Scuola
Internazionale
Superiore di
Studi Avanzati

Mathematics Area – PhD course in
Mathematical Analysis, Modelling, and Applications

On the relaxed area of maps from the plane to itself taking three values

Candidate:
ALAA ELSHORBAGY

Advisor:
PROF. GIOVANNI BELLETTINI

Academic Year 2018-19

Il presente lavoro costituisce la tesi presentata da Alaa Elshorbagy, sotto la direzione del Prof. Giovanni Bellettini, al fine di ottenere l'attestato di ricerca post-universitaria *Doctor Philosophiæ* presso la SISSA, Curriculum in Matematica Applicata, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della Sissa pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica*.

Trieste, Anno Accademico 2018-2019.

Acknowledgements.

I wish to express my sincere gratitude to my supervisor Giovanni Bellettini for his patience, his precious teachings and for all the support accorded to me during the entire period of my Ph.D. studies. I thank the Mathematics group at SISSA, in particular I would like to thank Gianni Dal Maso who in these years has followed my research with interest, giving very useful advices, and Barbara Fantechi for supporting me during my first year and inspiring me as a female mathematician. I am grateful to the entire section of Mathematics at ICTP, in particular to Claudio Arezzo, Lothar Gottsche, Stefano Luzzatto, Emanuel Carneiro, Fernando Villegas, Adriana Sofer, Oliver Butterley, Tarig Abdelgadir, Lucia Simonelli, Victoria Farfan, Wagner Ranter and Zakarias Dyrefelt, for guiding me since I came to Italy both as a diploma and a Ph.D. student. I would like to thank Riccardo Scala for the valuable discussions. Special thank goes to Koutou Mabilo for all the warm hugs. I am very grateful to my friends, who made me believe that we are never apart maybe in distance but never in heart, Eman Ahmed, Marwa Almowafy, Amr Elmohtaseb, Abdelhamid Salah, Mai Ahmed and Mohamed Mostafa. Finally I would like to thank my parents for their prayers that fill my life with blessings.

Contents

Introduction

In this thesis we address two different problems; the first and the main one is the study of the relaxed area functional $\mathcal{A}(u, \Omega)$ of the graph of vector maps $u \in L^1(\Omega; \mathbb{R}^k)$ on an open set $\Omega \subset \mathbb{R}^n$, mainly for $n = k = 2$. The characterization of the relaxation is far from being understood, also because for it there is no way to find an integral representation on the whole of its domain (which is strictly contained in the space of functions of bounded variation, see (1.9)). Indeed, as conjectured by De Giorgi [20] and proved in [1], even for $n = k = 2$ and for piecewise constant maps taking only three values, $\mathcal{A}(u, \cdot)$ is not subadditive, hence it is not a measure, see (1.13). In the same paper De Giorgi observed that: *"It would be interesting to study in a systematic way the functional $\mathcal{A}(u, \Omega)$ when u takes only a finite number of values"*. This motivates the work in the first part of the present thesis. We focus on the problem of estimating from above the area of the graph of a singular map u taking a disk D (called source disk) to three vectors, the vertices of a triangle, and jumping along three $\mathcal{C}^2-$ embedded curves that meet transversely at only one point (called source triple junction) of the disk. We show that the singular part of the relaxed area can be estimated from above by the solution of a Plateau-type problem involving three entangled nonparametric area-minimizing surfaces. The idea is to "fill the hole" in the graph of the singular map with an approximating sequence of smooth two-codimensional surfaces of graph-type, by imagining three minimal surfaces, placed vertically over the jump of u, coupled together via a triple point (called target triple point) in the target triangle. Such a construction depends on the choice of a target triple point, and on a connection passing through it, which dictate the boundary condition for the three minimal surfaces. We show that the singular part of the relaxed area of u cannot be larger than what we obtain by minimizing over all possible target triple points and all corresponding connections. We point out that under a symmetry assumption, both in the source and the target, the relaxed area $\mathcal{A}(u_{\text{symm}}, D)$ is actually given by the above upper estimate [10, 36], see the next Section for definition of u_{symm} and Section 1.3.1.

We investigate the possibility of adopting similar techniques to study the same problem in more general and different settings for instance when u has several (finite or infinite) triple junctions, or when $n = 3$, or when $\mathbb{R}^{2\times 2}$ is endowed by a Riemannian metric. An interesting open problem that we plan to address in the future, is to investigate the lower bound inequality (for the moment known only for $u = u_{\text{symm}}$); this seems to involve a strong use of geometric measure theory and Cartesian currents.

The second part of the thesis is devoted to the problem of characterizing *arbitrary codimensional* smooth manifolds *with boundary* embedded in an open set $\Omega \subset \mathbb{R}^n$ using the square distance function and the signed distance function from the manifold and from its boundary, *i.e.,* we want to isolate a set of necessary and sufficient conditions to be satisfied by the signed distance function and the square distance function from a set $E \subset \Omega$ and from a set $L \subset \Omega$ so that $E \cup L$ is a smooth manifold with boundary L in Ω.
One motivation for this study came from the curvature flow of planar networks with triple junctions [34, 35, 33], which is, in some weak sense, related to the area problem stated in the first part. The hope, still not clear at the present moment, is to find a way to express the flow (before singularities) via the distance functions; we recall that this has been successfully done in [25] in case of one codimensional manifolds without boundary, and in [21, 6] for arbitrary codimensional manifolds without boundary.

Many of the results in this thesis are contained in [8] and [7]; and have been obtained during my Ph.D. at SISSA (Scuola Internazionale Superiore di Studi Avanzati) in Trieste.

The relaxed area of the graph of a singular map

Let $\Omega \subset \mathbb{R}^2$ be an open set and $v = (v_1, v_2) : \Omega \to \mathbb{R}^2$ a Lipschitz map. It is well known that the area of the graph of v is given by

$$\mathbb{A}(v,\Omega) = \int_\Omega \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + \Big(\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y} - \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x}\Big)^2}\, dxdy. \tag{0.1}$$

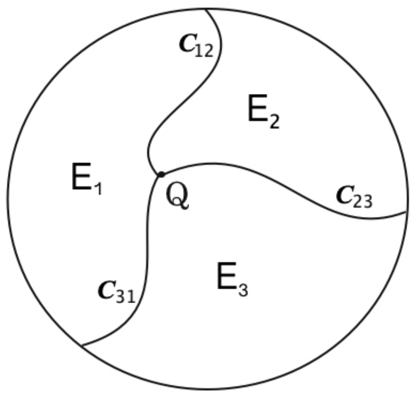
Extending to nonsmooth maps via relaxation the definition of the area is a difficult question [28], and is motivated by rather natural problems in calculus of variations: we can mention for example the use of direct methods to face the two-codimensional Plateau problem in $\mathbb{R}^4$ in cartesian form, and the study of lower semicontinuous envelopes of polyconvex functionals with nonstandard growth [1], [27]. A crucial issue is to decide which topology one has to consider in order to compute the relaxed functional of $\mathbb{A}(\cdot, \Omega)$: of course, the weakest the topology, the most difficult should be the computation of the relaxed functional, but the easiest becomes the coerciveness. We recall that when v is scalar valued, the natural choice is the $L^1(\Omega)$-convergence, and the relaxation problem is completely solved [18], [5]; the $L^1(\Omega)$-relaxed functional in this case consists, besides the absolutely continuous part, of a singular part which is the total variation of the jump and Cantor parts of the distributional derivative of v in Ω; in particular, the relaxed functional, when considered as a function of Ω, is a measure.
The case of interest here, namely when v takes values in $\mathbb{R}^2$, is much more involved, due to the *nonconvexity* of the integrand in (0.1), and to the *unilateral linear growth*

$$\mathbb{A}(v,\Omega) \geq \int_\Omega \sqrt{|\nabla v_1|^2 + |\nabla v_2|^2}\, dxdy.$$

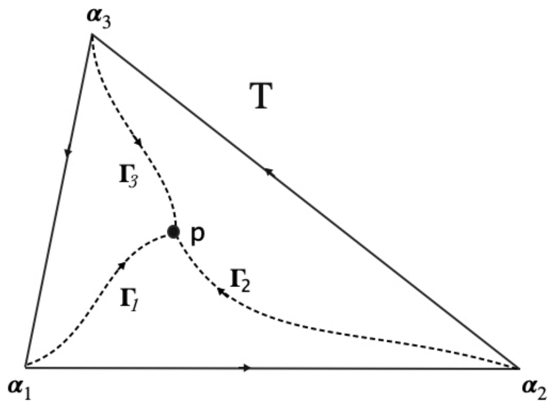
Choosing again the $L^1(\Omega;\mathbb{R}^2)$-convergence (as we shall do in this thesis), the relaxed functional $\mathcal{A}(\cdot,\Omega)$ of $\mathbb{A}(\cdot,\Omega)$, *i.e.*,

$$\mathcal{A}(v,\Omega) := \inf\left\{ \liminf_{\epsilon\to 0} \mathbb{A}(u^\epsilon,\Omega) : \{u^\epsilon\}\subset \mathrm{Lip}(\Omega;\mathbb{R}^2),\ u^\epsilon \to u \text{ in } L^1(\Omega;\mathbb{R}^2)\right\}, \quad (0.2)$$

is, for $v\in L^1(\Omega;\mathbb{R}^2)\backslash W^{1,2}(\Omega;\mathbb{R}^2)$, far from being understood, and exhibits surprising features. One of the few known facts that must be pointed out is that, for a large class of nonsmooth maps v, the function $\Omega \to \mathcal{A}(v,\Omega)$ cannot be written as an integral [1], [12], [13]; this interesting phenomenon, related to nonlocality, has at least two sources. For simplicity, let us focus our attention on nonsmooth functions with jumps, thus neglecting the case of vortices. The first source of nonlocality has been enlightened answering to a conjecture in [20]. Specifically, consider the symmetric triple junction map u_{symm}, *i.e.*, the singular map from a disk D of $\mathbb{R}^2_S=\mathbb{R}^2$ into $\mathbb{R}^2_T=\mathbb{R}^2$, taking only three values – the vertices of an equilateral triangle $T_{\text{eq}}\subset\mathbb{R}^2_T$ – and jumping along three segments meeting at the origin in a triple junction at equal 120° angles: then $\mathcal{A}(u_{\text{symm}},\cdot)$ is not subadditive. This result has been proven in [1]; subsequently in [11] it is shown that the value $\mathcal{A}(u_{\text{symm}},D)$ is related to the solution of three one-codimensional Plateau-type problems in cartesian form suitably entangled together through the Steiner point in the triangle T_{eq}. Due to the special symmetry of the map u_{symm}, the three-problems collapse together to only one one-codimensional Plateau-type problem in cartesian form, on a fixed rectangle R whose sides are the radius of D and the side of T_{eq}. Positioning three copies of this minimal surface "vertically" (in the space of graphs, *i.e.*, in $D\times\mathbb{R}^2$) over the jump of u_{symm} allows, in turn, to construct a sequence $\{u_\varepsilon\}$ of *Lipschitz* maps from D into $\mathbb{R}^2$ the limit area of which improves the upper estimate of [1]. Optimality of this construction has been shown in the recent paper [36], on the basis of a symmetrization procedure for currents.
It is one of the aims of the present thesis to inspect solutions of the above mentioned three Plateau-type problems in more general situations, in order to provide upper estimates for $\mathcal{A}(u,D)$, for suitable piecewise constant maps u.
A second source of nonlocality for the functional $\mathcal{A}(u,\Omega)$ is given by the interaction of the jump set of a discontinuous map u with the boundary of the domain Ω. This phenomenon, already observed in [1] for the map with one-vortex at the center of a suitable disk, appears also for functions with jump discontinuities not piecewise constant [13]. More surprisingly, it appears also for piecewise constant maps taking three values, provided the jump is sufficiently close to the boundary of Ω, as observed in [36], taking as Ω a sufficiently thin tubular neighbourhood of the jump itself. In this thesis we shall not be concerned with this second source of nonlocality. Also, we shall not study the relaxation on Sobolev functions in $W^{1,p}$, $p\in[1,2)$, in particular vortices. As already said, we are interested in estimating from above the area of the graph of a singular map u taking three (non collinear) values and jumping along three embedded curves of class $\mathcal{C}^2$ that meet transversely at only one point, see Figure 1. Let us state this in a more precise way, referring to Chapter 2 for all details. For simplicity from now on for the first part of the present thesis, we fix Ω to be an open disk D containing the origin 0_S in the source plane $\mathbb{R}^2=\mathbb{R}^2_{x,y}=\mathbb{R}^2_S$. Take



(a) The domain of u; $u = \alpha_i$ on E_i.



(b) A Lipschitz graph-type connection in the target triangle T. $\Gamma_1 \cup \Gamma_2$ (resp. $\Gamma_2 \cup \Gamma_3$, $\Gamma_3 \cup \Gamma_1$) is graph over the segment $\overline{\alpha_1\alpha_2}$ (resp. $\overline{\alpha_2\alpha_3}$, $\overline{\alpha_3\alpha_1}$) of a Lipschitz function φ_{12} (resp. φ_{23}, φ_{31}).

Figure 1

three non-overlapping non-empty two-dimensional connected regions E_1, E_2, E_3 of D such that

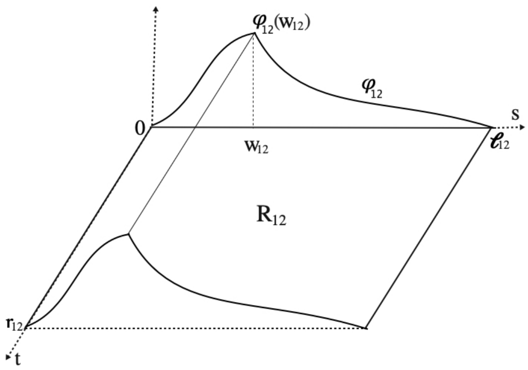
$$E_1 \cup E_2 \cup E_3 = D. \tag{0.3}$$

The three regions are separated by three embedded curves of class $\mathcal{C}^2$ (up to the boundary) of length r_{12}, r_{23}, r_{31} respectively, that meet only at Q (source triple junction); moreover, each curve is supposed to meet the boundary of D transversely and we assume also that Q is a transversal intersection for the three curves, see Figure 1a. Let α_1, α_2, α_3 be the vertices of a closed triangle T with non empty interior in the target plane.
Set

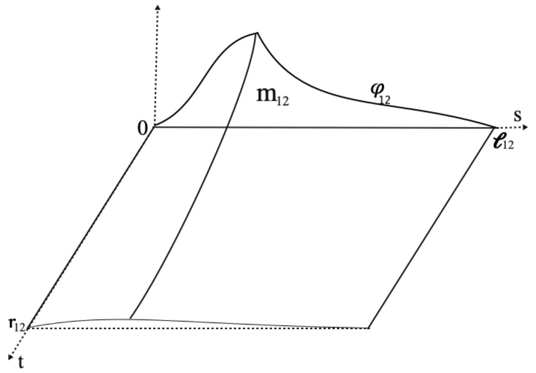
$$\ell_{12} := |\alpha_1 - \alpha_2|, \qquad \ell_{23} := |\alpha_2 - \alpha_3|, \qquad \ell_{31} := |\alpha_1 - \alpha_3|. \tag{0.4}$$

We suppose that T contains the origin 0_{T} in its interior.
Let us introduce the space X of connections (Definition 2.2 and (2.4), (2.5)); a connection $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ consists of three rectifiable curves in T, that connect the vertices of T to some point inside the triangle (called target triple point). We shall suppose that *each curve can be written as a graph, possibly with vertical parts, over the corresponding two sides of* T. When Γ consists of three Lipschitz graphs, we write $\Gamma \in X_{\mathrm{Lip}}$, and we say that Γ is a Lipschitz connection. We now show how to construct a new functional $\mathcal{G}$, consisting of the sum of the areas of three minimal surfaces – graphs of three suitable area-minimizing functions m_{12}, m_{23}, m_{31} defined on certain rectangles – coupled together by the connection considered as a Dirichlet boundary condition, see Definition 2.5.
Set

$$\mathrm{R}_{ij} := [0, \ell_{ij}] \times [0, r_{ij}], \qquad ij \in \{12, 23, 31\}. \tag{0.5}$$



(a) The graph of the function φ_{12} on R_{12}.



(b) The graph of m_{12} on R_{12}.

Figure 2

Assume $\Gamma \in X$. Then $\Gamma_{ij} := \Gamma_i \cup \Gamma_j$, $ij \in \{12, 23, 31\}$ are (generalized) graphs of functions φ_{ij} of bounded variation over $[0, \ell_{ij}]$. With a small abuse of notation, set

$$\varphi_{ij}(s,t) = \varphi_{ij}(s), \qquad (s,t) \in \mathrm{R}_{ij}, \qquad ij \in \{12, 23, 31\}. \tag{0.6}$$

The graph of φ_{12} on R_{12} is depicted in Figure 2a.
Let $m_{ij} = m_{ij}(\Gamma)$ be the unique solution of the Dirichlet-Neumann minimum problem, discussed in Section 1.4 and Section 2.2,

$$\min \left\{ \int_{\mathrm{R}_{ij}} \sqrt{1 + |\nabla f|^2}\, dsdt : \ f \in W^{1,1}(\mathrm{R}_{ij}),\ f = \varphi_{ij}\ \mathcal{H}^1 - a.e. \text{ on } \partial_D \mathrm{R}_{ij} \right\}, \tag{0.7}$$

where

$$\partial_D \mathrm{R}_{ij} = \partial \mathrm{R}_{ij} \setminus ([0, \ell_{ij}] \times \{r_{ij}\}), \qquad ij \in \{12, 23, 31\}. \tag{0.8}$$

Notice that the minimization is taken among all functions having a Dirichlet condition on three of the four sides of the rectangle R_{ij}; the missing side corresponds to the intersection points of the jump with the boundary of D.
From (0.6) it follows that the Dirichlet condition is zero on the sides $\{0\} \times [0, r_{ij}]$ and $\{\ell_{ij}\} \times [0, r_{ij}]$ of R_{ij}; see Figure 2b.
Set

$$\mathfrak{A}_{ij}(\Gamma) := \int_{\mathrm{R}_{ij}} \sqrt{1 + |\nabla m_{ij}|^2}\, dsdt, \qquad ij \in \{12, 23, 31\}. \tag{0.9}$$

The main result of the the first part of this thesis reads as follows (see Theorem 2.1 and Corollary 3.8).

Theorem 0.1. *Let $u : D \to \{\alpha_1, \alpha_2, \alpha_3\}$ be the discontinuous $BV(D; \mathbb{R}^2)$ function defined as*

$$u(x,y) := \begin{cases} \alpha_1 & \text{if } (x,y) \in E_1, \\ \alpha_2 & \text{if } (x,y) \in E_2, \\ \alpha_3 & \text{if } (x,y) \in E_3. \end{cases} \tag{0.10}$$

Then

$$\mathcal{A}(u, D) \leq |D| + \min\Big\{\mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma) : \Gamma \in X\Big\}. \tag{0.11}$$

This theorem says that the singular part of $\mathcal{A}(u, D)$ can be estimated from above by

$$\inf\Big\{\mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma) : \Gamma \in X\Big\} \tag{0.12}$$

and that such an infimum is a minimum. Intuitively, to “fill the hole” in the graph of u with smooth two-codimensional approximating surfaces of graph-type, we start to imagine three area-minimizing surfaces, placed vertically over the jump of u, coupled together via a triple point in the target triangle T (notice that the union of these three surfaces, viewed in $D \times \mathbb{R}^2$, is not smooth in correspondence of the source triple junction). Such a construction depends on the choice of a target triple point, and on a connection Γ passing through it, dictating the boundary condition for the three area-minimizing surfaces, over the sides of the triangle T. Theorem 0.1 asserts that the interesting part of the relaxed area of u, namely its singular part, cannot be larger than what we obtain by minimizing over all possible target triple points and all corresponding connections. As a direct consequence of the results in [11], [36], when $u = u_{\text{symm}}$ (and 0_S is the center of D), the inequality in (0.11) is an equality, and the infimum in (0.12) is achieved by the Steiner graph connecting the three vertices of T (the optimal triple point being the Steiner point, *i.e.*, the barycenter of T). This seems to be an interesting result that could be stated purely as a problem of three entangled area-minimizing surfaces (each of which lies in a half-space of $\mathbb{R}^4$, the three half-spaces having only $\{0\} \times \mathbb{R}^2$ in common) without referring to the relaxation of the functional $\mathbb{A}(\cdot, D)$. We do not know whether, in general, the Steiner graph is still the solution of the minimization problem in (0.12), when no symmetry assumptions (the case we are considering here) are required. However it is reasonable to expect that, if in the source we have symmetry, *i.e.*, the source triple junction is positioned at the center of D and u jumps along three segments meeting at equal 120° angles, and if the target triangle T is close to be equilateral, the inequality in (0.11) to be still an equality. In this respect, it is worthwhile to observe that showing a lower estimate, for instance showing that, in certain cases, the inequality in (0.11) is an equality, seems difficult. One of the main technical obstructions is due to the poor control on the tangential derivative of v^ε in proximity of the jump of a discontinuous L^1-limit function v (see [13]), where $\{v^\varepsilon\}$ is a sequence of Lipschitz maps converging in $L^1(\Omega; \mathbb{R}^2)$ to v, and satisfying the uniform bound $\sup_\varepsilon \mathbb{A}(v^\varepsilon, \Omega) < +\infty$. We also notice that the symmetrization methods of [36] cannot be applied anymore, in view of the lackness of symmetry.

It is worth mentioning that the restriction that we assume on the connections Γ, namely that each Γ_i is a graph (possibly with vertical parts) on the correspond-

ing two sides of T, cannot be avoided in our approach: indeed, only under this graphicality assumption we can solve the minimum problem in (0.11) in the class of surfaces which are graphs over the rectangles R_{ij}. In turn, the graphicality of such area-minimizing surfaces allows to construct the sequence $\{u^\varepsilon\}$, see (2.21), and find a uniform bound for the length in the space X of connections, see Proposition 3.3. Removing the graphicality assumption on Γ requires some change of perspective, and needs further investigation, see Section 4.2 for more details.
One more question that rise naturally is that whether or not the same result may apply to

$$\overline{u} : B \subset \mathbb{R}^3 \to \{\alpha_1, \alpha_2, \alpha_3\}, \qquad \overline{u}(\overline{E}_i) := \alpha_i, \quad i = 1, 2, 3,$$

providing that B is an open ball in $\mathbb{R}^3$ and $\overline{E}_i$, $i = 1, 2, 3$, are three partitions of B separated by three planes that meet at one line, see Figure 4.5. The problem is essentially what kind of boundary conditions that should be assigned to the corresponding one-codimensional minimizing problem, see (4.12) and Remark 4.1.

The square distance function from a manifold with boundary

It is well known that the smoothness of the boundary of a bounded open subset of $\mathbb{R}^n$ can be characterized$^{(1)}$ using the signed distance function (see for instance [26, 30, 29, 22]). This characterization is useful for several purposes, in particular is related to the study of Hamilton-Jacobi equations [15] and it can be used to face the mean curvature flow of a one-codimensional family of smooth embedded hypersurfaces without boundary [25].
For a compact smooth embedded manifold without boundary of arbitrary codimension, it turns out that the meaningful function to be considered is the square distance function: in [19] De Giorgi conjectured$^{(2)}$ that if E is a connected subset of an open set $\Omega \subseteq \mathbb{R}^n$ such that $E \cap \Omega = \overline{E} \cap \Omega$ and the $\frac{1}{2}$-square distance function from E,

$$\eta_E(x) := \frac{1}{2} \inf_{y \in E} |x - y|^2, \qquad x \in \mathbb{R}^n,$$

is smooth in a neighborhood of E, then E is an embedded smooth manifold$^{(3)}$ without boundary in Ω of codimension equal to rank $\left(\nabla^2 \eta_E\right)$. Such a conjecture has been proven in [6, 9] (see also [22]) and can be considered as one of the motivations of this paper.
Investigations on arbitrary codimensional mean curvature flow lead De Giorgi [21] to further express the motion using the Laplacian of the gradient of the $\frac{1}{2}$-square distance function from the evolving manifolds, and also to describe the flow passing to a level set formulation: we refer to [6] for more details, and to [10, 37, 4] for further applications.
The aim of the second part of this thesis is to characterize a smooth arbitrary codimensional manifold *with boundary* embedded in $\mathbb{R}^n$, using the distance functions.

$^{(1)}$see Theorem 1.27.
$^{(2)}$If $\mathcal{M}$ is a compact smooth embedded manifold without boundary then the square distance function $\eta_{\mathcal{M}}$ is smooth in a suitable tubular neighborhood of $\mathcal{M}$, see Theorem 1.29.
$^{(3)}$See Theorem 1.30.

The presence of the boundary is the novelty here, and indeed another motivation for our research came from the study of curvature flow of networks [35], where a sort of "boundary" (the triple points) is present in the evolution problem.
We start our discussion on the smoothness of the distance function from a manifold with boundary with a simple observation. Let E be a smooth compact curve in $\mathbb{R}^n$ with two end points (like the ones in Figure 5.1 for $n = 2$ or in Figure 5.4 for $n = 3$): then η_E turns out to be smooth in a sufficiently small neighborhood of E, excluding portions of a smooth hypersurface orthogonal to the boundary of E (the two dashed segments in Figure 5.1, and the two disks in Figure 5.4)(4). This suggests that we have to exclude the boundary and possibly some portions of a hypersurface containing it, if we are hoping to get some sort of regularity for the squared distance function from a manifold $\mathcal{M}$ with boundary. In fact, in Propositions 5.12 3) and 5.14 3) we show that, in general, $\eta_{\mathcal{M}}$ is smooth in a neighborhood of $\mathcal{M}$ out of a suitable hypersurface containing the boundary.
Supposing $\overline{\mathcal{M}} = \mathcal{M}$(5) is a smooth manifold with boundary, roughly speaking $\mathcal{M}$ is the union of two sets: the relative interior $\mathcal{M}^\circ$ (a relatively open subset of $\mathcal{M}$) and the boundary $\partial\mathcal{M}$ (a smooth submanifold of $\mathcal{M}$ of codimension one so that $\mathcal{M}$ lies locally "on one side" of $\partial\mathcal{M}$), joined smoothly; in particular $\mathcal{M}$ is contained in the relative interior of a larger smooth manifold of the same dimension.
We want to mimic the above properties for a pair of subsets of $\mathbb{R}^n$, making use only of the distance functions and their regularity properties. Therefore, let E and L be two subsets of $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. We want to isolate a set of necessary and sufficient conditions to be satisfied by the signed distance function and the square distance function from E and from L so that $E \cup L$ is a smooth manifold with boundary L in Ω. Our main Definition 5.4 reformulate the above properties as follows. We say that $E \cup L$ is a smooth manifold with boundary *in the sense of distance functions*, and we write $(E, L) \in D_h B\mathcal{C}^k(\Omega)$ (where h stands for the dimension of E and k for its smoothness degree), if:

- $\overline{L} \cap \Omega = L \cap \Omega$ and η_L is smooth in a neighborhood of L in Ω: this guarantees the smoothness of L;
- $\overline{E} \cap (\Omega \setminus L) = E \cap (\Omega \setminus L)$ and η_E is smooth in a neighborhood of E in $\Omega \setminus L$: this guarantees the smoothness of E in $\Omega \setminus L$;
- all points of L are accumulation points of E;
- there is a neighborhood B of $E \setminus L$ in Ω such that the signed distance function d_B from B (negative in B) is smooth in a neighborhood A of L: this guarantees the smoothness of the boundary of B in A. Such a boundary is, roughly, represented by the two dashed segments in Figure 5.1 and the two disks in Figure 5.4. Hence the set $E \setminus L$ must lie on one side of L. In particular points of L do not belong to the relative interior of $E \cup L$, see Figure 5.3;

(4)We can even consider the case $n = 1$: take a bounded closed interval $E = [a, b] \subset R$. Then $\eta_E \in \mathcal{C}^{1,1}$ but not $\mathcal{C}^2$ in any neighborhood of E in $\mathbb{R}$; however, η_E is smooth in $\mathbb{R} \setminus \{a, b\}$, see Figure 1.1.
(5)By $\overline{\mathcal{M}}$ we denote the closure of $\mathcal{M}$.

- there is a smooth extension of η_E in an open neighborhood of $\overline{B} \cap A$: this ensures that E and L join smoothly.

Notice that localization of Definition 5.1 (on which Definition 5.4 is based) in an open set is necessary: for instance, even in the simplest case $\Omega = \mathbb{R}^n$ in the list above, the regularity on η_E is required only in $\mathbb{R}^n \setminus L$, which is an open set.
The main result of the second part of this thesis reads as follows (See Theorems 5.11 and 5.15):

Theorem 0.2. *Let $k \in \mathbb{N}$, $k \geq 3$, or $k \in \{\infty, \omega\}$ and $h \in \{1, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set.*

a. If $\mathcal{M} \subset \mathbb{R}^n$ is an embedded $\mathcal{C}^k$-manifold of dimension $h \leq n$ with nonempty boundary $\partial_\Omega \mathcal{M}$ in Ω then $(\mathcal{M}, \partial_\Omega \mathcal{M}) \in D_h B\mathcal{C}^{k-1}(\Omega)$,

b. If $E, L \subset \mathbb{R}^n$ are such that $(E, L) \in D_h B\mathcal{C}^k(\Omega)$ then $(E \cup L) \cap \Omega$ is a h-dimensional $\mathcal{C}^{k-1}$-manifold with boundary $L \cap \Omega$ in Ω,

where $D_h B\mathcal{C}^k(\Omega)$ is the class of sets defined in Definition 5.4.

The content of the thesis is the following:
We start by a chapter of preliminaries. In Section 1.2 we recall some of the definitions and relevant results about the theory of functions of bounded variation, in particular, functions of one variable, the definition of generalized graph (formula (1.4)), and the chain rule. We define the relaxed area for vector valued functions and state some of the known results related to the characterization of such functional (Section 1.3). Solutions of a particular Plateau's problem will play an important role in the proofs, so we mention some of the results that will be needed (Section 1.4). In Section 1.5 we recall some properties of Cartesian currents carried by a BV-function. Finally we recall some of the known results relating the regularity of a set E with the regularity of the distance functions from it (Section 1.6).
In Chapter 2 the space of connections X and the functional $\mathcal{G}$, appearing on the right hand side of (0.11), are introduced (see Definition 2.2 and Definition 2.5 respectively). In Section 2.3 we show that

$$\mathcal{A}(u, D) \leq |D| + \inf \left\{ \mathcal{G}(\Gamma) : \Gamma \in X_{\mathrm{Lip}} \right\}, \tag{0.13}$$

see Theorem 2.1. The proof is rather involved, mainly due to technical difficulties: we first start by supposing that the jump of u consists of three segments (Proposition 2.8). Some work is required to define u^ε on an ε-strip around the jump of u and avoiding a neighbourhood of the source triple junction (formula (2.21)) and to define u^ε in the missing neighbourhood of the source triple junction (step 3 of the proof of Proposition 2.8): the construction must be done in such a way that u^ε remains Lipschitz, and turns out to be rather involved in the three triangles $T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon$, see Figure 2.2b.
In Chapter 3 we prove that the infimum in (0.12) is a minimum. The proof is achieved by defining a topology in the space X which allows to prove the density

of X_{Lip} in X (Lemma 3.2), the continuity of the functional $\mathcal{G}$ (Proposition 3.4) and the sequential compactness of X (Theorem 3.6). This latter result is also based on a uniform bound on the length of the connections (Proposition 3.3), which is a consequence of the bi-graphicality assumptions on the connections.
In Chapter 4 we introduce three different ways to extend the results obtained in Chapter 2 and Chapter 3, and we point out some of the difficulties that we may face. We start by introducing a possibility to adopt the former techniques to functions with a finite number of triple junctions; then we point out that there might be a way to construct a function u_* with an infinite number of triple junctions, accumulating at the origin, such that $\mathcal{A}(u_*, D) < \infty$. In Section 4.2 we discuss the possibility of removing the bi-graphicality condition in Definition 2.2. In Section 4.3 we change the domain of the piecewise constant map $\overline{u}$ to be an open ball B in $\mathbb{R}^3$, keeping the vertices of a triangle $\mathrm{T} \subset \mathbb{R}^2_T$ as a target, *i.e.,* the graph of $\overline{u}$ is of co-dimension two in $\mathbb{R}^5$. The last part of this chapter, Section 4.4, deals with the case where the domain and the target of u are endowed with Riemannian metrics.
Chapter 5 is dedicated to the study of the second problem that we aim to investigate in this thesis. We start by introducing the class $D_h\mathcal{C}^k(\Omega)$ of h-dimensional embedded $\mathcal{C}^k$-manifolds without boundary in Ω in the sense of distance functions (Definition 5.1). We recall the correspondence between the classical definition of manifolds without boundary and sets in $D_h\mathcal{C}^k(\Omega)$, based on the known results quoted in Subsection 1.6.1, see Remark 5.2. In Definition 5.4 we introduce the class $D_hB\mathcal{C}^k(\Omega)$; in Section 5.1 we illustrate the motivations behind this definition through several observations (Remark 5.5) and examples. In Section 5.2 we prove our first main result in this chapter (Theorem 5.11) showing that h-dimensional embedded $\mathcal{C}^k$-manifolds with boundary in Ω are elements of $D_hB\mathcal{C}^{k-1}(\Omega)$. In Section 5.3 we prove our second main result[(6)] (Theorem 5.15), showing that sets in $D_hB\mathcal{C}^k(\Omega)$ are h-dimensional embedded $\mathcal{C}^{k-1}$-manifolds with boundary. This concludes the proof of Theorem 0.2.

[(6)]In the C^∞ or analytic case, this is the converse of Theorem 5.11.

1. Preliminaries

1.1 Basic notation

$\mathbb{N}$	The set of positive natural numbers
$\mathbb{R}^n$	Euclidean space of dimension $n,\ n \in \mathbb{N}$
$\lvert \cdot \rvert$	The Euclidean norm of a vector or the Lebesgue measure of a set
Ω	Open set in $\mathbb{R}^n$
$\mathcal{B}(\Omega)$	The σ-algebra of Borel subsets of Ω
$B_\rho(x)$	Open ball of $\mathbb{R}^n$(resp. of $\mathbb{R}^h,\ h < n$) centered at x of radius $\rho > 0$
$\mathcal{C}(\Omega;\mathbb{R}^k)$	Space of continuous maps in Ω
$\mathcal{C}_c(\Omega;\mathbb{R}^k)$	Space of continuous maps with compact support in Ω
$\mathcal{C}^m(\Omega;\mathbb{R}^k)$	Space of maps with m-times continuous derivatives in Ω
$\mathcal{C}^{m,\alpha}(\Omega;\mathbb{R}^k)$	Maps in $\mathcal{C}^m(\Omega;\mathbb{R}^k)$ with m-th derivative Hölder continuous with exponent α
$\mathcal{C}_c^m(\Omega;\mathbb{R}^k)$	The space $\mathcal{C}^m(\Omega;\mathbb{R}^k) \cap \mathcal{C}_c(\Omega;\mathbb{R}^k)$
$\mathcal{C}^\omega(\Omega;\mathbb{R}^k)$	Space of real analytic functions in Ω
$\mathcal{C}_0(\Omega;\mathbb{R})$	The space of real-valued continuous functions on Ω which vanish at infinity
$\mathrm{Lip}(\Omega;\mathbb{R}^k)$	Space of Lipschitz maps in Ω
$L^p(\Omega;\mathbb{R}^k)$	The Lebesgue space of exponent $p,\ p \in [1, +\infty]$
$W^{1,p}(\Omega;\mathbb{R}^k)$	Space of maps belonging, with their distributional derivative, to $L^p(\Omega;\mathbb{R}^k)$
$\dot{f}$	The derivative of the function f, of one variable, in the classical sense
f'	The derivative of the function f, of one variable, in the sense of distribution
$\mathcal{L}^n$	The n-dimensional Lebesgue measure
$\mathcal{H}^n$	The n-dimensional Hausdorff measure
$\mathbb{1}_E$	The characteristic function of the set E
$F \triangle E$	The symmetric difference of the sets E and F
supp	The support of a function, of a measure, or of a current
∂^-	The reduced boundary of a set

We set $H^1(\Omega;\mathbb{R}^k) := W^{1,2}(\Omega;\mathbb{R}^k)$. If $k = 1$, we omit to indicate the target space (*e.g.* $\mathcal{C}(\Omega)$ in place of $\mathcal{C}(\Omega;\mathbb{R})$).

1.2 Functions of bounded variation

In this section we collect some of the definitions and relevant results about the theory of bounded variation maps. We refer to [5] for an exhaustive presentation.

Definition 1.1. *Let $u \in L^1(\Omega;\mathbb{R}^k)$; we say that u is a function of bounded variation in Ω if the distributional derivative Du of u is a finite Radon measure on Ω with values in the space $\mathbf{M}^{k\times n}$ of all $k \times n$ matrices and we write $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$.*

For any open set $A \subset \Omega$ we define *the variation* $|Du|(A)$ of $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ in A as

$$|Du|(A) := \sup\{\sum_{j=1}^{k} \int_A u^j \mathrm{div}\phi^j dx : \phi \in [\mathcal{C}^1_c(A)]^{kn}, ||\phi||_\infty \leq 1\}, \tag{1.1}$$

where we consider the Hilbert-Schmidt norm on $\mathbf{M}^{k\times n}$, i.e.,

$$||\phi||_\infty = \sup_{x\in A} |\phi(x)| \qquad \text{and} \qquad |\phi(x)|^2 = \sum_{\substack{j\in\{1,\cdots,k\}\\ h\in\{1,\cdots,n\}}} |\phi^j_h(x)|^2.$$

Then $|Du|(\cdot)$ can be extended to a Borel measure in Ω as follows

$$|Du|(B) := \inf\{|Du|(A) : B \subset A, \quad A \text{ open}\}, \qquad B \in \mathcal{B}(\Omega).$$

Moreover we have

- $u \in L^1(\Omega;\mathbb{R}^k)$ belongs to $\mathrm{BV}(\Omega;\mathbb{R}^k)$ if and only if $|Du|(\Omega) < \infty$,
- $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega;\mathbb{R}^k)$ with respect to the $L^1_{loc}(\Omega;\mathbb{R}^k)$ topology,
- $\mathrm{BV}(\Omega;\mathbb{R}^k)$, endowed with the norm

$$||u||_{\mathrm{BV}} := \int_\Omega |u|dx + |Du|(\Omega)$$

  is a Banach space; however, even for $k = 1$, the space $\mathcal{C}^1(\Omega)$ is not dense in $\mathrm{BV}(\Omega)$.

Definition 1.2. *We say that $\{u_h\} \subset \mathrm{BV}(\Omega;\mathbb{R}^k)$ weakly* converges in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ to $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ if $\{u_h\}$ converges to u in $L^1(\Omega;\mathbb{R}^k)$ and $\{Du_h\}$ weakly* converges to D_u in Ω, i.e.,*

$$\lim_{h\to\infty} \int_\Omega \phi dDu_h = \int_\Omega \phi dDu \qquad \forall \phi \in \mathcal{C}_0(\Omega).$$

It can be proved that

- $\{u_h\}$ weakly* converges to u in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ if and only if $\{u_h\}$ is bounded in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ and converges to u in $L^1(\Omega;\mathbb{R}^k)$,

- if $\{u_h\}$ converges to u in $L^1(\Omega;\mathbb{R}^k)$ and $|Du_h|(\Omega)$ converge to $|Du|(\Omega)$ then $\{u_h\}$ weakly* converges to u in $\mathrm{BV}(\Omega;\mathbb{R}^k)$, however the opposite is not true.

Theorem 1.3 (Compactness in BV). *Every $\{u_h\} \subset \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ satisfying*

$$\sup\{\int_A |u_h|dx + |Du_h|(A) : h \in \mathbb{N}\} < \infty \qquad \forall A \subset\subset \Omega \text{ open},$$

admits a subsequence $\{u_{h(k)}\}$ converging in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ to $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$. Moreover if Ω has compact Lipschitz boundary and the sequence is bounded in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ then $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and the subsequence weakly converges to u.*

For any measurable function $u : \Omega \to \mathbb{R}$ the *subgraph* of u is defined as the measurable subset of $\Omega \times \mathbb{R}$ given by

$$S\mathcal{G}_{u,\Omega} := \{(x,y) \in \Omega \times \mathbb{R} \mid y < u(x)\}. \tag{1.2}$$

Sets of finite perimeter

Definition 1.4. *Let $E \subset \Omega$ be $\mathcal{L}^n$-measurable set. The perimeter $P(E,\Omega)$ of E in Ω is the variation $|D\mathbb{1}_E|(\Omega)$, defined in (1.1), of $\mathbb{1}_E$ in Ω. We say that E is a set of finite perimeter in Ω if $P(E,\Omega) < +\infty$.*

Sets of finite perimeter can be approximated by open sets with smooth boundary in the following sense: If Ω is bounded with compact Lipschitz boundary and $P(E,\Omega) < +\infty$, then there exists a sequence $\{E_h\}$ of open sets with smooth boundary in $\mathbb{R}^n$ such that

$$|\Omega \cap (E_h \triangle E)| \to 0 \quad \text{and} \quad P(E_h, \overline{\Omega}) \to P(E,\Omega), \qquad \text{as } h \to +\infty.$$

If E is a set of finite perimeter in Ω we define the *reduced boundary* $\partial^- E$ of E in Ω as the set of points $x \in \mathrm{supp}|D\mathbb{1}_E| \cap \Omega$ such that the limit

$$\nu(x,E) := \lim_{\rho \downarrow 0} \frac{D\mathbb{1}_E(B_\rho(x))}{|D\mathbb{1}_E|(B_\rho(x))}, \tag{1.3}$$

exists in $\mathbb{R}^n$ and satisfies $|\nu(x,E)| = 1$.
The set $\partial^- E$ is a Borel set and the function $\nu(\cdot,E) : \partial^- E \to \mathbb{S}^{n-1}$ is a Borel map which we call the *generalized inner normal* to E. Moreover we have

$$P(E,\Omega) = \mathcal{H}^{n-1}(\partial^- E) \qquad \text{and} \qquad |D\mathbb{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^- E.$$

Finally the Gauss-Green formula still holds for sets of finite perimeter; if E is a set of finite perimeter in Ω then

$$\int_E \mathrm{div}\phi \, dx = -\int_{\partial^- E} < \nu(x,E), \phi > d\mathcal{H}^{n-1}(x) \qquad \forall \phi \in \mathcal{C}^1_c(\Omega;\mathbb{R}^n).$$

Theorem 1.5. *[28, Thm. 1. p. 371] Let $u \in L^1(\Omega)$. Then $u \in \mathrm{BV}(\Omega)$ if and only if the subgraph $S\mathcal{G}_{u,\Omega}$, defined in (1.2), of u has a finite perimeter in $\Omega \times \mathbb{R}$.*

BV functions of one variable

Definition 1.6. *For $\varphi : (a,b) \to \mathbb{R}^k$ we define the pointwise variation $pV(\varphi,(a,b))$ of φ in (a,b) as*

$$pV(\varphi,(a,b)) := \sup\{\sum_{i=1}^{n-1} |\varphi(s_{i+1}) - \varphi(s_i)| : n \geq 2, \quad a < s_1 < \cdots < s_n < b\}.$$

If $pV(\varphi,(a,b)) < \infty$ then φ is bounded, and it can be written as the difference of two monotone functions.
Changing φ, even at a single point, changes the value of $pV(\varphi,(a,b))$, hence we define the *essential variation* $eV(\varphi,(a,b)$ as

$$eV(\varphi,(a,b) := \inf\{pV(\overline{\varphi},(a,b)) : \quad \overline{\varphi} = \varphi \quad \mathcal{L}^1 - \text{a.e. in } (a,b)\},$$

and we have

Theorem 1.7. *Let $\varphi \in L^1_{\text{loc}}((a,b);\mathbb{R}^k)$. Then there exists $\bar{\varphi} = \varphi$ $\mathcal{L}^1-$a.e. such that*

$$pV(\bar{\varphi},(a,b)) = eV(\varphi,(a,b)) = |\varphi'|((a,b)),$$

where $\varphi' := D\varphi$.

Any $\bar{\varphi}$ satisfies the above property is called *a good representative.*

Theorem 1.8. *Let $\varphi \in BV((a,b);\mathbb{R}^k)$. Then*

i. there exists a unique $c \in \mathbb{R}^k$ such that

$$\varphi^l(s) := c + \varphi'((a,s)), \qquad \varphi^r(s) := c + \varphi'((a,s]) \qquad s \in (a,b)$$

are the left continuous and the right continuous good representatives of φ; any other good representative $\bar{\varphi}$ of φ satisfies

$$\bar{\varphi}(s) \in \{\theta\varphi^l(s) + (1-\theta)\varphi^r(s) : \ \theta \in [0,1]\} \qquad \forall s \in (a,b).$$

ii. Let $J_\varphi := \{s \in (a,b) : \varphi'(\{s\}) \neq 0\}$. Then J_φ is at most countable; and any good representative $\bar{\varphi}$ is continuous in $(a,b) \setminus J_\varphi$ and has a jump point at all the points of J_φ:

$$\bar{\varphi}(s_-) = \varphi^l(s) = \varphi^r(s_-), \qquad \bar{\varphi}(s_+) = \varphi^l(s_+) = \varphi^r(s) \qquad \forall s \in J_\varphi,$$

where $\varphi(s_\pm)$ are the right and left limits of φ. Moreover $\bar{\varphi}$ is differentiable at $\mathcal{L}^1$-a.e. and the derivative in the classical sense $\dot{\bar{\varphi}}$ is the density of φ' with respect to $\mathcal{L}^1$.

