Growth of Sobolev norms in time dependent semiclassical anharmonic oscillators

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Abstract

We consider the semiclassical Schrödinger equation on $\mathbb{R}^d$ given by

$$\mathrm{i}\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2}\Delta + W_l(x)\right)\psi + V(t,x)\psi,$$

where W_l is an anharmonic trapping of the form $W_l(x) = \frac{1}{2l}\sum_{j=1}^d x_j^{2l}$, $l \geq 2$ is an integer and $\hbar$ is a semiclassical small parameter. We construct a smooth potential $V(t,x)$, bounded in time with its derivatives, and an initial datum such that the Sobolev norms of the solution grow at a logarithmic speed for all times of order $\log^{\frac{1}{2}}(\hbar^{-1})$. The proof relies on two ingredients: first we construct an unbounded solution to a forced mechanical anharmonic oscillator, then we exploit semiclassical approximation with coherent states to obtain growth of Sobolev norms for the quantum system which are valid for semiclassical time scales.

1 Introduction and statement

In this paper we consider the semiclassical Schrödinger equation on $\mathbb{R}^d$, $d \geq 1$, given by

$$\mathrm{i}\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2}\Delta + W_l(x)\right)\psi + V(t,x)\,\psi\ , \qquad x \in \mathbb{R}^d\ , \tag{1.1}$$

where $W_l(x)$ is the anharmonic trapping potential

$$W_l(x) := \frac{1}{2l}\sum_{j=1}^d x_j^{2l}, \qquad l \in \mathbb{N}, \quad l \geq 2,$$

and $\hbar \in (0,1]$ is a semiclassical parameter. We construct a time dependent perturbation

$$V(t,x) := \beta(t)x_1, \tag{1.2}$$

with $\beta : \mathbb{R} \to \mathbb{R}$ smooth and bounded with its derivatives, so that (1.1) has a solution whose Sobolev-like norms grow at a logarithmic speed for all times of order $\log^{\frac{1}{2}}(K\hbar^{-1})$, which is a scale slightly shorter that the Ehrenfest time.

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The norms we use to measure the solution are the spectral ones associated with the anharmonic quantum oscillator

$$H_l := 1 - \hbar^2\Delta + W_l(x) \ . \tag{1.3}$$

More precisely we define the scale of Hilbert spaces $\mathcal{H}^r \equiv \mathcal{H}^r(\mathbb{R}^d) := D(H_l^{r/2})$ (domain of $H_l^{r/2}$) for $r \geq 0$, which we equip with the Sobolev norms[1]

$$\|u\|_r := \|H_l^{r/2} u\|_{L^2(\mathbb{R}^d)} < \infty, \qquad \forall r \geq 0. \tag{1.4}$$

The negative spaces $\mathcal{H}^{-r}$ are defined by duality with the $L^2(\mathbb{R}^d)$ scalar product. We also denote $\mathcal{H}^\infty := \cap_r \mathcal{H}^r$.

Our main result is the following one:

Theorem 1.1. *There exist $\psi_0 \in \mathcal{H}^\infty$ and a function $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ fulfilling*

$$\sup_{t\in\mathbb{R}} \left|\partial_t^\ell \beta(t)\right| < +\infty, \qquad \forall \ell \in \mathbb{N}_0, \tag{1.5}$$

such that the following is true. Denote by $\psi(t)$ the solution of equation (1.1) *with $V(t,x) = \beta(t)x_1$ and initial datum ψ_0. Fix an arbitrary $r \in \mathbb{N}$. Then there exist $\hbar_0, K_1, K_2, K_3 > 0$ such that $\forall \hbar \in (0, \hbar_0]$ one has*

$$\|\psi(t)\|_r \geq K_1 \Big[\log(2+t)\Big]^r \tag{1.6}$$

for all times

$$2 \leq t \leq K_2 \Big[\log\left(\frac{K_3}{\hbar}\right)\Big]^{\frac{1}{2}}. \tag{1.7}$$

While in the last few years there has been lot of activity aiming to obtain upper bounds on the growth of Sobolev norms [34, 8, 12, 31, 4, 32, 6], there are only few results [13, 3, 30] which give lower bounds: Theorem 1.1 goes in this direction, by exhibiting a solution of (1.1) whose norms increase for long but finite time.

The main difficulty in dealing with equation (1.1) is that very few is known on the spectrum of the unperturbed operator $-\frac{\hbar^2}{2}\Delta + W_l(x)$. In particular we are not aware of any asymptotic expansion of its eigenvalues (a property that plays an important role in [3, 30]).

In order to circumvent this problem, the idea is to exploit semiclassical approximation in a way that now we briefly describe. Equation (1.1) with $V(t,x) = \beta(t)x_1$ is the quantization of the classical Hamiltonian

$$H(t,q,p) = \frac{|p|^2}{2} + W_l(q) + \beta(t)q_1, \qquad q, p \in \mathbb{R}^d, \tag{1.8}$$

whose equations of motion are given by

$$\ddot{q}_1 + q_1^{2l-1} = -\beta(t), \qquad \ddot{q}_j + q_j^{2l-1} = 0, \quad \forall\, 2 \leq j \leq d. \tag{1.9}$$

We show, modyfing a construction of Levi and Zehnder [27], that it is possible to construct $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ bounded with all its derivatives and an initial datum $(q_0, p_0) \in \mathbb{R}^{2d}$ such that the solution of (1.9) with such an initial datum is unbounded; actually we show that the energy

$$E(q,p) := \frac{|p|^2}{2} + W_l(q) \tag{1.10}$$

[1] It turns out that such a space is equivalent to the space of functions

$$\left\{u \in L^2(\mathbb{R}^n) \colon \quad \|(1-\hbar\Delta)^{r/2}u\|_{L^2(\mathbb{R}^d)} + \|(1+|x|)^{rl}u\|_{L^2(\mathbb{R}^d)} < +\infty\right\},$$

see e.g. [37].

along such a solution grows at a logarithmic speed as $t \to \infty$.
The next step is to use the theory of semiclassical approximation with coherent states to convert dynamical information on the mechanical system (1.9) to the quantum system (1.1). This is done in two steps. First we construct an approximate solution of (1.1) using coherent states. A coherent state is a Gaussian packet which stays localized in the phase space along the trajectory of the mechanical system (1.9) till the Ehrenfest time (see e.g. [20, 11, 2, 9, 7]). As a consequence of the dynamics of (1.1), we are able to construct a coherent state which oscillates on longer and longer distances, provoking a growth of its Sobolev norms.
The second step is to show that there exists a solution of (1.1) which stays close, in the $\mathcal{H}^r$ topology, to such coherent state for all times in (1.7). This is done by extending classical results of semiclassical approximation [20, 11] to the $\mathcal{H}^r$ topology, a result which we think might be interesting in its own.

Theorem 1.1 extends partially to anharmonic oscillators a result of [3], which, in case of the quantum harmonic oscillators on $\mathbb{R}^d$, constructs solutions with unbounded path in Sobolev spaces. More precisely, in [3] it is proved that all the solutions of equation (1.1) with $l = 1$ (namely harmonic oscillators) and with

$$V(t,x) := \frac{a}{2}\sin(t)x_1, \qquad a \neq 0 \tag{1.11}$$

have Sobolev norms growing at a polynomial speed:

$$\|\psi(t)\|_r \geq C_r(1+t)^{2r}, \qquad \forall t \gg 1. \tag{1.12}$$

Remark that, in this case, the growth of Sobolev norms happens for all initial data, for all times and at a polynomial speed. The reason is that for system (1.1) with $l = 1$ and V as in (1.11) the classical–semiclassical correspondence is exact and valid for all times, a property first exploited by Enss and Veselić [15]. This is also the mechanism exploited in [3], which ultimately is based on the fact that (1.9) with $l = 1$ and $\beta(t) = \sin t$ is a resonant system, whose solutions are unbounded (see also [13, 30] for different examples of perturbations provoking growth of Sobolev norms).

In case $l \geq 2$, the classical-semiclassical correspondence is valid only for finite times, and the speed of growth of Sobolev norms is logarithmic and not polynomial in time. This is in accordance with the known upper bounds; in particular, in dimension $d = 1$, it is proved in [4] that each solution of (1.1) grows at most subpolynomially in time, in the sense that $\forall \epsilon, r > 0$, there exists a constant $C_{r,\epsilon} > 0$ such that each solution of (1.1) fufills

$$\|\psi(t)\|_r \leq C_{r,\epsilon}\,(1+|t|)^{\epsilon}, \quad \forall |t| \geq 1. \tag{1.13}$$

If the map $t \mapsto \beta(t)$ is real analytic in time, the subpolynomial bound (1.13) can be improved into a logarithmic one [31]:

$$\|\psi(t)\|_r \leq C_r\,\left[\log(1+|t|)\right]^{\frac{rl}{l-2}}, \quad \forall |t| \geq 1. \tag{1.14}$$

Remark that Theorem 1.1 almost saturates the upper bound, at least for finite but long time intervals. We are not aware of any results in which Sobolev norm explosion is achieved for all times.

