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Ph.D Course in Geometry and
Mathematical Physics

Ph.D. Thesis

Tau Functions:
Theory and Applications
to Matrix Models
and Enumerative Geometry

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Preface

In this thesis we study partition functions given by matrix integrals from the point of view of isomonodromic deformations, or more generally of Riemann–Hilbert problems depending on parameters.

The partition functions under investigation are relevant in particular because of their applications to Combinatorics and Enumerative Geometry. They are known to be tau functions in the usual sense of Integrable System Theory. Our method mainly differs in that we consider them as tau functions of *isomonodromic type*.

This approach proves to be an efficient way of studying these partition functions. Possible applications which are explored, to various extents, in this thesis can be outlined as follows.

- Direct and general derivation of *non-recursive effective formulæ* for the combinatorial/geometric content of the partition function.
- A new derivation of *Virasoro constraints* of the partition functions, directly connected with the action of the Witt algebra of infinitesimal conformal transformations of the plane.
- Rigorous study of *analytic aspects* of the matrix integrals, e.g. *large-size limits*, *resummation* of formal generating functions and their corresponding *nonlinear Stokes' phenomenon*.

The thesis contains several reviews of non-original results. The original contributions are based on the following works of the author (in chronological order).

- [BRc]: M. Bertola and GR. The Kontsevich–Penner matrix integral, isomonodromic tau functions and open intersection numbers. *Annales Henri Poincare*, 20(2):393–443, 2019. arxiv.org/abs/1711.03360
- [BRb]: M. Bertola and GR. The Brezin–Gross–Witten tau function and isomonodromic deformations. To appear in *Communications in Number Theory and Physics.* arxiv.org/abs/1812.02116
- [BRa]: M. Bertola and GR. Matrix models for stationary Gromov-Witten invariants of the Riemann sphere. In preparation.
- [GGR]: M. Gisonni, T. Grava and GR. Mixed correlators of the Laguerre Ensemble and monotone Hurwitz numbers. In preparation.

The organization of the thesis is as follows.

- In Chapters 1 and 2 we review the general theory of tau functions of isomonodromic type.
- In Chapter 3 we review the general theory of matrix models and associated orthogonal polynomials. We also present a review of recent results of [DYb] as well as original results of [GGR].
- In Chapter 4 we give a review of the Kontsevich–Witten tau function and of [BCa].
- In Chapter 5, 6 and 7 we report the original results of [BRb; BRc; BRa] respectively.
- Appendices A and B contain review of background material on integrable hierarchies and matrix integrals.

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Notations

Pauli matrices	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$
Elementary matrices	$(\mathrm{E}_{ab})_{i,j} = \delta_{a,i}\delta_{b,j}.$
Diagonal matrices	$\mathrm{diag}\,(x_1, ..., x_N) = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N \end{pmatrix}.$
Vandermonde determinant	$\Delta(x_1, ..., x_N) := \det\left(x_j^{N-k}\right)_{j,k=1}^{N} = \prod_{1 \le a < b \le N} (x_a - x_b)$
Rising factorial ($\alpha \in \mathbb{C}$, $\ell = 0, 1, 2, ...$)	$(\alpha)_\ell := \alpha(\alpha+1)\cdots(\alpha+\ell-1), \quad (\alpha)_{-1} := \frac{1}{\alpha - 1}.$
Double factorial ($k = 0, 1, 2, ...$)	$k!! := \prod_{j=1}^{\lfloor \frac{k}{2} \rfloor} (k - 2j), \quad (-1)!! = 1.$
Hypergeometric series	${}_pF_q\left({\alpha_1, ..., \alpha_p \atop \beta_1, ..., \beta_q} \middle| z \right) := \sum_{n \ge 0} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$
Symmetric group	$\mathfrak{S}_n$
Parity and sign of a permutation	$\lvert\sigma\rvert = 0, 1, \quad (-1)^{\lvert\sigma\rvert} = \pm 1, \qquad \sigma \in \mathfrak{S}_n$
Partitions	$\lambda = (\lambda_1, ..., \lambda_\ell) \in \mathbb{Y},\ \lambda_1 \ge \cdots \ge l_\ell > 0$

Introduction

Overview

Tau functions. There are many interrelated concepts of tau function, each appearing in specific, sometimes very far, branches of Mathematics. Among them we mention the following ones.

Tau functions of integrable systems. Historically, tau functions first occurred in the study of infinitely-dimensional integrable systems, viewed as families of commuting symmetries of an *integrable* equation. To give a simple example, the *Korteweg–de Vries equation*

$$u_t = 3uu_x + \frac{1}{4}u_{xxx} = \partial_x(L_1), \qquad L_1 = \frac{3u^2}{2} + \frac{u_{xx}}{4} \tag{1}$$

describes waves in shallow water and possesses an infinite set of symmetries ($x = t_0, t = t_1$)

$$u_{t_j} = \partial_x(L_j), \quad L_0 = u, \quad L_1 = \frac{3u^2}{2} + \frac{u_{xx}}{4}, \quad L_2 = \frac{5u^3}{2} + \frac{5u_x^2}{8} + \frac{5uu_{xx}}{4} + \frac{u_{4x}}{16} \tag{2}$$

where more generally the *Lenard–Magri differential polynomials* in u are defined by the recursion

$$\partial_x L_{j+1} = \left(\frac{1}{4}\partial_x^3 + 2u\partial_x + u_x\right) L_j. \tag{3}$$

Such symmetries of the KdV hierarchy are *in involution*, in the sense that they commute, i.e. $\partial_{t_j}\partial_{t_k}u = \partial_{t_k}\partial_{t_j}u$. The *tau function* in this case is defined by

$$2\partial_x^2 \log\tau = u \tag{4}$$

and can be regarded, along with u, as a function of the infinitely many variables $t_0, t_1, \ldots$ which takes the KdV equation, as well as all the equations of the hierarchy (2), into a bilinear form [Hc];

$$3\tau_{xx}^2 + 4\tau_x\tau_t - 4\tau_x\tau_{xxx} - 4\tau\tau_{xt} + \tau\tau_{4x} = 0. \tag{5}$$

This example is particular case of a much more general *universal* hierarchy, known as the *Kadomtsev–Petviashvili (KP) hierarchy*; it represents in a similar way an infinite set of commuting symmetries of the KP equation

$$u_{yy} + (4u_t + 12uu_x + u_{xxx})_x = 0. \tag{6}$$

and the tau function is defined exactly as in (4). Again, it takes all the equations of the KP hierarchy into a bilinear form. There is a beautiful description of the space of KP tau functions in terms of an infinite-dimensional grassmannian [SS; SWa], which is suggested by the analogy of this bilinear form of the equations with the classical *Plücker relations.*

Tau functions of isomonodromic systems. A fundamental idea, due to Riemann, is that of considering transcendents defined by linear ODEs with rational coefficients as functions of their *monodromy*, opportunely defined. The setting of Isomonodromy Theory is to consider deformations of these transcendents in such a way that their monodromy remains constant. This beautiful classical topic goes back to the beginning of the XXth century to the work, among the others, of P. Painlevé, R. Fuchs, B. Gambier. There was a great renewal of interest in this theory starting from the 1980's, because of the appearance of Painlevé transcendents in certain correlation functions of Conformal Field Theories, see e.g. [JMMS]. A cornerstone in the theory has been the introduction of *isomonodromic tau functions* in a very general setting [JMU]. Since then the theory of isomonodromic deformations has been a very active field of research, with fundamental applications to a wide variety of topics, ranging indeed from the study of correlation functions in several physical theories to Random Matrix Theory.

Tau functions as generating functions. Tau functions also appear in various contexts in Geometry and Combinatorics, as *formal generating functions.* It is well known indeed that generating functions are a fundamental tool in Mathematics, where deep nontrivial recursion relations are often encoded in simple form of (algebraic, differential, or difference, etc) equations involving generating functions. A plethora of examples where generating functions of interesting geometric/combinatorial invariants satisfy the same equations as a tau function of some integrable equation is known. The Witten conjecture [Wc] is one of the most notable examples, which has been greatly generalized with the theory of Gromov–Witten invariants [KM] and of Frobenius manifolds [Dd; DZa]. Many important cases are related with matrix models, and this has been the source of inspiration for the Topological Recursion Theory [EO].

Isomonodromic method. Much of the work done during the doctoral program at SISSA under the supervision of Prof. M. Bertola concerns the definition of suitable *isomonodromic systems* whose isomonodromic tau functions coincide with matrix integrals related to interesting combinatorial/geometric objects. Then one can exploit this underlying isomonodromic system to study these matrix integrals.

This was motivated by the description in [BCa] of the Kontsevich matrix integral [Kb] in terms of an isomonodromic tau function.

It turns out that this method can be applied to several interesting models (see below). Incidentally, let us comment on the interesting fact that such models which are of interest in Combinatorics and Geometry, coincide with models which are of interest in Random Matrix Theory. This connection was unveiled in [Ob] where intersection theory on the moduli spaces of curves was related with the edge-of-the-spectrum model, i.e. with the Airy kernel, and was actually used to provide formulæ for intersection numbers [Oc]. Let us point out that, in a similar way, the examples examined in this thesis present this double nature; on one side combinatorial/geometric quantities and on the other side Riemann–Hilbert problems related with Random Matrix Theory (e.g. Bessel and discrete Bessel process, appearing respectively in connection with Norbury intersection numbers and with stationary Gromov–Witten invariants of the Riemann sphere). We do not know a general explanation for this connection, and we believe that this point deserves further study.

