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Ph.D. Thesis

On the cohomology of moduli spaces of stable maps to Grassmannians

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Abstract

Let $\overline{M}_{0,n}(G(r,V),d)$ be the coarse moduli space that parametrizes stable maps of class $d \in \mathbb{Z}_{\geq 0}$ from n-pointed genus 0 curves to a Grassmann variety $G(r,V)$. We provide a recursive method for the computation of the Betti numbers and the Hodge numbers of $\overline{M}_{0,n}(G(r,V),d)$ for all n and d. Our method is a generalization of Getzler and Pandharipande's work [GP06]. First, we reduce our problem to the calculation of the Hodge numbers of the open locus $M_{0,n}(G(r,V),d)$ corresponding to maps from smooth curves. Then, we show that those can be determined by considering a suitable compactification of the space of degree d morphisms from $\mathbb{P}^1$ to $G(r,V)$, combined with previous results on the configuration space of n distinct points on $\mathbb{P}^1$.

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Contents

Introduction

Moduli spaces constitute a subject of remarkable importance in algebraic geometry. Indeed, the problem of classifying a certain class of geometric objects, up to some equivalence relation, is ubiquitous in geometry. The study of the existence of moduli spaces parametrizing these objects aims at solving such classification problems. In fact, the knowledge of the geometry of these spaces also provides a large amount of additional information on the parametrized objects. Furthermore, moduli spaces have many important applications, both in algebraic geometry and in other fields. Just to mention one, they are used in enumerative geometry to define some enumerative invariants, such as Gromov-Witten and Donaldson-Thomas invariants; the properties of these invariants reflect the geometric properties of the moduli spaces which they are built upon. Lastly, some important tools of algebraic geometry stemmed from the study of moduli spaces. For instance, the language of stacks was first introduced precisely to deal with moduli problems. All these reasons justify the extensive interest of algebraic geometers in the geometry of moduli spaces.

Projective spaces and, more generally, Grassmann varieties are examples of moduli spaces. Other very important examples are provided by Hilbert and Quot schemes, whose introduction dates back to Grothendieck [Gro61]. All these schemes are instances of fine moduli spaces, because each of them represents the functor associated to the moduli problem in question. This ideal situation does not always occur, though. Indeed, the presence of non-trivial automorphisms of the considered objects prevents the corresponding moduli functor from being representable. In order to deal with this kind of situations, one can either adopt the stack-theoretic point of view, or replace the representability condition with a weaker one, leading to the concept of coarse moduli spaces. For example, unless $g = 0$, the spaces $\overline{M}_{g,n}$ of Deligne-Mumford stable n-marked genus g curves (for $2g-2+n > 0$) are only coarse moduli spaces. We refer to [HM98] and [ACG11] for more details about the theory of moduli spaces, especially regarding moduli of curves.

In view of the connections with the moduli of stable maps, let us just recall some well-known facts about the geometry of $\overline{M}_{g,n}$ (over $\mathbb{C}$). The space $\overline{M}_{g,n}$ is an irreducible projective variety of dimension $3g-3+n$, with local quotient singularities; the corresponding moduli stack $\overline{\mathcal{M}}_{g,n}$ is a proper nonsingular Deligne-Mumford stack. For $n = 0$, $\overline{M}_{0,n}$ is actually a fine moduli space and a nonsingular variety. Up to a finite group quotient, the boundary of $\overline{M}_{g,n}$, i.e., the locus parametrizing singular curves, is a divisor with normal crossings. Moreover, $\overline{M}_{g,n}$ can be stratified by considering the topological type of the parametrized curves; each stratum corresponds to curves that have a fixed dual marked graph. Using the natural gluing morphisms $\overline{M}_{g,n+1} \times \overline{M}_{g',n'+1} \to \overline{M}_{g+g',n+n'}$ and $\overline{M}_{g,n+2} \to \overline{M}_{g+1,n}$, the boundary strata can be given a combinatorial description (see [ACG11, §10]). The combinatorial properties of the spaces $\overline{M}_{g,n}$ can be used to study their (co)homology or Chow ring. See, for instance, [Kee92] for a complete description of the Chow ring of $\overline{M}_{0,n}$. In this context, operad formalism comes into the picture, as a systematic way of encoding these combinatorial properties (see [GK98]). The theory of operads has revealed itself to be an important tool for studying the (co)homology of moduli spaces of stable curves (see [Get95b]).

In this thesis, we are primarily interested in moduli spaces of stable maps, which generalize the spaces of Deligne-Mumford stable curves recalled above. They were introduced by M. Kontsevich in [Kon95], in order to rigorously define Gromov-Witten invariants and to provide a complete proof of some enumerative geometry predictions which arose in string theory. These spaces are relevant not only in algebraic and enumerative geometry, but also in other fields of mathematics, such as symplectic topology, and of physics, such as string theory. Given $g, n \in \mathbb{Z}_{\geq 0}$, a nonsingular projective variety Y, and a curve class $\beta \in \mathrm{H}_2(Y, \mathbb{Z})$, we denote by $\overline{M}_{g,n}(Y, \beta)$ the coarse moduli space parametrizing stable maps of class β from genus g n-marked curves to Y. The complete definition of $\overline{M}_{g,n}(Y, \beta)$, together with its properties, can be found in Section 2.1. In particular, the object of our study are moduli spaces of stable maps from genus 0 curves. When Y is a convex variety, the space $\overline{M}_{0,n}(Y, \beta)$ shares many features of the spaces $\overline{M}_{g,n}$. Indeed, it is a normal projective variety which only has local quotient singularities, and the corresponding moduli stack $\overline{\mathcal{M}}_{0,n}(Y, \beta)$ is a proper nonsingular Deligne-Mumford stack. Its boundary, i.e., the locus of maps from reducible curves, is a divisor with normal crossings, up to a finite group quotient. Furthermore, $\overline{M}_{0,n}(Y, \beta)$ can be stratified by strata corresponding to isomorphism classes of stable (n, β)-trees, and its boundary strata can be given a combinatorial description similar to that of the boundary strata of $\overline{M}_{0,n}$ (see Section 2.2). As in the case of $\overline{M}_{0,n}$, these combinatorial properties of $\overline{M}_{0,n}(Y, \beta)$ can be used to study its cohomology.

Actually, computing the Betti numbers of $\overline{M}_{0,n}(Y, \beta)$ is already a nontrivial problem. For $Y = \mathbb{P}^r$, this problem was solved by E. Getzler and R. Pandharipande in [GP06], where they computed the Σ_n-equivariant Serre characteristic (i.e., a refined version of the E-polynomial) of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. Their main idea was to recursively reduce the computation of the Serre characteristic of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ to the computation of the Serre characteristics of the open loci $M_{0,m}(\mathbb{P}^r, \delta)$ (for $0 \leq m \leq \max\{n, d\}$ and $0 \leq \delta \leq d$) where the domain curves are smooth, using the connection between the geometry of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and the combinatorics of marked trees. Indeed, these combinatorial properties can be translated into recursive relations in the rings $(\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{Var}, \Sigma_n))[\![q]\!]$ and $(\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{MHS}, \Sigma_n))[\![q]\!]$, via the so-called composition structures of those rings. Here, $\mathrm{K}_0(\mathsf{Var}, \Sigma_n)$ (resp. $\mathrm{K}_0(\mathsf{MHS}, \Sigma_n)$) denotes the Grothendieck group of complex quasi-projective varieties (resp., of $\mathbb{Q}$-mixed Hodge structures) with an action of the symmetric group Σ_n. Once they reduced the problem to the computation of the Serre characteristic of $M_{0,m}(\mathbb{P}^r, \delta)$, they showed how to determine it explictly. This was done in terms of the Serre characteristic of $\mathrm{Mor}_\delta(\mathbb{P}^1, \mathbb{P}^r)$, which was computed by means of a certain projective space compactification, and of the Serre characteristic of the configuration space $\mathrm{F}(\mathbb{P}^1, m)$, which had already been determined (see [Get95a]). This way, they obtained a recursive method for the computation of the Serre characteristic of $\overline{M}_{0,n}(\mathbb{P}^r, d)$, from which its Hodge numbers can be recovered.

The aim of this thesis is to extend the method of [GP06] to the case of stable maps to a Grassmann variety $G(r, V)$. For some special values of n and d, there are some computations of the Poincaré polynomial of these spaces in the literature (see, for instance, [Ló14]), which are performed using a locally closed decomposition determined by a certain torus action (a so-called Białynicki-Birula decomposition). However, such computations are limited to $n = 0$ and $d \leq 3$. Therefore, in order to determine the Betti and the Hodge numbers of $\overline{M}_{0,n}(G(r, V), d)$ for all n and d, we developed an alternative method that generalizes [GP06]. The outcome of the thesis is a recursive algorithm which allows one to compute the Serre characteristic of $\overline{M}_{0,n}(G(r, V), d)$ for any V, r, n, d.

Organization of the thesis

In Chapter 1, we study more closely the composition structures introduced in [GP06]. In particular, we highlight the relation between those structures and analogous structures arising in operad theory, and we use this relation to present the detailed proofs of some properties that are only stated in [GP06]. We also introduce and study the rings where our calculations will take place, namely $(\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{Var}, \Sigma_n))[\![q]\!]$ and $(\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{MHS}, \Sigma_n))[\![q]\!]$, with a special focus on their composition structures. Finally, we recall the definition of the Serre characteristic $\mathfrak{e} : (\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{Var}, \Sigma_n))[\![q]\!] \to (\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{MHS}, \Sigma_n))[\![q]\!]$ and its properties. As we mentioned above, the Serre characteristic is a refined version of the E-polynomial; in particular, from the knowledge of the Serre characteristic of $\overline{M}_{0,n}(G(r, V), d)$ one can recover its Hodge numbers.

In Chapter 2, we introduce the moduli spaces of stable maps, and we recall some well-known facts about their geometry. In the case of stable maps from genus 0 curves to a convex variety, we examine the stratification of these spaces obtained by associating to each map a dual marked tree, and we describe the boundary strata of this decomposition.

In Chapter 3, we use the stratification studied in Chapter 2 and the composition structures of Chapter 1 to show that suitable recursive relations hold in $(\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{Var}, \Sigma_n))[\![q]\!]$ and $(\prod_{n=0}^{\infty} \mathrm{K}_0(\mathsf{MHS}, \Sigma_n))[\![q]\!]$. Those relations allow one to express $\mathfrak{e}(\overline{M}_{0,n}(G(r,V),d))$ in terms of $\mathfrak{e}(M_{0,m}(G(r,V),\delta))$ for $m \leq \max\{n,d\}$ and $\delta \leq d$, and they are analogous to those of [GP06] for the case of $\mathbb{P}^r$. In particular, this is a consequence of the fact that the natural evaluation morphism $\mathrm{ev}_{n+1} : \overline{M}_{0,n+1}(G(r,V),d) \to G(r,V)$ is Zariski locally trivial for any Grassmannian $G(r,V)$, and not only for projective spaces. After reducing our problem to the computation of $\mathfrak{e}(M_{0,m}(G(r,V),\delta))$, we show that the latter is determined by $\mathfrak{e}(\mathrm{Mor}_\delta(\mathbb{P}^1, G(r,V)))$ and $\mathfrak{e}(\mathrm{F}(\mathbb{P}^1,m))$. As $\mathfrak{e}(\mathrm{F}(\mathbb{P}^1,m))$ is known, this in turn reduces our problem to the calculation of $\mathfrak{e}(\mathrm{Mor}_\delta(\mathbb{P}^1, G(r,V)))$.

Chapter 4 is precisely devoted to the calculation of $\mathfrak{e}(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)))$. More generally, we compute the motive of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$, i.e., its class in the Grothendieck group $\mathrm{K}_0(\mathsf{Var}_{\mathbb{C}})$ of $\mathbb{C}$-varieties. Since each degree d morphism $\mathbb{P}^1 \to G(r,V)$ corresponds to a locally free quotient of $V \otimes \mathcal{O}_{\mathbb{P}^1}$ of rank r and degree d, the Quot scheme $\overline{Q}_d = \mathrm{Quot}^{(t+1)r+d}_{V\otimes\mathcal{O}_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{C}}$ is a compactification of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$, where the quotients may degenerate to non-locally free ones. We study this compactification, and we decompose it into locally closed subvarieties R_δ corresponding to quotients having a torsion subsheaf of fixed length $\delta \leq d$. The open locus in this decomposition is exactly $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$. Via power series manipulations, we can thus express the motive of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ in terms of the motives of R_δ and $\overline{Q}_\delta$ for $\delta \leq d$. Since the motive of $\overline{Q}_\delta$ can be recovered from [Str87] and that of R_δ can be computed using the results of the subsequent chapter, we obtain an explicit procedure for calculating the motive of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ for any d. Combined with the results of the previous chapters, this provides an effective recursive algorithm to determine the Serre characteristic of $\overline{M}_{0,n}(G(r,V),d)$, thus solving our original problem.

Chapter 5 is an independent part of our work. For the purpose of the thesis, its main outcome is a formula for the motive of the above-mentioned varieties R_δ. In fact, in Chapter 5 we consider a more general problem, namely we study the motive of Quot schemes of zero-dimensional quotients of a locally free sheaf on a smooth projective curve. Our main result in this context is the proof that this motive only depends on the rank of the sheaf and on the length of the quotients. This fact allows us to compute it explicitly assuming the sheaf to be trivial and using previous results of [Bif89].

Chapter 1

Composition structures

In this chapter, we study the algebraic setting where our calculations will take place, namely the Grothendieck groups of varieties and mixed Hodge structures with symmetric group actions. A particularly important structure on these groups is the so-called composition operation, which was introduced in [GP06]. This operation is fundamental in order to translate the operadic properties of the moduli spaces of stable maps into relations in the above-mentioned Grothendieck groups. We highlight the connection between the composition structures of [GP06] and some well-known structures which arise in the theory of operads, notably starting from [Kel05]. In addition to the clarification of this interesting connection, our categorical approach allows us to prove some assertions of [GP06] using simple categorical arguments. Having in mind a possible generalization of our work to spaces of stable maps to other targets, we mostly consider a general monoid H of curve classes in our dissertation.

Organization of the chapter. Section 1.1 is devoted to the study of functors from the permutation groupoid to a cosmos. The advantage of considering cosmoi is that some general categorical arguments can be used to treat this case. In §1.1.1 we recall the definition and some basic properties of coends, which we use in §1.1.2 to define a certain tensor product $*$ on the category of functors from P to a cosmos, following [Kel05]. In §1.1.3, we introduce all the other ingredients occuring in the definition of a composition operation. The properties of cosmoi allow us to prove easily that these elements interact with $*$ as required by the axioms defining the composition structures in [GP06].

In Section 1.2, the connection between the composition algebras of [GP06] and the structures introduced in Section 1.1 is made explicit. We prove that both the category Var of quasi-projective $\mathbb{C}$-varieties and the category MHS of $\mathbb{Q}$-mixed Hodge structures can be embedded in suitable cosmoi, so that the structures defined in [GP06] correspond to the ones introduced in Section 1.1 via this embedding. More generally, we also prove the analogous results in the corresponding categories of H-graded objects, where H is the monoid of curve classes of any smooth projective variety.

In Section 1.3, we study the Grothendieck ring of quasi-projective varieties with a symmetric group action, using the results of Section 1.2. In particular, we study the pre-λ-ring structure on this ring in §1.3.1. Similarly, we study the Grothendieck ring of mixed Hodge structures with a symmetric group action in Section 1.4. Finally, in Section 1.5, we introduce the Serre characteristic, and we describe some of its properties.

Notation. For all categories $\mathscr{C}$ and $\mathscr{D}$ such that $\mathscr{C}$ is small, $[\mathscr{C}, \mathscr{D}]$ denotes the category of functors from $\mathscr{C}$ to $\mathscr{D}$, with natural transformations of such functors as morphisms. For any object c of a category, $1_c : c \to c$ denotes the identity morphism of c.

We assume that if the (co)limit of a functor exists then a definite choice has been made of it. Given a collection $(c_j)_j$ of objects in a category $\mathscr{C}$, such that the coproduct $\bigsqcup_j c_j$ exists, the coprojection $c_k \to \bigsqcup_j c_j$ is denoted by i_k. For any cocone $(r, (f_j : c_j \to r)_j)$ the canonical morphism $\bigsqcup_j c_j \to r$ is denoted by $[f_j]$.

For any $n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$, Σ_n denotes the symmetric group on n elements.

1.1 The category of P-objects of a cosmos

1.1.1 Coends

Definition 1.1.1. Let $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$ be a functor. A cowedge of F consists of an object r of $\mathscr{D}$ equipped with a collection of arrows $(\alpha_c : F(c,c) \to r)_{c \in \mathscr{C}}$, such that for every arrow $f : c \to c'$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd}
F(c',c) \arrow[r, "F(1_{c'},f)"] \arrow[d, "F(f,1_c)"] & F(c',c') \arrow[d, "\alpha_{c'}"] \\
F(c,c) \arrow[r, "\alpha_c"] & r
\end{tikzcd}$$

commutes.

A coend of F is a universal cowedge, that is, a cowedge $(\alpha_c : F(c,c) \to r)_c$ such that for any other cowedge $(\beta_c : F(c,c) \to d)_c$, there exists a unique arrow $h : r \to d$ which makes all the diagrams

$$\begin{tikzcd}
F(c',c) \arrow[r, "F(1_{c'},f)"] \arrow[d, "F(f,1_c)"] & F(c',c') \arrow[d, "\alpha_{c'}"] \arrow[ddr, bend left, "\beta_{c'}"] & \\
F(c,c) \arrow[r, "\alpha_c"] \arrow[drr, bend right, "\beta_c"] & r \arrow[dr, "h"] & \\
 & & d
\end{tikzcd}$$

commute.

Because of its universal property, a coend of a functor is unique up to unique isomorphism. If (r, α) is the coend of F, then the object r is denoted by $\int^c F(c, c)$.

The universal property of coends also implies their functoriality. Indeed, let $F, G : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$ be two functors, with coends $(\int^c F(c, c), \alpha)$ and $(\int^c G(c, c), \beta)$, respectively. Then any natural transformation $\gamma : F \Rightarrow G$ induces a canonical morphism $h_\gamma : \int^c F(c, c) \to \int^c G(c, c)$, such that the diagram

$$\begin{tikzcd}
F(c,c) \arrow[r, "\gamma_{(c,c)}"] \arrow[d, "\alpha_c"] & G(c,c) \arrow[d, "\beta_c"] \\
\int^c F(c,c) \arrow[r, "h_\gamma"] & \int^c G(c,c)
\end{tikzcd}$$

commutes for any object c of $\mathscr{C}$.

Remark 1.1.2. There is a canonical isomorphism

$$\int^c F(c,c) \xrightarrow{\sim} \operatorname{coeq}\left(\coprod_{\substack{c,c' \in \mathscr{C} \\ f \in \mathscr{C}(c,c')}} F(c',c) \underset{[i_c \circ F(f,1_c)]}{\overset{[i_{c'} \circ F(1_{c'},f)]}{\rightrightarrows}} \coprod_{x \in \mathscr{C}} F(x,x) \right),$$

where the morphisms $i_c : F(c, c) \to \coprod_{x \in \mathscr{C}} F(x, x)$ are the coprojections.

1.1.2 Monoidal structures

Coends can be used to define two tensor products on the functor category $[\mathsf{P}, \mathscr{V}]$, where P is the permutation groupoid and $\mathscr{V}$ is a cosmos.

Recall that P is the category whose objects are the nonnegative integers $n \in \mathbb{N}$, with $\mathsf{P}(m, n) = \emptyset$ if $m \neq n$ and $\mathsf{P}(n, n) = \Sigma_n$, Σ_n being the symmetric group on n elements. The groupoid P has a symmetric monoidal structure, which is given by the sum of nonnegative integers; $\sigma + \tau : m + n \to m + n$ is the permutation which acts as σ on $\{1, \ldots, m\}$ and as $\tau(\bullet - m) + m$ on $\{m + 1, \ldots, m + n\}$.

Definition 1.1.3. A cosmos is a closed symmetric monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$ which is both complete and cocomplete; $\otimes$ denotes its tensor product, and $\mathbb{1}$ denotes its identity object.

This definition of a cosmos is the original one by J. Bénabou. Note that there are different nonequivalent definitions in the literature.

For any object A of $\mathscr{V}$, the functor $\bullet \otimes A : \mathscr{V} \to \mathscr{V}$ preserves colimits, because it has a right adjoint. Moreover, the hom-functor $\mathscr{V}(\mathbb{1}, \bullet) : \mathscr{V} \to \mathsf{Set}$

has a left adjoint $L : \mathsf{Set} \to \mathscr{V}$, which is defined as

$$LS := \coprod_S \mathbb{1}\,, \qquad L(f : S \to S') := [i_{f(s)}] : LS \to LS'\,.$$

There are canonical isomorphisms $L(S \times S') \cong LS \otimes LS'$ and $L\{\bullet\} \cong \mathbb{1}$. By Mac Lane's coherence theorem, we may assume that these isomorphisms are identities, without loss of generality. If S is a set and A is an object of $\mathscr{V}$, we write $S \otimes A$ for $LS \otimes A$.

Let $(\mathscr{V}, \otimes, \mathbb{1})$ be a fixed cosmos. Then for all functors $X, Y : \mathsf{P} \to \mathscr{V}$, one can define another functor $X * Y : \mathsf{P} \to \mathscr{V}$ as

$$X * Y := \int^{m,n} \mathsf{P}(m+n, \bullet) \otimes Xm \otimes Yn\,.$$

This is usually referred to as Day's convolution product.

Theorem 1.1.4 ([Day70])**.** *$([\mathsf{P}, \mathscr{V}], *)$ is a closed symmetric monoidal category, with identity object $\mathsf{P}(0, \bullet) \otimes \mathbb{1}$.*

Since $*$ is a symmetric tensor product on the category $[\mathsf{P}, \mathscr{V}]$, the assignment $(n, X) \mapsto X^{*n}$ gives a functor $\mathsf{P}^{\mathrm{op}} \times [\mathsf{P}, \mathscr{V}] \to [\mathsf{P}, \mathscr{V}]$. Therefore, one can define another bifunctor $\diamond : [\mathsf{P}, \mathscr{V}] \times [\mathsf{P}, \mathscr{V}] \to [\mathsf{P}, \mathscr{V}]$ by setting

$$X \diamond Y := \int^{n} Xn \otimes Y^{*n}$$

for all $X, Y : \mathsf{P} \to \mathscr{V}$.

Theorem 1.1.5 ([Kel05])**.** *$([\mathsf{P}, \mathscr{V}], \diamond)$ is a (non-symmetric) monoidal category, with identity object $\mathsf{P}(1, \bullet) \otimes \mathbb{1}$. Furthermore, this monoidal structure is left closed.*

By Remark (1.1.2), there is an useful description of both $*$ and $\diamond$ via coproducts and coequalizers. Day's convolution product $X * Y$ is canonically isomorphic to the coequalizer of

$$\coprod_{\substack{i,j \in \mathbb{N} \\ (\sigma,\tau) \in \Sigma_i \times \Sigma_j}} \mathsf{P}(i+j, \bullet) \otimes Xi \otimes Yj \;\underset{[1 \otimes X\sigma \otimes Y\tau]}{\overset{[\mathsf{P}(\sigma+\tau, \bullet) \otimes 1 \otimes 1]}{\rightrightarrows}}\; \coprod_{i,j \in \mathbb{N}} \mathsf{P}(i+j, \bullet) \otimes Xi \otimes Yj\,.$$

Since $\mathsf{P}(i+j, n) = \emptyset$ whenever $n \neq i+j$, it follows that $(X*Y)n$ is canonically isomorphic to

$$\coprod_{\substack{i,j \in \mathbb{N} \\ i+j=n}} \mathrm{coeq}\left(\coprod_{(\sigma,\tau) \in \Sigma_i \times \Sigma_j} \Sigma_n \otimes Xi \otimes Yj \;\underset{[1 \otimes X\sigma \otimes Y\tau]}{\overset{[\mathsf{P}(\sigma+\tau, n) \otimes 1 \otimes 1]}{\rightrightarrows}}\; \Sigma_n \otimes Xi \otimes Yj \right).$$

Similarly, $(X \diamond Y)n$ is canonically isomorphic to

$$\bigsqcup_{i\in\mathbb{N}} \mathrm{coeq}\left(\bigsqcup_{\sigma\in\Sigma_i} Xi \otimes Y^{*i}n \underset{[X\sigma\otimes 1]}{\overset{[1\otimes\langle\sigma\rangle_n]}{\rightrightarrows}} Xi \otimes Y^{*i}n \right),$$

where $\langle\sigma\rangle : Y^{*i} \to Y^{*i}$ is the iterated symmetry isomorphism.

We conclude this section considering a slight variation of the previous setting, which will be useful in the subsequent sections.

Definition 1.1.6. If $(H, +, 0)$ is a commutative monoid and $\mathscr{C}$ is a category, an H-graded object of $\mathscr{C}$ is a functor $\underline{H} \to \mathscr{C}$, where $\underline{H}$ is the discrete category whose set of objects is H.

Given a cosmos $(\mathscr{V}, \otimes, \mathbb{1})$, the category $\mathscr{V}' = [\underline{H}, \mathscr{V}]$ of H-graded objects of $\mathscr{V}$ is again a cosmos, with Day's convolution

$$F \otimes_{\mathscr{V}'} G = \int^{u,v} \underline{H}(u+v, \bullet) \otimes Fu \otimes Gv$$

as tensor product and $\mathbb{1}_{\mathscr{V}'} = \underline{H}(0, \bullet) \otimes \mathbb{1}$ as identity object (see [Day70]). Therefore, Theorem 1.1.4 and Theorem 1.1.5 apply to $[\mathsf{P}, \mathscr{V}']$.

Since $\underline{H}$ is a discrete category, there are canonical isomorphisms

$$(F \otimes_{\mathscr{V}'} G)w \cong \bigsqcup_{\substack{u,v\in H\\ u+v=w}} Fu \otimes Gv \qquad \text{and} \qquad (\mathbb{1}_{\mathscr{V}'})w \cong \begin{cases} \mathbb{1} & \text{if } w = 0 \\ \varnothing & \text{if } w \neq 0 \end{cases},$$

where $\varnothing$ is the initial object of $\mathscr{V}$. Thus, we obtain the following canonical isomorphism for the tensor products $*$ and $\diamond$ on $[\mathsf{P}, [\underline{H}, \mathscr{V}]]$:

$$(X * Y)(n, w) \cong \bigsqcup_{\substack{u,v\in H\\ u+v=w}} \int^{i,j} \mathsf{P}(i+j, n) \otimes X(i, u) \otimes Y(j, v)\,,$$

$$(X \diamond Y)(n, w) \cong \bigsqcup_{\substack{u,v\in H\\ u+v=w}} \int^{i} X(i, u) \otimes Y^{*i}(n, v)\,.$$

In particular, if H has indecomponsable zero, then the embedding of $[\mathsf{P}, \mathscr{V}]$ into $[\mathsf{P}, [\underline{H}, \mathscr{V}]]$ given by

$$X \mapsto X(n, w) = \begin{cases} Xn & \text{if } w = 0 \\ \varnothing & \text{if } w \neq 0 \end{cases}, \qquad \gamma \mapsto \gamma_{(n,w)} = \begin{cases} \gamma_n & \text{if } w = 0 \\ 1_\varnothing & \text{if } w \neq 0 \end{cases}$$

is a symmetric monoidal functor with respect to both $*$ and $\diamond$.

With a slight abuse of notation, we identify the (canonically isomorphic) categories $[\mathsf{P}, [\underline{H}, \mathscr{V}]]$, $[\underline{H}, [\mathsf{P}, \mathscr{V}]]$ and $[\mathsf{P} \times \underline{H}, \mathscr{V}]$.

1.1.3 Additional structures

By analogy with [GP06], we consider some additional structures on the category of functors from P to a category $\mathscr{C}$. We introduce them for general $\mathscr{C}$, and we study their relation with $*$ and $\diamond$ when $\mathscr{C}$ is a cosmos.

First, let $D : [\mathsf{P}, \mathscr{C}] \to [\mathsf{P}, \mathscr{C}]$ be the endofunctor defined by

$$DX := X(\bullet + 1)\,, \qquad (D\gamma)_\bullet := \gamma_{\bullet+1}$$

for all objects X and arrows γ of $[\mathsf{P}, \mathscr{C}]$. If $\mathscr{C}$ is a cosmos $(\mathscr{V}, \otimes, \mathbb{1})$, then $D : [\mathsf{P}, \mathscr{V}] \to [\mathsf{P}, \mathscr{V}]$ satisfies Leibniz's rule with respect to $*$, in the following sense.

Proposition 1.1.7. *For all functors $X, Y : \mathsf{P} \to \mathscr{V}$, there is a natural isomorphism*

$$D(X * Y) \cong (DX * Y) \sqcup (X * DY)\,.$$

Proof. By definition,

$$D(X * Y) = \int^{i,j} \mathsf{P}(i + j, \bullet + 1) \otimes Xi \otimes Yj\,.$$

The functor $\mathsf{P}(i + j, \bullet + 1) : \mathsf{P} \to \mathsf{Set}$ is isomorphic to

$$\int^{n} \mathsf{P}(n + j, \bullet) \times \mathsf{P}(i, n + 1) \sqcup \int^{n} \mathsf{P}(i + n, \bullet) \times \mathsf{P}(j, n + 1)\,,$$

and this isomorphism is natural in both i and j. Therefore

$$\begin{aligned} D(X * Y) \cong \int^{n,j} \mathsf{P}(n + j, \bullet) \otimes \int^{i} \mathsf{P}(i, n + 1) \otimes Xi \otimes Yj \\ \sqcup \int^{i,n} \mathsf{P}(i + n, \bullet) \otimes Xi \otimes \int^{j} \mathsf{P}(j, n + 1) \otimes Yj\,, \end{aligned}$$

because left adjoint functors preserve colimits. Using the fact that for any $Z : \mathsf{P} \to \mathscr{V}$ there is a canonical isomorphism $DZ \cong \int^{n} \mathsf{P}(n, \bullet + 1) \otimes Zn$, one gets

$$D(X * Y) \cong \int^{n,j} \mathsf{P}(n + j, \bullet) \otimes (DX)n \otimes Yj \sqcup \int^{i,n} \mathsf{P}(i + n, \bullet) \otimes Xi \otimes (DY)n\,,$$

that is, $D(X * Y) \cong (DX * Y) \sqcup (X * DY)$. $\square$

Using this result, we can also describe the behaviour of D with respect to $\diamond$. In particular, we see that $D : [\mathsf{P}, \mathscr{V}] \to [\mathsf{P}, \mathscr{V}]$ satisfies an analogue of the chain rule.

Proposition 1.1.8. *For all functors $X, Y : \mathsf{P} \to \mathscr{V}$, there is a natural isomorphism*

$$D(X \diamond Y) \cong (DX \diamond Y) * DY\,.$$

Proof. By Proposition 1.1.7, there is a natural isomorphism

$$D(Y^{*\bullet}) \cong \int^{n} \mathsf{P}(\bullet, n+1) \otimes Y^{*n} * DY$$

of functors $\mathsf{P}^{\mathrm{op}} \to [\mathsf{P}, \mathscr{V}]$. Therefore

$$\begin{aligned}
D(X \diamond Y) &= \int^{m} Xm \otimes Y^{*m}(\bullet + 1) \\
&\cong \int^{m} Xm \otimes D(Y^{*m}) \\
&\cong \int^{m} Xm \otimes \int^{n} \mathsf{P}(m, n+1) \otimes Y^{*n} * DY \\
&\cong \int^{i,j} \mathsf{P}(i+j, \bullet) \otimes \int^{n,m} \mathsf{P}(m, n+1) \otimes Xm \otimes Y^{*n}i \otimes (DY)j \\
&\cong \int^{i,j} \mathsf{P}(i+j, \bullet) \otimes \int^{n} (DX)n \otimes Y^{*n}i \otimes (DY)j \\
&\cong \int^{i,j} \mathsf{P}(i+j, \bullet) \otimes (DX \diamond Y)i \otimes (DY)j \\
&\cong (DX \diamond Y) * DY\,,
\end{aligned}$$

as claimed. □

Now, let us assume that $(\mathscr{C}, \otimes, \mathbb{1})$ is a symmetric monoidal category with an initial object $\varnothing$. Then there is a special sequence $(S_n)_{n \in \mathbb{N}}$ of objects of $[\mathsf{P}, \mathscr{C}]$: for any $n \in \mathbb{N}$, $S_n : \mathsf{P} \to \mathscr{C}$ is defined by

$$S_n m := \begin{cases} \mathbb{1} & \text{if } m = n \\ \varnothing & \text{if } m \neq n \end{cases}, \qquad S_n \sigma := \begin{cases} 1_{\mathbb{1}} & \text{if } m = n \\ 1_{\varnothing} & \text{if } m \neq n \end{cases}$$

for every $m \in \mathbb{N}$ and $\sigma \in \Sigma_m$. In particular, notice that $DS_n = S_{n-1}$ for all $n > 0$.

As before, when $\mathscr{C}$ is a cosmos $(\mathscr{V}, \otimes, \mathbb{1})$, these functors $S_n : \mathsf{P} \to \mathscr{V}$ have some interesting properties in relation to the tensor products $*$ and $\diamond$ on $[\mathsf{P}, \mathscr{V}]$.

Proposition 1.1.9. *(i) S_0 is the identity object for $*$.*

(ii) S_1 is the identity object for $\diamond$.

(iii) For all functors $X, Y : \mathsf{P} \to \mathscr{V}$ and all $n \in \mathbb{N}$, there is a natural isomorphism

$$S_n \diamond (X \sqcup Y) \cong \bigsqcup_{m=0}^{n} (S_m \diamond X) * (S_{n-m} \diamond Y)\,.$$

Proof. (i) $S_0 \cong \mathsf{P}(0,\bullet) \otimes I$.