In our opinion, our approach raises an interesting question: to which extent can dynamical properties of mechanical systems be converted into quantum analogous? Remark that mechanical systems of the form $\ddot{q}_1 + q_1^{2l-1} = \beta(t)$ or similar have been widely studied in the literature, and conditions on $\beta(t)$ are known to guarantee either the boundedness of all solutions, or the existence of unbounded ones, see e.g. [28, 27, 1, 24, 33, 14, 25, 26, 38, 36] and reference therein. For example if $t \mapsto \beta(t)$ is periodic or quasi-periodic in time with a Diophantine frequency vector and $d = 1$, then each orbit of (1.9) in bounded [38].

Before closing this introduction let us mention that the construction of unbounded orbits in *nonlinear* Schrödinger equations is an extremely difficult and challeging problem. A first breakthrough was achieved in [10], which constructs solutions of the cubic nonlinear Schrödinger equation on $\mathbb{T}^2$ whose Sobolev norms become arbitrary large (see also [21, 17, 23, 19, 18] for generalizations of this result). At the moment, existence of unbounded orbits has only been proved by Gérard and Grellier [16] for the cubic Szegő equation on $\mathbb{T}$, and by Hani, Pausader, Tzvetkov and Visciglia [22] for the cubic NLS on $\mathbb{R} \times \mathbb{T}^2$.

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2 Semiclassical pseudodifferential operators

We recall the definition and main properties of a class of semiclassical pseudodifferential operators adapted to study equation (1.1); the main reference for this part is [35]. We start by denoting

$$\mathtt{k}_0(x,\xi) := (1 + |x|^{2l} + |\xi|^2)^{\frac{l+1}{2l}} , \qquad \forall x, \xi \in \mathbb{R}^d. \tag{2.1}$$

The function $\mathtt{k}_0$ is a good weight, in the sense that there exists $C_l > 0$ such that

$$\mathtt{k}_0(z+w) \leq C_l \, \mathtt{k}_0(z) \, \mathtt{k}_0(w), \qquad \forall z, w \in \mathbb{R}^{2d}, \tag{2.2}$$

and moreover

$$\widetilde{c}_l (1 + E(x,\xi))^{\frac{l+1}{2l}} \leq \mathtt{k}_0(x,\xi) \leq \widetilde{C}_l (1 + E(x,\xi))^{\frac{l+1}{2l}}, \tag{2.3}$$

for some constants $\widetilde{c}_l, \widetilde{C}_l > 0$. We begin with the following definition.

Definition 2.1. *A smooth function $a(x,\xi)$ will be called a* symbol *in the class $\Sigma^m \equiv \Sigma^m(\mathbb{R}^{2d})$ if $\forall \alpha, \beta \in \mathbb{N}^d$ there exists $C_{\alpha\beta} > 0$ such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \leq C_{\alpha\beta} \, \mathtt{k}_0(x,\xi)^m \ .$$

Remark that we do not ask the derivatives of symbols to gain decay. With this definition of symbols, one has

$$x_j \in \Sigma^{\frac{1}{l+1}}, \quad \xi_j \in \Sigma^{\frac{l}{l+1}}, \quad |\xi|^2 + W_l(x) \in \Sigma^{\frac{2l}{l+1}}, \quad \mathtt{k}_0(x,\xi) \in \Sigma^1.$$

We endow Σ^m with the family of semi-norms defined for any $M \in \mathbb{N}_0$ by

$$\wp_M^m(a) := \sum_{|\alpha|+|\beta| \leq M} \sup_{x,\xi \in \mathbb{R}^d} \frac{\left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right|}{\mathtt{k}_0^m(x,\xi)} \ . \tag{2.4}$$

As we already mentioned, we work with semiclassical operators, thus we consider also symbols depending on the semiclassical parameter $\hbar \in (0,1]$.

Definition 2.2. *Let $a^\hbar$ be a family of symbols depending smoothly on $\hbar \in]0,1]$. We say that $a^\hbar \in \Sigma^m_u$ if $a^\hbar \in \Sigma^m$ for every $\hbar \in (0,1]$ and if*

$$\sup_{\hbar \in]0,1]} \wp^m_M(a^\hbar) < +\infty, \qquad \forall M \in \mathbb{N}^d_0.$$

Abusing notation, for a symbol $a^\hbar \in \Sigma^m_u$ we will denote by $\wp^m_M(a^\hbar)$ the seminorm (2.4) where the supremum is taken also on $\hbar \in (0,1]$. To any symbol function $a^\hbar \in \Sigma^m_u$ we associate its $\hbar$-Weyl quantization $\mathrm{Op}^w_\hbar\left(a^\hbar\right)$ by the rule

$$\mathrm{Op}^w_\hbar\left(a^\hbar\right)[\psi](x) := \frac{1}{(2\pi\hbar)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{\mathrm{i}}{\hbar}(x-y)\cdot\xi}\, a^\hbar\left(\frac{x+y}{2}, \xi\right) \psi(y)\, \mathrm{d}y \mathrm{d}\xi \ . \tag{2.5}$$

Sometimes we will write $a(x, \hbar D_x)$ to denote the operator $\mathrm{Op}^w_\hbar(a)$.

A classical result regards composition of pseudodifferential operators.

Theorem 2.3 (Symbolic calculus)**.** *Let $a^\hbar \in \Sigma^m_u$, $b^\hbar \in \Sigma^{m'}_u$ be symbols. Then there exists a symbol $c^\hbar \in \Sigma^{m+m'}_u$ such that $\mathrm{Op}^w_\hbar\left(a^\hbar\right) \circ \mathrm{Op}^w_\hbar\left(b^\hbar\right) = \mathrm{Op}^w_\hbar\left(c^\hbar\right)$. For every $j \in \mathbb{N}$, there exists a positive constant C and an integer $M \geq 1$ (both independent of $a^\hbar$ and $b^\hbar$) such that*

$$\wp^{m+m'}_j(c^\hbar) \leq C\, \wp^m_M(a^\hbar)\, \wp^{m'}_M(b^\hbar) \ .$$

The second result concerns the boundedness of pseudodifferential operators.

Theorem 2.4 (Calderon-Vaillancourt)**.** *Let $a^\hbar \in \Sigma^0_u$ be a symbol. Then $\mathrm{Op}^w_\hbar\left(a^\hbar\right)$ extends to a linear bounded operator from $L^2(\mathbb{R}^d)$ to itself. Moreover there exist constants $C, N > 0$ such that*

$$\sup_{\hbar \in (0,1]} \|\mathrm{Op}^w_\hbar\left(a^\hbar\right)\|_{\mathcal{L}(L^2)} \leq C\ \wp^0_N(a^\hbar) \ . \tag{2.6}$$

Finally we recall a result about functional calculus.

Theorem 2.5 (Functional calculus)**.** *Let $a^\hbar \in \Sigma^\rho_u$, $\rho \geq 0$, be real and positively bounded from below, i.e. $a^h(x,\xi) \geq \gamma_0 > 0$ $\forall x, \xi \in \mathbb{R}^d$, $\forall h \in [0, \hbar_0)$.*
Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$, supported in $[0,\infty)$, fulfill: $\exists r \in \mathbb{R}$ such that $\forall k \in \mathbb{N}$, $\exists C_k > 0$ such that

$$\left|\frac{d^k}{dt^k} f(t)\right| \leq C_k (1 + |t|)^{r-k}, \qquad \forall t \in \mathbb{R}.$$

Then $f(\mathrm{Op}^w_\hbar\left(a^\hbar\right))$, defined by functional calculus, is a pseudodifferential operator with symbol $a^h_f \in \Sigma^{r\rho}_u$.

As H_l is the Weyl quantization of the symbol $1 + |\xi|^2 + W_l(x) \in \Sigma^{\frac{2l}{l+1}}$, functional calculus implies that, $\forall r \in \mathbb{R}$, the operator H^r_l is a pseudodifferential operator with symbol in $\Sigma^{\frac{2lr}{l+1}}$. Using this fact, Calderon-Vaillancourt theorem and symbolic calculus, one obtains that if $a^\hbar \in \Sigma^m_u$, $m \in \mathbb{R}$, then $\mathrm{Op}^w_\hbar\left(a^\hbar\right)$ maps $\mathcal{H}^{r+\frac{m(l+1)}{l}}$ to $\mathcal{H}^r$ $\forall r \in \mathbb{R}$ with the quantitative bound

$$\sup_{\hbar \in (0,1]} \|\mathrm{Op}^w_\hbar\left(a^\hbar\right)\|_{\mathcal{L}\left(\mathcal{H}^{r+\frac{m(l+1)}{l}}, \mathcal{H}^r\right)} \leq C'\ \wp^m_{N'}(a^\hbar), \tag{2.7}$$

where C', N' are positive constants.

The next result is the exact Egorov theorem.

Proposition 2.6 (Exact Egorov)**.** *Let $\chi(t,x,\xi)$ be a polynomial function in x,ξ of degree at most two with smooth t-dependent coefficients. Let $U^\hbar_\chi(t,s)$ be the propagator of the Schrödinger equation $\mathrm{i}\hbar\dot\psi = \chi(t,x,\hbar D_x)\psi$. Then for every $a^\hbar \in \Sigma^m_u$, one has*

$$U^\hbar_\chi(t,0)^* \operatorname{Op}^w_\hbar\left(a^h\right) U^\hbar_\chi(t,0) = \operatorname{Op}^w_\hbar\left(a^\hbar_t\right), \qquad a^\hbar_t := a^\hbar \circ \phi^t_\chi,$$

where $\phi^t_\chi(x,\xi)$ is classical Hamiltonian flow at time t of $\chi(t,x,\xi)$ with initial datum (x,ξ) at time 0.