Relation with topological recursion theory. Many theoretical and computational aspects of the theory of tau functions are deeply connected with the *topological recursion* of Eynard and Orantin [EO]. Indeed very similar formulæ (in this context they are called *determinantal formulæ*) appear also in works related to the topological recursion, see e.g. [BE; BBE].

Let us point out that our approach is completely explicit and non-recursive and starts directly from the matrix models under investigation, therefore it is is completely independent from the topological recursion. Moreover, to the best of our knowledge, the determinantal formulæ of loc. cit. have never been used to derive *mixed correlators*, i.e. logarithmic derivatives of multi-pole tau functions with respect to time variables related to different poles, as we derive in Chap. 1 and apply in Chap. 3 to the Laguerre Unitary Ensemble.

Structure of the thesis. In Part I where we give an account of the general theory of tau functions, following the perspective which is most suited to our purposes. The content of this part is not new, but we felt necessary to give a fairly self-contained introduction to the subject. There are two main points of views explored in this part, one more formal (Chap. 1) and one more related to analytic aspects (Chap. 2). Part II is the core of the thesis and we present the isomonodromic method at work in several instances. Part III contains two appendices where we review some background material, on the Kadomtsev–Petviashvili hierarchy mentioned earlier in this introduction and on some standard techniques about matrix integrals.

More details on the contents of this thesis are described below, distinguishing between original and non original results.

Original contributions

Our original contributions mainly consist in the study of certain interesting tau functions by means of the aforementioned isomonodromic method, providing in particular explicit formulæ for them.

Laguerre Unitary Ensemble (LUE). According to a result of Bertola, Eynard, and Harnad [BEH], partition functions of one matrix models coincide with the isomonodromic tau function of the associated 2×2 ODE of the associated orthogonal polynomials. We apply this result (reviewed in general in Chap. 3) to study the LUE. The main result is Thm. 3.5.7, providing explicit formulæ for *mixed* correlators of the LUE. This result is contained in the work [GGR], which is in preparation, and is presented in Sec. 3.5.2.

Brézin–Gross–Witten tau function and Norbury intersection numbers. We have applied the isomonodromic method to the study of the Brézin–Gross–Witten tau function. The main result is given by the formulæ of Cor. 5.3.5 for Norbury intersection numbers[1]. In particular let us report the following simple expression for one-point intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,1}} \Theta_{g,1}\psi_1^{g-1} = \frac{(2g-1)!!(2g-3)!!}{8^g g!}. \tag{7}$$

Moreover, we make the small observation that the Brézin–Gross–Witten tau function provides a solution to the Painlevé XXXIV hierarchy, see Prop. 5.3.16. (This parallels the connection of the Kontsevich–Witten tau function with the Painlevé I hierarchy.)

This study is contained in the paper [BRb] and is presented in Chap. 5.

Kontsevich–Penner tau function and open intersection numbers. The isomonodromic approach can be applied to the Kontsevich–Penner tau function, a generalization of the Kontsevich–Witten tau function, whose algebro-geometric interpretation should be found, conjecturally, in the intersection theory on the moduli spaces of *open* Riemann surfaces. The main result is Thm. 6.3.3, providing explicit formulæ for open intersection numbers. Let us report by way of example the following formula for a generating function of one-point open intersection numbers

$$\sum_{\ell\geq 0}\left\langle \tau_{\frac{\ell}{2}-2}\right\rangle_{\mathsf{open}} x^{\frac{\ell}{2}} = \mathrm{e}^{\frac{x^3}{6}}\left({}_2F_2\left(\left.\begin{matrix}\frac12 - Q & \frac12+Q\\ \frac12 & \frac12\end{matrix}\right| -\frac{x^3}{8}\right) + Q\,x^{\frac32}\,{}_2F_2\left(\left.\begin{matrix}1-Q & 1+Q\\ 1 & \frac32\end{matrix}\right| -\frac{x^3}{8}\right)\right) \tag{8}$$

which generalizes the classical formula for (closed) intersection numbers $\langle\tau_{3g-2}\rangle = \frac{1}{24^g g!}$. An alternative formulation of the above result is

$$\sum_{\ell\geq 0}\left\langle \tau_{\frac{\ell}{2}-2}\right\rangle_{\mathsf{open}} x^{\frac{\ell}{2}} = \mathrm{e}^{\frac{x^3}{24}}\sum_{j\geq 0}\frac{A_j(Q)}{(j-1)!!}x^{\frac{3j}{2}} \tag{9}$$

where the coefficients $A_j(Q)$ are defined by

$$\left(\frac{2+x}{2-x}\right)^Q = \sum_{j\geq 0} A_j(Q)x^j \tag{10}$$

where again the reduction to the *closed* generating function $\mathrm{e}^{\frac{x^3}{24}}$ for $Q=0$ is manifest.

This study is contained in the paper [BRc] and is presented in Chap. 6.

Matrix models for stationary Gromov–Witten theory of the Riemann sphere. Explicit formulæ for stationary Gromov–Witten invariants of $\mathbb{P}^1$ have been recently discovered in [DYa], see also [Mc; DYZa]. Such formulæ can be very naturally identified with the general formulæ expressing logarithmic derivatives of a tau function of isomonodromic type. Applying then the isomonodromic method in

[1] The same formulæ are derived by a different approach (*matrix resolvent approach*) to tau functions of the KdV hierarchy in [DYZb].

reverse, we construct the following Kontsevich-like matrix model for stationary Gromov–Witten theory of $\mathbb{P}^1$, which is slightly different from the ones proposed in the literature.

Define the function $f(z;\epsilon)$ of the complex variable z, depending on a parameter $\epsilon > 0$;

$$f(z;\epsilon) := \frac{1}{\sqrt{2\pi\epsilon}} \int_{C_1} \exp\left(\frac{1}{\epsilon}\left(x - \frac{1}{x}\right) - \left(z + \frac{3}{2}\right)\log x\right) \mathrm{d}x. \tag{11}$$

The contour C_1 starts from 0 with $|\arg x| < \frac{\pi}{2}$ and arrives at ∞ with $\frac{\pi}{2} < \arg x < \pi$ (see Fig. 7.1). The function has the following asymptotic expansion as $z \to \infty$ within the sector $|\arg z| \le \frac{\pi}{2} - \delta$, $\delta > 0$;

$$\left(\frac{\epsilon z}{\mathrm{e}}\right)^{-z} f(z;\epsilon) \sim 1 + \frac{24-\epsilon^2}{24\epsilon^2 z} + \frac{\epsilon^4 + 528\epsilon^2 + 576}{1152\epsilon^4 z^2} + \frac{1003\epsilon^6 + 95400\epsilon^4 + 406080\epsilon^2 + 69120}{414720\epsilon^6 z^3} + \cdots. \tag{12}$$

Introduce

$$\tau_N(z_1, ..., z_N) := \frac{\det\left(\frac{1}{\epsilon^{k-1}}\left(\frac{\epsilon z_j}{\mathrm{e}}\right)^{-z_j} f(z_j + k - 1;\epsilon)\right)_{j,k=1}^N}{\prod_{1\le i<j\le N}(z_j - z_i)}. \tag{13}$$

Then the expansion of $\log \tau_N(z_1, ..., z_N)$, expressed in terms of the scaled Miwa variables

$$T_k := \frac{k!}{\epsilon^k}\left(\frac{1}{z_1^{k+1}} + \cdots + \frac{1}{z_N^{k+1}}\right) \tag{14}$$

stabilizes as $N \to \infty$ to the generating function (see (7.1)) of stationary GW invariants of $\mathbb{P}^1$.

This study is part of the work in preparation [BRa] and is presented in Chap. 7.

Review contributions

According with the spirit of this thesis outlined above, we have also included some (to a various extent) original proofs of non original results. In particular:

Averages of products and ratio of characteristic polynomials. In Thm. 3.4.1 we re-derive, from the general theory of Schlesinger transformations (reviewed in Chap. 2), the formula of [BHa; BDS] for expectation values of products and ratios of characteristic polynomials of random matrices.

Virasoro constraints. As a direct consequence of the Jimbo–Miwa–Ueno formula one may derive Virasoro constraints for an isomonodromic tau function. The complete proof of Virasoro constraints by this approach is presented for the case of the Brézin–Gross–Witten tau function in Chap. 5, where the Virasoro constraints where already known from [Ab]. However, the methods exposed there following [BRb] are of much more general applicability. A slightly different approach can be used to derive the first Virasoro constraints (corresponding to shifts and dilations in the plane), exploting translation and dilation covariance of the relevant Riemann–Hilbert problems, as illustrated in Chap. 6 following [BRc].

Gaussian Unitary Ensemble (GUE). The methods of Chap. 3 find a natural application to the study of the GUE partition function. We re-derive from this approach some results from [DYb] about the GUE in Sec. 3.5.1.

