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# A note on growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation

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**Abstract**

We present the recent result in [29] concerning strong nonlinear instability and growth of Sobolev norms near quasiperiodic finite-gap solutions of the defocusing cubic nonlinear Schrödinger equation (NLS) on the two-dimensional torus. The equation admits a special family of elliptic invariant quasiperiodic tori called finite-gap solutions. These are inherited from the integrable 1D model (cubic NLS on the circle) by considering solutions

that depend only on one variable. We construct solutions of the 2D cubic NLS that start arbitrarily close to such invariant tori in the $H^s$ topology ($0 < s < 1$) and whose $H^s$ norm can grow by any given factor.
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# 1 Introduction

A widely held principle in dynamical systems theory is that invariant quasiperiodic tori play an important role in understanding the complicated long-time behavior of Hamiltonian ODE and PDE. In addition to being important in their own right, the hope is that such quasiperiodic tori can play an important role in understanding other, possibly more generic, dynamics of the system by acting as *islands* in whose vicinity orbits might spend long periods of time before moving to other such islands. The construction of such invariant sets for Hamiltonian PDE has witnessed an explosion of activity over the past thirty years after the success of extending KAM techniques to infinite dimensions. However, the dynamics near such tori is still poorly understood, and often restricted to the linear theory. The purpose of this note is to present the recent result of [29], in which we construct non-trivial *nonlinear* dynamics in the vicinity of certain quasiperiodic solutions for the cubic defocusing NLS equation.

**1.1. The dynamical system and its quasiperiodic objects.** Consider the periodic cubic defocusing nonlinear Schrödinger equation (NLS),

$$\mathrm{i}\partial_t u + \Delta u = |u|^2 u \tag{2D-NLS}$$

where $(x, y) \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$. This is a Hamiltonian PDE with conserved quantities: i) the Hamiltonian

$$H_0(u) = \int_{\mathbb{T}^2} \left( |\nabla u(x, y)|^2 + \frac{1}{2}|u(x, y)|^4 \right) \mathrm{d}x\, \mathrm{d}y, \tag{1.1}$$

ii) the mass

$$M(u) = \int_{\mathbb{T}^2} |u(x, y)|^2 \mathrm{d}x\, \mathrm{d}y, \tag{1.2}$$

which is just the square of the $L^2$ norm of the solution, and iii) the momentum

$$P(u) = \mathrm{i} \int_{\mathbb{T}^2} \overline{u(x, y)} \nabla u(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{1.3}$$

We take the simplest non-trivial family of invariant quasiperiodic tori admitted by (2D-NLS), namely those inherited from its completely integrable 1D counterpart

$$\mathrm{i}\partial_t q = -\partial_{xx} q + |q|^2 q, \quad x \in \mathbb{T}. \tag{1D-NLS}$$

This is a subsystem of (2D-NLS) if we consider solutions that depend only on the first spatial variable. It is well known that equation (1D-NLS) is integrable and its phase space is foliated by tori of finite or infinite dimension with periodic, quasiperiodic, or almost periodic dynamics. The quasiperiodic orbits are usually called *finite-gap solutions.*

Such tori are Lyapunov stable (for all time!) as solutions of (1D-NLS) (as will be clear once we exhibit its integrable structure) and some of them are linearly stable as solutions of (2D-NLS), but we will be interested in their *long-time nonlinear stability* (or lack of it) as invariant objects for the 2D equation (2D-NLS). In fact, we prove in [29] that they are *nonlinearly unstable* (namely Lyapunov unstable) as solutions of (2D-NLS), and in a strong sense, in certain topologies and after very long times. Such instability is transversal in the sense that one drifts along the purely 2-dimensional directions: solutions which are initially very close to 1-dimensional become strongly 2-dimensional after some long time scales.

**1.2. Energy Cascade, Sobolev norm growth, and Lyapunov instability.** In addition to studying long-time dynamics close to invariant objects for NLS, another purpose of the work [29] is to make progress on a fundamental problem in nonlinear wave theory, which is the transfer of energy between characteristically different scales for a nonlinear dispersive PDE. This is called the *energy cascade* phenomenon. It is a purely nonlinear phenomenon (energy is static in frequency space for the linear system), and will be the underlying mechanism behind the long-time instability of the finite gap tori mentioned above.