We divide the measure φ' into two singular measures, the absolutely continuous and singular parts of φ' with respect to the Lebesgue measure, as follow

$$\varphi' := \varphi'^{(a)} + \varphi'^{(s)} = \dot{\bar{\varphi}} ds + \varphi'^{(s)},$$

where $\bar\varphi$ is any good representative of φ. We define the jump part $\varphi'^{(j)}$ and the Cantor part $\varphi'^{(c)}$ of φ' as

$$\varphi'^{(j)} = \varphi' \llcorner J_\varphi = \varphi'^{(s)} \llcorner J_\varphi \quad \text{and} \quad \varphi'^{(c)} = \varphi'^{(s)} \llcorner ((a,b) \setminus J_\varphi).$$

We say that $\varphi \in \mathrm{BV}((a,b);\mathbb{R}^k)$ is a *jump function* if $\varphi' = \varphi'^{(j)}$ and we say that φ is a *Cantor function* if $\varphi' = \varphi'^{(c)}$.
The following definition of generalized graph will be used to define the class of connections X in Definition 2.2.

Definition 1.9. *We define the generalized graph of $\varphi \in \mathrm{BV}((a,b))$ as*

$$\Gamma_\varphi := \{(s, \theta\bar\varphi(s_-) + (1-\theta)\bar\varphi(s_+)) : s \in (a,b),\ \theta \in [0,1]\}, \tag{1.4}$$

where $\bar\varphi$ is any good representative of φ.

We point out that the reduced boundary $\partial^- S\mathcal{G}_{\varphi,(a,b)}$ of the subgraph of φ, defined in (1.2), is a subset of Γ_φ and that

$$\mathcal{H}^1(\Gamma_\varphi \setminus \partial^- S\mathcal{G}_{\varphi,(a,b)}) = 0.$$

We set

$$\varphi_+(t) := \max\{\varphi^l(t), \varphi^r(t)\}, \qquad \varphi_-(t) := \min\{\varphi^l(t), \varphi^r(t)\}, \tag{1.5}$$

which are clearly good representatives of φ.
We conclude this brief section with three results that we shall refer to in the proof of Lemma 3.2 and Theorem 3.6. We state them as in [28, Thm. 4. p. 378, Thm. 5. p. 379, eqn. (5) and (6) p.486] taking Ω to be an open interval $I \subset \mathbb{R}$.

Theorem 1.10. *Let $\varphi \in \mathrm{BV}(I)$. Then*

(i) for $|\varphi'|$-a.e. $s \in I \setminus J_\varphi$ we have

$$\frac{d\mu(\varphi')}{d|\ \mu(\varphi')|}(x) = \nu((s, \varphi_+(s)), S\mathcal{G}_{\varphi,I})$$

(ii) for $|\varphi'|$-a.e. $s \in J_\varphi$ we have

$$\frac{d\mu(\varphi')}{d|\ \mu(\varphi')|}(s) = (\nu(s, J_\varphi), 0),$$

where $\mu(\varphi')$ is the vector valued measure defined by $\mu(\varphi') := (\varphi', -\mathcal{L}^1)$, $\nu(\cdot,\cdot)$ is the generalized inner normal defined in (1.3), in particular $\nu(s, J_\varphi) = \frac{\varphi(s_+)-\varphi(s_-)}{|\varphi(s_+)-\varphi(s_-)|}$.

Theorem 1.11. *Let $\varphi \in \mathrm{BV}(I)$. Then*

$$\begin{aligned}
\varphi'^{(a)} &= \varphi' \llcorner \{s \in I \mid \varphi_+(s) = \varphi_-(s),\ \nu^2((s,\varphi_+(s)), S\mathcal{G}_{\varphi,I}) < 0\},\\
\varphi'^{(c)} &= \varphi' \llcorner \{s \in I \mid \varphi_+(s) = \varphi_-(s),\ \nu^2((s,\varphi_+(s)), S\mathcal{G}_{\varphi,I}) = 0\},\\
\varphi'^{(j)} &= \varphi' \llcorner \{s \in I \mid \varphi_+(s) < \varphi_-(s)\}\\
&= \varphi' \llcorner \{s \in I \mid \varphi_+(s) < \varphi_-(s),\ \nu^2((s,\varphi_+(s)), S\mathcal{G}_{\varphi,I}) = 0\},
\end{aligned}$$

where $\nu = (\nu^1, \nu^2)$ is the generalized inner normal defined in (1.3).

Theorem 1.12. (Vol'pert chain rule)
Let $g \in \mathcal{C}^1(\mathbb{R})$ and $\varphi \in \mathrm{BV}((a,b))$. Then $g \circ \varphi \in \mathrm{BV}((a,b))$ and

$$(g \circ \varphi)' = g'(\varphi)\dot{\varphi}_+ ds + g'(\varphi)\varphi'^{(c)} \qquad \text{on } (a,b) \setminus J_\varphi$$
$$(g \circ \varphi)' = \sum_{s \in J_\varphi} \nu(s, J_\varphi)\Big[g(\varphi_+(s)) - g(\varphi_-(s))\Big]\delta_s \qquad \text{on } J_\varphi,$$

where $\nu(s, J_\varphi) = \frac{\varphi(s_+)-\varphi(s_-)}{|\varphi(s_+)-\varphi(s_-)|}$ and δ_s is the Dirac delta at s.

In the present thesis we shall always assume that any $\varphi \in \mathrm{BV}((a,b))$ is the good representative $\varphi = \varphi_+$.

1.3 The relaxed area functional $\mathcal{A}$

We start by recalling two versions of the area formula, the second one could be useful in an anisotropic setting.

Theorem 1.13. *Let $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be a Lipschitz function. Assume f is one-to-one on a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$. Then we have*

$$\mathcal{H}^n(f(E)) = \int_E \mathbf{J}_n(df_x)dx,$$

where $\mathbf{J}_n(df_x) = \sqrt{\det(df_x{}^* \circ df_x)} = \sqrt{\sum_B \det^2(B)}$, and B runs along all the $n \times n$ minors of the $(n+k) \times n$ matrix df_x.

Theorem 1.14. (anisotropic area formula for linear maps [3])
let W be a Banach space and let $f : \mathbb{R}^n \to W$ be a Lipschitz[(1)] function. Assume f is one-to-one on $B \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\mathcal{H}^n_W(f(B)) = \int_B \mathbf{J}_n(\mathrm{md}f_x)dx,$$

where $\mathcal{H}^n_W$ is the n-dimensional Hausdorff measure in W,

$$\mathbf{J}_n(\mathrm{md}f_x) := \frac{\omega_n}{\mathcal{H}^n(\{y \in \mathbb{R}^n : \mathrm{md}f_x(y) \leq 1\})},$$

and ω_n is the $\mathcal{L}^n$-measure of the unit ball in $\mathbb{R}^n$.

We shall not need the version for non injective maps.
Let $\Omega \subset \mathbb{R}^n$ be an open set and let $v \in \mathrm{Lip}(\Omega; \mathbb{R}^k)$; the area of the graph of v is given by

$$\mathbb{A}(v, \Omega) = \int_\Omega |\mathcal{M}(\nabla v)|dx,$$

(1) We recall that any Lipschitz function $f : \mathbb{R}^n \to W$ is metrically differentiable at $\mathcal{L}^n$- a.e. $x \in \mathbb{R}^n$, *i.e.,* for $\mathcal{L}^n$- a.e. x there exists a seminorm $||\cdot||_x$ in $\mathbb{R}^n$ such that $d(f(y), f(x)) - ||y-x||_x = o(|y-x|)$. This seminorm will be said to be the metric differential and denoted by $\mathrm{md}f_x$. If W is an Euclidean space then $\mathrm{md}f_x(\nu) = ||df_x(\nu)||$, $\nu \in \mathbb{R}^k$.

where $|\cdot|$ is the euclidean norm and $\mathcal{M}(\nabla v)$ is the vector whose components are the determinants of all minors, up to order $= \min\{n, k\}$, of the $k \times n$ matrix ∇v, including the minor of order 0, whose determinant, by convention, is set to be equal to 1.
In particular for $k = 1$ we have

$$|\mathcal{M}(\nabla v)| = \sqrt{1 + |\nabla v|^2},$$

and for $n = k = 2$ (the case of this thesis) we have

$$|\mathcal{M}(\nabla u)| = \sqrt{1 + |\nabla u_1|^2 + |\nabla u_2|^2 + (\partial_x u_1 \partial_y u_2 - \partial_y u_1 \partial_x u_2)^2}.$$

Extending the area functional for non-smooth maps is a natural question. Following a well established tradition we do that by extending the definition of $\mathbb{A}(v, \Omega)$ to $v \in L^1(\Omega; \mathbb{R}^k) \setminus \mathcal{C}^1(\Omega; \mathbb{R}^k)$ by setting $\mathbb{A}(v, \Omega) := +\infty$, then we consider the relaxed functional

$$\mathcal{A}(u, \Omega) := \inf \big\{ \liminf_{\epsilon \to 0} \mathbb{A}(u^\epsilon, \Omega) : \{u^\epsilon\} \subset \mathcal{C}^1(\Omega; \mathbb{R}^k),\ u^\epsilon \to u \text{ in } L^1(\Omega; \mathbb{R}^k) \big\},$$

which is the greatest lower semicontinuous functional on L^1 less than or equal to $\mathbb{A}$ (for more details of the relaxation in the calculus of variations see for instance [17] and [14])[(2)].
Moreover we have

$$\mathcal{A}(u, \Omega) = \inf \big\{ \liminf_{\epsilon \to 0} \mathbb{A}(u^\epsilon, \Omega) : \{u^\epsilon\} \subset \mathrm{Lip}(\Omega; \mathbb{R}^k),\ u^\epsilon \to u \text{ in } L^1(\Omega; \mathbb{R}^k) \big\}, \quad (1.6)$$

see [11, Step 1 p.377].
The scalar case is fully understood, the domain is characterized and we have an integral representation, more precisely

$$\{u \in L^1(\Omega) : \mathcal{A}(u, \Omega) < \infty\} = \mathrm{BV}(\Omega), \quad (1.7)$$

$$\mathcal{A}(u, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| dx + |D^s u|(\Omega), \qquad u \in \mathrm{BV}(\Omega), \quad (1.8)$$

where $Du = \nabla u dx + D^s u$ is the decomposition of the measure Du with respect to the Lebesgue measure, see *e.g.* [31], [5] and [14].
On the other hand the vectorial case is far from being understood. For instance, different from the equality in (1.7), we have only one strict[(3)] inclusion for the domain,

$$\{u \in L^1(\Omega; \mathbb{R}^k) : \mathcal{A}(u, \Omega) < \infty\} \subset \mathrm{BV}(\Omega; \mathbb{R}^k), \quad (1.9)$$

see [1, Remark 2.6.]. Moreover

$$\mathcal{A}(u, \Omega) \geq \int_\Omega |\mathcal{M}(\nabla u)| dx + |D^s u|(\Omega), \qquad u \in \mathrm{BV}(\Omega; \mathbb{R}^k), \quad (1.10)$$

see [1, Theorem 2.7] and we can not hope for the other inequality in (1.10) due to the non-subadditivity of $\mathcal{A}(u, \cdot)$ which we will discuss in the following subsection.

(2) The choice of the L^1 topology, in the vector context, is questionable; other notions of convergence could be considered as well, making the computation of the relaxation different in general.

(3) As a counter example take $u = \frac{x}{|x|^{3/2}}$ for x in the unit ball $B_1((1,0)) \subset \mathbb{R}^2$ centered at $(1,0)$ then $u \in BV(B_1((1,0)), \mathbb{R}^2)$ but u does not belong to the domain of $\mathcal{A}(\cdot, B_1((1,0)))$ [39].

1.3.1 Piecewise constant maps; Non-subadditivity of $\mathcal{A}(u,\cdot)$

We start by quoting the following result from [1, Theorem 3.14], setting $n = k = 2$, which says that if u is a piecewise constant map with a jump with no triple or multiple points then $\mathcal{A}(u,\cdot)$ is subadditive and has as an integral representative the right hand side of (1.10).

Theorem 1.15. *Let $(E_i)_{i\in I}$ be a finite partition of $\mathbb{R}^2$, such that each E_i is a locally finite perimeter set; let $(\alpha_i)_{i\in I}$ be a finite family of points of $\mathbb{R}^2$, and let $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ be the map defined as $u(\mathbf{x}) = \alpha_i$ if $\mathbf{x} \in E_i$. Suppose that for every $\mathbf{x} \in \overline{\Omega}$ there exists $r > 0$ such that $\mathcal{L}^2(B_r(\mathbf{x})\cap E_i) > 0$ for at most two distinct indices i. Then*

$$\begin{aligned}\mathcal{A}(u,\Omega) &= \mathcal{L}^2(\Omega) + \frac{1}{2}\sum_{i,j\in I}|\alpha_i-\alpha_j|\mathcal{H}^1(\partial^- E_i\cap\partial^- E_j\cap\Omega)\\ &= \int_\Omega |\mathcal{M}(\nabla u)|\,dx\,dy + |D^s u|(\Omega),\end{aligned}$$

provided that $\mathcal{L}^2(\partial\Omega) = 0$ and $\mathcal{H}^1(\partial^- E_i\cap\partial\Omega) = 0$ for every $i \in I$.

The problem appears once we remove the hypothesis that there are no triple points, the non-subadditive behavior of $\mathcal{A}(u,\cdot)$ arises, as conjectured by De Giorgi in [20]. More precisely in [1, Section 3] the authors showed that if $D_r \subset \mathbb{R}^2_S$ is a disk of radius $r > 0$, E_i, $i = 1,2,3$, are three angular partitions of D_r with angle equal to $120°$, and α_i, $i = 1,2,3$, are the vertices of an equilateral triangle $\mathrm{T}_{\mathrm{eq}} \subset \mathbb{R}^2_T$ of side length ℓ, then the symmetric singular map u_{symm}, defined as in (0.10), has one triple junction at the origin of D_r and satisfies the following properties

$$\mathcal{A}(u_{\mathrm{symm}}, D_r) \le |D_r| + 4rl, \tag{1.11}$$
$$\mathcal{A}(u_{\mathrm{symm}}, D_r) > |D_r| + 3rl. \tag{1.12}$$

Moreover there exist $\rho, s > 0$ with $0 < \rho < r < s$ such that

$$\mathcal{A}(u_{\mathrm{symm}}, D_r) > A(u_{\mathrm{symm}}, D_\rho) + A(u_{\mathrm{symm}}, \overline{D_s \setminus D_{\rho/2}}). \tag{1.13}$$

Thus u_{symm} is in the domain of $\mathcal{A}(\cdot, D_r)$, and the inequality in (1.10) is strict. Moreover, by (1.13), there is no way to find an integral representative for the functional $\mathcal{A}$ since $\mathcal{A}(u_{\mathrm{symm}},\cdot)$ is not a measure.
Aiming to find other ways to characterize $\mathcal{A}(\cdot,\Omega)$ at least for piecewise constant maps, the authors in [10] and [36] showed that the exact value of $\mathcal{A}(u_{\mathrm{symm}}, D_r)$ is given by:

$$\mathcal{A}(u_{\mathrm{symm}}, D_r) = |D_r| + 3\mathfrak{A},$$

where $\mathfrak{A}$ is the area of a particular area minimizer cartesian surface $m : \mathrm{R} := [0,\ell]\times[0,r] \to [0,+\infty)$. More precisely m is the unique solution of the Dirichlet-Neumann minimum problem:

$$\min\left\{\int_{\mathrm{R}} \sqrt{1+|\nabla f|^2}\,dsdt:\ f \in W^{1,1}(\mathrm{R}),\ f = \varphi\ \mathcal{H}^1 - a.e.\text{ on } \partial_D\mathrm{R} := \partial\mathrm{R}\setminus([0,\ell]\times\{r\})\right\},$$

where φ is the non-negative piecewise affine function that vanishes on the two parallel sides $\{0\} \times [0, r], \{\ell\} \times [0, r]$ of $\partial_D \mathrm{R}$ and its graph on the last side $[0, \ell] \times \{r\}$ consists of the two segments connecting the vertices α_i and α_j, $i, j \in \{1, 2, 3\}$, $i \neq j$, to the barycenter of the triangle T_{eq} (note that φ does not depend on the choice of i and j since T_{eq} is equilateral).
One of the aims of this thesis is to investigate the possibilities of having a suitable upper bound for the area functional for a larger class of piecewise constant maps.
Finally we would like to point out that piecewise constant maps are not the only source of non-subadditivity, for instance in [1] the authors give an example of a Sobolev function in $W^{1,p}$, $p \in [1, 2)$, the vortex map, where the non-subadditivity phenomenon is detected, however in this thesis we will not focus on such functions.

1.4 One-codimensional area-minimizing cartesian surfaces

In this section we collect some definitions and known results on Plateau's problem that will be needed to prove Theorems 2.1 and Proposition 3.4.
We start with the most regular situation: let $\Omega \subset \mathbb{R}^n$ be bounded open set with $\mathcal{C}^2$- boundary, and let $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$. We are interested in the problem of minimizing the area amongst all functions taking the prescribed value φ on $\partial\Omega$. Due to the smoothness of both the boundary of Ω and the initial datum φ one can show that the following minimization problem

$$\min \left\{ \int_\Omega \sqrt{1 + |\nabla f|^2} dx : \ f \in \mathrm{Lip}(\Omega), \ f = \varphi \text{ on } \partial\Omega \right\}, \tag{1.14}$$

has a unique solution and that it is analytic in Ω, see [30, Theorem 12.10 and Theorem 12.11].
Now we want to solve the same Dirichlet problem but for much less regular sets and initial data. To do that we use the direct method of calculus of variations, and the natural space to minimize is $\mathrm{BV}(\Omega)$.

Definition 1.16. *Let $\Omega \subset \mathbb{R}^n$, for $f \in \mathrm{BV}(\Omega)$, define*

$$\begin{aligned} A(f, \Omega) &:= \int_\Omega \sqrt{1 + |Df|^2} \\ &:= \sup \left\{ \int_\Omega (g_{n+1} + f D_i g_i) dx; \qquad g = (g_1. \cdots, g_{n+1}) \in \mathcal{C}^1_0(\Omega; \mathbb{R}^{n+1}); \quad |g| \leq 1 \right\}. \end{aligned}$$

We have

- if $f \in W^{1,1}(\Omega)$ then

$$\int_\Omega \sqrt{1 + |Df|^2} = \int_\Omega \sqrt{1 + |\nabla f|^2} dx,$$

- $A(f,\Omega)$ can be written in terms of a function taking values in $\mathbb{S}^0$ and jumping along the graph of f as follows:

$$A(f,\Omega) = P(S\mathcal{G}_{f,\Omega}, \Omega \times \mathbb{R}) = |D\mathbb{1}_{S\mathcal{G}_{f,\Omega}}|(\Omega \times \mathbb{R}), \tag{1.15}$$

where $S\mathcal{G}_{f,\Omega}$ is the subgraph defined in (1.2), see [30, Theorem 14.6],

- $A(\cdot,\Omega)$ is lower semi-continuous with respect to the L^1_{loc}-topology in Ω, *i.e.*,

$$A(f,\Omega) \le \liminf_{j\to\infty} A(f_j,\Omega); \qquad f_j \to f \text{ in } L^1_{\text{loc}}(\Omega),$$

see [30, Theorem 14.2].

For the rest of this section $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and $\varphi \in L^1(\partial\Omega)$. A known result by Gagliardo, see [30, Theorem 2.16], gurantees that if $\overline{\Omega} \subset B$, B some open ball in $\mathbb{R}^n$, then φ can be extend to a $W^{1,1}$ function on $B \setminus \overline{\Omega}$.
Consider the following weaker form of the Dirichlet problem:

$$\min\left\{A(f,\Omega) + \int_{\partial\Omega} |f-\varphi| d\mathcal{H}^{n-1} : \qquad f \in \mathrm{BV}(\Omega), \quad f = \varphi \text{ on } B \setminus \overline{\Omega}\right\}, \tag{1.16}$$

note that the last integral can be seen as a penalization for the trace of f not taking the boundary value φ on $\partial\Omega$.
Now we recall some interesting definition and results from [30].

Definition 1.17. *Let Ω be a finite perimeter set. We say that the mean curvature of $\partial\Omega$ is non-negative near $x \in \partial\Omega$ if there exists an open set B containing x such that*

$$\int_B |D\mathbb{1}_\Omega| \le \int_B |D\mathbb{1}_{\Omega\cup K}|, \qquad \text{for every compact set } K \subset B,$$

where $\mathbb{1}_\Omega$ and $\mathbb{1}_{\Omega\cup K}$ are the characteristic functions of Ω and $\Omega \cup K$ respectively.

From the lower semi-continuity of $A(\cdot,\Omega)$ and Theorem 1.3 we conclude the existence of a minimum. We have therefore:

Theorem 1.18. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set with Lipschitz boundary and let $\varphi \in L^1(\partial\Omega)$. Then the minimization problem* (1.16) *has a solution.*

Proof. See [30, Theorem 14.5]. □

Moreover we have

Theorem 1.19. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set with Lipschitz boundary, $\varphi \in L^1(\partial\Omega)$, and let $\overline{f}$ be a minimum of* (1.16)*. If $x \in \partial\Omega$ is a point of continuity of φ and Ω has non-negative mean curvature near x then*

$$\lim_{\overline{x}\to x} \overline{f}(\overline{x}) = \varphi(x).$$

Proof. See [30, Theorem 15.9]. □

In particular, if $\partial\Omega$ has non negative mean curvature $\mathcal{H}^{n-1}$- a.e. and φ is continuous a.e. then $\overline{f}$ attains the boundary condition $\mathcal{H}^{n-1}$-a.e., *i.e.* $\overline{f}$ is a minimizer of

$$\min\left\{A(f,\Omega): \qquad f\in \mathrm{Lip}_{\mathrm{loc}}(\Omega), \quad f=\varphi \text{ on } \partial\Omega\right\}, \tag{1.17}$$

by the following regularity Theorem:

Theorem 1.20. *If $\overline{f}\in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ minimize locally the functional $A(f,\Omega)$. Then $\overline{f}$ is locally Lipschitz and analytic in Ω.*

Proof. See [30, Theorem 14.13]. □

Moreover if Ω connected then $\overline{f}$ is the unique solution of (1.17), by the following result:

Theorem 1.21. *Let Ω be connected and let $\varphi\in L^1(\partial\Omega)$ and $f,\ g$ be two solutions of (1.16). Then $g=f+const$.*

Proof. See [30, Theorem 14.12]. □

To conclude:

> Let $\Omega\subset\mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary such that $\overline{\Omega}\subset B$ where B is an open ball in $\mathbb{R}^n$. Assume that $\partial\Omega$ has non negative mean curvature $\mathcal{H}^{n-1}$-a.e. and that $\varphi\in L^1(\partial\Omega)$ is continuous $\mathcal{H}^{n-1}$-a.e.. Then the minimization problem
>
> $$\begin{aligned}&\min\left\{A(f,\Omega): \qquad f\in \mathrm{Lip}_{\mathrm{loc}}(\Omega), \quad f=\varphi \text{ on } \partial\Omega\right\}\\ &\quad=\min\left\{A(f,\Omega)+\int_{\partial\Omega}|f-\varphi|d\mathcal{H}^{n-1}: \qquad f\in \mathrm{BV}(\Omega), \quad f=\varphi \text{ on } B\setminus\overline{\Omega}\right\},\end{aligned} \tag{1.18}$$
>
> has a unique solution and it is analytic in Ω.

1.5 On Cartesian currents

In this section we recall some of the definitions and results about currents in [28] that will be needed to prove the compactness of the space of connections X, see Theorem 3.6.
Let U open set in $\mathbb{R}^m$, $k\le m$. A $k-$*dimensional current* in U is a continuous linear functional on the space $D^k(U)$ of all infinitely differentiable compactly supported k-forms in U. We denote by $D_k(U)$ the space of all k-dimensional currents in U.
The *boundary* of $\mathcal{T}\in D_k(U)$ is the $(k-1)$-current

$$\partial\mathcal{T}(\eta):=\mathcal{T}(d\eta) \qquad \forall\eta\in D^{k-1}(U),$$

and we set $\partial\mathcal{T}=0$ for $\mathcal{T}\in D_0(U)$.

The ısupport of a current $\mathcal{T} \in D_k(U)$ is defined by

$$\text{supp } \mathcal{T} := \bigcap\{\mathcal{K} \subset U : \mathcal{K} \text{ relatively closed in } U, \quad \mathcal{T}(\omega) = 0 \text{ for all } \omega \in D_k(U) \text{ with supp } \omega \subset U \setminus \mathcal{K}\}.$$

A current $\mathcal{T} \in D_k(U)$ is called *rectifiable* if there exist an $\mathcal{H}^k$-measurable and countable k-rectifiable set[4] $\mathcal{K} \subset U$, an $\mathcal{H}^k$- measurable and and locally $\mathcal{H}^k \llcorner \mathcal{K}$-summable map $\theta : \mathcal{K} \to \mathbb{R}$ (called *the multipicity*), and an $\mathcal{H}^k$- measurable map $\xi : \mathcal{K} \to \Lambda_k\mathbb{R}^n$[5] with $||\xi|| = 1$ $\mathcal{H}^k \llcorner \mathcal{K}$-a.e and $\xi(x)$ is a simple k-vector associated to the tangent plane $T_x\mathcal{K}$ for $\mathcal{H}^k \llcorner \mathcal{K}$-a.e. x, such that

$$\mathcal{T}(\omega) = \int_{\mathcal{K}} < \xi, \omega > \theta d\mathcal{H}^k, \qquad \omega \in D^k(U),$$

where $< \cdot, \cdot >$ is the duality product between covectors and vectors. We write $\mathcal{T} = \tau(\mathcal{K}, \theta, \xi)$. If θ is integer-valued we say that $\mathcal{T}$ is *integer multiplicity rectifiable*, for short i.m. rectifiable. We define the *restriction of an i. m. rectifiable current* $\mathcal{T}$ on a Borel set $A \subset U$ as

$$\mathcal{T} \llcorner A := \int_A < \xi, \omega > \theta d\mathcal{H}^k.$$

If $\mathcal{T} \in D_m(\mathbb{R}^m)$ be i.m. rectifiable current of the form $\mathcal{T} = \{\mathcal{K}, 1, e_1 \wedge \cdots \wedge e_m\}$[6] then $\mathcal{T}$ is the integration over the set $\mathcal{K}$ and is denoted by

$$\mathcal{T} = [\![\mathcal{K}]\!].$$

We say a sequence $\{\mathcal{T}_h\} \subset D_k(U)$ *converges weakly* to $\mathcal{T} \in D_k(U)$, $\mathcal{T}_h \rightharpoonup \mathcal{T}$, if

$$\mathcal{T}_h(\omega) \to \mathcal{T}(\omega) \qquad \forall \omega \in D^k(U).$$

The *mass* of a current $\mathcal{T} \in D_k(U)$ in an open set $V \subset U$ is defined by

$$\mathbf{M}_V(\mathcal{T}) := \sup\{\mathcal{T}(\omega) \mid \omega \in D^k(U), \text{ supp } \omega \subset V, \ ||\omega(x)|| \leq 1 \ \forall x \in U\},$$

where $|| \cdot ||$ is the comass norm[7]. For short we write $\mathbf{M}(\mathcal{T}) = \mathbf{M}_U(\mathcal{T})$.
The mass is lower semicontinuous with respect to the weak convergence, i.e., if $\mathcal{T}_h \rightharpoonup \mathcal{T}$ in $V \subset U$ then

$$\mathbf{M}_V(\mathcal{T}) \leq \liminf_{h \to \infty} \mathbf{M}_V(\mathcal{T}_h). \tag{1.19}$$

[4] A k-rectifiable set is a countable union of subsets of k-dimensional Lipschitz surfaces (up to a set of $\mathcal{H}^k$ zero measure).
[5] We denote by $\Lambda_k\mathbb{R}^n$ the space of k-vectors of $\mathbb{R}^n$.
[6] We denote by $\{e_i\}_{i \leq m}$ the canonical basis of $\mathbb{R}^m$.
[7] The comass norm of a covector $\omega \in \Lambda^k\mathbb{R}^n$ is defined as $||\omega|| := \sup\{< \xi, \omega >: \xi \in \Lambda_k\mathbb{R}^n, |\xi| \leq 1, \xi \text{ simple k-vector}\}$.

Theorem 1.22. (Compactness theorem)[28, Thm. 2. p. 141] *Let* $\{\mathcal{T}_j\} \subset D_k(U)$ *be a sequence of i. m. rectifiable* k*-currents in an open set* $U \subset \mathbb{R}^m$, $k \leq m$, *satisfying*

$$\sup_j \left[\mathbf{M}_V(\mathcal{T}_j) + \mathbf{M}_V(\partial \mathcal{T}_j)\right] < \infty, \qquad \forall V \subset\subset U.$$

Then there exists a (not-relabeled) subsequence $\{\mathcal{T}_j\}$ *and an i. m. rectifiable* k*-current* $\mathcal{T}$ *in* U *such that*

$$\mathcal{T}_j \rightharpoonup \mathcal{T}.$$

Given $\Omega \subset \mathbb{R}^n$ open set and $F : U \to \Omega$ smooth map. The *push-forward* of a current $\mathcal{T} \in D_k(U)$ by F is defined as

$$F_\sharp \mathcal{T}(\omega) := \mathcal{T}(F^\sharp \omega) \qquad \text{for } \omega \in D^k(\Omega),$$

where $F^\sharp \omega$ is the pull-back of the form ω through F.

Definition 1.23. *Let* $U = \Omega \times \mathbb{R}^k$, *where* Ω *open set in* $\mathbb{R}^n$ *and* $\pi : U \to \Omega$ *be the canonical projection on* $\mathbb{R}^n$. *An i. m. rectifiable* n*-current* $\mathcal{T}$ *is called a Cartesian current if*

- $\partial \mathcal{T} = 0$, $\mathbf{M}(T) < \infty$, $\pi_\sharp \mathcal{T} = [\![\Omega]\!]$,

- $\mathcal{T}^{\hat{0}0} \geq 0$, *where* $T^{\hat{0}0}$ *is the distribution defined by* $T^{\hat{0}0}(f) := \mathcal{T}(f dx^1 \wedge \cdots \wedge dx^n)$(8) *for every* $f \in \mathcal{C}_c^\infty(U)$,

- $\sup \left\{\mathcal{T}(f(x,y)|y|\ dx^1 \cdots dx^n) \mid f \in \mathcal{C}_c^\infty(U), |f| \leq 1\right\} < \infty.$

Let $I \subset \mathbb{R}$ be a bounded open interval and $\varphi \in \mathrm{BV}(I)$. Then

$$[\![S\mathcal{G}_{\varphi,I}]\!] \in \mathcal{D}^2(\mathbb{R}^2).$$

The current $[\![S\mathcal{G}_{\varphi,I}]\!] \llcorner I \times \mathbb{R}$ can be also identified with an integer multiplicity current in $I \times \mathbb{R}$; moreover $S\mathcal{G}_{\varphi,I}$ has finite perimeter in $I \times \mathbb{R}$ so, if $\partial [\![S\mathcal{G}_{\varphi,I}]\!] \llcorner I \times \mathbb{R}$ denotes the 1-current in $I \times \mathbb{R}$ defined as the restriction to $I \times \mathbb{R}$ of the boundary of $[\![S\mathcal{G}_{\varphi,I}]\!]$, this results of finite mass.

For future purposes we recall the following result, see [28, Section 4.2.4].

(8) We use the coordinates $x^i, i = 1, \cdots, n$, in Ω and $y^i, i = 1, \cdots, k$ in $\mathbb{R}^k$.

Theorem 1.24. *Let $\varphi \in \mathrm{BV}(I)$ and $\mathcal{T}$ be the current defined by*

$$\mathcal{T} := -\partial[\![S\mathcal{G}_{\varphi,I}]\!] \llcorner I \times \mathbb{R}. \tag{1.20}$$

Then $\mathcal{T} \in \mathcal{D}_1(I \times \mathbb{R})$ is a Cartesian current, and

$$\mathcal{T}(\omega) = -\int < \omega(x), *\nu(x, S\mathcal{G}_{\varphi,I}) > d\mathcal{H}^1 \llcorner \partial^- S\mathcal{G}_{\varphi,I}(x) \qquad \forall \omega \in \mathcal{D}^1(I \times \mathbb{R}), \tag{1.21}$$

where $*$ is the Hodge operator and $\nu(\cdot, S\mathcal{G}_{\varphi,I})$ is the inward generalized unit normal. Moreover $\mathcal{T}$ can be decomposed into three mutually singular currents

$$\mathcal{T} = \mathcal{T}^{(a)} + \mathcal{T}^{(j)} + \mathcal{T}^{(c)}, \tag{1.22}$$

such that

$$\mathcal{T}^{(a)}(\omega) = \int_I [\omega_1(s, \varphi(s)) + \omega_2(s, \varphi(s))\dot{\varphi}(s)] ds, \tag{1.23}$$

$$\mathcal{T}^{(j)}(\omega) = \sum_{s \in J_\varphi} \nu(s, J_\varphi) \int_{\varphi_-(s)}^{\varphi_+(s)} \omega_2(s, \sigma) d\sigma, \tag{1.24}$$

$$\mathcal{T}^{(c)}(\omega) = \int_I \omega_2(s, \varphi(s))\dot{\varphi}^{(c)}, \tag{1.25}$$

where $\omega = \omega_1 ds + \omega_2 d\sigma$.

1.6 Distance functions

We start by recalling the definition of distance functions and some of their regularity properties.
Given $E \subseteq \mathbb{R}^n$, we will denote by

$$\mathrm{dist}(x, E) := \inf_{y \in E} |x - y|,$$

the *distnce function* from E, and we take the convention $\inf \emptyset := +\infty$.
The *signed distance function* and the *square distance function* from E are defined as

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbb{R}^n \setminus E), \qquad x \in \mathbb{R}^n,$$

and

$$\eta_E(x) := \frac{1}{2}(\mathrm{dist}(x, E))^2, \qquad x \in \mathbb{R}^n.$$

Notice that

$$d_E = d_{\overline{E}},$$

and

$$d_E(x) = \begin{cases} -\mathrm{dist}\left(x, (\overline{E} \setminus \mathrm{int}\ E)\right), & \text{if } x \in E \\ \mathrm{dist}\left(x, (\overline{E} \setminus \mathrm{int}\ E)\right), & \text{if } x \in \mathbb{R}^n \setminus E, \end{cases}$$

where $\overline{E}$ and int E denote the closure and the interior of E in $\mathbb{R}^n$. Hence if E has empty interior then $d_E(\cdot) = \mathrm{dist}(\cdot, E)$.
It is readily shown that $\mathrm{dist}(\cdot, E)$ and d_E are one-Lipschitz functions. Indeed if $x, y \in \mathbb{R}^n$, without loss of generality we assume E to be closed, hence there is $\xi \in E$ such that $|\xi - x| = \mathrm{dist}(x, E)$. Then

$$\mathrm{dist}(y, E) \le |\xi - y| \le |x - y| + \mathrm{dist}(x, E),$$

interchanging x and y we have

$$|\mathrm{dist}(x, E) - \mathrm{dist}(y, E)| \le |x - y|.$$

Theorem 1.25. *Let E be a non empty closed set.*

(a) If $\xi \in \mathbb{R}^n \setminus E$. Then, $\mathrm{dist}(\cdot, E)$ is differentiable at ξ if and only if there exists a unique $x \in E$ such that $\mathrm{dist}(\xi, E) = |\xi - x|$. Moreover

$$\nabla \mathrm{dist}(\xi, E) = \frac{\xi - x}{|\xi - x|} = \frac{\xi - x}{\mathrm{dist}(\xi, E)}.$$

(b) The squared distance function η_E is differentiable on $E = \{\eta_E = 0\}$ and satisfies the identity

$$|\nabla \eta_E(\xi)|^2 = 2\eta_E(\xi),$$

at any differentiability point $\xi \in \mathbb{R}^n$, in particular $E = \{\nabla \eta_E = 0\}$.

Proof. See [2, Theorem 1] and [9, Lemma 2.1]. □

Remark 1.26. i. If $\mathrm{dist}(\cdot, E)$ is differentiable at $\xi \in \mathbb{R}^n \setminus E$ then $|\nabla \mathrm{dist}(\xi, E)| = 1$,

ii. If $\xi \in \mathbb{R}^n \setminus E$ then $x_\xi := \xi - \mathrm{dist}(\xi, E)\nabla \mathrm{dist}(\xi, E)$ is the nearest point on E and if $z \in \overline{\xi x_\xi}$ then $\mathrm{dist}(\cdot, E)$ is differentiable at z, $\nabla \mathrm{dist}(z, E) = \nabla \mathrm{dist}(\xi, E)$, and x_ξ is the unique point such that $|z - x_\xi| = \mathrm{dist}(z, E)$.

iii. A similar result to Theorem 1.25 (a), applies to $d_E(\cdot)$ on the complement of the topological boundary of E defined as $\overline{E} \setminus \mathrm{int}\{E\}$.

The regularity of d_E, η_E and the smoothness of the topological boundary of E are related, see Figure 1.1. We start by recalling the following result for open sets.

Theorem 1.27. *Let $A \subset \mathbb{R}^n$ a bounded open set.*

(I) If the boundary of A of class $\mathcal{C}^k$, $k \ge 2$, then the signed distance function d_A is of class $\mathcal{C}^k$ in a tubular neighborhood of the boundary of A.

(II) Conversely, if the signed distance function d_A is of class $\mathcal{C}^k$, $k \ge 2$, in a suitable neighborhood of the boundary of A then the boundary of A is of class $\mathcal{C}^k$.

Proof. See [29, Lemma 14.16]. □

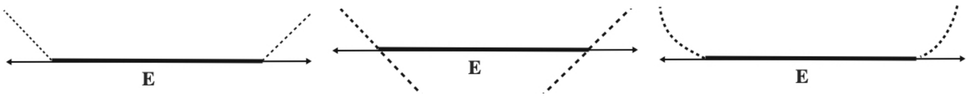


Figure 1.1: E is a segment in $\mathbb{R}$. Left: the graph of $\mathrm{dist}(\cdot, E)$. Middle: the graph of d_E. Right: the graph of η_E (note that η_E is $\mathcal{C}^{1,1}$ but not $\mathcal{C}^2$ at the two end points).

Notation: If $\rho > 0$, we set

$$E_\rho^+ := \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, E) < \rho\},$$

and if $\rho : E \to (0, +\infty]$ is a function, we set $E_{\rho(\cdot)}^+ := \bigcup_{x \in E} \{\xi \in \mathbb{R}^n : |\xi - x| < \rho(x)\}$.

1.6.1 Manifolds without boundary and distance functions

In this subsection we recall the notion of smooth (resp. analytic) manifold without boundary of arbitrary codimension, using the distance function, and the relation with the classical definition of smooth manifold.
Let us recall the definition of the class of h-dimensional embedded $\mathcal{C}^k$-manifolds(9) without boundary in a nonempty open set $\Omega \subset \mathbb{R}^n$ (see for instance [38, 19]).

Definition 1.28 (Smooth embedded manifold without boundary). *Let $k \in \mathbb{N} \cup \{\infty, \omega\}$ and $h \in \{1, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. We say that $\Gamma \subset \mathbb{R}^n$ is a h-dimensional embedded $\mathcal{C}^k$-manifold without boundary in Ω if*

$$\Gamma \cap \Omega = \overline{\Gamma} \cap \Omega, \tag{1.26}$$

and for all $x \in \Gamma \cap \Omega$ there exist an open set $R \subset \mathbb{R}^n$, an open set $G \subset \mathbb{R}^h$, and maps $\phi \in \mathcal{C}^k(G; \mathbb{R}^n)$, $\psi \in \mathcal{C}^k(R; \mathbb{R}^h)$ such that

$$x \in R, \ \psi(\phi(y)) = y \quad \forall y \in G,$$

$$\Gamma \cap R = \{\phi(y) : y \in G\}. \tag{1.27}$$

Theorem 1.29. *Let $k \in \mathbb{N}$, $k \geq 3$, or $k \in \{\infty, \omega\}$ and $h \in \{1, \ldots, n\}$. Let $\Gamma \subset \mathbb{R}^n$ be a compact h-dimensional embedded $\mathcal{C}^k$-manifold without boundary in $\mathbb{R}^n$. Then η_Γ is $\mathcal{C}^{k-1}$ in a tubular neighbourhood Γ_ρ^+ of Γ and $\eta_\Gamma(x + p) = \frac{1}{2}|p|^2$ for any $x \in \Gamma$ and any p in the normal space $N_x\Gamma$ to Γ at x, with $x + p \in \Gamma_\rho^+$. In particular the matrix $\nabla^2 \eta_\Gamma(x)$ represents the orthogonal projection on $N_x\Gamma$.*

Proof. See [2, Theorem 2] and [22]. □

Theorem 1.29 is still valid if Γ is a h-dimensional embedded $\mathcal{C}^k$-manifold without boundary (not necessarily compact) in some open set Ω, provided that Γ_ρ^+ becomes a neighborhood $\Gamma_{\rho(\cdot)}^+ \subset \Omega$ of $\Gamma \cap \Omega$. Indeed, following the same proof of Theorem

(9) k stands for a positive natural number. We also consider the cases $k = +\infty$ or $k = \omega$ (analytic manifolds), in these cases $k - 1 = +\infty$ (resp. ω).

1.29 in [2] it follows that that for any $x \in \Gamma \cap \Omega$ there exists $\rho(x) > 0$ such that $B_{\rho(x)}(x) \subset \Omega$, $\eta_\Gamma \in \mathcal{C}^{k-1}(B_{\rho(x)}(x))$, $\eta(x+p) = \frac{1}{2}|p|^2$ for any $p \in N_x\Gamma$ such that $x + p \in B_{\rho(x)}(x)$, and $\operatorname{rank}(\nabla^2 \eta_\Gamma(x)) = n - h$. Defining $\Gamma^+_{\rho(\cdot)} := \cup_{x \in \Gamma} B_{\rho(x)}(x)$ we get the assertion.

Theorem 1.30. *Let $k \in \mathbb{N}$, $k \geq 3$ or $k \in \{\infty, \omega\}$. Let $A \subseteq \mathbb{R}^n$ be an open set, $E \subset \mathbb{R}^n$ a closed subset and suppose that $\eta_E \in \mathcal{C}^k(A)$. Then any connected component of $E \cap A$ is an embedded $\mathcal{C}^{k-1}$-manifold without boundary in A.*

Proof. See [9, Theorem 2.4]. □

Notice that if $\operatorname{rank}(\nabla^2 \eta_E(x)) = n - h$ for any x in a connected component of $E \cap A$, then such a connected component must have dimension h. Furthermore it is sufficient to have E closed in A to get the thesis of Theorem 1.30. Indeed it is enough to apply Theorem 1.30 to $\overline{E}$, hence any connected component of $\overline{E} \cap A$ $(= E \cap A)$ is a manifold without boundary.

2. The functional $\mathcal{G}$, and an upper bound of $\mathcal{A}(u, D)$

In order to prove the main result (Theorem 0.1) of the first part of the present thesis we need some preparation. In this chapter we state precisely what we mean by a connection and define the functional $\mathcal{G}$ over the class of connections X. Then we show the main result of this chapter which provides an upper bound of the relaxed area $\mathcal{A}(u, D)$ and reads as follows:

Theorem 2.1. *Let $u \in BV(D; \{\alpha_1, \alpha_2, \alpha_3\})$ be the function defined in* (0.10). *Then*

$$\mathcal{A}(u, D) \leq |D| + \inf\left\{\mathcal{G}(\Gamma) : \Gamma \in X_{\mathrm{Lip}}\right\}. \tag{2.1}$$

2.1 The class of connections X

Take three open non-overlapping non-empty connected regions E_1, E_2, E_3 of an open disk D, each E_i with non empty interior and with $\overline{E_1} \cup \overline{E_2} \cup \overline{E_3} = \overline{D}$, and let C_{ij} be their boundaries in D as in the introduction.
Let α_1, α_2, α_3 be the vertices of a closed triangle T as in the introduction; we suppose that T contains the origin 0_{T} in its interior, and let ℓ_{ij} be as in (0.4).
For any $(a, b) \subset \mathbb{R}$ bounded open interval and any $\varphi \in \mathrm{BV}((a, b))$, we shall always assume that φ is a good representative in its L^1 class such that

$$\varphi(s) = \varphi_+(s) := \max\{\varphi(s_+), \varphi(s_-)\} \quad \text{for all } s \in (a, b),$$

where $\varphi(s_\pm)$ are the right and left limits of φ; hence the pointwise variation of φ is equal to the total variation $|\varphi'|((a, b))$. We conventionally set $\varphi(a_-) = 0$, $\varphi(b_+) = 0$; in this case we can define the generalized graph Γ_φ of φ as in (1.4) with (a, b) replaced by $[a, b]$, hence the generalized graph will always pass through the end points of the interval (with possibly vertical parts over a and b).

Definition 2.2 (**Connections in** T). *We say that $\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3)$ is a* BV *graph-type (resp.* Lip *graph-type) connection in* T *if Γ_i, $i \in \{1, 2, 3\}$, are subsets of* T *such that $\Gamma_1 \cap \Gamma_2 = \Gamma_2 \cap \Gamma_3 = \Gamma_3 \cap \Gamma_1$ is one point p of* T *called target triple point of Γ, $\alpha_i \in \Gamma_i$ for any $i = 1, 2, 3$, and*

$$\Gamma_{ij} := \Gamma_i \cup \Gamma_j, \qquad ij \in \{12, 23, 31\},$$

can be written as the generalized graph (resp. graph) of a function of bounded vari-

ation (resp. Lipschitz function) over the closed segment $\overline{\alpha_i\alpha_j}$ (see Figures 1b and 3.5).