(ii) $S_1 \cong \mathsf{P}(1,\bullet) \otimes I$.

(iii) We have

$$\begin{aligned} S_n \diamond (X \sqcup Y) &= \int^m S_n m \otimes (X \sqcup Y)^{*m} \\ &\cong \int^m S_n m \otimes \int^{i,j} \mathsf{P}(m, i+j) \otimes X^{*i} * Y^{*j} \\ &\cong \int^{i,j} S_n(i+j) \otimes X^{*i} * Y^{*j} \\ &\cong \bigsqcup_{m=0}^{n} \int^{i,j} S_m i \otimes S_{n-m} j \otimes X^{*i} * Y^{*j} \\ &\cong \bigsqcup_{m=0}^{n} (S_m \diamond X) * (S_{n-m} \diamond Y)\,, \end{aligned}$$

as claimed. □

1.2 Graded P-objects of Var and MHS

Let Var be the category of quasi-projective varieties over $\mathbb{C}$. For brevity, we refer to functors $\mathsf{P} \to \mathsf{Var}$ as P-varieties. Following our terminology, for any monoid H, a functor $\underline{H} \to [\mathsf{P}, \mathsf{Var}]$ (equivalently, $\mathsf{P} \times \underline{H} \to \mathsf{Var}$) will then be called an H-graded P-variety.

Let us fix a commutative monoid $(H, +, 0)$ with the following properties:

$(H1)$ for any $w \in H$, the set $A_i(w) = \{(w_1, \ldots, w_i) \in H^i \mid w = w_1 + \cdots + w_i\}$ is finite for all $i \geq 2$;

$(H2)$ for any $w \in H$, $A_i^\star(w) = \{(w_1, \ldots, w_i) \in A_i(w) \mid w_j \neq 0\ \forall j\}$ is empty for almost all i.

For example, if Y is a smooth projective variety over $\mathbb{C}$, then

$$\mathrm{H}_2(Y)^+ = \{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(Y), \mathbb{Z}) \mid \varphi(\mathcal{L}) \geq 0 \text{ whenever } \mathcal{L} \text{ is ample}\}$$

is such a monoid. In particular, note that H has indecomposable zero.

We study the category of H-graded P-varieties. As a motivation for studying such objects, note that they arise when considering moduli spaces of stable maps. Indeed, for Y a smooth projective variety, $g, n \in \mathbb{N}$ and $\beta \in \mathrm{H}_2(Y)^+$, the coarse moduli space $\overline{M}_{g,n}(Y, \beta)$ of stable maps from genus g, n-pointed curves to Y of class β carries an action of Σ_n, which is given on closed points by permuting the markings. Therefore, varying $n \in \mathbb{N}$ and $\beta \in H = \mathrm{H}_2(Y)^+$, we get exactly a functor

$$\overline{M}_{g,-}(Y,\bullet) : \mathsf{P} \times \underline{H} \to \mathsf{Var}\,.$$

In fact, $\mathbb{N}$-graded P-varieties appear in [GP06], where stable maps to a complex projective space $\mathbb{P}^r$ (note that $\mathbb{N} = \mathrm{H}_2(\mathbb{P}^r)^+$) are considered.

In [GP06], two operations $\boxtimes$ and $\circ$ between $\mathbb{N}$-graded P-varieties are defined: for all $n, d \in \mathbb{N}$,

$$\begin{aligned}
(X \boxtimes Y)(n,d) &:= \bigsqcup_{\delta=0}^{d} \bigsqcup_{m=0}^{n} \mathrm{Ind}_{\Sigma_m \times \Sigma_{m-n}}^{\Sigma_n} (X(m, d-\delta) \times Y(n-m, \delta))\,, \\
(X \circ Y)(n,d) &:= \bigsqcup_{\delta=0}^{d} \bigsqcup_{i=0}^{\infty} \big(X(i, d-\delta) \times Y^{\boxtimes i}(n,\delta)\big)/\Sigma_i\,.
\end{aligned}$$

In order for $\circ$ to be well-defined, one needs to assume that $Y(0,0) = \emptyset$.

These two operations have a straightforward generalization when H is a monoid as above. Let X and Y be H-graded P-varieties. For every $i, j \in \mathbb{N}$ and $u, v \in H$, the morphism

$$\begin{array}{c}
\Sigma_i \times \Sigma_j \times \Sigma_{i+j} \times X(i,u) \times Y(j,v) = \displaystyle\bigsqcup_{(\sigma,\tau)\in\Sigma_i\times\Sigma_j} \Sigma_{i+j} \times X(i,u) \times Y(j,v) \\
\Big\downarrow {\scriptstyle a_{\boxtimes} = \left[\left(\bullet \circ (\sigma+\tau)^{-1}\right) \times X(\sigma, 1_u) \times Y(\tau, 1_v)\right]} \\
\Sigma_{i+j} \times X(i,u) \times Y(j,v)
\end{array}$$

defines a free action of $\Sigma_i \times \Sigma_j$ on $\Sigma_{i+j} \times X(i,u) \times Y(j,v)$; on closed points of $\Sigma_{i+j} \times X(i,u) \times Y(j,v)$, this is given by

$$(\sigma, \tau) \cdot (\rho, x, y) = \big(\rho(\sigma+\tau)^{-1}, \sigma \cdot x, \tau \cdot y\big)\,.$$

Since $\Sigma_{i+j} \times X(i,u) \times Y(j,v)$ is a quasi-projective variety, the quotient

$$\mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} (X(i,u) \times Y(j,v)) = (\Sigma_{i+j} \times X(i,u) \times Y(j,v))/(\Sigma_i \times \Sigma_j)$$

exists and is a quasi-projective variety too.

There is also an action of Σ_{i+j} on $\Sigma_{i+j} \times X(i,u) \times Y(j,v)$, which is given on closed points by

$$\rho' \cdot (\rho, x, y) = (\rho'\rho, x, y)\,.$$

Since this action commutes with that of $\Sigma_i \times \Sigma_j$, $\mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} (X(i,u) \times Y(j,v))$ has an induced Σ_{i+j}-action. We define the H-graded P-variety $X \boxtimes Y$ by

$$(X \boxtimes Y)(n,w) := \bigsqcup_{\substack{u,v\in H \\ u+v=w}} \bigsqcup_{\substack{i,j\in\mathbb{N} \\ i+j=n}} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (X(i,u) \times Y(j,v))\,. \tag{1.1}$$

for all $n \in \mathbb{N}$ and $w \in H$. Notice that the coproduct on the right side is finite because H has the finite decomposition property.

This definition is clearly functorial in X and Y, so that we have a functor $\boxtimes : [\mathsf{P} \times \underline{H}, \mathsf{Var}] \times [\mathsf{P} \times \underline{H}, \mathsf{Var}] \to [\mathsf{P} \times \underline{H}, \mathsf{Var}]$. Let us prove that $\boxtimes$ is a symmetric tensor product, using the results of Section 1.1. However, the Cartesian monoidal category $(\mathsf{Var}, \times, \mathrm{Spec}(\mathbb{C}))$ is not a cosmos, because it is not bicomplete nor closed. Therefore, we first need to embed it into a larger category which is a cosmos.

Let Aff be the category of affine schemes over $\mathbb{C}$. The category $\mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})$ of sheaves over Aff in the fppf topology is a cosmos, because it is Cartesian closed and bicomplete. There is a fully faithful functor $J_0 : \mathsf{Var} \to \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})$ which associates to each variety X its functor of points $h_X : \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$. The functor J_0 is also symmetric monoidal with respect to the Cartesian monoidal structures on Var and $\mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})$. Let

$$J : [\mathsf{P} \times \underline{H}, \mathsf{Var}] \to [\mathsf{P} \times \underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]$$

be the fully faithful functor obtained by composing with J_0.

As we have seen in Section 1.1.3, $D : [\mathsf{P}, \mathscr{C}] \to [\mathsf{P}, \mathscr{C}]$ and the objects S_n of $[\mathsf{P}, \mathscr{C}]$ can be defined for both $\mathscr{C} = [\underline{H}, \mathsf{Var}]$ and $\mathscr{C} = [\underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]$. In order to differentiate them, we keep the notations D, S_n when $\mathscr{C} = [\underline{H}, \mathsf{Var}]$, while we write D', S'_n when $\mathscr{C} = [\underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]]$. Then we have

$$J \circ D = D' \circ J\,, \qquad S'_n = J(S_n) \tag{1.2}$$

for each $n \in \mathbb{N}$. In particular, $J(S_0) = S'_0$ is the identity object for $*$ and $J(S_1) = S'_1$ is the identity object for $\diamond$.

Using the embedding J, we can relate the operation $\boxtimes$ on $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$ to the tensor product $*$ on $[\mathsf{P} \times \underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})] = [\mathsf{P}, [\underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]]$ and prove the following result.

Theorem 1.2.1. *The category of H-graded P-varieties, together with the tensor product $\boxtimes$ and the identity object S_0, is a symmetric monoidal category.*

Proof. The proof is based on two lemmas.

Lemma 1.2.2. *For all H-graded P-varieties X, Y, there is an isomorphism*

$$J(X \boxtimes Y) \cong JX * JY \tag{1.3}$$

which is natural in both X and Y.

Proof of Lemma 1.2.2. Recall from Section 1.1.2 that for every $n \in \mathbb{N}$ and $w \in H$, $(JX * JY)(n,w)$ is canonically isomorphic to

$$\coprod_{\substack{u,v\in H\\ u+v=w}} \coprod_{\substack{i,j\in\mathbb{N}\\ i+j=n}} \operatorname{coeq}\left(\begin{array}{c} \displaystyle\coprod_{(\sigma,\tau)\in\Sigma_i\times\Sigma_j} \Sigma_n \times (JX)(i,u) \times (JY)(j,v) \\ {\scriptstyle [\mathsf{P}(\sigma+\tau,n)\times 1\times 1]}\Big\downarrow \Big\downarrow{\scriptstyle [1\times (JX)(\sigma,1)\times (JY)(\tau,1)]} \\ \Sigma_n \times (JX)(i,u) \times (JY)(j,v)\,. \end{array}\right). \tag{1.4}$$

Since $J_0 : \mathsf{Var} \to \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})$ preserves finite products and finite coproducts, the coequalizer appearing in (1.4) is in turn canonically isomorphic to the coequalizer of $(J_0 f, J_0 g)$, where

$$\begin{array}{c} \displaystyle\coprod_{(\sigma,\tau)\in\Sigma_i\times\Sigma_j} \Sigma_n \times X(i,u) \times Y(j,v) \\ {\scriptstyle f=[\mathsf{P}(\sigma+\tau,n)\times 1\times 1]}\Big\downarrow \Big\downarrow{\scriptstyle g=[1\times X(\sigma,1)\times Y(\tau,1)]} \\ \Sigma_n \times X(i,u) \times Y(j,v)\,. \end{array} \tag{1.5}$$

Notice that the precomposition of (1.5) with the isomorphism

$$\begin{array}{c} \displaystyle\coprod_{(\sigma,\tau)\in\Sigma_i\times\Sigma_j} \Sigma_n \times X(i,u) \times Y(j,v) \\ \Big\downarrow{\scriptstyle \left[i_{(\sigma,\tau)}\circ\left(\mathsf{P}\left((\sigma+\tau)^{-1},n\right)\times 1\times 1\right)\right]} \\ \displaystyle\coprod_{(\sigma,\tau)\in\Sigma_i\times\Sigma_j} \Sigma_n \times X(i,u) \times Y(j,v) \end{array}$$

yields the diagram

$$\Sigma_i \times \Sigma_j \times \Sigma_n \times X(i,u) \times Y(j,v) \underset{a_{\boxtimes}}{\overset{\mathrm{pr}}{\rightrightarrows}} \Sigma_n \times X(i,u) \times Y(j,v)\,,$$

where pr is the projection and $a_{\boxtimes}$ is the $(\Sigma_i \times \Sigma_j)$-action defined above. This implies that $\operatorname{coeq}(J_0 f, J_0 g) \cong \operatorname{coeq}(J_0(\mathrm{pr}), J_0(a_{\boxtimes}))$ canonically.

Finally, since the action $a_{\boxtimes}$ is free, we also have a canonical isomorphism

$$\operatorname{coeq}(J_0(\mathrm{pr}), J_0(a_{\boxtimes})) \cong J_0(\operatorname{coeq}(\mathrm{pr}, a_{\boxtimes}))$$

(see [Gab70]), where $\operatorname{coeq}(\mathrm{pr}, a_{\boxtimes})$ is exactly $\operatorname{Ind}_{\Sigma_i\times\Sigma_j}^{\Sigma_n}(X(i,u) \times Y(j,v))$. Therefore, using again the fact that J_0 preserves finite coproducts, we obtain canonical isomorphisms

$$\gamma_{(n,z)} : (JX * JY)(n,w) \xrightarrow{\sim} J_0((X \boxtimes Y)(n,w)) = J(X \boxtimes Y)(n,w)\,,$$

for all $n \in \mathbb{N}$ and $w \in H$.

All the isomorphisms involved in the proof are natural in (n, w), X and Y, therefore $\gamma = (\gamma_{(n,w)}) : JX * JY \Rightarrow J(X \boxtimes Y)$ is indeed an isomorphism in $[\mathsf{P} \times \underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]$, which is natural in X and Y. □

Now, the theorem is a direct corollary of the following lemma, whose proof is immediate.

Lemma 1.2.3. *Let $\mathscr{C}$ be a category, $\boxtimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ a functor, and $\mathbb{1}$ an object of $\mathscr{C}$. Suppose that there exists a fully faithful functor $F : \mathscr{C} \to \mathscr{C}'$ to a (symmetric) monoidal category $(\mathscr{C}', \otimes, \mathbb{1}')$, such that*

(i) there is a natural isomorphism $\gamma : F(\bullet \boxtimes \bullet) \Rightarrow F(\bullet) \otimes F(\bullet)$, and

(ii) $F(\mathbb{1}) = \mathbb{1}'$.

Then $(\mathscr{C}, \boxtimes, \mathbb{1})$ is a (symmetric) monoidal category.

This ends the proof of Theorem 1.2.1. □

As a consequence of Theorem 1.2.1, if Y is an H-graded P-variety, then for any $i \in \mathbb{N}$ and $\sigma \in \Sigma_i$ there is a natural isomorphism $\langle \sigma \rangle^{\boxtimes} : Y^{\boxtimes i} \to Y^{\boxtimes i}$. Thus, if X is another H-graded P-variety, then for any $i, n \in \mathbb{N}$ and $u, v \in H$ the morphism

$$\begin{array}{c} \Sigma_i \times X(i,u) \times Y^{\boxtimes i}(n,v) = \coprod_{\sigma \in \Sigma_i} X(i,u) \times Y^{\boxtimes i}(n,v) \\ \big\downarrow {\scriptstyle a_\circ = \left[X(\sigma, 1_u) \times \langle \sigma^{-1} \rangle^{\boxtimes}_{(n,v)}\right]} \\ X(i,u) \times Y^{\boxtimes i}(n,v) \end{array}$$

defines an action of Σ_i on $X(i,u) \times Y^{\boxtimes i}(n,v)$, which commutes with the action of Σ_n. Therefore, the quotient $\big(X(i,u) \times Y^{\boxtimes i}(n,v)\big)/\Sigma_i$ is a quasi-projective variety with an induced Σ_n-action.

Assume that $Y(0,0) = \emptyset$. We define the H-graded P-variety $X \circ Y$ by

$$(X \circ Y)(n,w) := \coprod_{\substack{u,v \in H \\ u+v=w}} \coprod_{i \in \mathbb{N}} \big(X(i,u) \times Y^{\boxtimes i}(n,v)\big)/\Sigma_i \tag{1.6}$$

for all $n \in \mathbb{N}$ and $w \in H$. The condition $Y(0,0) = \emptyset$, together with our assumptions on H, ensures that the coproduct on the right side is finite.

Let $[\mathsf{P} \times \underline{H}, \mathsf{Var}]_1$ be the full subcategory of $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$ on the objects Y such that $Y(0,0) = \emptyset$. The definition of $\circ$ is functorial in X and Y, so that we get a functor $\circ : [\mathsf{P} \times \underline{H}, \mathsf{Var}] \times [\mathsf{P} \times \underline{H}, \mathsf{Var}]_1 \to [\mathsf{P} \times \underline{H}, \mathsf{Var}]$. This time, $\circ$ is not an actual tensor product on $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$, because it is not well-defined for all objects. However, $([\mathsf{P} \times \underline{H}, \mathsf{Var}], \circ, S_1)$ shares the same features of a monoidal category: there are (partially defined) natural isomorphisms

$(\bullet\circ\bullet)\circ\bullet \Rightarrow \bullet\circ(\bullet\circ\bullet)$, $S_1\circ\bullet \Rightarrow 1_{[\mathsf{P}\times\underline{H},\mathsf{Var}]}$ and $\bullet\circ S_1 \Rightarrow 1_{[\mathsf{P}\times\underline{H},\mathsf{Var}]}$, which satisfy the usual coherence conditions, whenever those make sense. We summarize these facts in the following assertion, which we prove using the embedding J, as we did for $\boxtimes$.

Theorem 1.2.4. *The functor* $\circ : [\mathsf{P}\times\underline{H},\mathsf{Var}]\times[\mathsf{P}\times\underline{H},\mathsf{Var}]_1 \to [\mathsf{P}\times\underline{H},\mathsf{Var}]$, *together with the identity object S_1, is a partially defined tensor product on* $[\mathsf{P}\times\underline{H},\mathsf{Var}]$. *It is an actual tensor product when restricted to* $[\mathsf{P}\times\underline{H},\mathsf{Var}]_1$.

Proof. The proof relies on the following lemma.

Lemma 1.2.5. *For all H-graded P-varieties X, Y such that $Y(0,0)=\emptyset$, there is an isomorphism*

$$J(X\circ Y)\cong JX\diamond JY$$

which is natural in both X and Y.

Proof of Lemma 1.2.5. The proof follows the same steps as that of Lemma 1.2.2. For every $n\in\mathbb{N}$ and $w\in H$, $(JX\circ JY)(n,w)$ is canonically isomorphic to

$$\coprod_{\substack{u,v\in H\\ u+v=w}}\coprod_{i\in\mathbb{N}}\mathrm{coeq}\left(\begin{array}{c}\displaystyle\coprod_{\sigma\in\Sigma_i}(JX)(i,u)\times(JY)^{*i}(n,v)\\ {\scriptstyle[1\otimes\langle\sigma\rangle_{(n,v)}]}\Big\downarrow\;\Big\downarrow{\scriptstyle[(JX)(\sigma,1)\times 1]}\\ (JX)(i,u)\times(JY)^{*i}(n,v)\,.\end{array}\right). \tag{1.7}$$

Since $J_0:\mathsf{Var}\to\mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})$ preserves finite products and finite coproducts, and $J(Y)^{*i}\cong J(Y^{\boxtimes i})$ canonically, the coequalizer appearing in (1.7) is in turn canonically isomorphic to the coequalizer of (J_0f, J_0g), where

$$\coprod_{\sigma\in\Sigma_i}X(i,u)\times Y^{\boxtimes i}(n,v)\ \underset{g=[X(\sigma,1)\times 1]}{\overset{f=\left[1\times\langle\sigma\rangle^{\boxtimes}_{(n,v)}\right]}{\rightrightarrows}}\ X(i,u)\times Y^{\boxtimes i}(n,v)\,. \tag{1.8}$$

Notice that the precomposition of (1.8) with the isomorphism

$$\coprod_{\sigma\in\Sigma_i}X(i,u)\times Y^{\boxtimes i}(n,v)\xrightarrow{\left[i_\sigma\circ\left(1\times\langle\sigma^{-1}\rangle^{\boxtimes}_{(n,v)}\right)\right]}\coprod_{\sigma\in\Sigma_i}X(i,u)\times Y^{\boxtimes i}(n,v)$$

yields the diagram

$$\Sigma_i\times X(i,u)\times Y^{\boxtimes i}(n,v)\ \underset{a_\circ}{\overset{\mathrm{pr}}{\rightrightarrows}}\ X(i,u)\times Y^{\boxtimes i}(n,v)\,,$$

where pr is the projection and $a_\circ$ is the Σ_i-action defined above. This implies that $\mathrm{coeq}(J_0f,J_0g)\cong\mathrm{coeq}(J_0(\mathrm{pr}),J_0(a_\circ))$ canonically.

Finally, since the action $a_\circ$ is free, we have a canonical isomorphism

$$\mathrm{coeq}(J_0(\mathrm{pr}), J_0(a_\circ)) \cong J_0(\mathrm{coeq}(\mathrm{pr}, a_\circ))\,,$$

where $\mathrm{coeq}(\mathrm{pr}, a_\circ)$ is exactly $\big(X(i,u) \times Y^{\boxtimes i}(n,v)\big)/\Sigma_i$. As J_0 preserves finite coproducts, we thus obtain canonical isomorphisms

$$\delta_{(n,w)} : (JX \diamond JY)(n,w) \xrightarrow{\sim} J_0((X \circ Y)(n,w)) = J(X \circ Y)(n,w)\,,$$

for all $n \in \mathbb{N}$ and $w \in H$.

All the isomorphisms involved in the proof are natural in (n,w), X and Y, therefore $\delta = (\delta_{(n,w)}) : JX \diamond JY \Rightarrow J(X \circ Y)$ is indeed an isomorphism in $[\mathsf{P} \times \underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]$, which is natural in X and Y. $\square$

Once Lemma 1.2.5 has been established, Theorem 1.2.4 follows from the fact that J is a fully faithful functor and $J(S_1) = S_1'$ is the identity object for $\diamond$. $\square$

Lemma 1.2.2 and Lemma 1.2.5 also allow us to extend the results of Section 1.1.3 to $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$.

Proposition 1.2.6. *Let X, Y, Z, Z' be H-graded P-varieties, and assume that $Z(0,0) = Z'(0,0) = \emptyset$. Then there are isomorphisms*

$$\begin{aligned} D(X \boxtimes Y) &\cong (DX \boxtimes Y) \sqcup (X \boxtimes DY)\,, \\ D(X \circ Z) &\cong (DX \circ Z) \boxtimes DZ\,, \\ S_n \circ (Z \sqcup Z') &\cong \bigsqcup_{m=0}^{n} (S_m \circ Z) \boxtimes (S_{n-m} \circ Z')\,, \end{aligned}$$

which are natural in X, Y, Z, Z'.

Proof. Via J, we embed $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$ into $[\mathsf{P} \times \underline{H}, \mathsf{Sh}(\mathsf{Aff}_{\mathrm{fppf}})]$, where Proposition 1.1.7, Proposition 1.1.8 and Proposition 1.1.9 hold. Since J is fully faithful and preserves finite coproducts, the proposition directly follows from Lemma 1.2.2 and Lemma 1.2.5, using (1.2). $\square$

It is clear that the same procedure we followed for Var can be used to deal with H-graded P-objects of other categories. For instance, one can replace $(\mathsf{Var}, \times, \mathrm{Spec}(\mathbb{C}))$ by an abelian symmetric monoidal category $(\mathscr{A}, \otimes, \mathbb{1})$ whose tensor product $\otimes$ is exact. An example of such $\mathscr{A}$ is the category MHS of mixed Hodge structures over $\mathbb{Q}$. Again, H-graded P-objects of MHS arise when studying moduli spaces of stable maps (see [GP06] for the case $H = \mathbb{N}$).

We define the operations $\boxtimes$ and $\circ$ in $[\mathsf{P} \times \underline{H}, \mathscr{A}]$ as in (1.1) and (1.6), replacing $\times$ with $\otimes$. As above, the definition of $\circ$ requires the condition $Y(0,0) = 0$ to be satisfied, since infinite coproducts may not exist in $\mathscr{A}$. Let

$[\mathsf{P} \times \underline{H}, \mathscr{A}]_1$ denote the full subcategory of $[\mathsf{P} \times \underline{H}, \mathscr{A}]$ on the objects Y such that $Y(0,0) = 0$. Then we have the following analogue of Theorem 1.2.1, Theorem 1.2.4 and Proposition 1.2.6.

Theorem 1.2.7. *(i)* $([\mathsf{P} \times \underline{H}, \mathscr{A}], \boxtimes, S_0)$ *is a symmetric monoidal category.*

(ii) The functor $\circ : [\mathsf{P} \times \underline{H}, \mathscr{A}] \times [\mathsf{P} \times \underline{H}, \mathscr{A}]_1 \to [\mathsf{P} \times \underline{H}, \mathscr{A}]$*, together with the identity object* S_1*, is a partially defined tensor product on* $[\mathsf{P} \times \underline{H}, \mathscr{A}]$*.*

(iii) If X, Y, Z, Z' *are objects of* $[\mathsf{P} \times \underline{H}, \mathscr{A}]$ *and* $Z(0,0) = Z'(0,0) = 0$*, then there are isomorphisms*

$$
\begin{aligned}
D(X \boxtimes Y) &\cong (DX \boxtimes Y) \oplus (X \boxtimes DY)\,, \\
D(X \circ Z) &\cong (DX \circ Z) \boxtimes DZ\,, \\
S_n \circ (Z \oplus Z') &\cong \bigoplus_{m=0}^{n} (S_m \circ Z) \boxtimes (S_{n-m} \circ Z')\,,
\end{aligned}
$$

which are natural in X, Y, Z, Z'*.*

The proof technique is the same as in the case of Var. First, we embed $\mathscr{A}$ into the category $\mathsf{Ind}(\mathscr{A})$ of ind-objects of $\mathscr{A}$, via the canonical inclusion J_0. The category $\mathsf{Ind}(\mathscr{A})$ can be given a symmetric monoidal structure such that J_0 becomes a symmetric monoidal functor (see [Hồ02]). With this structure, $\mathsf{Ind}(\mathscr{A})$ is a comos; in particular, the results of Section 1.1 apply to $\mathsf{Ind}(\mathscr{A})$.

Secondly, let $J : [\mathsf{P} \times \underline{H}, \mathscr{A}] \to [\mathsf{P} \times \underline{H}, \mathsf{Ind}(\mathscr{A})]$ be the fully faithful functor induced by J_0. Since J_0 preserves finite limits and finite colimits, the proofs of Lemma 1.2.2 and Lemma 1.2.5 also work in this case, yielding the following result.

Lemma 1.2.8. *(i) For all objects* X, Y *of* $[\mathsf{P} \times \underline{H}, \mathscr{A}]$*, there is an isomorphism* $J(X \boxtimes Y) \cong JX * JY$ *which is natural in* X *and* Y*.*

(ii) For all objects X, Y *of* $[\mathsf{P} \times \underline{H}, \mathscr{A}]$ *such that* $Y(0,0) = 0$*, there is an isomorphism* $J(X \circ Y) \cong JX \diamond JY$ *which is natural in* X *and* Y*.*

Finally, Theorem 1.2.7 is a consequence of Lemma 1.2.8 and of the fully faithfulness of J.

1.3 Grothendieck groups of varieties

Let $\Bbbk$ be an algebraically closed field. By a $\Bbbk$-variety we mean a reduced separated scheme of finite type over $\Bbbk$, not necessarily irreducible.

Definition 1.3.1. The Grothendieck group of $\Bbbk$-varieties $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$ is the quotient of the free abelian group on the isomorphism classes of $\Bbbk$-varieties, by the subgroup generated by

$$\{[X]-[Y]-[X\setminus Y] \mid X \ \Bbbk\text{-variety}, \ Y\subseteq X \text{ closed subvariety}\}\,.$$

By defining

$$[X][Y] := [X\times Y]\,,$$

where $X\times Y = X\times_\Bbbk Y$, $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$ in fact becomes a commutative ring.

If X is a $\Bbbk$-variety, its class $[X]\in \mathrm{K}_0(\mathsf{Var}_\Bbbk)$ is sometimes called the motive of X. Following the customary notation, we denote by $\mathbf{L}\in \mathrm{K}_0(\mathsf{Var}_\Bbbk)$ the class of the affine line $\mathbb{A}^1_\Bbbk$.

The following is an important well-known property of $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$, which we will often use in the next chapters.

Proposition 1.3.2. *Let $f: X\to Y$ be a morphism of $\Bbbk$-varieties. If f is a locally trivial fibration in the Zariski topology, with fiber F, then $[X]=[Y][F]$ in $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$.*

Since we want to consider quotients by finite groups, in most of the thesis we will restrict our attention to quasi-projective varieties. Indeed, recall that for any quasi-projective variety X with an action of a finite group G, X can be covered by affine open subsets which are invariant under the action of G. This guarantees that the quotient X/G exists and is again a quasi-projective variety. Since we will mostly work over $\mathbb{C}$, the following definition is given only in the case $\Bbbk=\mathbb{C}$.

Definition 1.3.3. The Grothendieck group of complex quasi-projective varieties $\mathrm{K}_0(\mathsf{Var})$ is the quotient of the free abelian group on the isomorphism classes of quasi-projective varieties over $\mathbb{C}$, by the subgroup generated by

$$\{[X]-[Y]-[X\setminus Y] \mid X \text{ quasi-projective } \mathbb{C}\text{-variety}, \ Y\subseteq X \text{ closed subvariety}\}.$$

As above, $\mathrm{K}_0(\mathsf{Var})$ is in fact a commutative ring, where the product is given by $[X][Y] := [X\times Y]$.

There is a natural injective ring homomorphism $\mathrm{K}_0(\mathsf{Var}) \hookrightarrow \mathrm{K}_0(\mathsf{Var}_\mathbb{C})$. Via this inclusion, each equality involving only classes of quasi-projective varieties in $\mathrm{K}_0(\mathsf{Var}_\mathbb{C})$ holds in $\mathrm{K}_0(\mathsf{Var})$, too.

We will also consider some different Grothendieck groups, whose definition we recall below. Here, varieties are understood to be quasi-projective varieties over $\mathbb{C}$. As in Section 1.2, we fix a commutative monoid H with the properties $(H1)$ and $(H2)$.

Definition 1.3.4. For any $n\in\mathbb{N}$, the Grothendieck group $\mathrm{K}_0^{\Sigma_n}(\mathsf{Var})$ is the quotient of the free abelian group on the isomorphism classes of varieties with a left action of Σ_n, by the relations $[X]=[Y]+[X\setminus Y]$ whenever $Y\subseteq X$ is a closed Σ_n-invariant subvariety.

The Grothendieck group $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ is the quotient of the free abelian group on the isomorphism classes of P-varieties, by the relations $[X] = [Y] + [X \setminus Y]$ whenever $Yn \subseteq Xn$ is a closed Σ_n-invariant subvariety for all $n \in \mathbb{N}$.

The Grothendieck group $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ is the quotient of the free abelian group on the isomorphism classes of H-graded P-varieties, by the relations $[X] = [Y] + [X \setminus Y]$ whenever $Y(n, w) \subseteq X(n, w)$ is a closed Σ_n-invariant subvariety for all $n \in \mathbb{N}$ and $w \in H$.

Finally, the Grothendieck group $\mathrm{K}_0(\mathsf{Var}^H)$ of H-graded varieties is the quotient of the free abelian group on the isomorphism classes of H-graded varieties, by the relations $[X] = [Y] + [X \setminus Y]$ whenever $Yw \subseteq Xw$ is a closed subvariety for all $w \in H$.

In particular, we have $\mathrm{K}_0^{\Sigma_0}(\mathsf{Var}) = \mathrm{K}_0(\mathsf{Var})$. Note that there is a natural isomorphism

$$\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}) \cong \prod_{n \in \mathbb{N}} \mathrm{K}_0^{\Sigma_n}(\mathsf{Var})\,.$$

Furthermore, there are natural inclusions

$$\begin{tikzcd}[column sep=small] & \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}) \arrow[dr, hook] & \\ \mathrm{K}_0(\mathsf{Var}) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)\,. \\ & \mathrm{K}_0(\mathsf{Var}^H) \arrow[ur, hook] & \end{tikzcd} \tag{1.9}$$

We apply the results of Section 1.2 to the study of these Grothendieck groups. As a consequence of the properties of $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$, $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ is not only an abelian group, but it has a richer algebraic structure. By Theorem 1.2.1, a commutative ring structure on $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ can be defined by setting

$$[X][Y] := [X \boxtimes Y]$$

for all H-graded P-varieties X and Y, and by linearly extending this operation to all elements of $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. This linear extension is possible because $\boxtimes : [\mathsf{P} \times \underline{H}, \mathsf{Var}] \times [\mathsf{P} \times \underline{H}, \mathsf{Var}] \to [\mathsf{P} \times \underline{H}, \mathsf{Var}]$ preserves coproducts in both arguments. As a ring, $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ is isomorphic to the ring of generalized power series $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![H]\!]$, via the isomorphism

$$[X] \mapsto \sum_{w \in H} [X(\bullet, w)] q^w\,.$$

In the diagram (1.9), each Grothendieck group is closed with respect to this product, hence inherits a commutative ring structure. The inherited ring structure on $\mathrm{K}_0(\mathsf{Var})$ is the same as the one defined above. In particular, the inclusion $\mathrm{K}_0(\mathsf{Var}^H) \hookrightarrow \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ defines a commutative $\mathrm{K}_0(\mathsf{Var}^H)$-algebra structure on $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. This algebra inherits another operation from the functor $\circ : [\mathsf{P} \times \underline{H}, \mathsf{Var}] \times [\mathsf{P} \times \underline{H}, \mathsf{Var}]_1 \to [\mathsf{P} \times \underline{H}, \mathsf{Var}]$. As pointed out in [GP06], this operation is a composition operation when $H = \mathbb{N}$. We conjecture that the same holds for any H.