In [29] we exhibit solutions whose energy moves from very high frequencies towards low frequencies (*backward or inverse cascade*), as well as ones that exhibit cascade in the opposite direction (*forward or direct cascade*). Such cascade phenomena have attracted a lot of attention in the past few years as they are central aspects of various theories of turbulence for nonlinear systems. For dispersive PDE, this goes by the name of *wave turbulence theory* which predicts the existence of solutions (and statistical states) of (2D-NLS) that exhibit a cascade of energy between very different length-scales. In the mathematical community, Bourgain drew attention to such questions of energy cascade by first noting that it can be captured in a quantitative way by studying the behavior of the Sobolev norms of the solution

$$\|u\|_{H^s} = \left( \sum_{n\in\mathbb{Z}^2} (1+|n|)^{2s} |\widehat{u}_n|^2 \right)^{\frac{1}{2}}.$$

In his list of Problems on Hamiltonian PDE [11], Bourgain asked whether there exist solutions that exhibit a quantitative version of the forward energy cascade, namely solutions whose Sobolev norms $H^s$, with $s > 1$, are unbounded in time

$$\sup_{t\geq 0} \|u(t)\|_{H^s} = +\infty, \qquad s > 1. \tag{1.4}$$

(Note that finite time blow up is not possible for (2D-NLS) and therefore the supremum is reached as $t \to +\infty$).

We should point out here that such growth cannot happen for $s = 0$ or $s = 1$ due to the conservation laws of the equations. For other Sobolev indices, there exist polynomial upper bounds for the growth of Sobolev norms (cf. [8, 58, 17, 12, 62, 13, 55, 57, 56, 16, 47]). Nevertheless, results proving actual growth of Sobolev norms are much more scarce. After seminal works by Bourgain himself [8] and Kuksin [38, 37, 39], the landmark result in [15] played a fundamental importance in the recent progress, including our work: It showed that for any $s > 1$, $\delta \ll 1$, $K \gg 1$, there exist solutions $u$ of (2D-NLS) such that

$$\|u(0)\|_{H^s} \leq \delta \quad \text{and} \quad \|u(T)\|_{H^s} \geq K \tag{1.5}$$

for some $T > 0$. Even if not mentioned in that paper, the same techniques also lead to the same result for $s \in (0, 1)$. The paper [15] induced a lot of activity in the area [32, 33, 28, 34, 35, 30] (see also [24, 18, 48, 25, 49, 26, 42] on results about growth of Sobolev norms with different techniques). Despite all that, Bourgain's question about solutions exhibiting (1.4) remains open on $\mathbb{T}^d$ (however a positive answer holds for the cylindrical domain $\mathbb{R} \times \mathbb{T}^d$ with $d \geq 2$, [34]).

The above-cited works revealed an intimate connection between Lyapunov instability and Sobolev norm growth. Indeed, the solution $u = 0$ of (2D-NLS) is an elliptic critical point and is linearly stable in all $H^s$. From this point of view, the result in [15] given in (1.5) can be interpreted as a strong form of Lyapunov instability in $H^s$, $s \neq 1$, of the elliptic critical point $u = 0$ (the first integrals (1.1) and (1.2) imply Lyapunov stability in the $H^1$ and $L^2$ topology). It turns out that this connection runs further, particularly in relation to the question of finding solutions exhibiting (1.4). As was observed in [33], one way to prove the existence of such solutions is to prove that, for sufficiently many $\phi \in H^s$, an instability similar to that in (1.5) holds, but with $\|u(0) - \phi\|_{H^s} \leq \delta$. In other words, proving long-time instability as in (1.5) but with solutions starting $\delta-$close to $\phi$, and for sufficiently many $\phi \in H^s$ implies the existence (and possible genericity) of unbounded orbits satisfying (1.4). Such a program (based on a Baire-Category argument) was applied successfully for the Szegő equation on $\mathbb{T}$ in [26].

Motivated by this, one is naturally led to studying the Lyapunov instability of more general invariant objects of (2D-NLS) (or other Hamiltonian PDEs), or equivalently to investigate whether one can achieve Sobolev norm explosion starting arbitrarily close to a given invariant object. The first work in this direction is by one of the authors [33]. He considers the plane waves $u(t, x) = Ae^{\mathrm{i}(mx-\omega t)}$ with $\omega = m^2 + A^2$, periodic orbits of (2D-NLS), and proves that there are orbits which start $\delta$-close to them and undergo $H^s$ Sobolev norm explosion, $0 < s < 1$. This implies that the plane waves are Lyapunov unstable in these topologies. Stability results for plane waves in $H^s$, $s > 1$, on shorter time scales are provided in [22].

The next step in this program would be to study such instability phenomena near higher dimensional invariant objects, namely quasiperiodic orbits. This is the purpose of [29], in which we addressed this question for the family of finite-gap tori of (1D-NLS) as solutions to the (2D-NLS). To control the linearized dynamics around such tori, we impose some Diophantine (strongly non-resonant) conditions on the quasiperiodic frequency parameters. This allows us to obtain a stable linearized operator (at least with respect to the perturbations that we consider), which is crucial to control the delicate construction of the unstable nonlinear dynamics.