Note that the case $p \in \partial\mathrm{T}$ is not excluded. However, by definition, if $\pi_{ij} : \mathrm{T} \to \mathbb{R}_{\overline{\alpha_i\alpha_j}}$, $ij \in \{12, 23, 31\}$, is the orthogonal projection on the line $\mathbb{R}_{\overline{\alpha_i\alpha_j}}$ containing $\overline{\alpha_i\alpha_j}$, then $\pi_{ij}(p) \in \overline{\alpha_i\alpha_j}$. Set

$$w_{ij} := |\alpha_i - \pi_{ij}(p)|. \tag{2.2}$$

If necessary, in the sequel we will often identify Γ_{ij} with the (generalized) graph $\Gamma_{\varphi_{ij}}$ of a function

$$\varphi_{ij} : [0, \ell_{ij}] \to [0, \mathrm{diam}\mathrm{T}], \qquad \varphi_{ij} = \varphi_{ij}(\Gamma_{ij}), \tag{2.3}$$

of bounded variation. If T is acute, choosing a suitable cartesian coordinate system where the s-axis is the line $\mathbb{R}_{\overline{\alpha_i\alpha_j}}$, we necessarily have $\varphi_{ij}(0) = \varphi_{ij}(\ell_{ij}) = 0$. In contrast, if the angle of T at α_i is greater than or equal to $\frac{\pi}{2}$ then φ_{ij} might have a vertical part over α_i and $\varphi_{ij}(0_+) > 0$. In the sequel it will be often convenient to consider an extension of φ_{ij} on $(-\infty, 0) \cup (\ell_{ij}, +\infty)$. This extension is denoted by $\tilde{\varphi}_{ij}$. In the case of acute triangle $\tilde{\varphi}_{ij}$ is always set equal to 0 on $(-\infty, 0) \cup (\ell_{ij}, +\infty)$.

Remark 2.3. If for any $ij \in \{12, 23, 31\}$, w_{ij} in (2.2) is a point of continuity of φ_{ij} then the intersection of the generalized graph of φ_{ki} with the set $[w_{ki}, \ell_{ki}] \times \mathbb{R}$ coincides with Γ_i which is also the intersection of the generalized graph of φ_{ij} with the set $[0, w_{ij}] \times \mathbb{R}$, where $ij, ki \in \{12, 23, 31\}$, $ij \neq ki$. If w_{ij} is a jump point of φ_{ij} this is in general not true, as in Figure 3.6b, when $i = 2$.

Remark 2.4. Assume that an angle of T is greater than $\frac{\pi}{2}$, say for instance the angle at α_1; as already said, the generalized graphs composing a connection Γ are allowed to have vertical parts over α_1. The target triple point p of any connection Γ belongs to $T_{int} \subset T$, the part of the triangle T which is enclosed between the two lines passing through α_1 and orthogonal to $\overline{\alpha_1\alpha_2}$ and $\overline{\alpha_1\alpha_3}$ respectively.

Define the classes:

$$X_{\mathrm{Lip}} := \big\{\Gamma : \Gamma \text{ Lip graph} - \text{type connection in T}\big\}, \tag{2.4}$$

$$X := \big\{\Gamma : \Gamma \text{ BV graph} - \text{type connection in T}\big\}. \tag{2.5}$$

Obviously $X_{\mathrm{Lip}} \subset X$.

2.2 The functional $\mathcal{G}$

Let R_{ij} be as in (0.5), and $\Gamma \in X$. Then Γ_{ij}, $ij \in \{12, 23, 31\}$, are (generalized) graphs of functions φ_{ij} of bounded variation over $[0, \ell_{ij}]$. Let $B \subset \mathbb{R}^2$ be an open disk containing the doubled rectangle $\widehat{\mathrm{R}}_{ij}$ defined as

$$\widehat{\mathrm{R}}_{ij} := [0, \ell_{ij}] \times [0, 2r_{ij}], \qquad ij \in \{12, 23, 31\}. \tag{2.6}$$

We use for simplicity the same notation φ_{ij} for the extension of φ_{ij} to $\widehat{\mathrm{R}}_{ij}$, defined as -

$$\varphi_{ij}(s,t) = \varphi_{ij}(s), \qquad (s,t) \in \widehat{\mathrm{R}}_{ij} \qquad ij \in \{12, 23, 31\}, \tag{2.7}$$

and for the extension of φ_{ij} to a $W^{1,1}$ function on $B \setminus \widehat{\mathrm{R}}_{ij}$ as discussed in Section 1.4.

Let $\widehat{m}_{ij} = \widehat{m}_{ij}(\Gamma)$, $ij \in \{12, 23, 31\}$, be a solution of following Dirichlet minimum problem:

$$\min \left\{ \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1 + |Df|^2} \; + \int_{\partial\widehat{\mathrm{R}}_{ij}} |f - \varphi_{ij}| d\mathcal{H}^1 : \; f \in \mathrm{BV}(B), \; f = \varphi_{ij} \text{ on } B \setminus \widehat{\mathrm{R}}_{ij} \right\}, \tag{2.8}$$

where $\int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1 + |Df|^2}$ is the extension of the area functional to $\mathrm{BV}(\widehat{\mathrm{R}}_{ij})$ in Definition 1.16 and $\widehat{m}_{ij}$ is given by Theorem 1.18.

From the conclusion in Section 1.4 and the fact that the restriction of φ_{ij} to $\partial\widehat{\mathrm{R}}_{ij}$ is continuous up to a countable set of points, it follows that $\widehat{m}_{ij}$ is the unique solution of

$$\min \left\{ \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1 + |\nabla f|^2} \; dsdt : \; f \in W^{1,1}(\widehat{\mathrm{R}}_{ij}), \; f = \varphi_{ij} \; \mathcal{H}^1 - a.e. \text{ on } \partial\widehat{\mathrm{R}}_{ij} \right\}, \tag{2.9}$$

and it is analytic in $\widehat{\mathrm{R}}_{ij}$. Let $m_{ij} = m_{ij}(\Gamma)$ be the restriction of $\widehat{m}_{ij}$ to R_{ij}. Then, by the symmetry of φ_{ij} with respect to the line $\{t = r_{ij}\}$, m_{ij} is the unique solution of the Dirichlet-Neumann minimum problem (0.7). From (0.6) it follows that the Dirichlet condition is zero on the sides $\{0\} \times [0, r_{ij}]$ and $\{\ell_{ij}\} \times [0, r_{ij}]$ of the rectangle R_{ij}. Note that m_{ij} is analytic in the interior of R_{ij} but *not* necessarily Lipschitz in R_{ij}, see Theorem 1.20 and Figure 2b.

Definition 2.5 (**The functional** $\mathcal{G}$)**.** *We define the functional* $\mathcal{G} : X \longrightarrow [0, +\infty)$ *as*

$$\mathcal{G}(\Gamma) := \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma), \tag{2.10}$$

where $\mathfrak{A}_{ij}(\Gamma)$ *are as in* (0.9)*.*

The properties of the functional $\mathcal{G}$ will be discussed in Section 3.

2.3 Infimum of $\mathcal{G}$ as an upper bound of $\mathcal{A}(u, D)$

Lemma 2.6. *Let* $\ell \geq 0$, $\mathrm{p} \geq 0$, $\varphi \in \mathrm{Lip}([0, \ell]; [0, +\infty))$ *be such that* $\varphi(0) = \varphi(\ell) = 0$ *and* $w \in [0, \ell]$ *so that* $\varphi(w) = \mathrm{p}$*. Then there exists a sequence* $\{\varphi^\sigma\}$ *of* $\mathcal{C}^\infty$ *equi-Lipschitz functions in* $[0, \ell]$*, converging to* φ *uniformly on* $[0, \ell]$ *as* $\sigma \to 0^+$*, such that*

$$\varphi^\sigma(0) = \varphi^\sigma(\ell) = 0, \qquad \varphi^\sigma(w) = \mathrm{p}, \qquad \text{for any } \sigma > 0 \text{ sufficiently small.}$$

Proof. Let us extend φ in $\mathbb{R}$ such that $\varphi(s) = 0$ in $\mathbb{R}\backslash[0, \ell]$, so that the extension (still denoted by φ) belongs to $\mathrm{Lip}(\mathbb{R})$. Let $\widehat{\varphi}^\sigma(s) := \eta_\sigma * \varphi$ in $\mathbb{R}$, where $\{\eta_\sigma\}$ is a standard sequence of mollifiers. Then $\widehat{\varphi}^\sigma \in \mathcal{C}^\infty(\mathbb{R})$, $\mathrm{Lip}(\widehat{\varphi}^\sigma) \leq \mathrm{Lip}(\varphi)$ and the sequence $\{\widehat{\varphi}^\sigma\}$ converges uniformly to φ on compact subsets of $\mathbb{R}$. Without loss of generality we may assume $\widehat{\varphi}^\sigma(s) = 0$ in $\mathbb{R}\setminus(-\sigma/2, \ell+\sigma/2)$ and $\widehat{\varphi}^\sigma(\frac{\ell+2\sigma}{\ell}w-\sigma) = \mathrm{p}+c_\sigma$, $c_\sigma = o(1)$. Let us first suppose $\mathrm{p} \neq 0$. We define

$$\varphi^\sigma : [0, \ell] \to [0, +\infty), \qquad \varphi^\sigma(s) := \frac{\mathrm{p}}{\mathrm{p} + c_\sigma}\, \widehat{\varphi}^\sigma\big(\frac{\ell + 2\sigma}{\ell}\, s - \sigma\big). \tag{2.11}$$

It is easy to see that $\varphi^\sigma \in \mathcal{C}^\infty([0, \ell])$, $\varphi^\sigma(0) = \varphi^\sigma(\ell) = 0$, $\varphi^\sigma(w) = \mathrm{p}$, φ^σ are equi-Lipschitz, and $\{\varphi^\sigma\}$ converges to φ uniformaly as $\sigma \to 0^+$. Notice that the obtained approximation is constantly null in a neighborhood of 0 and ℓ.
In the case $\mathrm{p} = 0$, we argue differently. We consider the two intervals $[0, w]$ and $[w, \ell]$ and we repeat the same approximation above in the single intervals; more precisely we choose two points $w_1 \in (0, w)$ and $w_2 \in (w, \ell)$ with $\varphi(w_1) > 0$, $\varphi(w_2) > 0$ (if these points do not exist it means that the functions are constantly 0 and they are already smooth, so there is nothing to prove). Then we approximate the two functions $\varphi\llcorner(0, w)$ and $\varphi\llcorner[w, \ell]$ as before, and we glue them along w. Note that the glued function is smooth in w since both the two smooth approximations are constantly 0 in a neighborhood of w.

□

To prove Theorem 2.1 we use the three area-minimizing functions m_{ij} introduced in Section 2.2, to construct a sequence $\{u^\varepsilon\}$ of Lipschitz functions that converges to u in $L^1(D; \mathbb{R}^2)$. However m_{ij} are only locally Lipschitz so we need the following smoothing lemma.

Lemma 2.7. *Let $\Gamma \in X_{\mathrm{Lip}}$, $ij \in \{12, 23, 31\}$. Let $\varphi_{ij} = \varphi_{ij}(\Gamma_{ij}) \in \mathrm{Lip}([0, \ell_{ij}])$, $m_{ij} = m_{ij}(\Gamma_{ij}) \in W^{1,1}(\mathrm{R}_{ij})$, be defined as in Section 2.2. Then there exists a sequence $\{m^\sigma_{ij}\}$ of Lipschitz functions such that $m^\sigma_{ij} : \mathrm{R}_{ij} \to \mathbb{R}$, $m^\sigma_{ij} = \varphi_{ij}$ on $\partial_D \mathrm{R}_{ij}$, and*

$$\left| \int_{\mathrm{R}_{ij}} \sqrt{1 + |\nabla m_{ij}|^2}\, dsdt - \int_{\mathrm{R}_{ij}} \sqrt{1 + |\nabla m^\sigma_{ij}|^2}\, dsdt \right| \leq O(\sigma). \tag{2.12}$$

Proof. This can be proved using an argument similar to the one in [11, p. 378; p. 381], and using also Lemma 2.6 with the choice $w = w_{ij}$ and $\mathrm{p} = \varphi_{ij}(w_{ij})$. □

We start to prove Theorem 2.1 in the special case of a piecewise linear jump, as in Figure 2.1.

Proposition 2.8. *Let $u \in BV(D; \{\alpha_1, \alpha_2, \alpha_3\})$ be the map defined in* (0.10) *and assume that the jump set of u consists of three distinct segments that meet at the origin. Then* (2.1) *holds.*

Proof. Let $\Gamma \in X_{\mathrm{Lip}}$ be a connection passing through $p \in \mathrm{T}$ and $\mathcal{G}(\Gamma) = \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma)$. To prove the proposition it is sufficient to construct a sequence $\{u^\varepsilon\} \subset \mathrm{Lip}(D; \mathbb{R}^2)$ converging to u in $L^1(D; \mathbb{R}^2)$ such that

$$\lim_{\varepsilon \to 0} \mathcal{A}(u^\varepsilon, D) \leq |D| + \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma). \tag{2.13}$$

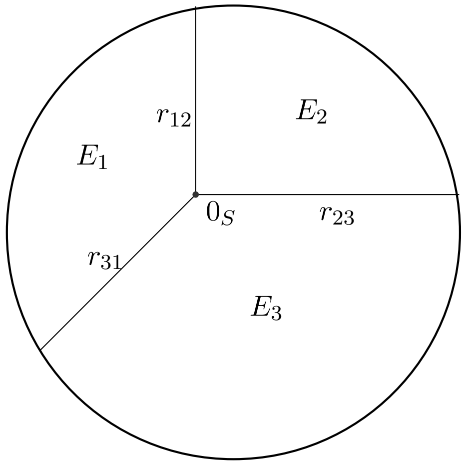


Figure 2.1: $E_1,\ E_2,\ E_3$ are separated by three segments of length r_{12}, r_{23}, r_{31} that meet at the origin.

Case 1. Assume that the segments separating $E_1,\ E_2,\ E_3$ meet at the origin with angles less than π, as in Figure 2.1.
To simplify the computation we may assume that $p = 0_{\mathrm{T}}$, see Figure 1b. The idea of the proof is similar to the one used in [11], with however new difficulties, in particular in T^ε (step 3). We will specify various subsets of D and define the sequence $\{u^\varepsilon\}$ on each of these sets. Let $\varepsilon > 0$ be sufficiently small and $\delta_\varepsilon > 0$ be such that $\delta_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$. Define T^ε to be the triangle with the origin 0_S in its interior, with vertices $\zeta^1 = \zeta^1_\varepsilon$, $\zeta^2 = \zeta^2_\varepsilon$, and $\zeta^3 = \zeta^3_\varepsilon$, and sides of lengths $\varepsilon_{12},\ \varepsilon_{23},\ \varepsilon_{31}$, $\varepsilon_{ij} := |\zeta^i - \zeta^j|$; the sides of T^ε are perpendicular to the lines containing r_{12}, r_{23}, r_{31} (respectively) and their distance from the origin 0_S equals δ_ε. Define three cygar-shaped sets $S^\varepsilon_{23},\ S^\varepsilon_{31}$ and S^ε_{12} as in Figure 2.2a: if for instance y is a coordinate on r_{12} and x is the perpendicular coordinate, then S^ε_{12} is defined as

$$S^\varepsilon_{12} := \left\{(x, y) \in D : x \in (\zeta^1_1, \zeta^2_1),\ y \geq \delta_\varepsilon\right\}, \tag{2.14}$$

where

$$\zeta^i = (\zeta^i_1, \zeta^i_2), \quad i = 1, 2, 3.$$

Let us set

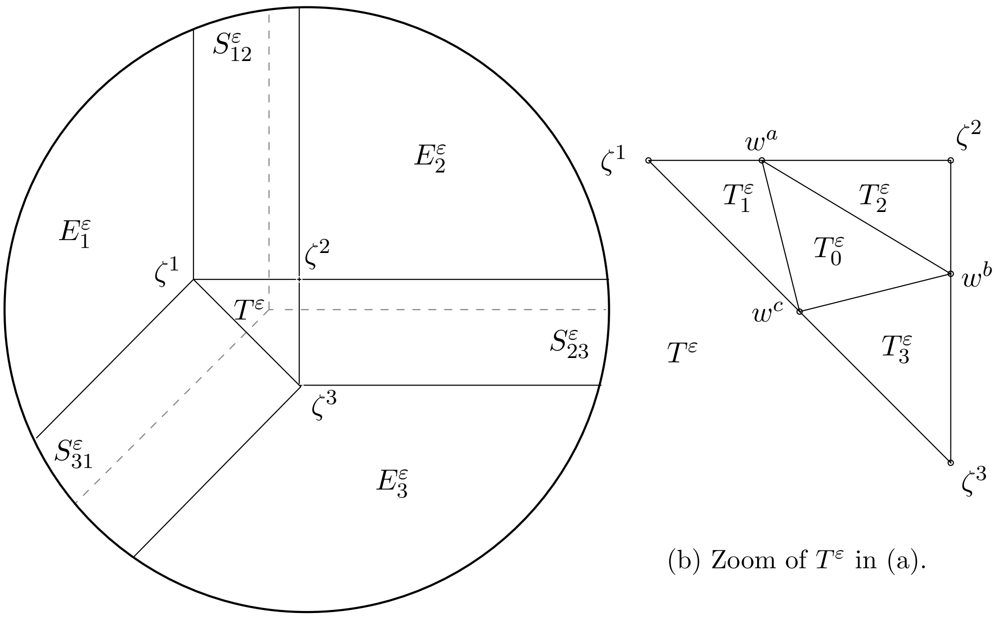
$$E^\varepsilon_1 := E_1 \setminus (S^\varepsilon_{31} \cup T^\varepsilon \cup S^\varepsilon_{12}), \quad E^\varepsilon_2 := E_2 \setminus (S^\varepsilon_{23} \cup T^\varepsilon \cup S^\varepsilon_{12}), \quad E^\varepsilon_3 := E_3 \setminus (S^\varepsilon_{23} \cup T^\varepsilon \cup S^\varepsilon_{31}). \tag{2.15}$$

Step 1. Definition of u^ε on $E^\varepsilon_1 \cup E^\varepsilon_2 \cup E^\varepsilon_3$. We define

$$u^\varepsilon := \begin{cases} \alpha_1 & \text{in } E^\varepsilon_1, \\ \alpha_2 & \text{in } E^\varepsilon_2, \\ \alpha_3 & \text{in } E^\varepsilon_3. \end{cases} \tag{2.16}$$

Note that $\mathcal{A}(u^\varepsilon, E^\varepsilon_1 \cup E^\varepsilon_2 \cup E^\varepsilon_3) = |E^\varepsilon_1| + |E^\varepsilon_2| + |E^\varepsilon_3|$, hence

$$\lim_{\varepsilon \to 0^+} \mathcal{A}(u^\varepsilon, E^\varepsilon_1 \cup E^\varepsilon_2 \cup E^\varepsilon_3) = |D|. \tag{2.17}$$



(b) Zoom of T^{ε} in (a).

(a) Case 1 of the proof of Proposition 2.8

Figure 2.2

Step 2. Definition of u^{ε} on $S_{23}^{\varepsilon} \cup S_{31}^{\varepsilon} \cup S_{12}^{\varepsilon}$.
We will start with the construction on S_{12}^{ε}. Set

$$\xi = (\xi_1, \xi_2) := \frac{\alpha_2 - \alpha_1}{\ell_{12}} \in \mathbb{S}^1, \qquad \eta = (\eta_1, \eta_2) := \xi^{\perp}, \tag{2.18}$$

where $^{\perp}$ denotes the counterclockwise rotation of $\pi/2$.
Let $\psi_{12}^{\varepsilon} : [\delta_{\varepsilon}, r_{12} + c_{\varepsilon}] \to [0, r_{12}]$ be linear, increasing, surjective, where $c_{\varepsilon} > 0$ is the smallest number such that

$$S_{12}^{\varepsilon} \subset [\zeta_1^1, \zeta_1^2] \times [\delta_{\varepsilon}, r_{12} + c_{\varepsilon}], \qquad \lim_{\varepsilon \to 0^+} c_{\varepsilon} = 0.$$

Note that for any $y \in [\delta_{\varepsilon}, r_{12} + c_{\varepsilon}]$ we have

$$(\psi_{12}^{\varepsilon})'(y) = \frac{r_{12}}{r_{12} + c_{\varepsilon} - \delta_{\varepsilon}} =: \kappa_{\varepsilon}, \qquad \lim_{\varepsilon \to 0^+} \kappa_{\varepsilon} = 1. \tag{2.19}$$

Let m_{12}^{σ} be the map defined in Lemma 2.7, whose area on R_{12} is by construction close to $\mathfrak{A}_{12}(\Gamma)$, with $\{\sigma_{\varepsilon}\} \subset (0, +\infty)$ a sequence such that

$$\lim_{\varepsilon \to 0^+} \sigma_{\varepsilon} = 0. \tag{2.20}$$

We set, with $\sigma = \sigma_{\varepsilon}$ for simplicity,

$$u^{\varepsilon}(x, y) := \alpha_1 + \left(\frac{x - \zeta_1^1}{\varepsilon_{12}} \right) \ell_{12} \xi + m_{12}^{\sigma} \left(\frac{x - \zeta_1^1}{\varepsilon_{12}} \ell_{12} \ , \ \psi_{12}^{\varepsilon}(y) \right) \eta, \qquad (x, y) \in S_{12}^{\varepsilon}. \tag{2.21}$$

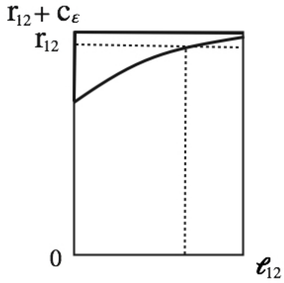


Figure 2.3: The set P_ε is bounded by the bold contour.

Observe that $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in \mathrm{Lip}(S_{12}^\varepsilon; \mathbb{R}^2)$, $u^\varepsilon = \alpha_1$ on $\{(x,y) \in S_{12}^\varepsilon : x = \zeta_1^1\}$, and $u^\varepsilon = \alpha_2$ on $\{(x,y) \in S_{12}^\varepsilon : x = \zeta_1^2\}$. By the definition of m_{12}^σ, it is uniquely defined the point (depending on ε) $w^a = (w_1^a, w_2^a) \in \overline{\zeta^1\zeta^2}$ such that $u^\varepsilon(w_1^a, w_2^a) = 0_T$ (see Figure 2.2b). Write for simplicity

$$\widetilde{m} = m_{12}^\sigma.$$

Using that $|\xi| = |\eta| = 1$, $\xi_1\eta_1 + \xi_2\eta_2 = 0$, and $\xi_1\eta_2 - \xi_2\eta_1 = 1$, we compute

$$1+|\nabla u_1^\varepsilon|^2+|\nabla u_2^\varepsilon|^2+\left(\frac{\partial u_1^\varepsilon}{\partial x}\frac{\partial u_2^\varepsilon}{\partial y} - \frac{\partial u_1^\varepsilon}{\partial y}\frac{\partial u_2^\varepsilon}{\partial x}\right)^2 = 1+\frac{\ell_{12}^2}{\varepsilon_{12}^2}\left(1 + \left(\widetilde{m}_s\right)^2 + \left(\widetilde{m}_t\right)^2\kappa_\varepsilon^2\left(1+\frac{\varepsilon_{12}^2}{\ell_{12}^2}\right)\right),$$

where $\widetilde{m}_s, \widetilde{m}_t$ denote, respectively, the partial derivatives of $\widetilde{m}$ with respect to $s := \frac{x-\zeta_1^1}{\varepsilon_{12}}\ell_{12}$ and $t := \psi_{12}^\varepsilon(y)$, and are evaluated at $\left(\frac{x-\zeta_1^1}{\varepsilon_{12}}\ell_{12}\ ,\ \psi_{12}^\varepsilon(y)\right)$. As a consequence

$$\begin{aligned}
&\mathcal{A}(u^\varepsilon, S_{12}^\varepsilon)\\
=&\frac{\ell_{12}}{\varepsilon_{12}}\int_{S_{12}^\varepsilon}\sqrt{1+\left[\widetilde{m}_s\left(\frac{x-\zeta_1^1}{\varepsilon_{12}}\ell_{12},\ \psi_{12}^\varepsilon(y)\right)\right]^2+\left[\widetilde{m}_t\left(\frac{x-\zeta_1^1}{\varepsilon_{12}}\ell_{12},\ \psi_{12}^\varepsilon(y)\right)\right]^2\kappa_\varepsilon^2\left(1+\frac{\varepsilon_{12}^2}{\ell_{12}^2}\right)+O(\varepsilon^2)}\ dxdy\\
=&\frac{1}{\kappa_\varepsilon}\int_{\mathrm{R}_{12}\setminus P_\varepsilon}\sqrt{1+\left[\widetilde{m}_s\left(s,t\right)\right]^2+\left[\widetilde{m}_t\left(s,t\right)\right]^2\kappa_\varepsilon^2\left(1+\frac{\varepsilon_{12}^2}{\ell_{12}^2}\right)+O(\varepsilon^2)}\ dsdt, \qquad (2.22)
\end{aligned}$$

where the last equality follows by the change of variables

$$\Phi : \mathrm{R}_{12} \ni (s,t) \mapsto \Phi(s,t) := \left(\frac{\varepsilon_{12}}{\ell_{12}}s + \zeta_1^1, {\psi_{12}^\varepsilon}^{-1}(t)\right) = (x,y) \in \left[\zeta_1^1, \zeta_1^2\right]\times[\delta_\varepsilon, r_{12}+c_\varepsilon] \supset S_{12}^\varepsilon, \tag{2.23}$$

and $P_\varepsilon := \mathrm{R}_{12} \setminus \Phi^{-1}(S_{12}^\varepsilon)$ (see Figure 2.3). Hence, recalling also (2.19), we conclude

$$\lim_{\varepsilon\to 0^+}\mathcal{A}(u^\varepsilon, S_{12}^\varepsilon) = \int_{\mathrm{R}_{12}}\sqrt{1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2}\ dsdt. \tag{2.24}$$

We recall that from (2.12) it follows that

$$\int_{\mathrm{R}_{12}}\sqrt{1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2}\ dsdt = \mathfrak{A}_{12}(\Gamma) + O(\varepsilon). \tag{2.25}$$

Hence, employing the same construction in the strips S_{23}^{ε} and S_{31}^{ε} we obtain

$$\lim_{\varepsilon \to 0^+} \mathcal{A}(u^\varepsilon, S_{23}^{\varepsilon} \cup S_{31}^{\varepsilon} \cup S_{12}^{\varepsilon}) = \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma). \tag{2.26}$$

Step 3. Definition of u^ε on T^ε. We divide T^ε into four closed triangles T_1^ε, T_2^ε, T_3^ε and T_0^ε as in Figure 2.2b. We set

$$u^\varepsilon := 0_{\mathrm{T}} \qquad \text{in } T_0^\varepsilon. \tag{2.27}$$

We first define u^ε on ∂T_1^ε as follows:

(i) the value of u^ε at ζ^1 is α_1;

(ii) the value of u^ε on the side $\overline{w^c w^a}$ is 0_{T}.

Note that u^ε is already defined on the edges $\overline{\zeta^1 w^a}$ and $\overline{\zeta^1 w^c}$ and its graph over both edges is given by a rescaled version of the curve Γ_1 suitably parametrized.
More precisely, we recall that $\pi_{12} : \Gamma_1 \to \overline{\alpha_1\alpha_2}$ and $\pi_{31} : \Gamma_1 \to \overline{\alpha_3\alpha_1}$ are the orthogonal projections onto the edges $\overline{\alpha_1\alpha_2}$ and $\overline{\alpha_3\alpha_1}$. Since Γ_1 is, by hypothesis, a part of a Lipschitz graph, the maps $\pi_{12}\llcorner\Gamma_1$ and $\pi_{31}\llcorner\Gamma_1$ are bi-Lipschitz bijections between Γ_1 and the segments $\overline{\alpha_1\pi_{12}(p)}$ and $\overline{\alpha_1\pi_{31}(p)}$, respectively. We know that if (s, t) are coordinates on T with respect to the system with s-axis $\overline{\alpha_1\alpha_2}$, then the inverse of $\pi_{12}\llcorner\Gamma_1$ is given by $\Phi_{12} : \overline{\alpha_1\alpha_2} \to \Gamma_1$,

$$\Phi_{12}((s, 0)) = (s, \varphi_{12}(s)).$$

Let us denote by $L_{12} = L_{12}^\varepsilon : \overline{\zeta^1 w^a} \subset \mathbb{R}^2_S \to \overline{\alpha_1\pi_{12}(p)} \subset \mathbb{R}^2_T$ and $L_{31} = L_{31}^\varepsilon : \overline{\zeta^1 w^a} \subset \mathbb{R}^2_S \to \overline{\alpha_1\pi_{31}(p)} \subset \mathbb{R}^2_T$ the linear bijective maps

$$L_{12}(Q) = \alpha_1 + \frac{|Q - \zeta^1|}{\varepsilon_{12}}(\alpha_2 - \alpha_1), \qquad L_{31}(Q') = \alpha_1 + \frac{|Q' - \zeta^1|}{\varepsilon_{31}}(\alpha_3 - \alpha_1).$$

Then we define

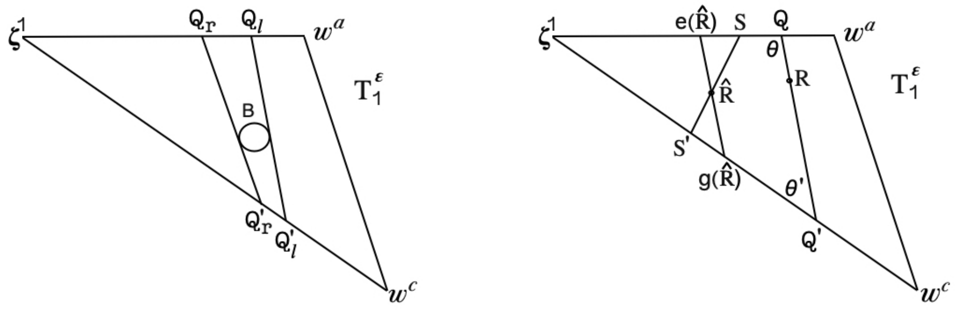
$$u^\varepsilon := \Phi_{12} \circ L_{12} \qquad \text{on} \quad \overline{\zeta^1 w^a}, \tag{2.28}$$

and

$$u^\varepsilon = \Phi_{31} \circ L_{31} \qquad \text{on} \quad \overline{\zeta^1 w^c}. \tag{2.29}$$

compare formula (2.21). Since Φ_{12} and Φ_{31} are Lipschitz with Lipschitz constants independent of ε, and the Lipschitz constants of L_{12} and L_{31} have order $\frac{1}{\varepsilon}$, it follows that the Lipschitz constants of u^ε over the segments $\overline{\zeta^1 w^a}$ and $\overline{\zeta^1 w^c}$ have order $\frac{1}{\varepsilon}$.
Now we want to define u^ε in the interior of T_1^ε. First we observe that the map $\pi_{31} \circ \Phi_{12} : \overline{\alpha_1\pi_{12}(p)} \subset \mathbb{R}^2_T \to \overline{\alpha_1\pi_{31}(p)} \subset \mathbb{R}^2_T$ is a bi-Lipschitz bijection, with constant independent of ε. A direct computation then provides that the map $\Psi : \overline{\zeta^1 w^a} \subset \mathbb{R}^2_S \to \overline{\zeta^1 w^c} \subset \mathbb{R}^2_S$ defined by

$$\Psi := (L_{31})^{-1} \circ \pi_{31} \circ \Phi_{12} \circ L_{12}, \tag{2.30}$$



(a) (b)

Figure 2.4: Surjectivity of the foliation in $T_1^\varepsilon \subset \mathbb{R}^2_S$.

is bi-Lipschitz between $\overline{\zeta^1 w^a}$ and $\overline{\zeta^1 w^c}$ with bi-Lipschitz constant of order 1 as $\varepsilon \to 0^+$. Given $Q \in \overline{\zeta^1 w^a}$ let $Q' := \Psi(Q) \in \overline{\zeta^1 w^c}$.

Now we show that T_1^ε is foliated by the segments $\overline{QQ'}$, *i.e.*, for any $R \in T_1^\varepsilon$ we can find a unique $Q \in \overline{\zeta^1 w^a}$ for which $R \in \overline{QQ'}$.

First we notice that $\overline{QQ'} \cap \overline{SS'} = \emptyset$ for any $Q \neq S \in \overline{\zeta^1 w^a}$ with $Q' = \Psi(Q)$ and $S' = \Psi(S)$. Indeed, thanks to the fact that Ψ is a homeomorphism and that it keeps ζ^1 fixed, it is easy to see that if $S \in \overline{\zeta^1 Q}$ then $S' \in \overline{\zeta^1 Q'}$, or if $Q \in \overline{\zeta^1 S}$ then $Q' \in \overline{\zeta^1 S'}$. Consider the function

$$f(q, \sigma) = q\tau + \sigma\nu(q), \qquad q \in [0, |w^a - \zeta^1|], \sigma \in [0, |\Psi(q\tau) - q\tau|],$$

where $\tau := \frac{w^a - \zeta^1}{|w^a - \zeta^1|}$ and $\nu(q) := \frac{\Psi(q\tau) - q\tau}{|\Psi(q\tau) - q\tau|}$. It is clear that the image of f is a closed set and $\mathrm{Im}(f) = \{\overline{QQ'} : Q \in \overline{\zeta^1 w^a}, Q' = \Psi(Q)\}$. Now we show that $\mathrm{Im}(f) = T_1^\varepsilon$. Assume by contradiction there is $R \in T_1^\varepsilon \setminus \mathrm{Im}(f)$ and take a disk $B \subset T_1^\varepsilon \setminus \mathrm{Im}(f)$ centered at R. Let $Q_r, Q_l \in \overline{\zeta^1 w^a}$ be such that $q_r := |Q_r - \zeta^1|$ (resp. $q_l = |Q_l - \zeta^1|$) be the supremum (resp. the infimum) parameter for which B lies on the right (resp. left) of $\overline{Q_r Q_r'}$ (resp. $\overline{Q_l Q_l'}$). Note that $Q_r \neq Q_l$ due to the injectivity of Ψ, thus for any $Q \in \overline{Q_r Q_l}$ the segment $\overline{QQ'}$ must intersect B, a contradiction, see Figure 2.4a. Hence we may define u^ε on T_1^ε as

$$u^\varepsilon(R) := u^\varepsilon(Q), \qquad R \in \overline{QQ'},\ Q \in \overline{\zeta^1 w^a}. \tag{2.31}$$

We want now to show that on T_1^ε, u^ε is Lipschitz continuous with Lipschitz constant of order $\frac{1}{\varepsilon}$. To prove this let us fix $R \in T_1^\varepsilon$. By definition $u^\varepsilon(R) = u^\varepsilon(Q)$ for some $Q \in \overline{\zeta^1 w^a}$ and u^ε is constant on the segment $\overline{QQ'} \ni R$.

Let $e : T_1^\varepsilon \to \overline{\zeta^1 w^a}$ be the function taking $(x, y) \in T_1^\varepsilon$ to the intersection point of $\overline{\zeta^1 w^a}$ and the line passing through (x, y) parallel to $\overline{QQ'}$. Let $g : T_1^\varepsilon \to \overline{\zeta^1 w^c}$ be the function taking $(x, y) \in T_1^\varepsilon$ to the intersection point of $\overline{\zeta^1 w^c}$ and the line passing through (x, y) parallel to $\overline{QQ'}$. Let $\hat{R} \in T_1^\varepsilon$ be a point in T_1^ε; we want to estimate the ratio

$$\frac{|u^\varepsilon(\hat{R}) - u^\varepsilon(R)|}{|\hat{R} - R|}.$$

Consider the two segments $\overline{Qe(\hat{R})}$ and $\overline{Q'g(\hat{R})}$. By definition $\hat{R} \in \overline{SS'}$ and $u^\varepsilon(\hat{R}) = u^\varepsilon(S) = u^\varepsilon(S')$ for two points $S \in \overline{\zeta^1 w^a}$ and $S' \in \overline{\zeta^1 w^c}$. It is straightforward that either $S \in \overline{Qe(\hat{R})}$ or $S' \in \overline{Q'g(\hat{R})}$. Without loss of generality suppose the first case holds, see Figure 2.4b.
Finally, denote by θ the angle between $\overline{QQ'}$ and $\overline{\zeta^1 w^a}$ and by θ' the angle between $\overline{QQ'}$ and $\overline{\zeta^1 w^c}$. Using the fact that the homeomorphism in (2.30) is bi-Lipschitz with constant of order 1 it is not difficult to see that there is a constant $\theta_0 > 0$ independent of ε such that $\min\{\theta, \theta'\} \geq \theta_0$. This is a consequence of the fact that the bi-Lipschitz constant of Ψ in (2.30) is of order 1. Indeed, if $L = \text{lip}(\Psi)$ and $1/L' = \text{lip}(\Psi^{-1})$, we see that

$$L' \leq \frac{|Q' - \zeta^1|}{|Q - \zeta^1|} \leq L,$$

hence

$$\frac{1/L + \cos\theta_{\zeta^1}}{\sin\theta_{\zeta^1}} \leq \frac{\cos\theta}{\sin\theta} \leq \frac{1/L' + \cos\theta_{\zeta^1}}{\sin\theta_{\zeta^1}},$$

where θ_{ζ^1} is the angle at ζ^1 (here we have used the law of sines and that $\theta' = \pi - \theta_{\zeta^1} - \theta$). A similar estimate holding for θ', this readily provides the boundedness from below of $\min\{\theta, \theta'\}$.
As a consequence we have

$$|\hat{R} - R| \geq |Q - e(\hat{R})||\sin\theta| \geq |Q - e(\hat{R})||\sin\theta_0|.$$

Thus, we compute

$$\frac{|u^\varepsilon(\hat{R}) - u^\varepsilon(R)|}{|\hat{R} - R|} \leq \frac{|u^\varepsilon(Q) - u^\varepsilon(S)|}{|Q - e(\hat{R})||\sin\theta|} \leq \frac{|u^\varepsilon(Q) - u^\varepsilon(S)|}{|Q - S||\sin\theta|} \leq \frac{1}{|\sin\theta_0|}\frac{|u^\varepsilon(Q) - u^\varepsilon(S)|}{|Q - S|}, \tag{2.32}$$

that is bounded by the Lipschitz constant of $\Phi_{12} \circ L_{12}$ which is of order $\frac{1}{\varepsilon}$.
Eventually we compute the Jacobian of u^ε in (2.31). By construction the image of T_1^ε by u^ε is exactly the curve Γ_1, which has zero Lebesgue measure in $\mathbb{R}^2$. By a standard application of the area formula it follows that the Jacobian of u^ε is vanishes a.e. in T_1^ε. We have concluded the definition of u^ε in T_1^ε. The constructions on T_2^ε and on T_3^ε are similar, and similar estimates of the derivatives and Jacobians hold. Using that the area of the triangle T^ε is of order ε^2, we have

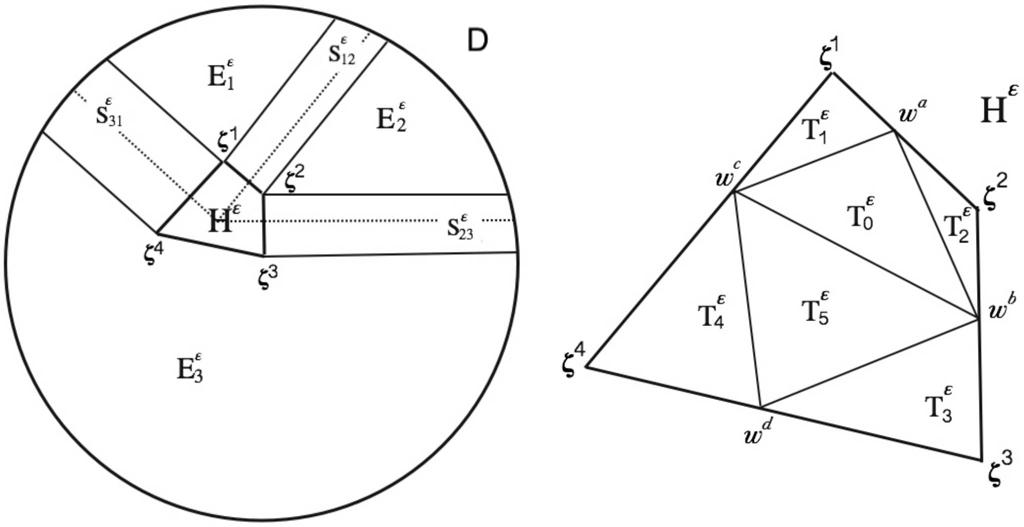
$$\lim_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon, T_\varepsilon) = \lim_{\varepsilon\to 0^+} (O(\varepsilon) + O(\varepsilon^2)) = 0. \tag{2.33}$$

From (2.16), (2.21), (2.27), (2.31), and the estimates above it follows that

$$\{u^\varepsilon\} \subset \text{Lip}(D; \mathbb{R}^2), \qquad \lim_{\varepsilon\to 0^+} \int_D |u^\varepsilon - u| \, dxdy = 0. \tag{2.34}$$

Moreover

$$\mathcal{A}(u^\varepsilon, D) = \mathcal{A}(u^\varepsilon, E_1^\varepsilon \cup E_2^\varepsilon \cup E_3^\varepsilon) + \mathcal{A}(u^\varepsilon, S_{23}^\varepsilon) + \mathcal{A}(u^\varepsilon, S_{31}^\varepsilon) + \mathcal{A}(u^\varepsilon, S_{12}^\varepsilon) + \mathcal{A}(u^\varepsilon, T^\varepsilon). \tag{2.35}$$



(a) The non acute case.

(b) Zoom of H^ε in (a).

Figure 2.5

Then (2.13) follows from (2.35), (2.17), (2.26), (2.20) and (2.33).

Case 2. Assume that two of the segments separating E_1, E_2, E_3 meet at the origin with an angle greater than or equal to π.
Similar to Case 1, we divide the domain D into a finite number of subsets and define the sequence $\{u^\varepsilon\}$ on each of these sets. Draw the normal to each segment at the point of distance δ_ε from the origin. The normal lines meet at two points ζ^1, ζ^2. Divide D into three cygar-shape subsets S_{23}^ε, S_{31}^ε, S_{12}^ε (with widths of order $\delta_\varepsilon = O(\varepsilon)$) and a quadrilateral H^ε as in Figure 2.5a. Let

$$E_1^\varepsilon := E_1 \backslash (S_{31}^\varepsilon \cup H^\varepsilon \cup S_{12}^\varepsilon)\,, \quad E_2^\varepsilon := E_2 \backslash (S_{23}^\varepsilon \cup H^\varepsilon \cup S_{12}^\varepsilon)\,, \quad E_3^\varepsilon := E_3 \backslash (S_{23}^\varepsilon \cup H^\varepsilon \cup S_{31}^\varepsilon)\,.$$

Set

$$u^\varepsilon := \begin{cases} \alpha_1 & \text{in } E_1^\varepsilon, \\ \alpha_2 & \text{in } E_2^\varepsilon, \\ \alpha_3 & \text{in } E_3^\varepsilon. \end{cases} \tag{2.36}$$

Define u^ε on $S_{23}^\varepsilon \cup S_{31}^\varepsilon \cup S_{12}^\varepsilon$ as in Step 2 in case 1. It remains to define u^ε on H^ε. Recall that by construction there exist uniquely determined three points $w^a \in \overline{\zeta^1\zeta^2}$, $w^b \in \overline{\zeta^2\zeta^3}$ and $w^c \in \overline{\zeta^1\zeta^4}$ such that

$$u^\varepsilon(w^a) = u^\varepsilon(w^b) = u^\varepsilon(w^c) = 0_{\mathrm{T}}.$$

Divide H^ε into six triangles T_0^ε, T_1^ε, T_2^ε, T_3^ε, T_4^ε, T_5^ε, as in Figure 2.5b, where w^d is any point in $\overline{\zeta^3\zeta^4}$ and $w^d \neq \zeta^3$, $w^d \neq \zeta^4$.
Set

$$u^\varepsilon := 0_{\mathrm{T}} \qquad \text{in} \quad T_0^\varepsilon.$$

We define u^ε in the triangles T_1^ε and T_2^ε as in Step 3; it remains to define u^ε on T_3^ε, T_4^ε, T_5^ε. Let us first define u^ε on the edges $\overline{w^c w^d}$ and $\overline{w^b w^d}$. The map u^ε is already defined on the other edges, and its graph over $\overline{\zeta^4 w^c}$ and $\overline{\zeta^3 w^b}$ is given by a suitable reparametrization of the curve Γ_3, whereas u^ε on $\overline{\zeta^4\zeta^3}$ is constantly α_3. Therefore it suffices to define u^ε in such a way that its graph over $\overline{w^c w^d}$ and $\overline{w^b w^d}$ coincides with Γ_3 as well, and then we can define u^ε inside T_3^ε and T_4^ε using the same construction for T_1^ε in step 3. Similarly, using that the graph of u^ε on $\overline{w^c w^d}$ and $\overline{w^b w^d}$ is again Γ_3, we can repeat the construction in the triangle T_5^ε. Following the computation as in case 1 we get (2.13). This concludes the proof.