Witten–Kontsevich tau functions. The isomonodromic interpretation of the Kontsevich matrix integral [BCa] was the first motivation for our investigations. We give a review of some of the results of loc. cit. in Chap. 4.

Part I

Tau Functions: Theory

CHAPTER 1

Tau differential

In this first chapter we introduce a formal notion of tau function, as the (logarithmic) potential of a closed differential. The latter is termed tau differential, and can be considered whenever a certain type of compatible deformations system arises; we review the important cases of isospectral and isomonodromic deformations and of Gelfand–Dickey hierarchies. The relation with Hirota bilinear equations is illustrated, and logarithmic derivatives of arbitrary order (in a subset of times) of the tau function are computed. More analytic aspects and more examples of the notion of tau function are explored in the next chapter.

The material of this chapter is mainly extracted from [BBT; BDYa; Db; JMU].

1.1 Tau differential and tau function

Let us start with a notation that will be employed everywhere in this work.

Notation 1.1.1. *z denotes a complex variable, and $\mathbf{t} = \{t_i\}$ a (possibly infinite) vector of parameters. We shall denote $' := \partial_z$ the derivative with respect to z, and we shall denote δ the exterior derivative in the parameters $\mathbf{t}$ (but not in z), i.e. $\delta f := \sum_i \frac{\partial f}{\partial t_i} \mathrm{d}t_i$.*

The ingredients to build a *tau differential* can be summarized as follows (see [Db, Chap.11] and [BBT, Chap.3]). Fix integers $N \geq 1$ and $m \geq 0$.

1. A diagonal $N \times N$ matrix valued function $\Xi = \Xi(z; \mathbf{t})$ (possibly multivalued in z) such that its z derivative Ξ' is rational in z with poles at some finite points $z = z_1, ..., z = z_m$ and at $z = \infty$ only. Concretely, we consider

$$\Xi = \sum_{\nu=1,\ldots,m,\infty} \Xi_\nu \tag{1.1}$$

where

$$\Xi_\nu = \begin{cases} \mathrm{diag}(\lambda_{\nu,1}, ..., \lambda_{\nu,N}) \log(z - z_\nu) + \sum\limits_{k\geq 1} \mathrm{diag}(t_{\nu,k,1}, ..., t_{\nu,k,N}) \frac{1}{(z-z_\nu)^k} & \nu = 1, ..., m \\ \\ \mathrm{diag}(\lambda_{\infty,1}, ..., \lambda_{\infty,N}) \log\left(\frac{1}{z}\right) + \sum\limits_{k\geq 1} \mathrm{diag}(t_{\infty,k,1}, ..., t_{\infty,k,N}) z^k & \nu = \infty. \end{cases} \tag{1.2}$$

We agree that the set of parameters $\mathbf{t}$ comprises the points $z_1, ..., z_m$ and the $t_{\nu,k,\alpha}$'s for $\nu = 1, ..., m, \infty$, $k \geq 1$, and $\alpha = 1, ..., N$, and not the $\lambda_{\nu,\alpha}$'s; i.e. we assume that the variation δ does not involve the parameters $\lambda_{\nu,\alpha}$, $\delta\lambda_{\nu,\alpha} = 0$.

2. A collection $\Gamma_\nu = \Gamma_\nu(z; \mathbf{t})$ of formal matrix valued series in z for $\nu = 1, ..., m, \infty$;

$$\begin{cases} \Gamma_\nu(z; \mathbf{t}) = G_\nu(\mathbf{t}) + O(z - z_\nu) = G_\nu(\mathbf{t}) \left(\mathbf{1} + \sum_{j\geq 1} \Gamma_\nu^{(j)}(\mathbf{t})(z - z_\nu)^j\right) & \nu = 1, ..., m \\ \Gamma_\infty(z; \mathbf{t}) = \mathbf{1} + \mathcal{O}(z^{-1}) = \mathbf{1} + \sum_{j\geq 1} \Gamma_\infty^{(j)}(\mathbf{t}) \frac{1}{z^j} & \nu = \infty \end{cases} \tag{1.3}$$

where $G_\nu(\mathbf{t})$ are invertible.

The type of dependence of $\Gamma_\nu(\mathbf{t})$ on the times $\mathbf{t}$ is formal in this general discussion; however in the following chapters we will also consider $\Gamma_\nu(\mathbf{t})$ as analytic functions of $\mathbf{t}$.

We require that the following equation is satisfied;

$$\delta\Gamma_\nu = \mathcal{M}\Gamma_\nu - \Gamma_\nu \delta\Xi_\nu. \tag{1.4}$$

Here $\mathcal{M}$ is an $N\times N$ matrix valued differential in $\mathbf{t}$ reading

$$\mathcal{M}(z;\mathbf{t}) = \sum_\nu \operatorname*{res}_{\xi=z_\nu} \Gamma_\nu(\xi;\mathbf{t})\delta\Xi_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t})\frac{\mathrm{d}\xi}{z-\xi} \tag{1.5}$$

where we introduce notations

$$z_\infty := \infty, \qquad \sum_\nu := \sum_{\nu=1,\dots,m,\infty} \tag{1.6}$$

which will occur frequently in the following.

Examples where equations (1.4) appear naturally are considered below, see Sec. 1.4. Let us point out that the residues in (1.5) extract the irregular parts of the series $\Gamma_\nu\delta\Xi\Gamma_\nu^{-1}$; we remind that if $f(z)$ is a formal series of the form $f(z) = \sum_{j\in\mathbb{Z}} f_j(z-z_*)^j$ then

$$\operatorname*{res}_{\xi=z_*} f(\xi)\frac{\mathrm{d}\xi}{z-\xi} = \sum_{j<0} f_j(z-z_*)^j \tag{1.7}$$

and if $f(z)$ is a formal series of the form $f(z) = \sum_{j\in\mathbb{Z}} f_j z^j$ then

$$\operatorname*{res}_{\xi=\infty} f(\xi)\frac{\mathrm{d}\xi}{z-\xi} = \sum_{j\geq 0} f_j z^j. \tag{1.8}$$

Let us make a few comments on this setting.

1. The system (1.4) is compatible[1]. This follows from the zero-curvature condition $\delta\mathcal{M} = \mathcal{M}\wedge\mathcal{M}$ (Prop. 1.1.2 below); indeed

$$\begin{aligned}\delta(\mathcal{M}\Gamma_\nu - \Gamma_\nu\delta\Xi_\nu) &= (\delta\mathcal{M})\,\Gamma_\nu - \mathcal{M}\wedge\delta\Gamma_\nu - \delta\Gamma_\nu\wedge\delta\Xi_\nu \\ &= \mathcal{M}\wedge\mathcal{M}\Gamma_\nu - \mathcal{M}\wedge\mathcal{M}\Gamma_\nu + \mathcal{M}\wedge\Gamma_\nu\delta\Xi_\nu - \mathcal{M}\Gamma_\nu\wedge\delta\Xi_\nu - \Gamma_\nu\delta\Xi_\nu\wedge\delta\Xi_\nu = 0\end{aligned}$$

   as in the last step the terms cancel pairwise and we note $\delta\Xi_\nu\wedge\delta\Xi_\nu = 0$ as Ξ_ν is diagonal.

2. In (1.2) we have implicitly set to zero the constant term of Ξ, as it would give rise to parameters $t_{\infty,0,\alpha}$ that can be absorbed by a common conjugation of the Γ_ν's. Indeed, replacing the definition of Ξ_∞ in (1.2) by $\sum\limits_{k\geq 0}\operatorname{diag}(t_{k,\infty,1},\dots,t_{k,\infty,N})z^k$ (note the range of summation including $k=0$), we have new flows

$$\frac{\partial}{\partial t_{\infty,0,\alpha}}\Gamma_\nu(z;\mathbf{t}) = \mathcal{M}\left(\frac{\partial}{\partial t_{\infty,0,\alpha}}\right)\Gamma_\nu(z;\mathbf{t}) - \Gamma_\nu(z;\mathbf{t})\mathrm{E}_{\alpha\alpha} = [\mathrm{E}_{\alpha\alpha},\Gamma_\nu] \tag{1.9}$$

   (where we use $\mathcal{M}\left(\frac{\partial}{\partial t_{\infty,0,\alpha}}\right) = \mathrm{E}_{\alpha\alpha}$) which are just global conjugations.

3. The ratio

$$\delta\Gamma_\nu\Gamma_\nu^{-1} = \mathcal{M} - \Gamma_\nu\delta\Xi_\nu\Gamma_\nu^{-1} \tag{1.10}$$

   is regular at $z = z_\nu$ for $\nu = 1,\dots,m$ and vanishes at $z=\infty$ for $\nu=\infty$; compare with (1.5).

4. There is a gauge freedom in (1.4) consisting of transformations multiplying each Γ_ν on the right by a diagonal matrix series in z constant in $\mathbf{t}$ (with constant term $\mathbf{1}$ when $\nu=\infty$).

5. One can introduce formal germs of matrix valued *wave function* $\Psi_\nu := \Gamma_\nu e^{\Xi_\nu}$; equation (1.4) is then equivalent to $\delta\Psi = \mathcal{M}\Psi$.

Proposition 1.1.2. *The zero-curvature condition $\delta\mathcal{M} = \mathcal{M}\wedge\mathcal{M}$ holds true for the matrix valued differential* (1.5).