**1.3. Statement of results.** Roughly speaking, we construct solutions to (2D-NLS) that start very close to the finite-gap tori in appropriate topologies, and exhibit either backward cascade of energy from high to low frequencies, or forward cascade of energy from low to high frequencies. In the former case, the solutions that exhibit backward cascade start in an arbitrarily small vicinity of a finite-gap torus in Sobolev spaces $H^s(\mathbb{T}^2)$ with $0 < s < 1$, but grow to become larger than any pre-assigned factor $K \gg 1$ in the same $H^s$ (higher Sobolev norms $H^s$ with $s > 1$ decrease, but they are large for all times). In the latter case, the solutions that exhibit forward cascade start in an arbitrarily small vicinity of a finite-gap torus in $L^2(\mathbb{T}^2)$, but their $H^s$ Sobolev norm (for $s > 1$) exhibits a growth by a large multiplicative factor $K \gg 1$ after a large time.

To state the result, we introduce the Birkhoff coordinates for equation 1D-NLS. Grébert and Kappeler showed in [27] that there exists a globally defined map, called the Birkhoff map, such that $\forall s \geq 0$

$$\begin{aligned} \Phi : H^s(\mathbb{T}) &\longrightarrow \ h^s(\mathbb{Z}) \times h^s(\mathbb{Z}) \\ q \quad &\longmapsto \ (z_m, \overline{z}_m)_{m \in \mathbb{Z}}, \end{aligned} \tag{1.6}$$

and equation (1D-NLS) is transformed in the new coordinates $(z_m, \overline{z}_m)_{m\in\mathbb{Z}} = \Phi(q)$ to:

$$\mathrm{i}\dot{z}_m = \alpha_m(I) z_m \tag{1.7}$$

where $I = (I_m)_{m\in\mathbb{Z}}$ and $I_m = |z_m|^2$ are the actions, which are conserved in time (since $\alpha_m(I) \in \mathbb{R}$). Therefore in these coordinates, called Birkhoff coordinates, equation (1D-NLS) becomes a trivially integrable chain of oscillators and it is clear that the phase space is foliated by finite and infinite dimensional tori with periodic, quasiperiodic or almost periodic dynamics, depending on how many of the actions $I_m$ (which are constant!) are nonzero and on the properties of rational dependence of the frequencies.

We are interested in the finite dimensional tori with quasiperiodic dynamics. Fix $\mathtt{d} \in \mathbb{N}$ and consider a set of modes

$$\mathcal{S}_0 = \{\mathtt{m}_1, \dots, \mathtt{m}_\mathtt{d}\} \subset \mathbb{Z} \times \{0\}. \tag{1.8}$$

Fix also a value for the actions $I_{\mathtt{m}_i} = I^0_{\mathtt{m}_i}$ for $i = 1, \dots \mathtt{d}$. Then we can define the $\mathtt{d}$-dimensional torus

$$\mathtt{T}^\mathtt{d} = \mathtt{T}^\mathtt{d}(\mathcal{S}_0, I^0_m) = \left\{ z \in \ell^2 : |z_{\mathtt{m}_i}|^2 = I^0_{\mathtt{m}_i}, \ \text{ for } \ i = 1, \dots, \mathtt{d}, \quad z_m = 0 \ \text{ for } \ m \notin \mathcal{S}_0 \right\}, \tag{1.9}$$

which is supported on the set $\mathcal{S}_0$. Any orbit on this torus is quasiperiodic (or periodic if the frequencies of the rigid rotation are completely resonant). We will impose conditions to have non-resonant quasiperiodic dynamics. This will imply that the orbits on $\mathtt{T}^{\mathtt{d}}$ are dense. By equation (1.7), it is clear that this torus, as an invariant object of equation (1D-NLS), is stable for this equation for all times in the sense of Lyapunov.

The torus (1.9) (actually, its pre-image $\Phi^{-1}(\mathtt{T}^{\mathtt{d}})$ through the Birkhoff map) is also an invariant object for the original equation (2D-NLS). Our main result shows, under certain assumptions on the choices of modes (1.8) and actions (1.9), the instability (in the sense of Lyapunov) of this invariant object. Even more, we prove the existence of orbits which start arbitrarily close to these tori and undergo an arbitrarily large $H^s$-norm explosion.

We will abuse notation, and identify $H^s(\mathbb{T})$ with the closed subspace of $H^s(\mathbb{T}^2)$ of functions depending only on the $x$ variable. Consequently, $\mathcal{T}^{\mathtt{d}} := \Phi^{-1}(\mathtt{T}^{\mathtt{d}})$ (see (1.6)) is a closed torus of $H^s(\mathbb{T}) \subset H^s(\mathbb{T}^2)$.

Before stating the theorem, we give a definition.