□

Proof of Theorem 2.1. We will suitably adapt the construction made in the proof of Proposition 2.8. By hypothesis the regions E_1, E_2, E_3 are enclosed by C^2-embedded curves C_{ij}, $ij \in \{12, 23, 31\}$, parametrized by arc length $c_{ij} : [0, r_{ij}] \to \mathbb{R}^2$, $ij \in \{12, 23, 31\}$. Moreover such curves meet ∂D transversely and intersect each other (transversely) only at one point Q. Suppose that the angles formed at Q by the three curves are all less than π (the other case is similarly adapted from the corresponding case in the proof of Proposition 2.8). We will divide the domain D into a finite number of subsets and define the sequence $\{u^\varepsilon\}$ on each of these sets.
Let $\delta_\varepsilon > 0$ be such that $\delta_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$. Let $\tau \in [0, r_{ij}]$ be an arc lenght parameter on C_{ij}, with orthogonal coordinate d that coincides with the signed distance from C_{ij} negative in E_i and positive in E_j. Let $Q_{ij} \in C_{ij}$ be the point with arc distance $\tau = \delta_\varepsilon$ from the origin Q. Consider the three lines normal to C_{ij} at Q_{ij}. For δ_ε sufficiently small, since the angles at the origin are less than π and the curves are of class C^2 up to the closure, these lines mutually meet at points ζ^1, ζ^2, and ζ^3. Let ε_{ij} be the length of $\overline{\zeta^i\zeta^j}$, which are of order ε. The tubular coordinates of the points ζ^1 and ζ^2 with respect to C_{12} are $(d_1, \delta_\varepsilon,)$ and $(d_2, \delta_\varepsilon)$, with $d_2 - d_1 = \varepsilon_{12}, d_1 < 0, d_2 > 0$. For δ_ε small enough we can consider the neighborhood of C_{12} defined as

$$S_{12}^\varepsilon := \{(x, y) \in D : \tau(x, y) \geq \delta_\varepsilon,\ d(x, y) \in (d_1, d_2)\}, \tag{2.37}$$

where we have prolonged C_{12} outside D for convenience. Similarly we define S_{23}^ε and S_{31}^ε. Let T^ε be the triangle with vertices ζ^1, ζ^2, and ζ^3.
Finally, let E_1^ε, E_2^ε, E_3^ε be defined as in (2.15), and u^ε as in (2.16).
Step 1. Definition of u^ε on $S_{12}^\varepsilon \cup S_{23}^\varepsilon \cup S_{31}^\varepsilon$. We do the construction on S_{12}^ε, and u^ε will be defined similarly on S_{23}^ε and S_{31}^ε. We know that $c_{12}([\delta_\varepsilon, r_{12}]) = C_{12} \cap S_{12}^\varepsilon$. The system of coordinates (d, τ) defines a C^1-diffeomorphism h between the rectangle $[d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]$ and its image $\mathcal{N}_{12}^{\varepsilon,\delta}$ which contains S_{12}^ε, namely

$$h : [d_1, d_2] \times [\delta_\varepsilon, \rho_{12}] \to \mathcal{N}_{12}^{\varepsilon,\delta}; \qquad h(d, \tau) := c_{12}(\tau) + d\bar{\nu}(\tau),$$

where $\bar{\nu}(\tau)$ is the unit normal vector pointing toward E_2 at $c_{12}(\tau)$ and $\rho_{12} = \rho_{12}^\varepsilon \geq r_{12}$ is the infimum of those ρ for which $S_{12}^\varepsilon \subset \mathcal{N}_{12}^{\varepsilon,\delta}$, see Figure 2.6b.
Since h is a C^1-diffeomorphism we have that

$$h^{-1} : \mathcal{N}_{12}^{\varepsilon,\delta} \to [d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]; \qquad h^{-1}(x, y) := (d(x, y), \tau(x, y)),$$

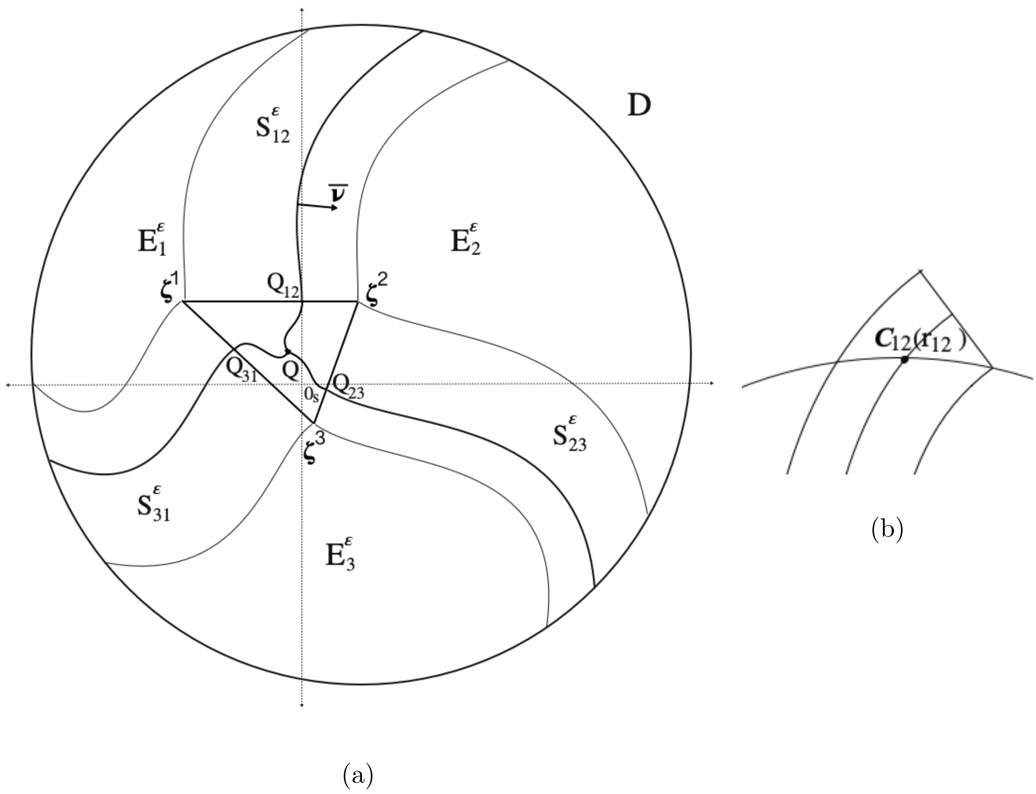


Figure 2.6: proof of Theorem 2.1.

is the inverse of h and is of class C^1. We want to estimate the Jacobian of h^{-1}. To this aim, we first see that $\nabla d(c_{12}(\tau)) = \bar{\nu}(\tau)$ since $c_{12}([\delta_\varepsilon, \rho_{12}])$ is the zero level set of d and, from [2, Rem. 3(1)], we have

$$\nabla d(h(d,\tau)) = \bar{\nu}(\tau), \qquad (d,\tau) \in [d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]. \tag{2.38}$$

Fix $\tau \in [\delta_\varepsilon, \rho_{12}]$; by definition of tubular coordinates the segment $\{c_{12}(\tau) + d\bar{\nu}(\tau) : d \in [d_1, d_2]\}$ is a level set of the function $\tau(\cdot)$, hence

$$\nabla\tau(x,y) \perp \bar{\nu}(\tau(x,y)), \tag{2.39}$$

therefore

$$\nabla\tau \cdot \nabla d \; = \; 0 \qquad \text{in } S_{12}^\varepsilon. \tag{2.40}$$

Thus the Jacobian of h^{-1} will be

$$j(h^{-1}) = |\nabla\tau||\nabla d| = |\nabla\tau|, \tag{2.41}$$

since $|\nabla d| = 1$ in $\mathcal{N}_{12}^{\varepsilon,\delta}$. Let us compute $\nabla\tau$; fix $d \in (d_1, d_2)$ and define $c_{12}^d(\tau) := c_{12}(\tau) + d\bar{\nu}(\tau)$. Now recall (2.39) and that $(c_{12}^d)'(\tau)$ is parallel to $\bar{\nu}^\perp(\tau)$, so that

$$|\nabla\tau| = \nabla\tau \cdot \bar{\nu}^\perp = \frac{\nabla\tau \cdot (c_{12}^d)'}{|(c_{12}^d)'|} \qquad \text{in } S_{12}^\varepsilon. \tag{2.42}$$

Let us recall that C_{12} is parametrized by arc length, *i.e.*, $|c_{12}'(\tau)| = 1$, so that $\bar{\nu}'(\tau) = |c_{12}''(\tau)|c_{12}'(\tau)$. Thus $\left(c_{12}^d\right)'(\tau) = \left(1 + d|c_{12}''(\tau)|\right) c_{12}'(\tau)$. Since $\tau \circ c_{12}^d = \mathrm{Id}$ it follows that $\nabla\tau(c_{12}^d(\tau))^T (c_{12}^d)'(\tau) = \nabla\tau(c_{12}^d(\tau)) \cdot (c_{12}^d)'(\tau) = 1$. Therefore, from (2.42), we deduce

$$|\nabla\tau| = \frac{1}{1 + d|c_{12}''(\tau)|} \qquad \text{in } S_{12}^\varepsilon, \tag{2.43}$$

and in particular $\lim_{d\to 0} |\nabla\tau| = 1$ uniformly in S_{12}^ε.
We are ready to define u^ε in S_{12}^ε. We first set ψ_{12}^ε as in (2.19) with $r_{12} + c_\varepsilon = \rho_{12}$, *i.e.*, $\psi_{12}^\varepsilon(\tau) = \overline{\kappa}_\varepsilon(z - \delta_\varepsilon)$, setting $\overline{\kappa}_\varepsilon := \frac{\rho_{12}}{\rho_{12} - \delta_\varepsilon}$. Then we define $\tilde{u}^\varepsilon$ on $[d_1, d_2] \times [\delta_\varepsilon, \rho_{12}]$ as in the right hand side of (2.21) and set

$$u^\varepsilon := \tilde{u}^\varepsilon \circ h^{-1} \qquad \text{in } S_{12}^\varepsilon. \tag{2.44}$$

Explicitly, recalling that $\xi = \frac{\alpha_2 - \alpha_1}{\ell_{12}}$ and $\eta = \xi^\perp$, for $(x,y) \in S_{12}^\varepsilon$ we have

$$u^\varepsilon(x,y) := \alpha_1 + \left(\frac{d(x,y) - d_1}{\varepsilon_{12}}\right)\ell_{12}\xi + m_{12}^\sigma\left(\frac{d(x,y) - d_1}{\varepsilon_{12}}\ell_{12} \;,\; \overline{\kappa}_\varepsilon\left(\tau(x,y) - \delta_\varepsilon\right)\right)\eta. \tag{2.45}$$

Observe that $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in \mathrm{Lip}(S_{12}^\varepsilon; \mathbb{R}^2)$, $u^\varepsilon = \alpha_1$ on $\{(x,y) \in S_{12}^\varepsilon : d(x,y) = d_1\}$, $u^\varepsilon = \alpha_2$ on $\{(x,y) \in S_{12}^\varepsilon : d(x,y) = d_2\}$, and by construction there exists $w_a \in h\left([d_1, d_2] \times \{\delta_\varepsilon\}\right)$ such that $u^\varepsilon(w_a) = 0_T$. Write for simplicity $\widetilde{m} = m_{12}^\sigma$. We have

$$\nabla u_1^\varepsilon = \left(\frac{\ell_{12}\xi^1}{\varepsilon_{12}}d_x + \frac{\ell_{12}\eta^1}{\varepsilon_{12}}\widetilde{m}_s d_x + \overline{\kappa}_\varepsilon \widetilde{m}_t \tau_x \eta^1 \;,\; \frac{\ell_{12}\xi^1}{\varepsilon_{12}}d_y + \frac{\ell_{12}\eta^1}{\varepsilon_{12}}\widetilde{m}_s d_y + \overline{\kappa}_\varepsilon \widetilde{m}_t \tau_y \eta^1\right),$$

$$\nabla u_2^\varepsilon = \left(\frac{\ell_{12}\xi^2}{\varepsilon_{12}}d_x + \frac{\ell_{12}\eta^2}{\varepsilon_{12}}\widetilde{m}_s d_x + \overline{\kappa}_\varepsilon \widetilde{m}_t \tau_x \eta^2 \;,\; \frac{\ell_{12}\xi^2}{\varepsilon_{12}}d_y + \frac{\ell_{12}\eta^2}{\varepsilon_{12}}\widetilde{m}_s d_y + \overline{\kappa}_\varepsilon \widetilde{m}_t \tau_y \eta^2\right),$$

where $\widetilde{m}_s, \widetilde{m}_t$ denote the partial derivatives of $\widetilde{m}$ with respect to $s = \frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}$ and $t = \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)$ respectively, and are evaluated at $\left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)\right)$. Hence

$$|\nabla u_1^\varepsilon|^2+|\nabla u_2^\varepsilon|^2 \;=\; \frac{\ell_{12}^2}{\varepsilon_{12}^2}|\nabla d|^2+\frac{\ell_{12}^2}{\varepsilon_{12}^2}|\nabla d|^2(\widetilde{m}_s)^2+\overline{\kappa}_\varepsilon^2|\nabla\tau|^2(\widetilde{m}_t)^2+\frac{2\ell_{12}}{\varepsilon_{12}}\overline{\kappa}_\varepsilon\left(\nabla d\cdot\nabla\tau\right)\widetilde{m}_s\widetilde{m}_t$$

where we have used $|\xi| = |\eta| = 1$ and $\xi_1\eta_1 + \xi_2\eta_2 = 0$. From (2.40) we have

$$|\nabla u_1^\varepsilon|^2+|\nabla u_2^\varepsilon|^2 \;=\; \frac{\ell_{12}^2}{\varepsilon_{12}^2}+\frac{\ell_{12}^2}{\varepsilon_{12}^2}(\widetilde{m}_s)^2+\overline{\kappa}_\varepsilon^2|\nabla\tau|^2(\widetilde{m}_t)^2. \tag{2.46}$$

Moreover

$$\left(\frac{\partial u_1^\varepsilon}{\partial x}\frac{\partial u_2^\varepsilon}{\partial y}-\frac{\partial u_1^\varepsilon}{\partial y}\frac{\partial u_2^\varepsilon}{\partial x}\right)^2 \;=\; \frac{\ell_{12}^2}{\varepsilon_{12}^2}\overline{\kappa}_\varepsilon^2(\widetilde{m}_t)^2\left(d_x\tau_y-d_y\tau_x\right)^2\left(\xi^1\eta^2-\xi^2\eta^1\right)^2=\frac{\ell_{12}^2}{\varepsilon_{12}^2}\overline{\kappa}_\varepsilon^2(\widetilde{m}_t)^2|\nabla\tau|^2, \tag{2.47}$$

where again $\widetilde{m}_s, \widetilde{m}_t$ are evaluated at $\left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)\right)$, and we have used (2.40), (2.41), and $\xi^1\eta^2-\xi^2\eta^1 = 1$. Therefore from (2.46) and (2.47) we obtain

$$\begin{aligned}&1+|\nabla u_1^\varepsilon|^2+|\nabla u_2^\varepsilon|^2+\left(\frac{\partial u_1^\varepsilon}{\partial x}\frac{\partial u_2^\varepsilon}{\partial y}-\frac{\partial u_1^\varepsilon}{\partial y}\frac{\partial u_2^\varepsilon}{\partial x}\right)^2\\ =&1+\frac{\ell_{12}^2}{\varepsilon_{12}^2}\left(1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2\overline{\kappa}_\varepsilon^2\left(1+\frac{\varepsilon_{12}^2}{\ell_{12}^2}\right)|\nabla\tau|^2\right).\end{aligned}$$

As a consequence

$$\begin{aligned}\mathbb{A}(u^\varepsilon,S_{12}^\varepsilon)=&\ \frac{\ell_{12}}{\varepsilon_{12}}\int_{S_{12}^\varepsilon}\sqrt{1+(\widetilde{m}_s)^2+(\widetilde{m}_t)^2\,\overline{\kappa}_\varepsilon^2\left(1+\frac{\varepsilon_{12}^2}{\ell_{12}^2}\right)|\nabla\tau|^2+O(\varepsilon^2)}\;dxdy\\ =&\frac{1}{\overline{\kappa}_\varepsilon}\int_{\mathrm{R}_{12}\setminus P_\varepsilon}\frac{1}{|\nabla\tau|}\sqrt{1+\left(\widetilde{m}_s\left(s,t\right)\right)^2+\left(\widetilde{m}_t\left(s,t\right)\right)^2\overline{\kappa}_\varepsilon^2\left(1+\frac{\varepsilon_{12}^2}{\ell_{12}^2}\right)|\nabla\tau|^2+O(\varepsilon^2)}\;dsdt,\end{aligned} \tag{2.48}$$

where $\widetilde{m}_s, \widetilde{m}_t$ in the first integral are evaluated at $\left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)\right)$, $\nabla\tau$ in the second integral is evaluated at $(x,y) = \Phi^{-1}(s,t)$ and the last equality follows from the change of variables

$$\Phi:(x,y)\in\mathcal{N}_{12}^\varepsilon\to\left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)\right)=(s,t)\in\mathrm{R}_{12}, \tag{2.49}$$

and $P_\varepsilon := \mathrm{R}_{12}\setminus\Phi(S_{12}^\varepsilon)$ (see Figure 2.3). Here one checks that $\Phi = H\circ h^{-1}$ with $H(d,\tau) = (\frac{d-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau-\delta_\varepsilon))$ so that, using (2.41), the Jacobian of the change of variable is $\frac{1}{|\nabla\tau(\Phi^{-1}(s,t))|}\frac{\varepsilon_{12}}{\ell_{12}\overline{\kappa}_\varepsilon}$. Hence, recalling (2.43) and that $\overline{\kappa}_\varepsilon\to 1$ as $\varepsilon\to 0^+$,

$$\lim_{\varepsilon\to 0^+}\mathbb{A}(u^\varepsilon,S_{12}^\varepsilon)=\int_{\mathrm{R}_{12}}\sqrt{1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2}\;dsdt. \tag{2.50}$$

Now, let us recall that $\tilde{m} = m_{12}^{\sigma}$ is the approximating function as in (2.12); it follows that

$$\int_{\mathrm{R}_{12}} \sqrt{1 + \left(\widetilde{m}_s\right)^2 + \left(\widetilde{m}_t\right)^2}\, dsdt = \mathfrak{A}_{12}(\Gamma) + O(\sigma). \tag{2.51}$$

Hence, employing the same construction in the strips S_{23}^{ε} and S_{31}^{ε}, and using (2.51) we obtain from a diagonal argument with $\sigma = \sigma_{\varepsilon} \to 0$ as $\varepsilon \to 0^+$,

$$\lim_{\varepsilon \to 0^+} \mathbb{A}(u^{\varepsilon}, S_{23}^{\varepsilon} \cup S_{31}^{\varepsilon} \cup S_{23}^{\varepsilon}) = \mathfrak{A}_{12}(\Gamma) + \mathfrak{A}_{23}(\Gamma) + \mathfrak{A}_{31}(\Gamma). \tag{2.52}$$

Step 2. Definition of u^{ε} on T^{ε}. This is identical to Step 3 of the proof of Proposition 2.8 and therefore $\{u^{\varepsilon}\} \subset \mathrm{Lip}(B_r; \mathbb{R}^2)$ and (2.34) holds. Following the same computations of Proposition 2.8 the conclusion follows.

Step 3. For the case where two of the curves $C_{ij}, ij \in \{12, 23, 31\}$ meet at Q with an angle larger than or equal to π we replace T^{ε} with H^{ε} defined in case 2 of Proposition 2.8, in the above construction.

□

Remark 2.9. It is not difficult to see, by truncating the area-minimizing surfaces graphs of m_{ij} with the lateral boundary of the prisms $[0, \ell_{ij}] \times \mathrm{T}$, that the infimum in (2.1) is the same as the infimum obtained without requiring in Definition 2.2 that $\Gamma_i \subset \mathrm{T}$, $i \in \{1, 2, 3\}$.

3. Existence of minimizers for the functional $\mathcal{G}$

This chapter is dedicated to the second part of the proof of Theorem 0.1. Let D be an open disk centered at the origin such that $E_1,\ E_2,\ E_3$ are circular sectors with $120°$ angles and let T be an equilateral triangle. Let p be the barycenter of T and $\widetilde{\Gamma}_i$ be the segment connecting α_i and p, $i \in \{1,2,3\}$. Hence $\widetilde{\Gamma} = (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\Gamma}_3) \in X_{\rm Lip}$ so that

$$\inf\left\{\mathcal{G}(\Gamma) : \Gamma \in X_{\rm Lip}\right\} \le \mathcal{G}(\widetilde{\Gamma}).$$

Moreover we have

$$|D| + \mathcal{G}(\widetilde{\Gamma}) = \mathcal{A}(u, D) \le |D| + \inf\left\{\mathcal{G}(\Gamma) : \Gamma \in X_{\rm Lip}\right\},$$

where $u = u_{\rm symm}$ (defined in the introduction), and the equality follows from [36, Section 3] and the inequality follows from Proposition 2.8. Thus

$$\mathcal{G}(\widetilde{\Gamma}) = \min\left\{\mathcal{G}(\Gamma) : \Gamma \in X_{\rm Lip}\right\}.$$

Hence in this symmetric situation the optimal connection is obtained through the Steiner graph connecting α_1, α_2 and α_3. This motivates the analysis of this section, which is carried on without symmetry assumptions.
We recall that given a connection $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \in X$ we denote by $\varphi_{ij} = \varphi_{ij}(\Gamma_{ij}) : [0, \ell_{ij}] \to [0, \mathrm{diamT}]$ the function whose graph is $\Gamma_{ij} = \Gamma_i \cup \Gamma_j$ (see (2.3)).

Definition 3.1 (**Convergence in** X)**.** *We say that a sequence* $\{\Gamma^n\} \subset X$ *converges to* $\Gamma \in X$ *in* X*, and we write* $\Gamma^n \to \Gamma$ *in* X*, if*

$$\varphi_{ij}(\Gamma^n_{ij}) \to \varphi_{ij}(\Gamma_{ij}) \quad \text{in } L^1([0, \ell_{ij}]), \qquad ij \in \{12, 23, 31\}. \tag{3.1}$$

3.1 Density and approximation

We start to show that a BV connection $\Gamma \in X$ can be approximated by Lipschitz connections; the difficulty is to keep graphicality of each branch of the approximating connections with respect to the two corresponding edges of T at the same time.

Recall that Γ_i is the branch of the connection Γ connecting α_i to p and that by Definition 2.2 we have

$$\Gamma_i \llcorner \mathrm{T} \setminus \overline{p\pi_{ij}(p)} \cup \overline{p\pi_{ki}(p)} = \Gamma_{\varphi_{ij}\llcorner[0,w_{ij})} = \Gamma_{\varphi_{ki}\llcorner(w_{ki},\ell_{ki}]}.$$

Note that we excluded the vertical parts over the points $\pi_{ij}(p)$, $ij \in \{12, 23, 31\}$, due to Remark 2.3; however we still have

$$\Gamma_i \cup \Gamma_j = \Gamma_{\varphi_{ij}}.$$

Lemma 3.2 (**Piecewise linear approximation**). *For any $\Gamma \in X$ with target triple point $p \in \mathrm{T}$ there exists a sequence $\{\Gamma^n\} \subset X_{\mathrm{Lip}}$ of connections with target triple point p such that $\varphi_{ij}(\Gamma^n_{ij}), ij \in \{12, 23, 31\}$, is a piecewise linear*[(1)] *function,*

$$\mathcal{H}^1(\Gamma^n_{ij}) \le \mathcal{H}^1(\Gamma_{ij})$$

and

$$\Gamma^n \to \Gamma \qquad \text{in } X. \tag{3.2}$$

Proof. Let $ij = 12$ and let w_{12} be defined as in (2.2). Let $n_{12} := (0, 1) \in \mathbb{R}^2$ be the inward unit normal to $\overline{\alpha_1\alpha_2}$, $n_{31} := (\alpha, \beta)$ be the inward unit normal to $\overline{\alpha_3\alpha_1}$, and $\nu(\bar{s}) := (\nu^1(\bar{s}), \nu^2(\bar{s}))$ be the generalized outward unit normal at the point $(\bar{s}, \varphi_{12}(\bar{s}))$ to the generalized graph $\Gamma_{\varphi_{12}\llcorner[0,w_{12}]}$ of $\varphi_{12}\llcorner[0, w_{12}]$ (for all $\bar{s}$ where it exists), see Figure 3.1. Without loss of generality we may assume $\Gamma_1 = \Gamma_{\varphi_{12}\llcorner[0,w_{12}]}$. We start to show that φ_{12} cannot have too negative slope, otherwise Γ_1 loses graphicality with respect to $\overline{\alpha_3\alpha_1}$.

Step 1. We claim that

$$\varphi'_{12}\llcorner[0, w_{12}] \ge \frac{\beta}{\alpha}$$

in the sense of measures, *i.e.*,

$$\varphi'_{12}(B) \ge \frac{\beta}{\alpha}\mathcal{L}^1(B), \qquad \forall B \subseteq [0, w_{12}] \text{ Borel set.} \tag{3.3}$$

From the graphicality with respect to $\overline{\pi_{31}(p)\alpha_1}$ we have, for all $\bar{s}$ where $\nu(\bar{s})$ exists,

$$\nu(\bar{s}) \cdot n_{31} \le 0. \tag{3.4}$$

Set

$$I^r := \{\bar{s} \in [0, w_{12}] : \nu(\bar{s}) \text{ is defined, and } \nu^2(\bar{s}) > 0\},$$
$$I^s := \{\bar{s} \in [0, w_{12}] : \nu(\bar{s}) \text{ is defined, and } \nu^2(\bar{s}) = 0\};$$

note that $\nu(\bar{s}) = \frac{1}{\sqrt{1+(\dot{\varphi}_{12}(\bar{s}))^2}}(-\dot{\varphi}_{12}(\bar{s}), 1)$ for any $\bar{s} \in I^r$. From Theorem 1.11, we have

$$\dot{\varphi}_{12}d\bar{s} = \varphi'_{12}\llcorner I^r, \qquad \dot{\varphi}^{(j)}_{12} + \dot{\varphi}^{(c)}_{12} = \varphi'_{12}\llcorner I^s. \tag{3.5}$$

[(1)]This means that it is Lipschitz piecewise linear with at most finitely many points of nondifferentiability.

From (3.4) it follows that

$$\nu(\bar{s}) = (-1, 0) \qquad \forall \bar{s} \in I^s \qquad \text{and} \qquad \dot{\varphi}_{12}(\bar{s}) \geq \frac{\beta}{\alpha} \qquad \forall \bar{s} \in I^r. \tag{3.6}$$

From Theorem 1.10, we have

$$\dot{\varphi}_{12}^{(j)} + \dot{\varphi}_{12}^{(c)} = -\nu^1 |\mu| \llcorner I^s = |\mu| \llcorner I^s,$$

where $\mu := (\varphi_{12}', -\mathcal{L}^1) = (-\nu^1, -\nu^2)|\mu|$ and the second equality follows from the first formula in (3.6).
For any Borel set $B \subseteq [0, w_{12}]$ we deduce

$$\varphi_{12}'(B) = \int_B \dot{\varphi}_{12} d\bar{s} + \dot{\varphi}_{12}^{(j)}(B) + \dot{\varphi}_{12}^{(c)}(B) \geq \frac{\beta}{\alpha}\, \mathcal{L}^1(B) + |\mu| \llcorner I^s(B) \geq \frac{\beta}{\alpha}\, \mathcal{L}^1(B).$$

Step 2. Given $\epsilon \in (0, 1)$, we choose $n = n(\epsilon) \in \mathbb{N}$ and points

$$\xi_0 = 0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = w_{12},$$

such that each $\xi_i,\ i \in \{1, \cdots, n-1\}$, is a point of continuity of φ_{12}, and if we define $\varphi^n \in \mathrm{Lip}([0, w_{12}])$ as the piecewise linear interpolation with

$$\varphi^n(\xi_i) = \varphi_{12}(\xi_i), \qquad i = 0, \cdots, n;$$

then, by taking n large enough,

$$\|\varphi^n - \varphi_{12}\|_{L^1((0,w_{12}))} < \epsilon.$$

The graph of φ^n may still have vertical parts over $\overline{\pi_{31}(p)\alpha_1}$. Indeed from Theorem 1.8, and the fact that ξ_i are continuity points of φ_{12} we have

$$\varphi^{n\prime}(\xi) = \frac{\varphi_{12}(\xi_i) - \varphi_{12}(\xi_{i-1})}{\xi_i - \xi_{i-1}} = \frac{\varphi_{12}'((\xi_{i-1}, \xi_i))}{\xi_i - \xi_{i-1}} \geq \frac{\beta}{\alpha}, \qquad \xi \in (\xi_{i-1}, \xi_i), \tag{3.7}$$

and equality may hold, hence the graph of φ^n over $\overline{\pi_{31}(p)\alpha_1}$ may have finitely many vertical parts. It is now sufficient to repeat the argument with φ^n in place of φ_{12}, choosing a suitable partition of $[w_{31}, \ell_{31}]$, so to ensure that (recall that $\varphi^{n\prime}$ exists out of finitely many points)

$$\varphi^{n\prime} > \frac{\beta}{\alpha} \qquad \text{at the differentiability points of } \ [0, w_{12}].$$

In this way φ^n is a Lipschitz graph also with respect to $\overline{\pi_{31}(p)\alpha_1}$.
Step 3. We have

$$\begin{aligned}
&\mathcal{H}^1(\Gamma_{\varphi^n}) = \sum_{i=1}^n \int_{\xi_{i-1}}^{\xi_i} \sqrt{1 + |\varphi^{n\prime}(s)|^2} ds = \sum_{i=1}^n |(\xi_i, \varphi^n(\xi_i)) - (\xi_{i-1}, \varphi^n(\xi_{i-1}))| \\
&\leq \sup \left\{ \sum_{i=1}^m |(\eta_i, \varphi_{12}(\eta_i)) - (\eta_{i-1}, \varphi_{12}(\eta_{i-1}))| : m \in \mathbb{N}, \eta_0 = 0 < \eta_1 < \cdots < \eta_{m-1} < \eta_m = w_{12} \right\} \\
&= \int_{[0,w_{12}]} |\Phi_{12}'| = \mathcal{H}^1(\Gamma_{\varphi_{12} \llcorner [0,w_{12}]}),
\end{aligned} \tag{3.8}$$

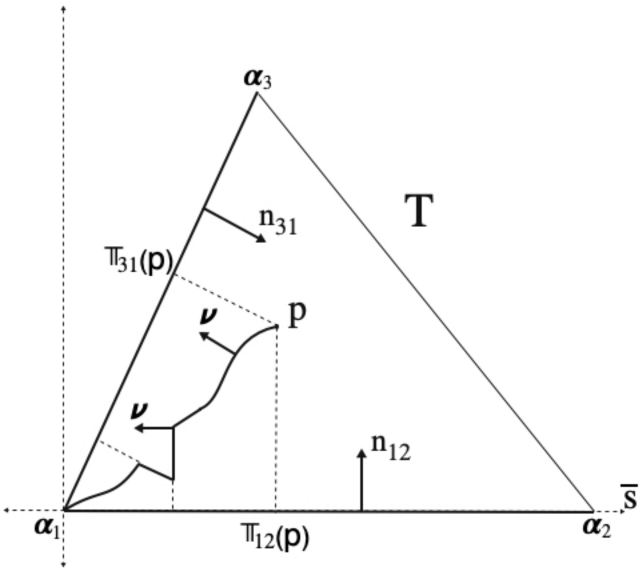


Figure 3.1: Proof of Lemma 3.2. For convenience we choose α_1 to be the origin.

where $\Phi_{12} \in \mathrm{BV}([0, w_{12}]; \mathbb{R}^2)$ is defined as $\Phi_{12}(\xi) := (\xi, \varphi_{12}(\xi))$, and the last equality follows from (1.15), Theorem 1.7, and the fact that φ_{12} is a good representative.
Step 4. Define

$$\Gamma_1^n := \Gamma_{\varphi^n}.$$

Similarly we define Γ_2^n and Γ_3^n, and we set $\Gamma_{ij}^n := \Gamma_i^n \cup \Gamma_j^n$. Then $\Gamma^n := (\Gamma_1^n, \Gamma_2^n, \Gamma_3^n)$ satisfies the required properties. □

Proposition 3.3 (Uniform estimate of the length). *There exists $c > 0$ depends on* T *such that for all $\Gamma \in X$ we have*

$$\mathcal{H}^1(\Gamma_{ij}) \leq c, \qquad ij \in \{12, 23, 31\}. \tag{3.9}$$

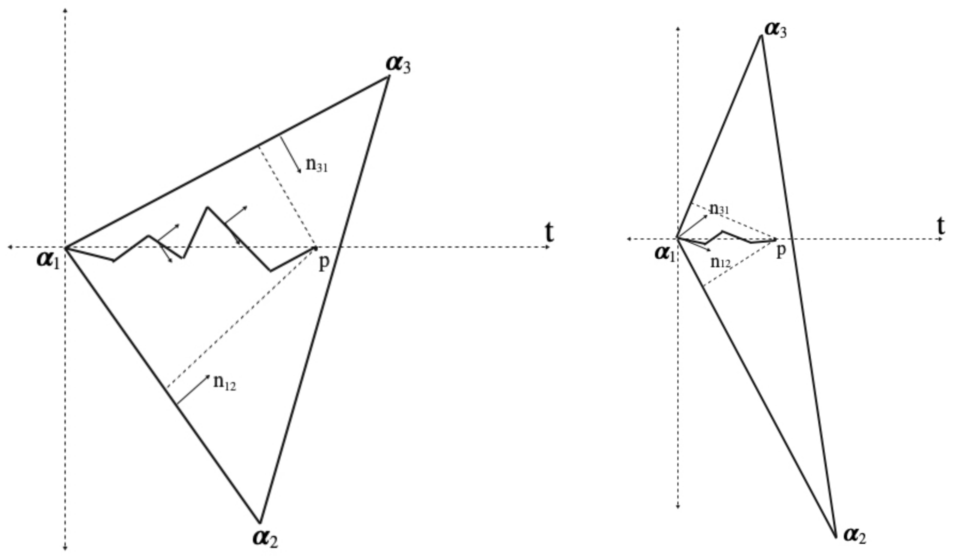
Proof. Let $\Gamma \in X$ be a connection through $p \in$ T. Without loss of generality we may assume that $p \neq \alpha_1$. From (3.8) we have

$$\mathcal{H}^1(\Gamma_1) = \sup \left\{ \sum_{i=1}^{m} |(\eta_i, \varphi_{12}(\eta_i)) - (\eta_{i-1}, \varphi_{12}(\eta_{i-1}))| : m \in \mathbb{N}, \eta_0 = 0 < \eta_1 < \cdots < \eta_{m-1} < \eta_m = w_{12} \right\}. \tag{3.10}$$

Choose a partition

$$\xi_0 = 0 < \xi_1 < \cdots < \xi_{h-1} < \xi_h = w_{12}.$$

Let Γ_1^h be the piecewise linear interpolation connecting $(\xi_{i-1}, \varphi_{12}(\xi_{i-1}))$ and $(\xi_i, \varphi_{12}(\xi_i))$, $i \in \{1, \cdots, h\}$. The unit tangent to Γ_1^h is enclosed in the angle formed by n_{12} and n_{31}, the unit normals to $\overline{\alpha_1\alpha_2}$ and $\overline{\alpha_3\alpha_1}$ (due to the graphicality condition with respect to $\overline{\alpha_1\alpha_2}$ and $\overline{\alpha_3\alpha_1}$), see Figure 3.2. It follows that Γ_1^h is the graph of a function



(a) T with angles less than or equal to $\frac{\pi}{2}$.

(b) T with an angle greater than $\frac{\pi}{2}$.

Figure 3.2

ϕ_{12}^h over the segment $\overline{\alpha_1 p}$. Fix a Cartesian coordinate system in which the t-axis is the line $\overline{\alpha_1 p}$ and the origin is α_1. For any $t \in [0, |\alpha_1 - p|]$ (up to a finite set) let $\tau(t)$ be the unit tangent to Γ_1^h at $(t, \phi_{12}^h(t))$ and let $n = (1,0)$ and $n^\perp = (0,1)$. Hence ${\phi_{12}^h}' = \frac{\tau \cdot n^\perp}{\tau \cdot n}$ satisfies

$$c_1^- := \frac{n_{31} \cdot n^\perp}{n_{31} \cdot n} \leq {\phi_{12}^h}' \leq \frac{n_{12} \cdot n^\perp}{n_{12} \cdot n} =: c_1^+.$$

Note that one between $|c_1^-|$ and $|c_1^+|$ might be $+\infty$, since one of the sides $\overline{\alpha_1 \alpha_2}$ or $\overline{\alpha_3 \alpha_1}$ can be horizontal (this happens only if the point p is on one side of the triangle). However we always have $c_1^- \leq 0$, $c_1^+ \geq 0$. Furthermore, when the angle $\hat{\alpha}_1$ in α_1 is less than or equal to $\frac{\pi}{2}$, it follows that $\tilde{c}_1 := \min\{|c_1^-|, |c_1^+|\} \leq |\tan(\frac{\pi}{2} - \frac{\hat{\alpha}_1}{2})|$. In the case $\hat{\alpha}_1 > \frac{\pi}{2}$, using that $p \in T_{int}$, we have $\max\{|c_1^-|, |c_1^+|\} \leq |\tan(\pi - \hat{\alpha}_1)|$. Thus the only difficulty to prove is that the length of Γ_1^h is controlled when $\hat{\alpha}_1 \leq \frac{\pi}{2}$. So let us assume this and in addition that $|c_1^-| = \tilde{c}_1$ (the other case is similar). Since $\phi_{12}^h(|\alpha_1 - p|) = \phi_{12}^h(0) = 0$ we have

$$0 = {\phi_{12}^h}'([0, |\alpha_1 - p|]) = ({\phi_{12}^h}')^+([0, |\alpha_1 - p|]) - ({\phi_{12}^h}')^-([0, |\alpha_1 - p|]),$$

where $({\phi_{12}^h}')^+$ and $({\phi_{12}^h}')^-$ are the positive and negative parts of the measure ${\phi_{12}^h}' = \dot{\phi}_{12}^h dt$. Thus we estimate

$$\begin{aligned} |\mathcal{H}^1(\Gamma_1^h)| &= \int_0^{|\alpha_1 - p|} \sqrt{1 + \dot{\phi}_{12}^h(t)^2} dt \\ &\leq |\alpha_1 - p| + |{\phi_{12}^h}'|([0, |\alpha_1 - p|]) = |\alpha_1 - p| + 2({\phi_{12}^h}')^-([0, |\alpha_1 - p|]) \\ &\leq |\alpha_1 - p| + 2|\alpha_1 - p|\tilde{c}_1 =: c_1. \end{aligned} \tag{3.11}$$

Then c_1 is a positive constant depending only on the geometry of T.
From (3.11) and (3.10) it then follows

$$\mathcal{H}^1(\Gamma_1) \le c_1. \tag{3.12}$$

Similarly we may show that $\mathcal{H}^1(\Gamma_2) \le c_2$ and $\mathcal{H}^1(\Gamma_3) \le c_3$ for $c_2, c_3 > 0$ depending only on T. This proves (3.9) with $c = c_1 + c_2 + c_3$. □

The next lemma shows continuity of the sum of the three areas of area minimizing surfaces defining $\mathcal{G}$ in (2.10), with respect to the L^1 convergence of the traces in T.

Proposition 3.4 (Continuity of $\mathcal{G}$). *Let $\Gamma \in X$, and let $\{\Gamma^n\} \subset X$ be a sequence converging to Γ in X. Then*

$$\lim_{n\to+\infty} \mathcal{G}(\Gamma^n) = \mathcal{G}(\Gamma). \tag{3.13}$$

Proof. Since $\Gamma \in X$ and $\{\Gamma^n\} \subset X$ we have $\varphi_{ij} \in \mathrm{BV}([0,\ell_{ij}])$ and $\{\varphi^n_{ij}\} \subset \mathrm{BV}([0,\ell_{ij}])$, where $\varphi_{ij} := \varphi_{ij}(\Gamma_{ij})$, $\varphi^n_{ij} := \varphi_{ij}(\Gamma^n_{ij})$.
Hence from (0.6) and Section 2.2 it follows that there exist $\widehat{m}_{ij}$, $\widehat{m}^n_{ij} \in W^{1,1}(\widehat{\mathrm{R}}_{ij})$ such that

$$\begin{aligned}2\mathfrak{A}_{ij}(\Gamma^n_{ij}) &= \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1+|\nabla \widehat{m}^n_{ij}|^2}\, dsdt \\ &= \min\Big\{ \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1+|Df|^2} + \int_{\partial\widehat{\mathrm{R}}_{ij}} |f-\varphi^n_{ij}|d\mathcal{H}^1 : f\in \mathrm{BV}(B),\ f=\varphi^n_{ij} \text{ on } B\setminus \widehat{\mathrm{R}}_{ij}\Big\},\end{aligned} \tag{3.14}$$

$$\begin{aligned}2\mathfrak{A}_{ij}(\Gamma_{ij}) &= \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1+|\nabla \widehat{m}_{ij}|^2}\, dsdt \\ &= \min\Big\{ \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1+|Df|^2} + \int_{\partial\widehat{\mathrm{R}}_{ij}} |f-\varphi_{ij}|d\mathcal{H}^1 : f\in \mathrm{BV}(B),\ f=\varphi_{ij} \text{ on } B\setminus \widehat{\mathrm{R}}_{ij}\Big\},\end{aligned} \tag{3.15}$$

where we recall that $\widehat{\mathrm{R}}_{ij}$ is the double rectangle defined in (2.6) and φ_{ij}, φ^n_{ij} are extended on a disk B containing $\widehat{\mathrm{R}}_{ij}$ as in Section 2.2.
Define $\widetilde{m}^n_{ij}$ and $\widetilde{m}_{ij}$ as

$$\widetilde{m}^n_{ij} := \begin{cases}\widehat{m}^n_{ij} \text{ in } \widehat{\mathrm{R}}_{ij},\\ \varphi_{ij} \text{ in } B\setminus\widehat{\mathrm{R}}_{ij},\end{cases} \qquad \widetilde{m}_{ij} := \begin{cases}\widehat{m}_{ij} \text{ in } \widehat{\mathrm{R}}_{ij},\\ \varphi^n_{ij} \text{ in } B\setminus\widehat{\mathrm{R}}_{ij},\end{cases}$$

so that $\widetilde{m}^n_{ij}$, $\widetilde{m}_{ij} \in \mathrm{BV}(B)$. Since $\widetilde{m}^n_{ij}$ is competitor in (3.15) and $\widetilde{m}_{ij}$ is competitor in (3.14) we have, recalling also the discussion leading to (2.9),

$$2\mathfrak{A}_{ij}(\Gamma_{ij}) \le \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1+|\nabla \widehat{m}^n_{ij}|^2}\, dsdt + \int_{\partial\widehat{\mathrm{R}}_{ij}} |\varphi^n_{ij}-\varphi_{ij}|d\mathcal{H}^1 = 2\mathfrak{A}^n_{ij} + \int_{\partial\widehat{\mathrm{R}}_{ij}} |\varphi^n_{ij}-\varphi_{ij}|d\mathcal{H}^1,$$

$$2\mathfrak{A}_{ij}(\Gamma^n_{ij}) \le \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{1+|\nabla \widehat{m}_{ij}|^2}\, dsdt + \int_{\partial\widehat{\mathrm{R}}_{ij}} |\varphi_{ij}-\varphi^n_{ij}|d\mathcal{H}^1 = 2\mathfrak{A}_{ij} + \int_{\partial\widehat{\mathrm{R}}_{ij}} |\varphi_{ij}-\varphi^n_{ij}|d\mathcal{H}^1.$$

Thus

$$|2\mathfrak{A}_{ij}(\Gamma^n_{ij}) - 2\mathfrak{A}_{ij}(\Gamma_{ij})| \leq \int_{\partial \widehat{\mathrm{R}}} |\varphi^n_{ij} - \varphi_{ij}| d\mathcal{H}^1. \tag{3.16}$$

Recall that m^n_{ij} (resp. m_{ij}) is the restriction of $\widehat{m}^n_{ij}$ (resp. $\widehat{m}_{ij}$) to R_{ij}. Hence, from (2.10), (3.1) and (3.16), formula (3.13) follows. □

Corollary 3.5. *We have*

$$\inf \{\mathcal{G}(\Gamma) : \Gamma \in X\} = \inf \{\mathcal{G}(\Gamma) : \Gamma \in X_{\mathrm{Lip}}\}. \tag{3.17}$$

3.2 Compactness of the class X

The aim of this section is to show that the infimum in (3.17) is attained. To do this we need the following result.

Theorem 3.6 (Compactness). *Any sequence $\{\Gamma^n\} \subset X$ admits a subsequence converging in X to some $\Gamma \in X$.*

Remark 3.7. In Definition 3.1 it is required convergence of $\{\Gamma^n\}$ to Γ in L^1. For this reason, if Γ^n has target triple point p_n, it is not guaranteed that the point $b := \lim_{n\to+\infty} p_n$ (it exists up to subsequences) still belongs to Γ_{ij} for all ij, see Figures 3.4 and 3.7a. As a consequence, if $\{\Gamma^n\}$ converges to Γ it is not true, in general, that $p_n \to p$, where p is the target triple point of Γ.

Let $\varphi \in \mathrm{BV}(I)$(2), $I = (a,b)$, and let $\mathcal{T}$ be the current defined in Theorem 1.24. Then $\mathcal{T}$ is boundaryless in $I \times \mathbb{R}$, namely $\partial\mathcal{T} = 0$. Furthermore, if Γ_φ is the generalized graph of φ as defined in (1.4), it turns out that

$$\partial^- S\mathcal{G}_{\varphi,I} \cap (I \times \mathbb{R}) \subseteq \Gamma_\varphi, \qquad \mathcal{H}^1(\Gamma_\varphi \setminus \partial^- S\mathcal{G}_{\varphi,I}) = 0,$$

where $\partial^- S\mathcal{G}_{\varphi,I}$ is the reduced boundary of the subgraph $S\mathcal{G}_{\varphi,I}$ of φ. It easily follows that the current $\mathcal{T}$ coincides with the integration over the rectifiable set Γ_φ (with the correct orientation). From now on, when the interval is clear from the context, we will simply denote $S\mathcal{G}_{\varphi,I}$ by $S\mathcal{G}_\varphi$.