[1] By compatible we always mean *integrable in the sense of Frobenius.*

It will be convenient hereafter to use the *graded* commutator

$$[\mathcal{A}, \mathcal{B}] := \mathcal{A} \wedge \mathcal{B} - (-1)^{ab} \mathcal{B} \wedge \mathcal{A} \tag{1.11}$$

for matrix valued differential forms $\mathcal{A}, \mathcal{B}$ of degrees a, b respectively.

Proof. First off we compute

$$\begin{aligned}\delta\mathcal{M}(z;\mathbf{t}) &= \sum_\nu \operatorname*{res}_{\xi=z_\nu} [\delta\Gamma_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t}), \Gamma_\nu(\xi;\mathbf{t})\delta\Xi_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi} \\ &= \sum_\nu \operatorname*{res}_{\xi=z_\nu} [\mathcal{M}(\xi;\mathbf{t}), \Gamma_\nu(\xi;\mathbf{t})\delta\Xi_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi}\end{aligned} \tag{1.12}$$

where we use $[\Gamma_\nu\delta\Xi_\nu\Gamma_\nu^{-1}, \Gamma_\nu\delta\Xi_\nu\Gamma_\nu^{-1}] = 0$. Now, since $\delta\Gamma_\nu\Gamma_\nu^{-1}$ is regular at $z_1, ..., z_m$ and vanishing at ∞ we have, for all $\nu = 1, ..., m, \infty$,

$$\begin{aligned}0 &= \operatorname*{res}_{\xi=z_\nu} [\delta\Gamma_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t}), \delta\Gamma_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi} \\ &= \operatorname*{res}_{\xi=z_\nu} [\mathcal{M}(\xi;\mathbf{t}) - \Gamma_\nu(\xi;\mathbf{t})\delta\Xi_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t}), \mathcal{M}(\xi;\mathbf{t}) - \Gamma_\nu(\xi;\mathbf{t})\delta\Xi_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi}\end{aligned}$$

which implies

$$\operatorname*{res}_{\xi=z_\nu} [\mathcal{M}(\xi;\mathbf{t}), \mathcal{M}(\xi;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi} = 2 \operatorname*{res}_{\xi=z_\nu} [\mathcal{M}(\xi;\mathbf{t}), \Gamma_\nu(\xi;\mathbf{t})\delta\Xi_\nu(\xi;\mathbf{t})\Gamma_\nu^{-1}(\xi;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi}. \tag{1.13}$$

Finally, summing over $\nu = 1, ..., m, \infty$ the left-hand side of the last equation gives, using the fact that the sum of residues of a globally defined meromorphic differential on the Riemann sphere vanishes,

$$\begin{aligned}\sum_\nu \operatorname*{res}_{\xi=z_\nu} [\mathcal{M}(\xi;\mathbf{t}), \mathcal{M}(z;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi} &= -\operatorname*{res}_{\xi=z} [\mathcal{M}(\xi;\mathbf{t}), \mathcal{M}(z;\mathbf{t})] \frac{\mathrm{d}\xi}{z-\xi} \\ &= [\mathcal{M}(z;\mathbf{t}), \mathcal{M}(z;\mathbf{t})] = 2\mathcal{M}(z;\mathbf{t}) \wedge \mathcal{M}(z;\mathbf{t})\end{aligned}$$

and the right hand side equals $2\delta\mathcal{M}(z;\mathbf{t})$ by virtue of (1.12), and the proof is complete. ■

Definition 1.1.3. *The tau differential* Ω *is the differential in the space of parameters* $\{\mathbf{t}\}$ *defined by*

$$\Omega := -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\Gamma_\nu^{-1}(z;\mathbf{t})\Gamma_\nu'(z;\mathbf{t})\delta\Xi_\nu(z;\mathbf{t})\right) \mathrm{d}z. \tag{1.14}$$

Theorem 1.1.4. *If the* Γ_ν*'s satisfy* (1.4)*, the tau differential is* δ*-closed;*

$$\delta\Gamma_\nu = \mathcal{M}\Gamma_\nu - \Gamma_\nu\delta\Xi_\nu \Rightarrow \delta\Omega = 0. \tag{1.15}$$

Proof. We compute $\delta\Omega$ using (1.4) and its differentiated form $\delta\Gamma_\nu' = \mathcal{M}'\Gamma_\nu + \mathcal{M}\Gamma_\nu' - \Gamma_\nu'\delta\Xi_\nu - \Gamma_\nu\delta\Xi_\nu'$

$$\begin{aligned}\delta\Omega &= -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(-\Gamma_\nu^{-1}\, \delta\Gamma_\nu\, \Gamma_\nu^{-1}\Gamma_\nu' \wedge \delta\Xi_\nu + \Gamma_\nu^{-1}\delta\Gamma_\nu' \wedge \delta\Xi_\nu\right) \mathrm{d}z \\ &= -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(-\Gamma_\nu^{-1}\mathcal{M}\Gamma_\nu' \wedge \delta\Xi_\nu + \delta\Xi_\nu\Gamma_\nu^{-1}\Gamma_\nu' \wedge \delta\Xi_\nu + \Gamma_\nu^{-1}\mathcal{M}'\Gamma_\nu \wedge \delta\Xi_\nu\right. \\ &\qquad\qquad \left.+\Gamma_\nu^{-1}\mathcal{M}\Gamma_\nu' \wedge \delta\Xi_\nu - \Gamma_\nu^{-1}\Gamma_\nu'\delta\Xi_\nu \wedge \delta\Xi_\nu - \delta\Xi_\nu' \wedge \delta\Xi_\nu\right) \mathrm{d}z \\ &= -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\mathcal{M}' \wedge \Gamma_\nu\delta\Xi_\nu\Gamma_\nu^{-1}\right) \mathrm{d}z\end{aligned} \tag{1.16}$$

where we have used the cyclic property of the trace together with $\delta\Xi_\nu \wedge \delta\Xi_\nu = 0$ and that $\delta\Xi_\nu' \wedge \delta\Xi_\nu$ is residueless. Hence

$$\begin{aligned}\delta\Omega &= -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\mathcal{M}' \wedge (\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1})\right) \mathrm{d}z \\ &= \underbrace{-\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\mathcal{M}' \wedge \mathcal{M}\right)}_{=0} + \sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\mathcal{M}' \wedge \delta\Gamma_\nu\Gamma_\nu^{-1}\right) \mathrm{d}z. \end{aligned} \tag{1.17}$$

In the last expression the first sum vanishes as it is the sum of residues of a (globally defined) meromorphic differential on the Riemann sphere. For the remaining term the crucial point is the following identity

$$\operatorname*{res}_{z=z_\nu} \operatorname{tr} \left((\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1})' \wedge (\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1})\right) \mathrm{d}z = \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left((\Gamma_\nu\delta\Xi_\nu\Gamma_\nu^{-1})' \wedge (\Gamma_\nu\delta\Xi_\nu\Gamma_\nu^{-1})\right) \mathrm{d}z = 0 \tag{1.18}$$

where we have used the cyclic property of the trace, the identity $\delta\Xi_\nu \wedge \delta\Xi_\nu = 0$ and the fact that $\delta\Xi_\nu' \wedge \delta\Xi_\nu$ is residueless. Finally, summing over $\nu = 1, ..., m, \infty$,

$$0 = \sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left((\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1})' \wedge (\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1})\right) \mathrm{d}z = -2\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\mathcal{M}' \wedge \delta\Gamma_\nu\Gamma_\nu^{-1}\right)$$

where we have used again that the sum of residues of the rational differential $\operatorname{tr}(\mathcal{M}' \wedge \mathcal{M})\,\mathrm{d}z$ vanishes, that $\delta\Gamma_\nu\Gamma_\nu^{-1}$ is regular at $z_1, ..., z_m$ and vanishes at ∞, and the following consequence of integration by parts

$$\operatorname*{res}_{z=z_\nu} \operatorname{tr} \left((\delta\Gamma_\nu\Gamma_\nu^{-1})' \wedge \mathcal{M}\right) \mathrm{d}z = \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\mathcal{M}' \wedge \delta\Gamma_\nu\Gamma_\nu^{-1}\right) \mathrm{d}z. \tag{1.19}$$

The proof is complete. ■

We close this section with the central definition.

Definition 1.1.5. *The* tau function $\tau(\mathbf{t})$ *is introduced according to*

$$\delta \log \tau = \Omega \tag{1.20}$$

where Ω *is the tau differential evaluated along* (1.4).

It is important to stress that the tau function just defined is not really a function; indeed (1.20) only defines $\tau(\mathbf{t})$ locally as a function of the parameters t; $\tau(\mathbf{t})$ is in general a multivalued function of the parameters $\mathbf{t}$.