Definition 1.3.5 ([GP06]). Let R be a commutative ring, and A a commutative R-algebra which is filtered by subalgebras $A = F_0A \supseteq F_1A \supseteq \dots$. A composition operation on A is an operation

$$\circ : A \times F_1A \to A$$

such that $F_nA \circ F_mA \subseteq F_{nm}A$, endowed with a derivation $D : F_nA \to F_{n-1}A$, and a sequence of elements $s_n \in F_nA$ for $n \geq 0$, satisfying the following axioms:

(i) for fixed $b \in F_1A$, the map $\bullet \circ b : A \to A$ is an R-algebra endomorphism;

(ii) $(a \circ b_1) \circ b_2 = a \circ (b_1 \circ b_2)$ for any $a \in A$, $b_1, b_2 \in F_1A$;

(iii) $D(a \circ b) = (Da \circ b)(Db)$ for any $a \in A$, $b \in F_1A$;

(iv) $s_0 = 1$, $s_1 \circ b = b$ for all $b \in F_1A$, $a \circ s_1 = a$ for all $a \in A$, $Ds_n = s_{n-1}$, and $s_n \circ (b_1 + b_2) = \sum_{m=0}^{n} (s_m \circ b_1)(s_{n-m} \circ b_2)$ for all $b_1, b_2 \in F_1A$.

Note that the structures and the properties appearing in this definition have been already defined and proved at a categorical level (namely for the category $[\mathsf{P} \times \underline{H}, \mathsf{Var}]$, before passing to the Grothendieck group) in the previous section. Now, we only have to show that these structures descend to $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$.

The $\mathrm{K}_0(\mathsf{Var}^H)$-algebra $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ is filtered by the subalgebras

$$F_n(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)) = \left\langle \bigcup_{i=0}^{n} \{[X] \mid X(m, w) = \emptyset \text{ if } m < n - i \text{ or } l_w < i\} \right\rangle,$$

where $l_w = \max(\{l \in \mathbb{N} \mid A_l^\star(w) \neq \emptyset\} \cup \{0\})$ (which exists by our assumption $(H2)$ on H). Note that $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ is complete with respect to this filtration.

The endofunctor $D : [\mathsf{P} \times \underline{H}, \mathsf{Var}] \to [\mathsf{P} \times \underline{H}, \mathsf{Var}]$ preserves coproducts, thus it descends to a group homomorphism $D : \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H) \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. By Proposition 1.2.6, D is a $\mathrm{K}_0(\mathsf{Var}^H)$-linear derivation. Moreover, we have $D(F_n(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H))) \subseteq F_{n-1}(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H))$.

For all $n \in \mathbb{N}$, let $s_n := [S_n]$. Then $s_0 = 1$, because S_0 is the identity object for $\boxtimes$. Furthermore, $Ds_n = s_{n-1}$ for all n.

Proposition 1.3.6. *The functor $\circ$ defined in Section 1.2 descends to a well-defined operation $\circ : \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H) \times F_1(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)) \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$, which is $\mathrm{K}_0(\mathsf{Var}^H)$-linear in the first argument and such that $[X] \circ [Y] = [X \circ Y]$ for all P-varieties X, Y.*

Proof. For simplicity of notation, we consider the case $H = \{0\}$, namely $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$. In the general case, the proof is exactly the same, and it is only notationally heavier.

Let us define $[X] \circ b \in \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ for X a P-variety and $b \in F_1(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}))$. This will extend to an operation $\circ : \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}) \times F_1(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})) \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ by linearity.

First, write b as $b = [Y] - [Z]$, where Y and Z are P-varieties. Let us consider the simplest case, i.e., when $Zn \subseteq Yn$ is a closed Σ_n-invariant subvariety of Yn for all $n \in \mathbb{N}$. Let $U = Y \setminus Z$. Then we have

$$[X] \circ [Y] = [X] \circ ([Z] + [U]) = [X \circ (Z \sqcup U)] \,.$$

Using the coend formalism, we have

$$X \circ (Z \sqcup U) \cong \int^{i,j} X(i+j) \times Z^{\boxtimes i} \boxtimes U^{\boxtimes j} \,,$$

which is in turn isomorphic to

$$(X \circ U) \sqcup \int^{i \geq 1, j} X(i+j) \times Z^{\boxtimes i} \boxtimes U^{\boxtimes j}$$

Therefore, $[X] \circ b = [X \circ (Y \setminus Z)]$ is equal to

$$[X \circ Y] - \left[\int^{i \geq 1, m, n} \mathsf{P}(m+n, \bullet) \times Z^{\boxtimes i} m \times \left(\int^{j} X(i+j) \times (Y \setminus Z)^{\boxtimes j} n \right) \right] .$$

Note that for fixed $k \in \mathbb{N}$, $([X] \circ b)_k \in \mathrm{K}_0^{\Sigma_k}(\mathsf{Var})$ depends on $(Y \setminus Z)^{\boxtimes j} n$ only for $n < k$. In particular, we have

$$\begin{aligned} ([X] \circ b)_0 &= [X0] \,, \\ ([X] \circ b)_1 &= [(X \circ Y)1] - [(X \circ Z)1] \,. \end{aligned}$$

This suggests a recursive approach for defining $[X] \circ b$ when $b = [Y] - [Z]$ and Z is not a subfunctor of Y.

Let us consider the Grothendieck group $\mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$ of functors $\mathsf{P} \times \mathsf{P} \to \mathsf{Var}$. For any element $a \in \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$, we write $a_{(i,n)}$ for the (i,n)-th component of a under the canonical isomorphism

$$\mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var}) \cong \prod_{(i,n) \in \mathbb{N}^2} \mathrm{K}_0^{\Sigma_i \times \Sigma_n}(\mathsf{Var}) \,.$$

Let us define an element $[X] \circ_k b \in \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$ for all $k \in \mathbb{N}$, by using a recursive construction. First, define an element $[X] \circ_0 b \in \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$ as follows:

$$\begin{aligned} ([X] \circ_0 b)_{(i,0)} &:= [X(i+0)] \\ ([X] \circ_0 b)_{(i,n)} &:= 0 \quad \forall n > 0 \,. \end{aligned}$$

Now, take $k > 0$ and suppose that we have already defined the element $[X] \circ_{k-1} b \in \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$. For any $T : \mathsf{P} \times \mathsf{P} \to \mathsf{Var}$, let $h_Z([T]) \in \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$ be the element whose (i,l)-th component is

$$h_Z([T])_{(i,l)} := \left[\int^{j \geq 1, m, n} \mathsf{P}(m+n, l) \times Z^{\boxtimes j} m \times T(i+j, n) \right] .$$

By linearity, this defines a group homomorphism $h_Z : \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var}) \to \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$. Now, define $[X] \circ_k b \in \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var})$ as

$$\begin{aligned}
([X] \circ_k b)_{(i,n)} &:= ([X] \circ_{k-1} b)_{(i,n)} \quad \forall\, n < k \\
([X] \circ_k b)_{(i,k)} &:= [(X(i + \bullet) \circ Y)k] - h_Z([X] \circ_{k-1} b)_{(i,k)} \\
([X] \circ_k b)_{(i,n)} &:= 0 \quad \forall\, n > k \,.
\end{aligned}$$

Finally, note that the functor $[\mathsf{P} \times \mathsf{P}, \mathsf{Var}] \to [\mathsf{P}, \mathsf{Var}]$ given by $T \mapsto T(0, \bullet)$ descends to a group homomorphism $g : \mathrm{K}_0^{\mathsf{P}^2}(\mathsf{Var}) \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$. Thus, we can define

$$[X] \circ b := \sum_{k=0}^{\infty} (g([X] \circ_k b))_k \in \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}) \,,$$

using the completeness of the $\mathrm{K}_0(\mathsf{Var})$-algebra $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$. □

Motivated by Theorem 1.2.4 and Proposition 1.2.6, we conjecture that this operation $\circ$ is in fact a composition operation on $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. Due to the complicated dependence of $\circ$ on the second argument, we have not been able to find a simple rigorous proof. Since this statement goes beyond the aim of the present thesis, we defer further investigation to future work. We refer to [GP06] for the cases $H = \{0\}$ and $H = \mathbb{N}$, which are the only ones that we will use in the next chapters.

Theorem 1.3.7 ([GP06])**.** *The Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^{\mathbb{N}}) \cong \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$, together with the filtration $F_n(\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^{\mathbb{N}}))$, the derivation D, and the sequence of elements (s_n) defined above, is a complete algebra with composition operation $\circ$ over $\mathrm{K}_0(\mathsf{Var}^{\mathbb{N}}) \cong \mathrm{K}_0(\mathsf{Var})[\![q]\!]$. Similarly, the Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ is a complete $\mathrm{K}_0(\mathsf{Var})$-algebra with composition operation.*

As a consequence of this theorem, there is an analogue of the function $\exp(\bullet) - 1$ for both $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ and $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$. Indeed, in any complete algebra A with composition operation, we can define $\mathrm{Exp} : F_1 A \to F_1 A$ as

$$\mathrm{Exp}(a) := \sum_{n=1}^{\infty} s_n \circ a \,,$$

for any $a \in F_1 A$. This operation has an inverse $\mathrm{Log} : F_1 A \to F_1 A$.

Proposition 1.3.8 ([GP06, Prop. 3.1])**.** *There exist elements $l_n \in F_n A$, $n \in \mathbb{N}_{>0}$, such that the operation*

$$\mathrm{Log}(a) := \sum_{n=1}^{\infty} l_n \circ a$$

is the inverse of Exp*.*

For instance, these operations allow us to compactly express the class of the configuration space of a variety in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ (see [GP06, Thm. 3.2] and Section 3.3).

1.3.1 Pre-λ-ring property

Recall the following definition.

Definition 1.3.9. A pre-λ-ring is a commutative ring R, together with operations $\lambda^n : R \to R$, for $n \in \mathbb{N}$, which satisfy the following conditions:

(i) $\lambda^0(x) = 1$ for all $x \in R$;

(ii) $\lambda^1(x) = x$ for all $x \in R$;

(iii) $\lambda^n(x+y) = \sum_{m=0}^{n} \lambda^m(x)\lambda^{m-n}(y)$ for all $x, y \in R$ and $n \in \mathbb{N}$.

In addition to the λ-operations, each pre-λ-ring R is equipped with the corresponding σ-operations $\sigma^n(x) := (-1)^n \lambda^n(-x)$, which satisfy the same conditions (i), (ii), (iii). It is customary to encode the λ-operations and the σ-operations in the formal power series

$$\lambda_q(x) := \sum_{n=0}^{\infty} \lambda^n(x) q^n \,, \qquad \sigma_q(x) := \sum_{n=0}^{\infty} \sigma^n(x) q^n \,.$$

The two series are related by the equality $\lambda_q(x) = \sigma_{-q}(x)^{-1}$ for all $x \in R$. Note that a pre-λ-ring can be equivalently defined by giving the σ-operations.

For instance, given any commutative ring R, $1 + qR[\![q]\!]$ has a natural structure of a pre-λ-ring (see [Knu73]). Another example of a pre-λ-ring structure is the one defined on $\mathrm{K}_0(\mathsf{Var}_{\mathbb{C}})$ by means of Kapranov's motivic zeta function

$$\zeta_{[X]}(q) = 1 + \sum_{n=1}^{\infty} [\mathrm{Sym}^n(X)] q^n \in \mathrm{K}_0(\mathsf{Var}_{\mathbb{C}})[\![q]\!] \,.$$

We refer to [GZLMH04] for more details about this pre-λ-ring structure and the power structure induced by it.

Proposition 1.3.10. *The Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$ is a pre-λ-ring.*

Proof. We will describe this pre-λ-ring structure via the σ-operations. Note that for any H-graded P-variety X, $[S_n \circ X]$ is a well-defined element of $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$, even if $X(0,0) \neq \emptyset$. Therefore, we can define

$$\sigma_q([X]) := \sum_{n=0}^{\infty} [S_n \circ X] q^n = 1 + [X] + \sum_{n=2}^{\infty} [S_n \circ X] q^n \,,$$

By Theorem 1.2.6, we have

$$\sigma_q([X] + [Y]) = \sigma_q([X])\, \sigma_q([Y])$$

for all X and Y, i.e., the σ-operations satisfy (iii) for all non-virtual elements $[X], [Y] \in \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. In order for (iii) to hold also for virtual elements $[X] - [Y] \in \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$, we are forced to define

$$\sigma_q([X] - [Y]) := \sigma_q([X])\, \sigma_q([Y])^{-1} \,.$$

With this definition, the σ-operations clearly satisfy the properties (i), (ii), (iii). Therefore, they determine a pre-λ-ring structure on $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. □

This pre-λ-ring structure is an extension of the one defined via the motivic zeta function on $\mathrm{K}_0(\mathsf{Var})$. Indeed, if we regard $\mathrm{K}_0(\mathsf{Var})$ as a subring of $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$, then $\sigma^n(\mathrm{K}_0(\mathsf{Var})) \subseteq \mathrm{K}_0(\mathsf{Var})$ for all n, and we have

$$\sigma_q([X]) = 1 + \sum_{n=1}^{\infty} [S_n \circ X] q^n = 1 + \sum_{n=1}^{\infty} [\mathrm{Sym}^n(X)] q^n = \zeta_{[X]}(q)$$

for any $[X] \in \mathrm{K}_0(\mathsf{Var})$.

In a general pre-λ-ring, the behaviour of the λ-operations with respect to the product is not prescribed, and the same happens for the composition of these operations. Conversely, in a so-called λ-ring, there are also explicit expressions for $\lambda^n(xy)$ and $\lambda^m(\lambda^n(x))$.

Definition 1.3.11. A pre-λ-ring R is a λ-ring if $\lambda_q : R \to 1 + qR[\![q]\!]$ is a morphism of λ-rings, that is, a ring homomorphism which commutes with the λ-operations.

Equivalently, for a pre-λ-ring R to be a λ-ring, we require the following conditions to hold:

(iv) $\lambda^n(xy) = P_n(\lambda^1(x), \dots, \lambda^n(x); \lambda^1(y), \dots, \lambda^n(y))$ for all $x, y \in R$ and $n \in \mathbb{N}$;

(v) $\lambda^m(\lambda^n(x)) = P_{mn}(\lambda^1(x), \dots, \lambda^n(x))$ for all $x \in R$ and $m, n \in \mathbb{N}$.

Here, $P_n \in \mathbb{Z}[t_1, \dots, t_n; u_1, \dots, u_n]$ and $P_{mn} \in \mathbb{Z}[t_1, \dots, t_{mn}]$ are certain universal polynomials, whose definition can be found in [Knu73, §1.2].

One of the most important examples of λ-rings is the ring of symmetric functions Λ, which is the the free λ-ring on one generator e_1, where e_1 is the elementary symmetric function of degree 1. In particular, for any λ-ring R and any $x \in R$, there is a unique morphism of λ-rings $f_x : \Lambda \to R$ such that $f_x(e_1) = x$. Let us consider the elementary symmetric functions $e_n \in \Lambda$ and the complete homogeneous symmetric functions $h_n \in \Lambda$. Then we have

$$\lambda^n(x) = f_x(e_n), \qquad \sigma^n(x) = f_x(h_n).$$

Note that $\Lambda \cong \mathrm{K}_0([\mathsf{P}, \mathsf{Vect}_{\mathbb{Q}}])$, where $\mathsf{Vect}_{\mathbb{Q}}$ is the category of finite dimensional $\mathbb{Q}$-vector spaces.

It is not known whether the pre-λ-ring $\mathrm{K}_0(\mathsf{Var})$ is actually a λ-ring. A fortiori, this is not known either for $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. We leave this question open for future research.

1.4 Grothendieck groups of $\mathbb{Q}$-linear abelian tensor categories

Let us recall the following general definition.

Definition 1.4.1. Let $\mathscr{A}$ be an abelian category which is essentially small, i.e., equivalent to a small category. The Grothendieck group $\mathrm{K}_0(\mathscr{A})$ of $\mathscr{A}$ is the quotient of the free abelian group on the isomorphism classes of objects of $\mathscr{A}$, by the relations $[X] = [Y] + [Z]$ whenever $0 \to Y \to X \to Z \to 0$ is a short exact sequence in $\mathscr{A}$. If $(\mathscr{A}, \otimes, \mathbb{1})$ is an essentially small, abelian symmetric monoidal category, whose tensor product $\otimes$ is exact, then the group $\mathrm{K}_0(\mathscr{A})$ is in fact a commutative ring, with the product given by $[X][Y] := [X \otimes Y]$.

Let $(\mathscr{A}, \otimes, \mathbb{1})$ be a fixed essentially small, $\mathbb{Q}$-linear, abelian symmetric monoidal category, with exact tensor product. The most relevant example for this thesis will be the category $\mathscr{A} = \mathsf{MHS}$ of mixed Hodge structures over $\mathbb{Q}$. Let $(H, +, 0)$ be a commutative monoid satisfying $(H1)$ and $(H2)$. By Theorem 1.2.7, $([\mathsf{P} \times \underline{H}, \mathscr{A}], \boxtimes, S_0)$ is again an essentially small, abelian symmetric monoidal category, and $\boxtimes$ is exact. Therefore, the previous definition associates to $[\mathsf{P} \times \underline{H}, \mathscr{A}]$ a corresponding Grothendieck ring. Similarly, it associates a corresponding Grothendieck group to the abelian categories $[\underline{H}, \mathscr{A}]$ and $[\underline{\Sigma_n}, \mathscr{A}]$, where $\underline{\Sigma_n}$ is the category with one object $\bullet$ and $\underline{\Sigma_n}(\bullet, \bullet) = \Sigma_n$.

Definition 1.4.2. The Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ of H-graded P-objects of $\mathscr{A}$ is the Grothendieck ring of $([\mathsf{P} \times \underline{H}, \mathscr{A}], \boxtimes, S_0)$. In particular, for $H = \{0\}$, we obtain the Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A})$ of P-objects of $\mathscr{A}$.

The Grothendieck group $\mathrm{K}_0(\mathscr{A}^H)$ of H-graded objects of $\mathscr{A}$ is defined as the Grothendieck group of $[\underline{H}, \mathscr{A}]$. Finally, the Grothendieck group $\mathrm{K}_0^{\Sigma_n}(\mathscr{A})$ is the Grothendieck group of $[\underline{\Sigma_n}, \mathscr{A}]$.

As in the case of Var, there is a natural isomorphism

$$\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}) \cong \prod_{n\in\mathbb{N}} \mathrm{K}_0^{\Sigma_n}(\mathscr{A}),$$

where $\mathrm{K}_0^{\Sigma_0}(\mathscr{A}) = \mathrm{K}_0(\mathscr{A})$, and there are natural inclusions

$$\begin{tikzcd}
 & \mathrm{K}_0^{\mathsf{P}}(\mathscr{A}) \arrow[dr, hook] & \\
\mathrm{K}_0(\mathscr{A}) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)\,. \\
 & \mathrm{K}_0(\mathscr{A}^H) \arrow[ur, hook] &
\end{tikzcd}$$

The inclusion homomorphism $\mathrm{K}_0(\mathscr{A}^H) \hookrightarrow \mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ defines a commutative $\mathrm{K}_0(\mathscr{A}^H)$-algebra structure on $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$. This algebra shares many features with $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$. It is filtered by the subalgebras

$$F_n(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)) = \left\langle \bigcup_{i=0}^{n} \{[X] \mid X(m,w) = 0 \text{ if } m < n-i \text{ or } l_w < i\} \right\rangle,$$

and it is complete with respect to this decreasing filtration. Moreover, there is a natural derivation of $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$, namely the group homomorphism $D : \mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H) \to \mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ induced by the endofunctor D of $[\mathsf{P} \times \underline{H}, \mathscr{A}]$. By Theorem 1.2.7, D is indeed a $\mathrm{K}_0(\mathscr{A}^H)$-linear derivation, and we have $D(F_n(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))) \subseteq F_{n-1}(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))$. Finally, we can define a sequence of elements $s_n \in F_n(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))$, by setting $s_n := [S_n]$ for all $n \in \mathbb{N}$. Then we have $s_0 = 1$, since S_0 is the identity object for $\boxtimes$, and $Ds_n = s_{n-1}$ for all n.

Exactly as we did for $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}^H)$, we can introduce a pre-λ-ring structure on $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$. If we define

$$\sigma_q([X]) := \sum_{n=0}^{\infty} [S_n \circ X] q^n$$

for all H-graded P-objects X of $\mathscr{A}$, then the operation σ^n is given by the coefficient of q^n in the power series

$$\sigma_q([X] - [Y]) := \sigma_q([X])\, \sigma_q([Y])^{-1},$$

for all $[X]-[Y] \in \mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$. By [Get95a], this pre-$\lambda$-ring structure is actually a λ-ring structure. Moreover, the equivalence of categories

$$[\mathsf{P}, \mathsf{Vect}_{\mathbb{Q}}] \otimes [\underline{H}, \mathscr{A}] \simeq [\mathsf{P}, [\underline{H}, \mathscr{A}]] \tag{1.10}$$

induces a natural isomorphism

$$\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H) \cong \big(\mathrm{K}_0(\mathscr{A}^H) \otimes \Lambda\big)^{\wedge} \tag{1.11}$$

of complete filtered λ-rings, where Λ is the filtered λ-ring of symmetric functions and $\big(\mathrm{K}_0(\mathscr{A}^H)\otimes\Lambda\big)^\wedge$ is the completion with respect to the induced filtration on $\mathrm{K}_0(\mathscr{A}^H)\otimes\Lambda$ (see [Get95a]). In particular, there is a natural identification

$$\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)\cong\mathrm{K}_0(\mathscr{A}^H)[\![s_1,s_2,\dots]\!]\,. \tag{1.12}$$

Under this identification and the equivalence (1.10), one can easily define an operation $\circ:\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)\times F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))\to\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ induced by the functor $\circ:[\mathsf{P}\times\underline{H},\mathscr{A}]\times[\mathsf{P}\times\underline{H},\mathscr{A}]_1\to[\mathsf{P}\times\underline{H},\mathscr{A}]$ of Theorem 1.2.7, i.e., such that $[X]\circ[Y]=[X\circ Y]$ for all H-graded P-objects X,Y of $\mathscr{A}$. Let us recall the construction from [Get95a].

For all n, let h_n be the complete homogeneous symmetric function of degree n. Since there is an isomorphism $\Lambda\cong\mathbb{Z}[\![h_1,h_2,\dots]\!]$, we can define a map $\varphi:\Lambda\times F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))\to\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ by $(h_{n_1}\cdots h_{n_i},a)\mapsto\sigma^{n_1}(a)\cdots\sigma^{n_i}(a)$, and extending it by linearity on the first argument. For any partition $\mu\vdash n$, let $s_\mu\in\Lambda$ be the corresponding Schur function, and let

$$\sigma^\mu:=\varphi(s_\mu,\bullet):F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))\to\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)\,.$$

If $f:\Lambda\to\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ is the unique homomorphism of λ-rings such that $f(h_n)=s_n$ for all n, let $s_\mu\in\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ be the image under f of the Schur polynomial $s_\mu\in\Lambda$. Via (1.12), we can write each element $a\in\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ as a series $\sum_\mu a_\mu s_\mu$ over all possible partitions μ, where $a_\mu\in\mathrm{K}_0(\mathscr{A}^H)$. For all $b\in F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))$, define

$$a\circ b:=\sum_\mu a_\mu\,\sigma^\mu(b)\,.$$

Via the equivalence (1.10), we see that the equality $[X]\circ[Y]=[X\circ Y]$ holds for all X,Y. Then by [Get95a] and Theorem 1.2.7, we obtain the following result (cf. [GP06, Thm. 5.1]).

Theorem 1.4.3. *The Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$, together with the filtration $F_n(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))$, the derivation D, and the sequence (s_n) defined above, is a complete algebra with composition operation $\circ$ over $\mathrm{K}_0(\mathscr{A}^H)$. In particular, the Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A})$ is a complete $\mathrm{K}_0(\mathscr{A})$-algebra with composition operation.*

The isomorphism (1.11) has other interesting consequences, which we recall from [GP06, §5]. Indeed, (1.11) allows us to transfer the theory of representations of symmetric groups in $\mathsf{Vect}_{\mathbb{Q}}$ to $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$. In particular, we have the equality

$$s_n\circ(ab)=\sum_{\mu\vdash n}(s_\mu\circ a)(s_\mu\circ b)\,.$$

For any $n\in\mathbb{N}$, let $p_n\in\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ be the image of the power sum symmetric function of degree n under f. Then there is an isomorphism

$$\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)\otimes\mathbb{Q}\cong(\mathrm{K}_0(\mathscr{A}^H)\otimes\mathbb{Q})[\![p_1,p_2,\dots]\!]\,,$$

and the equality $p_n \circ p_m = p_{mn}$ holds. By [GP06, Lemma 5.3], the map $F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)) \to F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))$ given by $a \mapsto p_n \circ a$ is an algebra homomorphism for every $n \in \mathbb{N}$. Moreover, we have $Dp_n = \delta_{n,1}$.

Finally, as $\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)$ is a complete algebra with composition operation, it is equipped with an operation $\mathrm{Exp} : F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H)) \to F_1(\mathrm{K}_0^{\mathsf{P}}(\mathscr{A}^H))$ and its inverse Log (see Section 1.3). The expression of Log in terms of the elements p_n yields a simple explicit formula for Log, which can be useful in computations. We refer to [GP06] for more details.

1.5 The Serre characteristic

Let us fix $\mathbb{C}$ as our ground field. All varieties and morphisms between them are assumed to be over $\mathbb{C}$.

Let X be a variety. By Deligne's work, we know that the compactly supported cohomology $\mathrm{H}_c^i(X, \mathbb{Q})$ carries a natural $\mathbb{Q}$-mixed Hodge structure $(\mathrm{H}_c^i(X, \mathbb{Q}), W_\bullet, F^\bullet)$. If X is a projective variety which has only finite quotient singularities, then $(\mathrm{H}^i(X, \mathbb{Q}), W_\bullet, F^\bullet)$ is actually a pure Hodge structure of weight i (cf. [PS08, Thm 2.43]).

Definition 1.5.1. The association

$$[X] \mapsto \sum_{i \in \mathbb{N}} (-1)^i [(\mathrm{H}_c^i(X, \mathbb{Q}), W_\bullet, F^\bullet)]$$

defines a ring homomorphism $\mathfrak{e} : \mathrm{K}_0(\mathsf{Var}_{\mathbb{C}}) \to \mathrm{K}_0(\mathsf{MHS})$, called the Serre characteristic.

Remark 1.5.2. The homomorphism $\mathfrak{e}$ has different names in the literature. We follow the terminology of [GP06]. For instance, $\mathfrak{e}$ is referred to as the Hodge-Grothendieck characteristic in [PS08]. The proof of the existence of $\mathfrak{e}$ and its properties first appeared in [DK86].

When $[X] \in \mathrm{K}_0(\mathsf{Var}_{\mathbb{C}})$ is the class of a variety, we simply write $\mathfrak{e}(X)$ for $\mathfrak{e}([X])$. Moreover, we use the same notation $\mathfrak{e}$ also for its restriction to $\mathrm{K}_0(\mathsf{Var})$. We denote by L the Serre characteristic $\mathfrak{e}(\mathbf{L})$ of $\mathbf{L} = [\mathbb{A}^1]$, namely $\mathsf{L} = [\mathrm{H}^2(\mathbb{P}^1, \mathbb{Q})] = [\mathbb{Q}(-1)]$.

The Serre characteristic $\mathfrak{e} : \mathrm{K}_0(\mathsf{Var}_{\mathbb{C}}) \to \mathrm{K}_0(\mathsf{MHS})$ is a refined version of the E-polynomial. Indeed, there is a natural homomorphism of rings $\mathfrak{h} : \mathrm{K}_0(\mathsf{MHS}) \to \mathbb{Z}[t, u, t^{-1}, u^{-1}]$ given by

$$[(V, W_\bullet, F^\bullet)] \mapsto \sum_{p,q \in \mathbb{Z}} \dim_{\mathbb{C}}\big(\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W (V_{\mathbb{C}})\big) t^p u^q\,,$$

and for any variety X, its E-polynomial is exactly $\mathfrak{h}(\mathfrak{e}(X))$.

If the variety X carries an action of the symmetric group Σ_n, then the Σ_n-representation $\mathrm{H}_c^i(X, \mathbb{Q})$ is compatible with the natural mixed Hodge structure on it, i.e., it is a Σ_n-representation in the category MHS. Therefore, we can give the following definition.

Definition 1.5.3. The (equivariant) Serre characteristic is the ring homomorphism $\mathfrak{e} : \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}) \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$ which is defined as

$$\mathfrak{e}(X)(n) := \sum_{i \in \mathbb{N}} (-1)^i [(\mathrm{H}_c^i(Xn, \mathbb{Q}), W_{\bullet}, F^{\bullet})] \in \mathrm{K}_0^{\Sigma_n}(\mathsf{MHS})$$

on the classes $[X] \in \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ of P-varieties.

Since the restriction of this homomorphism to the subring $\mathrm{K}_0(\mathsf{Var})$ is the same as the one defined above, we keep the same notation $\mathfrak{e}$. We will use the same notation and the same name also for the homomorphism of rings $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!] \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})[\![q]\!]$ defined by $\sum_{d=0}^{\infty} a_d q^d \mapsto \sum_{d=0}^{\infty} \mathfrak{e}(a_d) q^d$.

For the purpose of this thesis, the most important property of the Serre characteristic is the following.

Theorem 1.5.4 ([GP06])**.** *The Serre characteristic* $\mathfrak{e} : \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var}) \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$ *is a homomorphism of complete algebras with composition operation, and the same holds for* $\mathfrak{e} : \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!] \to \mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})[\![q]\!]$.

Now, we have all the algebraic setting that we need to calculate the Hodge numbers and the Betti numbers of the coarse moduli spaces of stable maps from genus 0 curves to a Grassmannian. In fact, since the i-th cohomology group of these spaces has a pure Hodge structure of weight i, their Serre characteristic determines their Hodge numbers. In the next chapters, we will use the properties of $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ and $\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$ (in particular, their composition operation) in order to compute the Serre characteristic of these moduli spaces.

Chapter 2

Moduli spaces of stable maps

The aim of this chapter is to present an overview of our main objects of interest, namely moduli spaces of stable maps.

Let Y be a nonsingular projective variety over $\mathbb{C}$. For any $g, n \in \mathbb{N}$ and $\beta \in \mathrm{H}_2(Y)^+$, we introduce the moduli stacks $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ and their corresponding coarse moduli spaces $\overline{M}_{g,n}(Y,\beta)$, which parametrize stable maps $f : (C, x_1, \ldots, x_n) \to Y$ from an n-pointed, genus g, prestable curve $(C, x_1, \ldots, x_n)$ to Y, such that $f_*[C] = \beta$. The stability condition on f ensures the finiteness of the automorphism group of f. For arbitrary g, $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ is a proper algebraic Deligne-Mumford stack, with a projective coarse moduli space $\overline{M}_{g,n}(Y,\beta)$. When $g = 0$ and Y is convex, $\overline{\mathcal{M}}_{0,n}(Y,\beta)$ is nonsingular and the coarse space $\overline{M}_{0,n}(Y,\beta)$ is a normal variety, which only has local quotient singularities. Furthermore, $\overline{M}_{0,n}(Y,\beta)$ can be decomposed into locally closed loci that correspond to certain kinds of trees, called stable (n,β)-trees.

Organization of the chapter. In Section 2.1, we recall the definition and the basic properties of moduli stacks of stable maps, and of their corresponding coarse spaces.

In Section 2.2, we study more closely the spaces of stable maps from genus 0 curves to convex varieties. In particular, we study their decomposition in locally closed subvarieties in terms of isomorphism classes of stable trees.

Throughout the chapter, we work over the field $\mathbb{C}$ of complex numbers.

2.1 Generalities on stable maps and their moduli

Stable maps were introduced by Kontsevich in [Kon95]. In what follows, we recall the main definitions and constructions from [BM96] and [FP97].

Definition 2.1.1. Let T be a scheme. An n-pointed (or n-marked) genus g prestable curve $(p : \mathcal{C} \to T, \{x_i\}_{1 \leq i \leq n})$ over T is a flat proper morphism $p : \mathcal{C} \to T$ whose geometric fibers are connected, reduced, at worst nodal curves of arithmetic genus g, together with n sections $x_i : T \to \mathcal{C}$, such that for any geometric point $t \in T$

- each $x_i(t)$ is in the smooth locus of $\mathcal{C}_t$, and
- $x_i(t) \neq x_j(t)$ whenever $i \neq j$.

Definition 2.1.2. If Y is a scheme having an ample invertible sheaf, let

$$\mathrm{H}_2(Y)^+ := \{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(Y), \mathbb{Z}) \mid \varphi(\mathcal{L}) \geq 0 \text{ whenever } \mathcal{L} \text{ is ample}\}\,.$$

For any morphism $f : \mathcal{C} \to Y$ from a prestable curve $\mathcal{C}$ over T, the homology class of f is the locally constant function $T \to \mathrm{H}_2(Y)^+$ given by

$$t \mapsto (\mathcal{L} \mapsto \chi((f^*\mathcal{L})_t) + g - 1)\,.$$

The homology class of f is denoted by $f_*[\mathcal{C}]$.

Hereafter, let Y be a nonsingular projective variety. Note that $\mathrm{H}_2(Y)^+$ can be identified with the monoid of curve classes in $\mathrm{H}_2(Y, \mathbb{Z})$. Under this identification, for any morphism $f : C \to Y$ from a prestable curve C over $\mathbb{C}$, the homology class of f is precisely the image of $[C]$ under the pushforward $f_* : \mathrm{H}_2(C, \mathbb{Z}) \to \mathrm{H}_2(Y, \mathbb{Z})$.

