal{J} \to \infty\,. \tag{4.12}$$

Since $\nabla_\zeta \widehat{H}_{\mathcal{J}}(s,\zeta_{\mathcal{J}}(s)) = \mathscr{P}_{\mathcal{J}}\nabla_\zeta \widehat{H}(s,\zeta_{\mathcal{J}}(s))$, the integral term in (4.11) satisfies

$$\int_0^t \Big\| J\mathscr{P}_{\mathcal{J}}\Big(\nabla_\zeta \widehat{H}(s,\zeta_{\mathcal{J}}(s)) - \nabla_\zeta \widehat{H}(s,\zeta_\infty(s))\Big)\Big\|\, ds \le L\int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_\infty(s)\|\, ds\,,$$

where L is a suitable Lipschitz constant. Summing up, we have

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le c_{\mathcal{J}} + L\int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_\infty(s)\|\, ds\,,$$

where $(c_{\mathcal{J}})_{\mathcal{J}}$ is a sequence, provided by the limits in (4.10) and (4.12), such that $\lim_{\mathcal{J}} c_{\mathcal{J}} = 0$. Hence, by Gronwall's Lemma,

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le c_{\mathcal{J}} e^{Lt}\,, \quad \text{for every } t \in [0,T]\,,$$

implying that $\zeta_{\mathcal{J}} \to \zeta_\infty$ uniformly on $[0,T]$. We conclude that $\zeta_\infty = \zeta^\natural$ on $[0,T]$, thus showing that $\zeta_\infty(0) = \zeta_\infty(T)$, so that ζ_∞ is a T-periodic solution of (3.2).

By the inverse change of variables

$$(\varphi(t), I(t), z(t)) = (\xi(t) + t\nabla\mathcal{K}(I^0), I(t), z(t))\,,$$

cf. (3.1), we have a solution of (2.1), satisfying (2.2). Moreover, condition (2.3) holds true, by Lemmas 4.1 and 4.4, suitably reducing, if necessary, the value of $\bar{\varepsilon}$. The proof of Theorem 2.1 is thus completed. □

5 Applications

5.1 Coupling second order with linear systems

We first state a simple lemma, which may be useful for the verification of the twist condition.

Lemma 5.1. *If there exists $I^0 \in X$ such that $\mathcal{K} : X \to \mathbb{R}$ is twice continuously differentiable at I^0 and $\mathcal{K}''(I^0) : X \to X$ is invertible, with bounded inverse, then there exist two positive constants $\bar{c}$, $\bar{\rho}$ such that*

$$\|I - I^0\| \le \bar{\rho} \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(y) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{K}''(I^0)(y - I^0)\big\rangle \ge \bar{c}\,\|y - I^0\|^2.$$

Moreover, if $\dim X = \infty$ and, with the usual notation, $\mathcal{K}(I) = \sum_{j=1}^\infty \mathcal{K}_j(\vec{I}_j)$, then condition **(Tw)** *holds.*

Proof. Since $\mathcal{B} := \mathcal{K}''(I^0) : X \to X$ is invertible with bounded inverse, there exists $\gamma > 0$ such that $\|\mathcal{B}\, I\| \geq \gamma \|I\|$ for every $I \in X$. Then,

$$\begin{aligned}
&\left\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \right\rangle = \\
&\quad = \int_0^1 \left\langle \mathcal{K}''\big(I^0 + s(I - I^0)\big)(I - I^0)\,,\, \mathcal{B}(I - I^0) \right\rangle\, ds \\
&\quad = \|\mathcal{B}(I - I^0)\|^2 + \int_0^1 \left\langle \left[\mathcal{K}''\big(I^0 + s(I - I^0)\big) - \mathcal{B}\right](I - I^0)\,,\, \mathcal{B}(I - I^0) \right\rangle\, ds \\
&\quad \geq \big(\gamma^2 - \|\mathcal{B}\| \cdot \|\mathcal{K}''\big(I^0 + s(I - I^0)\big) - \mathcal{B}\|\big)\, \|I - I^0\|^2\,.
\end{aligned}$$

Since $\mathcal{K}''$ is continuous at I^0, there exists $\bar{\rho} > 0$ such that, if $I \in X$ satisfies $\|I - I^0\| \leq \bar{\rho}$, then

$$\|\mathcal{K}''(I) - \mathcal{B}\| = \|\mathcal{K}''(I) - \mathcal{K}''(I^0)\| \leq \frac{\gamma^2}{2\|\mathcal{B}\|},$$

so

$$\left\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \right\rangle \geq \frac{\gamma^2}{2} \|I - I^0\|^2, \tag{5.1}$$

and the first part of the lemma is thus proved.