**Definition 1.1** ($\mathtt{L}-$genericity)**.** *Given $\mathtt{L} \in \mathbb{N}$, we say that $\mathcal{S}_0$ is* $\mathtt{L}$-generic *if it satisfies the condition*

$$\sum_{i=1}^{\mathtt{d}} \ell_i \mathtt{m}_i \neq 0 \qquad \forall\, 0 < |\ell| \leq \mathtt{L}. \tag{1.10}$$

The result that we prove in [29] is the following.

**Theorem 1.2** ([29])**.** *Fix a positive integer $\mathtt{d}$. For any choice of $\mathtt{d}$ modes $\mathcal{S}_0$ (see* (1.8)*) which is $\mathtt{L}$-generic (with $\mathtt{L}$ sufficiently large), there exists $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$ there exists a positive measure Cantor-like set $\mathcal{I} \subset (\varepsilon/2, \varepsilon)^{\mathtt{d}}$ of actions, for which the following holds true for any torus $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_m^0)$ with $I_m^0 \in \mathcal{I}$:*

1. *For any $s \in (0,1)$, $\delta > 0$ small enough, and $K > 0$ large enough, there exists an orbit $u(t)$ of* (2D-NLS) *and a time*
$$0 < T \leq e^{\left(\frac{K}{\delta}\right)^{\beta}}$$
*such that $u(0)$ is $\delta$-close to the torus $\mathcal{T}^{\mathtt{d}} := \Phi^{-1}(\mathtt{T}^{\mathtt{d}})$ in $H^s(\mathbb{T}^2)$ and $\|u(T)\|_{H^s} \geq K$. Here $\beta > 1$ is independent of $K, \delta$.*

2. *For any $s > 1$, and any $K > 0$ large enough, there exists an orbit $u(t)$ of* (2D-NLS) *and a time*
$$0 < T \leq e^{K^{\sigma}}$$
*such that*
$$\operatorname{dist}\left(u(0), \mathcal{T}^{\mathtt{d}}\right)_{L^2(\mathbb{T}^2)} \leq K^{-\sigma'} \qquad \text{and} \qquad \|u(T)\|_{H^s(\mathbb{T}^2)} \geq K \|u(0)\|_{H^s(\mathbb{T}^2)}.$$
*Here $\sigma, \sigma' > 0$ are independent of $K$.*

## 2 Comments and remarks on Theorem 1.2

1. *Why does the finite gap solution need to be small?* To prove Theorem 1.2 we need to analyze the linearization of equation (2D-NLS) at the finite gap solution. Roughly speaking, this leads to a Schrödinger equation with a small quasi-periodic potential. Luckily, such operators can be *reduced* to constant coefficients via a KAM scheme. This is known as perturbative *reducibility theory* which allows one to construct a change of variables that casts the linearized operator into an essentially constant coefficient diagonal one. This KAM scheme was carried out in [44], and requires the quasi-periodic potential, given by the finite gap solution here, to be small for the KAM iteration to converge. That being said, we suspect a similar result to be true for non-small finite gap solutions.

2. *Why do we need a finite gap solution?* As we explained above, we need to analyze the linearization of equation (2D-NLS) at the finite gap solution. Such operator is the linear, time dependent Schrödinger equation

$$\mathrm{i}\partial_t v = -\Delta v + 2|q(\omega t, x)|^2 v + q(\omega t, x)^2 \bar{v}. \tag{2.1}$$

Actually, we not only need to cast (2.1) to diagonal constant coefficients (as mentioned above), but we also need quite precise asymptotics on the resulting eigenvalues $\{\Omega_{\vec{\jmath}}\}_{\vec{\jmath}\in\mathbb{Z}^2\setminus\mathcal{S}_0}$. This is a notoriously difficult problem on $\mathbb{T}^2$; however, the fact that $q(\omega t, x)$ depends only on one spatial variable allowed us to extend to the two dimensional setting the techniques of reduction in orders developed in the one dimensional setting to deal with quasi-linear equations (see e.g. [1] and reference therein).

3. *Why do we need the* $\mathtt{L}$*-genericity condition?* We need to extract from the original dynamical system the resonant terms which are responsible for the instability. This is achieved by a normal form procedure which removes all the nonresonant interactions up to order four (in the Hamiltonian). It turns out that some of these interactions can be eliminated only exploiting (1.10), as we shall explain in section 3. Let us explain in which sense such condition is generic. Identify the set $\mathcal{S}_0 \subset \mathbb{Z}^{\mathtt{d}}$ with a point $\vec{m} \in \mathbb{C}^{\mathtt{d}}$ and fix a vector $\ell = (\ell_1, \dots, \ell_{\mathtt{d}})$ with $|\ell| \leq \mathtt{L}$. Then (1.10) means that $\vec{m}$ does not belong to the hyperplane orthogonal to the vector $\ell$. Therefore condition (1.10) is equivalent to say that the point $\vec{m}$ does not belong to a finite number of hyperplanes; this is a notion of algebraic genericity.