Definition 2.1.3. Let $\beta \in \mathrm{H}_2(Y)^+$. A stable map over T from n-pointed genus g curves to Y of class β is a triple

$$(p : \mathcal{C} \to T, \{x_i\}_{1 \leq i \leq n}, f : \mathcal{C} \to Y)\,,$$

where

- $(p : \mathcal{C} \to T, \{x_i\})$ is an n-pointed genus g prestable curve over T,
- $f : \mathcal{C} \to Y$ is a scheme morphism such that $f_*[\mathcal{C}] = \beta$, and
- the following stability condition is satisfied: for every geometric point $t \in T$, and for every irreducible component of $\mathcal{C}_t$ that is contracted to a point by f, with corresponding normalization C', we have
  (i) if C' has genus 0, then C' has at least three special points;
  (ii) if C' has genus 1, then C' has at least one special point.

Here, a point of C' is called special if it is mapped either to a node or to a marked point of $\mathcal{C}_t$ by the normalization morphism.

A stable map $(p : C \to \mathrm{Spec}(\mathbb{C}), \{x_i\}_{1 \leq i \leq n}, f)$ over $\mathbb{C}$ is simply called a stable map, and it is sometimes denoted by $f : (C, x_1, \ldots, x_n) \to Y$.

Let $(p : \mathcal{C} \to T, \{x_i\}, f)$ and $(p' : \mathcal{C}' \to T, \{x'_i\}, f')$ be stable maps over T from n-pointed genus g curves to Y of class β. An isomorphism of stable maps $(p : \mathcal{C} \to T, \{x_i\}, f) \xrightarrow{\sim} (p' : \mathcal{C}' \to T, \{x'_i\}, f')$ is an isomorphism $h : \mathcal{C} \to \mathcal{C}'$ of T-schemes, such that $h \circ x_i = x'_i$ for all i, and $f' \circ h = f$.

The stability condition is equivalent to the finiteness of the automorphism group of the map. In particular, it implies that constant maps (i.e., those of class 0) from unmarked, genus 1 curves are not stable. Whenever the map is not constant or $(g,n) \neq (1,0)$, condition (ii) is automatically satisfied, thus the important condition is (i).

Definition 2.1.4. For any $\beta \in \mathrm{H}_2(Y)^+$, we denote by $\overline{\mathcal{M}}_{g,n}(Y,\beta)(T)$ the groupoid whose objects are stable maps over T from n-pointed genus g curves to Y of class β, and whose morphisms are the isomorphisms of such stable maps. Letting T vary, we get a stack $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ on the category of $\mathbb{C}$-schemes with the fppf topology.

Note that for $Y = \mathrm{Spec}(\mathbb{C})$ and $\beta = 0$, one recovers the stack $\overline{\mathcal{M}}_{g,n}$ of Deligne-Mumford stable, n-pointed, genus g curves.

Theorem 2.1.5 ([BM96, Thm. 3.14])**.** *The stack $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ is a proper algebraic Deligne-Mumford stack over $\mathbb{C}$.*

Contrary to $\overline{\mathcal{M}}_{g,n}$, the stack $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ is singular for arbitrary g and Y. Nevertheless, it is virtually smooth (see [BF97]) with expected dimension

$$(1-g)\dim(Y) + \int_\beta c_1(T_Y) + 3g - 3 + n\,.$$

The construction of the stacks of stable maps in [BM96] highlights their combinatorial and operadic properties, which are related to the combinatorial properties of graphs. In the next section, we will investigate those properties further in the case $g = 0$. We refer to [BM96] for more details in the general case.

Recall that there are some important canonical morphisms from the stack of stable maps. If $2g-2+n > 0$, there is a natural forgetful morphism $\overline{\mathcal{M}}_{g,n}(Y,\beta) \to \overline{\mathcal{M}}_{g,n}$, which is given by forgetting the map and stabilizing the curve, i.e., contracting its unstable components.

There is also a natural forgetful morphism $\overline{\mathcal{M}}_{g,n+1}(Y,\beta) \to \overline{\mathcal{M}}_{g,n}(Y,\beta)$, which first forgets the $(n+1)$st section, and then contracts the component of the curve over which the map becomes unstable. As in the case of stable curves, the following result holds.

Proposition 2.1.6 ([BM96, Cor. 4.6])**.** *The natural forgetful morphism $\overline{\mathcal{M}}_{g,n+1}(Y,\beta) \to \overline{\mathcal{M}}_{g,n}(Y,\beta)$ is the universal family over $\overline{\mathcal{M}}_{g,n}(Y,\beta)$.*

Finally, there are natural evaluation morphisms $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(Y,\beta) \to Y$, for $i = 1,\dots,n$, which are given by associating to each stable map $(\mathcal{C}, \{x_j\}, f)$ over T the morphism $f \circ x_i : T \to Y$.

By [KM97], $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ admits a coarse moduli space $\overline{M}_{g,n}(Y,\beta)$, which has been explicitly constructed in [FP97]. In particular, we have the following theorem.

Theorem 2.1.7 ([FP97, Thm. 1]). *The coarse moduli space $\overline{M}_{g,n}(Y,\beta)$ is a projective scheme.*

All the morphisms described above induce natural morphisms between the corresponding coarse moduli spaces. Therefore, we have the forgetful morphisms $\overline{M}_{g,n}(Y,\beta) \to \overline{M}_{g,n}$ and $\overline{M}_{g,n+1}(Y,\beta) \to \overline{M}_{g,n}(Y,\beta)$, and the evaluation morphisms $\mathrm{ev}_i : \overline{M}_{g,n}(Y,\beta) \to Y$ (denoted as above, by abuse of notation) for $i = 1,\ldots,n$.

In the next section, we will study more closely the coarse moduli spaces $\overline{M}_{0,n}(Y,\beta)$ of genus 0 stable maps. Because of the purposes of this thesis, the stack-theoretic point of view will not be pursued.

2.2 Stable maps of genus 0 to convex varieties

Let Y be a nonsingular, projective, convex variety. Recall that Y is said to be convex if $\mathrm{H}^1(\mathbb{P}^1, f^*T_Y) = 0$ for any morphism $f : \mathbb{P}^1 \to Y$. In particular, this implies that the scheme $\mathrm{Mor}(\mathbb{P}^1, Y)$ is nonsingular.

Examples of convex varieties are homogeneous varieties $Y = G/P$, where G is a smooth, connected, algebraic group and P a parabolic subgroup. Indeed, in this case, T_Y is generated by global sections, therefore the same holds for f^*T_Y and $\mathrm{H}^1(\mathbb{P}^1, f^*T_Y) = 0$. In particular, Grassmann varieties are convex.

Moduli spaces of stable maps from genus 0 curves to Y have a particularly nice geometry, which we now recall.

Theorem 2.2.1 ([FP97, Thm. 2]). *The coarse moduli space $\overline{M}_{0,n}(Y,\beta)$ is a normal projective variety of pure dimension*

$$\dim(Y) + \int_\beta c_1(T_Y) + n - 3\,.$$

Furthermore, it is locally the quotient of a nonsingular variety by a finite group.

The singularity type of the boundary of $\overline{M}_{0,n}(Y,\beta)$, i.e., the locus of reducible domain curves, is the same as the singularity type of the boundary of $\overline{M}_g$ and $\overline{M}_{g,n}$.

Theorem 2.2.2 ([FP97, Thm. 3]). *Up to a finite group quotient, the boundary of $\overline{M}_{0,n}(Y,\beta)$ is a divisor with normal crossings.*

There is a more precise description of the boundary of $\overline{M}_{0,n}(Y,\beta)$. From [FP97], we know that it decomposes into a union of divisors $D(A,B;\beta_1,\beta_2)$ which are in bijection with quadruples $(A,B;\beta_A,\beta_B)$, where

- A and B form a partition of $\{1,\ldots,n\}$,
- $\beta_A, \beta_B \in \mathrm{H}_2(Y)^+$ and $\beta_A + \beta_B = \beta$, and

- $\#A \geq 2$ (resp. $\#B \geq 2$) whenever $\beta_A = 0$ (resp. $\beta_B = 0$).

Let $\overline{M}_A = \overline{M}_{0,A\cup\{\bullet\}}(Y,\beta_A)$ and $\overline{M}_B = \overline{M}_{0,B\cup\{\bullet\}}(Y,\beta_B)$, and consider the fiber product

$$\begin{array}{ccc} \overline{M}_A \times_Y \overline{M}_B & \longrightarrow & \overline{M}_B \\ \big\downarrow & & \big\downarrow \text{ev.} \\ \overline{M}_A & \xrightarrow{\text{ev.}} & Y \end{array}$$

with respect to the evaluation maps at the marked point $\bullet$. Then we have the following result.

Proposition 2.2.3 ([FP97]). *There is a natural morphism*

$$h : \overline{M}_A \times_Y \overline{M}_B \to D(A, B; \beta_1, \beta_2)\,,$$

which has the following properties:

(i) if $A \neq \emptyset \neq B$, then h is an isomorphism;

(ii) if $A \neq \emptyset$, or $B \neq \emptyset$, or $\beta_A \neq \beta_B$, then h is a birational morphism;

(iii) if $A = B = \emptyset$ and $\beta_A = \beta_B$, then h is generically 2 to 1.

As explained in [BM96], the geometry of moduli spaces of stable maps is strictly related to the combinatorics of graphs. In the genus 0, convex case, there is a decomposition of $\overline{M}_{0,n}(Y,\beta)$ into locally closed subvarieties that correspond to stable (n,β)-trees. Let us first recall some definitions from [BM96].

Definition 2.2.4. A graph τ is a triple (F_τ, j_τ, R_τ), where

- F_τ is a finite set, called the set of flags of τ,
- $j_\tau : F_\tau \to F_\tau$ is an involution, and
- $R_\tau \subseteq F_\tau \times F_\tau$ is an equivalence relation.

Associated to each graph $\tau = (F_\tau, j_\tau, R_\tau)$, there is a triple (V_τ, E_τ, L_τ) of sets, which are defined as follows:

- the set V_τ of vertices of τ is the quotient set of F_τ by R_τ;
- the set E_τ of edges of τ is the set of orbits of j_τ with two elements;
- the set L_τ of leaves of τ is the set of fixed points of j_τ.

Finally, we associate to every vertex $v \in V_\tau$

- the subset $F_\tau(v) \subseteq F_\tau$ of flags whose equivalence class under R_τ is v, and

- the valence $n(v) := \#F_\tau(v)$ of v.

An isomorphism of graphs $\varphi : \tau \to \sigma$ is a bijective map $\varphi_F : F_\tau \to F_\sigma$ such that $\varphi_F \circ j_\tau = j_\sigma \circ \varphi_F$ and $(\varphi_F \times \varphi_F)(R_\tau) = R_\sigma$. In particular, φ_F induces a bijection $\varphi_V : V_\tau \to V_\sigma$ commuting with the canonical surjections, and bijections $\varphi_E : E_\tau \to E_\sigma$ and $\varphi_L : L_\tau \to L_\sigma$.

For each graph τ, there is an associated topological space, called its geometric realization.

Definition 2.2.5. The geometric realization $|\tau|$ of a graph τ is the topological space constructed as follows. First, consider the disjoint union

$$T = \bigsqcup_{i \in L_\tau} [0_i, 1/2_i) \sqcup \bigsqcup_{i \in F_\tau \setminus L_\tau} [0_i, 1/2_i] \sqcup \bigsqcup_{v \in V_\tau} \{|v|\}\,.$$

For each $v \in V_\tau$, identify the elements of the set $\{0_i \mid i \in F_\tau(v)\}$ with $|v|$. For each edge $\{i_1, i_2\} \in E_\tau$, identify $1/2_{i_1}$ with $1/2_{i_2}$. In this way, one obtains a set $|\tau|$, together with a surjection $T \to |\tau|$. Define the topological space $|\tau|$ to be this set, endowed with its quotient topology.

Definition 2.2.6. A tree is a graph τ whose geometric realization $|\tau|$ is simply connected, i.e., $|\tau|$ is connected and $\#V_\tau = \#F_\tau + 1$.

An (n, β)-tree is a tree τ, together with a bijection $l_\tau : L_\tau \xrightarrow{\sim} \{1, \dots, n\}$ and a map $\beta_\tau : V_\tau \to \mathrm{H}_2(Y)^+$ such that $\sum_{v \in V_\tau} \beta_\tau(v) = \beta$. An isomorphism of (n, β)-trees $\varphi : (\tau, l_\tau, \beta_\tau) \to (\sigma, l_\sigma, \beta_\sigma)$ is an isomorphism of the underlying graphs $\varphi : \tau \to \tau'$ such that $l_\sigma \circ \varphi_L = l_\tau$ and $\beta_\sigma \circ \varphi_V = \beta_\tau$.

Following [GP06], to each stable map $f : (C; x_1, \dots, x_n) \to Y$ of genus 0 and class β, we can associate its dual (n, β)-tree τ_f. Let V_f be the set of irreducible components of C, E_f the set of double points of C, and L_f the set of marked points of C. For each $z \in E_f$, let $C_{z,1}$ and $C_{z,2}$ be the two irreducible components of C intersecting at z. Similarly, for each $x_i \in L_f$, let C_{x_i} be the irreducible component where x_i lies. Then the graph $\tau_f = (F_f, R_f, j_f)$ is defined as follows:

- $F_f := L_f \sqcup \bigsqcup_{z \in E_f} \{C_{z,1}, C_{z,2}\}$;
- R_f is the equivalence kernel of the surjective map $F_f \to V_f$ defined by $x_i \mapsto C_{x_i}$ and $C_{z,\lambda} \mapsto C_{z,\lambda}$;
- $j_f(x_i) := x_i$ and $j_f(C_{z,\lambda}) := C_{z,\mu}$ for each $\lambda \neq \mu$.

By construction, we have canonical identifications $V_{\tau_f} = V_f$, $E_{\tau_f} = E_f$ and $L_{\tau_f} = L_f$.

Since the curve C has arithmetic genus 0, τ_f is a tree. Now, define $l_f : L_f \to \{1, \dots, n\}$ as $l_f(x_i) := i$, and $\beta_f : V_f \to \mathrm{H}_2(Y)^+$ as $\beta_f(C') := \beta'$, where β' is the class of $f|_{C'}$. Then (τ_f, l_f, β_f) becomes an (n, β)-tree.

The stability condition on the map f corresponds to a stability condition on its dual graph τ_f.

Definition 2.2.7. An (n,β)-tree τ is stable if for each $v \in V_\tau$, either $\beta_\tau(v) \neq 0$ or $n(v) > 2$.

As a consequence of this definition, the dual graph τ_f of a stable map $f : (C, x_1, \ldots, x_n) \to Y$ of genus 0 and class β is a stable (n,β)-tree. Note that the isomorphism class $[\tau_f]$ only depends on the isomorphism class $[f]$.

Conversely, given an isomorphism class $[\tau]$ of stable (n,β)-trees, we can consider the locus $M(\tau)$ in $\overline{M}_{0,n}(Y,\beta)$ parametrizing maps whose dual graph is isomorphic to τ. This locus $M(\tau)$ is a locally closed subvariety of $\overline{M}_{0,n}(Y,\beta)$ of codimension $\#E_\tau$, which lies in the intersection of $\#E_\tau$ boundary divisors $D(A,B;\beta_A,\beta_B)$ (see [FP97]). Thus, we obtain a decomposition of $\overline{M}_{0,n}(Y,\beta)$ into locally closed subvarieties $M(\tau)$ parametrized by equivalence classes of stable (n,β)-trees τ. Note that this decomposition is finite.

Lemma 2.2.8 (cf. [GP06])**.** *The set $\Gamma_{0,n}(\beta)$ of isomorphism classes of stable (n,β)-trees is finite.*

Proof. For any (n,β)-tree τ, we have

$$\#V_\tau = \#E_\tau + 1 = \frac{1}{2}\left(\sum_{v \in V_\tau} n(v) - n\right) + 1\,,$$

therefore $\sum_{v \in V_\tau}(n(v) - 2) = n - 2$. In particular, the number of vertices $v \in V_\tau$ such that $n(v) > 2$ is bounded by $n - 2$. Now, the set

$$\left\{(\beta_1, \ldots, \beta_i \in (\mathrm{H}_2(Y)^+)^i \mid \beta_1 + \cdots + \beta_i = \beta\,,\ \beta_j \neq 0\ \forall\, j\right\}$$

is empty for almost all i; let m_β be the maximum integer for which it is nonempty. If τ is stable, then the maximum number of vertices $v \in V_\tau$ such that $n(v) \leq 2$ is m_β. Therefore, we have

$$\#F_\tau = \sum_{v \in V_\tau} n(v) = n - 2 + 2(\#V_\tau) \leq 3n + m_\beta - 4\,.$$

Since the isomorphism classes of trees with a fixed number of flags is finite, it follows that $\Gamma_{0,n}(\beta)$ is finite. □

Since $\Gamma_{0,n}(\beta)$ is a finite set, we obtain the equality

$$\left[\overline{M}_{0,n}(Y,\beta)\right] = \sum_{[\tau] \in \Gamma_{0,n}(\beta)} [M(\tau)]$$

in $\mathrm{K}_0(\mathsf{Var})$. Note that this does not hold in $\mathrm{K}_0^{\Sigma_n}(\mathsf{Var})$, because the loci $M(\tau)$ are not Σ_n-invariant. In $\mathrm{K}_0^{\Sigma_n}(\mathsf{Var})$, we only have the weaker equality

$$\left[\overline{M}_{0,n}(Y,\beta)\right] = \left[\bigsqcup_{[\tau] \in \Gamma_{0,n}(\beta)} M(\tau)\right].$$

As shown in [GP06], the loci $M(\tau)$ can be given a description in terms of gluing of stable maps from regular curves. For any finite set of labels L and any $\beta \in \mathrm{H}_2(Y)^+$ such that either $\beta \neq 0$ or $\#L > 2$, let $M_{0,L}(Y,\beta)$ be the coarse moduli space which parametrizes equivalence classes of stable maps $\mathbb{P}^1 \to Y$ of class β, together with an embedding $L \hookrightarrow \mathbb{P}^1$. There is a natural evaluation morphism $M_{0,L}(Y,\beta) \to Y^L$, which maps each geometric point $[f : \mathbb{P}^1 \to Y]$ to the composition $L \hookrightarrow \mathbb{P}^1 \xrightarrow{f} Y$. For any tree τ, we have $F_\tau = \bigsqcup_{v \in V_\tau} F_\tau(v)$, therefore there is a natural morphism

$$\prod_{v \in V_\tau} M_{0,F_\tau(v)}(Y, \beta_\tau(v)) \to Y^{F_\tau}.$$

By precomposition with the surjective map $F_\tau \to E_\tau \sqcup L_\tau$ that maps each flag to its orbit, we also get a morphism $Y^{E_\tau \sqcup L_\tau} \to Y^{F_\tau}$. Let $M_\square(\tau)$ be the fiber product

$$\begin{tikzcd}
M_\square(\tau) \arrow[r] \arrow[d] & \prod_{v \in V_\tau} M_{0,F_\tau(v)}(Y, \beta_\tau(v)) \arrow[d] \\
Y^{E_\tau \sqcup L_\tau} \arrow[r] & Y^{F_\tau}
\end{tikzcd}$$

with respect to these morphisms.

Proposition 2.2.9 (cf. [GP06]). *There is a canonical isomorphism*

$$M(\tau) \cong M_\square(\tau)/\operatorname{Aut}(\tau)$$

for each $[\tau] \in \Gamma_{0,n}(Y,\beta)$.

Proof. On geometric points, the isomorphism is obtained as follows. Let $z = ([f_v : \mathbb{P}^1_v \to Y; \iota_v : F_\tau(v) \hookrightarrow \mathbb{P}^1_v])_{v \in V_\tau}$ and $g : E_\tau \sqcup L_\tau \to Y$ be geometric points that map to the same point of Y^{F_τ}. For each $e = \{i_{1,e}, i_{2,e}\} \in E_\tau$, let $v_{1,e} = [i_{1,e}]$ and $v_{2,e} = [i_{2,e}]$. By gluing all the lines $\mathbb{P}^1_{v_{1,e}}, \mathbb{P}^1_{v_{2,e}}$ along the points $\iota_{v_{1,e}}(i_{1,e}), \iota_{v_{2,e}}(i_{2,e})$, we get a prestable curve C together with morphisms $p_v : \mathbb{P}^1_v \to C$ for each $v \in V_\tau$. In particular, the images of $F_\tau(v) \cap L_\tau$ under the morphisms $p_v \circ \iota_v$ are n distinct regular points of C. If we label those points according to the map $l_\tau : L_\tau \xrightarrow{\sim} \{1, \ldots, n\}$, then we obtain an n-pointed prestable curve $(C, x_1, \ldots, x_n)$. The condition that z and g have the same image in Y^{F_τ} assures that we can glue also the maps $f_v : \mathbb{P}^1_v \to Y$, so that we obtain a map $f : (C, x_1, \ldots, x_n) \to Y$ of class $\sum_{v \in V_\tau} \beta_\tau(v) = \beta$. The stability condition on τ is equivalent to the stability condition on the map f. Note that this construction is invariant under automorphisms of τ.

Conversely, let $[f : (C, x_1, \ldots, x_n) \to Y]$ be a geometric point of $M(\tau)$, and let $\varphi : \tau \xrightarrow{\sim} \tau_f$ be a fixed isomorphism. For each $v \in V_\tau \cong V_f$, let C_v be the corresponding connected component of the normalization C^ν of C.

Each C_v is a smooth projective curve of genus 0, hence isomorphic to $\mathbb{P}^1$; choose one such isomorphism $\mathbb{P}^1_v := \mathbb{P}^1 \xrightarrow{\sim} C_v$, and let p_v be the composition $\mathbb{P}^1_v \xrightarrow{\sim} C_v \hookrightarrow C^\nu \to C$ and $f_v = f \circ p_v$. Each $\mathbb{P}^1_v$ comes equipped with $\# F_\tau(v)$ distinct distinguished points, and these points are labelled by the elements of $F_\tau(v)$ via the isomorphism $\varphi_F : F_\tau \xrightarrow{\sim} F_f$. As a result, for every $v \in V_\tau$ we have a morphism $f_v : \mathbb{P}^1_v \to Y$ of class $\beta_\tau(v)$, together with an embedding $\iota_v : F_\tau(v) \hookrightarrow \mathbb{P}^1_v$. Moreover, the morphisms $f_v \circ \iota_v$ agree on the orbits of j_τ, thus they determine a morphism $g : E_\tau \sqcup L_\tau \to Y$. If we vary our choice of the isomorphism $\mathbb{P}^1_v \xrightarrow{\sim} C_v$, then we get a pair (f'_v, ι'_v) which is related to (f_v, ι_v) by an automorphism of $\mathbb{P}^1_v$ commuting with the maps f_v, f'_v and the embeddings ι_v, ι'_v; the morphism g does not depend on this choice. Therefore, once we have fixed an isomorphism $\varphi : \tau \xrightarrow{\sim} \tau_f$, we can associate to $[f]$ the points $([f_v, \iota_v])_{v \in V_\tau} \in \prod_{v \in V_\tau} M_{0,F_\tau(v)}(Y, \beta_\tau(v))$ and $g \in Y^{E_\tau \sqcup L_\tau}$ mapping to the same point of Y^{F_τ}, i.e., a point of $M_\square(\tau)$. Varying φ changes this point by an automorphism of τ, hence we have actually canonically associated to $[f]$ a point of $M_\square(\tau)/\operatorname{Aut}(\tau)$. Clearly, this construction is the inverse of the gluing process described above.

By [BM96, Prop. 2.4] and [FP97, §6]) (cf. also [ACG11, §12.10]), the mutually inverse constructions that we did for geometric points can be performed also in families, thus proving the proposition. $\square$

Chapter 3

The Serre characteristic of $\overline{M}_{0,n}(G(r,V),d)$

In this chapter, we focus our attention on moduli spaces of genus 0 stable maps to a Grassmann variety $G(r,V)$. We show that the combinatorial properties of such spaces determine recursive relations in the Grothendieck ring $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$, which can be used to reduce the computation of their Serre characteristic to that of the open loci parametrizing maps from smooth curves. In turn, the Serre characteristic of these loci is determined by that of the configuration space of $\mathbb{P}^1$ and that of the varieties parametrizing morphisms of fixed degree from $\mathbb{P}^1$ to $G(r,V)$. The approach that we follow is a direct generalization of [GP06, §4].

Organization of the chapter. In Section 3.1, we recall some well-known facts about Grassmann varieties, which we will use later. Furthermore, we prove an important local triviality result for the natural evaluation morphisms of $\overline{M}_{0,n}(G(r,V),d)$.

In Section 3.2, we use the stratification by stable trees introduced in Section 2.2 to obtain the desired recursive relations in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$.

Finally, Section 3.3 deals with the Serre characteristic of $M_{0,n}(G(r,V),d)$. In particular, we show how to reduce its computation to the computation of the Serre characteristic of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$, which we will carry out in the next chapter.

3.1 Grassmannians

Let V be a fixed $\mathbb{C}$-vector space of dimension k, and let $0 < r < k$ be an integer. We consider the Grassmannian $G(r,V)$ of r-dimensional quotients of V. The scheme $G(r,V)$ represents the functor

$$\mathfrak{Quot}^r_{V\otimes\mathcal{O}_{\mathbb{C}}/\mathbb{C}/\mathbb{C}} : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}\,,$$

which associates to any locally Noetherian $\mathbb{C}$-scheme T the set of equivalence classes of locally free quotients of $V \otimes \mathcal{O}_T$ of rank r. It is a nonsingular, projective, rational $\mathbb{C}$-variety of dimension $r(k-r)$, endowed with a transitive action of the group $\mathrm{GL}(V, \mathbb{C})$.

Recall that $G(r, V)$ has a cellular decomposition into so-called Schubert cells $W_{a_1,\dots,a_{k-r}}$, where $(a_1, \dots, a_{k-r})$ is a nonincreasing sequence of integers between 0 and r (see [GH94]). As a consequence, $\mathrm{H}_{2r(k-r)-2\sum a_\lambda}(G(r, V), \mathbb{Z})$ is freely generated by the cycles $\left[\overline{W_{a_1,\dots,a_{k-r}}}\right]$, and for each $0 \le i, j \le r(k-r)$ the (i, j)-th Hodge number of $G(r, V)$ is

$$h^{i,j}(G(r, V)) = \begin{cases} \#\{(a_1, \dots, a_{k-r}) \mid \sum a_\lambda = i\} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} . \tag{3.1}$$

In particular, $\{(a_1, \dots, a_{k-r}) \mid \sum a_\lambda = r(k-r)-1\} = \{(r, \dots, r, r-1)\}$, thus $\mathrm{H}_2(G(r, V), \mathbb{Z}) \cong \mathbb{Z}$. We identify the monoid $\mathrm{H}_2(G(r, V))^+$ of curve classes in $G(r, V)$ with $\mathbb{N}$: for any morphism $f : C \to G(r, V)$ from a prestable curve C, we have $f_*[C] = d \in \mathbb{N}$ if and only if $\deg(f^*\mathcal{Q}) = d$, where $\mathcal{Q}$ is the universal quotient sheaf on $G(r, V)$.

In addition to the Hodge polynomial of $G(r, V)$, (3.1) determines the class

$$[G(r, V)] = \sum_{i=0}^{r(k-r)} h^{i,i}(G(r, V))\mathbf{L}^i$$

in $\mathrm{K}_0(\mathsf{Var})$. It is also possible to compute $[G(r, V)]$ recursively. Indeed, we have $G(1, V) = \mathbb{P}(V) \cong \mathbb{P}^{k-1}$ and $G(k-1, V) = \mathbb{P}(V^\vee) \cong \mathbb{P}^{k-1}$. In these cases, using the locally closed decomposition of $\mathbb{P}^{k-1}$ as $\mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \dots \sqcup \mathbb{A}^{k-1}$, we get

$$[G(1, V)] = [G(k-1, V)] = [\mathbb{P}^{k-1}] = \sum_{i=0}^{k-1} \mathbf{L}^i .$$

In the other cases, we can use the following recursive relation.

Proposition 3.1.1 ([Mar16, Prop. 2.1])**.** *If $r \ge 2$, then the equality*

$$[G(r, V)] = [G(r, k-1)] + \mathbf{L}^{k-r}[G(r-1, k-1)]$$

holds in $\mathrm{K}_0(\mathsf{Var})$*.*

Examples. If $V = \mathbb{C}^4$ and $r = 2$, we have

$$[G(2, 4)] = [\mathbb{P}^2](1 + \mathbf{L}^2) = 1 + \mathbf{L} + 2\mathbf{L}^2 + \mathbf{L}^3 + \mathbf{L}^4 .$$

If $V = \mathbb{C}^5$ and $r \in \{2, 3\}$, we have

$$\begin{aligned}[G(3, 5)] = [G(2, 5)] &= [G(2, 4)] + \mathbf{L}^3[\mathbb{P}^3] \\ &= 1 + \mathbf{L} + 2\mathbf{L}^2 + 2\mathbf{L}^3 + 2\mathbf{L}^4 + \mathbf{L}^5 + \mathbf{L}^6 .\end{aligned}$$

Moduli spaces of stable maps from genus 0 curves to $G(r,V)$ have the following important property, which we will use in the next sections.

Proposition 3.1.2. *The evaluation* $\mathrm{ev}_{n+1} : \overline{M}_{0,n+1}(G(r,V),d) \to G(r,V)$ *and its restriction* $M_{0,n+1}(G(r,V),d) \to G(r,V)$ *are locally trivial fibrations in the Zariski topology.*

Proof. For notational simplicity, let us identify V with $\mathbb{C}^k$, via the choice of a basis of V. Throughout all the proof, by a point we mean a closed point. We represent each point $[V \twoheadrightarrow Q] \in G(r,V)$ by a $k \times (k-r)$ matrix, whose column vectors span $\ker(V \twoheadrightarrow Q)$. For any $k \times k$ matrix A and any $I, J \subseteq \{1, \dots, k\}$, we denote by $A_{(I,J)}$ the submatrix of A whose rows are indexed by I and whose columns are indexed by J.

For every $I = \{i_1, \dots, i_{k-r}\} \subseteq \{1, \dots, k\}$, let $U_I \cong \mathbb{A}^{r(k-r)}$ be the open affine subset of $G(r,V)$ whose points are quotients $[V \twoheadrightarrow Q]$ such that $\ker(V \twoheadrightarrow Q) \cap \langle \{e_j \mid j \notin I\} \rangle = \{0\}$. Here, e_j is the j-th canonical basis vector of $\mathbb{C}^k$. If $I^c = \{1, \dots, k\} \setminus I$, then U_I can be identified with the closed subgroup of $\mathrm{GL}(k,\mathbb{C})$ whose points are matrices A such that $A_{(I,I)} = \mathbb{1}_{k-r}$, $A_{(I^c,I^c)} = \mathbb{1}_r$ and $A_{(I,I^c)} = 0$. Under this identification, there is a left action of U_I on $G(r,V)$, and $U_I \subseteq G(r,V)$ is U_I-invariant. Furthermore, the action of U_I on itself corresponds to the translation in $\mathbb{A}^{r(k-r)}$; in particular, it is free and transitive. Hereafter, we use the additive notation $(U_I, +, 0_I)$ for the group U_I, and $a : \mathsf{Sch}(\bullet, U_I) \times \mathsf{Sch}(\bullet, G(r,V)) \Rightarrow \mathsf{Sch}(\bullet, G(r,V))$ denotes the action of U_I on $G(r,V)$. We will show that there is a canonical isomorphism

$$(\mathrm{ev}_{n+1})^{-1}(U_I) \cong U_I \times (\mathrm{ev}_{n+1})^{-1}(\{0_I\}) .$$

Let $(p : \mathcal{C} \to T, \{x_\lambda\}_{1 \le \lambda \le n+1}, f : \mathcal{C} \to G(r,V))$ be a stable map over T from $(n+1)$-pointed genus 0 curves to $G(r,V)$ of class d, such that $f \circ x_{n+1}$ factors through U_I. Then we can associate to $(p, \{x_\lambda\}, f)$ both the morphism $f \circ x_{n+1} : T \to U_I$ and the triple

$$\big(p : \mathcal{C} \to T, \{x_\lambda\}_{1 \le \lambda \le n+1}, f_0 = a_{\mathcal{C}}(-(f \circ x_{n+1} \circ p), f) : \mathcal{C} \to G(r,V)\big) ,$$

which is a stable map over T of class d, such that $f_0 \circ x_{n+1}$ factors through $\{0_I\}$. This correspondence respects the equivalence classes and is natural with respect to T, thus it determines a morphism

$$h : (\mathrm{ev}_{n+1})^{-1}(U_I) \to U_I \times (\mathrm{ev}_{n+1})^{-1}(\{0_I\}) .$$

Conversely, let $(p' : \mathcal{C} \to T, \{x'_\lambda\}_{1 \le \lambda \le n+1}, f' : \mathcal{C} \to G(r,V))$ be a stable map over T from $(n+1)$-pointed genus 0 curves to $G(r,V)$ of class d, such that $f' \circ x'_{n+1}$ factors through $\{0_I\}$, and let $g : T \to U_I$ be a morphism. Then

$$\big(p' : \mathcal{C} \to T, \{x'_\lambda\}_{1 \le \lambda \le n+1}, f_g = a_{\mathcal{C}}(g \circ p', f') : \mathcal{C} \to G(r,V)\big)$$

is a stable map over T of class d, such that $f_g \circ x'_{n+1} = g$. As above, this correspondence respects the equivalence classes and is natural with respect to T. Therefore, it determines a morphism

$$h' : U_I \times (\mathrm{ev}_{n+1})^{-1}(\{0_I\}) \to (\mathrm{ev}_{n+1})^{-1}(U_I)\,,$$

which is clearly the inverse of h.