We denote by $\mathcal{T}_\hbar(z)$ the Weyl operator

$$\left[\mathcal{T}_\hbar(z)\psi\right](x) := \left[\exp\left(-\frac{\mathrm{i}}{\hbar}(-p\cdot x + q\cdot\hbar D_x)\right)\psi\right](x) \tag{2.8}$$

remark that $\mathcal{T}_\hbar(z)$ is the time 1 flow of the Schrödinger equation $\mathrm{i}\hbar\dot\psi = \chi(z;x,\hbar D_x)\psi$, where $z := (p,q) \in \mathbb{R}^d\times$ and $\chi(z;x,\xi) := -p\cdot x + q\cdot\xi$ is a linear Hamiltonian. By Proposition 2.6 one gets

$$\mathcal{T}_\hbar(z)^* \operatorname{Op}^w_\hbar(a)\, \mathcal{T}_\hbar(z) = \operatorname{Op}^w_\hbar(a_z), \qquad a_z(x,\xi) := a(x+q,\xi+p). \tag{2.9}$$

We will also use the dilation operator

$$\left[\Lambda_\hbar\psi\right](x) := \frac{1}{\hbar^{d/4}}\,\psi\left(\frac{x}{\sqrt{\hbar}}\right), \qquad \text{for } \psi \in L^2(\mathbb{R}^d);$$

it is unitary on $L^2(\mathbb{R}^d)$ and conjugates pseudodifferential operators in the following way:

$$\Lambda_\hbar^{-1} \operatorname{Op}^w_\hbar(a)\, \Lambda_\hbar = \operatorname{Op}^w_1(b), \qquad b(x,\xi) := a\left(\sqrt{\hbar}x, \sqrt{\hbar}\xi\right). \tag{2.10}$$

3 Semiclassical approximation and coherent states

Consider the semiclassical Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi = H(t,x,\hbar D_x)\psi, \tag{3.1}$$

where $H(t,x,\hbar D_x)$ is the $\hbar$-Weyl quantization of a real valued Hamiltonian $H(t,x,\xi)$ with $x,\xi \in \mathbb{R}^d$. Through all the section we will make the following assumptions on both the classical symbol $H(t,x,\xi)$ and its Weyl quantization $H(t,x,\hbar D_x)$.

($\mathrm{H_{cl}}$) $H(t,x,\xi)$ is a C^∞–function in every variable. There exists $m>0$ such that $\forall t\in[0,T]$ the function $H(t,\cdot)\in\Sigma^m$. Its Hamiltonian flow, namely the solution $(x(t),\xi(t))$ of

$$\begin{cases} \dot x = \dfrac{\partial H}{\partial \xi}(t,x,\xi) \\ \dot\xi = -\dfrac{\partial H}{\partial x}(t,x,\xi) \end{cases}, \qquad x(0)=x_0, \quad \xi(0)=\xi_0 \tag{3.2}$$

exists for all $t\in[0,T]$ and any initial datum $(x_0,\xi_0)\in\mathbb{R}^{2d}$.

($\mathrm{H_{qu}}$) The Schrödinger equation (3.1) has a unique propagator $\mathcal{U}^\hbar(t,s)$, unitary in $L^2(\mathbb{R}^d)$ and fulfilling the group property $\mathcal{U}^\hbar(t,s)\mathcal{U}^\hbar(s,\tau) = \mathcal{U}^\hbar(t,\tau)$. The propagator $\mathcal{U}^\hbar(t,s)$ is bounded as a map from $\mathcal{H}^r$ to itself $\forall r$; moreover there exists $\mu>0$ and, for every $r>0$, a constant $C_r>0$ such that

$$\sup_{\hbar\in(0,1]} \|\mathcal{U}^\hbar(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \le C_r(1+|t-s|)^{r\mu}. \tag{3.3}$$

Remark that, in the case of equation (1.1), assumption ($\mathrm{H_{cl}}$) is easily checked, while assumption ($\mathrm{H_{qu}}$) follows by Theorem A.1, which is a semiclassical version of the abstract theorem of growth proved in [31].

We will construct an approximate solution of (3.1) using coherent states. Roughly speaking, a coherent state is a Gaussian packet concentrated in the phase space around a point $z = (q, p) \in \mathbb{R}^{2d}$. The theory of semiclassical approximation states that, if the initial datum of equation (3.1) is a coherent state concentrated near $z_0 = (q_0, p_0)$, then the true solution of (1.1) stays close, up to the Ehrenfest time, to a coherent state concentrated near the solution $z_t = (q(t), p(t))$ of the Hamiltonian equations of $H(t, q, p)$ with initial datum z_0.

To state rigorously this result we need to introduce some notation. Define for $z = (q, p) \in \mathbb{R}^{2d}$ the functions

$$\varphi_0(x) := \frac{1}{(\pi\hbar)^{d/4}}\, e^{-\frac{|x|^2}{2\hbar}}, \quad x \in \mathbb{R}^d, \tag{3.4}$$

$$\varphi_z := \mathcal{T}_\hbar(z)\varphi_0. \tag{3.5}$$

The function φ_z is called a *coherent state*; it is a Gaussian packet localized in the phase space around the point $z \in \mathbb{R}^{2d}$. It is normalized so that $\|\varphi_z\|_{L^2(\mathbb{R}^d)} = 1$.

Denote by $z_t = (q(t), p(t)) \in \mathbb{R}^{2d}$ the solution of the Hamiltonian equations of $H(t, q, p)$ with initial datum $z_0 \in \mathbb{R}^{2d}$; let M_t be the $2d \times 2d$ Hessian of the Hamiltonian computed at the solution z_t, namely

$$M_t := \left(\frac{\partial^2 H}{\partial z^2}\right)\Big|_{z=z_t}. \tag{3.6}$$

We use z_t and M_t to define the quadratic Hamiltonian

$$\begin{aligned} H_2(t, x, \xi) := H(t, z_t) &+ \left\langle x - q(t), \frac{\partial H}{\partial q}(t, z_t)\right\rangle_{\mathbb{R}^d} + \left\langle \xi - p(t), \frac{\partial H}{\partial p}(t, z_t)\right\rangle_{\mathbb{R}^d} \\ &+ \frac{1}{2}\left\langle M_t \begin{pmatrix} x - q(t) \\ \xi - p(t)\end{pmatrix}, \begin{pmatrix} x - q(t) \\ \xi - p(t)\end{pmatrix}\right\rangle_{\mathbb{R}^{2d}}, \end{aligned} \tag{3.7}$$

which is nothing but the Taylor expansion of order 2 of the Hamiltonian $H(t, q, p)$ around z_t. Its $\hbar$-quantization $H_2(t, x, \hbar D_x)$ generates a unitary propagator $\mathcal{U}_2^\hbar(t, s)$ in $L^2(\mathbb{R}^d)$.

We denote by F_t the solution of

$$\dot{F}_t = JM_tF_t\ , \qquad F_0 = \mathbb{1}, \tag{3.8}$$

where $J := \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$ is the standard Poisson tensor.

Lemma 3.1. *Let $a^\hbar \in \Sigma_u^m$, $m \in \mathbb{R}$. Then $\mathcal{U}_2^\hbar(t,0)^* \operatorname{Op}_\hbar^w\left(a^\hbar\right) \mathcal{U}_2^\hbar(t,0)$ is a pseudodifferential operator with symbol $a_t^\hbar$ given by*

$$a_t^\hbar(\zeta) = a^\hbar\left(z_t + F_t[\zeta - z_0]\right)\ , \quad \zeta = (x, \xi) \in \mathbb{R}^{2d}.$$

Proof. Since $H_2(t, x, \xi)$ is a quadratic polynomial in x, ξ, we can apply Proposition 2.6 and get $\mathcal{U}_2^\hbar(t,0)^* \operatorname{Op}_\hbar^w\left(a^\hbar\right) \mathcal{U}_2^\hbar(t,0) = \operatorname{Op}_\hbar^w\left(a^\hbar \circ \phi_{H_2}^t\right)$, where $\phi_{H_2}^t$ is the Hamiltonian flow of $H_2(t, x, \xi)$. We compute explicitly such a flow. Thus let $\zeta(t) := \phi_{H_2}^t(\zeta)$ be the solution of

$$\dot{\zeta} = JM_t\zeta + J\nabla H(t, z_t) - JM_tz_t, \qquad \zeta(0) = \zeta \in \mathbb{R}^{2d}.$$

By Duhamel's formula we get

$$\zeta(t) = F_t\zeta + F_t \int_0^t F_{-s} J\nabla H(s, z_s)\, \mathrm{d}s - F_t \int_0^t F_{-s} J M_s z_s\, \mathrm{d}s. \tag{3.9}$$

Now use that z_s is a solution of the Hamiltonian equations of $H(s, z)$ to write $J\nabla H(s, z_s) = \frac{d}{ds} z_s$; integrating by parts we obtain

$$\begin{aligned} F_t \int_0^t F_{-s} J\nabla H(s, z_s)\, \mathrm{d}s &= z_t - F_t z_0 - F_t \int_0^t \left(\frac{d}{ds} F_{-s}\right) z_s\, \mathrm{d}s \\ &= z_t - F_t z_0 + F_t \int_0^t F_{-s} J M_s z_s\, \mathrm{d}s, \end{aligned} \tag{3.10}$$

where in the last inequality we used that

$$\frac{d}{ds} F_{-s} = \frac{d}{ds} F_s^{-1} = -F_s^{-1} \left(\frac{d}{ds} F_s\right) F_s^{-1} = -F_{-s} J M_s.$$

Inserting (3.10) into (3.9) gives the result. □

Now fix $z_0 \in \mathbb{R}^{2d}$ and consider the solution of (3.1) with initial datum the coherent state φ_{z_0} defined in (3.5). The main result of the section is that the quantum evolution $\mathcal{U}^\hbar(t, 0)\varphi_{z_0}$ is well approximated by the dynamics of $\mathcal{U}_2^\hbar(t, 0)\varphi_{z_0}$ in the topology of $\mathcal{H}^r$, $\forall r \geq 0$. This extend to higher Sobolev spaces the results of [11]. To state the theorem precisely, let us introduce for any $T \geq 0$ the quantities

$$|F|_T := \sup_{0\leq t\leq T} |F_t|, \qquad \mathcal{E}_T := \sup_{0\leq t\leq T} E(z_t),$$

where $E(z) \equiv E(q, p)$ is the anharmonic energy defined in (1.10).