The tau function of Def. 1.1.5 is of course introduced up to multiplicative constants. Moreover, recall the gauge freedom of (1.4) mentioned above which consists in multiplying Γ_ν on the right by a diagonal matrix Λ_ν of unit determinant and constant in $\mathbf{t}$; writing $\widetilde{\Gamma}_\nu = \Gamma_\nu\Lambda_\nu$ we note that the tau differential transforms as

$$\begin{aligned}\widetilde{\Omega} &= -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\widetilde{\Gamma}_\nu^{-1}\widetilde{\Gamma}_\nu'\delta\Xi_\nu\right) \mathrm{d}z = -\sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left((\Gamma_\nu^{-1}\Gamma_\nu' + \Lambda_\nu^{-1}\Lambda_\nu')\delta\Xi_\nu\right) \mathrm{d}z \\ &= \Omega - \sum_\nu \operatorname*{res}_{z=z_\nu} \operatorname{tr} \left(\Lambda_\nu^{-1}\Lambda_\nu'\delta\Xi_\nu\right) \mathrm{d}z.\end{aligned}$$

Notice however that the difference $\Omega - \widetilde{\Omega}$ is a constant and hence this amounts to a transformation

$$\widetilde{\tau}(\mathbf{t}) = \mathrm{e}^{f(\mathbf{t})}\tau(\mathbf{t}) \tag{1.21}$$

for some $f(\mathbf{t})$ linear in the times; in particular, logarithmic derivatives of order ≥ 2 are unaffected by this gauge freedom (compare with Thm. 1.2.1 and Thm. 1.2.2).

1.2 Higher order derivatives of the tau function

Remarkably, as it was first discovered in [BDYa], arbitrary logarithmic derivatives of the tau function with respect to the times $t_{\nu,k,\alpha}$ can be expressed in terms of our main ingredients $\{\Gamma_\nu, \Xi_\nu\}$ and not of their $\mathbf{t}$-derivatives.

Introduce matrices

$$R_{\nu,\alpha}(z;\mathbf{t}) := \Gamma_\nu(z;\mathbf{t})\mathrm{E}_{\alpha\alpha}\Gamma_\nu^{-1}(z;\mathbf{t}) \tag{1.22}$$

for all $\alpha = 1, ..., N$ and $\nu = 1, ..., m, \infty$.

Theorem 1.2.1. *Second logarithmic derivatives of the tau function with respect to the $t_{\nu,k}^\alpha$'s can be expressed as*

$$\frac{\partial^2 \log \tau}{\partial t_{\nu_1,k_1,\alpha_1}\partial t_{\nu_2,k_2,\alpha_2}} = \operatorname*{res}_{\xi_1=z_{\nu_1}} \operatorname*{res}_{\xi_2=z_{\nu_2}} \frac{\operatorname{tr}\left(R_{\nu_1,\alpha_1}(\xi_1;\mathbf{t})R_{\nu_2,\alpha_2}(\xi_2;\mathbf{t})\right) - \delta_{\alpha_1,\alpha_2}}{(\xi_1-\xi_2)^2} \frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{(\xi_1-z_{\nu_1})^{k_1}(\xi_2-z_{\nu_2})^{k_2}} \tag{1.23}$$

where we agree that $\frac{1}{\xi - z_\nu} := \xi$ when $\nu = \infty$.

Let us comment on the fact that the order in which the residues are carried over in (1.23) is immaterial. This is in principle not clear when $\nu_1 = \nu_2 = \nu$; however note that the function

$$\operatorname{tr}\left(R_{\nu,\alpha_1}(\xi_1;\mathbf{t})R_{\nu,\alpha_2}(\xi_2;\mathbf{t})\right) - \delta_{\alpha_1,\alpha_2} \tag{1.24}$$

vanishes when $\xi_1 = \xi_2$, because $\operatorname{tr}\left(R_{\nu,\alpha_1}(\xi_1;\mathbf{t})R_{\nu,\alpha_2}(\xi_1;\mathbf{t})\right) = \operatorname{tr}\left(\mathrm{E}_{\alpha_1\alpha_1}\mathrm{E}_{\alpha_2\alpha_2}\right) = \delta_{\alpha_1,\alpha_2}$. Now (1.24) is symmteric in ξ_1, ξ_2, therefore it must vanish to second order when $\xi_1 = \xi_2$; hence the function whose residues have to be extracted at $\xi_1 = \xi_2 = z_\nu$ in (1.23) is regular along $\xi_1 = \xi_2$ and so the residues may be switched.

Proof. Let us denote $\partial_i := \frac{\partial}{\partial t_{\nu_i,k_i,\alpha_i}}$ for $i = 1, 2$. Repeating essentially the computation of $\delta\Omega$ in (1.16), we have

$$\begin{aligned}
\partial_1\partial_2 \log\tau &= -\partial_1 \operatorname*{res}_{\xi_2=z_{\nu_2}} \operatorname{tr}\left(\Gamma_{\nu_2}^{-1}(\xi_2;\mathbf{t})\Gamma_{\nu_2}'(\xi_2;\mathbf{t})\frac{\mathrm{E}_{\alpha_2\alpha_2}}{(\xi_2-z_{\nu_2})^{k_2}}\right)\mathrm{d}\xi_2 \\
&= -\operatorname*{res}_{\xi_2=z_{\nu_2}} \operatorname{tr}\left(-\Gamma_{\nu_2}^{-1}\partial_1\Gamma_{\nu_2}\Gamma_{\nu_2}^{-1}\Gamma_{\nu_2}'\frac{\mathrm{E}_{\alpha_2\alpha_2}}{(\xi_2-z_{\nu_2})^{k_2}} + \Gamma_{\nu_2}^{-1}(\xi_2;\mathbf{t})\partial_1\Gamma_{\nu_2}'(\xi_2;\mathbf{t})\frac{\mathrm{E}_{\alpha_2\alpha_2}}{(\xi_2-z_{\nu_2})^{k_2}}\right)\mathrm{d}\xi_2 \\
&= -\operatorname*{res}_{\xi_2=z_{\nu_2}} \operatorname{tr}\left(\mathcal{M}'(\xi_2;\mathbf{t})(\partial_1)\frac{\Gamma_{\nu_2}(\xi_2;\mathbf{t})\mathrm{E}_{\alpha_2\alpha_2}\Gamma_{\nu_2}^{-1}(\xi_2;\mathbf{t})}{(\xi_2-z_{\nu_2})^{k_2}} - \left(\frac{\mathrm{E}_{\alpha_1\alpha_1}}{(\xi_2-z_{\nu_1})^{k_1}}\right)'\frac{\mathrm{E}_{\alpha_2\alpha_2}}{(\xi_2-z_{\nu_2})^{k_2}}\right)\mathrm{d}\xi_2 \\
&= -\operatorname*{res}_{\xi_2=z_{\nu_2}} \operatorname{tr}\left(\mathcal{M}'(\xi_2;\mathbf{t})(\partial_1)\frac{R_{\nu_2,\alpha_2}(\xi_2;\mathbf{t})}{(\xi_2-z_{\nu_2})^{k_2}} + k_1\frac{\mathrm{E}_{\alpha_1\alpha_1}\mathrm{E}_{\alpha_2\alpha_2}}{(\xi_2-z_{\nu_1})^{k_1+1}(\xi_2-z_{\nu_2})^{k_2}}\right)\mathrm{d}\xi_2.
\end{aligned}$$

Directly from the definition (1.5) we find

$$\begin{aligned}
\mathcal{M}'(\xi_2;\mathbf{t})(\partial_1) &= -\operatorname*{res}_{\xi_1=z_{\nu_1}} \Gamma_{\nu_1}(\xi_1;\mathbf{t})\frac{\mathrm{E}_{\alpha_1\alpha_1}}{(\xi_1-z_{\nu_1})^{k_1}}\Gamma_{\nu_1}^{-1}(\xi_1;\mathbf{t})\frac{\mathrm{d}\xi_1}{(\xi_1-\xi_2)^2} \\
&= -\operatorname*{res}_{\xi_1=z_{\nu_1}} R_{\nu_1,\alpha_1}(\xi_1;\mathbf{t})\frac{1}{(\xi_1-z_{\nu_1})^{k_1}}\frac{\mathrm{d}\xi_1}{(\xi_1-\xi_2)^2}
\end{aligned}$$

and then using the identity

$$\operatorname{tr}\left(k_1\frac{\mathrm{E}_{\alpha_1\alpha_1}\mathrm{E}_{\alpha_2\alpha_2}}{(\xi_2-z_{\nu_1})^{k_1+1}}\right) = -\delta_{\alpha_1,\alpha_2}\operatorname*{res}_{\xi_1=z_{\nu_1}}\frac{1}{(\xi_1-\xi_2)^2}\frac{1}{(\xi_1-z_{\nu_1})^{k_1}} \tag{1.25}$$

we obtain the claimed formula. ■

It turns out that we can inductively compute higher order logarithmic derivatives of the tau function in the times $t_{\nu,k}^\alpha$. To this end it is convenient to introduce the functions

$$S_r\left(\vec{\nu},\vec{\alpha};\vec{\xi}\right) := -\frac{1}{r}\sum_{\pi\in\mathfrak{S}_r}\frac{\operatorname{tr}\left(R_{\nu_{\pi(1)},\alpha_{\nu_{\pi(1)}}}(\xi_{\pi(1)};\mathbf{t})\cdots R_{\nu_{\pi(r)},\alpha_{\nu_{\pi(r)}}}(\xi_{\pi(r)};\mathbf{t})\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_{\pi(r-1)}-\xi_{\pi(r)})(\xi_{\pi(r)}-\xi_{\pi(1)})} - \frac{\delta_{r,2}\delta_{\alpha_1,\alpha_2}}{(\xi_1-\xi_2)^2} \tag{1.26}$$

where we denote $\vec{\nu} = (\nu_1, ..., \nu_r), \vec{\alpha} = (\alpha_1, ..., \alpha_r), \vec{\xi} = (\xi_1, ..., \xi_r)$.