Assume now that $\mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j)$. We have that

$$\mathcal{B}\, I = (\mathcal{B}_1 \vec{I}_1, \ldots, \mathcal{B}_j \vec{I}_j, \ldots)\,,$$

where $\mathcal{B}_j = \mathcal{K}_j''(\vec{I}_j^{\,0})$. Then, **(Tw)** is verified directly from (5.1) defining, for every $j \in \{1, 2, \ldots\}$, the vector I as $\vec{I}_i = \vec{I}_i^{\,0}$ if $i \neq j$, once $\vec{I}_j$ has been chosen. □

We thus have the following.

Corollary 5.2. *Assume* **(L)**, **(P$_\tau$)**, **(P$_{bd}$)**, **(NR)**, **(Dec1)**, **(Dec2)** *and* **(C$_\tau$)** *hold. If $\mathcal{K} : X \to \mathbb{R}$ is twice continuously differentiable at I^0 and $\mathcal{K}''(I^0) : X \to X$ is invertible, with bounded inverse, then there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, system* (2.1) *has a T-periodic solution.*

Let us now consider an equation in an infinite-dimensional space of the type

$$\begin{cases} \dfrac{d}{dt}\,(\nabla\Phi \circ \dot{x}) = \varepsilon\, \nabla_x F(t, x, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t, x, z)\,. \end{cases} \tag{5.2}$$

Let, for definiteness, $\dim X = \infty$ and $\dim Z = \infty$. Concerning the bounded selfadjoint operator $\mathcal{A}$, we require the nonresonance assumption **(NR)** and that it decomposes as in **(Dec1)**. For the differential operator in the first equation, we suppose that there exists a sequence of positive integers $(N_j)_{j\geq 1}$ such that, writing any vector $y \in X$ as $y = (\vec{y}_1, \ldots, \vec{y}_j, \ldots)$, with $\vec{y}_j \in \mathbb{R}^{N_j}$,

$$\Phi(y) = \sum_{j=1}^{\infty} \Phi_j(\vec{y}_j)\,,$$

where each Φ_j is a continuous real valued strictly convex function defined on a closed ball $\overline{B}(0, a_j)$ in $\mathbb{R}^{N_j}$, continuously differentiable in the open ball $B(0, a_j)$, with $\nabla\Phi_j : B(0, a_j) \to X$ being a homeomorphism, and $\nabla\Phi_j(0) = 0$.

Denoting by Φ_j^* the Legendre–Fenchel transform of Φ_j, we have that $\Phi_j^* : X \to \mathbb{R}$ is strictly convex and coercive, with $\nabla\Phi^* = (\nabla\Phi)^{-1} : X \to B(0,a)$, cf. [11, Chapter 2]. We can define

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j)\,,$$

so that system (5.2) can be written as a Hamiltonian system

$$\begin{cases} \dot{x} = \nabla\Phi^*(y) \\ \dot{y} = \varepsilon\,\nabla_x F(t,x,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t,x,z)\,. \end{cases}$$

So, we are in the situation of system (2.1), taking $\mathcal{K}(I) = \Phi^*(I)$ and $P(t,\varphi,I,z) = F(t,\varphi,z)$.

An example is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \Big(1 - \sqrt{1 - \|\vec{y}_j\|^2}\,\Big)\,,$$

for which, writing $x = (\vec{x}_1, \dots, \vec{x}_j, \dots)$, system (5.2) becomes

$$\begin{cases} \dfrac{d}{dt}\,\dfrac{\dot{\vec{x}}_j}{\sqrt{1 - \|\dot{\vec{x}}_j\|^2}} = \varepsilon\,\nabla_{\vec{x}_j} F(t,x,z)\,, \qquad j = 1,2,\dots \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t,x,z)\,, \end{cases} \tag{5.3}$$

so that, in the first equation, we can see a kind of *relativistic operator*. We then have the following.