4. The relative measure of the set $\mathcal{I}$ of admissible actions can be taken as close to 1 as desired. Indeed, by taking smaller $\varepsilon_*$, one has that the relative measure satisfies

$$|1 - \mathrm{Meas}(\mathcal{I})| \leq C\varepsilon_*^{\kappa}$$

for some constant $C > 0$ and $0 < \kappa < 1$ independent of $\varepsilon_* > 0$. The genericity condition on the set $\mathcal{S}_0$ and the actions $(I_{\mathtt{m}})_{\mathtt{m}\in\mathcal{S}_0} \in \mathcal{I}$ ensures that the *linearized* dynamics around the resulting torus $\mathcal{T}^{\mathtt{d}}$ is stable under those perturbations that we use to induce the nonlinear instability. In fact, a subset of those tori is even linearly stable under much more general perturbations, as we remark below.

5. To put the complexity of this result in perspective, it is instructive to compare it with the stability result in [44]. In that paper, it is shown that a proper *subset* $\mathcal{I}' \subset \mathcal{I}$ of the tori considered in Theorem 1.2 are Lyapunov stable in $H^s$, $s > 1$, but for shorter time scales than those considered in this theorem. More precisely, all orbits that are initially $\delta$-close to $\mathcal{T}^{\mathtt{d}}$ in $H^s$ stay $C\delta$-close for some fixed $C > 0$ for time scales $t \sim \delta^{-2}$. The same stability result (with a completely identical proof) holds if we replace $H^s$ by $\mathcal{F}\ell_1$ norm (functions whose Fourier series is in $\ell^1$). In fact, by trivially modifying the proof, one could also prove stability on the $\delta^{-2}$ timescale in $\mathcal{F}\ell_1 \cap H^s$ for $0 < s < 1$. What this means is that the solutions in the first part of Theorem 1.2 remains within $C\delta$ of $\mathtt{T}^{\mathtt{d}}$ up to times $\sim \delta^{-2}$ but can diverge vigorously afterwards at much longer time scales.

It is also worth mentioning that the complementary subset $\mathcal{I}\setminus\mathcal{I}'$ has a positive measure subset where tori are linearly unstable since they possess a finite set of modes that exhibit hyperbolic behavior. In principle, hyperbolic directions are good for instability, but they are not useful for our purposes since they live at very low frequencies, and hence cannot be used (at least not by themselves alone) to produce a substantial growth of Sobolev norms. We avoid dealing with these linearly unstable directions by restricting our solution to an invariant subspace on which these modes are at rest.

6. It is expected that a similar statement to the first part of Theorem 1.2 is also true for $s > 1$. This would be a stronger instability compared to that in the second part (for which the initial perturbation is small in $L^2$ but not in $H^s$). Nevertheless, this case cannot be tackled with the techniques considered in this paper. Indeed, one of the key points in the proof is to perform a (partial) Birkhoff normal form up to order 4 around the finite gap solution. The terms which lead to the instabilities in Theorem 1.2 are quasi-resonant instead of being completely resonant. Working in the $H^s$ topology with $s \in (0, 1)$, such terms can be considered completely resonant with little error on the timescales where instability happens. However, this cannot be done for $s > 1$, for which one might be able to eliminate those terms by a higher order normal form ($s > 1$ gives a stronger topology and can thus handle worse small divisors). This would mean that one needs other resonant terms to achieve growth of Sobolev norms. The same difficulties were encountered in [33] to prove the instability of the plane waves of (2D-NLS).

7. For finite dimensional Hamiltonian dynamical systems, proving Lyapunov instability for quasi-periodic Diophantine elliptic (or maximal dimensional Lagrangian) tori is an extremely difficult task. Actually all the obtained results

[14, 31] deal with $C^r$ or $C^\infty$ Hamiltonians, and not a single example of such instability is known for analytic Hamiltonian systems. In fact, there are no results of instabilities in the vicinity of non-resonant Diophantine elliptic critical points or periodic orbits for analytic Hamiltonian systems (see [41, 20, 36] for results on the $C^\infty$ topology). The present paper proves the existence of unstable Diophantine elliptic tori in an analytic infinite dimensional Hamiltonian system. Obtaining such instabilities in infinite dimensions is, in some sense, easier: having infinite dimensions gives "more room" for instabilities.

8. It is well known that many Hamiltonian PDEs possess quasiperiodic invariant tori [61, 50, 40, 9, 6, 21, 23, 5, 60, 54, 7, 52, 51, 1]. Most of these tori are normally elliptic and thus linearly stable. It is widely expected that the behavior given by Theorem 1.2 also arises in the neighborhoods of (many of) those tori. Nevertheless, it is not clear how to apply the techniques of the present paper to these settings.