Note that due to the cyclic property of the trace and of the denominator in each summand, the sum in the right side of (1.26) involves only $(r-1)!$ terms. More precisely, we can alternatively sum over the permutations that fix r, i.e.

$$S_r\left(\vec{\nu},\vec{\alpha};\vec{\xi}\right) := -\sum_{\pi\in\mathfrak{S}_{r-1}} \frac{\mathrm{tr}\,\left(R_{\nu_{\pi(1)},\alpha_{\nu_{\pi(1)}}}(\xi_{\pi(1)};\mathbf{t})\cdots R_{\nu_{\pi(r-1)},\alpha_{\nu_{\pi(r-1)}}}(\xi_{\pi(r-1)};\mathbf{t})R_{\nu_r,\alpha_{\nu_r}}(\xi_r;\mathbf{t})\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_{\pi(r-1)}-\xi_r)(\xi_r-\xi_{\pi(1)})} - \frac{\delta_{r,2}\delta_{\alpha_1,\alpha_2}}{(\xi_1-\xi_2)^2}. \tag{1.27}$$

E.g., for $r=2,3$ we have

$$S_2(\vec{\nu},\vec{\alpha};\vec{\xi}) = \frac{\mathrm{tr}\,(R_{\nu_1,\alpha_1}(\xi_1;\mathbf{t})R_{\nu_2,\alpha_2}(\xi_2;\mathbf{t}))-\delta_{\alpha_1,\alpha_2}}{(\xi_1-\xi_2)^2}$$
$$S_3(\vec{\nu},\vec{\alpha};\vec{\xi}) = \frac{\mathrm{tr}\,\left(R_{\nu_1,\alpha_1}(\xi_1;\mathbf{t})R_{\nu_2,\alpha_2}(\xi_2;\mathbf{t})R^{\alpha_3}_{\nu_3}(\xi_3;\mathbf{t})-R_{\nu_2,\alpha_2}(\xi_2;\mathbf{t})R_{\nu_1,\alpha_1}(\xi_1;\mathbf{t})R_{\nu_3,\alpha_3}(\xi_3;\mathbf{t})\right)}{(\xi_1-\xi_2)(\xi_1-\xi_3)(\xi_2-\xi_3)}$$

Theorem 1.2.2. *Logarithmic derivatives of the tau function with respect to the $t_{\nu,k,\alpha}$'s can be expressed for $r\geq 2$ as*

$$\frac{\partial^r\log\tau}{\partial t_{\nu_1,k_1,\alpha_1}\cdots\partial t_{\nu_r,k_r,\alpha_r}} = \operatorname*{res}_{\xi_1=z_{\nu_1}}\cdots\operatorname*{res}_{\xi_1=z_{\nu_1}} S_r(\vec{\nu},\vec{\alpha};\vec{\xi})\frac{\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r}{(\xi_1-z_{\nu_1})^{k_1}\cdots(\xi_r-z_{\nu_r})^{k_r}} \tag{1.28}$$

where, as above, we agree that $\frac{1}{\xi-z_\nu}:=\xi$ for $\nu=\infty$.

The order in which the residues are carried out in the above formula is immaterial. For $r=2$ it was explained right after the statement of Thm. 1.2.1. For $r\geq 3$ we can reason as follows. The only case to consider is when some of the ν_i's coincide, hence let us assume that $\nu_{r-1}=\nu_r=\nu$ and we want to show that (1.26) is regular for $\xi_{r-1}=\xi_r$. We introduce the convenient notation

$$R_i := R_{\nu_i,\alpha_i}(\xi_i;\mathbf{t}) \tag{1.29}$$

and, looking at (1.27), we collect only summands in $S_r(\vec{\nu},\vec{\alpha};\vec{\xi})$ which are singular for $\xi_{r-1}=\xi_r$

$$\frac{1}{\xi_r-\xi_{r-1}}\sum_{\pi\in\mathfrak{S}_{r-2}}\left(\frac{\mathrm{tr}\,\left(R_{\pi(1)}\cdots R_{\pi(r-2)}R_{r-1}R_r\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_{\pi(r-2)}-\xi_{r-1})(\xi_r-\xi_{\pi(1)})}\right.$$
$$\left.-\frac{\mathrm{tr}\,\left(R_{r-1}R_{\pi(1)}\cdots R_{\pi(r-2)}R_r\right)}{(\xi_{r-1}-\xi_{\pi(1)})\cdots(\xi_{\pi(r-2)}-\xi_r)}\right)$$

but this is manifestly regular for $\xi_{r-1}=\xi_r$, as due to the cyclic property of the trace the two terms cancel exactly when $\xi_{r-1}=\xi_r$. Therefore the residues in (1.28) may be arbitrarily interchanged.

Proof. Let us use the short notation (1.29) and denote $\partial_i := \frac{\partial}{\partial t_{\nu_i,k_i,\alpha_i}}$. Preliminarily we note that

$$\partial_{r+1}R_j = \left[\mathcal{M}\left(\partial_{r+1}\right),R_j\right] = \operatorname*{res}_{\xi_{r+1}=z_{\nu_{r+1}}}\frac{[R_{r+1},R_j]}{\xi_j-\xi_{r+1}}\frac{\mathrm{d}\xi_{r+1}}{(\xi_{r+1}-z_{\nu_{r+1}})^{k_{r+1}}} \tag{1.30}$$

and so

$$\partial_{r+1}S_r = \sum_{\pi\in\mathfrak{S}_{r-1}}\frac{\partial_{r+1}\mathrm{tr}\,\left(R_{\pi(1)}\cdots R_{\pi(r-1)}R_r\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_r-\xi_{\pi(1)})}$$
$$= \operatorname*{res}_{\xi_{r+1}=z_{\nu_{r+1}}}\sum_{\pi\in\mathfrak{S}_{r-1}}\sum_{j=1}^{r}\frac{\mathrm{tr}\,\left(R_{\pi(1)}\cdots[R_{r+1},R_{\pi(j)}]\cdots R_r\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_r-\xi_{\pi(1)})(\xi_{\pi(j)}-\xi_{r+1})}\frac{\mathrm{d}\xi_{r+1}}{(\xi_{r+1}-z_{\nu_{r+1}})^{k_{r+1}}}$$

where we set $\pi(r):=r$ for notational convenience in the j-summation. Expanding the commutator $[R_{r+1},R_{\pi(j)}]=R_{r+1}R_{\pi(j)}-R_{\pi(j)}R_{r+1}$ we note that each expression $\mathrm{tr}\,\left(R_{\pi(1)}\cdots R_{r+1}R_{\pi(j)}\cdots R_r\right)$ appears twice, with different denominators. Collecting these pairs of terms gives

$$
\begin{aligned}
&\sum_{\pi\in\mathfrak{S}_{r-1}}\sum_{j=1}^{r}\frac{\mathrm{tr}\,\left(R_{\pi(1)}\cdots[R_{r+1},R_{\pi(j)}]\cdots R_r\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_r-\xi_{\pi(1)})(\xi_{\pi(j)}-\xi_{r+1})}\\
&\quad=\sum_{\pi\in\mathfrak{S}_{r-1}}\sum_{j=1}^{r}\frac{\mathrm{tr}\,\left(R_{\pi(1)}\cdots R_{r+1}R_{\pi(j)}\cdots R_r\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_r-\xi_{\pi(1)})}\left(\frac{1}{\xi_{\pi(j)}-\xi_{r+1}}-\frac{1}{\xi_{\pi(j-1)}-\xi_{r+1}}\right)\\
&\quad=\sum_{\pi\in\mathfrak{S}_{r-1}}\sum_{j=1}^{r}\frac{\mathrm{tr}\,\left(R_{\pi(1)}\cdots R_{r+1}R_{\pi(j)}\cdots R_r\right)}{(\xi_{\pi(1)}-\xi_{\pi(2)})\cdots(\xi_{\pi(j-1)}-\xi_{r+1})(\xi_{r+1}-\xi_{\pi(j)})\cdots(\xi_r-\xi_{\pi(1)})}\\
&\quad=\sum_{\pi'\in\mathfrak{S}_{r}}\sum_{j=1}^{r}\frac{\mathrm{tr}\,\left(R_{\pi'(1)}\cdots R_{\pi'(r)}R_{r+1}\right)}{(\xi_{\pi'(1)}-\xi_{\pi'(2)})\cdots(\xi_{r+1}-\xi_{\pi'(1)})}
\end{aligned}
$$

where the last step is just a re-parametrization of the sum, using the cyclicity of the trace; explicitly we are considering the bijection

$$\mathfrak{S}_{r-1}\times\{1,...,r\}\to\mathfrak{S}_r:(\pi,j)\mapsto\pi' \tag{1.31}$$

where $\pi'\in\mathfrak{S}_r$ is defined by

$$\pi'(\ell):=\begin{cases}\pi(\ell+j-1) & 1\le\ell\le r-j\\ r & \ell=r-j+1\\ \pi(\ell-r+j-1) & r-j+2\le\ell\le r.\end{cases} \tag{1.32}$$