Theorem 3.2. *Assume* ($\mathrm{H_{cl}}$) *and* ($\mathrm{H_{qu}}$). *Fix* $z_0 \in \mathbb{R}^{2d}$, $r \geq 0$ *and* $\kappa \in (0, 1]$. *Then there exists a constant* $\Gamma > 0$ *such that for any* $\hbar, T > 0$ *fulfilling*

$$\sqrt{\hbar}|F|_T \leq \kappa, \tag{3.11}$$

one has

$$\|\mathcal{U}^\hbar(t, 0)\varphi_{z_0} - \mathcal{U}_2^\hbar(t, 0)\varphi_{z_0}\|_r \leq \Gamma\, \hbar^{1/2}\, |F|_T^3 (1 + T)^{\mu r+1} (1 + \mathcal{E}_T)^{\frac{\overline{r}}{2}}, \qquad \forall 0 \leq t \leq T, \tag{3.12}$$

where $\overline{r} := r + 3 + m(l + 1)/l$.

Proof. One starts with Duhamel's formula

$$\mathcal{U}^\hbar(t, 0)\varphi_{z_0} - \mathcal{U}_2^\hbar(t, 0)\varphi_{z_0} = \frac{1}{\mathrm{i}\hbar} \int_0^t \mathcal{U}^\hbar(t, \tau)\, [H(\tau, x, \hbar D_x) - H_2(\tau, x, \hbar D_x)]\, \mathcal{U}_2^\hbar(\tau, 0)\varphi_{z_0}\, \mathrm{d}\tau. \tag{3.13}$$

Recall that $H_2(t, x, \xi)$ is the Taylor expansion at order two of $H(t, x, \xi)$ around the path z_t, thus

$$H(t, z) - H_2(t, z) = R(t, z - z_t), \quad z = (x, \xi) \in \mathbb{R}^{2d} \tag{3.14}$$

where

$$R(t, \zeta) = \sum_{\substack{\nu\in\mathbb{N}_0^{2d} \\ |\nu|=3}} R_\nu(t, \zeta) \cdot \zeta^\nu\ , \qquad R_\nu(t, \zeta) := \frac{1}{(\nu - 1)!} \int_0^1 H^{(\nu)}\,(t, z_t + \theta\zeta)\,(1 - \theta)^2\, \mathrm{d}\theta. \tag{3.15}$$

Since $H(t,\cdot) \in \Sigma^m$, one has $R(t,\zeta) \in \Sigma^{m+3l/(l+1)}$. Quantizing (3.14) we obtain

$$H(\tau, x, \hbar D_x) - H_2(\tau, x, \hbar D_x) = \mathrm{Op}_\hbar^w \left(R(\tau, \zeta - z_t) \right). \tag{3.16}$$

Inserting (3.16) into (3.13) and taking the $\mathcal{H}^r$ norm, we have that

$$\begin{aligned} \|\mathcal{U}^\hbar(t,0)\varphi_{z_0} - \mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\|_r &\leq \frac{1}{\hbar}\int_0^t \|\mathcal{U}^\hbar(t,\tau)\|_{\mathcal{L}(\mathcal{H}^r)} \|\mathrm{Op}_\hbar^w \left(R(\tau,\zeta - z_\tau)\right) \mathcal{U}_2^\hbar(\tau,0)\varphi_{z_0}\|_r \,\mathrm{d}\tau \\ &\overset{(3.3)}{\leq} C_r \frac{(1+|t|)^{1+r\mu}}{\hbar} \sup_{0\leq\tau\leq t} \|\mathrm{Op}_\hbar^w \left(R(\tau,\zeta - z_\tau)\right) \mathcal{U}_2^\hbar(\tau,0)\varphi_{z_0}\|_r \end{aligned}$$

To control the last term we proceed as following. First remark that $\mathcal{U}_2^\hbar(\tau,0)$ and $\mathcal{T}_\hbar(z_0)$ are isometry in $L^2(\mathbb{R}^d)$, so is

$$U^\hbar(t) := \mathcal{U}_2^\hbar(\tau,0)\mathcal{T}_\hbar(z_0). \tag{3.17}$$

Then, exploiting (3.5) and the identity

$$U^\hbar(t)^* \,\mathrm{Op}_\hbar^w(a)\, U^\hbar(t) = \mathrm{Op}_\hbar^w(b), \qquad b(t,\zeta) := a(z_t + F_t\zeta), \tag{3.18}$$

which follows by (2.9) and Lemma 3.1, we obtain

$$\begin{aligned} \|\mathrm{Op}_\hbar^w \left(R(\tau,\zeta - z_\tau)\right) \mathcal{U}_2^\hbar(\tau,0)\varphi_{z_0}\|_r &= \|H_l^{r/2}\mathrm{Op}_\hbar^w \left(R(\tau,\zeta - z_\tau)\right) U^\hbar(\tau)\varphi_0\|_0 \\ &= \|\left(U^\hbar(\tau)^*\, H_l^{r/2} U^\hbar(\tau)\right)\left(U^\hbar(\tau)^* \mathrm{Op}_\hbar^w \left(R(\tau,\zeta - z_\tau)\right) U^\hbar(\tau)\right)\varphi_0\|_0 \\ &= \|\mathrm{Op}_\hbar^w \left(\mathtt{h}_r(z_\tau + F_\tau\zeta)\right)\, \mathrm{Op}_\hbar^w \left(R(\tau, F_\tau\zeta)\right)\varphi_0\|_0. \end{aligned} \tag{3.19}$$

In the last line we denoted by $\mathtt{h}_r \in \Sigma^{\frac{lr}{l+1}}$ the symbol of $H_l^{r/2}$, i.e. $H_l^{r/2} = \mathrm{Op}_\hbar^w(\mathtt{h}_r)$. We are left with estimating (3.19). Let $\mathtt{h}_r(z_t;\zeta) := \mathtt{h}_r(\zeta + z_t)$. By (2.10)

$$\Lambda_\hbar^{-1}\mathrm{Op}_\hbar^w \left(\mathtt{h}_r(z_\tau; F_\tau\zeta)\right)\, \mathrm{Op}_\hbar^w \left(R(\tau, F_\tau\zeta)\right)\Lambda_\hbar = \mathrm{Op}_1^w \left(\mathtt{h}_r(z_\tau; \sqrt{\hbar}F_\tau\zeta)\right)\, \mathrm{Op}_1^w R(\tau, \sqrt{\hbar}F_\tau\zeta).$$

Thus, using that $\Lambda_\hbar$ is unitary in $L^2(\mathbb{R}^d)$ and writing $\varphi_0 = \Lambda_\hbar\varphi$, where $\varphi(x) := \frac{1}{\pi^{d/4}}e^{-|x|^2}$, we get

$$(3.19) = \|\mathrm{Op}_1^w \left(\mathtt{h}_r(z_\tau; \sqrt{\hbar}F_\tau\zeta)\right)\, \mathrm{Op}_1^w \left(R(\tau, \sqrt{\hbar}F_\tau\zeta)\right)\varphi\|_0.$$

Now remark that φ is a Schwartz function, so by Calderon-Vaillancourt theorem there exist $C, N > 0$ such that

$$(3.19) \leq C\, \wp_N^{lr/(l+1)}\left(\mathtt{h}_r(z_t; \sqrt{\hbar}F_t\zeta)\right)\; \wp_N^{m+3l/(l+1)}\left(R(\tau, \sqrt{\hbar}F_t\zeta)\right)\; \|\varphi\|_{\overline{r}}.$$

We are left with estimating the seminorms of the symbols. By assumption (3.11) we have

$$\sqrt{\hbar}|F_t| \leq \kappa \leq 1, \qquad \forall\, 0 \leq t \leq T,$$

therefore the seminorm of $\mathtt{h}_r$ is controlled by

$$\wp_N^{lr/(l+1)}\left(\mathtt{h}_r(z_t; \sqrt{\hbar}F_t\zeta)\right) \leq \wp_N^{lr/(l+1)}(\mathtt{h}_r) \sup_{\zeta\in\mathbb{R}^{2d}} \left|\frac{\mathtt{k}_0(\zeta + z_t)}{\mathtt{k}_0(\zeta)}\right|^{lr/(l+1)}.$$