9. A natural question is the role played by the integrability of (1D-NLS) in our problem. For example, could we reproduce our result for the quintic NLS on $\mathbb{T}^2$? Let us remark immediately that the one dimensional quintic NLS is not an integrable system, so finite gap solutions do not exist. However, the one dimensional quintic NLS has small invariant tori carrying quasi-periodic solutions, which can be constructed by KAM methods. Such tori are supported in Fourier coordinates only on a finite set of modes, and of course they are invariant objects, depending only on 1 spatial variable, of the quintic NLS on $\mathbb{T}^2$. Therefore at least the analysis of the linearized operator should be very similar to our case, and one should be able to prove long time Lyapunov stability (as in [44]). Concerning the strong instability, it might be possible to obtain some results exploiting the analysis of [35, 30].

10. The next step of the program would be to prove strong nonlinear instability close to more complicated invariant objects, for example KAM tori of NLS on $\mathbb{T}^2$. Such tori were constructed by Procesi-Procesi [51], and are filled with quasiperiodic solutions depending on both spatial variables. Moreover, [51] proves that the linearized NLS at a KAM torus can be reduced to diagonal constant coefficients. However, at the moment the asymptotics of the eigenvalues is not yet completely understood, and, at the present level, it seems very difficult to reproduce the instability mechanism.

11. The problem of giving upper and lower bounds for the Sobolev norms is interesting also in the case of linear *time dependent* Schrödinger equations. Upper bounds were given in [10, 59, 18, 45, 46, 2] for different Schrödinger-like equations. There are also results which engineer time dependent potentials which provoke unbounded growth of Sobolev norms of the solutions, see [10, 19, 3, 42]. It is not clear if and how one could exploit the ideas of these papers in a nonlinear setup.

## 3 Ideas of the proof

Let us explain the main steps to prove Theorem 1.2.

**1. Analysis of the 1-dimensional cubic Schrödinger equation.** We express the 1-dimensional cubic NLS in terms of the Birkhoff coordinates. We need a quite precise knowledge of the Birkhoff map (1.6). In particular, we need that it "behaves well" in $\ell^1$; this is proved in the paper [43], exploiting a quantitative Vey-type theorem proved in [4]. This property is needed since we need to extract from the original Hamiltonian a resonant normal form. Such a normal form is obtained after a series of transformations which are well defined in $B(r) \cap h^1$, where $B(r)$ is a ball of center zero and radius $r$ in the $\ell^1(\mathbb{Z}^2 \setminus \mathcal{S}_0)$ topology. Hence we need also the Birkhoff map to be well defined and analytic in this space.

In Birkhoff coordinates, the finite gap solutions are supported in a finite set of variables. We use such coordinates to express the Hamiltonian (1.1) in a more convenient way.

**2. Reducibility of the 2-dimensional cubic NLS around a finite gap solution.** We reduce the linearization of the vector field around the finite gap solutions to a constant coefficients diagonal (in Fourier space) operator with eigenvalues $\{\Omega_{\vec{\jmath}}\}_{\vec{\jmath} \in \mathbb{Z}^2 \setminus \mathcal{S}_0}$. It is very important to have good asymptotics of these eigenvalues. Roughly speaking, we prove that they look like

$$\Omega_{\vec{\jmath}} = |\vec{\jmath}|^2 + O(J^{-2}) \tag{3.1}$$

for frequencies $\vec{\jmath} = (m, n)$ satisfying $|m|, |n| \sim J$. This seemingly harmless $O(J^{-2})$ correction to the unperturbed Laplacian eigenvalues is sharp and will be responsible for the restriction to $s \in (0, 1)$ in the first part of Theorem 1.2 as we shall explain below.

**3. Degree three Birkhoff normal form around the finite gap solution.** We remove completely third order interactions. This requires third order nonresonance conditions of the form

$$|\omega \cdot \ell \pm \Omega_{\vec{\jmath}_1} \pm \Omega_{\vec{\jmath}_2} \pm \Omega_{\vec{\jmath}_3}| \geq \frac{\varepsilon \gamma}{\langle \ell \rangle^\tau}$$

where $\omega$ is the frequency of the finite gap and $\gamma, \tau$ are positive numbers. The fact that one can impose such conditions on a set of large measure of the actions is proved in [44]. Actually we shall need more precise information from this normal form that will be crucial for Steps 4 and 5 below. In particular we need to keep track of the fourth order terms which are generated by the third order Birkhoff normal form.