Proof of Theorem 3.6. Let $\{\Gamma^n\} \subset X$ and $\varphi^n_{ij} = \varphi_{ij}(\Gamma^n_{ij})$, $ij \in \{12, 23, 31\}$. From Proposition 3.3 it follows that $\{\varphi^n_{ij}\}$ is uniformly bounded in $\mathrm{BV}([0,\ell_{ij}])$ for any $ij \in \{12, 23, 31\}$. Thus, up to a not relabelled subsequence, there exists $\varphi_{ij} \in \mathrm{BV}([0,\ell_{ij}])$ such that

$$\varphi^n_{ij} \to \varphi_{ij} \qquad \text{in } L^1((0,\ell_{ij})) \text{ and pointwise a.e.,} \tag{3.18}$$

$$(\varphi^n_{ij})' \rightharpoonup \varphi'_{ij} \qquad \text{weakly}^* \text{ as measures,} \tag{3.19}$$

see Theorem 1.3. We shall adopt our usual convention

$$\varphi^n_{ij}(0_-) = \varphi_{ij}(0_-) = \varphi^n_{ij}(\ell_{ij+}) = \varphi_{ij}(\ell_{ij+}) = 0, \ \ \varphi^n_{ij} = \varphi^n_{ij+}, \ \ \varphi_{ij} = \varphi_{ij+}, \ \ \varphi = \varphi_+. \tag{3.20}$$

(2)Recall that we always assume $\varphi = \varphi_+$, the good representative defined in (1.5).

Denote by $\Gamma_{ij} \subset \mathbb{R}^2$ the limit graph over (the closed segment) $\overline{\alpha_i\alpha_j}$ that we identify with the generalized graph of φ_{ij} over $[0, \ell_{ij}]$. Since T is closed and convex, from (3.18), we have $\Gamma_{ij} \subset T$; moreover, by construction, α_i and α_j are the endpoints of Γ_{ij}. Notice that if we assume that T is acute, this excludes the presence of vertical parts over its vertices.
It remains to prove that the three obtained curves Γ_{ij}, $ij \in \{12, 23, 31\}$, form a BV connection; in particular that they intersect mutually in a unique well-defined point.
We claim that

> there exists a unique $p \in \bigcap_{ij} \Gamma_{ij}$ that divides each Γ_{ij} into two curves Γ^l_{ij} and Γ^r_{ij} such that
>
> $$\Gamma^l_{ij} = \Gamma^r_{ki}, \qquad ij, ki \in \{12, 23, 31\},\ ij \neq ki.$$

Let us denote by $\tilde{\varphi}^n_{ij}$ the extension to $\mathbb{R}$ of the function φ^n_{ij} vanishing in $(-\infty, 0) \cup (\ell_{ij}, +\infty)$. Similarly $\tilde{\varphi}_{ij}$ is the extension of φ_{ij} vanishing in $(-\infty, 0) \cup (\ell_{ij}, +\infty)$. Consider the sequence $\{[\![S\mathcal{G}_{\tilde{\varphi}^n_{ij}}]\!]\}_n \subset \mathcal{D}_2(\mathbb{R}^2)$ of 2-currents regarded in $\mathbb{R}^2$ and the 2-current $[\![S\mathcal{G}_{\tilde{\varphi}_{ij}}]\!]$. Their boundaries are the currents carried by the graphs of $\tilde{\varphi}^n_{ij}$ and $\tilde{\varphi}_{ij}$, respectively, as defined in Theorem 1.24. The 1-currents carried by the graph of φ^n_{ij} and φ_{ij}, by convention (3.20), coincide with the restrictions of $\partial[\![S\mathcal{G}_{\tilde{\varphi}^n_{ij}}]\!]$ and $\partial[\![SG_{\tilde{\varphi}_{ij}}]\!]$ to the closed set $[0, \ell_{ij}] \times \mathbb{R}$. Namely, if we denote by

$$[\![\Gamma^n_{ij}]\!] := \partial[\![S\mathcal{G}_{\tilde{\varphi}^n_{ij}}]\!] \llcorner [0, \ell_{ij}] \times \mathbb{R}, \qquad [\![\Gamma_{ij}]\!] := \partial[\![SG_{\tilde{\varphi}_{ij}}]\!] \llcorner [0, \ell_{ij}] \times \mathbb{R},$$

then

$$[\![\Gamma^n_{ij}]\!] = \partial[\![S\mathcal{G}_{\tilde{\varphi}^n_{ij}}]\!] - \mathcal{L}_{ij} \quad \text{and} \quad [\![\Gamma_{ij}]\!] = \partial[\![SG_{\tilde{\varphi}_{ij}}]\!] - \mathcal{L}_{ij}, \tag{3.21}$$

where $\mathcal{L}_{ij}$ is the 1-current given by integration over the two halflines $(-\infty, 0) \times \{0\} \cup (\ell_{ij}, +\infty) \times \{0\}$. The curves Γ^n_{ij} and Γ_{ij} coincide with the support of $[\![\Gamma^n_{ij}]\!]$ and $[\![\Gamma_{ij}]\!]$, respectively.
We now prove our claim in three steps.
Step 1. The currents $[\![\Gamma^n_{ij}]\!]$ converge (up to a not relabelled subsequence) weakly in the sense of currents to $[\![\Gamma_{ij}]\!]$, *i.e.*,

$$[\![\Gamma^n_{ij}]\!](\omega) \to [\![\Gamma_{ij}]\!](\omega) \qquad \forall \omega \in \mathcal{D}^1(\mathbb{R}^2). \tag{3.22}$$

Moreover

$$\mathcal{H}^1(\Gamma_{ij}) \leq c, \tag{3.23}$$

where $c > 0$ is the constant in (3.9).
There are two ways to prove step 1. The first, which is standard, more general, and shorter, is as follow:
The characteristic functions $\mathbb{1}_{S\mathcal{G}_{\varphi^n_{ij}}}$ converge to $\mathbb{1}_{S\mathcal{G}_{\varphi_{ij}}}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$, thanks to (3.18), hence

$$[\![S\mathcal{G}_{\tilde{\varphi}^n_{ij}}]\!] \rightharpoonup [\![S\mathcal{G}_{\tilde{\varphi}_{ij}}]\!] \text{ weakly as currents,}$$

since

$$\int_{S\mathcal{G}_{\tilde\varphi^n_{ij}}} \hat\omega(s,t)dsdt \to \int_{S\mathcal{G}_{\tilde\varphi_{ij}}} \hat\omega(s,t)dsdt \qquad \forall \hat\omega \in \mathcal{C}^\infty_c(\mathbb{R}^2).$$

This implies

$$\partial[\![S\mathcal{G}_{\tilde\varphi^n_{ij}}]\!] \rightharpoonup \partial[\![S\mathcal{G}_{\tilde\varphi_{ij}}]\!] \text{ weakly in the sense of currents,}$$

and (3.22) follows from (3.21). Which conclude the first proof.
The second proof of (3.22) goes as follow. For simplicity we assume T is an acute triangle, hence none of the currents $[\![\Gamma^n_{ij}]\!]$ and $[\![\Gamma_{ij}]\!]$ has vertical parts over the vertices.
Let

$$\omega(s,t) = \omega_1(s,t)ds + \omega_2(s,t)dt, \qquad \omega_1, \omega_2 \in \mathcal{C}^\infty_c\big((0,\ell_{ij}) \times \mathbb{R}\big).$$

We have

$$\begin{aligned} \lim_{n\to+\infty} [\![\Gamma^n_{ij}]\!]\big(\omega_1(s,t)ds\big) &= \lim_{n\to+\infty} \int_0^{\ell_{ij}} \omega_1(s, \varphi^n_{ij}(s))ds \\ &= \int_0^{\ell_{ij}} \omega_1(s,\varphi_{ij}(s))ds = [\![\Gamma_{ij}]\!]\big(\omega_1(s,t)ds\big), \end{aligned} \tag{3.24}$$

where the first and the last equalities follow from (1.23), and the second equality follows from Lebesgue dominated convergence theorem (since φ^n_{ij} are uniformly bounded, and $\omega_1 \in \mathcal{C}^\infty_c((0,\ell_{ij}) \times \mathbb{R})$). Let us show

$$\lim_{n\to+\infty} [\![\Gamma^n_{ij}]\!]\big(\omega_2(s,t)dt\big) = [\![\Gamma_{ij}]\!]\big(\omega_2(s,t)dt\big). \tag{3.25}$$

Assume first that ω_2 can be written as

$$\omega_2(s,t) = h(s)g(t), \qquad h \in \mathcal{C}^\infty_c((0,\ell_{ij})),\ g \in \mathcal{C}^\infty_c(\mathbb{R}). \tag{3.26}$$

Without loss of generality we may also assume that $g = G'$ for some $G \in \mathcal{C}^\infty(\mathbb{R})$. Thus we have

$$\begin{aligned} &[\![\Gamma_{ij}]\!]\big(\omega_2(s,t)dt\big) \\ &= \int_0^{\ell_{ij}} h(s)g(\varphi_{ij}(s))\dot{\varphi_{ij}}(s)ds + \int_0^{\ell_{ij}} h(s)g(\varphi_{ij}(s))\dot{\varphi_{ij}}^{(c)} + \sum_{s\in J_{\varphi_{ij}}} \nu(s, J_{\varphi_{ij}})h(s)\int_{\varphi_{ij-}(s)}^{\varphi_{ij+}(s)} g(t)dt \\ &= \int_0^{\ell_{ij}} h(s)G'(\varphi_{ij}(s))\dot{\varphi_{ij}}(s)ds + \int_0^{\ell_{ij}} h(s)G'(\varphi_{ij}(s))\dot{\varphi_{ij}}^{(c)} + \sum_{s\in J_{\varphi_{ij}}} \nu(s, J_{\varphi_{ij}})h(s)\int_{\varphi_{ij-}(s)}^{\varphi_{ij+}(s)} G'(t)dt \\ &= \int_0^{\ell_{ij}} h(s)G'(\varphi_{ij}(s))\dot{\varphi_{ij}}(s)ds + \int_0^{\ell_{ij}} h(s)G'(\varphi_{ij}(s))\dot{\varphi_{ij}}^{(c)} \\ &+ \sum_{s\in J_{\varphi_{ij}}} \nu(s, J_{\varphi_{ij}})h(s)[G(\varphi_{ij+}(s)) - G(\varphi_{ij-}(s))] \\ &= \int_0^{\ell_{ij}} h(s)(G\circ\varphi_{ij})', \end{aligned} \tag{3.27}$$

where the first equality follows from (1.23), (1.24), (1.25), and the last equality from Theorem 1.12. Similarly we have

$$[\![\Gamma_{ij}^n]\!]\big(\omega_2(s,t)dt\big) = \int_0^{\ell_{ij}} h(s)(G\circ\varphi_{ij}^n)'.$$

Hence

$$\begin{aligned}\lim_{n\to+\infty}[\![\Gamma_{ij}^n]\!]\big(\omega_2(s,t)dt\big) &= \lim_{n\to+\infty}\int_0^{\ell_{ij}} h(s)(G\circ\varphi_{ij}^n)' = -\lim_{n\to\infty}\int_0^{\ell_{ij}} h'(s)G(\varphi_{ij}^n(s))ds \\ &= -\int_0^{\ell_{ij}} h'(s)G(\varphi_{ij}(s))ds = \int_0^{\ell_{ij}} h(s)(G\circ\varphi_{ij})',\end{aligned} \tag{3.28}$$

where the second and last equalities follows from Theorem 1.12 and the third equality follows from Lebesgue dominated convergence theorem. Thus, from (3.28) and (3.27), we conclude that (3.25) holds for ω_2 that can be written as in (3.26). From linearity we also have that (3.25) holds for ω_2 of the form:

$$\omega_2(s,t) = \sum_{r=1}^{l} h_r(s)g_r(t), \qquad h_r\in\mathcal{C}_c^\infty((0,\ell_{ij})),\ g_r\in\mathcal{C}_c^\infty(\mathbb{R}), \tag{3.29}$$

where $l\in\mathbb{N}$. From one version of the Stone-Weiestrass Theorem [16, Corollary 8.3.] the space

$$\mathcal{A} = \left\{\sum_{r=1}^{l} h_r(s)g_r(t):\ l\in\mathbb{N},\ h_r\in\mathcal{C}_c^\infty((0,\ell_{ij})),\ g_r\in\mathcal{C}_c^\infty(\mathbb{R})\right\}$$

is dense[(3)] in the space $\mathcal{C}_0((0,\ell_{ij})\times\mathbb{R})$ of real-valued continuous functions on $(0,\ell_{ij})\times\mathbb{R}$ which vanish at infinity. In particular for any $\omega_2\in\mathcal{C}_c^\infty((0,\ell_{ij})\times\mathbb{R})$ there exists a sequence $\{\omega_2^m\}$ of functions of the form (3.29) that converges locally uniformly in $(0,\ell_{ij})\times\mathbb{R}$ to ω_2. We are now in a position to conclude the proof of (3.25). Fix $k\in\mathbb{N}$ then there exists $m(k)$ such that

$$\sup_{(0,\ell_{ij})\times\mathbb{R}}|\omega_2-\omega_2^m| \le \frac{1}{2kc}, \tag{3.30}$$

where $c>0$ is a constant such that $|\varphi_{ij}^n{}'|((0,\ell_{ij})) + |\varphi_{ij}{}'|((0,\ell_{ij}))\le c$ (recall that φ_{ij}^n and φ_{ij} have equibounded BV−norm, from Proposition 3.3). Since ω_2^m is of the form (3.26) then there exists $n=n(m,\epsilon)=n(k)$ such that

$$\Big|\int_0^{\ell_{ij}}\omega_2^m(s,\varphi_{ij}^n(s))\varphi_{ij}^n{}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}}\omega_2^m(s,\varphi_{ij})\varphi_{ij}' - \sum_{s\in J_{\varphi_{ij}}}\nu(s,J_{\varphi_{ij}})\int_{\varphi_{ij\,-}(s)}^{\varphi_{ij\,+}(s)}\omega_2^m(s,t)dt\Big| \le \frac{1}{2k}. \tag{3.31}$$

(3) To show that $\mathcal{A}$ separates points in $(0,\ell_{ij})\times\mathbb{R}$: recall that for any $x,\ y\in(0,\ell_{ij})\times\mathbb{R},\ x\neq y$, there exists open ball $B_\rho(x)$ such that $y\notin B_\rho(x)$ hence we may construct a smooth function ω_x with compact support equal to $B_\rho(x)$ and $\omega_x(x)=1$.

Moreover we have

$$
\begin{aligned}
&\Big|\int_0^{\ell_{ij}} \omega_2(s,\varphi_{ij}^n(s)){\varphi_{ij}^n}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}} \omega_2(s,\varphi_{ij}){\varphi_{ij}}' - \sum_{s\in J_{\varphi_{ij}}} \nu(s,J_{\varphi_{ij}}) \int_{{\varphi_{ij}}_-(s)}^{{\varphi_{ij}}_+(s)} \omega_2(s,t)dt\Big| \\
=&\Big|\int_0^{\ell_{ij}} (\omega_2(s,\varphi_{ij}^n(s)-\omega_2^m(s,\varphi_{ij}^n(s)){\varphi_{ij}^n}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}} (\omega_2(s,\varphi_{ij})-\omega_2^m(s,\varphi_{ij}))\varphi_{ij}' \\
&\quad - \sum_{s\in J_{\varphi_{ij}}} \nu(s,J_{\varphi_{ij}}) \int_{{\varphi_{ij}}_-(s)}^{{\varphi_{ij}}_+(s)} (\omega_2(s,t)-\omega_2^m(s,t))dt \\
&\quad + \int_0^{\ell_{ij}} \omega_2^m(s,\varphi_{ij}^n(s){\varphi_{ij}^n}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}} \omega_2^m(s,\varphi_{ij})\varphi_{ij}' - \sum_{s\in J_{\varphi_{ij}}} \nu(s,J_{\varphi_{ij}}) \int_{{\varphi_{ij}}_-(s)}^{{\varphi_{ij}}_+(s)} \omega_2^m(s,t)dt\Big| \\
\le& \sup|\omega_2-\omega_2^m|\big\{|{\varphi_{ij}^n}'|((0,\ell_{ij}))+|\varphi_{ij}'|((0,\ell_{ij}))\big\} \\
&\quad +\Big|\int_0^{\ell_{ij}} \omega_2^m(s,\varphi_{ij}^n(s)){\varphi_{ij}^n}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}} \omega_2^m(s,\varphi_{ij})\varphi_{ij}' - \sum_{s\in J_{\varphi_{ij}}} \nu(s,J_{\varphi_{ij}}) \int_{{\varphi_{ij}}_-(s)}^{{\varphi_{ij}}_+(s)} \omega_2^m(s,t)dt\Big| \\
\le& \frac{1}{2k} +\Big|\int_0^{\ell_{ij}} \omega_2^m(s,\varphi_{ij}^n(s)){\varphi_{ij}^n}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}} \omega_2^m(s,\varphi_{ij})\varphi_{ij}' - \sum_{s\in J_{\varphi_{ij}}} \nu(s,J_{\varphi_{ij}}) \int_{{\varphi_{ij}}_-(s)}^{{\varphi_{ij}}_+(s)} \omega_2^m(s,t)dt\Big|.
\end{aligned}
\tag{3.32}
$$

Hence from (3.30), (3.31), (3.32) and a diagonal argument it follows that (up to a not relabelled subsequence)

$$
\lim_{n\to+\infty}\Big|\int_0^{\ell_{ij}} \omega_2(s,\varphi_{ij}^n(s)){\varphi_{ij}^n}' - \int_{(0,\ell_{ij})\setminus J_{\varphi_{ij}}} \omega_2(s,\varphi_{ij}){\varphi_{ij}}' - \sum_{s\in J_{\varphi_{ij}}} \nu(s,J_{\varphi_{ij}}) \int_{{\varphi_{ij}}_-(s)}^{{\varphi_{ij}}_+(s)} \omega_2(s,t)dt\Big| = 0.
\tag{3.33}
$$

Which conclude the second proof of (3.22). Finally (3.23) follows from Lemma 3.3 and the weak lower semicontinuity of the mass of currents, (1.19), and the proof of step 1 is concluded.

It is not restrictive to assume that $w_{ij}^n = |\alpha_i - \pi_{ij}(p_n)|$ is a point of continuity of φ_{ij}^n for all $n\in\mathbb{N}$ and all $ij\in\{12,23,31\}$. Indeed given a sequence $\{\Gamma^n\}\subset X$ converging to Γ, from Lemma 3.2 for all n we can assign a sequence $\{\Gamma^{m,n}\}\subset X_{\rm Lip}$ such that $\Gamma^{m,n}\to\Gamma^n$ as $m\to+\infty$. Thus by a diagonal argument, we find a sequence $\{\Gamma^{m(n),n}\}\subset X_{\rm Lip}$ which tends to Γ and satisfies the above requirement (we can also assume that Γ^n is Lipschitz, but this will not be needed in the proof).

Without loss of generality (up to a not relabeled subsequence) we may further assume

$$
p_n \to b\in \mathrm{T},
$$

$\{w_{ij}^n\}$ is a monotone sequence, and

$$
w_{ij}^n \to w_{ij} := |\alpha_i - \pi_{ij}(b)|, \qquad ij\in\{12,23,31\}.
$$

Before passing to the second step, it is convenient to divide the target triangle T into various regions.

Assume first that T is acute. The point b, together with the heights

$$h_{ij} := \overline{\pi_{ij}(b)b}, \qquad ij \in \{12, 23, 31\}, \tag{3.34}$$

divides T into three regions $\mathcal{P}_i$, $i \in \{1, 2, 3\}$, as shown in Figure 3.3*a*; precisely, if $\overline{\mathcal{P}}_i$ denotes the closed region enclosed by h_{ij}, h_{ki}, $\overline{\alpha_i\pi_{ij}(b)}$ and $\overline{\alpha_i\pi_{ki}(b)}$, then $\mathcal{P}_i$ is defined by

$$\mathcal{P}_i := \overline{\mathcal{P}}_i \setminus (h_{ij} \cup h_{ki}), \qquad i = 1, 2, 3. \tag{3.35}$$

Similarly we define h^n_{ij} and $\mathcal{P}^n_i$ by replacing b with p_n in (3.34) and (3.35).
Assume now that T is not acute. Without loss of generality we may assume that the angle at α_1 is greater than $\frac{\pi}{2}$. The only difference here is with the definition of $\mathcal{P}^n_1$ and $\mathcal{P}_1$, since each Γ_{ij} has to satisfy the graphicality condition with respect to $\overline{\alpha_i\alpha_j}$; hence we define $\overline{\mathcal{P}_1}$ as the closed quadrilateral bounded by h_{12}, h_{31}, n_{12} and n_{31}, where n_{12} and n_{31} are the normals to $\overline{\alpha_1\alpha_2}$ and $\overline{\alpha_3\alpha_1}$, respectively, passing through α_1 (see Figure 3.3b). Similarly we define $\overline{\mathcal{P}^n_1}$. Finally we set $\mathcal{P}_1 := \overline{\mathcal{P}_1} \setminus (h_{12} \cup h_{31})$ and $\mathcal{P}^n_1 := \overline{\mathcal{P}^n_1} \setminus (h_{12} \cup h_{31})$.
Step 2. We will prove that we can decompose $\Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}$ as three currents meeting at a point p.
It is easy to see that the sets $\overline{\mathcal{P}^n_i}$ are converging to $\overline{\mathcal{P}}_i$ with respect to the Hausdorff distance. It is not true in general that $\{\Gamma^n_{ij}\}$ is converging to Γ_{ij} in the Hausdorff distance (see Figure 3.4); however, since

$$\Gamma^n_i = \Gamma^n_{ij} \cap \overline{\mathcal{P}^n_i} = \Gamma^n_{ki} \cap \overline{\mathcal{P}^n_i}, \qquad \Gamma^n_i \subset \overline{\mathcal{P}^n_i}, \tag{3.36}$$

for all $ij, ki \in \{12, 23, 31\}, ij \neq ki$, it is readily seen that

$$\begin{aligned}&\Gamma_{ij} \subset \mathrm{T} \setminus \mathcal{P}_k,\\ &[\![\Gamma_{ij}]\!] = [\![\Gamma_{ij}]\!] \llcorner \mathcal{P}_i + [\![\Gamma_{ij}]\!] \llcorner \mathcal{P}_j + [\![\Gamma_{ij}]\!] \llcorner h_{ki} + [\![\Gamma_{ij}]\!] \llcorner h_{ij} + [\![\Gamma_{ij}]\!] \llcorner h_{jk}.\end{aligned} \tag{3.37}$$

For $i \in \{1, 2, 3\}$, the integral 1-current $[\![\Gamma^n_i]\!] = [\![\Gamma^n_{ij}]\!] \llcorner [0, w_{ij}] \times \mathbb{R}$ has boundary $\delta_{p_n} - \delta_{\alpha_i}$ in $\mathbb{R}^2$. By the compactness theorem for integral currents, see Theorem 1.22, there exists an integral current $\mathcal{T}_i \in \mathcal{D}_1(\mathbb{R}^2)$, $i = 1, 2, 3$, such that, up to a not relabeled subsequence,

$$[\![\Gamma^n_i]\!](\omega) \to \mathcal{T}_i(\omega) \qquad \forall \omega \in \mathcal{D}^1(\mathbb{R}^2). \tag{3.38}$$

Clearly

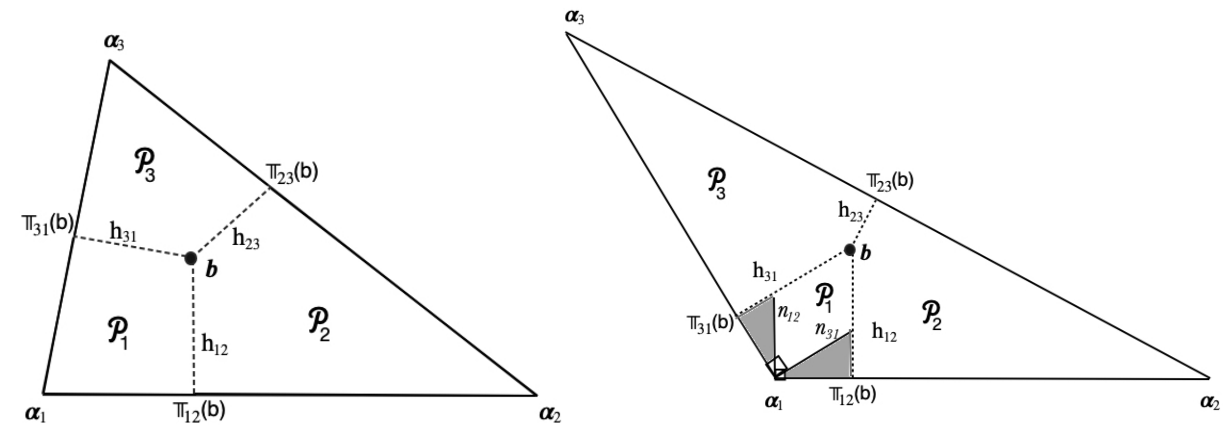
$$\partial \mathcal{T}_i = \delta_b - \delta_{\alpha_i}. \tag{3.39}$$

From (3.36) and the convergence of $\overline{\mathcal{P}^n_i}$ to $\overline{\mathcal{P}}_i$ in the Hausdorff distance, we infer

$$\operatorname{supp} \mathcal{T}_i \subset \overline{\mathcal{P}_i}, \qquad \text{hence} \quad \mathcal{T}_i = \mathcal{T}_i \llcorner \mathcal{P}_i + \mathcal{T}_i \llcorner h_{ij} + \mathcal{T}_i \llcorner h_{ki}, \tag{3.40}$$

where $ij, ki \in \{12, 23, 31\}$. Note that $\mathcal{T}_i$ is not necessarily equal to $[\![\Gamma_{ij}]\!] \llcorner \overline{\mathcal{P}_i}$, due to a possible cancellation of a vertical part over $\pi_{ij}(b)$, $ij \in \{12, 23, 31\}$ (that is, on h_{ij}), see Figure 3.4. However from $[\![\Gamma^n_{ij}]\!] = [\![\Gamma^n_i]\!] + [\![\Gamma^n_j]\!]$ and (3.38) we have

$$[\![\Gamma_{ij}]\!] = \mathcal{T}_i - \mathcal{T}_j, \qquad ij \in \{12, 23, 31\}, \tag{3.41}$$



(a) Partitions of T into $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and three segments.

(b) Partitions of T minus the two grey triangles into $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and three segments for a non acute triangle.

Figure 3.3

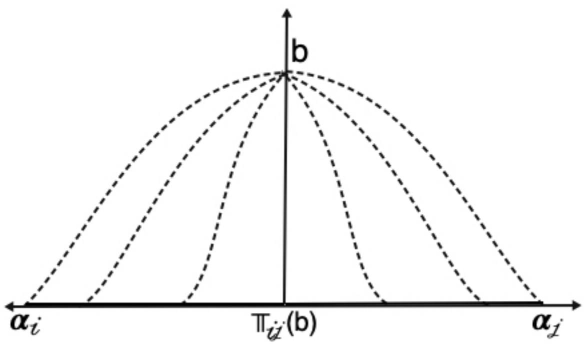


Figure 3.4: In dots a sequence of graphs $[\![\Gamma_{ij}^n]\!]$ of functions that pass through a fixed point $b \in T$. In bold the graph of the limit function (the horizontal segment) $[\![\Gamma_{ij}]\!]$. The limit in the sense of currents of the left branches of the sequence $\{\Gamma_i^n\}$ is $\overline{\alpha_i \pi_{ij}(b)} \cup \overline{\pi_{ij}(b) b}$ while the limit of the right branches $\{\Gamma_j^n\}$ is $\overline{\pi_{ij}(b) b} \cup \overline{\pi_{ij}(b) \alpha_j}$.

as currents in $\mathbb{R}^2$. Notice that $\mathcal{T}_i$ and $\mathcal{T}_j$ have multiplicity one, and in (3.41) they contribute with opposite orientation. This allows, if necessary, to identify $\mathcal{T}_i$, $i = 1, 2, 3$, with its support. Note also that $\mathcal{T}_i$ may have vertical part over α_i, see Figure 3.5. Now, since $[\![\Gamma_i^n]\!]$ is Cartesian with respect to both the edges $\overline{\alpha_i \alpha_j}$ and $\overline{\alpha_k \alpha_i}$, from (3.38) it follows that $\mathcal{T}_i \llcorner \mathcal{P}_i$ is part of two generalized graphs over the same edges, *i.e.*,

$$\mathcal{T}_i \llcorner \mathcal{P}_i = [\![\Gamma_{ij}]\!] \llcorner \mathcal{P}_i = -[\![\Gamma_{ki}]\!] \llcorner \mathcal{P}_i. \tag{3.42}$$

Moreover, we infer that $\mathcal{T}_i$ cannot have vertical part over h_{ij} and h_{ki} at the same time; in other words once the current $\mathcal{T}_i$ touches one of the heights h_{ij} or h_{ki} it

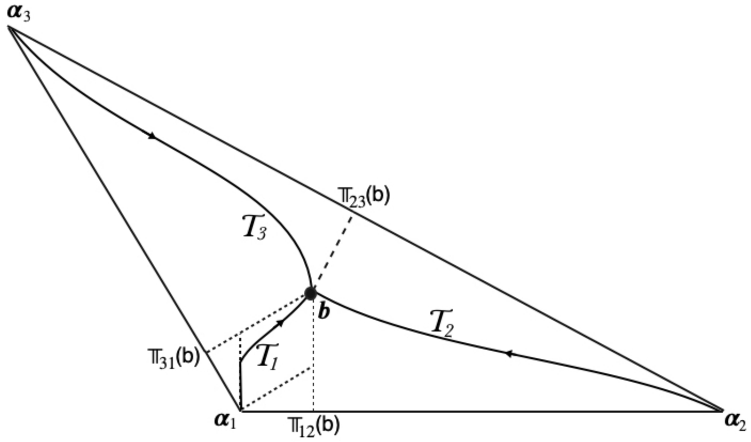


Figure 3.5: $\mathcal{T}_i$ has vertical part over α_i when $i = 1$.

stays there until it reaches b, and $\mathcal{T}_i$ cannot have a nonempty support in more than one height, see Figures 3.6b-3.7b. We conclude the following statement:

(A) The supports of the three currents $\mathcal{T}_i$, $i = 1, 2, 3$, have as common point b. Moreover, if there are $i \neq j$ such that the supports of $\mathcal{T}_i$ and $\mathcal{T}_j$ intersect in a point different from b, then this intersection occurs on the mutual height h_{ij}. Finally, if the supports of $\mathcal{T}_i$ and $\mathcal{T}_j$ intersect on h_{ij} outside b, then they intersect on a closed segment and the intersection of the supports of $\mathcal{T}_i$ and $\mathcal{T}_j$ with $\mathcal{T}_k$ is only the point b.

Step 3. To conclude the proof of our claim we now analyse the possible cases arising from (A).
Case (i). Assume that the three supports of the currents $\mathcal{T}_i$, $i = 1, 2, 3$, intersect only at the point b. This includes the case

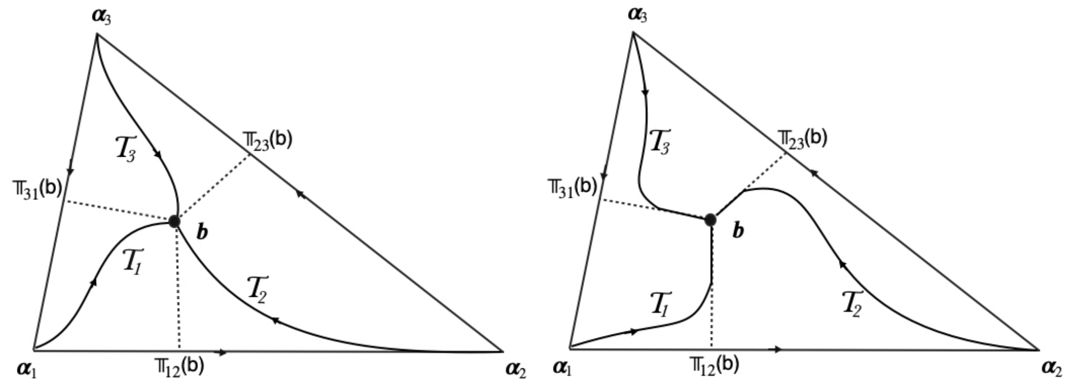
$$\mathcal{T}_i \llcorner h_{ij} = \mathcal{T}_j \llcorner h_{ij} = 0 \qquad \text{for all } ij \in \{12, 23, 31\},$$

as in Figure 3.6a. But it may also happen that $\mathcal{T}_i$ has vertical part over h_{ij}, provided that $\mathcal{T}_j$ does not have vertical part over the same height (see for instance Figure 3.6b). In any case we may set

$$p := b, \qquad \Gamma^l_{ij} = \mathcal{T}_i\,, \qquad \Gamma^r_{ij} := -\mathcal{T}_j\,, \qquad ij \in \{12, 23, 31\},$$

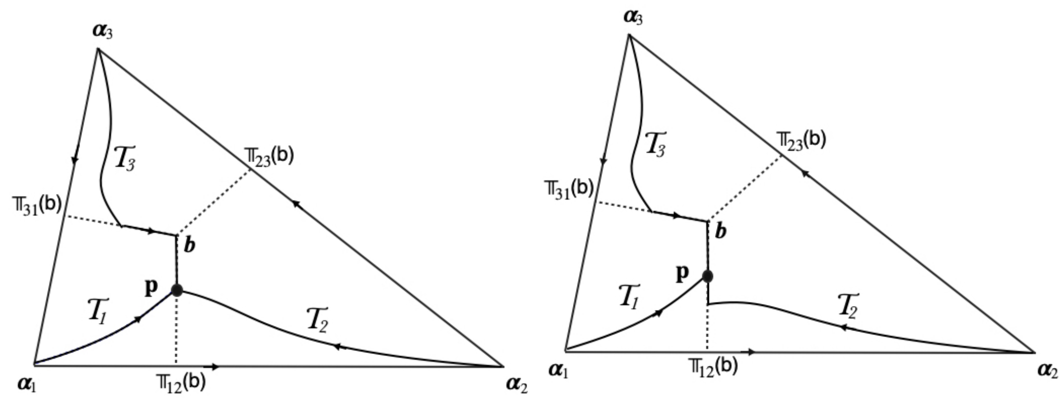
where we have identified the currents $\mathcal{T}_i$ with their supports. By (3.39) and (3.42), the claim is achieved.
Case (ii). The second case to be discussed is the one considering possible overlapping of the support of the currents $\mathcal{T}_i$. By condition (A) such overlapping, giving rise to cancellations, can occur only on one height h_{ij}. Hence, assume there exists one (and only one) $ij \in \{12, 23, 31\}$ such that

$$\mathcal{T}_i \llcorner h_{ij} \neq 0 \quad \text{and} \quad \mathcal{T}_j \llcorner h_{ij} \neq 0.$$



(a) (b)

Figure 3.6: Case (i) of step 3 in the proof of Theorem 3.6.



(a) $\mathcal{T}_1$ and $\mathcal{T}_2$ coincide only on $\overline{bp}$.

(b) $\mathcal{T}_1$ and $\mathcal{T}_2$ overlap on $\overline{bp}$ and $\mathcal{T}_1 \llcorner \overline{bp} \subset \mathcal{T}_2 \llcorner \overline{bp}$.

Figure 3.7: Case (ii) of step 3 in the proof of Theorem 3.6.

Thus we have $\mathcal{T}_i \llcorner h_{ki} = 0$ and $\mathcal{T}_j \llcorner h_{jk} = 0$.
First assume that $[\![\Gamma_{ij}]\!] \llcorner h_{ij} = 0$, *i.e.*, φ_{ij} is continuous at w_{ij}. Then $\mathcal{T}_i \llcorner h_{ij} = \mathcal{T}_j \llcorner h_{ij}$, see Figure 3.7a. We set, identifying $\mathcal{T}_i$ with its support,

$$
\begin{aligned}
&p := \varphi_{ij}(w_{ij}), \\
&\Gamma^l_{ij} := \mathcal{T}_i \llcorner \mathcal{P}_i, && \Gamma^r_{ij} := \mathcal{T}_j \llcorner \mathcal{P}_j, \\
&\Gamma^l_{jk} := \mathcal{T}_j \llcorner \mathcal{P}_i, && \Gamma^r_{jk} := \mathcal{T}_k \cup \mathcal{T}_j \llcorner h_{ij}, \\
&\Gamma^l_{ki} := \mathcal{T}_k \cup \mathcal{T}_i \llcorner h_{ij}, && \Gamma^r_{ki} := \mathcal{T}_i \llcorner \mathcal{P}_i.
\end{aligned}
$$

One checks that the connection built above is a BV graph type connection, addressing the claim.
Now assume that

$$[\![\Gamma_{ij}]\!] \llcorner h_{ij} \neq 0,$$

i.e., φ_{ij} jumps at w_{ij}. Thus either $\operatorname{supp}[\![\Gamma_{ij}]\!] \llcorner h_{ij} \subseteq \operatorname{supp} \mathcal{T}_i \llcorner h_{ij}$ or $\operatorname{supp}[\![\Gamma_{ij}]\!] \llcorner h_{ij} \subseteq \operatorname{supp} \mathcal{T}_j \llcorner h_{ij}$. Without loss of generality we may assume that $\operatorname{supp}[\![\Gamma_{ij}]\!] \llcorner h_{ij} \subseteq \operatorname{supp} \mathcal{T}_i \llcorner h_{ij}$, hence $\mathcal{T}_i \llcorner (h_{ij} \setminus \operatorname{supp}[\![\Gamma_{ij}]\!]) = -\mathcal{T}_j \llcorner h_{ij} \neq 0$ (note that $\operatorname{supp}[\![\Gamma_{ij}]\!] \llcorner h_{ij} = \{t\varphi_{ij}(w_{ij+}) + (1-t)\varphi_{ij}(w_{ij-}) : t \in [0,1]\}$). We set

$$
\begin{aligned}
&p := \varphi_{ij}(w_{ij}) = \varphi_{ij+}(w_{ij}), \\
&\Gamma^l_{ij} := \mathcal{T}_i \llcorner \mathcal{P}_i \cup (h_{ij} \cap \operatorname{supp}[\![\Gamma_{ij}]\!]), \qquad && \Gamma^r_{ij} := \mathcal{T}_j \llcorner \mathcal{P}_j, \\
&\Gamma^l_{jk} := \mathcal{T}_j \llcorner \mathcal{P}_i, && \Gamma^r_{jk} := \mathcal{T}_k \cup \mathcal{T}_j \llcorner h_{ij}, \\
&\Gamma^l_{ki} := \mathcal{T}_k \cup \mathcal{T}_j \llcorner h_{ij}, && \Gamma^r_{ki} := \mathcal{T}_i \llcorner \mathcal{P}_i \cup (h_{ij} \cap \operatorname{supp}[\![\Gamma^n_{ij}]\!]),
\end{aligned}
$$

see Figure 3.7b. Also in this case the conclusion follows.
In the end it is enough to define

$$
\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3), \qquad \Gamma_i := \Gamma^l_{ij} = \Gamma^r_{ki}, \qquad i = 1, 2, 3. \tag{3.43}
$$

□

From the compactness of the space of BV connection, combining with Proposition 3.4, we see that the infimum in (2.1) is attained. As a consequence, we can conclude the proof of Theorem 0.1.

Corollary 3.8. *We have*

$$
\mathcal{A}(u, D) \leq |D| + \inf\{\mathcal{G}(\Gamma) : \Gamma \in X_{\mathrm{Lip}}\} = |D| + \min\{\mathcal{G}(\Gamma) : \Gamma \in X\}. \tag{3.44}
$$

4. Outlook: several triple junctions; parametric approach; higher dimension; Riemannian metrics

The aim of this chapter is to introduce three different possible extensions of the results obtained in Chapter 2 and Chapter 3 that we plan to investigate in the future; and to point out some of the difficulties that we may face.

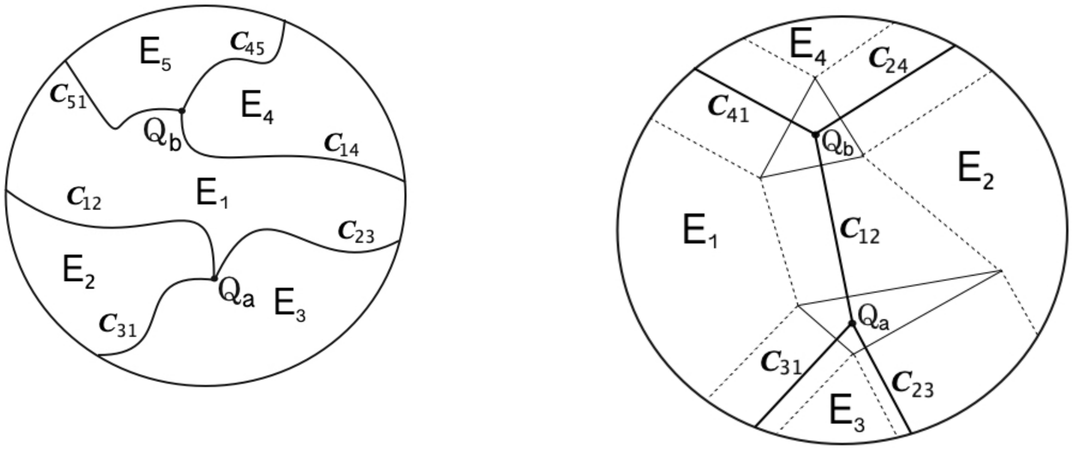
4.1 Several triple junctions

Assume that $u : D \to \{\alpha_1, \alpha_2, \alpha_3\}$ has two triple junctions Q_a and Q_b in a source disk D. If the jump curves of each triple junction are disjoint up to the boundary of D, see Figure 4.1a, then we can apply the former results on each triple junction separately and the upper bound $\mathcal{A}(u, D)$ will have two (non-related) minimization problems (similar to (0.12)) one for each junction, i.e., the two triple junctions do not really see each other (we may get two different connections as a solution for each minimization problem). The same apply for a finite number of triple junctions with disjoint jump curves up to the boundary.

Now assume that Q_a and Q_b share a common jump segment, say C_{12}, and the rest of the jump segments are disjoint up to the boundary of D. In this case the two triple junctions affect each other. We apply the former techniques for all the jump segments except for C_{12}, where we need to rephrase the one-codimensional minimum problem corresponding to it. First the domain of integration in (2.9) may not be a rectangle anymore but a quadrilateral, see Figure 4.1b. Second the connections, in the target triangle, used to construct u^ε on a neighborhood of each triple junction are not necessarily the same, hence the boundary condition on the quadrilateral depends on the two connections and not only one of them. Thus the upper bound of $\mathcal{A}(u, D)$ will consist of one minimization problem, in which the two triple junctions interact, of the form

$$\min\left\{\overline{\mathfrak{A}}_{12}(\Gamma^a \cup \Gamma^b) + \mathfrak{A}_{23}(\Gamma^a) + \mathfrak{A}_{31}(\Gamma^a) + \mathfrak{A}_{24}(\Gamma^b) + \mathfrak{A}_{41}(\Gamma^b) : (\Gamma^a, \Gamma^b) \in X \times X\right\}, \quad (4.1)$$

where $\overline{\mathfrak{A}}_{12}(\Gamma^a \cup \Gamma^b)$ is the area of the surface over the quadrilateral in the neighbor-



(a) The source disk with two triple junctions Q_a and Q_b; and disjoint jump curves. $u(E_1) = \alpha_1,\ u(E_2) = u(E_4) = \alpha_2,\ u(E_3) = u(E_5) = \alpha_3$.

(b) The source disk with two triple junctions Q_a and Q_b; and one jump segment C_{12} in common.

Figure 4.1

hood of C_{12}, and $\mathfrak{A}_{ij}(\Gamma^a)$, $\mathfrak{A}_{kl}(\Gamma^b)$, $ij \in \{23, 31\}$, $kl \in \{24, 41\}$, are as defined in (0.9).
The study of $\mathcal{A}(u, D)$ for u taking only three values can be difficult: Now we want to indicate a possible configuration of a map u_*, with an infinite number of triple junctions accumulating at the center of a disk, such that $\mathcal{A}(u_*, D)$ is finite (no claim on the actual value of $\mathcal{A}(u_*, D)$ is made).
Assume that D is the unit disk centered at the origin. Consider the countable set TJ of triple junctions defined as follows

$$TJ := \left\{ Q_{k,l} = \frac{1}{k^2}\, e^{\frac{(3+l)}{6}\pi i} : \quad l \in \{0, 1, 2, 3, 4, 5\}, \quad k \in \mathbb{N},\ k > 1 \right\} \subset D.$$

Points in TJ divides D into a countable number of partitions as in Figure 4.2. For simplicity we set

$$E_i := {u_*}^{-1}(\alpha_i), \qquad i = 1, 2, 3.$$

Each triple junction $Q_{k,l}$, $\{k, l\} \notin \{\{2, 0\}, \{2, 2\}, \{2, 4\}\}$, share all of its jump curves $C^{k,l}_{ij}, ij \in \{12, 23, 31\}$, with other three triple junctions; on the other hand each triple junction $Q_{2,l}$, $l = 0, 2, 4$, share only two of its jump curves with another two triple junctions, i.e., among all the jump curves of u_* only three curves are connected to only one triple junction, see Figure 4.2.
Note that the length of each curve in the jump set of u_* is either $\frac{\pi}{3k^2}$ or $(\frac{1}{k^2} -$

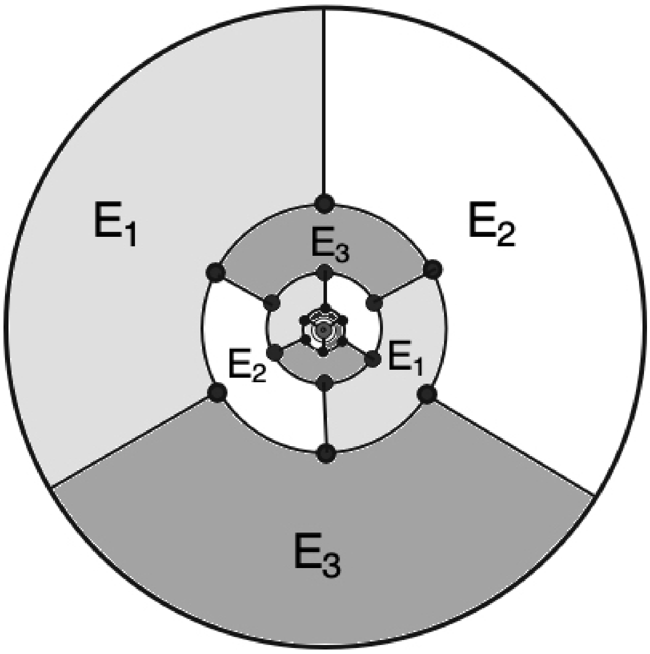


Figure 4.2: The source disk D divided by the jump set (of finite length) of u_* into a countable number of partitions. The black thick dots are the triple junctions of u_*, each circle of radius $\frac{1}{k^2}$, $k \in \mathbb{N}, k > 1$, has six triple junctions given by the rotation of $\frac{1}{k^2}\ e^{\frac{\pi}{2}i}$ with angle $\frac{\pi}{3}$. Only three jump curves reach the boundary of D.