By (2.2) and (2.3), for any $t \in [0,T]$ we bound

$$\sup_{\zeta\in\mathbb{R}^{2d}} \left|\frac{\mathtt{k}_0(\zeta+z_t)}{\mathtt{k}_0(\zeta)}\right|^{lr/(l+1)} \le C_l'\mathtt{k}_0(z_t)^{lr/(l+1)} \le C_l'\widetilde{C}_l'(1+E(z_t))^{\frac{r}{2}} \le C(1+\mathcal{E}_T)^{\frac{r}{2}}.$$

Thus we proved

$$\wp_N^{lr/(l+1)}\left(\mathtt{h}_r(z_t;\sqrt{\hbar}F_t\zeta)\right) \le C(1+\mathcal{E}_T)^{\frac{r}{2}}, \qquad \forall\, 0\le t\le T. \tag{3.20}$$

Consider now the seminorm of $R(\tau,\sqrt{\hbar}F_t\zeta)$. Proceeding as above and using the definition of R in (3.15) we obtain

$$\wp_N^{m+3l/(l+1)}\left(R(\tau,\sqrt{\hbar}F_t\zeta)\right) \le C(\sqrt{\hbar}|F|_T)^3(1+\mathcal{E}_T)^{\frac{m(l+1)}{2l}+\frac{3}{2}}, \qquad \forall\, 0\le t\le T. \tag{3.21}$$

Combining all estimates we have

$$\|\mathcal{U}^\hbar(t,0)\varphi_{z_0} - \mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\|_r \le \Gamma\frac{(1+T)^{1+r\mu}}{\hbar}(\sqrt{\hbar}|F|_T)^3(1+\mathcal{E}_T)^{\frac{\overline{r}}{2}}, \qquad \forall\, 0\le t\le T$$

which proves (3.12). □

Theorem 3.2 tells that it is possible to approximate, in the $\mathcal{H}^r$ topology, the quantum dynamics of a coherent state with the approximate flow generated by a quadratic Hamiltonian. In the next proposition we show that it is easy to compute the values of observables along the approximate flow.

Proposition 3.3. *Assume* ($\mathrm{H_{cl}}$) *and* ($\mathrm{H_{qu}}$)*. Fix* $z_0\in\mathbb{R}^{2d}$ *and* $\kappa\in(0,1]$*. Furthermore assume that* $a\in\Sigma^\rho$*,* $\rho\ge 0$*, fulfills the condition*

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta}\ \mathtt{k}_0(x,\xi)^{\rho-\frac{l\beta+\alpha}{l+1}}, \quad \forall|\alpha|+|\beta|\le 1. \tag{3.22}$$

Then there exist a constant $\Gamma_1>0$ *and for any* $\hbar_0, T>0$ *fulfilling*

$$\sqrt{\hbar_0}\,|F|_T \le \kappa, \tag{3.23}$$

a smooth function $b:(0,\hbar_0]\times[0,T]\to\mathbb{R}$ *such that*

$$\langle \mathrm{Op}_\hbar^w(a)\,\mathcal{U}_2^\hbar(t,0)\varphi_{z_0}, \mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\rangle = a(z_t)+b(\hbar,t), \tag{3.24}$$

and moreover

$$|b(\hbar,t)| \le \Gamma_1\,\hbar^{\frac12}\,|F_t|\,(1+\mathcal{E}_t)^{\frac{(l+1)\rho-1}{2l}}, \qquad \forall 0\le t\le T, \quad \forall\hbar\in(0,\hbar_0]. \tag{3.25}$$

Proof. With $U^\hbar(t)$ defined in (3.17) and exploiting (3.18) we get

$$\left\langle \mathrm{Op}_\hbar^w(a)\,\mathcal{U}_2^\hbar(t,0)\varphi_{z_0}, \mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\right\rangle = \left\langle U^\hbar(t)^*\mathrm{Op}_\hbar^w(a)\,U^\hbar(t)\varphi_0,\varphi_0\right\rangle = \langle \mathrm{Op}_\hbar^w(a(F_t\zeta+z_t))\,\varphi_0,\varphi_0\rangle$$

To compute the last scalar product we proceed as following. Denote by Ψ the orthogonal projector on φ_0, $\Psi u := \langle u,\varphi_0\rangle\varphi_0$; it is a pseudodifferential operator with $\hbar$-Weyl symbol given by the Wigner function

$$\mathcal{W}_{\varphi_0}(x,\xi) = 2^d\, e^{-\frac{|x|^2+|\xi|^2}{\hbar}},$$

see e.g. [11]. Now remark that for any operator A one has

$$\langle A\varphi_0, \varphi_0\rangle = \langle A\Psi\varphi_0, \varphi_0\rangle = \sum_{j\geq 0} \langle A\Psi\varphi_j, \varphi_j\rangle = \mathrm{tr}(A\Psi),$$

where $\{\varphi_j\}_{j\geq 0}$ is any orthonormal basis of $L^2(\mathbb{R}^d)$ that completes φ_0.
If $A = \mathrm{Op}^w_\hbar(a)$ is a pseudodifferential operator, trace formula (see [35, Proposition II–56]) assures that

$$\mathrm{tr}(A\Psi) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^{2d}} a(\zeta)\mathcal{W}_{\varphi_0}(\zeta)\,\mathrm{d}\zeta.$$

In our case we obtain

$$\begin{aligned} \langle \mathrm{Op}^w_\hbar\left(a(F_t\zeta + z_t)\right)\varphi_0, \varphi_0\rangle &= \frac{1}{(\pi\hbar)^d}\int_{\mathbb{R}^{2d}} a(F_t\zeta + z_t)\, e^{-\frac{|\zeta|^2}{\hbar}}\mathrm{d}\zeta \\ &= \pi^{-d}\int_{\mathbb{R}^{2d}} a(\sqrt{\hbar}F_t\zeta + z_t)\, e^{-|\zeta|^2}\mathrm{d}\zeta \end{aligned} \tag{3.26}$$

Now we write

$$a\left(\sqrt{\hbar}F_t\zeta + z_t\right) = a(z_t) + \mathtt{b}(\hbar, t, \zeta)\ , \qquad \mathtt{b}(\hbar, t, \zeta) := a\left(\sqrt{\hbar}F_t\zeta + z_t\right) - a(z_t). \tag{3.27}$$

Inserting (3.27) in (3.26) gives formula (3.24) with $b(\hbar, t) := \pi^{-d}\int \mathtt{b}(\hbar, t, \zeta)e^{-|\zeta|^2}\mathrm{d}\zeta$. We prove now (3.25). By Lagrange mean value theorem and assumption (3.22) we get

$$\begin{aligned} |\mathtt{b}(\hbar, t, \zeta)| &\leq C\hbar^{\frac{1}{2}}\,|F_t|\,\|\zeta\| \sup_{0\leq s\leq 1} \mathtt{k}_0\left(\hbar^{\frac{1}{2}}F_t\zeta + sz_t\right)^{\rho - \frac{1}{l+1}} \\ &\leq C'\hbar^{\frac{1}{2}}\,|F_t|\,\|\zeta\|\,\mathtt{k}_0\left(\hbar^{\frac{1}{2}}F_t\zeta\right)^{\rho - \frac{1}{l+1}}\mathtt{k}_0(z_t)^{\rho - \frac{1}{l+1}} \end{aligned}$$

Now insert the last estimate in (3.26), and use (3.23) and the inequality (2.3) to obtain the claimed result. □

4 Application to anharmonic oscillators

In this section we apply Theorem 3.2 to construct a solution of equation (1.1) whose Sobolev norms grow for long but finite time.

4.1 Unbounded orbits for classical anharmonic oscillator

The first step is to consider the mechanical system (1.9) and construct a forcing term $\beta(t)$, smooth and bounded with its derivatives, so that there exists at least one unbounded solution.

This is the content of the next result.

Proposition 4.1. *There exists a smooth function $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ fulfilling* (1.5)*, such that equation* (1.9) *possesses an unbounded solution $q(t)$. Moreover there exist $C_1, C_2 > 0$ s.t.*

$$C_1\,[\log(2+t)]^2 \leq E\left(q(t), p(t)\right) \leq C_2\,[\log(2+t)]^2 \qquad \forall t \geq 0. \tag{4.1}$$

To prove the result we follow the strategy of [27]. First remark that the dynamic of (1.9) is decoupled into one dimensional systems. Since $q_j = p_j = 0\ \ \forall j \geq 2$ is an invariant subspace, we take an initial datum with $q_j(0) = p_j(0) = 0 \ \forall j \geq 2$. Then the dynamics of (1.9) becomes one dimensional and restricted to the variables (q_1, p_1).

The idea is to create $\beta(t)$ by giving a particular solution $q(t)$ of (1.9) a "helping kick" to the right direction each time the solution passes through the interval $-1 \leq q_1 \leq 1$ from left to right, and make $\beta(t) = 0$ at all other times. With such a $\beta(t)$ the energy along the solution will increase at each passage from -1 to 1 while remaining constant between consecutive passages.