Finally, note that $(\mathrm{ev}_{n+1})^{-1}(\{0_I\})$ does not depend on the choice of I. Indeed, if $I' \subseteq \{1,\ldots,k\}$ is another subset with $\#I' = k-r$, then an isomorphism $(\mathrm{ev}_{n+1})^{-1}(\{0_I\}) \cong (\mathrm{ev}_{n+1})^{-1}(\{0_{I'}\})$ is obtained via the action of row-switching elementary matrices of $\mathrm{GL}(k,\mathbb{C})$ on $G(r,V)$. Since the subsets U_I form an open cover of $G(r,V)$, we obtain the thesis. □

The fiber of ev_{n+1} will be denoted by $\overline{\Phi}_{0,n}(G(r,V),d)$. Similarly, the fiber of its restriction $\mathrm{ev}_{n+1}|_{M_{0,n+1}(G(r,V),d)}$ will be denoted by $\Phi_{0,n}(G(r,V),d)$.

3.2 Recursive relations in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$

The variety $\overline{M}_{0,n}(G(r,V),d)$ has a natural left action of the symmetric group Σ_n, which is given by permutation of the marked points. Its subvariety $M_{0,n}(G(r,V),d)$, which parametrizes maps from smooth curves, is clearly invariant under this action. If we view Σ_n as the subgroup of Σ_{n+1} given by $\{\sigma \in \Sigma_{n+1} \mid \sigma(n+1) = n+1\}$, then $\overline{\Phi}_{0,n}(G(r,V),d)$ and $\Phi_{0,n}(G(r,V),d)$ are invariant under the restriction of the Σ_{n+1}-action on $\overline{M}_{0,n+1}(G(r,V),d)$ and $M_{0,n+1}(G(r,V),d)$ to Σ_n. Thus, we can consider the $\mathbb{N}$-graded P-varieties $\overline{M}, M : \mathsf{P} \times \underline{\mathbb{N}} \to \mathsf{Var}$ and $\overline{\Phi}, \Phi : \mathsf{P} \times \underline{\mathbb{N}} \to \mathsf{Var}$, defined as

$$\begin{aligned} \overline{M}(n,d) &:= \overline{M}_{0,n}(G(r,V),d)\,, & M(n,d) &:= M_{0,n}(G(r,V),d)\,,\\ \overline{\Phi}(n,d) &:= \overline{\Phi}_{0,n}(G(r,V),d)\,, & \Phi(n,d) &:= \Phi_{0,n}(G(r,V),d)\,. \end{aligned}$$

Here, $\underline{\mathbb{N}}$ denotes the discrete monoidal category whose set of objects is $\mathbb{N}$. Using Proposition 1.3.2, Proposition 3.1.2 directly implies the next result.

Corollary 3.2.1. *The equalities*

$$D[\overline{M}] = [G(r,V)][\overline{\Phi}]\,, \qquad D[M] = [G(r,V)][\Phi]$$

hold in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$.

Following [GP06], we will find a formula which relates the classes of $\overline{M}, M, \overline{\Phi}, \Phi$ in the Grothendieck group of $\mathbb{N}$-graded P-varieties $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$. First, we need some preliminary results.

Let us fix a representative for every equivalence class in $\Gamma_{0,n}(d)$; the set of these representatives is denoted by $\tilde{\Gamma}_{0,n}(d)$. For any $\tau \in \tilde{\Gamma}_{0,n}(d)$, let us also fix representatives for each equivalence class in $V_\tau/\operatorname{Aut}(\tau)$, $E_\tau/\operatorname{Aut}(\tau)$ and

$F_\tau/\operatorname{Aut}(\tau)$; the corresponding sets are denoted by $\tilde{V}_\tau$, $\tilde{E}_\tau$ and $\tilde{F}_\tau$, respectively. Notice that $L_\tau/\operatorname{Aut}(\tau) = L_\tau$. Then we can consider the following sets:

$$\begin{aligned}
\Gamma^V_{0,n}(d) &= \{(\tau, v) \mid \tau \in \tilde{\Gamma}_{0,n}(d)\,, v \in \tilde{V}_\tau\}\,,\\
\Gamma^E_{0,n}(d) &= \{(\tau, e) \mid \tau \in \tilde{\Gamma}_{0,n}(d)\,, e \in \tilde{E}_\tau\}\,,\\
\Gamma^L_{0,n}(d) &= \{(\tau, l) \mid \tau \in \tilde{\Gamma}_{0,n}(d)\,, l \in L_\tau\}\,,\\
\Gamma^F_{0,n}(d) &= \{(\tau, i) \mid \tau \in \tilde{\Gamma}_{0,n}(d)\,, i \in \tilde{F}_\tau\}\,.
\end{aligned}$$

If $a \in V_\tau \sqcup E_\tau \sqcup L_\tau \sqcup F_\tau$, we write $\operatorname{Aut}(\tau, a)$ for the group of automorphisms of τ preserving a.

Proposition 3.2.2 (cf. [GP06, Lemma 4.]). *The equalities*

$$\sum_{d=0}^{\infty} q^d \sum_{n=0}^{\infty} \left[\bigsqcup_{(\tau,v)\in\Gamma^V_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, v) \right] = [M] \circ (s_1 + [\overline{\Phi}])\,, \tag{3.2}$$

$$\sum_{d=0}^{\infty} q^d \sum_{n=0}^{\infty} \left[\bigsqcup_{(\tau,e)\in\Gamma^E_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, e) \right] = [G(r, V)](s_2 \circ [\overline{\Phi}])\,, \tag{3.3}$$

$$\sum_{d=0}^{\infty} q^d \sum_{n=0}^{\infty} \left[\bigsqcup_{(\tau,l)\in\Gamma^L_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, l) \right] = s_1 D[\overline{M}]\,, \tag{3.4}$$

$$\sum_{d=0}^{\infty} q^d \sum_{n=0}^{\infty} \left[\bigsqcup_{(\tau,i)\in\Gamma^F_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, i) \right] = [G(r, V)][\overline{\Phi}]^2 + s_1 D[\overline{M}] \tag{3.5}$$

hold in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$.

Proof. The equalities are obtained by the same gluing process described in Proposition 2.2.9, using [BM96, Prop. 2.4] and the definition of $\circ$. We show their geometric interpretation on closed points.

The closed points of $M_\square(\tau)/\operatorname{Aut}(\tau, v)$ correspond to equivalence classes $[f, C_v]$ of stable maps $(C, \{x_\lambda\}, f)$ with dual (n, d)-tree isomorphic to τ, together with a distinguished irreducible component C_v of C. Then we obtain a Σ_n-equivariant isomorphism

$$\begin{gathered}
\bigsqcup_{(\tau,v)\in\Gamma^V_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, v)\\
\Big\downarrow \cong\\
\bigsqcup_{\delta=0}^{d} \bigsqcup_{j=0}^{\infty} \big(M_{0,j}(G(r, V), \delta) \times (S_1 \sqcup \overline{\Phi})^{\boxtimes j}(n, d-\delta)\big)/\Sigma_j
\end{gathered}$$

by viewing $(C, \{x_\lambda\}, f)$ as the result of gluing $(C_v^\nu, \{x_i^\nu\}_{i\in F_\tau(v)}, f_v^\nu)$ with a stable map along each x_i^ν, $i \in F_\tau(v) \setminus L_\tau$, and with a marked point along each x_i^ν, $i \in F_\tau(v) \cap L_\tau$. Here, C_v^ν is the normalization of C_v, x_i^ν are the corresponding special points on it, and $f_v^\nu : C_v^\nu \to G(r, V)$ is the morphism which factors through $f|_{C_v}$. The existence of this isomorphism implies (3.2).

The closed points of $M_\square(\tau)/\operatorname{Aut}(\tau, e)$ correspond to equivalence classes $[f, z_e]$ of stable maps $(C, \{x_\lambda\}, f)$ with dual (n, d)-tree isomorphic to τ, with a distinguished double point $z_e \in C$. Then a Σ_n-equivariant isomorphism

$$\bigsqcup_{(\tau,e)\in\Gamma^E_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, e) \xrightarrow{\sim} G(r, V) \times \big((\overline{\Phi} \boxtimes \overline{\Phi})(n, d)/\Sigma_2\big)$$

is obtained by viewing $(C, \{x_\lambda\}, f)$ as the gluing of two stable maps having an additional marked point which they map to the same point $f(z_e) \in G(r, V)$, along these additional points. The existence of this isomorphism implies (3.3).

The closed points of $M_\square(\tau)/\operatorname{Aut}(\tau, l)$ correspond to equivalence classes $[f, x_l]$ of stable maps $(C, \{x_\lambda\}, f)$ with dual (n, d)-tree isomorphic to τ, with a distinguished marked point $x_l \in \{x_1, \ldots, x_n\}$. Since the Σ_n-action which permutes the points $x_1, \ldots, x_n$ is the same as the one induced by the action of Σ_{n-1} permuting $x_1, \ldots, \hat{x}_l, \ldots, x_n$, we get a Σ_n-equivariant isomorphism

$$\bigsqcup_{(\tau,l)\in\Gamma^L_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau, l) \xrightarrow{\sim} (S_1 \boxtimes D\overline{M})(n, d)\,.$$

The existence of this isomorphism implies (3.4).

Finally, let $i \in F_\tau$. If $i = l \in L_\tau$, then we have the same result as above. If $i \in F_\tau \setminus L_\tau$, then the closed points of $M_\square(\tau)/\operatorname{Aut}(\tau, i)$ correspond to equivalence classes $[f, C_i^\nu, z_i^\nu]$ of stable maps $(C, \{x_\lambda\}, f)$ with dual (n, d)-tree isomorphic to τ, together with a distinguished connected component C_i^ν of C^ν and a distinguished marked point $z_i^\nu \in C^\nu$, which is mapped to a double point $z_i \in C$. As in the case of edges, we can view f as the result of gluing two stable maps having an additional marked point which they map to the same point $f(z_i) \in G(r, V)$, along these additional points. However, unlike the case of edges, the triples (f, C_i^ν, z_i^ν), $(f, C^\nu_{j_\tau(i)}, z^\nu_{j_\tau(i)})$ are not identified in $M_\square(\tau)/\operatorname{Aut}(\tau, i)$. Therefore, we obtain a Σ_n-equivariant isomorphism

$$\bigsqcup_{\substack{([\tau],i)\in\Gamma^F_{0,n}(d)\\ i\in F_\tau\setminus L_\tau}} M_\square(\tau)/\operatorname{Aut}(\tau, i) \xrightarrow{\sim} G(r, V) \times (\overline{\Phi} \boxtimes \overline{\Phi})(n, d)\,,$$

where we do not quotient by Σ_2. It follows that there is a Σ_n-equivariant isomorphism

$$\begin{gathered}\bigsqcup_{(\tau,i)\in\Gamma^F_{0,n}(d)} M_\Box(\tau)/\operatorname{Aut}(\tau,i)\\ \big\downarrow\cong\\ \big(G(r,V)\times(\overline{\Phi}\boxtimes\overline{\Phi})(n,d)\big)\sqcup(S_1\boxtimes D\overline{M})(n,d)\,,\end{gathered}$$

whose existence implies (3.5). $\square$

The following technical results will be needed in the proof of the main theorem of this section.

Lemma 3.2.3 ([GP06, Lemma 4.8])**.** *For any $[\tau]\in\Gamma_{0,n}(d)$, there is a canonical injective map $\iota : F_\tau \hookrightarrow V_\tau \sqcup E_\tau \sqcup L_\tau$.*

The proof of the next lemma is straightforward.

Lemma 3.2.4. *Let X be a quasi-projective variety, and let A be a finite set. Let G be a finite group acting on both X and A. Then any choice of representatives for the equivalence classes in A/G determines an isomorphism*

$$(A\times X)/G \cong \bigsqcup_{[a]\in A/G} X/G_a\,,$$

where $G_a=\{g\in G \mid ga=a\}$ is the stabilizer subgroup of G with respect to the chosen representative a of $[a]$.

Now, we can finally prove our main theorem.

Theorem 3.2.5 (cf. [GP06, Thm. 4.5])**.** *The equality*

$$[\overline{M}] = [M]\circ(s_1+[\overline{\Phi}]) + [G(r,V)]\big(s_2\circ[\overline{\Phi}]-[\overline{\Phi}]^2\big)$$

holds in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$.

Proof. By Lemma 3.2.4, we have the equalities

$$\left[\bigsqcup_{(\tau,v)\in\Gamma^V_{0,n}(d)} M_\Box(\tau)/\operatorname{Aut}(\tau,v)\right] = \left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} (V_\tau\times M_\Box(\tau))/\operatorname{Aut}(\tau)\right],$$

$$\left[\bigsqcup_{(\tau,e)\in\Gamma^E_{0,n}(d)} M_\Box(\tau)/\operatorname{Aut}(\tau,e)\right] = \left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} (E_\tau\times M_\Box(\tau))/\operatorname{Aut}(\tau)\right],$$

$$\left[\bigsqcup_{(\tau,l)\in\Gamma^L_{0,n}(d)} M_\Box(\tau)/\operatorname{Aut}(\tau,l)\right] = \left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} (L_\tau\times M_\Box(\tau))/\operatorname{Aut}(\tau)\right].$$

The sum of the right hand sides is equal to

$$\left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} ((V_\tau \sqcup E_\tau \sqcup L_\tau) \times M_\square(\tau))/\operatorname{Aut}(\tau)\right]. \tag{3.6}$$

The class (3.6) can be rewritten using Lemma 3.2.3. Indeed, let us consider the canonical injection $\iota : F_\tau \hookrightarrow V_\tau \sqcup E_\tau \sqcup L_\tau$. Since τ is a tree, we have $\#F_\tau = \#(V_\tau \sqcup E_\tau \sqcup L_\tau) - 1$, thus the injection ι induces a canonical bijection $F_\tau \sqcup \{\bullet\} \xrightarrow{\sim} V_\tau \sqcup E_\tau \sqcup L_\tau$. As a consequence, we can rewrite (3.6) as

$$\left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} ((F_\tau \sqcup \{\bullet\}) \times M_\square(\tau))/\operatorname{Aut}(\tau)\right],$$

which in turn is equal to

$$\left[\bigsqcup_{(\tau,i)\in\Gamma^F_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau,i)\right] + \left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau)\right],$$

again by Lemma 3.2.4. Finally, from Proposition 2.2.9 we know that

$$\left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} M_\square(\tau)/\operatorname{Aut}(\tau)\right] = \left[\bigsqcup_{[\tau]\in\Gamma_{0,n}(d)} M(\tau)\right] = \left[\overline{M}_{0,n}(G(r,V),d)\right].$$

Combining all these equalities and applying Proposition 3.2.2, we conclude that the sum

$$[M] \circ (s_1 + [\overline{\Phi}]) + [G(r,V)](s_2 \circ [\overline{\Phi}]) + s_1 D[\overline{M}]$$

equals

$$[G(r,V)][\overline{\Phi}]^2 + s_1 D[\overline{M}] + [\overline{M}]$$

in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$, as claimed. $\square$

Corollary 3.2.6. *The equality*

$$[G(r,V)]\big([\overline{\Phi}] - [\Phi] \circ (s_1 + [\overline{\Phi}])\big) = 0$$

holds in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$.

Proof. Let us apply the derivation D to both sides of the equality in Theorem 3.2.5. The left hand side becomes

$$D[\overline{M}] = [G(r,V)][\overline{\Phi}].$$

Since D is additive, we can apply D separately on each summand on the right hand side. Using the properties of D, we get

$$\begin{aligned} D\big([M]\circ(s_1+[\overline{\Phi}])\big) &= \big(D[M]\circ(s_1+[\overline{\Phi}])\big)\, D(s_1+[\overline{\Phi}]) \\ &= \big(([G(r,V)][\Phi])\circ(s_1+[\overline{\Phi}])\big)(1+D[\overline{\Phi}]) \\ &= [G(r,V)]\big([\Phi]\circ(s_1+[\overline{\Phi}])\big)(1+D[\overline{\Phi}])\,, \end{aligned}$$

then

$$\begin{aligned} D\big([G(r,V)](s_2\circ[\overline{\Phi}])\big) &= [G(r,V)]\, D(s_2\circ[\overline{\Phi}]) \\ &= [G(r,V)]\,(s_1\circ[\overline{\Phi}])\, D[\overline{\Phi}] \\ &= [G(r,V)][\overline{\Phi}]\, D[\overline{\Phi}]\,, \end{aligned}$$

and finally

$$D\big([G(r,V)][\overline{\Phi}]^2\big) = 2[G(r,V)][\overline{\Phi}]\, D[\overline{\Phi}]\,.$$

Therefore, combining all the equalities, we see that

$$[G(r,V)](1+D[\overline{\Phi}])\big([\overline{\Phi}]-[\Phi]\circ(s_1+[\overline{\Phi}])\big)=0\,.$$

Since $(1+D[\overline{\Phi}])$ is not a zero divisor in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]$, we get the corollary. $\square$

By Theorem 1.5.4, the Serre characteristic $\mathfrak{e}:\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})[\![q]\!]\to\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})[\![q]\!]$ is a morphism of complete algebras with composition operation. Therefore, by applying $\mathfrak{e}$ to both Theorem 3.2.5 and Corollary 3.2.6, we obtain the following result.

Corollary 3.2.7. *The equalities*

$$\mathfrak{e}(\overline{M})=\mathfrak{e}(M)\circ\big(s_1+\mathfrak{e}(\overline{\Phi})\big)+\mathfrak{e}(G(r,V))\big(s_2\circ\mathfrak{e}(\overline{\Phi})-\mathfrak{e}(\overline{\Phi})^2\big)$$

and

$$\mathfrak{e}(\overline{\Phi})=\mathfrak{e}(\Phi)\circ\big(s_1+\mathfrak{e}(\overline{\Phi})\big)$$

hold in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})[\![q]\!]$.

The first equality of Corollary 3.2.7 shows that the Serre characteristic $\mathfrak{e}(\overline{M}_{0,n}(G(r,V),d))\in\mathrm{K}_0^{\Sigma_n}(\mathsf{MHS})$ can be obtained by knowing

(i) $\mathfrak{e}(G(r,V))\in\mathrm{K}_0(\mathsf{MHS})$,

(ii) $\mathfrak{e}(\overline{\Phi}_{0,m}(G(r,V),\delta))\in\mathrm{K}_0^{\Sigma_m}(\mathsf{MHS})$ for $m\le n$ and $\delta\le d$, and

(iii) $\mathfrak{e}(M_{0,m}(G(r,V),\delta))\in\mathrm{K}_0^{\Sigma_m}(\mathsf{MHS})$ for $m\le\max\{n,d\}$ and $\delta\le d$.

The second equality yields a recursive algorithm for the computation of $\mathfrak{e}(\overline{\Phi}_{0,m}(G(r,V),\delta))$. Thus, since $[G(r,V)]$ (and hence its Serre characteristic) can be computed as in Section 3.1, we see that the problem of determining $\mathfrak{e}(\overline{M}_{0,n}(G(r,V),d))$ boils down to the calculation of $\mathfrak{e}(M_{0,m}(G(r,V),\delta))$.

3.3 The Serre characteristic of $M_{0,n}(G(r,V),d)$

The open locus $M_{0,n}(G(r,V),d) \hookrightarrow \overline{M}_{0,n}(G(r,V),d)$, which parametrizes maps from nonsingular curves (i.e., curves isomorphic to $\mathbb{P}^1$), corresponds to $M(\tau)$, where $[\tau] \in \Gamma_{0,n}(d)$ is the equivalence class of stable (n,d)-trees with a single vertex and n leaves.

Let $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ be the space of morphisms from $\mathbb{P}^1$ to $G(r,V)$ of class d, and let

$$\mathrm{F}(\mathbb{P}^1,n) := \left\{(x_1,\dots,x_n) \in \left(\mathbb{P}^1\right)^n \mid x_i \neq x_j \ \forall\, i \neq j\right\}$$

be the configuration space of n distinct points in $\mathbb{P}^1$. The symmetric group Σ_n acts on $\mathrm{F}(\mathbb{P}^1,n)$ on the left by permuting the n-tuples of points. If $d > 0$ or $n \geq 3$, there is a canonical Σ_n-equivariant isomorphism

$$M_{0,n}(G(r,V),d) \cong \left(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)) \times \mathrm{F}(\mathbb{P}^1,n)\right)/\,\mathrm{Aut}(\mathbb{P}^1)\,, \tag{3.7}$$

where $\mathrm{Aut}(\mathbb{P}^1)$ acts on both $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ and $\mathrm{F}(\mathbb{P}^1,n)$ via its action on $\mathbb{P}^1$. If we set $\mathrm{F}(\mathbb{P}^1,0) := \mathrm{Spec}(\mathbb{C})$, then this isomorphism holds also for $n = 0$ (when $d > 0$). In the remaining cases, namely for $d = 0$ and $n \in \{0,1,2\}$, the stability condition implies that $M_{0,n}(G(r,V),0) = \emptyset$.

Let $\mathrm{F}(\mathbb{P}^1)$ be the P-variety defined as $\mathrm{F}(\mathbb{P}^1)(n) := \mathrm{F}(\mathbb{P}^1,n)$. From [GP06], we know the class of $\mathrm{F}(\mathbb{P}^1)$ in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$.

Proposition 3.3.1 ([GP06, Thm. 3.2])**.** *The equality*

$$[\mathrm{F}(\mathbb{P}^1)] = 1 + \mathrm{Exp}\big([\mathbb{P}^1]\,\mathrm{Log}(s_1)\big)$$

holds in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$.

The corresponding formula for the Serre characteristic $\mathfrak{e}(\mathrm{F}(\mathbb{P}^1))$ can be simplified using the properties of $\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$.

Corollary 3.3.2 ([GP06])**.** *Let μ denote the Möbius function. Then the equality*

$$\mathfrak{e}(\mathrm{F}(\mathbb{P}^1)) = (1+p_1)\prod_{n=1}^{\infty}(1+p_n)^{(1/n)\sum_{k|n}\mu(n/k)\mathsf{L}^k}$$

holds in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$.

We also know the class of $\mathrm{Aut}(\mathbb{P}^1)$ in $\mathrm{K}_0(\mathsf{Var})$, and its corresponding Serre characteristic.

Proposition 3.3.3. *We have*

$$[\mathrm{Aut}(\mathbb{P}^1)] = \mathbf{L}^3 - \mathbf{L}\,, \qquad \mathfrak{e}(\mathrm{Aut}(\mathbb{P}^1)) = \mathsf{L}^3 - \mathsf{L}$$

in $\mathrm{K}_0(\mathsf{Var})$ *and* $\mathrm{K}_0(\mathsf{MHS})$*, respectively.*

Proof. Since $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{Mor}_1(\mathbb{P}^1, \mathbb{P}^1)$, the equalities follow from the explicit formula for $[\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^r)] \in \mathrm{K}_0(\mathsf{Var})$ of [GP06, Prop. 3.3]. $\square$

Since the morphism $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)) \times \mathrm{F}(\mathbb{P}^1, n) \to M_{0,n}(G(r,V), d)$ is not a locally trivial fibration in the Zariski topology, we cannot express the class $[M(\bullet, d)] \in \mathrm{K}_0^{\mathsf{P}}(\mathsf{Var})$ in terms of the classes $[\mathrm{F}(\mathbb{P}^1)]$ and $[\mathrm{Aut}(\mathbb{P}^1)]$, using the isomorphism (3.7). However, the next result tells us that we can overcome this problem by considering the Serre characteristic.

Theorem 3.3.4 ([GP06, Thm. 5.4])**.** *Let $X : \mathsf{P} \to \mathsf{Var}$ be a P-variety, and let G be a connected algebraic group which acts on each Xn. Assume that the actions of Σ_n and G on each Xn commute. If the action of G has finite stabilizers, then $\mathfrak{e}(X) = \mathfrak{e}(G)\,\mathfrak{e}(X/G)$ in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$.*

As a consequence of this theorem, for all $d > 0$, we obtain the equality

$$\mathfrak{e}(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)))\,\mathfrak{e}(\mathrm{F}(\mathbb{P}^1)) = \mathfrak{e}(\mathrm{Aut}(\mathbb{P}^1))\,\mathfrak{e}(M(\bullet, d))$$

in $\mathrm{K}_0^{\mathsf{P}}(\mathsf{MHS})$. We write

$$\mathfrak{e}(M(\bullet, d)) = \frac{\mathfrak{e}(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)))\,\mathfrak{e}(\mathrm{F}(\mathbb{P}^1))}{\mathfrak{e}(\mathrm{Aut}(\mathbb{P}^1))}\,, \tag{3.8}$$

meaning that $\mathfrak{e}(M(\bullet, d))$ is the unique element of the form $\mathfrak{e}(X)$, for X a P-variety, such that $\mathfrak{e}(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)))\,\mathfrak{e}(\mathrm{F}(\mathbb{P}^1)) = \mathfrak{e}(\mathrm{Aut}(\mathbb{P}^1))\,\mathfrak{e}(X)$. We also get the equality

$$\mathfrak{e}(M(\bullet, 0)) = \frac{\mathfrak{e}(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V)))}{\mathfrak{e}(\mathrm{Aut}(\mathbb{P}^1))}\left(\mathfrak{e}(\mathrm{F}(\mathbb{P}^1)) - \sum_{i=0}^{2}\mathfrak{e}(\mathrm{F}(\mathbb{P}^1, i))\right). \tag{3.9}$$

Therefore, in order to compute $\mathfrak{e}(M(\bullet, d))$ (and hence $\mathfrak{e}(\overline{M}(\bullet, d))$, by the results of Section 3.2), the only missing part is the computation of $\mathfrak{e}(\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))) \in \mathrm{K}_0(\mathsf{Var})$. In the next chapter, we will show how this can be performed.

Chapter 4

The motive of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$

The aim of this chapter is to show how the class of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$ can be explicitly determined, in terms of the motives of certain well-studied Quot schemes.

Organization of the chapter. In Section 4.1, we introduce the Quot scheme compactification $\overline{Q}_d$ of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$, and we study a certain locally closed decomposition of $\overline{Q}_d$.

The class of each of these strata in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$ is determined in Section 4.2. In Section 4.3, these classes are used to obtain a formula for the motive of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$.

Section 4.4 contains some examples, which show how to compute the Serre characteristic of $\overline{M}_{0,n}(G(r,V),d)$ using the method presented in this thesis. In particular, we calculate the Serre characteristic of $\overline{M}_{0,n}(G(2,4),2)$ for $n \in \{0,1\}$.

Notation. We work over a fixed algebraically closed field $\Bbbk$ of characteristic 0. Morphisms of $\Bbbk$-schemes are tacitly assumed to be $\Bbbk$-morphisms. Throughout the chapter, $\mathbb{P}^1 = \mathbb{P}^1_{\Bbbk}$, and V denotes a fixed $\Bbbk$-vector space of dimension k.

For any Cartesian diagram

$$\begin{tikzcd} X \times_Y T \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

and any morphism $\psi : \mathcal{F} \to \mathcal{G}$ of sheaves on X, the pullback of ψ to $X \times_Y T$ is denoted by $\psi_T : \mathcal{F}_T \to \mathcal{G}_T$. In particular, if $\mathrm{Spec}(\kappa(y)) \to Y$ is the morphism determined by a point $y \in Y$, then the pullback of ψ to X_y is denoted by $\psi_y : \mathcal{F}_y \to \mathcal{G}_y$.

4.1 Quot compactification and its decomposition

In this section, we consider the scheme $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ that parametrizes morphisms of degree d from $\mathbb{P}^1$ to $G(r,V)$. We examine a compactification of $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ given by a certain Quot scheme $\overline{Q}_d$, which we will use in Section 4.3 to compute its class in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$.

As shown in [Nit05], $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ can be naturally realized as an open subscheme of the Hilbert scheme of $\mathbb{P}^1 \times G(r,V)$, by associating to each morphism $f : \mathbb{P}^1 \to G(r,V)$ its graph $\Gamma_f \subseteq \mathbb{P}^1 \times G(r,V)$.

A different compactification is provided by the Quot scheme

$$\overline{Q}_d := \mathrm{Quot}^{(t+1)r+d}_{V\otimes\mathcal{O}_{\mathbb{P}^1}/\mathbb{P}^1/\Bbbk},$$

which parametrizes equivalence classes of coherent quotients of $V \otimes \mathcal{O}_{\mathbb{P}^1}$ of rank r and degree d, i.e., with Hilbert polynomial equal to $(t+1)r+d \in \Bbbk[t]$. Inside $\overline{Q}_d$, $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$ is the open locus Q_d corresponding to locally free quotients of $V \otimes \mathcal{O}_{\mathbb{P}^1}$.

The compactification $\overline{Q}_d$ was studied in [Str87]. In particular, we recall the following result.

Theorem 4.1.1 ([Str87])**.** *The Quot scheme $\overline{Q}_d$ is an irreducible, rational, nonsingular, projective $\Bbbk$-variety of dimension $kd + r(k-r)$.*

Our aim is to study a certain locally closed decomposition of $\overline{Q}_d$.

Definition 4.1.2. Let Y be an algebraic $\Bbbk$-scheme. A locally closed decomposition of Y is a morphism $f : T \to Y$ of algebraic $\Bbbk$-schemes, that satisfies the following conditions:

(i) the restriction of f to each connected component of T is a locally closed immersion;

(ii) f is bijective on closed points.

Remark 4.1.3. Let Y be a $\Bbbk$-variety, and let $f : T \to Y$ be a locally closed decomposition of Y. If $T_1, \ldots, T_m \subseteq T$ are the connected components of T, then the equality

$$[Y] = \sum_{i=1}^{m} [f(T_i)]$$

holds in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$.

The starting point in constructing this decomposition is the following observation. Since any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$ splits as the direct sum of its torsion subsheaf $\mathcal{T}(\mathcal{F})$ and of the locally free sheaf $\mathcal{F}/\mathcal{T}(\mathcal{F})$, the geometric points of $\overline{Q}_d$ parametrizing quotients which are not locally free correspond to morphisms $\mathbb{P}^1 \to G(r,V)$ of lower degree, by considering only the locally free part of the quotients. This fact suggests the possibility of finding a decomposition of $\overline{Q}_d$ such that each of its members correspond to morphisms $\mathbb{P}^1 \to G(r,V)$ of some fixed degree $\delta \leq d$.

Motivated by this observation, we consider the schemes

$$\overline{Q}_\delta = \mathrm{Quot}^{(t+1)r+\delta}_{V\otimes\mathcal{O}_{\mathbb{P}^1}/\mathbb{P}^1/\Bbbk}$$

for any $\delta \in \mathbb{Z}$ such that $0 \leq \delta \leq d$. Let $[\pi_\delta : V \otimes \mathcal{O}_{\mathbb{P}^1\times\overline{Q}_\delta} \to \mathcal{F}_\delta]$ be the universal quotient on $\mathbb{P}^1 \times \overline{Q}_\delta$, $\mathcal{E}_\delta = \ker(\pi_\delta)$ and $\iota_\delta : \mathcal{E}_\delta \to V \otimes \mathcal{O}_{\mathbb{P}^1\times\overline{Q}_\delta}$ the inclusion. We then have the following short exact sequence of coherent $\mathcal{O}_{\mathbb{P}^1\times\overline{Q}_\delta}$-modules:

$$0 \to \mathcal{E}_\delta \xrightarrow{\iota_\delta} V \otimes \mathcal{O}_{\mathbb{P}^1\times\overline{Q}_\delta} \xrightarrow{\pi_\delta} \mathcal{F}_\delta \to 0\,. \tag{4.1}$$

The exactness of this sequence is preserved by any base change $T \to \overline{Q}_\delta$, as the following lemma shows.

Lemma 4.1.4. *Let Y be a locally Noetherian scheme, let X be a Y-scheme locally of finite type, and let*

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is Y-flat, then $\mathcal{H}$ is Y-flat if and only if for every morphism $T \to Y$ of locally Noetherian schemes, the pulled-back morphism $\varphi_T : \mathcal{F}_T \to \mathcal{G}_T$ is injective.

Proof. Assume that for every morphism $T \to Y$, $\varphi_T : \mathcal{F}_T \to \mathcal{G}_T$ is injective. In particular, for each point $y \in Y$, $\varphi_y : \mathcal{F}_y \to \mathcal{G}_y$ is injective. Then $\mathcal{H}$ is Y-flat by [Mat80, Thm. 49, Appl. 2].

Conversely, assume that $\mathcal{H}$ is Y-flat. We shall prove that φ_T is injective on each stalk. Let $z = (x,t,y,\mathfrak{p}) \in X \times_Y T$, where y is a point of Y, $x \in X$ and $t \in T$ are in the fibers over y, and $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_{X,x} \otimes_{O_{Y,y}} \mathcal{O}_{T,t}$ such that $\mathfrak{p} \cap \mathcal{O}_{X,x} = \mathfrak{m}_x$ and $\mathfrak{p} \cap \mathcal{O}_{T,t} = \mathfrak{m}_t$. Then $(\varphi_T)_z = (\varphi_x \otimes_{\mathcal{O}_{Y,y}} \mathrm{id}_{\mathcal{O}_{T,t}})_\mathfrak{p}$. Since φ_x is injective and $\mathcal{H}_x$ is $\mathcal{O}_{Y,y}$-flat, the homomorphism $\varphi_x \otimes_{\mathcal{O}_{Y,y}} \mathrm{id}_{\kappa(y)}$ is injective, by [Mat80, Thm. 49, Appl. 2]. Thus, $(\varphi_x \otimes_{\mathcal{O}_{Y,y}} \mathrm{id}_{\kappa(t)})_\mathfrak{p}$ is injective, too. As $(\bullet \otimes_{\mathcal{O}_{Y,y}} \kappa(t))_\mathfrak{p}$ and $(\bullet \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{T,t})_\mathfrak{p} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ are naturally isomorphic functors of $\mathcal{O}_{X,x}$-modules, it follows that $(\varphi_x \otimes_{\mathcal{O}_{Y,y}} \mathrm{id}_{\mathcal{O}_{T,t}})_\mathfrak{p} \otimes_{\mathcal{O}_{T,t}} \mathrm{id}_{\kappa(t)}$ is injective. Again by [Mat80, Thm. 49, Appl. 2], this implies the injectivity of $(\varphi_x \otimes_{\mathcal{O}_{Y,y}} \mathrm{id}_{\mathcal{O}_{T,t}})_\mathfrak{p}$. $\square$

Now, we introduce the relative Quot schemes

$$\overline{R}_\delta := \mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta/\mathbb{P}^1\times\overline{Q}_\delta/\overline{Q}_\delta}$$

over $\overline{Q}_\delta$.