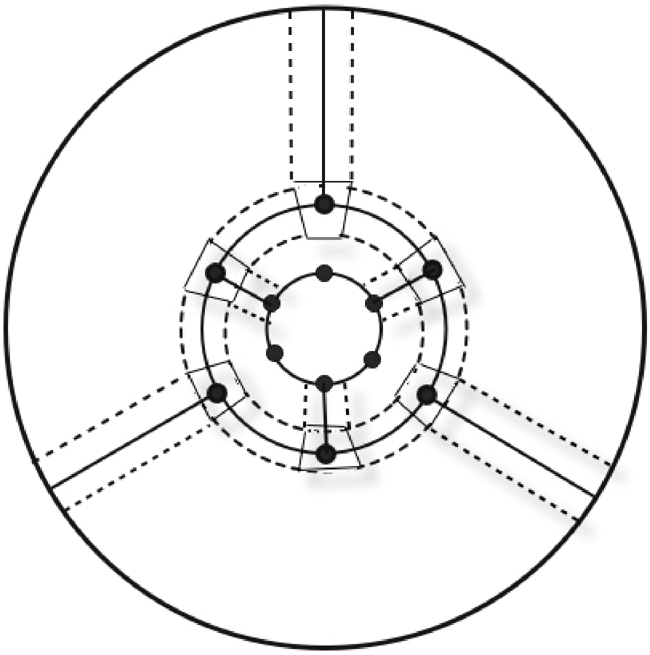


Figure 4.3: Divide D into finite number of partitions: write D as a union of a disk of radius $\bar{\varepsilon}$ and an annulus. Then divide the annulus into a finite number of partitions. Two of the jump curves of each triple junction meets at angles equal to π hence the neighborhoods of each triple junction has to be quadrilateral (as explained in Case 2 in the proof of Proposition 2.8). We define u^ε on each partition by adopting the former techniques.

$\frac{1}{(k+1)^2}),\ k \in \mathbb{N},\ k > 1$, hence we have

$$\sum_{k=1}^{\infty} c_k < +\infty \qquad \text{where } c_k := \frac{\pi}{3k^2} + \frac{1}{k^2} - \frac{1}{(k+1)^2}. \tag{4.2}$$

A construction of a sequence $\{u^\varepsilon\}$ of Lipschitz maps, converging to $u_* \in \mathrm{BV}(D, \mathbb{R}^2)$ in $L^1(D; \mathbb{R}^2)$ such that $\lim_{\varepsilon \to 0} \mathcal{A}(u^\varepsilon, D) < +\infty$, seems to be doable. We may start by defining u^ε in the annuls N centered at the origin with radii 1 and $\bar{\varepsilon}(u^\varepsilon),\ \bar{\varepsilon}(u^\varepsilon) > 0$.

Divide N into finite number of open sets as in Figure 4.3. Let p be any point in the target triangle T and let Γ be the connection given by the segments connecting p to the vertices $\alpha_1, \alpha_2, \alpha_3$ of T.

On the (cygar-shape) neighborhoods of the three jump curves connected to only one triple junction (resp. on the quadrilateral neighborhoods of the triple junctions) we define u^ε as in Step 2 (resp. Case 2) in the proof of Proposition 2.8.

On the neighborhoods of the jump curves shared by two triple junctions we may consider the solution given by a Dirichlet minimum problem similar to (2.9). Hence

$$\begin{aligned}\mathcal{A}(u^\varepsilon, N) \leq & |N| + \mathfrak{A}^{2,0}_{12}(\Gamma) + \mathfrak{A}^{2,4}_{23}(\Gamma) + \mathfrak{A}^{2,2}_{31}(\Gamma) \\ & + \sum_{\substack{k>1,\ l\in\{0,1,2,3,4,5\} \\ \{k,l\}\notin\{\{2,0\},\{2,2\},\{2,4\}\}}} \left\{ \overline{\mathfrak{A}}^{k,l}_{12}(\Gamma) + \overline{\mathfrak{A}}^{k,l}_{23}(\Gamma) + \overline{\mathfrak{A}}^{k,l}_{31}(\Gamma) \right\} + O(\varepsilon), \end{aligned} \tag{4.3}$$

where $\mathfrak{A}^{2,l}_{ij}(\Gamma)$ is the area of the solution of the corresponding Dirichlet-Neumann minimum problem (of type (0.7)) defined in (0.9), and $\overline{\mathfrak{A}}^{k,l}_{ij}(\Gamma)$ is the area of the solution of the corresponding Dirichlet minimum problem (of type (2.9)). Note that the inequality in (4.3) is due to the fact that some of the surfaces are counted twice, otherwise we get equality.
Along each jump curve of a triple junction $Q_{k,l}$ the piecewise affine function $\varphi^{k,l}_{ij} = \varphi^{k,l}_{ij}(\Gamma)$, defined in (2.7), is one of the competitor in the minimization problem (of type (0.7) or (2.9)). Moreover we have

$$\mathfrak{A}^{k,l}_{ij}(\Gamma) < 4c_k \text{ diam } T, \tag{4.4}$$

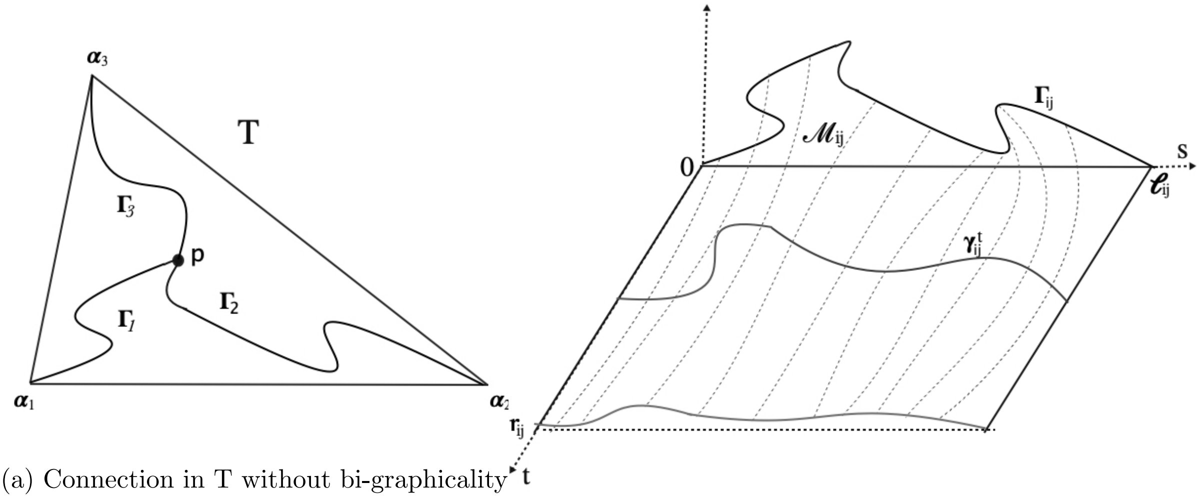
where diam T is the diameter of the target triangle T.
It remains to define u^ε on the disk of radius $\bar{\varepsilon}(u^\varepsilon)$. We will not do it but one way could be defining u^ε to be constant in a neighborhood of the origin and use the intermediate region to glue the constant, in a Lipschitz way, with the value of u^ε on the annulus N.
From (4.2), (4.3), (4.4) and using a diagonal argument we expect that

$$\lim_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon, D) < +\infty.$$

4.2 Parametric approach under a semi-cartesian hypothesis

If we look for a lower bound for the area functional $\mathcal{A}(u, D)$ without symmetry assumptions on u, the first remark is that in principle there is no reason for the upper bound on the right hand side of (0.11) to be optimal. For instance, in order to solve the one-codimensional area minimizing problems, the connections do not necessarily need not to satisfy the bi-graphicality conditions in Definition 2.2, see Figure 4.4a, which indicates that we may be able to find a better upper bound.
In this section, we discuss the possibility of constructing a sequence $\{u^\varepsilon\}$ of Lipschitz maps that converges to u in $L^1(D;\mathbb{R}^2)$ for connections in the target triangle just with finite length and without self-intersections.
We start by assuming that the surfaces $\mathcal{M}_{ij}$, $ij \in \{12, 23, 31\}$, that solve the one-codimensional area minimizing problems[(1)], are semi-cartesian, *i.e.*, for each $ij \in \{12, 23, 31\}$, the intersection between $\mathcal{M}_{ij}$ and the vertical plane (parallel to the target triangle) at level t is given by a simple curve $\gamma^t_{ij} : [0, \ell_{ij}] \to \mathbb{R}^3$ connecting

(1) In the parametric sense, as image of a disk [23].



(a) Connection in T without bi-graphicality conditions (with respect to $[\alpha_1, \alpha_2]$ and $[\alpha_2, \alpha_3]$).

(b) $\mathcal{M}_{12}$ area minimizing surface spanning Γ_{12} and the two long sides of R_{12}.

Figure 4.4

the two sides of the rectangle R_{ij}, see (4.5) and Figure 4.4b[(2)]. We define u^ε on the cygar-shape regions S^ε_{ij} by allowing the first component of γ^t_{ij} to be the component of u^ε along the $\overline{\alpha_i\alpha_j}$ direction and letting the second (non constant) component of γ^t_{ij} to be the component of u^ε along the direction normal to $\overline{\alpha_i\alpha_j}$, see m^1_{12} and m^3_{12} in (4.6) respectively.
Let $u \in \mathrm{BV}(D;\mathbb{R}^2)$ be the function defined in (0.10). Let p be a point in T and let $\Gamma_i \subset \mathrm{T}$, $i \in \{1,2,3\}$, be three rectifiable curves in T, with no self-intersections and finite-length, that connects α_i to p, $i \in \{1,2,3\}$, moreover we assume $\Gamma_i \cap \Gamma_j = \{p\}$ for $i \neq j$. Let

$$\Gamma_{ij} := \Gamma_i \cup \Gamma_j, \qquad ij \in \{12, 23, 31\},$$

(note that $\Gamma := (\Gamma_1, \Gamma_2, \Gamma_3)$ may not be a connection in the sense of Definition 2.2 since Γ_{ij} need not to be a generalized graph of a BV-function over $\overline{\alpha_i\alpha_j}$).
Let R_{ij} and $\widehat{\mathrm{R}}_{ij}$ be as defined in (0.5) and (2.6) respectively, and let $\Gamma^0_{ij} \subset \widehat{\mathrm{R}}_{ij} \times \mathbb{R}$ (resp. Γ^ℓ_{ij}) be the copy of Γ_{ij} in $[0,\ell_{ij}] \times \{0\} \times \mathbb{R}$ (resp. $[0,\ell_{ij}] \times \{2r_{ij}\} \times \mathbb{R}$) that connects the two points $(0,0,0)$ and $(\ell_{ij},0,0)$ (resp. $(0,2r_{ij},0)$ and $(\ell_{ij},2r_{ij},0)$).
Set

$$\bar{\Gamma}_{ij} := \Gamma^0_{ij} \cup \big(\{\ell_{ij}\} \times [0,r_{ij}] \times \{0\}\big) \cup \big(\{0\} \times [0,r_{ij}] \times \{0\}\big) \subset \mathrm{R}_{ij} \times \mathbb{R},$$

and define the closed Jordan curve $\hat{\Gamma}_{ij} \subset \widehat{\mathrm{R}}_{ij} \times \mathbb{R}$ as

$$\hat{\Gamma}_{ij} := \Gamma^0_{ij} \cup \big(\{\ell_{ij}\} \times [0,2r_{ij}] \times \{0\}\big) \cup \Gamma^\ell_{ij} \cup \big(\{0\} \times [0,2r_{ij}] \times \{0\}\big).$$

(2)The existence of a semi-cartesian parametrization of a disk-type area minimizing surface is an interesting problem that requires further investigation. Since $\mathcal{M}_{ij}$ has a zero boundary condition along a preferred direction t and the connections have no self-intersections, it is reasonable to expect the existence of a semi-cartesian parametrization in this context.

Let $\mathcal{M}_{ij} = \mathcal{M}_{ij}(\Gamma_{ij})$ be a disk-type solution[(3)] for the Plateau problem corresponding to $\hat{\Gamma}_{ij}$, moreover we assume that $\mathcal{M}_{ij}$ admits a semi-cartesian parametrization, *i.e.*, $\mathcal{M}_{ij}$ can be parametrized by a map

$$\hat{m}_{ij} : \widehat{\mathrm{R}}_{ij} \to \mathcal{M}_{ij} \subset \widehat{\mathrm{R}}_{ij} \times \mathbb{R}, \qquad \hat{m}_{ij}(s,t) := \big(\hat{m}^1_{ij}(s,t), t, \hat{m}^3_{ij}(s,t)\big), \tag{4.5}$$

with the following properties

$$\begin{aligned}
&\hat{m}_{ij} \in \mathcal{C}(\widehat{\mathrm{R}}_{ij}; \mathbb{R}^3) \cap \mathcal{C}^2(\mathrm{int}(\widehat{\mathrm{R}}_{ij}); \mathbb{R}^3), && \hat{m}_{ij}(\widehat{\mathrm{R}}_{ij}) = \mathcal{M}_{ij},\\
&\hat{m}_{ij}([0,\ell_{ij}] \times \{0\}) = \Gamma^0_{ij}, && \hat{m}_{ij}([0,\ell_{ij}] \times \{2r_{ij}\}) = \Gamma^\ell_{ij},\\
&\hat{m}_{ij}(0,t) = (0,t,0), && \hat{m}_{ij}(\ell_{ij},t) = (\ell_{ij},t,0).
\end{aligned}$$

Let $m_{ij} : \mathrm{R}_{ij} \to \mathbb{R}^3$ be the restriction of $\hat{m}_{ij}$ to the rectangle R_{ij}, $ij \in \{12, 23, 31\}$, hence

$$m_{ij}(\partial_D \mathrm{R}_{ij}) = \bar{\Gamma}_{ij},$$

where $\partial_D \mathrm{R}_{ij}$ is defined in (0.8).
Of course our previous discussion refers to the case $m^1_{ij} = s$.
Let $\mathfrak{A}_{ij} = \mathfrak{A}_{ij}(\Gamma_{ij})$ be the area of $\mathcal{M}_{ij}$; then by the area formula we have

$$\mathfrak{A}_{ij} = \int_{\widehat{\mathrm{R}}_{ij}} \sqrt{\left|\frac{\partial m^1_{ij}}{\partial s}\right|^2 + \left|\frac{\partial m^3_{ij}}{\partial s}\right|^2 + \left(\frac{\partial m^1_{ij}}{\partial s}\frac{\partial m^3_{ij}}{\partial t} - \frac{\partial m^1_{ij}}{\partial t}\frac{\partial m^3_{ij}}{\partial s}\right)^2}\, dsdt.$$

Divide D into seven open sets $E^\varepsilon_i, S^\varepsilon_{ij}, T^\varepsilon$, $ij \in \{12, 23, 31\}$, (or $E^\varepsilon_i, S^\varepsilon_{ij}, H^\varepsilon$, $ij \in \{12, 23, 31\}$) like we did in the proof of Theorem 2.1, in particular S^ε_{12} is as defined in (2.37).
Define u^ε on E^ε_i, $i = 1, 2, 3$ as in (2.16).
We show that even in this semicartesian setting we can produce an admissible sequence $\{u^\varepsilon\}$. We restrict ourselves to define u^ε on S^ε_{12} as follows:

$$\begin{aligned}
u^\varepsilon(x,y) := \alpha_1 + m^1_{12}\left(\frac{d(x,y) - d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon\left(\tau(x,y) - \delta_\varepsilon\right)\right)\xi\\
+ m^3_{12}\left(\frac{d(x,y) - d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon\left(\tau(x,y) - \delta_\varepsilon\right)\right)\eta,
\end{aligned} \tag{4.6}$$

where $d, d_1, \overline{\kappa}_\varepsilon, \tau, \delta_\varepsilon, \varepsilon_{12}, \xi, \eta$ as in Chapter 2. Observe that $u^\varepsilon = \alpha_1$ on $\{(x,y) \in S^\varepsilon_{12} : d(x,y) = d_1\}$, $u^\varepsilon = \alpha_2$ on $\{(x,y) \in S^\varepsilon_{12} : d(x,y) = d_2\}$.
Of course $u^\varepsilon := (u^\varepsilon_1, u^\varepsilon_2)$ may not to be Lipschitz up to the boundary of S^ε_{12} and we need to regularize it as we did in the non-parametric approach, see Lemma 2.7, but we will not insist on this now.
Write for simplicity

$$\widetilde{m}^1 := m^1_{12} \qquad \text{and} \qquad \widetilde{m}^3 := m^3_{12}.$$

(3) For more details on disk-type (parametric) solutions of the Plateau problem we refer to [23].

We have

$$\nabla u_1^\varepsilon = \Big(\frac{\ell_{12}}{\varepsilon_{12}}\xi^1\widetilde{m}_s^1 d_x + \overline{\kappa}_\varepsilon\xi^1\widetilde{m}_t^1\tau_x + \frac{\ell_{12}}{\varepsilon_{12}}\eta^1\widetilde{m}_s^3 d_x + \overline{\kappa}_\varepsilon\eta^1\widetilde{m}_t^3\tau_x\ , \\ \frac{\ell_{12}}{\varepsilon_{12}}\xi^1\widetilde{m}_s^1 d_y + \overline{\kappa}_\varepsilon\xi^1\widetilde{m}_t^1\tau_y + \frac{\ell_{12}}{\varepsilon_{12}}\eta^1\widetilde{m}_s^3 d_y + \overline{\kappa}_\varepsilon\eta^1\widetilde{m}_t^3\tau_y\Big),$$

$$\nabla u_2^\varepsilon = \Big(\frac{\ell_{12}}{\varepsilon_{12}}\xi^2\widetilde{m}_s^1 d_x + \overline{\kappa}_\varepsilon\xi^2\widetilde{m}_t^1\tau_x + \frac{\ell_{12}}{\varepsilon_{12}}\eta^2\widetilde{m}_s^3 d_x + \overline{\kappa}_\varepsilon\eta^2\widetilde{m}_t^3\tau_x\ , \\ \frac{\ell_{12}}{\varepsilon_{12}}\xi^2\widetilde{m}_s^1 d_y + \overline{\kappa}_\varepsilon\xi^2\widetilde{m}_t^1\tau_y + \frac{\ell_{12}}{\varepsilon_{12}}\eta^2\widetilde{m}_s^3 d_y + \overline{\kappa}_\varepsilon\eta^2\widetilde{m}_t^3\tau_y\Big),$$

where $\widetilde{m}_s^i, \widetilde{m}_t^i$, $i = 1, 2$, denote the partial derivatives of $\widetilde{m}^i$ with respect to $s = \frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}$ and $t = \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)$ respectively, and are evaluated at $\Big(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)\Big)$.
Hence

$$|\nabla u_1^\varepsilon|^2+|\nabla u_2^\varepsilon|^2 \ = \ (\frac{\ell_{12}}{\varepsilon_{12}})^2(\widetilde{m}_s^1)^2+\overline{\kappa}_\varepsilon^2|\nabla\tau|^2(\widetilde{m}_t^1)^2+(\frac{\ell_{12}}{\varepsilon_{12}})^2(\widetilde{m}_s^3)^2+\overline{\kappa}_\varepsilon^2|\nabla\tau|^2(\widetilde{m}_t^3)^2, \quad (4.7)$$

where we have used $|\xi| = |\eta| = 1$, $\xi_1\eta_1 + \xi_2\eta_2 = 0$, $|\nabla d| = 1$ and (2.40).
Moreover

$$\left(\frac{\partial u_1^\varepsilon}{\partial x}\frac{\partial u_2^\varepsilon}{\partial y} - \frac{\partial u_1^\varepsilon}{\partial y}\frac{\partial u_2^\varepsilon}{\partial x}\right)^2 \ = (\frac{\ell_{12}}{\varepsilon_{12}})^2\overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\widetilde{m}_s^1\widetilde{m}_t^3 - \widetilde{m}_t^1\widetilde{m}_s^3\right)^2, \quad (4.8)$$

where again $\widetilde{m}_s^i, \widetilde{m}_t^i$, $i = 1, 3$, are evaluated at $\Big(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y)-\delta_\varepsilon)\Big)$, and we have used (2.40), (2.41), and $\xi^1\eta^2 - \xi^2\eta^1 = 1$.
Therefore from (4.7) and (4.8) we obtain

$$1 + |\nabla u_1^\varepsilon|^2 + |\nabla u_2^\varepsilon|^2 + \left(\frac{\partial u_1^\varepsilon}{\partial x}\frac{\partial u_2^\varepsilon}{\partial y} - \frac{\partial u_1^\varepsilon}{\partial y}\frac{\partial u_2^\varepsilon}{\partial x}\right)^2 \\ =1 + (\frac{\ell_{12}}{\varepsilon_{12}})^2\left(\left(\widetilde{m}_s^1\right)^2 + \left(\widetilde{m}_s^3\right)^2\right) + \overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\left(\widetilde{m}_t^1\right)^2 + \left(\widetilde{m}_t^3\right)^2\right) + (\frac{\ell_{12}}{\varepsilon_{12}})^2\overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\widetilde{m}_s^1\widetilde{m}_t^3 - \widetilde{m}_t^1\widetilde{m}_s^3\right)^2.$$

As a consequence

$$\mathbb{A}(u^\varepsilon, S_{12}^\varepsilon) \ = \ \frac{\ell_{12}}{\varepsilon_{12}}\int_{S_{12}^\varepsilon}\Big[\left(\left(\widetilde{m}_s^1\right)^2 + \left(\widetilde{m}_s^3\right)^2\right) + (\frac{\varepsilon_{12}}{\ell_{12}})^2\overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\left(\widetilde{m}_t^1\right)^2 + \left(\widetilde{m}_t^3\right)^2\right) \\ + \overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\widetilde{m}_s^1\widetilde{m}_t^3 - \widetilde{m}_t^1\widetilde{m}_s^3\right)^2 + O(\varepsilon^2)\Big]^{\frac{1}{2}}\ dxdy \\ = \ \frac{1}{\overline{\kappa}_\varepsilon}\int_{\mathrm{R}_{12}\setminus P_\varepsilon}\frac{1}{|\nabla\tau|}\Big[\left(\left(\widetilde{m}_s^1(s,t)\right)^2 + \left(\widetilde{m}_s^3(s,t)\right)^2\right) + (\frac{\varepsilon_{12}}{\ell_{12}})^2\overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\left(\widetilde{m}_t^1(s,t)\right)^2 + \left(\widetilde{m}_t^3(s,t)\right)^2\right) \\ +\overline{\kappa}_\varepsilon^2|\nabla\tau|^2\left(\widetilde{m}_s^1(s,t)\widetilde{m}_t^3(s,t) - \widetilde{m}_t^1(s,t)\widetilde{m}_s^3(s,t)\right)^2 + O(\varepsilon^2)\Big]^{\frac{1}{2}}\ dsdt,$$

where $\widetilde{m}_s^i, \widetilde{m}_t^i$, $i = 1, 3$, in the first integral are evaluated at $\left(\frac{d(x,y)-d_1}{\varepsilon_{12}}\ell_{12}\ ,\ \overline{\kappa}_\varepsilon(\tau(x,y) - \delta_\varepsilon)\right)$ and $\nabla\tau$ in the second integral is evaluated at $(x, y) = \Phi^{-1}(s, t)$ where Φ is the change of variables defined in (2.49).
Hence, recalling (2.43), and that $\overline{\kappa}_\varepsilon \to 1$ as $\varepsilon \to 0^+$, we have

$$\lim_{\varepsilon\to 0^+} \mathbb{A}(u^\varepsilon, S_{12}^\varepsilon) = \int_{\mathrm{R}_{12}} \sqrt{\left|\frac{\partial m_{ij}^1}{\partial s}\right|^2 + \left|\frac{\partial m_{ij}^3}{\partial s}\right|^2 + \left(\frac{\partial m_{ij}^1}{\partial s}\frac{\partial m_{ij}^3}{\partial t} - \frac{\partial m_{ij}^1}{\partial t}\frac{\partial m_{ij}^3}{\partial s}\right)^2}\, dsdt.$$
$$= \mathfrak{A}_{ij}.$$

Notice that we do not define u^ε on T^ε, and so we can not check that, in the limit, there is zero contribution of the mass over the triple junction point. Recall that in Step 3 of the proof of Proposition 2.8 the bi-graphicalty condition was used, for instance, to define the bi-Lipschitz map Ψ in (2.30). We also point out that we needed the bi-graphicality condition to get the uniform estimate of the length of the connections in Proposition 3.3.

4.3 Higher dimension: singular $\overline{u}$ from a ball in $\mathbb{R}^3$ to $\mathbb{R}^2$

In this section we will try to investigate the possibility of applying the former techniques used in Chapter 2 and Chapter 3 to piecewise constant maps defined in $\mathbb{R}^3$. For simplicity we will consider the symmetric case in both the domain and the target[(4)]. Let $B \subset \mathbb{R}^3_{xyz}$ be an open ball centered at the origin of radius r, then

$$D := B \cap \{z = 0\} \subset \mathbb{R}^2_{xy}$$

is an open disk. Divide D into three circular regions $E_i,\ i \in \{1, 2, 3\}$, separated by three segments meeting at the origin with an angle of 120°, and let $\alpha_1,\ \alpha_2,\ \alpha_3$ be the vertices of an equilateral triangle in $\mathbb{R}^2_T$, hence

$$\ell := \ell_{12} = \ell_{23} = \ell_{31}, \qquad \varepsilon := \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31},$$

where $\ell_{ij},\ \varepsilon_{ij},\ ij \in \{12, 23, 31\}$, are defined in the proof of Proposition 2.8.
Let $\pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2_{xy}$ be the orthogonal projection to the xy- plane and set

$$\overline{E}_i := \pi_{xy}^{-1}(E_i) \cap B, \qquad i \in \{1, 2, 3\},$$

i.e., $\overline{E}_i,\ i \in \{1, 2, 3\}$, are three partitions of B formed by three planes meeting at a common line, z-axis, at equal 120° angles, see Figure 4.5.
Define $\overline{u} : B \to \mathbb{R}^2$ such that

$$\overline{u}(x, y, z) := \alpha_i, \qquad (x, y, z) \in E_i,$$

(4) The arguments in this section seem to be possibly adopted for the case with no symmetry assumptions.

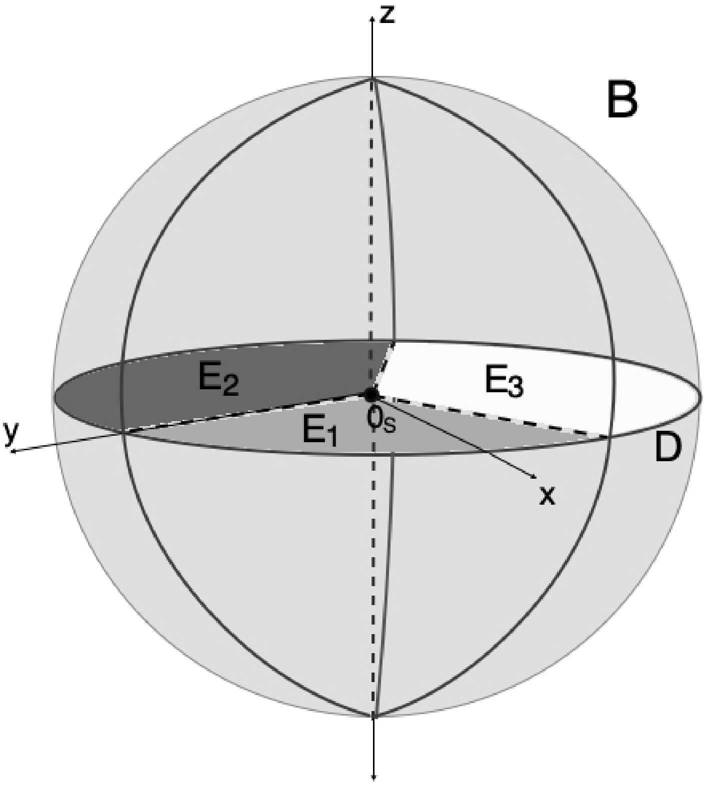
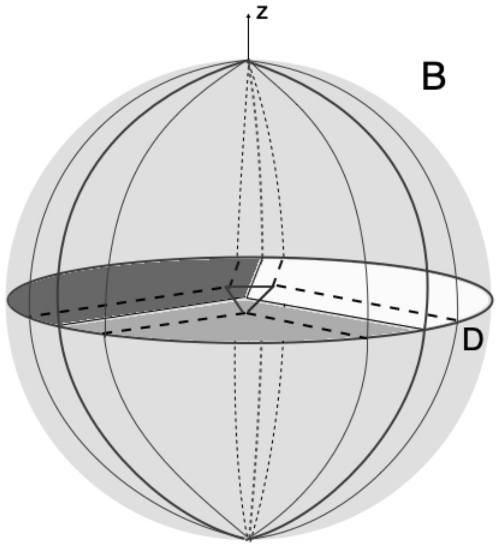
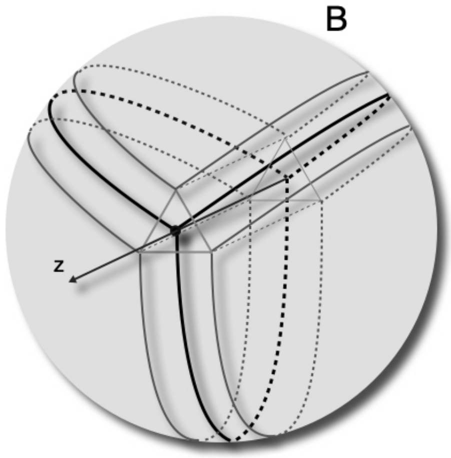


Figure 4.5: The division of a ball $B \subset \mathbb{R}^3$ of radius r into three partitions $\overline{E}_i,\ i = 1, 2, 3$, by three planes passing through the z-axis and normal to the xy-plane. The orthogonal projection of the partitions on the xy-plane divide the disk D into three regions $E_i,\ i = 1, 2, 3$, meet at the origin with angle 120°.



(a)



(b)

Figure 4.6: Divide the disk D into seven sets. The inverse image of the map $\pi_{xy}|_B$ of the partitions of D gives seven partitions of the ball B itself.

i.e., $\overline{u}$ is a singular piecewise constant map jumping along three planes meeting at the z-axis, a triple junction line passing through the origin, at equal 120° angles. Recall that for $v = (v_1, v_2) \in \mathcal{C}^1(B \subset \mathbb{R}^3;\ \mathbb{R}^2)$, the area of the graph of v is given by

$$\mathbb{A}(v, B) = \int_B \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + \sum_{\{k,l\}\in\{\{x,y\},\{y,z\},\{z,x\}\}} \Big(\frac{\partial v_1}{\partial k}\frac{\partial v_2}{\partial l} - \frac{\partial v_1}{\partial l}\frac{\partial v_2}{\partial k}\Big)^2}\, dxdydz. \tag{4.9}$$

We extend the definition of area to non-smooth function in $L^1(B;\mathbb{R}^2)$ as in (1.6), and we aim to find a suitable upper bound of $\mathcal{A}(\overline{u}, B)$ following the same techniques in Proposition 2.8: Divide B into regions and define a sequence $\{\overline{u}^\varepsilon\} \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ on each of these regions such that u^ε converges to u strongly in $L^1(B;\mathbb{R}^2)$. First, divide the disk D into seven regions $E_i^\varepsilon,\ T^\varepsilon,\ S_{ij}^\varepsilon,\ ij \in \{12, 23, 31\}$ as in the proof of Proposition 2.8 then define

$$\overline{E}_i^\varepsilon := \{\pi_{xy}^{-1}(E_i^\varepsilon)\} \cap B, \quad \overline{S}_{ij}^\varepsilon := \{\pi_{xy}^{-1}(S_{ij}^\varepsilon)\} \cap B, \quad \overline{T}^\varepsilon := \{\pi_{xy}^{-1}(T^\varepsilon)\} \cap B,$$

where $ij \in \{12, 23, 31\}$, see Figure 4.6.
Set

$$\overline{u}^\varepsilon(x, y, z) := \alpha_i \qquad (x, y, z) \in \overline{E}_i^\varepsilon,\ i \in \{1, 2, 3\}, \tag{4.10}$$

$$\overline{u}^\varepsilon(x, y, z) := u^\varepsilon(x, y) \qquad (x, y, z) \in \overline{T}, \tag{4.11}$$

where u^ε is as in step 3 of the proof of Proposition 2.8.

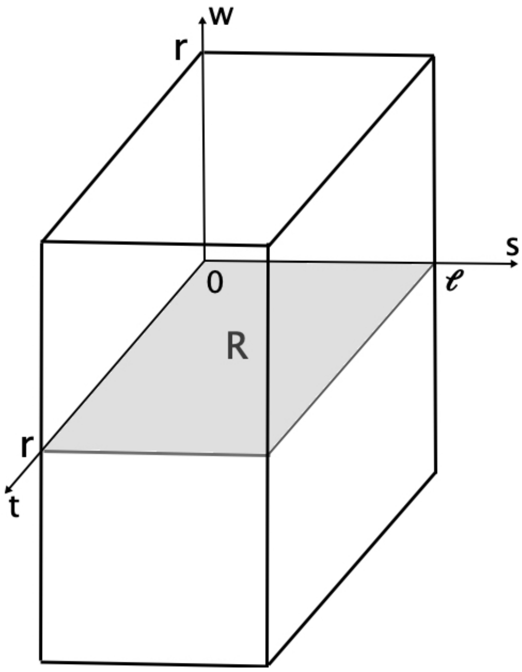


Figure 4.7: The rectangular cuboid $\overline{\mathrm{R}}$.

Now we deal with the challenging part, defining $\overline{u}^\varepsilon$ on $\overline{S}^\varepsilon_{ij}$, $ij \in \{12, 21, 31\}$. Let $\overline{\mathrm{R}}$ be the rectangular cuboid (parallelepiped) defined as

$$\overline{\mathrm{R}} := [0,\ell] \times [0,r] \times [-r,r] := \mathrm{R} \times [-r,r],$$

see Figure 4.7, and let

$$\widehat{\overline{\mathrm{R}}} := [0,\ell] \times [0,2r] \times [-r,r] =: \widehat{\mathrm{R}} \times [-r,r],$$

be the double rectangular cuboid.
Recall $\widetilde{\Gamma} = (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\Gamma}_3)$ is the connection defined at the beginning of Chapter 3; and $\widehat{m}$ is the solution of (2.9).
Set

$$\widehat{\overline{m}} = \widehat{\overline{m}}(\widetilde{\Gamma}) : \widehat{\overline{\mathrm{R}}} \to [0, +\infty], \qquad (s,t,w) \to \widehat{\overline{m}}(s,t,w),$$

be the unique solution[(5)] of the following Dirichlet minimum problem:

$$\min \left\{ \int_{\widehat{\overline{\mathrm{R}}}} \sqrt{1 + |\nabla f|^2}\, dsdtdw : \ f \in W^{1,1}(\widehat{\overline{\mathrm{R}}}),\ f = \overline{\varphi}\ \mathcal{H}^1 - a.e. \text{ on } \partial\widehat{\overline{\mathrm{R}}} \right\}, \tag{4.12}$$

where

$$\overline{\varphi}(s,t,w) := \begin{cases} 0, & \text{for } s = 0 \text{ and } s = \ell \\ \varphi(s), & \text{for } t = 0 \text{ and } t = 2r \\ \widehat{m}(s,t), & \text{for } w = -r \text{ and } w = r \end{cases},$$

(5) see Section 1.4.

and

$$\varphi := \varphi_{12}(\widetilde{\Gamma}_{12}) = \varphi_{23}(\widetilde{\Gamma}_{23}) = \varphi_{31}(\widetilde{\Gamma}_{31}), \qquad \widehat{m} := \widehat{m}_{12}(\widetilde{\Gamma}) = \widehat{m}_{23}(\widetilde{\Gamma}) = \widehat{m}_{31}(\widetilde{\Gamma}).$$

Notice that due to the special symmetry of the map $\overline{u}$, the three-problems collapse together to only one one-codimensional Plateau-type problem in cartesian form, on a fixed rectangular cuboid $\widehat{\overline{\mathrm{R}}}$.
Note also that $\widehat{\overline{m}}$ may not be smooth enough to construct the sequence $\{\overline{u}^\varepsilon\}$ and we need to regularize it first like we did in Lemma 2.7. However we will continue the computations assuming that $\widehat{\overline{m}}$ is Lipschitze up to the boundary of $\widehat{\overline{\mathrm{R}}}$.
Let $\overline{m}$ be the restriction of $\widehat{\overline{m}}$ to $\overline{\mathrm{R}}$, i.e. $\overline{m} : \overline{\mathrm{R}} \to [0, +\infty]$ is the unique solution of the Dirichlet-Neumann minimum problem:

$$\min\left\{\int_{\widehat{\overline{\mathrm{R}}}} \sqrt{1+|\nabla f|^2}\, dsdtdw : \ f \in W^{1,1}(\overline{\mathrm{R}}),\ f = \overline{\varphi}\ \mathcal{H}^1 - a.e. \text{ on } \partial_D\overline{\mathrm{R}}\right\} =: \overline{\mathfrak{A}}(\widetilde{\Gamma}), \tag{4.13}$$

where $\partial_D\overline{\mathrm{R}} := \partial\overline{\mathrm{R}} \setminus ([0,\ell] \times \{r\} \times [-r,r])$.
Recall

$$\overline{S}^\varepsilon_{12} = \{(x,y,z) \in B : \quad x \in (-\varepsilon,\varepsilon),\ y \in (\delta_\varepsilon, r),\ z \in (-r,r)\}.$$

We define $\overline{u}^\varepsilon$ on $\overline{S}^\varepsilon_{12}$ as follows

$$\overline{u}^\varepsilon(x,y,z) := \alpha_1 + \left(\frac{x+(\varepsilon/2)}{\varepsilon}\right)\ell\xi + \overline{m}\left(\frac{x+(\varepsilon/2)}{\varepsilon}\ell\ ,\ \psi^\varepsilon_{12}(y)\ ,\ z\right)\eta,$$

where ξ, η are as in (2.18) and $\psi^\varepsilon : [\delta_\varepsilon, r] \to [0,r]$ the linear, increasing, surjective map, hence for any $y \in [\delta_\varepsilon, r]$ we have $(\psi^\varepsilon)'(y) = \kappa_\varepsilon$; κ_ε is as defined in (2.19).
We have

$$\nabla\overline{u}^\varepsilon_1 = \left(\frac{\ell\xi^1}{\varepsilon} + \frac{\ell\eta^1}{\varepsilon}\overline{m}_s\ ,\ \kappa_\varepsilon\eta^1\overline{m}_t\ ,\ \eta^1\overline{m}_w\right),$$
$$\nabla\overline{u}^\varepsilon_2 = \left(\frac{\ell\xi^2}{\varepsilon} + \frac{\ell\eta^2}{\varepsilon}\overline{m}_s\ ,\ \kappa_\varepsilon\eta^2\overline{m}_t\ ,\ \eta^2\overline{m}_w\right),$$

where $\overline{m}_s, \overline{m}_t$, and $\overline{m}_w$ denote the partial derivatives of $\overline{m}$ with respect to s, t,and w respectively, and are evaluated at $\left(\frac{x+(\varepsilon/2)}{\varepsilon}\ell\ ,\ \psi^\varepsilon(y)\ ,\ z\right)$.
Hence

$$|\nabla\overline{u}^\varepsilon_1|^2 + |\nabla\overline{u}^\varepsilon_2|^2 \ = \ \frac{\ell^2}{\varepsilon^2} + \frac{\ell^2}{\varepsilon^2}(\overline{m}_s)^2 + (\kappa_\varepsilon)^2(\overline{m}_t)^2 + (\overline{m}_w)^2, \tag{4.14}$$

where we have used $|\xi| = |\eta| = 1$ and $\xi_1\eta_1 + \xi_2\eta_2 = 0$, moreover

$$\begin{aligned}
\left(\frac{\partial\overline{u}^\varepsilon_1}{\partial x}\frac{\partial\overline{u}_2}{\partial y} - \frac{\partial\overline{u}^\varepsilon_1}{\partial y}\frac{\partial\overline{u}^\varepsilon_2}{\partial x}\right)^2 &= \frac{\ell^2}{\varepsilon^2}(\kappa_\varepsilon)^2(\overline{m}_t)^2,\\
\left(\frac{\partial\overline{u}^\varepsilon_1}{\partial x}\frac{\partial\overline{u}_2}{\partial z} - \frac{\partial\overline{u}^\varepsilon_1}{\partial z}\frac{\partial\overline{u}^\varepsilon_2}{\partial x}\right)^2 &= \frac{\ell^2}{\varepsilon^2}(\overline{m}_w)^2,\\
\left(\frac{\partial\overline{u}^\varepsilon_1}{\partial y}\frac{\partial\overline{u}_2}{\partial z} - \frac{\partial\overline{u}^\varepsilon_1}{\partial z}\frac{\partial\overline{u}^\varepsilon_2}{\partial y}\right)^2 &= 0,
\end{aligned} \tag{4.15}$$

where again $\overline{m}_s, \overline{m}_t$, and $\overline{m}_w$ are evaluated at $\left(\frac{x+(\varepsilon/2)}{\varepsilon}\ell\ ,\ \psi^\varepsilon(y)\ ,\ z\right)$, and we have used $\xi^1\eta^2 - \xi^2\eta^1 = 1$. Therefore from (4.9), (4.14) and (4.15) we obtain

$$\begin{aligned}\mathbb{A}(\overline{u}^\varepsilon, \overline{S}^\varepsilon_{12}) = &\frac{\ell}{\varepsilon}\int_{\overline{S}^\varepsilon_{12}} \sqrt{1 + (\overline{m}_s)^2 + (\overline{m}_t)^2\,(\kappa_\varepsilon)^2\left(1+\frac{\varepsilon^2}{\ell^2}\right) + (\overline{m}_w)^2 + O(\varepsilon^2)}\ dxdydz\\ =&\frac{1}{\kappa_\varepsilon}\int_{\overline{\mathrm{R}}\setminus\overline{P}_\varepsilon} \sqrt{1 + (\overline{m}_s)^2 + (\overline{m}_t)^2\,(\kappa_\varepsilon)^2\left(1+\frac{\varepsilon^2}{\ell^2}\right) + (\overline{m}_w)^2 + O(\varepsilon^2)}\ \ dsdtdw,\end{aligned}$$

where $\overline{m}_s, \overline{m}_t$, and $\overline{m}_w$ in the first integral are evaluated at $\left(\frac{x+(\varepsilon/2)}{\varepsilon}\ell\ ,\ \psi^\varepsilon(y)\ ,\ z\right)$ and the last equality follows from the change of variables

$$\begin{aligned}\Phi : \overline{\mathrm{R}} \ni (s,t,w) \mapsto \Phi(s,t,w) :=& \left(\frac{\varepsilon}{\ell}s - \frac{\varepsilon}{2}, (\psi^\varepsilon)^{-1}(t), w\right)\\ =& (x,y,z) \in \left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]\times[\delta_\varepsilon, r]\times[-r,r] \supset \overline{S}^\varepsilon_{12},\end{aligned}$$

and $\overline{P}_\varepsilon := \overline{\mathrm{R}} \setminus \Phi^{-1}(\overline{S}^\varepsilon_{12})$. Hence, recalling also (2.19) and (4.13), we conclude

$$\lim_{\varepsilon\to 0^+} \mathcal{A}(\overline{u}^\varepsilon, \overline{S}^\varepsilon_{12}) = \int_{\overline{\mathrm{R}}_{12}} \sqrt{1 + \left(\overline{m}_s\right)^2 + \left(\overline{m}_t\right)^2 + \left(\overline{m}_w\right)^2}\ dsdt = \overline{\mathfrak{A}}(\widetilde{\Gamma}),$$

where $\overline{\mathfrak{A}}(\widetilde{\Gamma})$ is defined in (4.13). Employing the same construction in the strips $\overline{S}^\varepsilon_{23}$ and $\overline{S}^\varepsilon_{31}$ we obtain

$$\lim_{\varepsilon\to 0^+} \mathcal{A}(\overline{u}^\varepsilon, \overline{S}^\varepsilon_{23} \cup \overline{S}^\varepsilon_{31} \cup \overline{S}^\varepsilon_{12}) = 3\overline{\mathfrak{A}}(\widetilde{\Gamma}). \tag{4.16}$$

As a consequence of (4.9), (4.10), (4.11), (2.33), and (4.16) we have

$$\mathcal{A}(\overline{u}, B) \le \lim_{\varepsilon\to 0^+} \mathcal{A}(\overline{u}^\varepsilon, B) = |B| + 3\overline{\mathfrak{A}}(\widetilde{\Gamma}).$$

Remark 4.1. Putting $\widehat{m}$ as a boundary condition on the two sides $\widehat{\mathrm{R}} \times \{\pm r\}$ of $\widehat{\widehat{\mathrm{R}}}$ in (4.12) is questionable; on the other side putting a Neumann condition is another option. This is related to a solution of the following Dirichlet-Neumann problem:

$$\min\left\{\int_{\widehat{\widehat{\mathrm{R}}}} \sqrt{1+|\nabla f|^2}\ dsdt :\ f \in W^{1,1}(\widehat{\widehat{\mathrm{R}}}),\ f = \overline{\varphi}\ \mathcal{H}^1 - a.e.\text{ on } \partial_d\widehat{\widehat{\mathrm{R}}}\right\},$$

where

$$\partial_d\widehat{\widehat{\mathrm{R}}} := \widehat{\widehat{\mathrm{R}}} \setminus (\widehat{\mathrm{R}} \times \{\pm r\}),$$

and

$$\overline{\varphi}(s,t,w) := \begin{cases} 0, & \text{for } s = 0 \text{ and } s = \ell,\\ \varphi(s), & \text{for } t = 0 \text{ and } t = 2r.\end{cases}$$

which possibly may give better upper bound to $\mathcal{A}(\overline{u}, B)$.