We can summarize the computation above as

$$\partial_{r+1}S_r(\vec{\nu};\vec{\alpha};\vec{\xi})=\operatorname*{res}_{\xi_{r+1}=z_{\nu_{r+1}}}S_{r+1}(\vec{\nu},\nu_{r+1};\vec{\alpha},\alpha_{r+1};\vec{\xi},\xi_{r+1})\frac{\mathrm{d}\xi_{r+1}}{(\xi_{r+1}-z_{\nu_{r+1}})^{k_{r+1}}} \tag{1.33}$$

and now the proof is straightforward by induction, the induction base $r=2$ being proven in Thm. 1.2.1. ■

1.3 Sato formula

There are classical formulæ computing the entries of Γ_ν in terms of the tau function. For example for diagonal entries we have

$$\frac{\tau\left(\left\{t_{\nu,k,\alpha}-\frac{\delta_{\alpha\beta}}{k(z-z_\nu)^k}\right\}\right)}{\tau\left(\{t_{\nu,k,\alpha}\}\right)}=\left(\Gamma_\nu(z;\mathbf{t})\right)_{\beta\beta} \tag{1.34}$$

for all $\beta=1,...,N$. Formulæ of this type computing all entries of Γ_ν in terms of the tau function exist, however we will deduce them as special ("elementary") cases of *Schlesinger transformations* in the next chapter, see Sec. 2.4, in particular (2.48). For the same reason we also omit the proof at this formal level, which can be found e.g. in [BBT]. Let us mention that these formulæ are relevant, among other reasons, as one can use them to establish the connection with the KP hierarchy and in general with hierarchies of *Hirota bilinear equations*, see e.g. [BBT, Sec. 8.9]. (Compare also with App. A, in particular Cor. A.2.2.)

1.4 Examples

We consider here some examples where the system of equations (1.4) arises naturally, leading to the introduction of a tau function.

1.4.1 Isospectral deformations

Lax equations. The *Lax equation* is

$$\partial_t L(z;t) = [M(z;t), L(z;t)] \tag{1.35}$$

for $N \times N$ matrices L, M which are rational in z. It defines an *isospectral deformation* in the sense that the spectrum of L is an integral of motion, as we have

$$\partial_t \mathrm{tr}\, L^k = k \mathrm{tr}\, [M, L^{k-1}] = 0. \tag{1.36}$$

In particular, the spectral curve $\det(L(z) - w) = 0$ is a constant of motion of (1.35).

Suppose that the poles of L and those of M are at $z = z_1, ..., z_m, \infty$ only, hence we decompose

$$L = \sum_\nu L_\nu, \qquad L_\nu = \operatorname*{res}_{\xi = z_\nu} \frac{L(\xi)\mathrm{d}\xi}{z - \xi} = \begin{cases} \frac{L_1^\nu}{z - z_\nu} + \cdots + \frac{L_{\ell_\nu}^\nu}{(z - z_\nu)^{\ell_\nu}} & \nu = 1, ..., m \\ L_0^\infty + L_1^\infty z + \cdots + L_{\ell_\infty}^\infty z^{\ell_\infty} & \nu = \infty \end{cases} \tag{1.37}$$

$$M = \sum_\nu M_\nu, \qquad M_\nu = \operatorname*{res}_{\xi = z_\nu} \frac{M(\xi)\mathrm{d}\xi}{z - \xi}. \tag{1.38}$$

Assuming that the leading orders $L_{\ell_\nu}^\nu$ are semisimple (we can lift this assumption, see Rem. 1.4.2) we can find analytic germs[2]

$$\Gamma_\nu(z) = \begin{cases} G_\nu(\mathbf{t}) + \mathcal{O}(z - z_\nu) & \nu = 1, ..., m \\ \mathbf{1} + \mathcal{O}(z^{-1}) & \nu = \infty \end{cases} \tag{1.39}$$

which are analytically invertible and such that

$$L_\nu = \Gamma_\nu A_\nu \Gamma_\nu^{-1} \tag{1.40}$$

where A_ν is a meromorphic germ of diagonal matrix at z_ν, with a pole of order ℓ_ν. Note that to have $\Gamma_\infty = \mathbf{1} + \mathcal{O}(z^{-1})$ we must preliminarily perform a constant global gauge transformation on L, M diagonalizing $L_{\ell_\infty}^\infty$.

The local gauge transformation $M \mapsto \Gamma_\nu B_\nu \Gamma_\nu^{-1} - \partial_t \Gamma_\nu \Gamma_\nu^{-1}$ maps (1.35) into

$$\partial_t A_\nu = [B_\nu, A_\nu] \tag{1.41}$$

hence B_ν must be diagonal (otherwise $[A_\nu, B_\nu]$ would have nonzero off-diagonal entries), hence

$$\partial_t A_\nu = 0 \tag{1.42}$$

as expected, and $M = \sum_\nu M_\nu$ in (1.38) is given by

$$M_\nu(z) = \operatorname*{res}_{\xi = z_\nu} \Gamma_\nu(\xi) B_\nu(\xi) \Gamma_\nu^{-1}(\xi) \frac{\mathrm{d}\xi}{z - \xi} \tag{1.43}$$

as the term $\partial_t \Gamma_\nu \Gamma_\nu^{-1}$ is regular at z_ν and vanishes at ∞ due to (1.39).

Elementary isospectral deformations. Summarizing, the Lax equation (1.35), for rational matrices L, M with poles at $z = z_1, ..., z_m, \infty$, implies that M is diagonalized by the same local gauge transformations as L, assuming that the leading orders $L_{\ell_\nu}^\nu$ are semisimple. M is then equal to the sum of the irregular parts at the poles $z_1, ..., z_m, \infty$ (including the constant at ∞), computed as in (1.43). This suggests to consider *elementary isospectral deformations*, where M has the simplest possible form

$$M = \operatorname*{res}_{\xi = z_\nu} \Gamma_\nu(\xi; \mathbf{t}) \frac{\mathrm{E}_{\alpha\alpha}}{(\xi - z_\nu)^k} \Gamma_\nu^{-1}(\xi; \mathbf{t}) \frac{\mathrm{d}\xi}{z - \xi} \tag{1.44}$$

for some choice of $\nu = 1, ..., m, \infty$. In the notations introduced above, such elementary isospectral deformations can be written all together as

$$\delta L = [\mathcal{M}, L], \qquad \delta z_1 = ... = \delta z_m = 0 \tag{1.45}$$

where $\mathcal{M}$ is given in (1.5), and we set $\lambda_{\nu,\alpha} = 0$, see (1.2). Note that the poles z_ν are not deformation parameters (this will be the case for *isomonodromic* deformations, see below), and that the zero-curvature condition $\delta \mathcal{M} = \mathcal{M} \wedge \mathcal{M}$ established in Prop. 1.1.2 also ensures compatibility of the system (1.45); for, we have

$$\delta[\mathcal{M}, L] = [\delta\mathcal{M}, L] - [\mathcal{M}, \delta L] = [\delta\mathcal{M}, L] - [\mathcal{M}, [\mathcal{M}, L]] = [\delta\mathcal{M} - \mathcal{M} \wedge \mathcal{M}, L] = 0. \tag{1.46}$$

[2] In this setting the Γ_ν's are not just formal series in z, they are germs of analytic functions, which extend up to the nearest branch point of the spectral curve $\det(L(z) - w) = 0$.

The isospectral tau function. Let us prove that the Lax equation (1.45) can be written in terms of Γ_ν as (1.4). To this end let us note that

$$[\mathcal{M}, L] = \delta L = [\delta\Gamma_\nu\Gamma_\nu^{-1}, L] \tag{1.47}$$

and so

$$[\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1}, L] = 0 \tag{1.48}$$

which in turn implies, as we are assuming semisimplicity of L near z_ν, that

$$\mathcal{M} - \delta\Gamma_\nu\Gamma_\nu^{-1} = \Gamma_\nu\mathcal{D}_\nu\Gamma_\nu^{-1} \tag{1.49}$$

for some diagonal differentials $\mathcal{D}_\nu$. More precisely, $\mathcal{D}_\nu$ is a formal Laurent series in $z - z_\nu$ for $\nu = 1, ..., m$ and in z^{-1} for $\nu = \infty$, whose irregular part coincides with $\delta\Xi_\nu$, see (1.49) and (1.5);

$$\operatorname*{res}_{\xi=z_\nu} \mathcal{D}_\nu(\xi;\mathbf{t})\frac{\mathrm{d}\xi}{z-\xi} = \delta\Xi_\nu(z;\mathbf{t}). \tag{1.50}$$

The zero curvature condition $\delta\mathcal{M} = \mathcal{M}\wedge\mathcal{M}$ implies $\delta\mathcal{D}_\nu = \mathcal{D}_\nu\wedge\mathcal{D}_\nu = 0$, hence $\mathcal{D}_\nu = \delta\mathcal{E}_\nu$ for some $\mathcal{E}_\nu$ diagonal analytic series in z (of unit determinant). Note that the Γ_ν's are defined only by the requirement that $\Gamma_\nu^{-1}L_\nu\Gamma_\nu$ is diagonal and so they are defined only up to right multiplication by a diagonal matrix. This gauge freedom $\Gamma_\nu \mapsto \Gamma_\nu\Lambda_\nu$ implies the following gauge freedom on $\mathcal{D}_\nu$

$$\mathcal{D}_\nu \mapsto \mathcal{D}_\nu + \Lambda_\nu^{-1}\delta\Lambda_\nu \tag{1.51}$$

and therefore also the following gauge freedom on $\mathcal{E}_\nu$

$$\mathcal{E}_\nu \mapsto \mathcal{E}_\nu + \log\Lambda_\nu. \tag{1.52}$$

Then it is clear that we can choose Λ_ν so to kill the regular part of $\mathcal{E}_\nu$ and due to (1.50) and (1.49) we conclude that in this gauge the Γ_ν's must satisfy (1.4). Therefore, under this gauge fixing of the Γ_ν's, we can introduce a tau function by Thm. 1.1.4.