Proposition 4.1.5. *The Quot scheme $\overline{R}_\delta$ is projective and smooth over $\overline{Q}_\delta$. In particular, $\overline{R}_\delta$ is nonsingular.*

Proof. Since the projection $\mathbb{P}^1 \times \overline{Q}_\delta \to \overline{Q}_\delta$ is a projective morphism, $\overline{R}_\delta$ is a projective $\overline{Q}_\delta$-scheme (see [Nit05]).

If $y \in \overline{Q}_\delta$ is a closed point, let $0 \to \mathcal{K} \to (\mathcal{E}_\delta)_y \to \mathcal{G}$ be the corresponding short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules. Then

$$\mathrm{Ext}^1(\mathcal{K}, \mathcal{G}) \cong \mathrm{H}^1(\mathbb{P}^1, \mathcal{K}^\vee \otimes \mathcal{G}) = 0\,,$$

because $\mathcal{G}$ has 0-dimensional support. Therefore, from [Leh98] it follows that $\overline{R}_\delta$ is a smooth $\overline{Q}_\delta$-scheme.

Finally, since $\overline{Q}_\delta$ is a smooth $\Bbbk$-scheme, $\overline{R}_\delta$ is smooth over $\Bbbk$ and thus nonsingular, because $\Bbbk$ is a perfect field. □

There is a Cartesian diagram

$$\begin{tikzcd}
\mathbb{P}^1 \times \overline{R}_\delta \arrow[r, "f'_\delta"] \arrow[d] & \mathbb{P}^1 \times \overline{Q}_\delta \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d] \\
\overline{R}_\delta \arrow[r, "f_\delta"] & \overline{Q}_\delta \arrow[r] & \mathrm{Spec}(\Bbbk)
\end{tikzcd}\,,$$

where $f_\delta : \overline{R}_\delta \to \overline{Q}_\delta$ is the structure morphism and $f'_\delta = \mathrm{id}_{\mathbb{P}^1} \times f_\delta$. Let

$$0 \to (\mathcal{E}_\delta)_{\overline{R}_\delta} \xrightarrow{(\iota_\delta)_{\overline{R}_\delta}} V \otimes \mathcal{O}_{\mathbb{P}^1 \times \overline{R}_\delta} \xrightarrow{(\pi_\delta)_{\overline{R}_\delta}} (\mathcal{F}_\delta)_{\overline{R}_\delta} \to 0$$

be the pullback of (4.1) by f'_δ. Moreover, let $[\rho_\delta : (\mathcal{E}_\delta)_{\overline{R}_\delta} \to \mathcal{G}_\delta]$ be the universal quotient on $\mathbb{P}^1 \times \overline{R}_\delta$, and let $\mathcal{H}_\delta$ be the cokernel of the inclusion $\ker(\rho_\delta) \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1 \times \overline{R}_\delta}$. Then we have the following commutative diagram of coherent $\mathcal{O}_{\mathbb{P}^1 \times \overline{R}_\delta}$-modules, with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d, dotted] & \\
0 \arrow[r] & \mathcal{K}_\delta := \ker(\rho_\delta) \arrow[r] \arrow[d, equal] & (\mathcal{E}_\delta)_{\overline{R}_\delta} \arrow[r, "\rho_\delta"] \arrow[d, "(\iota_\delta)_{\overline{R}_\delta}"] & \mathcal{G}_\delta \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathcal{K}_\delta \arrow[r] \arrow[d] & V \otimes \mathcal{O}_{\mathbb{P}^1 \times \overline{R}_\delta} \arrow[r] \arrow[d, "(\pi_\delta)_{\overline{R}_\delta}"] & \mathcal{H}_\delta \arrow[r] \arrow[d, dotted] & 0 \\
 & 0 \arrow[r, dotted] & (\mathcal{F}_\delta)_{\overline{R}_\delta} \arrow[r, dotted, "\cong"] \arrow[d] & \mathcal{H}_\delta/\mathcal{G}_\delta \arrow[r, dotted] \arrow[d, dotted] & 0 \\
 & & 0 & 0 &
\end{tikzcd}\,. \tag{4.2}$$

The existence of the dotted arrows follows from the universal properties of kernels and cokernels. The exactness of all rows and columns is a consequence of the Four lemma and of the Nine lemma.

Since both $\mathcal{G}_\delta$ and $\mathcal{H}_\delta/\mathcal{G}_\delta$ are flat over $\overline{R}_\delta$, $\mathcal{H}_\delta$ is $\overline{R}_\delta$-flat. Thus, for any $x \in \overline{R}_\delta$ the sequence of $\mathcal{O}_{\mathbb{P}^1_x}$-modules

$$0 \to (\mathcal{G}_\delta)_x \to (\mathcal{H}_\delta)_x \to (\mathcal{H}_\delta/\mathcal{G}_\delta)_x \to 0$$

is exact. Since $(\mathcal{H}_\delta/\mathcal{G}_\delta)_x \cong (\mathcal{F}_\delta)_{f_\delta(x)}$, we obtain that the Hilbert polynomial of $(\mathcal{H}_\delta)_x$ is

$$d - \delta + (t+1)r + \delta = (t+1)r + d \in \Bbbk[t]\,.$$

Therefore, by the universal property of $\overline{Q}_d$, there exists a unique morphism $g_\delta \colon \overline{R}_\delta \to \overline{Q}_d$ such that the quotient $V \otimes \mathcal{O}_{\mathbb{P}\times\overline{R}_\delta} \to \mathcal{H}_\delta$ is equivalent to the pullback of $\pi_d \colon V \otimes \mathcal{O}_{\mathbb{P}^1\times\overline{Q}_d} \to \mathcal{F}_d$ by $\mathrm{id}_{\mathbb{P}^1} \times g_\delta$.

Remark 4.1.6. By Lemma 4.1.4, the diagram (4.2) is stable under any base change $T \to \overline{R}_\delta$, because all sheaves appearing in it are flat over $\overline{R}_\delta$. In particular, for any $x \in \overline{R}_\delta$, the fiber $(\mathcal{G}_\delta)_x$ is (isomorphic to) a 0-dimensional subsheaf of $(\mathcal{H}_\delta)_x$, so that there is an injective morphism from $(\mathcal{G}_\delta)_x$ into the torsion subsheaf $\mathcal{T}((\mathcal{H}_\delta)_x)$ of $(\mathcal{H}_\delta)_x$. As a consequence, the length of $\mathcal{T}((\mathcal{H}_\delta)_x)$ is at least $d-\delta$.

Let Q_δ be the open locus in $\overline{Q}_\delta$ corresponding to locally free quotients of $V \otimes \mathcal{O}_{\mathbb{P}^1}$; Q_δ is the locus corresponding to $\mathrm{Mor}_\delta(\mathbb{P}^1, G(r,V))$ inside $\overline{Q}_\delta$. Let R_δ be the preimage $f_\delta^{-1}(Q_\delta)$, with its open subscheme structure.

Proposition 4.1.7. *For each $0 \le \delta \le d$, the set-theoretic image $g_\delta(R_\delta)$ is the constructible subset of $\overline{Q}_d$ whose closed points are the elements of*

$$\left\{ y \in \overline{Q}_d(\Bbbk) \mid \operatorname{length} \mathcal{T}((\mathcal{F}_d)_y) = d - \delta \right\}.$$

In particular, there is a set-theoretic decomposition $\overline{Q}_d = \bigsqcup_{0\le\delta\le d} g_\delta(R_\delta)$.

Proof. For any closed point y of $g_\delta(R_\delta)$, let $x \in R_\delta(\Bbbk)$ such that $g_\delta(x) = y$. Let us consider the pullback of (4.2) to the fiber $\mathbb{P}^1_x$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\mathcal{K}_\delta)_x \arrow[r] \arrow[d, equal] & (\mathcal{E}_\delta)_{f_\delta(x)} \arrow[r] \arrow[d] & (\mathcal{G}_\delta)_x \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{K}_\delta)_x \arrow[r] \arrow[d] & V \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & (\mathcal{H}_\delta)_x \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & (\mathcal{F}_\delta)_{f_\delta(x)} \arrow[r, "\cong"] \arrow[d] & (\mathcal{H}_\delta/\mathcal{G}_\delta)_x \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By Remark 4.1.6, $(\mathcal{G}_\delta)_x$ is a subsheaf of $\mathcal{T}((\mathcal{H}_\delta)_x)$. Moreover, since $x \in R_\delta$, the quotient sheaf $(\mathcal{H}_\delta)_x/(\mathcal{G}_\delta)_x \cong (\mathcal{H}_\delta/\mathcal{G}_\delta)_x \cong (\mathcal{F}_\delta)_{f_\delta(x)}$ is locally free. As a consequence, $(\mathcal{G}_\delta)_x$ actually coincides with $\mathcal{T}((\mathcal{H}_\delta)_x)$. It follows that the length of $\mathcal{T}((\mathcal{H}_\delta)_x)$ is exactly $d-\delta$. By construction of the morphism g_δ, we have $(\mathcal{H}_\delta)_x \cong (\mathcal{F}_d)_y$, therefore the length of $\mathcal{T}((\mathcal{F}_d)_y)$ is $d-\delta$.

Conversely, let $y \in \overline{Q}_d$ be a closed point such that $\mathcal{T} = \mathcal{T}((\mathcal{F}_d)_y)$ has length $d-\delta$. Then we have a quotient

$$V \otimes \mathcal{O}_{\mathbb{P}^1} \to (\mathcal{F}_d)_y \to (\mathcal{F}_d)_y/\mathcal{T}\,,$$

where $(\mathcal{F}_d)_y/\mathcal{T}$ is a locally free $\mathcal{O}_{\mathbb{P}^1}$-module of rank r and degree δ. Let $z \in Q_\delta(\Bbbk)$ be the point corresponding to this quotient. There is a diagram

$$\begin{tikzcd}
 & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dotted] & \ker(\rho) \arrow[r, dotted] \arrow[d, dotted, "\cong"] & (\mathcal{E}_\delta)_z \arrow[r, dotted, "\rho"] \arrow[d] & \mathcal{T} \arrow[r, dotted] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{E}_d)_y \arrow[r] \arrow[d, dotted] & V \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & (\mathcal{F}_d)_y \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r, dotted] & (\mathcal{F}_\delta)_z \arrow[r, "\cong"] \arrow[d] & (\mathcal{F}_d)_y/\mathcal{T} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

which commutes and has exact rows and columns; this is a consequence of the Four lemma and of the Nine lemma. In particular, $\mathcal{T}$ is a 0-dimensional quotient of $(\mathcal{E}_\delta)_z$ of length $d-\delta$. Therefore, $\rho : (\mathcal{E}_\delta)_z \to \mathcal{T}$ corresponds to a closed point $x \in R_\delta$ such that $f_\delta(x) = z$. Moreover, we have $g_\delta(x) = y$, hence $y \in g_\delta(R_\delta)$, as claimed.

Note that for any $y \in \overline{Q}_d$, the length of $\mathcal{T} = \mathcal{T}((\mathcal{F}_d)_y)$ cannot be greater than d. If it was, then from the quotient $V \otimes \mathcal{O}_{\mathbb{P}^1_y} \to (\mathcal{F}_d)_y/\mathcal{T}$ we would get an invertible sheaf on $\mathbb{P}^1$ with negative degree and a non-zero section. Thus, every closed point $y \in \overline{Q}_d$ lies in some $g_\delta(R_\delta)$.

By Chevalley's Theorem, each $g_\delta(R_\delta)$ is a constructible subset of $\overline{Q}_d$. Since $\Bbbk$ is algebraically closed and $\overline{Q}_d$ is of finite type over $\Bbbk$, the constructible subsets of $\overline{Q}_d$ are uniquely determined by their intersection with $\overline{Q}_d(\Bbbk)$. Therefore, from the above discussion we deduce that $\overline{Q}_d = \bigcup_{0\le\delta\le d} g_\delta(R_\delta)$ and that $g_\delta(R_\delta) \cap g_\varepsilon(R_\varepsilon) = \emptyset$ whenever $\delta \neq \varepsilon$. We thus get a set-theoretic decomposition $\overline{Q}_d = \bigsqcup_{0\le\delta\le d} g_\delta(R_\delta)$. $\square$

For every $0 \le \delta \le d$, define

$$U_\delta := \overline{Q}_d \setminus \bigsqcup_{\varepsilon<\delta} g_\varepsilon(R_\varepsilon)\,.$$

In particular, $U_0 = \overline{Q}_d$. Let us recursively construct morphisms $j_\delta : R_\delta \to U_\delta$ as follows.

For $\delta = 0$, we have $R_0 = \overline{R}_0$. Indeed, if $x \in \overline{R}_0(\Bbbk)$ then the subsheaf $(\mathcal{G}_0)_x \subseteq \mathcal{T}((\mathcal{H}_0)_x)$ has maximal length d, hence it coincides with $\mathcal{T}((\mathcal{H}_0)_x)$. Therefore, $(\mathcal{F}_0)_{f_0(x)} \cong (\mathcal{H}_0)_x/(\mathcal{G}_0)_x$ is locally free on $\mathbb{P}^1$, i.e., $x \in R_0(\Bbbk)$. Let $j_0 := g_0 : R_0 \to U_0$. Since the morphisms g_δ are projective, the image $g_0(R_0)$ is closed in $\overline{Q}_d$. Thus, $U_1 = \overline{Q}_d \setminus g_0(R_0)$ is an open subvariety of $\overline{Q}_d$.

Now, for $\delta > 0$, assume that we have already defined $j_{\delta-1}$ and proved that U_δ is an open subvariety of $\overline{Q}_d$. Since $g_\delta(R_\delta) \subseteq U_\delta$, $g_\delta|_{R_\delta} : R_\delta \to \overline{Q}_d$ factors through a morphism $j_\delta : R_\delta \to U_\delta$.

Lemma 4.1.8. *The commutative diagram*

$$\begin{tikzcd} R_\delta \arrow[r, hook] \arrow[d, "j_\delta"] & \overline{R}_\delta \arrow[d, "g_\delta"] \\ U_\delta \arrow[r, hook] & \overline{Q}_d \end{tikzcd}$$

is a Cartesian diagram. As a consequence, $j_\delta : R_\delta \to U_\delta$ is projective.

Proof. Let

$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "v"] & \overline{R}_\delta \arrow[d, "g_\delta"] \\ U_\delta \arrow[r, hook] & \overline{Q}_d \end{tikzcd}$$

be a commutative diagram of $\Bbbk$-schemes of finite type. Let us consider the pullback of (4.2) by $u' = \mathrm{id}_{\mathbb{P}^1} \times u : \mathbb{P}^1 \times T \to \mathbb{P}^1 \times \overline{R}_\delta$. For any closed point $t \in T$, $\mathcal{T}((u'^*\mathcal{H}_\delta)_t)$ contains the subsheaf $(u'^*\mathcal{G}_\delta)_t$, which has length $d - \delta$. On the other hand, since $g_\delta \circ u$ factors through U_δ, the length of $\mathcal{T}((u'^*\mathcal{H}_\delta)_t)$ cannot be greater than $d-\delta$, thus it is exactly $d-\delta$. Therefore, $\mathcal{T}((u'^*\mathcal{H}_\delta)_t)$ coincides with its subsheaf $(u'^*\mathcal{G}_\delta)_t$. As a consequence, the sheaf $(u'^*(\mathcal{H}_\delta/\mathcal{G}_\delta))_t \cong (u'^*\mathcal{H}_\delta)_t/(u'^*\mathcal{G}_\delta)_t$ is locally free.

Now, the set $\{t \in T \mid (u'^*(\mathcal{H}_\delta/\mathcal{G}_\delta))_t$ is locally free$\}$ is an open subset of T, because $u'^*(\mathcal{H}_\delta/\mathcal{G}_\delta)$ is T-flat. Since it contains all closed points of T, it is actually equal to T. Therefore, for any $t \in T$, $((f'_\delta \circ u')^*\mathcal{F}_\delta)_t \cong (u'^*(\mathcal{H}_\delta/\mathcal{G}_\delta))_t$ is locally free. It follows that $u : T \to \overline{R}_\delta$ factors through a unique morphism $h : T \to R_\delta$. Since the immersion $U_\delta \hookrightarrow \overline{Q}_d$ is a monomorphism, we also have $j_\delta \circ h = v$. $\square$

By Lemma 4.1.8, the image $j_\delta(R_\delta)$ is closed in U_δ. Thus, the complement $U_\delta \setminus j_\delta(R_\delta)$ is an open subvariety of $\overline{Q}_d$, because $U_\delta \subseteq \overline{Q}_d$ is open. Note that we exactly have $U_{\delta+1} = U_\delta \setminus j_\delta(R_\delta)$. Thus, we can proceed recursively and define the morphisms $j_\delta : R_\delta \to U_\delta$ for all $0 \leq \delta \leq d$.

Lemma 4.1.9. *For every $0 \leq \delta \leq d$, $j_\delta : R_\delta \to U_\delta$ is a monomorphism.*

Proof. By [Gro67, Prop. 17.2.6], j_δ is a monomorphism if and only if it is radicial (equivalently, universally injective) and formally unramified. Since $j_\delta : R_\delta \to U_\delta$ is a morphism of finite type of algebraic $\Bbbk$-schemes, it suffices to prove that j_δ is injective on closed points and unramified.

Let $x_1, x_2 \in R_\delta$ be closed points such that $j_\delta(x_1) = j_\delta(x_2)$. For each i, pulling-back (4.2) to $\mathbb{P}^1_{x_i}$, we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\mathcal{K}_\delta)_{x_i} \arrow[r] \arrow[d, equal] & ((\mathcal{E}_\delta)_{R_\delta})_{x_i} \arrow[r] \arrow[d] & (\mathcal{G}_\delta)_{x_i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{K}_\delta)_{x_i} \arrow[r] \arrow[d] & V \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & (\mathcal{H}_\delta)_{x_i} \arrow[r] \arrow[d] & 0 \cdot \\
 & 0 \arrow[r] & ((\mathcal{F}_\delta)_{R_\delta})_{x_i} \arrow[r, "\cong"] \arrow[d] & (\mathcal{H}_\delta/\mathcal{G}_\delta)_{x_i} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $j_\delta(x_1) = j_\delta(x_2)$, there is an isomorphism $(\mathcal{H}_\delta)_{x_1} \cong (\mathcal{H}_\delta)_{x_2}$ which commutes with the quotients $V \otimes \mathcal{O}_{\mathbb{P}^1} \to (\mathcal{H}_\delta)_{x_i}$. Equivalently, we have $(\mathcal{K}_\delta)_{x_1} = (\mathcal{K}_\delta)_{x_2}$ as subsheaves of $\mathcal{O}_{\mathbb{P}^1}$. The sheaves $((\mathcal{F}_\delta)_{R_\delta})_{x_i}$ are locally free, because $f_\delta(x_i) \in Q_\delta$. Therefore, each $\mathcal{T}((\mathcal{H}_\delta)_{x_i})$ coincides with its subsheaf $(\mathcal{G}_\delta)_{x_i}$. As a consequence, we obtain that $((\mathcal{E}_\delta)_{R_\delta})_{x_1} = ((\mathcal{E}_\delta)_{R_\delta})_{x_2}$ as subsheaves of $V \otimes \mathcal{O}_{\mathbb{P}^1}$, because both of them are the saturation of the same subsheaf $(\mathcal{K}_\delta)_{x_i}$. Moreover, there is an isomorphism $(\mathcal{G}_\delta)_{x_1} \cong (\mathcal{G}_\delta)_{x_2}$ commuting with the quotients $((\mathcal{E}_\delta)_{R_\delta})_{x_i} \to (\mathcal{G}_\delta)_{x_i}$, i.e., $x_1 = x_2$. Thus, j_δ is injective on closed points.

Let us now prove that j_δ is unramified. Since $f_\delta|_{R_\delta} : R_\delta \to Q_\delta$ is smooth, there is a short exact sequence of locally free $\mathcal{O}_{R_\delta}$-modules

$$0 \to (f_\delta|_{R_\delta})^* \Omega_{Q_\delta/\Bbbk} \to \Omega_{R_\delta/\Bbbk} \to \Omega_{R_\delta/Q_\delta} \to 0\,. \tag{4.3}$$

Moreover, we have the exact sequence of $\mathcal{O}_{R_\delta}$-modules

$$g_\delta^*(\Omega_{U_\delta/\Bbbk}) \to \Omega_{R_\delta/\Bbbk} \to \Omega_{R_\delta/U_\delta} \to 0\,. \tag{4.4}$$

By [Leh98], for any closed point $x \in R_\delta$, we have

$$\begin{aligned}
((f_\delta|_{R_\delta})^* \Omega_{Q_\delta/\Bbbk}) \otimes \kappa(x) &\cong \operatorname{Hom}\big(((\mathcal{E}_\delta)_{R_\delta})_x, ((\mathcal{F}_\delta)_{R_\delta})_x\big)^\vee\,,\\
\Omega_{R_\delta/Q_\delta} \otimes \kappa(x) &\cong \operatorname{Hom}((\mathcal{K}_\delta)_x, (\mathcal{G}_\delta)_x)^\vee\,,\\
g_\delta^*(\Omega_{U_\delta/\Bbbk}) \otimes \kappa(x) &\cong \operatorname{Hom}((\mathcal{K}_\delta)_x, (\mathcal{H}_\delta)_x)^\vee\,.
\end{aligned}$$

The vector space $\mathrm{Hom}((\mathcal{G}_\delta)_x, (\mathcal{H}_\delta)/\mathcal{G}_\delta)_x)$ vanishes, because $(\mathcal{G}_\delta)_x$ is a torsion sheaf and $(\mathcal{H}_\delta)/\mathcal{G}_\delta)_x$ is torsion-free. Therefore, using the long exact Ext sequences associated to the pullback of (4.2) to $\mathbb{P}^1_x$, (4.3) implies that

$$\Omega_{R_\delta/\Bbbk} \otimes \kappa(x) \cong \mathrm{coker}(\alpha)^\vee ,$$

where $\alpha = \bullet \circ (\rho_\delta)_x : \mathrm{Hom}((\mathcal{G}_\delta)_x, (\mathcal{H}_\delta)_x) \to \mathrm{Hom}(((\mathcal{E}_\delta)_{R_\delta})_x, (\mathcal{H}_\delta)_x)$. Moreover, the pullback of the morphism $g_\delta^*(\Omega_{U_\delta/\Bbbk}) \to \Omega_{R_\delta/\Bbbk}$ of (4.4) to $\mathbb{P}^1_x$ is the dual of the canonical injection $\mathrm{coker}(\alpha) \hookrightarrow \mathrm{Hom}((\mathcal{K}_\delta)_x, (\mathcal{H}_\delta)_x)$, hence it is surjective. As a consequence, we obtain the vanishing of $\Omega_{R_\delta/U_\delta} \otimes \kappa(x)$. Since $\Omega_{R_\delta/U_\delta} \otimes \kappa(x) = 0$ for each closed point $x \in R_\delta$ and R_δ is of finite type over $\Bbbk$, we have $\Omega_{R_\delta/U_\delta} = 0$, i.e., j_δ is unramified. □

Theorem 4.1.10. *The morphism*

$$j := \bigsqcup_{0 \leq \delta \leq d} \left(R_\delta \xrightarrow{j_\delta} U_\delta \hookrightarrow \overline{Q}_d \right) : \bigsqcup_{0 \leq \delta \leq d} R_\delta \to \overline{Q}_d$$

is a locally closed decomposition of $\overline{Q}_d$.

Proof. By Proposition 4.1.7, since $j_\delta(R_\delta) = g_\delta(R_\delta)$, j is bijective on closed points. From Lemma 4.1.8 and Lemma 4.1.9 it follows that $j_\delta : R_\delta \to U_\delta$ is a proper monomorphism, hence a closed immersion by [Gro67, Cor. 18.12.6]. Therefore, each composition $R_\delta \xrightarrow{j_\delta} U_\delta \hookrightarrow \overline{Q}_d$ is a locally closed immersion and j is a locally closed decomposition of $\overline{Q}_d$. □

Using Remark 4.1.3, we immediately obtain the following result.

Corollary 4.1.11. *The equality*

$$[\overline{Q}_d] = \sum_{0 \leq \delta \leq d} [R_\delta]$$

holds in $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$.

4.2 The motive of R_δ

Let $0 \leq \delta \leq d$ be fixed, and let R_δ be the scheme defined in Section 4.1, which comes equipped with a structure morphism $f_\delta|_{R_\delta} : R_\delta \to Q_\delta$ over $Q_\delta \cong \mathrm{Mor}_\delta(\mathbb{P}^1, G(r, V))$. In this section, we describe $[R_\delta] \in \mathrm{K}_0(\mathsf{Var}_\Bbbk)$ in terms of the classes of certain locally closed subschemes $Q_\delta^{\mathbf{a}}$ of Q_δ.

Let $s := k - r$. By Grothendieck's theorem on vector bundles on $\mathbb{P}^1$, for every $y \in Q_\delta$ there is an isomorphism

$$(\mathcal{E}_\delta)_y \cong \bigoplus_{i=0}^{s} \mathcal{O}_{\mathbb{P}^1_y}(a_i(y)) ,$$

where $a_1(y) \leq \cdots \leq a_s(y)$ is a uniquely determined sequence of integers such that $a_1(y) + \cdots + a_s(y) = -\delta$. Note that each $a_i(y)$ is ≤ 0, because $(\mathcal{E}_\delta)_y$ is a subsheaf of $V \otimes \mathcal{O}_{\mathbb{P}^1_y}$.

Let $A_\delta = \{\mathbf{a} \in \mathbb{Z}^s \mid a_1 \leq \cdots \leq a_s \leq 0\,,\ |\mathbf{a}| = -\delta\}$. By [Sha77], for any $\mathbf{a} \in A_\delta$ the locus

$$Q_\delta^{\mathbf{a}} := \{y \in Q_\delta \mid (a_1(y), \ldots, a_s(y)) = \mathbf{a}\}$$

is locally closed in Q_δ, because $\mathcal{E}_\delta$ is flat over Q_δ. Let $R_\delta^{\mathbf{a}}$ be the preimage $f_\delta^{-1}(Q_\delta^{\mathbf{a}})$, with its reduced induced scheme structure.

The main theorem of this section is the following.

Theorem 4.2.1. *For every $\mathbf{a} \in A_\delta$, let*

$$\mathcal{E}_\delta^{\mathbf{a}} := \bigoplus_{i=0}^{s} \mathcal{O}_{\mathbb{P}^1}(a_i)\,.$$

Then the restriction $f_\delta|_{R_\delta^{\mathbf{a}}} : R_\delta^{\mathbf{a}} \to Q_\delta^{\mathbf{a}}$ is a locally trivial fibration in the Zariski topology, with fiber $\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk}$.

The proof of Theorem 4.2.1 essentially relies on the following result.

Proposition 4.2.2. *Let T be a smooth $\Bbbk$-scheme, and let $\mathcal{E}$ be a coherent locally free sheaf of rank s on $\mathbb{P}^1 \times T$, flat over T. Assume that for every $t \in T$, there is an isomorphism*

$$\mathcal{E}_t \cong \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^1_t}(a_i)\,,$$

where the integers $a_1 \leq \cdots \leq a_s$ are independent of t. Then, Zariski locally on T, $\mathcal{E}$ is isomorphic to the pullback of $\bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ by the projection $\mathbb{P}^1 \times T \to \mathbb{P}^1$.

Proof. Without loss of generality, we may assume that T is integral. Let $p : \mathbb{P}^1 \times T \to \mathbb{P}^1$ and $q : \mathbb{P}^1 \times T \to T$ be the two projections. If $\mathcal{H}$ is a sheaf on $\mathbb{P}^1 \times T$, we write $\mathcal{H}(a)$ for $\mathcal{H} \otimes p^*\mathcal{O}_{\mathbb{P}^1}(a)$. We will proceed by induction on s.

If $s = 1$, let us consider the coherent $\mathcal{O}_T$-module $\mathcal{F} = q_*(\mathcal{E}(-a_1))$. By the results on cohomology and base change, $\mathcal{F}$ is a locally free sheaf of rank $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1$, which commutes with any base change. The counit of the adjunction $q^* \dashv q_*$ gives an isomorphism of invertible sheaves $q^*\mathcal{F} \xrightarrow{\sim} \mathcal{E}(-a_1)$. Therefore, we can cover T with open subsets over which $q^*\mathcal{F}$ trivializes, so that we get an isomorphism $p^*\mathcal{O}_{\mathbb{P}^1}(a_1) \xrightarrow{\sim} \mathcal{E}$.

If $s > 1$, let $\mathcal{F} = q_*(\mathcal{E}(-a_s))$. By Grauert's theorem on base change, $\mathcal{F}$ is a locally free $\mathcal{O}_T$-module of rank $s' = \#\{i \mid a_i = a_s\}$. As above, the counit of the adjunction $q^* \dashv q_*$ gives a morphism of locally free sheaves

$q^*\mathcal{F} \to \mathcal{E}(-a_s)$. On each fiber $\mathbb{P}^1_t$ over $t \in T$, this morphism is the evaluation $\mathrm{H}^0(\mathbb{P}^1_t, \mathcal{E}_t(-a_s)) \otimes \mathcal{O}_{\mathbb{P}^1_t} \to \mathcal{E}_t(-a_s)$, which is an injection of vector bundles. Therefore, we have an injection of locally free sheaves $(q^*\mathcal{F})(a_s) \to \mathcal{E}$ with locally free quotient $\mathcal{Q}$:

$$0 \to (q^*\mathcal{F})(a_s) \to \mathcal{E} \to \mathcal{Q} \to 0\,. \tag{4.5}$$

On each fiber $\mathbb{P}^1_t$, we have

$$(q^*\mathcal{F})(a_s)_t \cong \bigoplus_{i|a_i=a_s} \mathcal{O}_{\mathbb{P}^1_t}(a_i)\,, \qquad \mathcal{Q}_t \cong \bigoplus_{i|a_i\neq a_s} \mathcal{O}_{\mathbb{P}^1_t}(a_i)\,.$$

By the induction hypothesis, since the rank of $\mathcal{Q}$ is strictly less than s, we can cover T with open subsets U, such that $\mathcal{Q}|_U$ is isomorphic to the pullback of $\bigoplus_{i|a_i\neq a_s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ by the projection $\mathbb{P}^1 \times U \to U$. By shrinking T if necessary, we may thus assume that this happens globally on T.

In order to prove that the sequence (4.5) splits (Zariski locally), consider

$$\mathrm{Ext}^1(\mathcal{Q}, (q^*\mathcal{F})(a_s)) \cong \mathrm{H}^1\big(\mathbb{P}^1 \times T, \mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s)\big)\,.$$

Let us apply the Leray spectral sequence $\mathrm{H}^i(T, \mathrm{R}^j q_*(\bullet)) \Rightarrow \mathrm{H}^{i+j}(\mathbb{P}^1 \times T, \bullet)$ to the sheaf $\mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s)$. Since $\dim(\mathbb{P}^1) = 1$, $\mathrm{R}^i q_*(\mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s)) = 0$ for every $i > 1$. For $i = 1$, using the projection formula, we obtain

$$\mathrm{R}^1 q_*(\mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s)) \cong \mathcal{F} \otimes \mathrm{R}^1 q_*(\mathcal{Q}^\vee(a_s))\,.$$

The sheaf $\mathcal{Q}^\vee(a_s)$ is the pullback of $\bigoplus_{j|a_j\neq a_s} \mathcal{O}_{\mathbb{P}^1}(a_s - a_j)$, whose H^1 vanishes because $a_s > a_j$, thus $\mathrm{R}^1 q_*(\mathcal{Q}^\vee(a_s)) = 0$. Finally, the last contribution to $\mathrm{H}^1(\mathbb{P}^1 \times T, \mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s))$ coming from the Leray spectral sequence is

$$\mathrm{H}^1\big(T, q_*(\mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s))\big) \cong \mathrm{H}^1(T, \mathcal{F} \otimes q_*(\mathcal{Q}^\vee(a_s))\,.$$

If U is an affine open subset of T, then $\mathrm{H}^1(U, \mathcal{F} \otimes q_*(\mathcal{Q}^\vee(a_s)) = 0$. It follows that

$$\mathrm{H}^1\big(\mathbb{P}^1 \times U, \mathcal{Q}^\vee \otimes (q^*\mathcal{F})(a_s)\big) = 0\,,$$

hence the sequence (4.5) splits over U. By considering an affine open cover $\{U_\lambda\}_\lambda$, such that $\mathcal{F}$ is trivial on each U_λ, we get the desired isomorphism. □

Proof of Theorem 4.2.1. By Proposition 4.2.2, there exists a Zariski open cover $\{U_\lambda\}_\lambda$ of $Q^{\mathbf{a}}_\delta$ such that $\mathcal{E}_\delta|_{\mathbb{P}^1\times U_\lambda} \cong p^*(\mathcal{E}^{\mathbf{a}}_\delta)$ for all λ, where the morphism $p : \mathbb{P}^1 \times U_\lambda \to \mathbb{P}^1$ is the projection. Therefore, by the base change property of the Quot scheme, the preimage $f_\delta^{-1}(U_\lambda) \subseteq R^{\mathbf{a}}_\delta$ is naturally isomorphic to

$$\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta/\mathbb{P}^1\times Q^{\mathbf{a}}_\delta/Q^{\mathbf{a}}_\delta} \times_{Q^{\mathbf{a}}_\delta} U_\lambda \cong \mathrm{Quot}^{d-\delta}_{p^*(\mathcal{E}^{\mathbf{a}}_\delta)/\mathbb{P}^1\times U_\lambda/U_\lambda} \cong \mathrm{Quot}^{d-\delta}_{\mathcal{E}^{\mathbf{a}}_\delta/\mathbb{P}^1/\mathbb{k}} \times U_\lambda\,.$$

Thus $f_\delta|_{R^{\mathbf{a}}_\delta} : R^{\mathbf{a}}_\delta \to Q^{\mathbf{a}}_\delta$ is a locally trivial fibration. □

By means of Proposition 1.3.2, Theorem 4.2.1 has the following direct consequence.