4.4 Graphs of non-smooth maps in Riemannian manifolds

The motivation behind this section is an attempt to extend the results in Chapters 2 and 3 to maps from $\mathbb{S}^2$ to $\{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{S}^2$, with the final aim of hoping formally to understand better the expression of $\mathcal{A}$ in codimension two in an anisotropic setting. We replace the Euclidean metric in the source and target of the map $u : D \subset \mathbb{R}^2_S \to \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{R}^2_T$ with a Riemannian metric and we compute the contribution of the area on the cygar-shape region S^ε_{12} trying to guess what is the corresponding one-codimensional problem that may replace (0.7) in the former arguments (see the right hand side of (4.23)). We shall see that even in the simplest case (see (4.18) and (4.19)) the corresponding one-codimensional problem is not as expected: the space $\mathbb{R}^3$ where the graph of m lives is constructed by the product of $\mathbb{R}_x \subset \mathbb{R}^2_S$ and $\mathbb{R}^2_T$ which is not the metric we get in (4.24).
Assume that in the source we have $(\mathbb{R}^2_S, g_S)$ where g_S is the Riemannian metric defined as

$$g_S|_{(x,y)} = \begin{bmatrix} E_S(x,y) & F_S(x,y) \\ F_S(x,y) & G_S(x,y) \end{bmatrix}, \qquad (x,y) \in \mathbb{R}^2_S,$$

and in the target we have $(\mathbb{R}^2_T, g_T)$ where g_T is the Riemannian metric defined as

$$g_T|_{(u,v)} = \begin{bmatrix} E_T(u,v) & F_T(u,v) \\ F_T(u,v) & G_T(u,v) \end{bmatrix}, \qquad (u,v) \in \mathbb{R}^2_T.$$

Consider the product space $\mathbb{R}^2_S \times \mathbb{R}^2_T$ with the product metric

$$g|_{(x,y,u,v)} = \begin{bmatrix} g_S|_{(x,y)} & 0 \\ 0 & g_T|_{(u,v)} \end{bmatrix}, \qquad (x,y,u,v) \in \mathbb{R}^2_S \times \mathbb{R}^2_T.$$

Thus

$$|z|^2_g = |\pi_S z|^2_{g_S} + |\pi_T z|^2_{g_T}, \qquad z \in \mathbb{R}^2_S \times \mathbb{R}^2_T,$$

see [24, p.42].
Let $u \in \mathcal{C}^1(\Omega \subset \mathbb{R}^2_S; \mathbb{R}^2_T)$. The graph of u is a 2-dimension surface in the space $\mathbb{R}^2_S \times \mathbb{R}^2_T$ parametrized by

$$\Phi : \Omega \subset \mathbb{R}^2_E \to \mathbb{R}^2_S \times \mathbb{R}^2_T, \qquad \phi(x,y) = (x, y, u(x,y)),$$

where $\mathbb{R}^2_E$ is $\mathbb{R}^2$ with the Euclidean metric. Hence the area of the graph of u is given by

$$\mathbb{A}(u, \Omega) := \int_\Omega \sqrt{\det g^\phi_{ij}} dxdy,$$

where

$$g^\phi_{11}(p) := g(\Phi_x(p), \Phi_x(p)), \quad g^\phi_{12}(p) := g(\Phi_x(p), \Phi_y(p)), \quad g^\phi_{22}(p) := g(\Phi_y(p), \Phi_y(p)), \qquad p \in \Omega,$$

see [24, p.44].

Hence, we have

$$\begin{aligned}\det g_{ij}^{\phi} &= (E_S G_S - F_S^2) + G_S E_T (u_x^1)^2 + E_S E_T (u_y^1)^2 + G_S G_T (u_x^2)^2 + E_S G_T (u_y^2)^2 + 2F_S F_T u_x^1 u_x^2 \\ &\quad + 2E_S F_T u_y^1 u_y^2 + (E_S G_S - F_S^2)(u_x^1 u_y^2 - u_y^1 u_x^2)^2 - 2E_T F_S u_x^1 u_y^1 - 2G_T F_S u_x^2 u_y^2 - 2F_T F_S u_x^1 u_y^2 \\ &\quad - 2F_T F_S u_y^1 u_x^2 \\ &= (\det g_S) + G_S |u_x|_{g_T}^2 + E_S |u_y|_{g_T}^2 - 2F_S g_T(u_x, u_y) + (\det g_T)(u_x^1 u_y^2 - u_y^1 u_x^2)^2 \\ &= (\det g_S) + G_S |u_x|_{g_T}^2 + E_S |u_y|_{g_T}^2 - 2F_S g_T(u_x, u_y) + (\det g_T)(\det du)^2.\end{aligned}$$

Thus

$$\mathbb{A}(u,\Omega) = \int_{\Omega} \sqrt{\det g_S + G_S |u_x|_{g_T}^2 + E_S |u_y|_{g_T}^2 - 2F_S g_T(u_x, u_y) + \det g_T (\det du)^2}\, dxdy. \tag{4.17}$$

From now on for simplicity we assume the following

$$F_S = F_T = 0; \tag{4.18}$$

$$E_S, G_S, E_T, \text{ and } G_T \text{ are constant functions.} \tag{4.19}$$

Let $\Omega = D$ and let $S_{12}^{\varepsilon}, S_{23}^{\varepsilon}, S_{31}^{\varepsilon}$ and T^{ε} be as defined in the proof of Proposition 2.8. From (4.17), the area of the graph of $u \in \mathrm{Lip}(S_{ij}^{\varepsilon}; \mathbb{R}_T^2)$ is given by

$$\mathbb{A}(u, S_{ij}^{\varepsilon}) = \int_{S_{ij}^{\varepsilon}} \sqrt{\det g_S + G_S |u_x|_{g_T}^2 + E_S |u_y|_{g_T}^2 + \det g_T (\det du)^2}\, dxdy.$$

Let

$$\ell_{12} := |\alpha_2 - \alpha_1|_{g_T}, \qquad \xi = (\xi^1, \xi^2) := \frac{\alpha_2 - \alpha_1}{\ell_{12}},$$

and

$$\eta := (\eta^1, \eta^2) := \sqrt{\det g_T}\Big(\frac{-\xi^2}{E_T}, \frac{\xi^1}{G_T}\Big) = \sqrt{E_T G_T}\Big(\frac{-\xi^2}{E_T}, \frac{\xi^1}{G_T}\Big).$$

Thus we have

$$|\xi|_{g_T} = |\eta|_{g_T} = 1, \qquad g_T(\xi, \eta) = 0, \tag{4.20}$$

$$\xi_1 \eta^2 - \xi^2 \eta^1 = \frac{1}{\sqrt{\det g_T}} = \frac{1}{\sqrt{E_T G_T}}. \tag{4.21}$$

Let

$$\mathrm{R}_{12} := [0,1] \times [0, r_{12}],$$

and

$$m_{12} \in \mathcal{C}^1(\mathrm{R}_{12} \subset \mathbb{R}^2; \mathbb{R}); \qquad m_{12}(0,t) = m_{12}(1,t) = 0.$$

Define u^{ε} on S_{12}^{ε} as follows

$$u^{\varepsilon}(x,y) := \alpha_1 + \left(\frac{x - \zeta_1^1}{\varepsilon_{12}}\right) \ell_{12} \xi + m_{12}\left(\frac{x - \zeta_1^1}{\varepsilon_{12}}\ ,\ \psi_{12}^{\varepsilon}(y)\right) \eta, \qquad (x,y) \in S_{12}^{\varepsilon}, \tag{4.22}$$

where ψ_{12}^{ε} and ζ^1 are as defined in the proof of Proposition 2.8.

Write for simplicity

$$\widetilde{m} = m_{12}.$$

Hence

$$u^\varepsilon_x = \frac{1}{\varepsilon_{12}}\left(\ell_{12}\xi^1 + \widetilde{m}_s\eta^1\ ,\ \ell_{12}\xi^2 + \widetilde{m}_s\eta^2\right), \qquad u^\varepsilon_y = \left(\overline{\kappa}_\varepsilon\widetilde{m}_t\eta^1\ ,\ \overline{\kappa}_\varepsilon\widetilde{m}_t\eta^2\right),$$

where $\widetilde{m}_s, \widetilde{m}_t$ denote, respectively, the partial derivatives of $\widetilde{m}$ with respect to $s := \frac{x-\zeta^1_1}{\varepsilon_{12}}$ and $t := \psi^\varepsilon_{12}(y)$, and are evaluated at $\left(\frac{x-\zeta^1_1}{\varepsilon_{12}}\ ,\ \psi^\varepsilon_{12}(y)\right)$.
Hence we have

$$\begin{aligned}
|u^\varepsilon_x|^2_{g_T} &= (\frac{\ell_{12}}{\varepsilon_{12}})^2\left[|\xi|^2_{g_T} + (\frac{1}{\ell_{12}})^2|\eta|^2_{g_T}\widetilde{m}^2_s + 2\frac{1}{\ell_{12}}g_T(\xi,\eta)\widetilde{m}_s\right],\\
|u^\varepsilon_y|^2_{g_T} &= \overline{\kappa}^2_\varepsilon\widetilde{m}^2_t|\eta|^2_{g_T},\\
\det du^\varepsilon &= \frac{\ell_{12}}{\varepsilon_{12}}\overline{\kappa}_\varepsilon\widetilde{m}_t(\xi^1\eta^2 - \xi^2\eta^2).
\end{aligned}$$

Thus, from (4.20), we have

$$\begin{aligned}
|u^\varepsilon_x|^2_{g_T} &= (\frac{\ell_{12}}{\varepsilon_{12}})^2\left[1 + (\frac{1}{\ell_{12}})^2\widetilde{m}^2_s\right], \qquad |u^\varepsilon_y|^2_{g_T} = \overline{\kappa}^2_\varepsilon\widetilde{m}^2_t,\\
\det du^\varepsilon &= \frac{\ell_{12}}{\varepsilon_{12}\sqrt{\det g_T}}\overline{\kappa}_\varepsilon\widetilde{m}_t.
\end{aligned}$$

As a consequence

$$\begin{aligned}
\mathcal{A}(u^\varepsilon, S^\varepsilon_{12}) \ &= \ \frac{\ell_{12}}{\varepsilon_{12}}\int_{S^\varepsilon_{12}}\left[G_S + (\frac{1}{\ell_{12}})^2 G_S\left[\widetilde{m}_s\left(\frac{x-\zeta^1_1}{\varepsilon_{12}},\ \psi^\varepsilon_{12}(y)\right)\right]^2\right.\\
&\qquad \left. + E_S\kappa^2_\varepsilon\left(1 + \frac{\varepsilon^2_{12}}{\ell^2_{12}}\right)\left[\widetilde{m}_t\left(\frac{x-\zeta^1_1}{\varepsilon_{12}},\ \psi^\varepsilon_{12}(y)\right)\right]^2 + O(\varepsilon^2)\right]^{\frac{1}{2}} dxdy\\
&= \ \frac{\ell_{12}}{\kappa_\varepsilon}\int_{\mathrm{R}_{12}\setminus P_\varepsilon}\sqrt{G_S + (\frac{1}{\ell_{12}})^2 G_S\left[\widetilde{m}_s\,(s,t)\right]^2 + E_S\kappa^2_\varepsilon\left(1 + \frac{\varepsilon^2_{12}}{\ell^2_{12}}\right)\left[\widetilde{m}_t\,(s,t)\right]^2 + O(\varepsilon^2)}\ dsdt,
\end{aligned}$$

where the last equality follows by the change of variables Φ defined in (2.23), and $P_\varepsilon := \mathrm{R}_{12} \setminus \Phi^{-1}(S^\varepsilon_{12})$.
Hence, recalling also that $\overline{\kappa}_\varepsilon \to 1$ as $\varepsilon \to 0$, we conclude

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\mathcal{A}(u^\varepsilon, S^\varepsilon_{12}) &= \int_{\mathrm{R}_{12}}\ell_{12}\sqrt{G_S + (\frac{1}{\ell_{12}})^2 G_S\big(\widetilde{m}_s\big)^2 + E_S\big(\widetilde{m}_t\big)^2}\ dsdt\\
&= \int_{\mathrm{R}_{12}}\sqrt{\ell^2_{12}G_S + G_S\big(\widetilde{m}_s\big)^2 + \ell^2_{12}E_S\big(\widetilde{m}_t\big)^2}\ dsdt. \qquad (4.23)
\end{aligned}$$

We are lead therefore to believe that the graph of $\widetilde{m}$ lives in $\mathbb{R}^3$ with the metric

$$\begin{bmatrix} \ell^2_{12}\sqrt{E_S} & 0 & 0\\ 0 & \frac{G_S}{\sqrt{E_S}} & 0\\ 0 & 0 & \sqrt{E_S}\end{bmatrix}. \qquad (4.24)$$

5. Characterization of manifolds with boundary

The discussion in Section 1.6.1 suggests to introduce the following class of sets (which we shall consider as the class of h-dimensional embedded $\mathcal{C}^k$-manifolds without boundary in the sense of distance functions):

Definition 5.1 (The class $D_h\mathcal{C}^k(\Omega)$). *Let $k \in \mathbb{N}$, $k \geq 2$, or $k \in \{\infty, \omega\}$ and $h \in \{0, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, $E \subset \mathbb{R}^n$. We write $E \in D_h\mathcal{C}^k(\Omega)$ if*

(i) $E \cap \Omega = \{x \in \Omega : \eta_E(x) = 0\}$;

(ii) there exists an open set $A \subseteq \Omega$ with $E \cap \Omega \subseteq A$ such that $\eta_E \in \mathcal{C}^k(A)$;

(iii) $\mathrm{rank}(\nabla^2 \eta_E(x)) = n - h$ *for any $x \in E \cap \Omega$.*

Remark 5.2. (I) If $E \in D_h\mathcal{C}^k(\Omega)$, $k \geq 3$ or $k \in \{\infty, \omega\}$ then E is closed in Ω and $E \cap \Omega$ is a $h-$dimensional embedded manifold of class $\mathcal{C}^{k-1}(\Omega)$ without boundary in Ω. Conversely, if Γ is a $h-$dimensional embedded manifold of class $\mathcal{C}^k(\Omega)$, $k \geq 3$ or $k \in \{\infty, \omega\}$, without boundary in Ω then $\Gamma \in D_h\mathcal{C}^{k-1}(\Omega)$.

(II) $E = \emptyset \in D_h\mathcal{C}^k(\Omega)$ for any h, k and any open set Ω.

(III) If $\overline{E} = E \subseteq \Omega$ then $E \in D_h\mathcal{C}^k(\Omega)$ implies $E \in D_h\mathcal{C}^k(\Omega')$ for any open set $\Omega' \supset \Omega$.

5.1 Manifolds with boundary and distance functions

We start this section by defining what we mean by an embedded h-dimensional $\mathcal{C}^k$-manifold in an open set with boundary in the sense of distance functions. But first we recall the classical definition (see for instance [38, 19]).

Definition 5.3 (**Smooth embedded manifold with boundary**). *Let $k \in \mathbb{N} \cup \{\infty, \omega\}$ and $h \in \{1, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. We say that $\mathcal{M} \subseteq \mathbb{R}^n$ is a h-dimensional embedded manifold of class $\mathcal{C}^k$ with boundary of class $\mathcal{C}^k$ in Ω (a h-dimensional $\mathcal{C}^k$-manifold in Ω with boundary, for short) if*

$$\mathcal{M} \cap \Omega = \overline{\mathcal{M}} \cap \Omega$$

and for all $x \in \mathcal{M} \cap \Omega$ there exist an open set $R \subseteq \mathbb{R}^n$, an open set $G \subseteq \mathbb{R}^h$, maps $\phi \in \mathcal{C}^k(G; \mathbb{R}^n)$, $\psi \in \mathcal{C}^k(R; \mathbb{R}^h)$ and a point $z \in \mathbb{R}^h$ such that

$$x \in R, \qquad \psi(\phi(y)) = y \quad \forall y \in G,$$

$$\mathcal{M} \cap R = \{\phi(y) : y \in G, \langle y, z \rangle \geq 0\} . \tag{5.1}$$

The boundary of $\mathcal{M}$ in Ω, denoted

$$\partial_\Omega \mathcal{M} \quad (\partial \mathcal{M} \text{ when } \Omega = \mathbb{R}^n),$$

is the set of all points $\overline{x} \in \mathcal{M} \cap \Omega$ such that

$$\overline{x} = \phi(\overline{y}), \qquad \overline{y} \in G, \ \ \langle \overline{y}, z \rangle = 0.$$

We denote by $\mathcal{M}^\circ$ the (relative) interior of $\mathcal{M}$ defined as $\mathcal{M} \setminus \partial_\Omega \mathcal{M}$ and by $T_x\mathcal{M}$ (resp. $N_x\mathcal{M}$) the tangent space (resp. the normal space) to $\mathcal{M}$ at $x \in \mathcal{M}$.
Our main definition of smooth manifold with boundary using the distance functions reads as follows.

Definition 5.4 (**The class $D_hB\mathcal{C}^k(\Omega)$**). *Let $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$ and $h \in \{1, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, and $E, L \subseteq \mathbb{R}^n$. We write $(E, L) \in D_hB\mathcal{C}^k(\Omega)$ if:*

(i) $L \in D_{h-1}\mathcal{C}^k(\Omega)$ *and* $E \in D_h\mathcal{C}^k(\Omega \setminus L)$;

(ii) $d_{(E \setminus L)\cap\Omega}(x) \leq d_{L\cap\Omega}(x)$ *for any* $x \in \mathbb{R}^n$;

(iii) if we define

$$B := \{x \in \Omega : d_{(E \setminus L)\cap\Omega}(x) < d_{L\cap\Omega}(x)\}, \tag{5.2}$$

then there exists an open set $A \subseteq \Omega$ with $L \cap \Omega \subseteq A$ such that $d_B \in \mathcal{C}^k(A)$;

(iv) we have[1] $\eta_E \in \mathcal{C}^k\big(\{x \in A : d_B(x) \leq 0\}\big)$.

Since Definition 5.4 is crucial, some comments are in order. Informally the set $L \cap \Omega$ should be considered as the "boundary" of $E \cup L$ in Ω, and by condition (i) it must satisfy Definition 5.1, with $h - 1$ in place of h, while E must satisfy Definition 5.1 *not in the whole of* Ω, *but only in the open set* $\Omega \setminus L$ (remember that L is closed in Ω by condition (i) in Definition 5.1), see Figure 5.1 for an elementary example.

(1) If $C \subset \mathbb{R}^n$, we say that $f \in \mathcal{C}^k(C)$ if there exist an open set $\widehat{C} \supset C$ and a function $\widehat{f} \in \mathcal{C}^k(\widehat{C})$ such that $\widehat{f} = f$ on C.

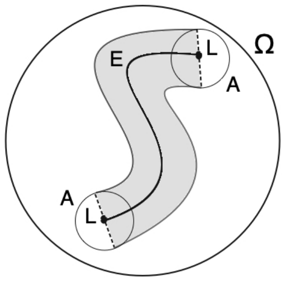


Figure 5.1: $\Omega = B_1(0)$ is an open disk in $\mathbb{R}^2$, $k = \infty$, $h = 1$, E is the bold curve (including the two endpoints), L consists of the two end points of E; $d_{(E \setminus L) \cap \Omega}(\cdot) = \operatorname{dist}(\cdot, (E \setminus L) \cap \Omega) = \operatorname{dist}(\cdot, E)$, $d_{L \cap \Omega}(\cdot) = \operatorname{dist}(\cdot, L \cap \Omega)$, B contains the shaded region, A is the union of the two small disks containing L. In Section 5.2 it will also useful to consider $H := A \cap \partial B$, which in this case consists of two dashed segments containing L and normal to E. Finally, $\{x \in A : d_B(x) \leq 0\}$ consists of the grey areas inside the two disks, including the dashed segments.

To understand condition (iii), which is a regularity requirement on ∂B, we refer to Examples 5.7 and 5.10.

Condition (iv) says that $E \cap \Omega$ is smooth up to $L \cap \Omega$. Note carefully that, in general, η_E is not of class $\mathcal{C}^k$ in an open neighborhood of $E \cup L$. For instance, if $n = 1 = h,\ E = [-1, 1] \subset \mathbb{R},\ L = \{\pm 1\}$ then $\eta_E \in \mathcal{C}^{1,1}$ but not $\mathcal{C}^2$ in a neighborhood of L.

Note that $(E, \emptyset) \in D_h B\mathcal{C}^k(\Omega)$ if and only if $E \in D_h \mathcal{C}^k(\Omega)$. Moreover if $\overline{E} = E,\ \overline{L} = L$, and $E \cup L \subseteq \Omega$ then $(E, L) \in D_h B\mathcal{C}^k(\Omega)$ implies $(E, L) \in D_h B\mathcal{C}^k(\Omega')$ for every open set $\Omega' \supseteq \Omega$.

Remark 5.5. Suppose $k \in \mathbb{N}$, $k \geq 3$ or $k \in \{\infty, \omega\}$ and $(E, L) \in D_h B\mathcal{C}^k(\Omega)$.

(I) By Definition 5.4 (i) we have $L \in D_{h-1}\mathcal{C}^k(\Omega)$ hence, recalling Remark 5.2, we have that L is an embedded $(h-1)$-dimensional $\mathcal{C}^{k-1}$-manifold without boundary in Ω. Also, since $E \in D_h\mathcal{C}^k(\Omega \setminus L)$, E is an embedded h-dimensional $\mathcal{C}^{k-1}$-manifold without boundary in $\Omega \setminus L$.

(II) In Definition 5.4, we do not specify whether or not points of L belong to E. However, condition (ii) says that (if L is nonempty) all points of $L \cap \Omega$ are accumulation points of $(E \setminus L) \cap \Omega$. Indeed, if $x \in L \cap \Omega$ then $d_{L \cap \Omega}(x) = 0$, hence (ii) implies

$$d_{(E \setminus L) \cap \Omega}(x) \leq 0,$$

and so $x \in \overline{(E \setminus L) \cap \Omega}$.

(III) We have

$$\overline{(E \cup L)} \cap \Omega = (E \cup L) \cap \Omega.$$

Indeed Definition 5.4(i) implies

$$\overline{L} \cap \Omega = L \cap \Omega \quad \text{and} \quad \overline{E} \cap (\Omega \setminus L) = E \cap (\Omega \setminus L).$$

Take $x \in \overline{E \cup L} \cap \Omega$. If $x \in \overline{L} \cap \Omega$ then $x \in L \cap \Omega \subseteq (E \cup L) \cap \Omega$. If $x \notin \overline{L} \cap \Omega$, then

$$x \in \overline{E} \cap (\Omega \setminus \overline{L}) \subseteq \overline{E} \cap (\Omega \setminus L) = E \cap (\Omega \setminus L) \subseteq (E \cup L) \cap \Omega.$$

(IV) We have

$$(E \cup L) \cap \Omega = \{x \in \Omega : \eta_E(x) = 0\},$$

i.e.,

$$\overline{E} \cap \Omega = (E \cup L) \cap \Omega. \tag{5.3}$$

Indeed, from (II) it follows $(E \cup L) \cap \Omega \subseteq \overline{E} \cap \Omega$. Now take $x \in (\overline{E} \setminus E) \cap \Omega$, and select a sequence $(x_j) \subseteq E \cap \Omega$ with $x_j \to x$. But $x_j \in (E \cup L) \cap \Omega$ which is closed in Ω by (III). Therefore $x \in (E \cup L) \cap \Omega$.

(V) Recalling (5.2), we have

$$(E \setminus L) \cap \Omega \subseteq B. \tag{5.4}$$

Indeed let $x \in (E \setminus L) \cap \Omega$ so that $d_{(E \setminus L) \cap \Omega}(x) \leq 0$. Since L is closed in Ω we have $\text{dist}(x, L \cap \Omega) > 0$ and therefore $d_{(E \setminus L) \cap \Omega}(x) < \text{dist}(x, L \cap \Omega) = d_{L \cap \Omega}(x)$.

(VI) We have

$$L \cap \Omega \subset \text{topological boundary of } B. \tag{5.5}$$

Let $x \in L \cap \Omega$; from (II), $x \in \overline{(E \setminus L) \cap \Omega}$, hence $x \in \overline{B}$ from (5.4). Since B is open, it remains to show that $x \notin B$, *i.e.*, that $d_{(E \setminus L) \cap \Omega}(x) = d_{L \cap \Omega}(x)$. Since $\dim(L \cap \Omega) < n$, $d_{L \cap \Omega}(\cdot) = \text{dist}(\cdot, L \cap \Omega)$. By Definition 5.4(ii), $d_{(E \setminus L) \cap \Omega}(x) \leq d_{L \cap \Omega}(x) = \text{dist}(x, L \cap \Omega) = 0$ and since $x \notin (E \setminus L)$ we have $d_{(E \setminus L) \cap \Omega}(x) = \text{dist}(x, (E \setminus L) \cap \Omega) \geq 0$. Thus $d_{(E \setminus L) \cap \Omega}(x) = \text{dist}(x, L \cap \Omega) = 0$. Notice that from (5.5) it follows

$$L \cap \Omega \subseteq \{x \in A : d_B(x) \leq 0\}.$$

(VII) In a neighborhood of $L \cap \Omega$, the topological boundary of B is an embedded hypersurface of class $\mathcal{C}^{k-1}$. Indeed since B is an open set and there exists an open set $A \supset L \cap \Omega$ such that $d_B \in \mathcal{C}^k(A)$, it follows from Theorem 1.27, that in A the topological boundary of B is a $\mathcal{C}^{k-1}$ hypersurface. Consistently with our notation in Definition 5.3, we indicate by $\partial_A B$ the boundary of B in A.

(VIII) For $h = n$ we have

$$B = (E \setminus L) \cap \Omega. \tag{5.6}$$

The inclusion $(E \setminus L) \cap \Omega \subseteq B$ is in (5.4). To show the converse inclusion we argue by contradiction. Assume that $B \not\subseteq (E \setminus L) \cap \Omega$. From (I) we know that $(E \setminus L) \cap \Omega$ is an open set and $L \cap \Omega$ is a hypersurface, moreover $L \cap \Omega$ is the topological boundary of $(E \setminus L)$ in Ω from (5.3). Hence $\overline{(E \setminus L) \cap \Omega} \cap B \neq \emptyset$; it follows $L \cap B \neq \emptyset$ which contradicts (5.5).

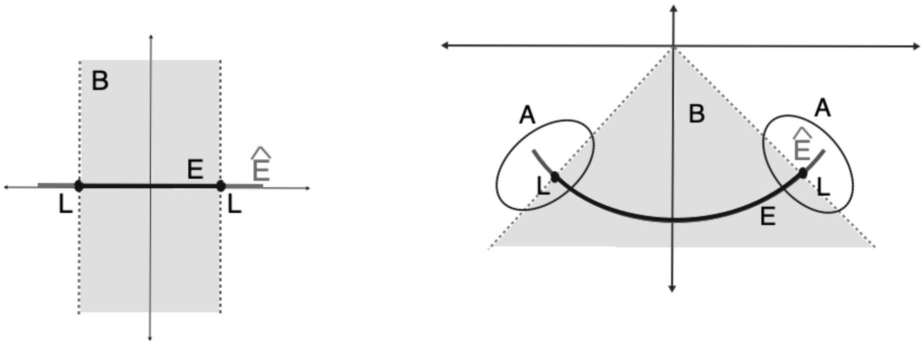


Figure 5.2: Left: E is a segment in $\mathbb{R}^2$. Right: E is an arc of a circle in $\mathbb{R}^2$ and L its two end points.

Example 5.6. We start from the simplest nontrivial case (Figure 5.2, left): we take $n = 2$, $h = 1$, $k \in \{\infty, \omega\}$, $\Omega = \mathbb{R}^2$, $L = \{(\pm 1, 0)\}$, and $E = (-1, 1) \times \{0\}$ ($E = (-1, 1] \times \{0\}$ or $E = [-1, 1) \times \{0\}$ or $E = [-1, 1] \times \{0\}$ would not affect the discussion). In this case it is immediate to verify that the set B in condition (iii) equals $B = (-1, 1) \times \mathbb{R}$; the largest A fulfilling condition (iii) can be taken to be $A = \mathbb{R}^2 \setminus \{x_1 = 0\}$, and $\{(x_1, x_2) \in A : d_B((x_1, x_2)) \leq 0\} = ([-1, 1] \times \mathbb{R}) \setminus \{x_1 = 0\}$. Finally, in order to fulfill (iv), it is sufficient to take $\widehat{\eta}_E = \eta_{\widehat{E}}$, where $\widehat{E} = (-1 - \delta, 1 + \delta) \times \{0\}$, for any $\delta > 0$ so that $\eta_{\widehat{E}} \in \mathcal{C}^k([-1, 1] \times \mathbb{R})$. Note that η_E is not even $\mathcal{C}^2$ on $\{x = \pm 1\}$.
If we choose L to be only one point of the two points $\{(\pm 1, 0)\}$, say $L = \{(1, 0)\}$, then $E = (-1, 1) \times \{0\}$ is no longer closed in $\Omega \setminus L$ hence it does not belong to $D_1\mathcal{C}^k(\Omega \setminus L)$. On the other hand $E = (-1, 1] \times \{0\}$ is closed in $\Omega \setminus L$ but condition (ii) of Definition 5.1 (with Ω replaced by $\Omega \setminus L$) is not satisfied, hence E does not belong to $D_1\mathcal{C}^k(\Omega \setminus L)$.

Example 5.7. Take $n = 2$, $h = 1$, $k \in \{\infty, \omega\}$, $E = (\cos\theta, \sin\theta), \theta \in (\frac{5\pi}{4}, \frac{7\pi}{4})$, $\Omega = \mathbb{R}^2$, and $L = \{(\frac{\pm 1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$, see Figure 5.2. We have $B \cap (\mathbb{R} \times (-\infty, 0)) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < |x_2|, x_2 < 0\}$. A can be taken to be any open subset of $\mathbb{R} \times (-\infty, 0)$ containing L that does not contain the origin. Finally, taking $\widehat{\eta}_E = \eta_{\widehat{E}}$ where $\widehat{E} = (\cos\theta, \sin\theta)$, $\theta \in (\frac{5\pi}{4} - \delta, \frac{7\pi}{4} + \delta)$, $\delta < \frac{\pi}{4}$, condition (iv) is fulfilled. Note that ∂B is smooth close to L, but not necessarily far from L (for instance at the origin).

Example 5.8. Take $n = h \geq 1$, $k \in \{\infty, \omega\}$, $E = \overline{B_1(0)}$, $\Omega = \mathbb{R}^n$, and $L = \partial B_1(0)$. Note that $\eta_E \in \mathcal{C}^1(B_{1+\epsilon}(0)) \setminus \mathcal{C}^2(B_{1+\epsilon}(0))$ for any $\epsilon > 0$. $\overline{B_1(0)} \in D_n\mathcal{C}^k(\mathbb{R}^n \setminus \partial B_1(0))$: indeed E is closed in $\mathbb{R}^n \setminus \partial B_1(0)$, $\eta_E|_E = 0$ thus $\eta_E \in \mathcal{C}^k(E \setminus L)$ and $\text{rank}(\nabla^2 \eta_E(x)) = 0$ for all $x \in E \setminus L$. Moreover $L \in D_{n-1}C^k(\mathbb{R}^n)$ from Remark 5.2 (I). Hence condition (i) is fulfilled; condition (ii) is immediate and we also have $B = B_1(0)$ and $A = \mathbb{R}^n \setminus \{0\}$. Finally, $\widehat{\eta}_E = 0$ in $\mathbb{R}^n$ allows to check condition (iv).

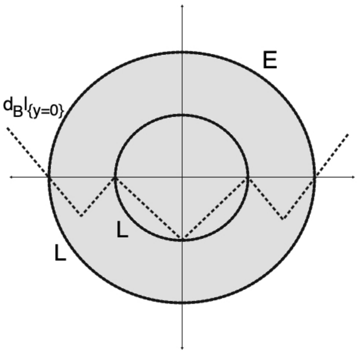


Figure 5.3: Example 5.10: L is the union of the two bold circles, one being included in the larger open disk, B is the grey region. Dashed segments: graph of the signed distance function d_B along $\{y = 0\}$.

Example 5.9. Take $n = 2$, $h = 1$, $k \in \{\infty, \omega\}$, $E = \mathbb{S}^1$ the unit circle centered at the origin, $\Omega = \mathbb{R}^2$, and $L = \emptyset$. Then condition (i) is immediate. Notice that $d_L \equiv +\infty$ hence $B = \mathbb{R}^2$, $d_B \equiv -\infty$, and $A = \emptyset$ so that also condition (iv) is trivially satisfied.

Example 5.10. Take $n = h = 2$, $E = \overline{B_2(0)}$, $\Omega = \mathbb{R}^2$, and $L = \partial B_1(0) \cup \partial B_2(0)$. Then (i) and (ii) of Definition 5.4 are fulfilled. $B = B_2(0) \setminus L$, moreover there is no $A \supset \partial B_1(0)$ such that $d_B \in \mathcal{C}^1(A)$ hence (iii) is not satisfied (note that $\eta_E = 0$ in $\overline{B_2(0)}$, *i.e.,* fulfilling (iv) also depends on the existence of A), see Figure 5.3.

5.2 Smooth manifolds with boundary are in $D_h B\mathcal{C}^k(\Omega)$

In this section we show that smooth manifolds with boundary in the classical sense (Definition 5.3) are smooth manifolds with boundary in the sense of distance functions (Definition 5.4), more precisely:

Theorem 5.11. *Let $k \in \mathbb{N}$, $k \geq 3$, or $k \in \{\infty, \omega\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and $\mathcal{M} \subset \mathbb{R}^n$ be an embedded $\mathcal{C}^k$-manifold of dimension $h \leq n$ with nonempty boundary in Ω. Then $(\mathcal{M}, \partial_\Omega \mathcal{M}) \in D_h B\mathcal{C}^{k-1}(\Omega)$.*

First we need the following result.

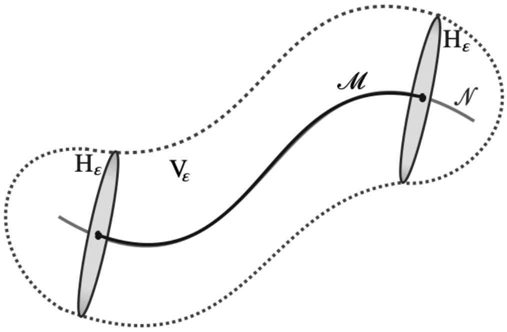


Figure 5.4: $\mathcal{M}$ is a curve (smooth up to the boundary) embedded in $\mathbb{R}^3$, $\mathcal{N}$ is a smooth extension of $\mathcal{M}$, and H_ϵ consists of two open disks normal to $\mathcal{M}$ at the endpoints (the boundary of $\mathcal{M}$).

Proposition 5.12. *Let $k \in \mathbb{N}$, $k \geq 2$, or $k \in \{\infty, \omega\}$, and $h \in \{1, \dots, n\}$. Let $\mathcal{M} \subset \mathbb{R}^n$ be a compact embedded $\mathcal{C}^k$-manifold of dimension h with nonempty boundary in $\mathbb{R}^n$. Then there exists $\epsilon > 0$ such that, setting*

$$H_\epsilon := \bigcup_{x \in \partial\mathcal{M}} B_\epsilon(x) \cap N_x\mathcal{M}, \tag{5.7}$$

the following properties hold:

1) $\partial\mathcal{M} \subseteq H_\epsilon \subseteq \bigcup_{x \in \partial\mathcal{M}} N_x\mathcal{M}$;

2) H_ε is an embedded $\mathcal{C}^{k-1}$-hypersurface without boundary in $\mathcal{M}_\epsilon^+$, and $N_x H_\epsilon \subseteq T_x\mathcal{M}$ for any $x \in \partial\mathcal{M}$;

3) $\eta_\mathcal{M} \in \mathcal{C}^{k-1}(\mathcal{M}_\epsilon^+ \setminus H_\epsilon)$.

Proof. Since we can work separately on each connected component of $\mathcal{M}$, from now on we suppose that $\mathcal{M}$ is connected. Suppose first $h = n$. In this case the interior of $\mathcal{M}$ is a nonempty open set with $\mathcal{C}^k$-boundary, and [30, 29, 22] if $\varepsilon > 0$ is sufficiently small, $d_\mathcal{M}$ is of class $\mathcal{C}^k$ in the tubular neighborhood $(\partial\mathcal{M})_\epsilon^+$ of $\partial\mathcal{M}$. Define $H_\epsilon := \partial\mathcal{M}$. Then 1) holds (with the equalities in place of the inequalities), and also 2) holds because $T_x\mathcal{M} = \mathbb{R}^n$ for any $x \in \partial\mathcal{M}$. Moreover $\mathrm{dist}(\cdot, \mathcal{M}) \in \mathcal{C}^k(\mathcal{M}_\epsilon^+ \setminus H_\epsilon)$, since $\mathrm{dist}(\cdot, \mathcal{M}) = 0$ in the interior of $\mathcal{M}$ and $\mathrm{dist}(\cdot, \mathcal{M}) = d_\mathcal{M}(\cdot)$ in $\mathcal{M}_\epsilon^+ \setminus \mathcal{M}$, hence also 3) follows.
Now suppose $h \in \{1, \dots, n-1\}$. We divide the proof into 3 steps.

Step 1. There exists $\epsilon_1 > 0$ such that

$$\eta_\mathcal{M} \in \mathcal{C}^{k-1}(V_{\epsilon_1}), \tag{5.8}$$

where V_{ϵ_1} is the neighborhood of the relative interior $\mathcal{M}^\circ$ of $\mathcal{M}$ defined as

$$V_{\epsilon_1} := \bigcup_{x\in\mathcal{M}^\circ} B_{\epsilon_1}(x) \cap N_x\mathcal{M}^\circ. \tag{5.9}$$

The initial part of the proof of this step is rather standard, see for instance [2]. Take any $x_\circ \in \mathcal{M}^\circ$. Since $\mathcal{M}$ is a smooth manifold embedded in $\mathbb{R}^n$, there exists $\rho = \rho(x_\circ) > 0$ such that

$$B_\rho(x_\circ) \cap \mathcal{M} = B_\rho(x_\circ) \cap \mathcal{M}^\circ, \tag{5.10}$$

and there are smooth orthonormal vector fields $\nu^1(x), \ldots, \nu^{n-h}(x)$ spanning $N_x\mathcal{M}^\circ$ for any $x \in B_\rho(x_\circ) \cap \mathcal{M}^\circ$. Consider the function

$$\widetilde{\Phi} = \widetilde{\Phi}_{\mathcal{M}^\circ} : (B_\rho(x_\circ) \cap \mathcal{M}^\circ) \times \mathbb{R}^{n-h} \longrightarrow \mathbb{R}^n, \qquad \widetilde{\Phi}(x,\alpha) := x + \sum_{i=1}^{n-h} \alpha_i \nu^i(x), \tag{5.11}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-h}) \in \mathbb{R}^{n-h}$. Let $G \subset \mathbb{R}^h$ be an open set and $f : G \to B_\rho(x_\circ) \cap \mathcal{M}^\circ$ be a local parametrization of $\mathcal{M}^\circ$ with $f(y_\circ) = x_\circ$, $y_\circ \in G$. Then $\widetilde{\Phi}$ in local coordinates becomes

$$\Phi : G \times \mathbb{R}^{n-h} \to \mathbb{R}^n, \quad \Phi(y,\alpha) := \widetilde{\Phi}(f(y),\alpha) = f(y) + \sum_{i=1}^{n-h} \alpha_i \nu^i(f(y)).$$

Clearly Φ is $\mathcal{C}^{k-1}$ and therefore $d\Phi_{(y_\circ,0)}$ is represented by a matrix with columns

$$f_{y_1}(y_\circ),\ f_{y_2}(y_\circ),\ \ldots,\ f_{y_h}(y_\circ),\ \nu^1(f(y_\circ)),\ \nu^2(f(y_\circ)),\ \ldots,\ \nu^{n-h}(f(y_\circ)),$$

where $y = (y_1, \ldots, y_h)$ and $f_{y_i} = \frac{\partial}{\partial y_i}$. Since $\operatorname{span}\{f_{y_1}(y_\circ), \ldots, f_{y_h}(y_\circ)\} = T_{x_\circ}\mathcal{M}$, the columns of $d\Phi_{(y_\circ,0)}$ are linearly independent. Hence, by the implicit function theorem, Φ is locally invertible with inverse of class $\mathcal{C}^{k-1}$. Let

$$O := (B_{r_\circ}(x_\circ) \cap \mathcal{M}^\circ) \times B^{n-h}_{r_\circ}(0),$$

where $0 < r_\circ = r(x_\circ) \le \rho$ is so that the implicit function theorem holds, and let

$$\Psi : \widetilde{\Phi}(O) \subset \mathbb{R}^n \to O, \qquad \Psi(\xi) = (x(\xi), \alpha(\xi)),$$

be the local inverse of $\widetilde{\Phi}$. Take $\delta_\circ \in (0, r_\circ/2)$ and $\xi \in B_{\delta_\circ}(x_\circ) \subset \widetilde{\Phi}(O)$, and let $x \in \mathcal{M}$ be so that $\operatorname{dist}(\xi, \mathcal{M}) = |x - \xi|$, recall that $\mathcal{M}$ is closed by Definition 5.3. Since $|x - \xi| \le |x_\circ - \xi| < \delta_\circ$ it follows $x \in B_{r_\circ}(x_\circ) \cap \mathcal{M}^\circ$ (recall (5.10) and $r_\circ \le \rho$), hence $x = x(\xi)$[(2)] and $\operatorname{dist}(\xi, \mathcal{M}) = |\alpha(\xi)|$. Thus,

$$\eta_{\mathcal{M}}(\xi) = \frac{1}{2}|\alpha(\xi)|^2 = \frac{1}{2}\sum_{i=1}^{n-h} (\alpha_i(\xi))^2 \qquad \forall \xi \in B_{\delta_\circ}(x_\circ),$$

[(2)] Indeed $\xi \in N_x\mathcal{M}^\circ$, *i.e.*, for any $\omega \in T_x\mathcal{M}$ we have $< x - \xi, \omega >= 0$. To prove that, consider a local chart f around x such that $f(p) = x$ and $df_p\tau = \omega$. Since p is a minimum point for the function $|\xi - f(p + \sigma\tau)|^2$ where $|\sigma|$ is small enough then $0 = \frac{d}{d\sigma}|\xi - f(p+\sigma\tau)|^2|_{\sigma=0} =< \xi - x, \omega >$.

where $\alpha(\xi) = (\alpha_1(\xi)\ldots\alpha_{n-h}(\xi))$. Therefore $\eta_{\mathcal{M}} \in \mathcal{C}^{k-1}(B_{\delta_\circ}(x_\circ))$.
Now we deal with points on $\partial\mathcal{M}$. Since $\mathcal{M}$ is an embedded $\mathcal{C}^k$-manifold with boundary, it can be extended[(3)] to a connected $\mathcal{C}^k$-manifold with boundary $\mathcal{N}$ of the same dimension such that $\partial\mathcal{M} \subset \mathcal{N}^\circ$. Let $\bar{x} \in \partial\mathcal{M} \subset \mathcal{N}$ and repeat the argument at the beginning of this step with $\mathcal{M}$ replaced by $\mathcal{N}$, to conclude that $\eta_{\mathcal{N}}$ is $\mathcal{C}^{k-1}$ in $B_\delta(\bar{x}) \subset \widetilde{\Phi}_{\mathcal{N}^\circ}((B_r(\bar{x}) \cap \mathcal{N}^\circ) \times B_r^{n-h}(0))$ for $r > 0$ sufficient small and $\delta \in (0, \frac{r}{2})$. Consider the open set

$$\mathcal{W} = \mathcal{W}(\bar{x}) := B_\delta(\bar{x}) \cap \left(\widetilde{\Phi}_{\mathcal{N}^\circ}((B_r(x) \cap \mathcal{M}^\circ) \times B_r^{n-h}(0))\right).$$

We claim that

$$\eta_{\mathcal{M}} = \eta_{\mathcal{N}} \quad \text{on} \ \ \mathcal{W},$$

and hence $\eta_{\mathcal{M}} \in \mathcal{C}^{k-1}(\mathcal{W})$. Indeed, take $\xi \in \mathcal{W}$; then $\xi \in B_\delta(\bar{x})$ implies the existence of a unique $x(\xi) \in \mathcal{N}$ such that $\mathrm{dist}(\xi, \mathcal{N}) = |x(\xi) - \xi|$ and $\xi \in N_{x(\xi)}\mathcal{N}$ (clearly $N_x\mathcal{M}^\circ = N_x\mathcal{N}^\circ$ at $x \in \mathcal{M}^\circ$). Moreover $\xi \in \widetilde{\Phi}_{\mathcal{N}^\circ}\big((B_r(\bar{x}) \cap \mathcal{M}^\circ) \times B_r^{n-h}(0)\big)$ implies $x(\xi) \in \mathcal{M}^\circ$ by the definition of $\widetilde{\Phi}_{\mathcal{N}^\circ}$. In particular, any point of $\mathcal{W}$ has a unique point of minimal distance to $\mathcal{N}$ on $B_r(\bar{x}) \cap \mathcal{M}^\circ$.