**4. Partial normal form of degree four.** We remove all degree four monomials which are not (too close to) resonant. This requires fourth order nonresonance conditions of the form

$$|\omega \cdot \ell \pm \Omega_{\vec{\jmath}_1} \pm \Omega_{\vec{\jmath}_2} \pm \Omega_{\vec{\jmath}_3} \pm \Omega_{\vec{\jmath}_4}| \geq \frac{\varepsilon\gamma}{\langle \ell \rangle^\tau}. \tag{3.2}$$

It is here that we need the $\mathtt{L}$-genericity. Let us explain briefly why. Consider a monomial of the form $e^{i\theta\cdot\ell} a_{\vec{\jmath}_1}^{\sigma_1} a_{\vec{\jmath}_2}^{\sigma_2} a_{\vec{\jmath}_3}^{\sigma_3} a_{\vec{\jmath}_4}^{\sigma_4}$ whose indexes $\vec{\jmath}_1$, $\vec{\jmath}_2$, $\vec{\jmath}_3$, $\vec{\jmath}_4$ have the first components in the set $\mathcal{S}_0$. Here, for a complex number $z$, we used the notation

$$z^\sigma = \begin{cases} z & \text{if } \sigma = +1, \\ \bar{z} & \text{if } \sigma = -1. \end{cases}$$

We prove that any eigenvalue corresponding to a site $\vec{\jmath} = (\mathtt{m}, n) \in \mathcal{S}_0 \times \mathbb{Z}$ has the form $\Omega_{\vec{\jmath}} = \varepsilon \mu_{\mathtt{m}}(I) + n^2 + O(\varepsilon^2)$, where $\mu_{\mathtt{m}}(I)$ is an algebraic function of the actions. If we want to remove the monomial $e^{i\theta\cdot\ell} a_{\vec{\jmath}_1}^{\sigma_1} a_{\vec{\jmath}_2}^{\sigma_2} a_{\vec{\jmath}_3}^{\sigma_3} a_{\vec{\jmath}_4}^{\sigma_4}$ we must check that (3.2) holds for the corresponding $(\ell, \vec{\jmath}_1, \vec{\jmath}_2, \vec{\jmath}_3, \vec{\jmath}_4)$. A necessary condition is that the term of order $\varepsilon$ in the l.h.s. of (3.2) does not vanish identically. We prove in [29] that this is equivalent to requiring that the function

$$-I \cdot \ell + \sigma_1 \mu_{\mathtt{m}_{i_1}}(I) + \sigma_2 \mu_{\mathtt{m}_{i_2}}(I) + \sigma_3 \mu_{\mathtt{m}_{i_3}}(I) + \sigma_4 \mu_{\mathtt{m}_{i_4}}(I) \tag{3.3}$$

is not identically zero as a function of the actions $I$, for any $\ell \in \mathbb{Z}^d$ so that

$$\sum_{i=1}^{\mathtt{d}} \mathtt{m}_i \ell_i = 0. \tag{3.4}$$

Unfortunately, we do not have the explicit expression of the $\mu_{\mathtt{m}_i}(I)$ (we only know they are roots of an irreducible polynomial), so we argue following [53]. Any choice of $\mathtt{m}_{i_1}, \ldots, \mathtt{m}_{i_4}$ and of $\sigma_1, \ldots, \sigma_4$ for which (3.3) is identically zero automatically fixes $\ell$ uniquely. But then it is sufficient to choose $(\mathtt{m}_1, \ldots, \mathtt{m}_{\mathtt{d}})$ so that condition (3.4) is not achieved. We have to repeat this argument for any (possible but finite) choice of $\mathtt{m}_i$ and $\sigma_i$ in (3.3), which explains condition (1.10).

A further remark is that if the indexes $(\vec{\jmath}_1, \vec{\jmath}_2, \vec{\jmath}_3, \vec{\jmath}_4)$ form a rectangle in $\mathbb{Z}^2$ and all their components have large modulus, then the monomial $a_{\vec{\jmath}_1} \overline{a_{\vec{\jmath}_2}} a_{\vec{\jmath}_3} \overline{a_{\vec{\jmath}_4}}$ is almost resonant (see formula (3.1)). This is a key point of our argument: the normal form around a finite gap solution is somehow "close" to the resonant form of NLS around the zero solution. This allows us to exploit in our model the instability mechanism of [15].