Corollary 4.2.3. *For each $0 \leq \delta \leq d$ and each $\mathbf{a} \in A_\delta$, the equality*

$$[R_\delta] = \sum_{\mathbf{a} \in A_\delta} [Q_\delta^{\mathbf{a}}] \left[\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk} \right]$$

holds in $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$.

Proof. The subschemes $R_\delta^{\mathbf{a}}$, for $\mathbf{a} \in A_\delta$, give a locally closed decomposition of R_δ. Therefore, we have

$$[R_\delta] = \sum_{\mathbf{a} \in A_\delta} [R_\delta^{\mathbf{a}}] \, .$$

By Theorem 4.2.1, each $f_\delta|_{R_\delta^{\mathbf{a}}} : R_\delta^{\mathbf{a}} \to Q_\delta^{\mathbf{a}}$ is a locally trivial fibration in the Zariski topology, with fiber $\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk}$. Thus, from Proposition 1.3.2 it follows that

$$[R_\delta^{\mathbf{a}}] = [Q_\delta^{\mathbf{a}}] \left[\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk} \right] .$$

Combining the two equalities, we obtain the desired result. □

We are now faced with the problem of computing the class of $\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk}$ in $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$. In principle, this class could depend on $\mathbf{a}$. However, we will show in Chapter 5 that it only depends on the rank of the locally free sheaf $\mathcal{E}_\delta^{\mathbf{a}}$, hence it is independent of $\mathbf{a}$.

Actually, the same result holds also for the Quot scheme of 0-dimensional quotients of a locally free sheaf on an arbitrary smooth projective curve. Since the study of these Quot schemes would constitute a little digression from the main purpose of this chapter, we chose to postpone it until Chapter 5. Nevertheless, we will use some of those results in the subsequent sections.

In this context, the most important outcome of Chapter 5 is the following corollary of Proposition 5.3.5.

Proposition 4.2.4. *For any integer $0 \leq \delta \leq d$, the equality*

$$\left[\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk} \right] = \sum_{\mathbf{m} \in \mathbb{N}^s, \ |\mathbf{m}| = d-\delta} \frac{(1-\mathbf{L}^{m_1+1}) \cdots (1-\mathbf{L}^{m_s+1})}{(1-\mathbf{L})^s} \mathbf{L}^{d_{\mathbf{m}}} \tag{4.6}$$

holds in $\mathrm{K}_0(\mathsf{Var}_\Bbbk)$*, where* $d_{\mathbf{m}} = \sum_{i=1}^s (i-1)m_i$*. In particular, this class does not depend on* $\mathbf{a} \in A_\delta$.

Proof. By Proposition 5.3.5, we have

$$\left[\mathrm{Quot}^{d-\delta}_{\mathcal{E}_\delta^{\mathbf{a}}/\mathbb{P}^1/\Bbbk} \right] = \sum_{\mathbf{m} \in \mathbb{N}^s, \ |\mathbf{m}| = d-\delta} [\mathbb{P}^{m_1}] \cdots [\mathbb{P}^{m_s}] \, \mathbf{L}^{d_{\mathbf{m}}} \, .$$

Since for each m_i there is a locally closed decomposition $\bigsqcup_{j=0}^{m_i} \mathbb{A}^j \to \mathbb{P}^{m_i}$, we also have

$$[\mathbb{P}^{m_i}] = \sum_{j=0}^{m_i} \mathbf{L}^j = \frac{1-\mathbf{L}^{m_i+1}}{1-\mathbf{L}} . \tag{4.7}$$

The combination of the two equalities yields the result. $\square$

The class appearing in (4.6), which depends only on $s = k - r$ and $d - \delta$, will be denoted by $\mathbf{R}^{\mathrm{fbr}}_{d-\delta}$.

Remark 4.2.5. In (4.6), we are not claiming that $1 - \mathbf{L}$ is invertible in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$. The right hand side of (4.6) must be understood as a formal expression, as in (4.7).

4.3 The class of $\mathrm{Mor}_d(\mathbb{P}^1, G(r, V))$ in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$

Now, we have all the ingredients to find a method for computing the motive of $\mathrm{Mor}_d(\mathbb{P}^1, G(r, V)$. This is exactly the content of this section.

Let $\mathbf{Q}_d := [Q_d] = [\mathrm{Mor}_d(\mathbb{P}^1, G(r, V))] \in \mathrm{K}_0(\mathsf{Var}_{\Bbbk})$. Using the results of the previous sections, we obtain the following equality.

Theorem 4.3.1. *The equality*

$$\sum_{d=0}^{\infty} \mathbf{Q}_d q^d = \left(\sum_{d=0}^{\infty} [\overline{Q}_d] q^d\right) \left(\sum_{d=0}^{\infty} \mathbf{R}^{\mathrm{fbr}}_d q^d\right)^{-1} \tag{4.8}$$

holds in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})[\![q]\!]$.

Proof. Applying Corollary 4.2.3 and Proposition 4.2.4, we get the equality

$$[R_\delta] = \sum_{\mathbf{a} \in A_\delta} [Q^{\mathbf{a}}_\delta] \left[\mathrm{Quot}^{d-\delta}_{\mathcal{E}^{\mathbf{a}}_\delta/\mathbb{P}^1/\Bbbk}\right] = \mathbf{R}^{\mathrm{fbr}}_{d-\delta} \left(\sum_{\mathbf{a} \in A_\delta} [Q^{\mathbf{a}}_\delta]\right) = \mathbf{R}^{\mathrm{fbr}}_{d-\delta} [Q_\delta] .$$

in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$. By Corollary 4.1.11, we thus have

$$[\overline{Q}_d] = \sum_{\delta=0}^{d} [R_\delta] = \sum_{\delta=0}^{d} \mathbf{R}^{\mathrm{fbr}}_{d-\delta} [Q_\delta] ,$$

therefore

$$\sum_{d=0}^{\infty} [\overline{Q}_d] q^d = \left(\sum_{d=0}^{\infty} \mathbf{Q}_d q^d\right) \left(\sum_{d=0}^{\infty} \mathbf{R}^{\mathrm{fbr}}_d q^d\right) .$$

Since $\mathbf{R}^{\mathrm{fbr}}_0 = 1$ the last power series is invertible, thus we obtain the claimed equality. $\square$

Theorem 4.3.1 provides a recursive formula for the class $\mathbf{Q}_d$ in terms of the classes $[\overline{Q}_\delta]$ and $\mathbf{R}^{\mathrm{fbr}}_\delta$ for $\delta \le d$:

$$\mathbf{T}_0 := 1\,, \quad \mathbf{T}_j := -\sum_{i=0}^{j-1} \mathbf{T}_i \mathbf{R}^{\mathrm{fbr}}_{j-i}\,, \quad \mathbf{Q}_d = \sum_{j=0}^{d} \mathbf{T}_j [\overline{Q}_{d-j}]\,. \tag{4.9}$$

We have already calculated $\mathbf{R}^{\mathrm{fbr}}_\delta$ in Proposition 4.2.4. Thus, in order for (4.9) to be effective for computing $\mathbf{Q}_d$, we only need to calculate $[\overline{Q}_\delta]$.

From [Str87], we know that $\overline{Q}_\delta$ has a Białynicki-Birula decomposition. In particular, this is a locally closed decomposition into subvarieties, each one of which is isomorphic to some affine space $\mathbb{A}^i$. For every fixed i, let $m_{\delta,i}$ be the number of i-dimensional cells of this decomposition. Then we have

$$[\overline{Q}_\delta] = \sum_{i=0}^{\dim(\overline{Q}_\delta)} m_{\delta,i} [\mathbb{A}^i] = \sum_{i=0}^{k\delta + r(k-r)} m_{\delta,i} \mathbf{L}^i\,. \tag{4.10}$$

Strømme also computed the numbers $m_{\delta,i}$. Let $F \subseteq \mathbb{Z}^s \times \mathbb{Z}^{s+1} \times \mathbb{Z}^s$ be the subset whose elements are triples ($\mathbf{a} = (a_1, \dots, a_s)$, $\mathbf{b} = (b_0, \dots, b_s)$, $\mathbf{c} = (c_1, \dots, c_s)$) such that

$$b_0 = 0 \le a_1 \le b_1 \le a_2 \le \cdots \le b_{s-1} \le a_s \le \delta = b_s\,,$$
$$0 \le c_1 \le \cdots \le c_s \le r\,.$$

Then the following result holds.

Theorem 4.3.2 ([Str87, Thm. 5.4])**.** *For all $0 \le i \le k\delta + r(k-r)$, $m_{\delta,i}$ is equal to the number of elements $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in F$ such that the equality*

$$\sum_{j=1}^{s} (a_j + c_j(1 + b_j - b_{j-1})) = i$$

holds.

Using this theorem, it is possible to calculate all the coefficients $m_{\delta,i}$ in (4.10). Therefore, (4.10) becomes an explicit formula for the computation of $[\overline{Q}_\delta] \in \mathrm{K}_0(\mathsf{Var}_{\Bbbk})$. In particular, this completely solves the problem of calculating $\mathbf{Q}_d$: after computing $[\overline{Q}_\delta]$ for all $\delta \le d$ via (4.10), one can use the recursive equalities (4.9) and Proposition 4.2.4 to determine $\mathbf{Q}_d$.

4.4 Examples

Let $\Bbbk = \mathbb{C}$. Following the method explained in Section 4.3, it is possible to compute $\mathbf{Q}_d = [\mathrm{Mor}_d(\mathbb{P}^1, G(r, V))] \in \mathrm{K}_0(\mathsf{Var})$ and thus $\mathfrak{e}(\mathbf{Q}_d) \in \mathrm{K}_0(\mathsf{MHS})$. We can then insert the result in (3.8) and (3.9) to obtain $\mathfrak{e}(M(\bullet, d))$. Finally, the knowledge of $\mathfrak{e}(M_{0,m}(G(r,V),\delta))$ for all $m \leq \max\{n,d\}$ and $\delta \leq d$ allows us to determine $\mathfrak{e}(\overline{M}_{0,n}(G(r,V),d))$, via the recursive formulas of Corollary 3.2.7. In this way, we have obtained a general algorithm for computing the Serre characteristic (and hence the Hodge numbers) of $\overline{M}_{0,n}(G(r,V),d)$ for any $n, d \in \mathbb{N}$ and any Grassmannian $G(r,V)$.

Remark 4.4.1. As a consequence of our procedure, the cohomology group $\mathrm{H}^i(\overline{M}_{0,n}(G(r,V),d))$ vanishes for i odd, whereas its class in $\mathrm{K}_0(\mathsf{MHS})$ is a multiple of $\mathsf{L}^{i/2}$ for i even. This agrees with the results of [Opr06].

In this section, we examine some examples, in order to show how the algorithm effectively works in practice.

4.4.1 The Serre characteristic of $\overline{M}_{0,0}(G(2,4),2)$

We consider the Grassmannian $G(2,4)$, for which $(k,r,s) = (4,2,2)$. Recall that

$$[G(2,4)] = (\mathbf{L}^2+1)(\mathbf{L}^2+\mathbf{L}+1) \in \mathrm{K}_0(\mathsf{Var})\,.$$

First, let us compute $\mathbf{R}_i^{\mathrm{fbr}}$ for all $0 \leq i \leq 2$. For any $\mathbf{m} = (m_1, m_2) \in \mathbb{N}^2$, we have $d_{\mathbf{m}} = m_2$. Therefore, from (4.6) we obtain

$$\begin{aligned}
\mathbf{R}_0^{\mathrm{fbr}} &= 1\,,\\
\mathbf{R}_1^{\mathrm{fbr}} &= \frac{(\mathbf{L}^2-1)(\mathbf{L}-1)}{(\mathbf{L}-1)^2} + \frac{(\mathbf{L}-1)(\mathbf{L}^2-1)}{(\mathbf{L}-1)^2}\mathbf{L} = (\mathbf{L}+1)^2\,,\\
\mathbf{R}_2^{\mathrm{fbr}} &= \frac{(\mathbf{L}^3-1)(\mathbf{L}-1)}{(\mathbf{L}-1)^2} + \frac{(\mathbf{L}^2-1)^2}{(\mathbf{L}-1)^2}\mathbf{L} + \frac{(\mathbf{L}-1)(\mathbf{L}^3-1)}{(\mathbf{L}-1)^2}\mathbf{L}^2\\
&= \mathbf{L}^4 + 2\mathbf{L}^3 + 4\mathbf{L}^2 + 2\mathbf{L} + 1\,.
\end{aligned}$$

Now, we can compute $\mathbf{T}_i$ for all $0 \leq i \leq 2$:

$$\begin{aligned}
\mathbf{T}_0 &= 1\,,\\
\mathbf{T}_1 &= -\mathbf{R}_1^{\mathrm{fbr}} = -(\mathbf{L}+1)^2\,,\\
\mathbf{T}_2 &= -\mathbf{R}_2^{\mathrm{fbr}} - \mathbf{R}_1^{\mathrm{fbr}}\mathbf{T}_1 = -\mathbf{R}_2^{\mathrm{fbr}} + (\mathbf{R}_1^{\mathrm{fbr}})^2 = 2\mathbf{L}(\mathbf{L}^2+\mathbf{L}+1)\,.
\end{aligned}$$

Using Theorem 4.3.2, we can also calculate $[\overline{Q}_d]$ for any $0 \leq d \leq 2$. We have

$$\begin{aligned}
[\overline{Q}_0] &= (\mathbf{L}^2+1)(\mathbf{L}^2+\mathbf{L}+1)\,,\\
[\overline{Q}_1] &= (\mathbf{L}+1)^2(\mathbf{L}^2+1)(\mathbf{L}^2-\mathbf{L}+1)(\mathbf{L}^2+\mathbf{L}+1)\,,\\
[\overline{Q}_2] &= (\mathbf{L}^2+1)(\mathbf{L}^2+\mathbf{L}+1)(\mathbf{L}^4+1)(\mathbf{L}^4+\mathbf{L}^3+\mathbf{L}^2+\mathbf{L}+1)\,.
\end{aligned}$$

Combining the previous equalities, we get

$$\begin{aligned}
\mathbf{Q}_0 &= \mathbf{T}_0[\overline{Q}_0] = (\mathbf{L}^2+1)(\mathbf{L}^2+\mathbf{L}+1)\,,\\
\mathbf{Q}_1 &= \mathbf{T}_0[\overline{Q}_1] + \mathbf{T}_1[\overline{Q}_0] = \mathbf{L}(\mathbf{L}-1)(\mathbf{L}+1)^2(\mathbf{L}^2+1)(\mathbf{L}^2+\mathbf{L}+1)\,,\\
\mathbf{Q}_2 &= \mathbf{T}_0[\overline{Q}_2] + \mathbf{T}_1[\overline{Q}_1] + \mathbf{T}_2[\overline{Q}_0]\\
&= \mathbf{L}^3(\mathbf{L}-1)(\mathbf{L}+1)(\mathbf{L}^2+1)(\mathbf{L}^2+\mathbf{L}+1)(\mathbf{L}^3+\mathbf{L}^2+\mathbf{L}-1)\,.
\end{aligned}$$

The knowledge of $\mathfrak{e}(\mathbf{Q}_d)$ allows us to determine $\mathfrak{e}(M_{0,0}(G(2,4),d))$ for all n, via the equalities (3.8) and (3.9). For notational simplicity, let us write $Y = G(2,4)$. For $n = 0$ we have $\overline{M}_{0,0}(Y,0) = \emptyset$ and

$$\mathfrak{e}(M_{0,0}(Y,d)) = \frac{\mathfrak{e}(\mathbf{Q}_d)}{\mathsf{L}^3 - \mathsf{L}}$$

for all $d > 0$, therefore

$$\begin{aligned}
\mathfrak{e}(M_{0,0}(Y,0)) &= \mathfrak{e}(\overline{M}_{0,0}(Y,0)) = 0\,, \\
\mathfrak{e}(M_{0,0}(Y,1)) &= \mathfrak{e}(\overline{M}_{0,0}(Y,1)) = (\mathsf{L}+1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)\,, \\
\mathfrak{e}(M_{0,0}(Y,2)) &= \mathsf{L}^2(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)(\mathsf{L}^3+\mathsf{L}^2+\mathsf{L}-1)\,.
\end{aligned}$$

For $n = 1$ we have $\overline{M}_{0,1}(Y,0) = \emptyset$ and

$$\mathfrak{e}(M_{0,1}(Y,d)) = \frac{\mathfrak{e}(\mathbf{Q}_d)}{\mathsf{L}^3 - \mathsf{L}}(\mathsf{L}+1)s_1 = \frac{\mathfrak{e}(\mathbf{Q}_d)}{\mathsf{L}(\mathsf{L}-1)}s_1$$

for all $d > 0$, because $\mathfrak{e}(\mathrm{F}(\mathbb{P}^1,1)) = (\mathsf{L}+1)s_1$ by Corollary 3.3.2. Thus

$$\begin{aligned}
\mathfrak{e}(M_{0,1}(Y,0)) &= \mathfrak{e}(\overline{M}_{0,1}(Y,0)) = 0\,, \\
\mathfrak{e}(M_{0,1}(Y,1)) &= \mathfrak{e}(\overline{M}_{0,1}(Y,1)) = (\mathsf{L}+1)^2(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_1\,, \\
\mathfrak{e}(M_{0,1}(Y,2)) &= \mathsf{L}^2(\mathsf{L}+1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)(\mathsf{L}^3+\mathsf{L}^2+\mathsf{L}-1)s_1\,.
\end{aligned}$$

For $n = 2$ we have $\overline{M}_{0,2}(Y,0) = \emptyset$ and

$$\mathfrak{e}(\overline{M}_{0,2}(Y,d)) = \frac{\mathfrak{e}(\mathbf{Q}_d)}{\mathsf{L}^3 - \mathsf{L}}\big((\mathsf{L}^2-\mathsf{L})s_2 + \mathsf{L}s_1^2\big) = \frac{\mathfrak{e}(\mathbf{Q}_d)}{\mathsf{L}^2 - 1}\big((\mathsf{L}-1)s_2 + s_1^2\big)$$

for all $d > 0$, because $\mathfrak{e}(\mathrm{F}(\mathbb{P}^1,2)) = (\mathsf{L}^2-\mathsf{L})s_2 + \mathsf{L}s_1^2$. Therefore

$$\begin{aligned}
\mathfrak{e}(M_{0,2}(Y,0)) &= \mathfrak{e}(\overline{M}_{0,2}(Y,0)) = 0\,, \\
\mathfrak{e}(M_{0,2}(Y,1)) &= \mathsf{L}(\mathsf{L}+1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)\big((\mathsf{L}-1)s_2+s_1^2\big)\,, \\
\mathfrak{e}(M_{0,2}(Y,2)) &= \mathsf{L}^3(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)(\mathsf{L}^3+\mathsf{L}^2+\mathsf{L}-1)\big((\mathsf{L}-1)s_2+s_1^2\big)\,.
\end{aligned}$$

We can also compute $\mathfrak{e}(\Phi_{0,n}(Y,d))$, via Corollary 3.2.1. For $n = 0$ we have

$$\mathfrak{e}(\Phi_{0,0}(Y,d)) = \frac{D(\mathfrak{e}(M_{0,1}(Y,d)))}{\mathfrak{e}(Y)} = \frac{\mathfrak{e}(\mathbf{Q}_d)}{\mathsf{L}(\mathsf{L}-1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)}\,,$$

hence

$$\begin{aligned}
\mathfrak{e}(\Phi_{0,0}(Y,0)) &= \mathfrak{e}(\overline{\Phi}_{0,0}(Y,0)) = 0\,, \\
\mathfrak{e}(\Phi_{0,0}(Y,1)) &= \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1)) = (\mathsf{L}+1)^2\,, \\
\mathfrak{e}(\Phi_{0,0}(Y,2)) &= \mathsf{L}^2(\mathsf{L}+1)(\mathsf{L}^3+\mathsf{L}^2+\mathsf{L}-1)\,.
\end{aligned}$$

For $n = 1$ we have

$$\begin{aligned}\mathfrak{e}(\Phi_{0,1}(Y,d)) = \frac{D(\mathfrak{e}(M_{0,2}(Y,d)))}{\mathfrak{e}(Y)} &= \frac{\mathfrak{e}(\mathbf{Q}_d)\, D\big((\mathsf{L}-1)s_2 + s_1^2\big)}{(\mathsf{L}^2-1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)} \\ &= \frac{\mathfrak{e}(\mathbf{Q}_d)}{(\mathsf{L}-1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)} s_1 \,,\end{aligned}$$

hence

$$\begin{aligned}\mathfrak{e}(\Phi_{0,1}(Y,0)) &= \mathfrak{e}(\overline{\Phi}_{0,1}(Y,0)) = 0 \,, \\ \mathfrak{e}(\Phi_{0,1}(Y,1)) &= \mathsf{L}(\mathsf{L}+1)^2 s_1 \,, \\ \mathfrak{e}(\Phi_{0,1}(Y,2)) &= \mathsf{L}^3(\mathsf{L}+1)(\mathsf{L}^3+\mathsf{L}^2+\mathsf{L}-1)s_1 \,.\end{aligned}$$

Now, recall that we have the equalities $s_1 = p_1$ and $2s_2 = p_2 + p_1^2$, and that $p_n \circ \bullet$ is an algebra homomorphism, for any n. Then Corollary 3.2.7 implies that

$$\begin{aligned}\mathfrak{e}(\overline{M}_{0,0}(Y,2)) = \mathfrak{e}(M_{0,0}(Y,2)) &+ \frac{\mathfrak{e}(M_{0,1}(Y,1))}{s_1}\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1)) \\ &+ \mathfrak{e}(Y)\frac{p_2 \circ \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1)) - \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))^2}{2} \,. \qquad (4.11)\end{aligned}$$

By the previous computations, we have

$$\frac{\mathfrak{e}(M_{0,1}(Y,1))}{s_1}\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1)) = (\mathsf{L}+1)^4(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1) \,.$$

Moreover, we have

$$\frac{p_2 \circ \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1)) - \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))^2}{2} = \frac{(\mathsf{L}^2+1)^2 - (\mathsf{L}+1)^4}{2} = -2\mathsf{L}(\mathsf{L}^2+\mathsf{L}+1) \,,$$

therefore

$$\mathfrak{e}(Y)\frac{p_2 \circ \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1)) - \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))^2}{2} = -2\mathsf{L}(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)^2 \,.$$

Finally, the sum of all contributions in (4.11) yields the Serre characteristic of $\overline{M}_{0,0}(Y,2)$:

$$\mathfrak{e}(\overline{M}_{0,0}(Y,2)) = \mathsf{L}^9 + 3\mathsf{L}^8 + 7\mathsf{L}^7 + 11\mathsf{L}^6 + 14\mathsf{L}^5 + 14\mathsf{L}^4 + 11\mathsf{L}^3 + 7\mathsf{L}^2 + 3\mathsf{L} + 1 \,.$$

In particular, the E-polynomial of $\overline{M}_{0,0}(Y,2)$ is

$$t^9u^9 + 3t^8u^8 + 7t^7u^7 + 11t^6u^6 + 14t^5u^5 + 14t^4u^4 + 11t^3u^3 + 7t^2u^2 + 3tu + 1$$

in agreement with [Ló14, Thm. 3.1].

4.4.2 The Serre characteristic of $\overline{M}_{0,1}(G(2,4),2)$

As above, we write $Y = G(2,4)$. In addition to the previous computations, we first need to determine $\mathfrak{e}(M_{0,3}(Y,0))$ and $\mathfrak{e}(\overline{\Phi}_{0,1}(Y,1))$.

Since $\mathfrak{e}(\mathrm{F}(\mathbb{P}^1,3)) = (\mathsf{L}^3 - \mathsf{L})s_3$, by (3.9) we have

$$\mathfrak{e}(M_{0,3}(Y,0)) = \frac{\mathfrak{e}(\mathbf{Q}_0)}{\mathsf{L}^3 - \mathsf{L}}(\mathsf{L}^3 - \mathsf{L})s_3 = (\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_3\,.$$

As a consequence, we also have

$$\mathfrak{e}(\Phi_{0,2}(Y,0)) = \frac{D(\mathfrak{e}(M_{0,3}(Y,0)))}{\mathfrak{e}(Y)} = s_2\,.$$

Thus, from Corollary 3.2.7 it follows that

$$\begin{aligned}\mathfrak{e}(\overline{\Phi}_{0,1}(Y,1)) &= \mathfrak{e}(\Phi_{0,1}(Y,1)) + \frac{\mathfrak{e}(\Phi_{0,2}(Y,0))}{s_2}\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))s_1 \\ &= (\mathsf{L}+1)^3 s_1\,.\end{aligned}$$

We are now able to compute $\mathfrak{e}(\overline{M}_{0,1}(Y,2))$. Indeed, via the equalities $s_1 = p_1$, $2s_2 = p_2 + p_1^2$ and $6s_3 = 2p_3 + 3p_1p_2 + p_1^3$, Corollary 3.2.7 implies that $\mathfrak{e}(\overline{M}_{0,1}(Y,2))$ is the sum of the following terms:

$$\begin{aligned}
\mathfrak{e}(M_{0,1}(Y,2)) &= \mathsf{L}^2(\mathsf{L}+1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)(\mathsf{L}^3+\mathsf{L}^2+\mathsf{L}-1)s_1\,; \\
\frac{\mathfrak{e}(M_{0,1}(Y,1))}{s_1}\mathfrak{e}(\overline{\Phi}_{0,1}(Y,1)) &= (\mathsf{L}+1)^5(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_1\,; \\
\{\mathfrak{e}(\overline{M}_{0,2}(Y,1))\}_{s_2}\,\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))s_1 &= \mathsf{L}(\mathsf{L}+1)^3(\mathsf{L}-1)(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_1\,; \\
2\{\mathfrak{e}(\overline{M}_{0,2}(Y,1))\}_{s_1^2}\,\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))s_1 &= 2\mathsf{L}(\mathsf{L}+1)^3(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_1\,; \\
\frac{\mathfrak{e}(M_{0,3}(Y,0))}{2s_3}\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))^2 s_1 &= \frac{1}{2}(\mathsf{L}+1)^4(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_1\,; \\
\frac{\mathfrak{e}(M_{0,3}(Y,0))}{2s_3}\big(p_2 \circ \mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))\big)s_1 &= \frac{1}{2}(\mathsf{L}^2+1)^3(\mathsf{L}^2+\mathsf{L}+1)s_1\,; \\
-\mathfrak{e}(Y)\,\mathfrak{e}(\overline{\Phi}_{0,0}(Y,1))\,\mathfrak{e}(\overline{\Phi}_{0,1}(Y,1)) &= -(\mathsf{L}+1)^5(\mathsf{L}^2+1)(\mathsf{L}^2+\mathsf{L}+1)s_1\,.
\end{aligned}$$

Here, $\{\mathfrak{e}(\overline{M}_{0,2}(Y,1))\}_{s_2}$ (resp. $\{\mathfrak{e}(\overline{M}_{0,2}(Y,1))\}_{s_1^2}$) denotes the coefficient of s_2 (resp. s_1^2) in $\mathfrak{e}(\overline{M}_{0,2}(Y,1))$. Summing all the terms, we obtain that the Serre characteristic of $\overline{M}_{0,1}(Y,2)$ is equal to

$$(\mathsf{L}^{10} + 4\mathsf{L}^9 + 12\mathsf{L}^8 + 22\mathsf{L}^7 + 33\mathsf{L}^6 + 36\mathsf{L}^5 + 33\mathsf{L}^4 + 22\mathsf{L}^3 + 12\mathsf{L}^2 + 4\mathsf{L} + 1)s_1\,.$$

In particular, its E-polynomial is

$$\begin{aligned}t^{10}u^{10} + 4t^9u^9 + 12t^8u^8 + 22t^7u^7 + 33t^6u^6 + 36t^5u^5 + 33t^4u^4 + 22t^3u^3 \\ + 12t^2u^2 + 4tu + 1\,.\end{aligned}$$

Chapter 5

Motive of Quot schemes of zero-dimensional quotients on a curve

This chapter is based on a joint work with B. Fantechi and F. Perroni [BFP19].

Let C be a smooth projective curve over an algebraically closed ground field $\mathbb{k}$, and let $\mathcal{E}$ be a locally free sheaf of rank r on C. For any $n \in \mathbb{Z}_{>0}$, let $\mathrm{Quot}^n_{\mathcal{E}/C/\mathbb{k}}$ be the Quot scheme which parametrizes coherent quotients of $\mathcal{E}$ with finite support and n-dimensional space of global sections. This Quot scheme is a smooth projective $\mathbb{k}$-variety, and it follows easily from its definition that there is a natural isomorphism

$$\mathrm{Quot}^n_{\mathcal{E}/C/\mathbb{k}} \cong \mathrm{Quot}^n_{\mathcal{E}\otimes\mathcal{L}/C/\mathbb{k}}$$

for any invertible sheaf $\mathcal{L}$ on C. In particular, when $r = 1$, $\mathrm{Quot}^n_{\mathcal{E}/C/\mathbb{k}}$ is isomorphic to $\mathrm{Quot}^n_{\mathcal{O}/C/\mathbb{k}}$. In the case where $r > 1$, the isomorphism class of $\mathrm{Quot}^n_{\mathcal{E}/C/\mathbb{k}}$ depends on $\mathcal{E}$, as one already sees when $n = 1$, in which case

$$\mathrm{Quot}^1_{\mathcal{E}/C/\mathbb{k}} \cong \mathbb{P}(\mathcal{E}),$$

the projective space bundle associated to $\mathcal{E}$ (see Section 5.1.3).

The aims of this chapter are to study the class of $\mathrm{Quot}^n_{\mathcal{E}/C/\mathbb{k}}$ (where $r \geq 2$) in the Grothendieck ring of $\mathbb{k}$-varieties and to compute it in terms of the classes $[\mathrm{Sym}^m(C)]$, for $m \geq 0$. In particular, we prove the following result.

Main Theorem. *Under the previous hypotheses, the equality*

$$\left[\mathrm{Quot}^n_{\mathcal{E}/C/\mathbb{k}}\right] = \left[\mathrm{Quot}^n_{\mathcal{O}^{\oplus r}/C/\mathbb{k}}\right]$$

holds true in the Grothendieck ring $\mathrm{K}_0(\mathsf{Var}_{\mathbb{k}})$ *of* $\mathbb{k}$*-varieties.*

Organization of the chapter. In Section 5.1, we present some well-known facts about Quot schemes on smooth projective curves. In particular, in §5.1.1 we recall the existence of a natural morphism $\sigma : \mathrm{Quot}^n_{\mathcal{E}/C/\Bbbk} \to \mathrm{Sym}^n(C)$, which we describe explicitly in §5.1.2. Finally, in §5.1.3 we consider $\mathrm{Quot}^1_{\mathcal{E}/C/\Bbbk}$, and we show that, in general, it is not isomorphic to $\mathrm{Quot}^1_{\mathcal{O}^{\oplus r}/C/\Bbbk}$.

A more detailed study of the morphism σ is the subject of Section 5.2, where we show that the fibers of σ only depend (up to isomorphism) on the rank of the locally free sheaf $\mathcal{E}$.

Section 5.3 contains the proof of the main theorem of this chapter. As an application, we explicitly compute $[\mathrm{Quot}^n_{\mathcal{E}/C/\Bbbk}] \in \mathrm{K}_0(\mathsf{Var}_\Bbbk)$ and we prove that $\mathrm{Quot}^n_{\mathcal{E}/C/\Bbbk}$ is irreducible.

5.1 Notations and basic results

In this section, we recall some basic results that are relevant for us and we fix the notation. For the proofs and for more details we refer to [Gro61]. Throughout the chapter, we work over an algebraically closed ground field $\Bbbk$. All schemes and morphisms between them are assumed to be over $\Bbbk$.

Let X be a projective scheme. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X, let $P \in \mathbb{Q}[\zeta]$ be a polynomial with rational coefficients, and let $\mathcal{F}$ be a coherent sheaf on X. We denote by $\mathrm{Quot}^P_{\mathcal{F}/X/\Bbbk}$ the Quot scheme that parametrizes coherent quotients of $\mathcal{F}$ with Hilbert polynomial P.

Recall that $\mathrm{Quot}^P_{\mathcal{F}/X/\Bbbk}$ is a projective scheme, which represents the contravariant functor that associates to any locally noetherian scheme S the set of isomorphism classes of S-flat coherent quotients $\pi : \mathcal{F}_S \to \mathcal{H}$, such that the Hilbert polynomial of $\mathcal{H}_s$ is equal to P, for all $s \in S$. Here, $\mathcal{F}_S$ (respectively $\mathcal{H}_s$) is the pullback of $\mathcal{F}$ to $S \times X$ under the projection onto the second factor (respectively the pullback of $\mathcal{H}$ to X_s). In particular, the identity morphism of $\mathrm{Quot}^P_{X/k}(\mathcal{F})$ corresponds to the universal quotient $\rho : \mathcal{F}_{\mathrm{Quot}^P_{\mathcal{F}/X/\Bbbk}} \to \mathcal{Q}$.