By the compactness of $\mathcal{M}$, we can select $\epsilon_1 > 0$ such that:

- (5.8) holds;
- for any $\xi \in V_{\epsilon_1}$ there exists a unique $x(\xi) \in \mathcal{M}^\circ$ such that $\mathrm{dist}(\xi, \mathcal{M}) = |\xi - x(\xi)|$, in particular
$$\mathrm{dist}(\cdot, \mathcal{M}) < \mathrm{dist}(\cdot, \partial\mathcal{M}) \qquad \text{in} \quad V_{\epsilon_1}; \tag{5.12}$$
- by construction $V_{\epsilon_1} \subset \mathcal{M}^+_{\epsilon_1}$ and the topological boundary of V_{ϵ_1} is $K \cup H_{\epsilon_1}$, where $K \subset \partial(\mathcal{M}^+_{\epsilon_1})$ and H_{ϵ_1} is defined in (5.7) with ϵ replaced by ϵ_1. Hence the closure of V_{ϵ_1} in $\mathcal{M}^+_{\epsilon_1}$ is H_{ϵ_1} (see Figure 5.4);
- $\eta_{\mathcal{N}} \in \mathcal{C}^{k-1}(\mathcal{M}^+_{\epsilon_1})$ and
$$\eta_{\mathcal{N}} = \eta_{\mathcal{M}} \qquad \text{in} \quad V_{\epsilon_1} \cup H_{\epsilon_1}. \tag{5.13}$$

Step 2. For $\epsilon_2 > 0$ small enough, H_{ϵ_2} is a $\mathcal{C}^{k-1}$ embedded hypersurface without boundary in $\mathcal{M}^+_{\varepsilon_2}$.
Let $\bar{x} \in \partial\mathcal{M}$ and $g : G'_{\bar{x}} \subset \mathbb{R}^{h-1} \longrightarrow B_\rho(\bar{x}) \cap \partial\mathcal{M}$, $\rho > 0$, be a local chart on $\partial\mathcal{M}$. Define

$$X(y', \alpha) = g(y') + \sum_i^{n-h} \alpha_i \nu^i(g(y')), \qquad y' \in G'_{\bar{x}}, \alpha \in B^{n-h}_{\epsilon(\bar{x})}(0), \tag{5.14}$$

where $\{\nu^i(g(y'))\}_{i=1,\ldots,n-h}$ are orthonormal vector fields of class $\mathcal{C}^{k-1}$ spanning the normal space to $\mathcal{M}$ at $g(y') \in \partial\mathcal{M}$ and $\epsilon(\bar{x}) > 0$. Clearly X is $\mathcal{C}^{k-1}$ and $dX_{(y',0)}$ is non singular and X is a local homeomorphism onto its image. Now, we use the

[(3)] This directly follows from Definition 5.3.

compactness of $\partial\mathcal{M}$ to get a finite subcovering $\bigcup_{i=1}^{l} g_i(G'_{\bar{x}_i}) = \bigcup_{i=1}^{l} B_{\rho_i}(\bar{x}_i)\cap\partial\mathcal{M} = \partial\mathcal{M}, \bar{x}_i \in \partial\mathcal{M}$, and define

$$\epsilon_2 := \min\{\epsilon(\bar{x}_i) : i = 1, \dots, l\} > 0, \qquad H_{\epsilon_2} = \bigcup_{i=1}^{l} X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0)).$$

It remains to prove that, possibly reducing the value of ϵ_2, any $\zeta \in H_{\epsilon_2}$ has an open neighborhood $\mathcal{V}$ such that $\mathcal{V} \cap H_{\epsilon_2}$ is exactly the image of one of the charts $X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0))$ (that is, H_{ϵ_2} has no self-intersections). Assume that $\epsilon_2 > 0$ is small enough so that

$$H_{\epsilon_2} \subset (\phi\mathcal{M})^+_{\epsilon_2},$$

and

- for every $\xi \in (\phi\mathcal{M})^+_{\epsilon_2}$ there exists unique $x_\xi \in \partial\mathcal{M}$ such that $\xi \in N_{x_\xi}\partial\mathcal{M}$ and $\mathrm{dist}(\xi, \partial\mathcal{M}) = |\xi - x_\xi|$;
- the projection map $P : (\partial\mathcal{M})^+_{\epsilon_2} \to \partial\mathcal{M}, P(\xi) = x_\xi$ is $\mathcal{C}^{k-1}$, see [32].

Now let $\zeta \in H_{\epsilon_2}$ and $\bar{x}_i$ be such that $x_\zeta \in B_{\rho_i}(\bar{x}_i)\cap\partial\mathcal{M}$. Define $\mathcal{V} := P^{-1}(B_{\rho_i}(\bar{x}_i)\cap\partial\mathcal{M})$ which is an open neighborhood of ζ in $\mathbb{R}^n$. We have

$$\mathcal{V} \cap H_{\epsilon_2} = X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0)).$$

Indeed, if $\xi \in \mathcal{V} \cap H_{\epsilon_2}$ then $x_\xi \in B_{\rho_i}(\bar{x}_i) \cap \partial\mathcal{M}$, $|\xi - x_\xi| < \epsilon_2$ and $\xi \in N_{x_\xi}M$, *i.e.*, $\xi \in X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0))$ by the definition of X in (5.14). On the other hand the inclusion $\mathcal{V} \cap H_{\epsilon_2} \supseteq X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0))$ is immediate.
Note that assertions 1) and 2) of the proposition follow immediately from (5.7), with ϵ replaced by ϵ_2.

Step 3. There exists $\epsilon \in (0, \min(\epsilon_1, \epsilon_2))$ such that

$$\eta_\mathcal{M} \in \mathcal{C}^{k-1}(\mathcal{M}^+_\epsilon \setminus H_\epsilon). \tag{5.15}$$

From Theorem 1.29, we may assume that $\eta_{\partial\mathcal{M}}$ is $\mathcal{C}^{k-1}$ in a tubular neighborhood $(\partial\mathcal{M})^+_{\epsilon_3}$ of $\partial\mathcal{M}$ of radius $\epsilon_3 > 0$. Define

$$\epsilon := \min\{\epsilon_1, \epsilon_2, \epsilon_3\}. \tag{5.16}$$

If V_ϵ is as in step 1, we have $V_\epsilon \subset \mathcal{M}^+_\epsilon$ and, from step 1, $\eta_\mathcal{M}$ is $\mathcal{C}^{k-1}$ in V_ϵ. We claim that

$$\eta_\mathcal{M} = \eta_{\partial\mathcal{M}} \quad \text{in} \ \ \mathcal{M}^+_\epsilon \setminus V_\epsilon. \tag{5.17}$$

Since $\partial\mathcal{M} \subset \mathcal{M}$, $\mathrm{dist}(\cdot, \mathcal{M}) \leq \mathrm{dist}(\cdot, \partial\mathcal{M})$ hence $\eta_\mathcal{M} \leq \eta_{\partial\mathcal{M}}$. Assume by contradiction that there exists $\xi \in \mathcal{M}^+_\epsilon \setminus V_\epsilon$ such that $\mathrm{dist}(\xi, \mathcal{M}) < \mathrm{dist}(\xi, \partial\mathcal{M})$. Then there exists $x \in \mathcal{M} \setminus \partial\mathcal{M}$ such that $|\xi - x| = \mathrm{dist}(\xi, \mathcal{M}) < \epsilon$ which, by the definition of V_ϵ, implies $\xi \in V_\epsilon$, a contradiction.
Since the closure of V_ϵ in $\mathcal{M}^+_\epsilon$ is H_ϵ (see the end of step 1) it follows that $\mathcal{M}^+_\epsilon \setminus (V_\epsilon \cup H_\epsilon)$ is an open subset of $(\partial\mathcal{M})^+_{\epsilon_3}$ in which $\eta_\mathcal{M}$ is $\mathcal{C}^{k-1}$. Hence assertion 3) is proven. □

Now, we prove Theorem 5.11 when $\Omega = \mathbb{R}^n$ and supposing that the manifold is compact.

Theorem 5.13. *Let $k \in \mathbb{N}$, $k \geq 3$, or $k \in \{\infty, \omega\}$, and $h \in \{1, \dots, n\}$. Let $\mathcal{M} \subset \mathbb{R}^n$ be an embedded compact $\mathcal{C}^k$-manifold of dimension h with nonempty boundary $\partial\mathcal{M}$ in $\mathbb{R}^n$. Then*

$$(\mathcal{M}, \partial\mathcal{M}) \in D_hB\mathcal{C}^{k-1}(\mathbb{R}^n).$$

Proof. We have to check conditions (i)-(iv) of Definition 5.4.
Suppose $h = n$. By Remark 5.2 (I) it follows $\partial\mathcal{M} \in D_{n-1}\mathcal{C}^{k-1}(\mathbb{R}^n)$. One also immediately checks that $\mathcal{M} \in D_n\mathcal{C}^{k-1}(\mathbb{R}^n \backslash \partial\mathcal{M})$. Moreover $d_{\mathcal{M}}(\cdot) = \pm\text{dist}(\cdot, \partial\mathcal{M}) \leq \text{dist}(\cdot, \partial\mathcal{M}) = d_{\partial\mathcal{M}}(\cdot)$ and $B = \mathcal{M} \setminus \partial\mathcal{M}$, hence [30, 29, 2] the function d_B is $\mathcal{C}^k$ in a tubular neighborhood of $\partial\mathcal{M}$, which shows condition (iii). Clearly $\{x \in \mathbb{R}^n : d_B(x) \leq 0\} = \mathcal{M}$; thus $\widehat{\eta} \equiv 0$ is a $\mathcal{C}^k(\mathbb{R}^n)$ extension of $\eta_{\mathcal{M}}$, so that condition (iv) is fulfilled.
Now suppose $h < n$. From Remark 5.2(I) we have

$$\partial\mathcal{M} \in D_{h-1}\mathcal{C}^{k-1}(\mathbb{R}^n).$$

Moreover, since $\mathcal{M}$ is a $\mathcal{C}^k$-manifold without boundary in $\mathbb{R}^n \setminus \partial\mathcal{M}$ then, again by Remark 5.2(I),

$$\mathcal{M} \in D_h\mathcal{C}^{k-1}(\mathbb{R}^n \setminus \partial\mathcal{M}),$$

which shows (i). We also have $d_{\mathcal{M}}(\cdot) = \text{dist}(\cdot, \mathcal{M}) \leq \text{dist}(\cdot, \partial\mathcal{M}) = d_{\partial\mathcal{M}}(\cdot)$, which shows (ii).
Let B be defined as in (5.2) with $\Omega = \mathbb{R}^n$, and $(E, L) := (\mathcal{M}, \partial\mathcal{M})$. Then by (5.12) and the last comments in step 1 in Proposition 5.12 we have

$$B \cap \mathcal{M}_\epsilon^+ = V_\epsilon \qquad \text{and} \qquad \mathcal{M}_\epsilon^+ \cap \partial B = H_\epsilon,$$

where ϵ, V_ϵ and H_ϵ are as in (5.16), (5.9) and (5.7), respectively. Since by Proposition 5.12 2) H_ϵ is an embedded hypersurface (without boundary) of class $\mathcal{C}^{k-1}$ in $\mathcal{M}_\epsilon^+$ then, following the same argument in the comment after Theorem 1.29 and using [30, 29, 2], there exists an open neighborhood $A \subset \mathcal{M}_\epsilon^+$ of H_ϵ such that $d_B \in \mathcal{C}^{k-1}(A)$. Hence condition (iii) is satisfied.
Finally, since $\{x \in A : d_B(x) \leq 0\} \subset V_\epsilon \cup H_\epsilon$, condition (iv) follows from (5.13). □

Now we generalize Proposition 5.12: dropping the compactness of $\mathcal{M}$ implies that we can not take tubular neighborhoods of constant width.

Proposition 5.14. *Let $k \in \mathbb{N}$, $k \geq 2$, or $k \in \{\infty, \omega\}$, and $h \in \{1, \dots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. Let $\mathcal{M} \subset \mathbb{R}^n$ be an embedded $\mathcal{C}^k$-manifold of dimension h with nonempty boundary in Ω. Then there exists a function $\epsilon : \mathcal{M} \cap \Omega \to (0, +\infty]$ such that, setting*

$$H_{\bar{\epsilon}(\cdot)} := \bigcup_{\bar{x} \in \partial_\Omega \mathcal{M}} B_{\bar{\epsilon}(\bar{x})}(\bar{x}) \cap N_{\bar{x}}\mathcal{M} \tag{5.18}$$

where $\bar{\varepsilon}(\bar{x}) := \sup\{\rho > 0 : B_\rho(\bar{x}) \cap N_{\bar{x}}\mathcal{M} \subset (\mathcal{M} \cap \Omega)^+_{\epsilon(\cdot)}\}$, the following properties hold:

1) $\partial_\Omega \mathcal{M} \subseteq H_{\bar{\epsilon}(\cdot)} \subseteq \bigcup_{\bar{x} \in \partial_\Omega \mathcal{M}} N_{\bar{x}}\mathcal{M}$;

2) *$H_{\bar{\epsilon}(\cdot)}$ is an embedded $\mathcal{C}^{k-1}$-hypersurface without boundary in $(\mathcal{M} \cap \Omega)^+_{\epsilon(\cdot)}$, and $N_{\bar{x}} H_{\bar{\epsilon}(\cdot)} \subseteq T_{\bar{x}}\mathcal{M}$ for any $\bar{x} \in \partial_\Omega \mathcal{M}$;*

3) $\eta_\mathcal{M} \in \mathcal{C}^{k-1}((\mathcal{M} \cap \Omega)^+_{\epsilon(\cdot)} \setminus H_{\bar{\epsilon}(\cdot)})$.

Proof. The proof is similar to the proof of Proposition 5.12 with slight modifications. We suppose that $\mathcal{M} \cap \Omega$ is connected. Let us write for simplicity $\epsilon_x = \epsilon(x)$, $\bar{\epsilon}_x = \bar{\epsilon}(x)$ and so on.
Assume first $h = n$; then the interior of $\mathcal{M} \cap \Omega$ is a nonempty open set with $\mathcal{C}^k$-boundary in Ω, and for every $\bar{x} \in \partial_\Omega \mathcal{M}$ there exists $\epsilon_{\bar{x}} > 0$ such that $B_{\epsilon_{\bar{x}}}(\bar{x}) \subset \Omega$ and $d_\mathcal{M} \in \mathcal{C}^k(B_{\epsilon_{\bar{x}}}(\bar{x}))$ [30, 29, 22], hence $d_\mathcal{M}$ is of class $\mathcal{C}^k$ in $(\partial_\Omega \mathcal{M})^+_{\epsilon(\cdot)} \subset \Omega$. Define $H_{\bar{\epsilon}(\cdot)} := \partial_\Omega \mathcal{M}$. Then assertions 1)-3) follow as in the proof of Proposition 5.12.

Now suppose $h \in \{1, \dots, n-1\}$. Following the same argument in step 1 in the proof of Proposition 5.12 it follows that:

- for every $x_\circ \in (\mathcal{M} \cap \Omega)^\circ$ there exists $\epsilon^1_{x_\circ} > 0$ such that $B_{\epsilon^1_{x_\circ}}(x_\circ) \subset \Omega$, $B_{\epsilon^1_{x_\circ}}(x_\circ) \cap \partial_\Omega \mathcal{M} = \emptyset$ and $\eta_\mathcal{M} \in \mathcal{C}^{k-1}(B_{\epsilon^1_{x_\circ}}(x_\circ))$;

- if $\mathcal{N} \subset \Omega$ is a connected embedded $\mathcal{C}^k$-manifold of dimension h containing $\mathcal{M}$ and so that $\partial_\Omega \mathcal{M} \subset \mathcal{N}^o$ then, for every $\bar{x} \in \partial_\Omega \mathcal{M}$, there exists $\epsilon^1_{\bar{x}} > 0$ such that $B_{\epsilon^1_{\bar{x}}}(\bar{x}) \subset \Omega$ and

$$\begin{aligned} \mathcal{W}_{\epsilon^1_{\bar{x}}} = \mathcal{W}_{\epsilon^1_{\bar{x}}}(\bar{x}) &:= \{\xi \in B_{\epsilon^1_{\bar{x}}}(\bar{x}) : \operatorname{dist}(\xi, \mathcal{N}) = |\xi - x_\xi|,\ x_\xi \in (\mathcal{M} \cap \Omega)^\circ\} \\ &= \bigcup_{y \in (\mathcal{M} \cap \Omega)^\circ} B_{\epsilon^1_{\bar{x}}}(\bar{x}) \cap N_y\mathcal{M} \end{aligned}$$

  is an open subset of Ω, and $\overline{\mathcal{W}_{\epsilon^1_{\bar{x}}}} \cap B_{\epsilon^1_{\bar{x}}}(\bar{x}) = \mathcal{W}_{\epsilon^1_{\bar{x}}} \cup \Big(\bigcup_{y \in \partial_\Omega \mathcal{M}} (B_{\epsilon^1_{\bar{x}}}(\bar{x}) \cap N_y\mathcal{M}) \Big)$;

- $\eta_\mathcal{M} = \eta_\mathcal{N}$ in $\overline{\mathcal{W}_{\epsilon^1_{\bar{x}}}}$, hence $\eta_\mathcal{M} \in \mathcal{C}^{k-1}\big(\overline{\mathcal{W}_{\epsilon^1_{\bar{x}}}} \cap B_{\epsilon^1_{\bar{x}}}(\bar{x})\big)$.

Define

$$V_{\epsilon^1(\cdot)} := \Big(\bigcup_{x_\circ \in (\mathcal{M} \cap \Omega)^\circ} B_{\epsilon^1_{x_\circ}}(x_\circ) \Big) \cup \Big(\bigcup_{\bar{x} \in \partial_\Omega \mathcal{M}} \mathcal{W}_{\epsilon^1_{\bar{x}}} \Big).$$

The presence of the set $\bigcup_{\bar{x}\in\partial_\Omega\mathcal{M}} \mathcal{W}_{\epsilon^1_{\bar{x}}}$ is due to the fact that, when $\mathcal{M}$ is not compact, it could happen that, as $x_\circ \in \mathcal{M}^\circ$ converges to a point of $\partial_\Omega\mathcal{M}$, the corresponding $\epsilon^1_{x_\circ}$ converges to zero.
By construction $V_{\epsilon^1(\cdot)} \subset (\mathcal{M}\cap\Omega)^+_{\epsilon^1(\cdot)}$ and the topological boundary of $V_{\epsilon^1(\cdot)}$ is $K \cup H_{\bar{\epsilon}^1(\cdot)}$, where K is subset of the topological boundary of $((\mathcal{M}\cap\Omega)^+_{\epsilon^1(\cdot)})$ and $H_{\bar{\epsilon}^1(\cdot)}$ are as in (5.18) with $\bar{\epsilon}$ replaced by $\bar{\epsilon}^1$. Hence the topological boundary of $V_{\epsilon^1(\cdot)}$ in $(\mathcal{M}\cap\Omega)^+_{\epsilon^1(\cdot)}$ is $H_{\bar{\epsilon}^1(\cdot)}$. Moreover $\eta_{\mathcal{N}} \in \mathcal{C}^{k-1}((\mathcal{M}\cap\Omega)^+_{\epsilon^1(\cdot)})$ and $\eta_{\mathcal{N}} = \eta_{\mathcal{M}}$ in $V_{\epsilon^1(\cdot)} \cup H_{\bar{\epsilon}^1(\cdot)}$. We conclude that $\eta_{\mathcal{M}} \in \mathcal{C}^{k-1}(V_{\epsilon^1(\cdot)})$.
Following the same arguments in step 1 and step 2 in the proof of Proposition 5.12 it follows that for each $\bar{x} \in \partial_\Omega\mathcal{M}$ there exist $\epsilon^2_{\bar{x}} > 0$, $G'_{\bar{x}} \subset \mathbb{R}^{h-1}$, $B_{\bar{x}} \subset \mathbb{R}^{n-h+1}$ open sets and a $\mathcal{C}^{k-1}$-diffeomorphism

$$\overline{X} : G'_{\bar{x}} \times B_{\bar{x}} \to B_{\epsilon^2_{\bar{x}}}(\bar{x}) \subset \mathbb{R}^n, \qquad \overline{X}(y',\overline{\alpha}) := g(y') + \sum_{i=1}^{n-h+1} \overline{\alpha}_i \nu^i(g(y')),$$

where $g : G'_{\bar{x}} \subset \mathbb{R}^{h-1} \longrightarrow B_{\epsilon^2_{\bar{x}}}(\bar{x})\cap\partial_\Omega\mathcal{M}$ is a local chart on $\partial_\Omega\mathcal{M}$ and $\{\nu^i(g(y'))\}_{i=1,\dots,n-h+1}$, $\nu^{n-h+1}(g(y')) \in T_{g(y')}\mathcal{M}$, are orthonormal vector fields of class $\mathcal{C}^{k-1}$ spanning the normal space to $\partial_\Omega\mathcal{M}$ at $g(y') \in \partial_\Omega\mathcal{M}$. Thus

$$(\partial_\Omega\mathcal{M})^+_{\epsilon^2(\cdot)} = \bigcup_{\bar{x}\in\partial_\Omega\mathcal{M}} \overline{X}(G'_{\bar{x}} \times B_{\bar{x}}).$$

Let

$$X : G'_{\bar{x}} \times B^{n-h}_{\bar{x}} \to B_{\epsilon^2_{\bar{x}}}(\bar{x}), \qquad X(y',\alpha) := g(y') + \sum_{i=1}^{n-h} \alpha_i \nu^i(g(y')),$$

where $B^{n-h}_{\bar{x}} := \{(\overline{\alpha}_1,\cdots,\overline{\alpha}_{n-h+1}) \in B_{\bar{x}} : \overline{\alpha}_{n-h+1} = 0\} \subset \mathbb{R}^{n-h}$ (note that X equals the restriction of $\overline{X}$ in $G'_{\bar{x}} \times B^{n-h}_{\bar{x}}$, hence it is a $\mathcal{C}^{k-1}$-diffeomorphism). Setting $H_{\bar{\epsilon}^2(\cdot)}$ as in (5.18) with $\bar{\epsilon}$ replaced by $\bar{\epsilon}^2$, we have

$$H_{\bar{\epsilon}^2(\cdot)} = \bigcup_{\bar{x}\in\partial_\Omega\mathcal{M}} X(G'_{\bar{x}} \times B^{n-h}_{\bar{x}}).$$

Thus for each $\zeta \in H_{\bar{\epsilon}^2(\cdot)}$ there exists $\bar{x} \in \partial_\Omega\mathcal{M}$ such that $\zeta \in X(G'_{\bar{x}} \times B^{n-h}_{\bar{x}})$. Letting $\mathcal{V} := \overline{X}(G'_{\bar{x}} \times B_{\bar{x}})$, following the argument in step 2 of Proposition 5.12, we may show that the maps $X(G'_{\bar{x}} \times B^{n-h}_{\bar{x}})$ are local charts covering $H_{\bar{\epsilon}^2(\cdot)}$. This completes the proof of assertion 2).
To prove 3) we assume that $\eta_{\partial\mathcal{M}}$ is $\mathcal{C}^{k-1}$ in a neighborhood $(\partial_\Omega\mathcal{M})^+_{\epsilon^3(\cdot)}$ of $\partial_\Omega\mathcal{M}$, see the comment after Theorem 1.29, and we define $\varepsilon(x) := \min\{\epsilon^1_x, \epsilon^2_x, \epsilon^3_x\}$. Again following step 3 in the proof of Proposition 5.12 we have $\eta_{\mathcal{M}} \in \mathcal{C}^{k-1}((\mathcal{M}\cap\Omega)^+_{\epsilon(\cdot)} \setminus H_{\bar{\epsilon}(\cdot)})$. □

Conclusion of the proof of Theorem 5.11. It follows from Proposition 5.14 the same way the proof of Theorem 5.13 follows from Proposition 5.12. □

5.3 Sets in $D_hB\mathcal{C}^k(\Omega)$ are smooth manifolds with boundary

The goal of this section is to prove a sort of converse[(4)] of Theorem 5.11. That is, we want to show the following:

Theorem 5.15. *Let $k \in \mathbb{N}$, $k \geq 3$, or $k \in \{\infty, \omega\}$ and $h \in \{1, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, and let $E, L \subset \mathbb{R}^n$ be such that $(E, L) \in D_hB\mathcal{C}^k(\Omega)$. Then $(E \cup L) \cap \Omega$ is a h-dimensional $\mathcal{C}^{k-1}$-manifold in Ω with boundary $L \cap \Omega$.*

Proof. Let us assume first $\Omega = \mathbb{R}^n$ (which includes the converse of Theorem 5.13). We can suppose

$$L \neq \emptyset,$$

since if $L = \emptyset$ the result follows from Remark 5.2 (I).
Recall from Remark 5.5 (I) that L is an embedded $\mathcal{C}^{k-1}$-manifold in $\mathbb{R}^n$ without boundary of dimension $h-1$, and E is an embedded $\mathcal{C}^{k-1}$-manifold without boundary in $\mathbb{R}^n \setminus L$ of dimension h.
Moreover from condition (iv) in Definition 5.4, following the notation of (iii) in particular concerning the sets B and A, if we call

$$C := \{x \in A : d_B(x) \leq 0\}, \tag{5.19}$$

then there exist an open set $\widehat{C} \subset \mathbb{R}^n$ containing C and a function $\widehat{\eta} \in \mathcal{C}^k(\widehat{C})$ such that

$$\widehat{\eta} = \eta_E \quad \text{on} \quad C. \tag{5.20}$$

We divide the proof of the theorem into five steps.

Step 1. We have

$$\overline{E} \cap C \subseteq \{x \in \widehat{C} : \nabla\widehat{\eta}(x) = 0\}. \tag{5.21}$$

From Theorem 1.25(b), η_E is differentiable on $\overline{E}$ and

$$\overline{E} = \{x \in \mathbb{R}^n : \nabla\eta_E(x) = 0\}.$$

Hence, since from (5.3) we have $\overline{E} = E \cup L$, it follows

$$\overline{E} \cap C = (E \cup L) \cap C = \{x \in C : \nabla\eta_E(x) = 0\}. \tag{5.22}$$

Now we show that

$$\nabla\widehat{\eta} = \nabla\eta_E \quad \text{on} \quad (E \cup L) \cap C. \tag{5.23}$$

We split the proof into two cases. If $x \in (E \setminus L) \cap C$ then from (5.4) and (5.19) it follows

$$x \in B \cap C = \{x \in A : d_B(x) < 0\} = B \cap A \subset C. \tag{5.24}$$

Hence, since $B \cap A$ is open, x is an interior point of C, and from (5.20) we deduce

$$\nabla\widehat{\eta}(x) = \nabla\eta_E(x).$$

(4) In the C^∞ or analytic case, it is the converse.

Now, let $x \in L = L \cap C$; recall from Remark 5.5 (VII) that the topological boundary of B is of class $\mathcal{C}^{k-1}$ in a neighborhood of x. Then $x \in \partial_A B = \{x \in A : d_B(x) = 0\}$ from (5.5). We shall show that

$$\nabla\widehat{\eta}(x)\nu = \nabla\eta_E(x)\nu \qquad \forall \nu \in \mathbb{R}^n. \tag{5.25}$$

Take $\nu \in \mathbb{R}^n \setminus \{0\}$. Let $n \geq 2$ (the case $n = 1$ being trivial); if $\nu \in T_x\partial_A B$ then there exist $\epsilon > 0$ and $\alpha : (-\epsilon, \epsilon) \to \partial_A B$ of class $\mathcal{C}^1$ such that $\alpha(0) = x$, $\alpha'(0) = \nu$. Hence, using also (5.20),

$$\nabla\eta_E(x)\nu = \frac{d}{dt}\eta_E(\alpha(t))|_{t=0} = \frac{d}{dt}\widehat{\eta}(\alpha(t))|_{t=0} = \nabla\widehat{\eta}(x)\nu.$$

If $\nu \in N_x\partial_A B$ then we can select $\beta : (-\epsilon, \epsilon) \to \mathbb{R}^n$ of class $\mathcal{C}^1$ such that $\beta(0) = x$, $\beta'(0) = \nu$ and $\beta((-\epsilon, 0))$ is contained in the interior of C. Hence, denoting by $\frac{d}{dt_-}$ the left derivative,

$$\nabla\widehat{\eta}_E(x)\nu = \frac{d}{dt}\eta_E(\beta(t))|_{t=0} = \frac{d}{dt_-}\eta_E(\beta(t))|_{t=0} = \frac{d}{dt_-}\widehat{\eta}(\beta(t))|_{t=0} = \frac{d}{dt}\widehat{\eta}(\beta(t))|_{t=0} = \nabla\widehat{\eta}(x)\nu.$$

This concludes the proof of (5.25), and then (5.21) follows from (5.22) and (5.23).

Step 2. We have

$$\operatorname{rank}\big(\nabla^2\widehat{\eta}(x)\big) = n - h \qquad \text{for any } x \in (E \cup L) \cap C. \tag{5.26}$$

From Definition 5.4 (i), Definition 5.1 (iii) and (5.24) we have

$$\operatorname{rank}\big(\nabla^2\widehat{\eta}(x)\big) = n - h \qquad \text{for any} \quad x \in (E \setminus L) \cap C. \tag{5.27}$$

Now we observe that (5.27) holds also for $x \in L$. Indeed, if $x \in L$, from (5.3) we can select a sequence $\{x_m\} \subset (E \setminus L) \cap C$ converging to x. Then, by the continuity of $\nabla^2\widehat{\eta}$ at x, it follows $\operatorname{rank}\big(\nabla^2\widehat{\eta}(x)\big) = n - h$.

Step 3. There exists an embedded h-dimensional manifold $\mathcal{N}$ of class $\mathcal{C}^{k-1}$, without boundary in a sufficiently small neighborhood of $E \cup L$, such that

$$E \cup L \subset \mathcal{N}.$$

For $h = n$, it is sufficient to take $\mathcal{N} = \mathbb{R}^n$. Hence, suppose $h < n$. Take

$$\overline{x} \in L$$

and, recalling (5.26), let $\{\overline{\nu}^1, \overline{\nu}^2, \ldots, \overline{\nu}^{n-h}\}$ be an orthonormal basis of $\operatorname{Im}\big(\nabla^2\widehat{\eta}(\overline{x})\big)$. Define

$$F_i(x) := \langle \nabla\widehat{\eta}(x), \overline{\nu}^i \rangle, \qquad i = 1, \ldots, n - h, \quad x \in \widehat{C},$$

and set

$$F : \widehat{C} \longrightarrow \mathbb{R}^{n-h}, \qquad F := (F_1, F_2, \ldots, F_{n-h}).$$

From (5.3),(5.22) and (5.23) we have

$$(E \cup L) \cap C = \overline{E} \cap C \subseteq \{x \in \widehat{C} : \nabla \widehat{\eta}(x) = 0\} \subseteq \{x \in \widehat{C} : F(x) = 0\}. \tag{5.28}$$

Observe that $F \in \mathcal{C}^{k-1}(\widehat{C}; \mathbb{R}^{n-h})$. Moreover, if we denote by $JF(\overline{x})$ the Jacobian of F at $\overline{x}$, then

$$JF(\overline{x}) = Q^T \nabla^2 \widehat{\eta}(\overline{x}), \tag{5.29}$$

where Q^T is the transposed of the $n \times (n-h)$ matrix $Q := \left[\overline{\nu}^1 \overline{\nu}^2 \dots \overline{\nu}^{n-h}\right]$ having as columns the linear independent vectors $(\overline{\nu}^i)_{i=1,\dots,n-h}$. Recalling the definition of $\overline{\nu}^1, \dots \overline{\nu}^{n-h}$, by construction $JF(\overline{x})$ has rank $n-h$. Choose $\sigma = \sigma(\overline{x}) > 0$ so that the Jacobian of F has constant rank $n-h$ on $B_\sigma(\overline{x})$. Let

$$\Gamma_{\overline{x}} := B_\sigma(\overline{x}) \cap \{x \in \widehat{C} : F(x) = 0\}.$$

Then the implicit function theorem ensures that $\Gamma_{\overline{x}}$ is an embedded $h-$dimensional manifold (without boundary in $B_\sigma(\overline{x})$) of class $\mathcal{C}^{k-1}$.
Note that $B_\sigma(\overline{x}) \cap ((E \setminus L) \cap C)$ (which is nonempty by (5.4) and (5.19)) is a manifold without boundary in $B_\sigma(\overline{x}) \setminus L$ of dimension h (Remark 5.5 (I)) and it is contained in $\Gamma_{\overline{x}}$ by (5.28). Hence $\Gamma_{\overline{x}}$ is an extension of $(E \setminus L) \cap C$ in $B_\sigma(\overline{x})$.
Defining

$$\mathcal{N} := E \cup \bigcup_{\overline{x} \in L} \Gamma_{\overline{x}},$$

we have that $\mathcal{N}$ satisfies the assertion.

Step 4. $E \cup L$ is an embedded h-dimensional $\mathcal{C}^{k-1}$-manifold in $\mathbb{R}^n$ with boundary.
We need to check that Definition 5.3 is satisfied. Recall from Remark 5.5 (III) that $\overline{E \cup L} = E \cup L$. Now, let $x \in E \setminus L$; in this case there is nothing to prove, since $E \setminus L$ is a manifold without boundary in $\mathbb{R}^n \setminus L$ of dimension h (Remark 5.5 (I)).
Let $\bar{x} \in L$. Since L is a $\mathcal{C}^{k-1}$ embedded submanifold of $\mathcal{N}$ of codimension 1 (step 3), there exist an open neighborhood $R \subset \mathbb{R}^n$ of $\bar{x}$ and a $\mathcal{C}^{k-1}$ local parametrization

$$\phi : G := B_1^h(0) \to U := R \cap \mathcal{N} \subset \mathbb{R}^n \tag{5.30}$$

such that

$$R \cap L = \{\phi(y) : y = (y_1, \dots, y_h) \in G, y_h = 0\}. \tag{5.31}$$

Hence $U \cap L$ divides U into two relatively open connected components U^+ and U^- defined as

$$U^\pm := \{\phi(y) : y \in G,\ \langle y, \pm e_h \rangle > 0\}, \tag{5.32}$$

where $e_h := (0, \dots, 0, 1) \in \mathbb{R}^h$ (note that $(E \setminus L) \cap (U \setminus L) \neq \emptyset$). Clearly

$$L \cap U^+ = L \cap U^- = \emptyset. \tag{5.33}$$

Let us show

$$U^\pm \cap (E \setminus L) \neq \emptyset \qquad \Rightarrow \qquad U^\pm \cap (E \setminus L) = U^\pm. \tag{5.34}$$

Assume $U^+ \cap (E \setminus L) \neq \emptyset$ and suppose by contradiction that $U^+ \setminus \big(U^+ \cap (E \setminus L)\big)$ is nonempty (the case for U^- being similar).
Recalling that U^+ is connected and that both sets $E \setminus L$ and U^+ are relatively open in $\mathcal{N}$, we have

$$U^+ \cap (E \setminus L) \subsetneq \overline{U^+ \cap (E \setminus L)} \cap U^+. \tag{5.35}$$

Thus

$$\overline{U^+ \cap (E \setminus L)} \cap U^+ \subseteq U^+ \cap \overline{(E \setminus L)} = (U^+ \cap (E \setminus L)) \cup (L \cap U^+), \tag{5.36}$$

where the equality follows from Remark 5.5 (III). From (5.35) and (5.36) we deduce $L \cap U^+ \neq \emptyset$, which contradicts (5.33).
Case 1. $U^- \cap (E \setminus L) = \emptyset$. Then from (5.34) it follows $U \cap (E \setminus L) = U^+$, and (5.1) (with $\mathcal{M}$ replaced by $E \cup L$) is a consequence of (5.31) and (5.32). We argue similarly in the case $U^+ \cap (E \setminus L) = \emptyset$.
Case 2. $U^\pm \cap (E \setminus L) \neq \emptyset$. Then from (5.34) it follows $U \cap (E \setminus L) = U \setminus L$, and (1.27) follows from (5.31) and (5.32).
This concludes the proof of step 4.

Step 5. We have

$$\partial(E \cup L) = L.$$

Since E is a $\mathcal{C}^{k-1}$-manifold without boundary in $\mathbb{R}^n \setminus L$ of dimension h (Remark 5.5 (I)), we have

$$\partial(E \cup L) \subseteq L.$$

To prove the converse inclusion, recalling also the proof of step 4 (see (5.30), (5.31) and (5.34)), it is sufficient to show that for any $\overline{x} \in L$ there is no relatively open neighborhood U of $\overline{x}$ in $\mathcal{N}$ such that $U \cap (E \setminus L) = U \setminus L$.
Let $\overline{x} \in L$, and recall once more the definition of B in (5.2), and that $L \subset \partial_A B$ (see (5.5)). From condition (iii) of Definition 5.4 we know that d_B is of class $\mathcal{C}^k$ in a neighborhood of $\overline{x}$. Hence there exist a neighborhhood $R \subset \mathbb{R}^n$ of $\overline{x}$, $\delta > 0$, and a map $\psi \in \mathcal{C}^k(R; \mathbb{R}^n)$ such that $\psi(R) = B_\delta(0)$, $\psi(R \cap B) = B_\delta(0) \cap \{x_n > 0\}$ and $\psi(R \cap \partial B) = B_\delta(0) \cap \{x_n = 0\}$ (in particular B locally lies on one side of ∂B).
If $h = n$, our assertion follows from the fact that $B = E \setminus L$ by (5.6).
Assume now $h < n$. Suppose by contradiction that there exist $\overline{x} \in L$ and a neighborhood U of $\overline{x}$ in $\mathcal{N}$ such that $U \setminus L = U \cap (E \setminus L)$. Since B is locally on one side of ∂B and $\overline{x} \in L \subset \partial_A B$, recalling also (5.4), we have

$$U \setminus L \subset B.$$

Moreover since U is relatively open in $\mathcal{N}$ and by (5.5) we have $L \subset \partial_A B$, we get

$$T_{\overline{x}}\mathcal{N} = T_{\overline{x}}U \subset T_{\overline{x}}\partial_A B.$$

Take $\xi \in B \setminus \mathcal{N}$ such that $\mathrm{dist}(\xi, \mathcal{N}) = |\xi - \overline{x}|$. Then

$$d_L(\xi) = \mathrm{dist}(\xi, L) = \mathrm{dist}(\xi, E \setminus L) = d_{E\setminus L}(\xi), \tag{5.37}$$

where the second equality follows from Remark 5.5 (III) and $\bar{x} \in L$, and the last equality follows from the fact that $E \setminus L \subset \mathcal{N}$ and $\xi \notin \mathcal{N}$, so that $\xi \notin E \setminus L$. Then (5.37) contradicts the inclusion $\xi \in B = \{z \in \mathbb{R}^n : d_{(E \setminus L) \cap \Omega}(z) < d_{L \cap \Omega}(z)\}$.
This concludes the proof when $\Omega = \mathbb{R}^n$. The proof when Ω is a nonempty open subset of $\mathbb{R}^n$ follows by replacing E with $E \cap \Omega$, L with $L \cap \Omega$, and $\mathbb{R}^n$ with Ω in the above arguments. □

Bibliography

[1] E. Acerbi, G. Dal Maso, *New lower semicontinuity results for polyconvex integrals*, Calc. Var. Partial Differential Equations **2** (1994), 329–371.

[2] L. Ambrosio, *Geometric evolution problems, distance function and viscosity solutions,* Calc. Var. Partial Differential Equations (Pisa,1996), Springer, Berlin (2000), 5–93.

[3] L. Ambrosio, B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), 527–555.

[4] L. Ambrosio, C. Mantegazza, *Curvature and distance function from a manifold*, J. Geom. Anal. **8** (1998), 723–748.

[5] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford, Clarendon Press (2000).

[6] L. Ambrosio, H. M. Soner, *Level set approach to mean curvature flow in arbitrary codimension*, J. Differential Geom. **43** (1996), 693–737.

[7] G. Bellettini, A. Elshorbagy, *On the square distance function from a manifold with boundary*, Submitted.

[8] G. Bellettini, A. Elshorbagy, M. Paolini, R. Scala, *On the relaxed area of the graph of discontinuous maps from the plane to the plane taking three values with no symmetry assumptions*, Annali di Matematica Pura ed Applicata (2019), DOI 10.1007/s10231-019-00887-0.

[9] G. Bellettini, M. Masala, M. Novaga, *On a conjecture of De Giorgi on the squared distance function*, J. Convex Anal. **14** (2007), 353–359.

[10] G. Bellettini, M. Novaga, M. Paolini, *An example of three-dimensional fattening for linked space curves evolving by curvature*, Comm. Partial Differential Equations **23** (1998), 1475–1492 .

[11] G. Bellettini, M. Paolini, *On the area of the graph of a singular map from the plane to the plane taking three values*, Adv. Calc. Var. **3** (2010), 371-386.

[12] G. Bellettini, M. Paolini, L. Tealdi, *On the area of the graph of a piecewise smooth map from the plane to the plane with a curve discontinuity*, ESAIM: Control, Optimization and Calculus of Variations **22** (2015), 29–63.

[13] G. Bellettini, M. Paolini, L. Tealdi, *Semicartesian surfaces and the relaxed area of maps from the plane to the plane with a line discontinuity*, Ann. Mat. Pura Appl. **195** (2016), 2131-2170.

[14] G. Buttazzo, Semicontinuity, Relaxation and Integral Representation Problems in the Calculus of Variations. Springer-Verlag, Berlin (1989).

[15] P. Cannarsa, C. Sinestrari, Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Progress in Nonlinear Differential Equations and Their Applications, Boston, Birkhäuser **58** (2004).

[16] J. B. Conway, A Course in Functional Analysis, Springer (1994).

[17] B. Dacorogna, Direct Methods in the Calculus of Variations, Berlin, Springer-Verlag (1989).

[18] G. Dal Maso, *Integral representation on* $\mathrm{BV}(\Omega)$ *of* Γ*-limits of variational integrals*, Manuscripta Mathematica **30** (1980), 387-416.

[19] E. De Giorgi, *Introduzione ai problemi di discontinuita libera*, Symmetry in Nature I, Scuola Norm. Sup. Pisa (1989), 265-285.

[20] E. De Giorgi, *On the relaxation of functionals defined on Cartesian manifolds*, in Developments in Partial Differential Equations and Applications in Mathematical Physics (Ferrara 1992), Plenum Press, New York (1992), 33-38.

[21] E. De Giorgi, Barriers, boundaries, motion of manifolds. Conference held at Department of Mathematics of Pavia, March 18 (1994).

[22] M. C. Delfour, J.-P. Zolésio, Shapes and Geometries. Analysis, Differential Calculus, and Optimization, Advances in Design and Control. 4. Philadelphia, PA: SIAM, (2011).

[23] U. Dierkes, S. Hildebrandt, F. Sauvigny, Minimal Surfaces, Grundlehren der mathematischen Wissenschaften, Springer **339** (2010).

[24] M. Do Carmo, Riemannian Geometry, Birkhäuser (1992).

[25] L.C. Evans, J. Spruck, *Motion of level sets by mean curvature II*, Trans. Amer. Math. Soc. **330** (1992), 321–332.

[26] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[27] N. Fusco, J.E. Hutchinson, *A direct proof for lower semicontinuity of polyconvex functionals*, Manuscripta Mat. **87** (1995), 35-30.

[28] M. Giaquinta, G. Modica, J. Souček, Cartesian currents in the calculus of variations I, Ergebnisse Math. Grenzgebiete (III Ser.) **37** (1998).

[29] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin Heidelberg New York (2001).

[30] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, Boston (1984).

[31] C. Goffman, J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Mathematical Journal **31.1** (1964), 159-178.

[32] V. Guillemin, A. Pollack, Differential Topology, Prentice-Hall, Englewood Cliffs, N.J. (1974).

[33] A. Magni, C. Mantegazza, M. Novaga, *Motion by curvature of planar networks II*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **XV** (2016), 117-144.

[34] C. Mantegazza, M. Novaga, A. Pluda, *Lectures on curvature flow of networks*, Springer INdAM Series, **33** (2019), 369–417.

[35] C. Mantegazza, M. Novaga, V.M. Tortorelli, *Motion by curvature of planar networks*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **3** (2004), 235–324.

[36] R. Scala, *Optimal estimates for the triple junction function and other surprising aspects of the area functional*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **20(2)** (2020).

[37] D. Slepcev, *On level set approach to motion of manifolds of arbitrary codimension*, Interfaces Free Bound. **5** (2003), 417–458.

[38] M. Spivak, Calculus on Manifolds, Westview Press (1965).

[39] L. Tealdi, The relaxed area of maps from the plane to the plane with a line discontinuity, and the role of semicartesian surfaces, Ph.D. Thesis, SISSA (2015).