Furthermore it is important to weaken the "kicks" at every passage, otherwise the external force $\beta(t)$ will have some derivatives which are unbounded in t.

In order to construct $\beta(t)$ we use an auxiliary nonlinear equation. First define g_1 and g_2 to be positive cut-off functions on $\mathbb{R}$ s.t.

$$g_1(y) := \begin{cases} 1\ , & |y| \leq 1/2 \\ 0\ , & |y| \geq 1 \end{cases} \quad , \qquad g_2(y) := \begin{cases} 0\ , & y \leq 0 \\ 1\ , & y \geq 1 \end{cases} \quad , \qquad y \in \mathbb{R}.$$

Then consider the nonlinear equation

$$\ddot{y} + y^{2l-1} = f(y, \dot{y}), \qquad y, \dot{y}, \ddot{y} \in \mathbb{R} \tag{4.2}$$

with

$$f(y, \dot{y}) := g_1(y)\, g_2(\dot{y})\, e^{-\dot{y}}\ . \tag{4.3}$$

Abusing notation, we denote again by $E(y, \dot{y})$ the mechanical energy

$$E(y, \dot{y}) := \frac{\dot{y}^2}{2} + \frac{y^{2l}}{2l}.$$

Proposition 4.2. *Consider equation* (4.2)*. The solution with initial datum* $y(0) = \dot{y}(0) = 1$ *is globally defined and unbounded. More precisely there exist* $C_1, C_2 > 0$ *s.t.*

$$C_1 \log(1+t)^2 \leq E(y(t), \dot{y}(t)) \leq C_2 \log(1+t)^2\ , \qquad \forall t \geq 1\ . \tag{4.4}$$

Proof. Along a solution of (4.2) the function $E(t) \equiv E(y(t), \dot{y}(t))$ fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = \dot{y}\, f(y, \dot{y}) \geq 0\ . \tag{4.5}$$

More precisely $\frac{\mathrm{d}}{\mathrm{d}t}E(t) > 0$ when $|y(t)| < 1$ and $\dot{y}(t) > 0$, otherwise $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$.

Set $t_0 = 0$, define the increasing sequence of all times $0 < t_1 < t_2 < \ldots$ such that $y(t_n) = \pm 1$, $\dot{y}(t_n) > 0$ for $n \geq 1$, and denote $E_n := E(t_{2n}) \ \forall n \geq 0$. It is easy to verify that such a sequence is well defined and that $y(t_n) = -1$ for n odd and $y(t_n) = 1$ for n even (see Figure 1 for a sketch of the phase portrait).

By (4.5) and the definition of $f(y, \dot{y})$ we have that E_n is monotone increasing and furthermore

$$E(t) = E_n\ , \qquad \forall t_{2n} \leq t \leq t_{2n+1}, \qquad \forall n \geq 0\ . \tag{4.6}$$

We shall now prove, with a quantitative bound from below, that $E(t)$ increases when $t \in [t_{2n+1}, t_{2n+2}]$. Observe that $E_n > \frac{1}{2l}$ for all $n \geq 0$. Using that

$$E_n \leq E(t) \leq E_{n+1}\ , \qquad \forall\ t_{2n+1} \leq t \leq t_{2n+2},$$

and $|y| \leq 1$ one obtains the bound

$$\sqrt{2E_n - \frac{1}{l}} \leq \dot{y}(t) \leq \sqrt{2E_{n+1}}\ , \quad \forall\ t_{2n+1} \leq t \leq t_{2n+2}. \tag{4.7}$$

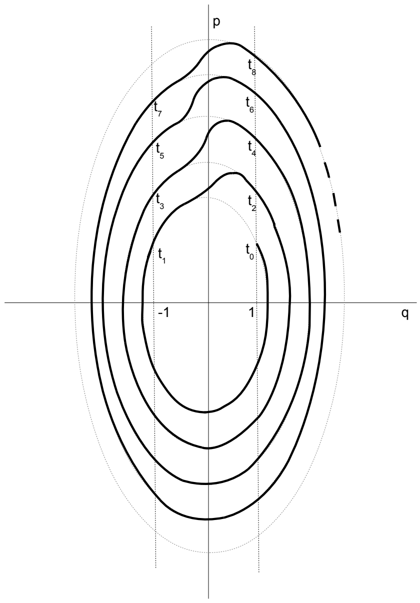

Figure 1: A sketch of the phase portrait: the oscillator follows the conservative dynamics from t_{2n} to t_{2n+1}, then the "helping kick" pushes the trajectory to a higher energy level between t_{2n+1} and t_{2n+2}.

Next write

$$E_{n+1} - E_n = \int_{t_{2n}}^{t_{2n+2}} \dot{y}(t)\, f(y(t), \dot{y}(t))\, \mathrm{d}t = \int_{t_{2n+1}}^{t_{2n+2}} \dot{y}(t)\, f(y(t), \dot{y}(t))\, \mathrm{d}t = \int_{-1}^{1} g_1(y) g_2(\dot{y}) e^{-\dot{y}}\, \mathrm{d}y,$$

where the last integral is obtained via the change of variable $t \rightsquigarrow y(t)$ and $\dot{y}$ has to be thought of as a function of y (this can be done since, when $t \in [t_{2n+1}, t_{2n+2}]$, the function $t \mapsto y(t)$ is strictly increasing, see (4.7)). This integral can be estimated by (4.7) obtaining

$$ce^{-\sqrt{2E_{n+1}}} \leq E_{n+1} - E_n \leq 2e^{-\sqrt{2E_n - \frac{1}{l}}} \leq 2e^{\sqrt{\frac{1}{l}}} e^{-\sqrt{2E_n}} , \qquad \forall n \geq 0 , \tag{4.8}$$

for some constant $c > 0$ depending only on the choice of the cutoff functions g_1, g_2. We claim that

$$\lim_{n\to\infty} E_n = +\infty .$$

Indeed, the limit exists since $\{E_n\}_{n\geq 1}$ is an increasing sequence. Assuming that $\lim\limits_{n\to\infty} E_n = E_\infty < \infty$, one gets a contradiction when passing to the limit in (4.8) (recall that $E_n > \frac{1}{2l}$).

Now use (4.8) and the fact that $E_n \geq \frac{1}{2l}$ $\forall n$, to get that $1 \leq E_{n+1}/E_n \leq K := 1 + 4l$, which implies that

$$ce^{-\sqrt{2KE_n}} \leq E_{n+1} - E_n \leq 2e^{\sqrt{\frac{1}{l}}} e^{-\sqrt{2E_n}} , \qquad \forall n \geq 0 . \tag{4.9}$$

To estimate E_n we define the interpolating function

$$\eta(\theta) = (\theta - n)E_{n+1} + (1 + n - \theta)E_n , \quad n \leq \theta \leq n + 1,$$

so that the right derivative D_+ of η fulfills

$$c e^{-\sqrt{2K\eta(\theta)}} \leq D_+\eta(\theta) \leq 2e^{\sqrt{\frac{1}{l}}} e^{-\sqrt{2\eta(\theta)/K}} \ , \qquad \forall \theta \geq 0 \ . \tag{4.10}$$

To estimate η we will use the method of the super and sub solutions. In particular, for any

$$K' > 2K = 2(1+4l)$$

there exists $c_{K'} > 0$ so that

$$c_{K'} \sqrt{\eta(\theta)}\, e^{-\sqrt{K'\eta(\theta)}} \leq D_+\eta(\theta) \leq 2e^{\sqrt{\frac{1}{l}}} \sqrt{2l\eta(\theta)}\, e^{-\sqrt{2\eta(\theta)/K}} \ , \qquad \forall \theta \geq 0 \ , \tag{4.11}$$

where we used also that $\eta(\theta) \geq \frac{1}{2l}$. The solution of this differential inequality can be estimated by the supersolution and subsolution method: in particular consider the differential equations

$$\xi'(\theta) = c_{K'} \sqrt{\xi(\theta)}\, e^{-\sqrt{K'\xi(\theta)}} \ , \quad \zeta'(\theta) = 2e^{\sqrt{\frac{1}{l}}} \sqrt{2l\zeta(\theta)}\, e^{-\sqrt{2\zeta(\theta)/K}}$$

and initial condition $\xi(0) = \zeta(0) = \eta(0)$. Then one has $\xi(\theta) \leq \eta(\theta) \leq \zeta(\theta)$ for all $\theta \geq 0$. A simple computation shows that

$$\frac{1}{K'} \left[\log \left(e^{\sqrt{K'}\eta(0)} + \frac{c_{K'}\sqrt{K'}}{2}\theta \right) \right]^2 \leq \eta(\theta) \leq \frac{K}{2} \left[\log \left(e^{\sqrt{2\eta(0)/K}} + 2\sqrt{l/K} e^{\sqrt{1/l}}\, \theta \right) \right]^2 \ , \qquad \forall \theta \geq 0 \ .$$