A similar result holds true if X is replaced by a quasi-projective scheme U. In this case, one defines a functor as before, with the additional requirement that $\mathcal{H}$ has proper support over S. Then, this functor is representable by a quasi-projective scheme $\mathrm{Quot}^P_{\mathcal{F}/U/\Bbbk}$. The relation between the two constructions is given by the following result (see also [Nit05]).

Theorem 5.1.1. *Let X and $\mathcal{F}$ be as before, and let $U \subseteq X$ be an open subscheme. Then $\mathrm{Quot}^P_{\mathcal{F}|_U/U/\Bbbk}$ is naturally an open subscheme of $\mathrm{Quot}^P_{\mathcal{F}/X/\Bbbk}$.*

Now, let us consider the case where P is a constant polynomial equal to $n \in \mathbb{Z}_{>0}$. Then $\mathrm{Quot}^n_{\mathcal{F}/X/\Bbbk}$ parametrizes coherent quotients of $\mathcal{F}$ with finite support and such that the dimension of the space of sections is equal to n. Therefore, $\mathrm{Quot}^n_{\mathcal{F}/X/\Bbbk}$ is independent of $\mathcal{O}_X(1)$.

In this chapter, the main object of study is $\mathrm{Quot}^n_{\mathcal{E}/C/\Bbbk}$, where C is a smooth projective curve and $\mathcal{E}$ is locally free.

Notation. Throughout the chapter, $\mathrm{Quot}^n_{\mathcal{E}/C/\Bbbk}$ will be denoted by $Q^n_C(\mathcal{E})$. Accordingly, the corresponding universal quotient on $C \times Q^n_C(\mathcal{E})$ will be denoted by $\rho : \mathcal{E}_{Q^n_C(\mathcal{E})} \to \mathcal{Q}$. Whenever $U \subseteq C$ is an open subscheme, we will write $Q^n_U(\mathcal{E})$ for the Quot scheme $\mathrm{Quot}^n_{\mathcal{E}|_U/U/\Bbbk}$, and $\rho_0 : \mathcal{E}_{Q^n_U(\mathcal{E})} \to \mathcal{Q}_0$ for its universal quotient.

Let us first recall the following fact.

Lemma 5.1.2. *Let C be a smooth projective curve, and let $\mathcal{E}$ be a locally free coherent sheaf of rank r on C. Then $Q^n_C(\mathcal{E})$ is a smooth variety of dimension nr.*

Proof. Let $[\pi : \mathcal{E} \to \mathcal{H}]$ be a $\Bbbk$-rational point of $Q^n_C(\mathcal{E})$. Since the support of $\mathcal{H}$ is 0-dimensional, we have that

$$\mathrm{Ext}^1(\ker(\pi), \mathcal{H}) \cong \mathrm{H}^1(C, \ker(\pi)^\vee \otimes \mathcal{H}) = 0\,.$$

The smoothness now follows from [HL10, Prop. 2.2.8]. Moreover, the dimension coincides with that of the Zariski tangent space at the point $[\pi : \mathcal{E} \to \mathcal{H}]$, which is equal to $\dim_\Bbbk \mathrm{H}^0(C, \ker(\pi)^\vee \otimes \mathcal{H}) = nr$. $\square$

As a consequence of Theorem 5.1.1 and Lemma 5.1.2, $Q^n_U(\mathcal{E})$ is a smooth quasi-projective variety, for any $U \subseteq C$ open.

Remark 5.1.3. Under the above hypotheses, $Q^n_C(\mathcal{E})$ is also irreducible (see Corollary 5.3.7).

5.1.1 The morphism σ

In the proof of the Main Theorem we will use the morphism $\mathfrak{N}_{X/\Bbbk}$ defined in [Gro61, §6], which will be denoted σ in this article. The following result is a special case of Grothendieck's construction.

Proposition 5.1.4. *Let $\mathcal{F}$ be a coherent sheaf on C. Then there exists a canonical morphism $\sigma : Q^n_C(\mathcal{F}) \to \mathrm{Sym}^n(C)$ that maps any $\Bbbk$-rational point $x = [\pi : \mathcal{F} \to \mathcal{H}]$ to the effective divisor*

$$\mathrm{div}(\mathcal{H}) := \sum_{x \in C} \dim_\Bbbk(\mathcal{H}_x)\, x\,,$$

where $\mathcal{H}_x$ is the stalk of $\mathcal{H}$ at x.

Remark 5.1.5. If $U \subseteq C$ is an open subset, then $\sigma^{-1}(\mathrm{Sym}^n(U))$ can be naturally identified with $Q^n_U(\mathcal{E})$. Hereafter, the morphism induced by σ will be denoted by $\sigma_U \colon Q^n_U(\mathcal{E}) \to \mathrm{Sym}^n(U)$.

When $\mathcal{E} = \mathcal{O}_C$, $Q^n_U(\mathcal{O}_C) = \mathrm{Hilb}^n_{U/\Bbbk}$ and this morphism is exactly the Hilbert-Chow morphism $\mathrm{HC}_U : \mathrm{Hilb}^n_{U/\Bbbk} \to \mathrm{Sym}^n(U)$ of [FG05]. Notice that HC_U is an isomorphism. Therefore, for any $\mathcal{E}$, σ_U factors through HC_U via the morphism $h_U := \mathrm{HC}_U^{-1} \circ \sigma_U : Q^n_U(\mathcal{E}) \to \mathrm{Hilb}^n_{U/\Bbbk}$.

5.1.2 Explicit construction of h_U

For later use, we provide here an explicit construction of h_U.

Let us consider the universal quotient $\rho_0 : \mathcal{E}_{Q^n_U(\mathcal{E})} \to \mathcal{Q}_0$ associated to $Q^n_U(\mathcal{E})$. If $\mathcal{K} = \ker(\rho_0)$ and $\iota : \mathcal{K} \to \mathcal{E}_{Q^n_U(\mathcal{E})}$ is the inclusion, then we have the short exact sequence

$$0 \to \mathcal{K} \xrightarrow{\iota} \mathcal{E}_{Q^n_U(\mathcal{E})} \xrightarrow{\rho_0} \mathcal{Q}_0 \to 0\,.$$

Since both $\mathcal{E}_{Q^n_U(\mathcal{E})}$ and $\mathcal{Q}_0$ are flat over $Q^n_U(\mathcal{E})$, $\mathcal{K}$ is flat over $Q^n_U(\mathcal{E})$, too. Moreover, the restriction of $\mathcal{K}$ to U_x is locally free, for all $x \in Q^n_U(\mathcal{E})$. It follows that $\mathcal{K}$ is a locally free sheaf of rank $r = \mathrm{rk}(\mathcal{E})$.

Let $\wedge^r(\iota) : \wedge^r(\mathcal{K}) \to \wedge^r(\mathcal{E}_{Q^n_U(\mathcal{E})})$ be the r-th exterior power of ι. By tensoring it with $\wedge^r(\mathcal{E}_{Q^n_U(\mathcal{E})})^\vee$, we get a short exact sequence

$$0 \to \wedge^r(\mathcal{K}) \otimes \wedge^r(\mathcal{E}_{Q^n_U(\mathcal{E})})^\vee \to \mathcal{O}_{Q^n_U(\mathcal{E})\times U} \to \mathcal{G} \to 0\,.$$

Notice that $\mathcal{G}$ is flat over $Q^n_U(\mathcal{E})$, since $\mathcal{O}_{Q^n_U(\mathcal{E})\times U}$ is $Q^n_U(\mathcal{E})$-flat and $\wedge^r(\iota)$ remains injective when restricted to every fiber (see [Mat80, Thm. 49 and its corollaries]). Moreover, the Hilbert polynomial of the restriction of $\mathcal{G}$ to every fiber is equal to n; indeed, the elementary divisor theorem for PIDs implies that the restriction of $\mathcal{G}$ to every fiber is isomorphic to the restriction of $\mathcal{Q}_0$ to the same fiber. Therefore, the quotient $\mathcal{O}_{Q^n_U(\mathcal{E})\times U} \to \mathcal{G}$ corresponds to a morphism $Q^n_U(\mathcal{E}) \to \mathrm{Hilb}^n_{U/k}$, which is exactly the morphism h_U defined in Section 5.1.1.

5.1.3 The case $n = 1$

The following result should be well known, but we include it here for lack of a suitable reference.

Proposition 5.1.6. *The Quot scheme $Q^1_C(\mathcal{E})$ is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{E})$.*

Proof. In order to simplify the notation, let us denote $Q^1_C(\mathcal{E})$ by Q, and the universal quotient over $Q \times C$ by $\rho : \mathcal{E}_Q \to \mathcal{Q}$.

Let $\sigma : Q \to C$ be the morphism introduced in Section 5.1.1 and let $f = (\mathrm{id}_Q, \sigma) : Q \to Q \times C$ be the morphism with components the identity of Q and σ, respectively. Then the pullback of ρ via f gives a quotient $f^*\rho : f^*\mathcal{E}_Q \to f^*\mathcal{Q}$. Notice that $f^*\mathcal{Q} \otimes \kappa(x) \cong \kappa(x)$ for any $x \in Q$ (where

$\kappa(x)$ is the residue field of x), therefore $f^*\mathcal{Q}$ is locally free of rank 1. Since $f^*\mathcal{E}_Q = \sigma^*\mathcal{E}$, by [Har77, Prop. 7.12] we obtain a morphism $g : Q \to \mathbb{P}(\mathcal{E})$.

On the other hand, let $p : \mathbb{P}(\mathcal{E}) \to C$ be the projection associated to $\mathcal{E}$, and let $p^*\mathcal{E} \to \mathcal{L}$ be the universal quotient over $\mathbb{P}(\mathcal{E})$. Let us consider the subscheme $\tilde{\Delta} \subset \mathbb{P}(\mathcal{E}) \times C$ whose structure sheaf is $(p \times \mathrm{id}_C)^*\mathcal{O}_\Delta$, where $p \times \mathrm{id}_C : \mathbb{P}(\mathcal{E}) \times C \to C \times C$ is the morphism with components p and the identity of C respectively, and $\Delta \subset C \times C$ is the diagonal. Taking the tensor product of the quotient $\mathcal{O}_{\mathbb{P}(\mathcal{E})\times C} \to \mathcal{O}_{\tilde{\Delta}}$ with $\mathcal{E}_{\mathbb{P}(\mathcal{E})}$, we obtain a surjection $\pi : \mathcal{E}_{\mathbb{P}(\mathcal{E})} \to \mathcal{E}_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\tilde{\Delta}}$.

If $\mathrm{pr}_1 : \mathbb{P}(\mathcal{E}) \times C \to \mathbb{P}(\mathcal{E})$ is the projection, then there is an isomorphism $\mathcal{E}_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\tilde{\Delta}} \cong (\mathrm{pr}_1)^*p^*\mathcal{E} \otimes \mathcal{O}_{\tilde{\Delta}}$. Therefore, we can compose π with the morphism $(\mathrm{pr}_1)^*p^*\mathcal{E} \otimes \mathcal{O}_{\tilde{\Delta}} \to (\mathrm{pr}_1)^*\mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}$, and we get a surjective morphism $\mathcal{E}_{\mathbb{P}(\mathcal{E})} \to (\mathrm{pr}_1)^*\mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}$. Since $(\mathrm{pr}_1)^*\mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}$ is flat over $\mathbb{P}(\mathcal{E})$ and has constant Hilbert polynomial 1, this surjection corresponds to a morphism $\mathbb{P}(\mathcal{E}) \to Q$, which is the inverse of g by construction. □

We conclude this section with the following result, from which we deduce that in general $Q^n_C(\mathcal{E})$ depends on $\mathcal{E}$, if $\mathrm{rk}(\mathcal{E}) \geq 2$.

Proposition 5.1.7. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two locally free coherent $\mathcal{O}_C$-modules of the same rank $r \geq 2$. Assume that at least one of the following conditions holds true:*

(i) the genus of C is greater than or equal to 1;

(ii) $r > 2$.

Then $Q^1_C(\mathcal{E}) \cong Q^1_C(\mathcal{E}')$ if and only if there exists an automorphism f of C and an invertible sheaf $\mathcal{L}$ on C, such that $\mathcal{E}' \cong f^*\mathcal{E} \otimes \mathcal{L}$.

Proof. First, assume that $Q^1_C(\mathcal{E}) \cong Q^1_C(\mathcal{E}')$. By Proposition 5.1.6, we thus have an isomorphism $g : \mathbb{P}(\mathcal{E}') \xrightarrow{\sim} \mathbb{P}(\mathcal{E})$. Let us denote the projections of these bundles by $p : \mathbb{P}(\mathcal{E}) \to C$ and $p' : \mathbb{P}(\mathcal{E}') \to C$. Under the hypothesis (i) or (ii), the morphism $p \circ g : \mathbb{P}(\mathcal{E}') \to C$ is constant on the fibers of p'. Therefore, there exists an automorphism f of C such that the diagram

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}') \arrow[r, "g"] \arrow[d, "p'"] & \mathbb{P}(\mathcal{E}) \arrow[d, "p"] \\ C \arrow[r, "f"] & C \end{tikzcd}$$

commutes. Since there is also a cartesian diagram

$$\begin{tikzcd} \mathbb{P}(f^*\mathcal{E}) \arrow[r, "\cong"] \arrow[d, "q"] & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\ C \arrow[r, "f"] & C \end{tikzcd} \tag{5.1}$$

where q is the canonical projection, it follows that $p' : \mathbb{P}(\mathcal{E}') \to C$ and $q : \mathbb{P}(f^*\mathcal{E}) \to C$ are isomorphic over C. By [Har77, §2.7, Exer. 7.9], we deduce that $\mathcal{E}' \cong f^*\mathcal{E} \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$ on C.

The inverse implication directly follows from [Har77, §2.7, Exer. 7.9] and the diagram (5.1). $\square$

5.2 The fibers of σ

In this section, we describe the fibers of the morphism σ introduced in Section 5.1.1. Throughout the section, C denotes a smooth projective curve over $\Bbbk$ and $\mathcal{E}$ is a coherent locally free $\mathcal{O}_C$-module of rank r.

Proposition 5.2.1. *The fiber of the morphism $\sigma : Q^n_C(\mathcal{E}) \to \mathrm{Sym}^n(C)$ over a point $D \in \mathrm{Sym}^n(C)$ is isomorphic to the fiber of the analogous morphism $\mathrm{Q}^n_C(\mathcal{O}_C^{\oplus r}) \to \mathrm{Sym}^n(C)$ over the same point.*

Proof. From Remark 5.1.5 we have that $\sigma^{-1}(\mathrm{Sym}^n(U))$ depends only on $\mathcal{E}|_U$, for any $U \subseteq C$ open. Then the proposition follows from the fact that for any $D \in \mathrm{Sym}^n(C)$, there exists an open subset $U \subseteq C$ such that $D \in \mathrm{Sym}^n(U)$ and $\mathcal{E}|_U$ is trivial.

In order to see this, let V be an open affine subset of C, such that $D \in \mathrm{Sym}^n(V)$. Then, by [Ser58, Thm. 1], we have $\mathcal{E}|_V \cong \mathcal{O}_V^{\oplus(r-1)} \oplus \mathcal{L}$, where $\mathcal{L}$ is an invertible $\mathcal{O}_V$-module. Let us consider the short exact sequence $0 \to \mathcal{L}(-D) \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_D \to 0$, and the associated exact cohomology sequence, $0 \to \mathrm{H}^0(V, \mathcal{L}(-D)) \to \mathrm{H}^0(V, \mathcal{L}) \to \mathrm{H}^0(V, \mathcal{L} \otimes \mathcal{O}_D) \to 0$. We deduce that there exists $s \in \mathrm{H}^0(V, \mathcal{L})$ such that $s(y) \neq 0$, for all $y \in \mathrm{Supp}(D)$. Hence $\mathcal{L}$ (and consequently $\mathcal{E}$) is trivial on the open set $U = V \setminus \mathrm{Supp}(s)$. $\square$

Definition 5.2.2. For any $y \in C$, let us define $F_{n,r}(y) := \sigma^{-1}(ny)$. More generally, for any $D \in \mathrm{Sym}^n(C)$, we define $F_r(D) := \sigma^{-1}(D)$.

Proposition 5.2.3. *Let $D = a_1y_1 + \ldots + a_my_m \in \mathrm{Sym}^n(C)$, with $y_1, \ldots, y_m$ pairwise distinct. Then*

$$F_r(D) \cong F_{a_1,r}(y_1) \times \cdots \times F_{a_m,r}(y_m)\,.$$

Proof. There is a natural morphism $F_r(D) \to F_{a_1,r}(y_1) \times \cdots \times F_{a_m,r}(y_m)$, which is defined in the following way on $\Bbbk$-rational points. For any quotient $[\pi : \mathcal{E} \to \mathcal{H}]$ in $F_r(D)$, we have a splitting $\mathcal{H} = \oplus_{i=1}^m \mathcal{H}_{y_i}$, where $\mathcal{H}_{y_i}$ is a skyscraper sheaf on C, which is supported in $\{y_i\}$. Therefore $\pi = \oplus_{i=1}^m \pi_i$, with each $\pi_i : \mathcal{E} \to \mathcal{H}_{y_i}$. Then the point $[\pi : \mathcal{E} \to \mathcal{H}] \in F_r(D)$ is mapped to $[\pi_1 : \mathcal{E} \to \mathcal{H}_{y_1}] \times \cdots \times [\pi_m : \mathcal{E} \to \mathcal{H}_{y_m}] \in F_{a_1,r}(y_1) \times \cdots \times F_{a_m,r}(y_m)$. Clearly this is an isomorphism. $\square$

5.3 The class of $Q^n_C(\mathcal{E})$ in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$

In this section we prove our main theorem.

Theorem 5.3.1. *Let C be a smooth projective curve over $\Bbbk$. Let $\mathcal{E}$ be a coherent locally free $\mathcal{O}_C$-module of rank r. Then, for any non-negative integer n, the equality*

$$[Q^n_C(\mathcal{E})] = [Q^n_C(\mathcal{O}^{\oplus r})]$$

holds true in the Grothendieck group $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$ of $\Bbbk$-varieties.

The proof will be divided into several steps.

Step 1. In order to make the proof clearer, we first fix our notation (see also Section 5.1). Let $U \subseteq C$ be a fixed open subset such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$, and let $C \setminus U = \{y_1, \dots, y_N\}$. Then $\mathrm{Sym}^n(C)$ is the set-theoretic disjoint union of the locally closed subsets

$$Z_{\mathbf{a}} := \{E \in \mathrm{Sym}^n(C) \mid \mathrm{Supp}(E - a_1y_1 - \ldots - a_Ny_N) \subseteq U\},$$

for $\mathbf{a} \in A := \{(a_1, \dots, a_N) \in \mathbb{N}^N \mid a_1 + \ldots + a_N \leq n\}$. Notice that

$$Z_{\mathbf{a}} \cong \mathrm{Sym}^{n-|\mathbf{a}|}(U),$$

where $|\mathbf{a}| := a_1 + \cdots + a_N$.

For any $\mathbf{a} \in A$, we denote by $Q_{\mathbf{a}}(\mathcal{E})$ the preimage of $Z_{\mathbf{a}}$ under the morphism $\sigma : Q^n_C(\mathcal{E}) \to \mathrm{Sym}^n(C)$ of Section 5.1.1, with the reduced subscheme structure.

Remark 5.3.2. Using the relations in the Grothendieck group of varieties, the decomposition of $Q^n_C(\mathcal{E})$ into its locally closed subsets $Q_{\mathbf{a}}(\mathcal{E})$ yields the equality

$$[Q^n_C(\mathcal{E})] = \sum_{\mathbf{a} \in A} [Q_{\mathbf{a}}(\mathcal{E})]$$

in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$.

Finally, we denote by D the divisor $a_1y_1 + \cdots + a_Ny_N \in \mathrm{Sym}^{|\mathbf{a}|}(C)$ corresponding to $\mathbf{a} \in A$. Associated to this effective divisor we have the fiber $F_r(D) \subset Q^{|\mathbf{a}|}_C(\mathcal{E})$, as in Definition 5.2.2.

Step 2. The core of our proof is the following proposition.

Proposition 5.3.3. *For any $\mathbf{a} \in A$, there is a natural isomorphism*

$$Q_{\mathbf{a}}(\mathcal{E}) \xrightarrow{\sim} Q_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D).$$

The idea behind this proposition is that any quotient of $\mathcal{E}$ in $Q_{\mathbf{a}}(\mathcal{E})$ can be obtained by gluing a quotient supported in U and a quotient supported on $\{y_1, \dots, y_N\}$.

Before proving Proposition (5.3.3) in Step 3, we need the following result in order to define the morphism $Q_{\mathbf{a}}(\mathcal{E}) \to Q_U^{n-|\mathbf{a}|}(\mathcal{E})$.

Lemma 5.3.4. *Let $\mathcal{Q}$ be the universal quotient associated to $Q_C^n(\mathcal{E})$, and let $i : U \times Q_{\mathbf{a}}(\mathcal{E}) \hookrightarrow C \times Q_C^n(\mathcal{E})$ be the inclusion. For any $\mathbf{a} \in A$, the support $\mathrm{Supp}(i^*\mathcal{Q})$ is proper over $Q_{\mathbf{a}}(\mathcal{E})$.*

Proof. Let us apply the valuative criterion of properness to the restriction $f : \mathrm{Supp}(i^*\mathcal{Q}) \to Q_{\mathbf{a}}(\mathcal{E})$ of the projection $U \times Q_{\mathbf{a}}(\mathcal{E}) \to \mathrm{Q}_{\mathbf{a}}(\mathcal{E})$. Let R be a valuation ring, and let K be its quotient field. Assume we are given a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Supp}(i^*\mathcal{Q}) \arrow[d, "f"] \\
\mathrm{Spec}(R) \arrow[r] & \mathrm{Q}_{\mathbf{a}}(\mathcal{E})
\end{tikzcd}$$

where $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ is the morphism induced by $R \hookrightarrow K$.

Since $\mathrm{Supp}(i^*\mathcal{Q}) \subset C \times Q_{\mathbf{a}}(\mathcal{E})$ and $C \times Q_{\mathbf{a}}(\mathcal{E})$ is proper over $Q_{\mathbf{a}}(\mathcal{E})$, there is a unique morphism $g : \mathrm{Spec}(R) \to C \times Q_{\mathbf{a}}(\mathcal{E})$ such that the diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Supp}(i^*\mathcal{Q}) \arrow[r, hook] \arrow[d, "f"] & C \times Q_{\mathbf{a}}(\mathcal{E}) \arrow[dl] \\
\mathrm{Spec}(R) \arrow[r] \arrow[urr, dotted, "g"] & Q_{\mathbf{a}}(\mathcal{E}) &
\end{tikzcd}$$

commutes. The claim follows, if we prove that the image of g is contained in $U \times Q_{\mathbf{a}}(\mathcal{E})$.

Let us consider the composition

$$g' : \mathrm{Spec}(R) \xrightarrow{g} C \times Q_{\mathbf{a}}(\mathcal{E}) \xrightarrow{\mathrm{id}_C \times (h|_{Q_{\mathbf{a}}(\mathcal{E})})} C \times \mathrm{HC}^{-1}(Z_{\mathbf{a}}),$$

where $h : Q_C^n(\mathcal{E}) \to \mathrm{Hilb}^n_{C/\Bbbk}$ is the morphism defined in Section 5.1.1. The universal ideal sheaf of $\mathrm{Hilb}^n_{C/\Bbbk}$ restricted to $C \times \mathrm{HC}^{-1}(Z_{\mathbf{a}})$ is of the form

$$\mathcal{O}_{C \times \mathrm{HC}^{-1}(Z_{\mathbf{a}})}(-\mathcal{D} - \mathcal{D}'),$$

for $\mathcal{D} = a_1\{y_1\} \times \mathrm{HC}^{-1}(Z_{\mathbf{a}}) + \dots + a_N\{y_N\} \times \mathrm{HC}^{-1}(Z_{\mathbf{a}})$ and $\mathcal{D}'$ an effective Weil divisor (notice that $C \times \mathrm{HC}^{-1}(Z_{\mathbf{a}})$ is smooth), such that $\mathcal{D} \cap \mathcal{D}' = \emptyset$. Since the image of the composition

$$\mathrm{Spec}(K) \to \mathrm{Supp}(i^*\mathcal{Q}) \hookrightarrow C \times Q_{\mathbf{a}}(\mathcal{E}) \xrightarrow{\mathrm{id}_C \times (h|_{Q_{\mathbf{a}}(\mathcal{E})})} C \times \mathrm{HC}^{-1}(Z_{\mathbf{a}})$$

is contained in $\mathrm{Supp}(\mathcal{D}') \subset U \times \mathrm{HC}^{-1}(Z_{\mathbf{a}})$, the same holds for the image of g'. Therefore, the image of g lies in $U \times Q_{\mathbf{a}}(\mathcal{E})$, as claimed. $\square$

Step 3. Using Lemma 5.3.4, we can now prove Proposition 5.3.3.

Proof of Proposition 5.3.3. As above, let $\rho : \mathcal{E}_{Q^n_C(\mathcal{E})} \to \mathcal{Q}$ be the universal quotient associated to $Q^n_C(\mathcal{E})$, and let $i : U \times Q_{\mathbf{a}}(\mathcal{E}) \hookrightarrow C \times Q^n_C(\mathcal{E})$ be the inclusion. By Lemma 5.3.4, the quotient $i^*\rho : i^*(\mathcal{E}_{Q^n_C(\mathcal{E})}) \to i^*\mathcal{Q}$ yields a natural morphism $f_0 : Q_{\mathbf{a}}(\mathcal{E}) \to Q^{n-|\mathbf{a}|}_U(\mathcal{E})$.

In order to define a morphism $f_\infty : Q_{\mathbf{a}}(\mathcal{E}) \to F_r(D)$, let us consider the open neighbourhood $(C \times Q_{\mathbf{a}}(\mathcal{E})) \setminus \operatorname{Supp}(i^*\mathcal{Q})$ of $\cup_{\lambda=1}^N \{y_\lambda\} \times Q_{\mathbf{a}}(\mathcal{E})$, together with its inclusion j into $C \times Q_{\mathbf{a}}(\mathcal{E})$. By composing the pullback of ρ to $Q_{\mathbf{a}}(\mathcal{E})$ with the unit of the adjunction $j^* \dashv j_*$, we get a surjective morphism

$$\pi : \mathcal{E}_{Q_{\mathbf{a}}(\mathcal{E})} \to \mathcal{Q}_{Q_{\mathbf{a}}(\mathcal{E})} \to j_*j^*(\mathcal{Q}_{Q_{\mathbf{a}}(\mathcal{E})})$$

of coherent $\mathcal{O}_{C \times Q_{\mathbf{a}}(\mathcal{E})}$-modules. Note that $j_*j^*(\mathcal{Q}_{Q_{\mathbf{a}}(\mathcal{E})})$ has constant Hilbert polynomial $|\mathbf{a}|$, hence it is flat over $Q_{\mathbf{a}}(\mathcal{E})$. Therefore, π is associated to a natural morphism $Q_{\mathbf{a}}(\mathcal{E}) \to Q^{|\mathbf{a}|}_C(\mathcal{E})$, whose image is contained in $F_r(D)$. Thus we obtain a morphism $f_\infty : Q_{\mathbf{a}}(\mathcal{E}) \to F_r(D)$, and the morphism in (5.3.3) is (f_0, f_∞).

In order to prove that (f_0, f_∞) is an isomorphism, we exhibit its inverse, as follows. Let $\rho_0 : \mathcal{E}_{Q^{n-|\mathbf{a}|}_U(\mathcal{E})} \to \mathcal{Q}_0$ be the universal quotient of $Q^{n-|\mathbf{a}|}_U(\mathcal{E})$, and let $\rho_\infty : \mathcal{E}_{F_r(D)} \to \mathcal{Q}_\infty$ be the pullback of the universal quotient of $Q^{|\mathbf{a}|}_C(\mathcal{E})$ to $C \times F_r(D) \subseteq C \times Q^{|\mathbf{a}|}_C(\mathcal{E})$. In the following, we view ρ_0 as a family of quotients of $\mathcal{E}$ supported in U (see Lemma 5.1.1). Let us denote the projections by $\mathrm{pr}_{12} : C \times Q^{n-|\mathbf{a}|}_U(\mathcal{E}) \times F_r(D) \to C \times Q^{n-|\mathbf{a}|}_U(\mathcal{E})$ and $\mathrm{pr}_{13} : C \times Q^{n-|\mathbf{a}|}_U(\mathcal{E}) \times F_r(D) \to C \times F_r(D)$. Then

$$(\mathrm{pr}_{12})^*\rho_0 \oplus (\mathrm{pr}_{13})^*\rho_\infty : \mathcal{E}_{Q^{n-|\mathbf{a}|}_U(\mathcal{E}) \times F_r(D)} \to (\mathrm{pr}_{12})^*\mathcal{Q}_0 \oplus (\mathrm{pr}_{13})^*\mathcal{Q}_\infty$$

is a family of quotients of $\mathcal{E}$, parametrized by $Q^{n-|\mathbf{a}|}_U(\mathcal{E}) \times F_r(D)$, whose Hilbert polynomial is constantly equal to n. The associated morphism $Q^{n-|\mathbf{a}|}_U(\mathcal{E}) \times F_r(D) \to Q^n_C(\mathcal{E})$ is the inverse morphism of (f_0, f_∞). $\square$

Step 4. We can finally conclude the proof of Theorem 5.3.1.

By Remark 5.3.2 and Proposition 5.3.3, we have the following equalities in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$:

$$[Q^n_C(\mathcal{E})] = \sum_{\mathbf{a} \in A} [Q_{\mathbf{a}}(\mathcal{E})] = \sum_{\mathbf{a} \in A} [Q^{n-\mathbf{a}}_U(\mathcal{E})][F_r(D)]\,.$$

In particular, this is true also for $\mathcal{E} = \mathcal{O}_C^{\oplus r}$.

Now, $\mathcal{E}|_U$ is trivial, therefore $[Q^{n-\mathbf{a}}_U(\mathcal{E})] = [Q^{n-\mathbf{a}}_U(\mathcal{O}^{\oplus r})]$. Theorem 5.3.1 thus follows from Proposition 5.2.1.

5.3.1 Explicit computation

We provide an explicit formula for the class $[Q^n_C(\mathcal{E})] \in \mathrm{K}_0(\mathsf{Var}_{\Bbbk})$ in terms of the classes $[\mathrm{Sym}^m(C)]$.

Proposition 5.3.5. *For any non-negative integer n, the equality*

$$[Q^n_C(\mathcal{E})] = \sum_{\mathbf{n}\in\mathbb{N}^r,\,|\mathbf{n}|=n} [\mathrm{Sym}^{n_1}(C)]\cdots[\mathrm{Sym}^{n_r}(C)]\,\mathbf{L}^{d_{\mathbf{n}}}$$

holds true in $\mathrm{K}_0(\mathsf{Var}_{\Bbbk})$, where $d_{\mathbf{n}} := \sum_{i=1}^r (i-1)n_i$.

Proof. From Theorem 5.3.1 we have that $[Q^n_C(\mathcal{E})] = [Q^n_C(\mathcal{O}^{\oplus r})]$. Then the result follows directly from [Bif89]. $\square$

Remark 5.3.6. From the previous formula we can determine the Poincaré polynomial of $Q^n_C(\mathcal{E})$ (for ℓ-adic cohomology, where $\ell \neq \mathrm{char}(\Bbbk)$ is a prime) as follows (cf. also [BGL94]). By [Mac62], we know that the Poincaré polynomial $P(\mathrm{Sym}^m(C); t)$ of $\mathrm{Sym}^m(C)$ is the coefficient of u^m in the expansion of

$$\frac{(1+tu)^{2g}}{(1-u)(1-t^2u)}\,,$$

where g is the genus of C. Then, for $E(t,u) := \sum_{n=0}^{\infty} P(Q^n_C(\mathcal{E}); t)\,u^n$, we have:

$$\begin{aligned}
E(t,u) &= \sum_{n=0}^{\infty}\sum_{\mathbf{n}\in\mathbb{N}^r,\,|\mathbf{n}|=n} P(\mathrm{Sym}^{n_1}(C);t)\cdots P(\mathrm{Sym}^{n_r}(C);t)\,t^{2d_{\mathbf{n}}}u^n \\
&= \sum_{\mathbf{n}\in\mathbb{N}^r}\prod_{i=1}^{r} P(\mathrm{Sym}^{n_i}(C);t)\,t^{2(i-1)n_i}u^{n_i} \\
&= \prod_{i=0}^{r-1}\frac{(1+t^{2i+1}u)^{2g}}{(1-t^{2i}u)(1-t^{2i+2}u)}\,.
\end{aligned}$$

Corollary 5.3.7. *The Quot scheme $Q^n_C(\mathcal{E})$ is irreducible.*

Proof. Since $Q^n_C(\mathcal{E})$ is smooth, it suffices to show that the coefficient of t^0 in the Poincaré polynomial of $Q^n_C(\mathcal{E})$ is 1. To this aim, notice that, for any $\mathbf{n}\in\mathbb{N}^r$ with $|\mathbf{n}| = n$, we have that $d_{\mathbf{n}} = \sum_{i=1}^r (i-1)n_i = 0$ only for $\mathbf{n} = (n, 0, \ldots, 0)$. The claim now follows from Proposition 5.3.5. $\square$

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