Corollary 5.3. *In the above setting, assume moreover the following conditions:*

(L) *for every $R > 0$ there exists a positive constant L_R such that*

$$\|\nabla_u F(t,u') - \nabla_u F(t,u'')\| \le L_R \|u' - u''\|\,,$$

for every $t \in [0,T]$ and $u' = (x',z'), u'' = (x'',z'') \in X \times Z$ with $\|u'\| < R$ and $\|u''\| < R$;

($\mathbf{F}_\tau$) *the function $F(t,x,z)$ is τ_k-periodic in each x_k, and the sequence $(\tau_k)_k$ belongs to ℓ^2;*

($\mathbf{F}_{bd}$) *there exist $(\alpha_k^\star)_k$ and $(\alpha_l^\sharp)_l$ in ℓ^2 such that, for every $k, l = 1, 2, \dots$,*

$$\left|\frac{\partial F}{\partial x_k}(t,x,z)\right| \le \alpha_k^\star\,, \qquad \|\nabla_{z_l} F(t,x,z)\| \le \alpha_l^\sharp\,,$$

for every $(t,x,z) \in [0,T] \times X \times Z$.

Then, there exists $\bar\varepsilon > 0$ such that, if $|\varepsilon| \le \bar\varepsilon$, system (5.3) *has a T-periodic solution.*

Proof. Taking $I^0 = 0$, we have that $\nabla\Phi^*(0) = 0$ and $(\Phi^*)''(0) = \mathrm{Id}$. So, assumption **($\mathbf{C}_\tau$)** is fulfilled taking $m_1 = m_2 = \cdots = 0$ and, in view of Lemma 5.1, we can apply Theorem 2.1 to conclude. □

We have thus obtained an extension to infinite-dimensional systems of a result in [10].

Another possible situation where Theorem 2.1 applies is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \left(\sqrt{1 + \|\vec{y}_j\|^2} - 1 \right).$$

In this case, we find

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j) = \sum_{j=1}^{\infty} \left(1 - \sqrt{1 - \|\vec{y}_j\|^2} \right),$$

and the first equation in system (5.2) becomes

$$\frac{d}{dt} \frac{\dot{\vec{x}}_j}{\sqrt{1 + \|\dot{\vec{x}}_j\|^2}} = \varepsilon \nabla_{\vec{x}_j} F(t, x, z), \qquad j = 1, 2, \dots$$

involving a kind of *mean curvature operator*.

Since each $\nabla \Phi_j^*$ is defined only on the open ball $B(0,1)$, we must first modify and extend the Hamiltonian function outside a ball $B(0,r)$, with $r \in]0,1[$, and then be careful that the $\vec{y}_j$ component of the T-periodic solution we find remains in $B(0,r)$. We omit the details, for briefness. Stating the analogue of Corollary 5.3, we thus obtain an infinite-dimensional version of some results obtained in [8, 9] (see also [13], where bounded variation solutions are considered).

5.2 Perturbations of "superintegrable" systems

In this section we study a slightly different situation with respect to system (2.1). We are going to consider the Hamiltonian system

$$\begin{cases} \dot{\varphi} = \nabla \mathcal{K}(I) + \eta^2 \nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \eta^2 \nabla_\varphi P(t, \varphi, I, z) \\ J\dot{z} = \eta \mathcal{A} z + \eta^2 \nabla_z P(t, \varphi, I, z), \end{cases} \tag{5.4}$$

with Hamiltonian function

$$H(t, \varphi, I, z) = \mathcal{K}(I) + \frac{\eta}{2} \langle \mathcal{A} z, z \rangle + \eta^2 P(t, \varphi, I, z).$$

The following result extends to an infinite-dimensional setting [7, Theorem 4.1], which was motivated by the study of perturbations of superintegrable systems, cf. [12].

Theorem 5.4. *Assume* **(L)**, **(P_τ)**, **(P_{bd})**, **(Dec1)**, **(Dec2)**, **(Tw)** *and* **(C_τ)**. *Moreover let the operator $\mathcal{A}$ be invertible with a bounded inverse. Then, for every $\sigma > 0$ there exists $\bar{\eta} > 0$ such that, if $|\eta| \le \bar{\eta}$, system* (5.4) *has a solution satisfying* (2.2) *and* (2.3).

Notice that the nonresonance assumption **(NR)** is not required here.