Evaluating this expression at $\theta = n$, one has

$$\mathtt{a}[\log(2+n)]^2 \leq E_n \leq \mathtt{b}[\log(2+n)]^2 \ , \qquad \forall n \geq 0 \tag{4.12}$$

for some positive constants $\mathtt{a}, \mathtt{b}$. Now we need to relate n with t_n. To do so, denote by $T(E)$ the period of oscillation of the solutions of $\ddot{y} + y^{2l-1} = 0$ with energy $E > 0$. It is given by

$$T(E) = c_l\, E^{-\frac{l-1}{2l}} \tag{4.13}$$

for some constant $c_l > 0$ (in particular, $T(E)$ is a strictly decreasing function of E). Moreover in our case

$$\frac{T(E_{n+1})}{2} \leq \frac{T(E_n)}{2} \leq t_{2n+2} - t_{2n} \leq T(E_n) \ , \qquad \forall n \geq 0. \tag{4.14}$$

To see that $\frac{T(E_n)}{2} \leq t_{2n+2} - t_{2n}$, observe that the time interval $[t_{2n}, t_{2n+2}]$ contains (more than) a half oscillation at energy E_n. The last inequality $t_{2n+2} - t_{2n} \leq T(E_n)$ is deduced by comparison with the conservative system at energy E_n and by observing that in the forced ODE the particle travels at energy E_n when $t \in [t_{2n}, t_{2n+1}]$ and undergoes a further forward acceleration for $t \in [t_{2n+1}, t_{2n+2}]$. Using (4.12) and the explicit expression (4.13), one has

$$\mathtt{a}\, [\log(2+n)]^{-\frac{l-1}{l}} \leq t_{2n+2} - t_{2n} \leq \mathtt{b}\, [\log(2+n)]^{-\frac{l-1}{l}} \ , \qquad \forall n \geq 0 \ . \tag{4.15}$$

for some new constants $\mathtt{a}, \mathtt{b}$ different from those in (4.12). Now write $t_{2n} = \sum_{m=0}^{n-1} t_{2m+2} - t_{2m}$, thus (4.15) and the estimates

$$c_1 \frac{n}{[\log(2+n)]^\alpha} \leq \sum_{m=0}^{n-1} \frac{1}{[\log(2+m)]^\alpha} \leq c_2 \frac{n}{[\log(2+n)]^\alpha}, \qquad \forall \alpha \in [0,1], \quad \forall n \geq 1$$

show that

$$\frac{\tilde{\mathtt{a}}\, n}{[\log(2+n)]^{(l-1)/l}} \leq t_{2n} \leq \frac{\tilde{\mathtt{b}}\, n}{[\log(2+n)]^{(l-1)/l}}, \qquad \forall n \geq 1. \tag{4.16}$$

By (4.16) and (4.12), we get that

$$C_1 \leq \frac{E_n}{[\log(2+t_{2n})]^2} \leq C_2 \ , \qquad \forall n \geq 0 \ , \tag{4.17}$$

which implies (4.4). $\square$

Proof of Proposition 4.1. Let $y(t)$ be the solution of (4.2) with initial datum $(y(0), \dot{y}(0)) = (1, 1)$. Define $\beta(t)$ as

$$\beta(t) := -f\left(y(t), \dot{y}(t)\right) \ . \tag{4.18}$$

Then $q(t) := (y(t), 0, \dots, 0)$, $\dot{q}(t) := (\dot{y}(t), 0, \dots, 0)$ is the solution of (1.9) with initial datum $q(0) = (1, 0, \dots, 0)$ and $\dot{q}(0) = (1, 0, \dots, 0)$. By Proposition 4.2, the energy along $q(t)$ increases, and (4.1) holds (remark that $p(t) = \dot{q}(t)$).

To prove (1.5), first observe that $\beta(t) = -g_1(y)g_2(\dot{y})e^{-\dot{y}}$ is a bounded function of t. Next, we estimate $\dot{\beta}(t)$: we have

$$\begin{aligned} \dot{\beta}(t) &= -e^{-\dot{y}}\{\dot{y}g_1'(y)g_2(\dot{y}) + \ddot{y}g_1(y)g_2'(\dot{y}) - \ddot{y}g_1(y)g_2(\dot{y})\} \\ &= -e^{-\dot{y}}\{\dot{y}g_1'(y)g_2(\dot{y}) - [y^{2l-1} + \beta(t)][g_1(y)g_2'(\dot{y}) - g_1(y)g_2(\dot{y})]\}, \end{aligned}$$

where we have used the ODE (4.2) to obtain the last equality. Now, notice that $g_1(y) \equiv 0$ for $|y| \geq 1/2$, that $g_2(\dot{y}) = g_2'(\dot{y}) \equiv 0$ for $\dot{y} \leq 0$ and that $\dot{y}e^{-\dot{y}}$ is bounded for $\dot{y} \geq 0$. The estimate for higher order derivatives is obtained similarly via Faà di Bruno's formula. $\square$

Remark 4.3. *Combining* (4.4) *and* (4.18), *one sees easily that* $\beta(t) \to 0$ *as* $t \to \infty$. *If instead* $t \mapsto \beta(t)$ *is periodic, it is known that all the solutions of* $\ddot{q}_1 + q_1^{2l-1} = -\beta(t)$ *are bounded in time [33, 14, 25, 26, 38]. Remark that, for autonomous system, the phenomenon of having all solutions bounded is very interesting and often associated to some sort of integrability, for example as it happens in the defocusing cubic NLS on* $\mathbb{T}$ *or the Toda lattice (see e.g. [29, 5]).*

Finally we need to estimate the norm of F_t, which in this case is defined as the flow of the linearized Hamiltonian (1.8) along the solution $(q(t), p(t))$ of Proposition (4.1). By (3.8), F_t solves the equation

$$\dot{F}_t = \begin{pmatrix} 0_d & \mathbb{1}_d \\ -\Upsilon(t) & 0_d \end{pmatrix} F_t \ , \qquad F_0 = \mathbb{1}_{2d} \tag{4.19}$$

where $\mathbb{1}_d$ is the $d \times d$ identity matrix, 0_d the $d \times d$ zero matrix, and $\Upsilon(t)$ the $d \times d$ diagonal matrix defined by

$$\Upsilon(t) := \operatorname{diag}\Big((2l-2)(q_1(t))^{2l-2}, 0, \dots, 0\Big)$$

where $q_1(t) \equiv y(t)$ is first component of the unbounded solution constructed in Proposition 4.1.

Lemma 4.4. *Consider equation* (4.19). *There exists* $\mathtt{c} > 0$ *such that, for all* $T > 0$, *one has*

$$\sup_{0 \leq t \leq T} |F_t| \leq \exp\Big(\mathtt{c}T\,[\log(2+T)]^{\varsigma}\Big), \qquad \varsigma := 2\left(1 - \frac{1}{l}\right). \tag{4.20}$$

Proof. Using the results of Proposition 4.1, one gets

$$|\Upsilon(t)| \leq C_2\,(2l-2)\,[\log(2+t)]^{\varsigma}\,, \quad \forall t \geq 1,$$

therefore

$$|F_t| \leq \exp\left(\int_0^t (1+|\Upsilon(s)|)\mathrm{d}s\right) \leq \exp\left(\mathtt{c}t\,[\log(2+t)]^{\varsigma}\right),$$

which gives the thesis. □

4.2 Growth of Sobolev norms

In this section we apply the semiclassical approximation to the quantum Hamiltonian (1.1). The idea is that the coherent state stays localized in the phase space around the solution $(q(t), p(t))$ of (1.8), and therefore oscillates more and more, increasing its Sobolev norms.

Lemma 4.5. *Let $z_t := (q(t), p(t))$ be the unbounded solution of Proposition 4.1, and denote by $z_0 := (q(0), p(0))$ its initial datum. Fix an arbitrary $r \in \mathbb{N}$ and $0 < \epsilon < 1$. Then there exist $\kappa, \hbar_0, \mathtt{C}_1, \mathtt{C}_2 > 0$ such that $\forall \hbar \in (0, \hbar_0]$, one has*

$$\|\mathcal{U}_2^{\hbar}(t,0)\varphi_{z_0}\|_r \geq \mathtt{C}_1\,[\log(2+t)]^r \tag{4.21}$$

for all times

$$2 \leq t \leq \mathtt{C}_2 \left[\log\left(\frac{\kappa}{\sqrt{\hbar}}\right)\right]^{1-\epsilon}. \tag{4.22}$$

Proof. The result is an application of Proposition 3.3, which requires the condition $\sqrt{\hbar}|F|_T \leq \kappa$ to be fulfilled. Having fixed $\kappa, \hbar_0 > 0$ sufficiently small (to be specified later), and estimating $|F|_T$ by Lemma 4.4, we obtain that T is constrained by the condition

$$0 \leq T \leq \mathtt{C}_2 \left[\log\left(\frac{\kappa}{\sqrt{\hbar_0}}\right)\right]^{1-\epsilon}, \tag{4.23}$$

where $\epsilon > 0$ is an arbitrarily small number and $\mathtt{C}_2 \equiv \mathtt{C}_2(\epsilon) > 0$. Define

$$\mathcal{E}_r(x,\xi) := (E(x,\xi))^r \equiv \left(\frac{|\xi|^2}{2} + W_l(x)\right)^r.$$

The function $\mathcal{E}_r$ is a symbol in $\Sigma^{2lr/(l+1)}$ fulfilling (3.22); moreover estimate (2.7) implies that

$$\|\mathcal{E}_r(x,\hbar D_x)^{1/2}\psi\|_0 \leq C'\|\psi\|_r, \quad \forall \psi \in \mathcal{H}^r. \tag{4.24}$$