**5. Reduction to the Toy model.** We follow the paradigm set forth in [15, 32] to construct solutions to the truncated Hamiltonian consisting of the (close to) resonant degree-four terms isolated above, and then afterwards to the full Hamiltonian by an approximation argument. This construction will be done at frequencies $\vec{\jmath} = (m, n)$ such that $|m|, |n| \sim J$ with $J$ very large, and for which the dynamics is effectively given by the following system of ODEs

$$\begin{cases} \mathrm{i}\dot{a}_{\vec{\jmath}} & = -|a_{\vec{\jmath}}|^2 a_{\vec{\jmath}} + \sum_{\mathcal{R}(\vec{\jmath})} C_{\vec{\jmath}_1\vec{\jmath}_2\vec{\jmath}_3\vec{\jmath}}\, a_{\vec{\jmath}_1} \overline{a_{\vec{\jmath}_2}} a_{\vec{\jmath}_3} e^{\mathrm{i}\Gamma t} \\ \mathcal{R}(\vec{\jmath}) & := \{(\vec{\jmath}_1, \vec{\jmath}_2, \vec{\jmath}_3) \in \mathbb{Z}^2 \setminus \mathcal{S}_0 : \vec{\jmath}_1, \vec{\jmath}_3 \neq \vec{\jmath}, \quad \vec{\jmath}_1 - \vec{\jmath}_2 + \vec{\jmath}_3 = \vec{\jmath}, \\ & \qquad |\vec{\jmath}_1|^2 - |\vec{\jmath}_2|^2 + |\vec{\jmath}_3|^2 = |\vec{\jmath}|^2\} \\ \Gamma & := \Omega_{\vec{\jmath}_1} - \Omega_{\vec{\jmath}_2} + \Omega_{\vec{\jmath}_3} - \Omega_{\vec{\jmath}}. \end{cases} \tag{3.5}$$

We remark that the conditions of the set $\mathcal{R}(\vec{\jmath})$ are essentially equivalent to saying that $(\vec{\jmath}_1, \vec{\jmath}_2, \vec{\jmath}_3, \vec{\jmath})$ form a rectangle in $\mathbb{Z}^2$. Also note that by the asymptotics of $\Omega_{\vec{\jmath}}$ mentioned above in (3.1), one obtains that $\Gamma = O(J^{-2})$ if all the frequencies involved are in $\mathcal{R}(\vec{\jmath})$ and satisfy $|m|, |n| \sim J$. The idea now is to reduce this system into a finite dimensional system called the "Toy Model" which is tractable enough for us to construct a solution that cascades energy. The first step of this reduction is to carefully select a frequency set $\Lambda \subset \cup_{\vec{\jmath}} \mathcal{R}_{\vec{\jmath}}$, invariant by the dynamics of (3.5) and whose frequencies have $|m|, |n| \sim J$, in such a way that the system of ODEs (3.5) collapses to the Toy model of [15].

There are two obstructions to this plan. The first is represented by the presence of the coefficients $C_{\vec{\jmath}_1\vec{\jmath}_2\vec{\jmath}_3\vec{\jmath}}$ in front of the nonlinearity. In order to reduce the system of ODEs to the Toy Model of [15], we need $C_{\vec{\jmath}_1\vec{\jmath}_2\vec{\jmath}_3\vec{\jmath}} \approx 1$. This is achieved by inspecting carefully the terms which are generated by the third order Birkhoff normal form (see Step 3) and by some combinatorial arguments on the choice of the sites in $\Lambda$. For example, we ask that two points $\vec{\jmath}_1, \vec{\jmath}_2 \in \Lambda$ form a right angle with a point $(m, 0) \in \mathbb{Z} \times \{0\}$ only if $|m| \geq J$. This analysis allows us to prove that $C_{\vec{\jmath}_1\vec{\jmath}_2\vec{\jmath}_3\vec{\jmath}} = 1 + O(J^{-2})$.

A second obstruction is presented by the presence of the oscillating factor $e^{\mathrm{i}\Gamma t}$ for which $\Gamma$ is not zero (in contrast to [15]) but rather $O(J^{-2})$. The way we deal with this problem is to approximate $e^{\mathrm{i}\Gamma t} \sim 1$ which is only possible provided $J^{-2}T \ll 1$. The solution coming from the Toy Model is supported on a finite number of modes $\vec{\jmath} \in \mathbb{Z}^2 \setminus \mathcal{S}_0$ satisfying $|j| \sim J$, and the time it takes for the energy to diffuse across its modes is $T \sim O(\nu^{-2})$ where $\nu$ is the characteristic size of the modes in $\ell^1$ norm. Requiring the solution to be initially close in $H^s$ to the finite gap would necessitate that $\nu J^s \lesssim \delta$ which gives that $T \gtrsim_\delta J^{-2s}$, and hence the condition $J^{-2}T \ll 1$ translates into the condition $s < 1$. This explains the restriction to $s < 1$ in the first part of Theorem 1.2. If we only require our solutions to be close to the finite gap in $L^2$, then no such restriction on $\nu$ is needed, and hence there is no restriction on $s$ beyond being $s > 0$ and $s \neq 1$, which is the second part of the theorem.

**6. Approximation arguments.** As a last step we perform the approximation argument allowing to shadow the Toy Model solution mentioned above with a solution of (2D-NLS) exhibiting the needed norm growth, thus completing the proof of Theorem 1.2.

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