Proof. Arguing as above we can perform the change of variable (3.1) and set without loss of generality $I^0 = 0$, so to obtain

$$\begin{cases} \dot{\xi} = \nabla \mathcal{K}(I) - \nabla \mathcal{K}(0) + \eta^2 \nabla_I \widehat{P}(t, \xi, I, z) \\ -\dot{I} = \eta^2 \nabla_\xi \widehat{P}(t, \xi, I, z) \\ J\dot{z} = \eta \mathcal{A} z + \eta^2 \nabla_z \widehat{P}(t, \xi, I, z), \end{cases} \tag{5.5}$$

and, for every index $\mathcal{J} \geq 1$, its approximation

$$\begin{cases} \dot{\xi} = \Pi_{S^\star_{\mathcal{J}}}[\nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \eta^2\nabla_I\widehat{P}(t,\xi,I,z)] \\ -\dot{I} = \Pi_{S^\star_{\mathcal{J}}}[\eta^2\nabla_\xi\widehat{P}(t,\xi,I,z)] \\ J\dot{z} = \Pi_{S^\sharp_{\mathcal{J}}}[\eta\mathcal{A}z + \eta^2\nabla_z\widehat{P}(t,\xi,I,z)]\,. \end{cases} \tag{5.6}$$

Lemmas 4.1 and 4.2 holds again, simply replacing $|\varepsilon|$ with η^2 and $\bar{\varepsilon}$ with $\bar{\eta}^2$. The statement and the proof of Lemma 4.4, however, must be modified as follows.

Lemma 5.5. *There exists a sequence $(r_j)_j \in \ell^2$ of positive constants such that, for every T-periodic solution $\zeta(t) = (\xi(t), I(t), z(t))$ of* (5.5) *we have*

$$\|\vec{z}_j\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \leq |\eta| r_j\,,$$

for every $j \geq 1$. The same conclusion holds for every solution of (5.6)*, when $j = 1, \dots, \mathcal{J}$.*

Proof. Fix $j \geq 1$ and consider the j-th block of the third equation in (5.6), i.e.

$$\mathcal{L}_j\vec{z}_j = \eta\mathcal{A}_j\vec{z}_j + \eta^2\nabla_{\vec{z}_j}\widehat{P}(t,\zeta)\,, \tag{5.7}$$

where $\mathcal{L}_j$ denotes the j-th block of the linear operator $\mathcal{L}$, cf. (4.5). From hypothesis **(Dec1)**, we have that $\sigma(\mathcal{L}_j - \eta\mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \eta\mathcal{A})$. We set $\eta_0 = \min\{1, \frac{\pi}{T\|\mathcal{A}\|}\}$ and, recalling that $0 \notin \sigma(\mathcal{A})$, we choose $\delta \in (0, \frac{\pi}{T})$ such that $\sigma(\mathcal{A}) \cap [-\delta, \delta] = \varnothing$.

Claim. When $|\eta| < \eta_0$, every $\lambda \in \sigma(\mathcal{L} - \eta\mathcal{A})$ satisfies $|\lambda| > \delta|\eta|$.

In order to prove this Claim, notice that, if $\lambda \in \sigma(\mathcal{L}-\eta\mathcal{A})$, there exists a non-trivial T-periodic solution z of $Jz' = (\eta\mathcal{A} - \lambda I)z$, so

$$\sigma\big(J(\eta\mathcal{A} - \lambda I)\big) \cap \frac{2\pi}{T}i\mathbb{Z} \neq \varnothing\,. \tag{5.8}$$

If $|\lambda| \geq \pi/T$, then $|\lambda| > \delta > \delta|\eta|$. So, we can assume $|\lambda| < \pi/T$. In this case, we have

$$\|J(\eta\mathcal{A} - \lambda I)\| \leq |\eta|\,\|\mathcal{A}\| + |\lambda| < \frac{2\pi}{T}\,,$$

so,

$$\mu \in \sigma\big(J(\eta\mathcal{A} - \lambda I)\big) \quad \Rightarrow \quad |\mu| \leq \|J(\eta\mathcal{A} - \lambda I)\| < \frac{2\pi}{T}\,.$$

By (5.8), we have that $0 \in \sigma(J(\eta\mathcal{A} - \lambda I))$ and, since J is invertible, $0 \in \sigma(\eta\mathcal{A} - \lambda I)$. Hence, $\frac{\lambda}{\eta} \in \sigma(\mathcal{A})$ and so $|\frac{\lambda}{\eta}| > \delta$, thus proving the Claim.