By Proposition 3.3 we have, for every $t \in [0,T]$, the equality

$$\left\langle \mathcal{E}_r(x,\hbar D_x)\,\mathcal{U}_2^{\hbar}(t,0)\varphi_{z_0}, \mathcal{U}_2^{\hbar}(t,0)\varphi_{z_0}\right\rangle = \mathcal{E}_r(q(t),p(t)) + b(\hbar,t), \tag{4.25}$$

where $b(\hbar,t)$ fulfills, by (3.25) and (4.1)

$$|b(\hbar,t)| \leq \Gamma_1 \hbar^{\frac{1}{2}}\,|F_t|\,(1+E(z_t))^{r-\frac{1}{2l}} \leq \Gamma_1 C_2^{r-\frac{1}{2l}}\kappa\Big[\log(2+t)\Big]^{2r-\frac{1}{l}}, \qquad \forall(\hbar,t) \in (0,\hbar_0]\times[0,T]. \tag{4.26}$$

The function $\mathcal{E}_r(q(t),p(t))$ grows in time at a logarithmic speed; indeed by Proposition 4.1

$$\mathcal{E}_r(q(t),p(t)) \geq C_1\Big[\log(2+t)\Big]^{2r}. \tag{4.27}$$

Therefore collecting estimates (4.24)–(4.26) we obtain

$$\|\mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\|_r^2 \geq \frac{1}{C'^2}\left(C_1\Big[\log(2+t)\Big]^{2r} - \Gamma_1 C_2^{r-\frac{1}{2l}}\kappa\Big[\log(2+t)\Big]^{2r-\frac{1}{l}}\right) \geq \frac{C_1}{2C'^2}\Big[\log(2+t)\Big]^{2r},$$

for all time t provided

$$\mathsf{e}(\kappa) \leq t \leq T, \qquad \mathsf{e}(\kappa) := \exp\left[\left(\frac{2\Gamma_1 C_2^{r-\frac{1}{2l}}\kappa}{C_1}\right)^l\right].$$

Now fix $\kappa > 0$ so small that $\mathsf{e}(\kappa) \leq 2$, and $\hbar_0$ small enough so that 2 is smaller than the r.h.s. of (4.23). ☐

We can finally prove Theorem 1.1.

Proof of Theorem 1.1. The result is an application of Theorem 3.2 to system 1.1. Assumption ($\mathrm{H_{cl}}$) is trivially verified; to show that ($\mathrm{H_{qu}}$) holds note that by (2.7)

$$\sup_{\substack{\hbar\in(0,1]\\ t\in\mathbb{R}}} \frac{1}{\hbar}\|[\beta(t)x_1, H_l]\psi\|_r \leq C\|H_l^{\frac{1}{2}}\psi\|_r, \qquad \forall\psi\in\mathcal{H}^{r+1}.$$

Therefore condition (A.3) holds with $\tau = \frac{1}{2}$ and Theorem A.1 implies that

$$\sup_{\hbar\in(0,1]} \|\mathcal{U}^\hbar(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C'_r\,\langle t-s\rangle^r\,;$$

in particular condition (3.3) holds with $\mu = 1$.

Thus, by Theorem 3.2, Lemma 4.5 (with $\epsilon = 1/2$) and using also (4.1),(4.20), one finds constants $K_1, K_2, K_3 > 0$ such that

$$\|\mathcal{U}^\hbar(t,0)\varphi_{z_0}\|_r \geq \|\mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\|_r - \|\mathcal{U}^\hbar(t,0)\varphi_{z_0} - \mathcal{U}_2^\hbar(t,0)\varphi_{z_0}\|_r \geq K_1[\log(2+t)]^r$$

for all times

$$2 \leq t \leq K_2\left[\log\left(\frac{K_3}{\hbar}\right)\right]^{\frac{1}{2}}.$$

☐

A A semiclassical abstract theorem on growth of Sobolev norms

We prove here a semiclassical version of Thereom 1.5 of [31]. Thus consider an Hilbert space $\mathscr{H}^0$ and a positive, invertible, selfadjoint operator $K^\hbar$ (possibly $\hbar$-dependent) acting on it. Define the scale of spaces $\mathscr{H}^r := D\left((K^\hbar)^r\right)$, endowed with the norm $\|\psi\|_r \equiv \|(K^\hbar)^r\psi\|_{\mathscr{H}^0}$. Note that the norms might depend on $\hbar$ as well. On $\mathscr{H}^r$, consider the time dependent Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi(t) = L^\hbar(t)\,\psi(t)\,, \qquad \psi|_{t=s} = \psi_s \in \mathscr{H}^r \tag{A.1}$$

where $L^\hbar(t)$ is a selfadjoint operator in $C^0\Big([0,T], \mathcal{L}(\mathscr{H}^{r+m}, \mathscr{H}^r)\Big)$, $m\in\mathbb{R}$.

Theorem A.1. *Assume that there exists $\tau \in \mathbb{Q}$, $\tau < 1$ such that the following holds true: $\forall r \geq 0$, there exists $C_r > 0$ such that*

$$\sup_{\hbar\in(0,1]} \frac{1}{\hbar} \| \left[L^\hbar(t), K^\hbar\right] \, (K^\hbar)^{-\tau} \|_{\mathcal{L}(\mathscr{H}^r)} \leq C_r, \quad \forall t \in [0,T]. \tag{A.2}$$

Then equation (A.1) *has a unique propagator $\mathcal{U}^\hbar(t,s)$, $\forall t, s \in [0,T]$, unitary in $\mathscr{H}^0$ which restricts to a bounded operator from $\mathscr{H}^r$ to itself fulfilling*

$$\sup_{\hbar\in(0,1]} \|\mathcal{U}^\hbar(t,s)\|_{\mathcal{L}(\mathscr{H}^r)} \leq C'_r \left\langle t-s\right\rangle^{\frac{r}{1-\tau}} . \tag{A.3}$$

This result is proved in [31] for $\hbar = 1$; here we prove its extension to the semiclassical case.

Remark A.2. *In our application, we will set $K^\hbar = H_l$, and therefore $\mathscr{H}^r = \mathcal{H}^{2r}$.*

Proof. The existence of the propagator, its unitarity in $\mathscr{H}^0$ and the group property follow from Theorem 1.5 of [31]. To prove (A.3) we revisit the proof of that theorem. First by induction one verifies that $\forall k \in \mathbb{N}$

$$\sup_{\hbar\in(0,1]} \frac{1}{\hbar} \| \left[L^\hbar(t), (K^\hbar)^k\right] \, (K^\hbar)^{-(k-1+\tau)} \|_{\mathcal{L}(\mathscr{H}^0)} \leq C'_k. \tag{A.4}$$

(see e.g. [31, Lemma 2.1]). Now remark that $\mathcal{U}^\hbar(t,s)$ is an isometry in $\mathscr{H}^0$, so $\|\mathcal{U}^\hbar(t,s)\psi_s\|_k = \|[\mathcal{U}^\hbar(t,s)]^* \, (K^\hbar)^k \, \mathcal{U}^\hbar(t,s)\psi_s\|_0$. But we have

$$[\mathcal{U}^\hbar(t,s)]^* \, (K^\hbar)^k \, \mathcal{U}^\hbar(t,s)\psi_s = (K^\hbar)^k\psi_s + \frac{1}{\mathrm{i}\hbar}\int_s^t [\mathcal{U}^\hbar(t_1,s)]^* \, [L^\hbar(t_1), (K^\hbar)^k] \, \mathcal{U}^\hbar(t_1,s)\psi_s \, \mathrm{d}t_1$$

Hence using (A.4) we get the first estimate

$$\|\mathcal{U}^\hbar(t,s)\psi_s\|_k \leq \|\psi_s\|_k + C'_k \int_s^t \|\mathcal{U}^\hbar(t_1,s)\psi_s\|_{k-\theta} \mathrm{d}t_1, \quad \theta = 1-\tau. \tag{A.5}$$

After m iterations of (A.5), with other constants $C_{k,m}$, we get that $\|\mathcal{U}^\hbar(t,s)\psi_s\|_k$ is bounded by

$$C_{k,m}\sum_{j=0}^{m-1} \|\psi_s\|_{k-j\theta} \left\langle t-s\right\rangle^j + C_{k,m}\int_s^t\int_s^{t_1}\cdots\int_s^{t_{m-1}} \|\mathcal{U}^\hbar(t_m,s)\psi_s\|_{k-m\theta} \, \mathrm{d}t_m \mathrm{d}t_{m-1}\cdots\mathrm{d}t_1.$$

Since τ is rational, one can take k sufficiently large so that k/θ is integer, so choosing $m = k/\theta$ one has $\|\mathcal{U}^\hbar(t_m,s)\psi_s\|_{k-m\theta} = \|\mathcal{U}^\hbar(t_m,s)\psi_s\|_0 = \|\psi_s\|_0$, exploiting the fact that $\mathcal{U}^\hbar(t,s)$ is unitary in $\mathscr{H}^0$. Thus, one deduces (A.3) with $r = k$. The result for arbitrary $r \geq 0$ follows from linear interpolation, since $\mathcal{U}^\hbar(t,s)$ preserves the norm in $\mathscr{H}^0$ and k can be chosen as large as needed. □

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