From now on we assume $|\eta| < \eta_0$. By the Claim, in particular, $0 \notin \sigma(\mathcal{L} - \eta\mathcal{A})$ and so $\mathcal{L} - \eta\mathcal{A}$ is invertible, as well as $\mathcal{L}_j - \eta\mathcal{A}_j$, with bounded inverses. Hence, (5.7) is equivalent to

$$\vec{z}_j = \eta^2(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\nabla_{\vec{z}_j}\widehat{P}(t,\zeta)\,.$$

Moreover,

$$\|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| = \frac{1}{\operatorname{dist}(0, \sigma(\mathcal{L}_j - \eta\mathcal{A}_j))} \leq \frac{1}{\operatorname{dist}(0, \sigma(\mathcal{L} - \eta\mathcal{A}))} \leq \frac{1}{\delta|\eta|}\,,$$

and consequently

$$\|\vec{z}_j\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \leq \eta^2\,\|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\vec{z}_j}\widehat{P}\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \leq \frac{\eta^2\sqrt{T}a^\sharp_j}{\delta|\eta|} = |\eta|\frac{\sqrt{T}a^\sharp_j}{\delta}\,,$$

thus concluding the proof of the lemma. □

The proof of Theorem 5.4 can now be completed following again the lines of the proof of Theorem 2.1. □

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References

[1] A. Ambrosetti, V. Coti Zelati, and I. Ekeland, Symmetry breaking in Hamiltonian systems, J. Differential Equations 67 (1987), 165–184.

[2] D. Bernstein and A. Katok, Birkhoff periodic orbits for small perturbations of completely integrable Hamiltonian systems with convex Hamiltonians, Invent. Math. 88 (1987), 225–241.

[3] A. Boscaggin, A. Fonda and M. Garrione, An infinite-dimensional version of the Poincaré–Birkhoff theorem on the Hilbert cube, Ann. Sc. Norm. Super. Pisa, online first, DOI: 10.2422/2036-2145.201710_005.

[4] W.F. Chen, Birkhoff periodic orbits for small perturbations of completely integrable Hamiltonian systems with nondegenerate Hessian, in: Twist mappings and their applications, vol. 44 of IMA Vol. Math. Appl., Springer, Berlin, 1992, pp. 87–94.

[5] I. Ekeland, A perturbation theory near convex Hamiltonian systems, J. Differential Equations 50 (1983), 407–440.

[6] A. Fonda, M. Garrione and P. Gidoni, Periodic perturbations of Hamiltonian systems, Adv. Nonlinear Anal. 5 (2016), 367–382.

[7] A. Fonda and P. Gidoni, Coupling linearity and twist: an extension of the Poincaré–Birkhoff Theorem for Hamiltonian systems, preprint 2019, available at https://dmi.units.it/~fonda/p2019_Fonda-Gidoni_preprint.pdf

[8] A. Fonda and R. Toader, Periodic solutions of pendulum-like Hamiltonian systems in the plane, Adv. Nonlin. Stud. 12 (2012), 395–408.

[9] A. Fonda and A.J. Ureña, A higher dimensional Poincaré–Birkhoff theorem for Hamiltonian flows, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), 679–698.

[10] J. Mawhin, Multiplicity of solutions of variational systems involving ϕ-Laplacians with singular ϕ and periodic nonlinearities, Discrete Contin. Dyn. Syst. 32 (2012), 4015–4026.

[11] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, 1989.

[12] A.S. Mishchenko and A.T. Fomenko, Generalized Liouville method of integration of Hamiltonian systems, Funct. Anal. Appl. 12 (1978), 113–121.

[13] F. Obersnel and P. Omari, Multiple bounded variation solutions of a periodically perturbed sine-curvature equation, Commun. Contemp. Math. 13 (2011), 1–21.

[14] E. Zeidler, Nonlinear Functional Analysis and its Applications II/A, Springer, Berlin, 1990.

Authors' addresses:

Alessandro Fonda and Andrea Sfecci
Dipartimento di Matematica e Geoscienze
Università di Trieste
P.le Europa 1, I-34127 Trieste, Italy
e-mail: a.fonda@units.it, asfecci@units.it

Giuliano Klun
Scuola Internazionale Superiore di Studi Avanzati
Via Bonomea 265, I-34136 Trieste, Italy
e-mail: giuliano.klun@sissa.it

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