Hamiltonian aspects. The Lax equation (1.35) admits an hamiltonian representation. For simplicity we consider the case where the only pole is at $z = \infty$, but this discussion generalizes straightforwardly to the multipole case. Introduce the Lie algebra $\mathfrak{g} := \mathfrak{gl}_N \otimes z^{-1}\mathbb{C}[\![z^{-1}]\!]$. It is associated with the *loop group*, formed by the elements $\Gamma(z) = \mathbf{1} + \sum_{j\geq 1}\Gamma_j z^{-j}$ with the usual multiplication. The space $\mathfrak{g}^* := \mathfrak{gl}_N \otimes \mathbb{C}[z]$ can be injected into the linear dual of $\mathfrak{g}$ via the nondegenerate pairing

$$\langle L, X\rangle := \operatorname*{res}_{z=\infty} \operatorname{tr}\left(L(z)X(z)\right)\mathrm{d}z, \qquad L = L(z) \in \mathfrak{g}^*, X = X(z) \in \mathfrak{g}. \tag{1.53}$$

Therefore the Lax matrix $L(z)$ is an element of $\mathfrak{g}^*$. We claim that the Lax equation (1.35) is hamiltonian with respect to the Lie–Poisson bracket on $\mathfrak{g}^*$. Let us remind that the Lie–Poisson bracket on $\mathfrak{g}^*$ is defined by

$$\{f, g\}(L) := L([\mathrm{d}f, \mathrm{d}g]) \tag{1.54}$$

for all f, g smooth functions on $\mathfrak{g}^*$; indeed $\mathrm{d}f, \mathrm{d}g$ are linear functionals on $\mathfrak{g}^*$ hence they belong to $\mathfrak{g}$[3]. We claim that the hamiltonian

$$H = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(A\mathrm{E}_{\alpha\alpha}z^k\right)\mathrm{d}z \tag{1.55}$$

generates the Lax equation (1.35), with respect to the Lie–Poisson bracket. To prove this claim we first compute the differential of H as follows; given the variation $L \mapsto L + \mathrm{d}L$ we have

$$A \mapsto A + \mathrm{d}A, \qquad \mathrm{d}A = \Gamma^{-1}\mathrm{d}L\Gamma - [\Gamma^{-1}\mathrm{d}\Gamma, A] \tag{1.56}$$

and so

$$H \mapsto H + \mathrm{d}H, \qquad \mathrm{d}H = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\mathrm{E}_{\alpha\alpha}z^k\mathrm{d}A\right)\mathrm{d}z = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\Gamma\mathrm{E}_{\alpha\alpha}\Gamma^{-1}\mathrm{d}L\right)z^k\mathrm{d}z = \langle M, L\rangle \tag{1.57}$$

[3] There are some inconsequential subtleties and possibly misleading notations here, due to infinite-dimensionality. $\mathfrak{g}^*$ is not properly the linear dual of $\mathfrak{g}$, it just injects in it. However, $\mathfrak{g}$ is the linear dual of $\mathfrak{g}^*$, hence the Lie–Poisson bracket is well defined.

therefore $\mathrm{d}H = M$. Writing each entry of L as $L_{ab} = \sum_{k\geq 0} L^k_{ab} z^k$, the coordinates L^k_{ab} on $\mathfrak{g}^*$ are identified with $-\mathrm{E}_{ba} z^{-k-1} \in \mathfrak{g}$, as for the pairing (1.53) we have

$$L^k_{ab} = \langle L, -\mathrm{E}_{ba} z^{-k-1} \rangle. \tag{1.58}$$

Therefore the Hamilton equation

$$\begin{aligned}\partial_t L^k_{ab} = \{H, L^k_{ab}\} = \langle L, [\mathrm{d}H, -\mathrm{E}_{ba} z^{-k-1}]\rangle &= -\operatorname*{res}_{z=\infty} \operatorname{tr}\left(L[M, \mathrm{E}_{ba}]\right) z^{-k-1}\mathrm{d}z \\ &= -\operatorname*{res}_{z=\infty} \operatorname{tr}\left([M, L]\mathrm{E}_{ba}\right) z^{-k-1}\mathrm{d}z\end{aligned}$$

is equivalent to the Lax equation (1.35) $\partial_t L = [M, L]$, and the claim is proved.

It is well known that the Lie–Poisson bracket is degenerate, and that its symplectic leaves are the coadjoint orbits

$$\mathcal{O}^*_A = \left\{ \operatorname*{res}_{\xi=\infty} \Gamma(\xi) A(\xi) \Gamma^{-1}(\xi) \frac{\mathrm{d}\xi}{z-\xi} \right\} \tag{1.59}$$

where A is in $\mathfrak{g}^*$ and Γ in the loop group. This agrees with the fact that the spectrum of L is invariant under the Lax flow. The restriction of the Lie–Poisson bracket to a coadjoint orbit is nondegenerate so it comes associated with a symplectic form ω, which reads in general

$$\omega(\operatorname{ad}^*_x \eta, \eta^*_y \eta) = \eta([x, y]) \tag{1.60}$$

(denoting ad^* the infinitesimal coadjoint action) and it is called Kirillov–Konstant symplectic form. Parametrizing the coadjoint $\mathcal{O}^*_A$ by the loop group element Γ we can write the symplectic form at the point $L = \operatorname*{res}_{\xi=\infty} \Gamma(\xi) A(\xi) \Gamma^{-1}(\xi) \frac{\mathrm{d}\xi}{z-\xi}$ as

$$\omega = \langle \mathrm{d}\Gamma\Gamma^{-1} \wedge \mathrm{d}\Gamma\Gamma^{-1}, L\rangle = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(A\Gamma^{-1}\mathrm{d}\Gamma \wedge \Gamma^{-1}\mathrm{d}\Gamma\right)\mathrm{d}z. \tag{1.61}$$

The regular (i.e. L_ℓ semisimple) coadjoint orbit $\mathcal{O}^*_L$ through $L = L_0 + L_1 z + \cdots + L_\ell z^\ell$ can be parametrized by the germ $\Gamma = \mathbf{1} + \Gamma_1 z^{-1} + \cdots + \Gamma_\ell z^{-\ell}$ and up to diagonal germs of the same order, hence the dimension of the orbit is $N(N-1)\ell$ (in fact it is even).

One can further perform a symplectic reduction with respect to the conjugation of L by a *constant* (in z) diagonal matrix, which is a symmetry of every Lax equation (1.35). Letting $\mathfrak{d}$ the Lie algebra of traceless diagonal matrices, associated with the group of unit-determinant diagonal $N \times N$ matrices, the moment map $\mu :\in \mathcal{O}^*_L \to \mathfrak{d}^*$ of this hamiltonian action is given as

$$\mu_{\widetilde{L}} = \operatorname*{res}_{z=\infty} \operatorname{tr}(\widetilde{A}D)\mathrm{d}z \tag{1.62}$$

where $\widetilde{L} \in \mathcal{O}^*_L$ is $\widetilde{L}(z) = \operatorname*{res}_{\xi=\infty} \Gamma(\xi)\widetilde{A}(\xi)\Gamma^{-1}(\xi)\frac{\mathrm{d}\xi}{z-\xi}$, with $\widetilde{A}$ diagonal; this can be shown by the same computations above in the proof of the hamiltonian representation of the Lax equation. The symplectic quotient of the coadjoint orbit with respect to this moment map is the relevant *phase space* of isospectral deformations; the dimension of this phase space is computed as $N(N-1)\ell - 2(N-1) = (N-1)(N\ell-2)$. We will see an example below.

Isospectral tau functions and theta functions. It can be shown that the spectral curve $\det(L(z) - w) = 0$ (compactified as usual for algebraic plane curves) is a Riemann surface of genus $\frac{1}{2}(N-1)(N\ell-2)$, half the dimension of the phase space introduced above. We remind that this Riemann surface is invariant under the Lax flow (1.35). Moreover, it can be shown that the solution L, as well as the associated isospectral tau function, can be expressed in terms of theta functions on the spectral curve. For the general situation we refer to the literature [BBT, Chap. 5], and we content ourselves with one simple example.

Example 1.4.1. *Fix $N = 2$ and consider*

$$L(z) = L_0 + L_1 z + L_2 z^2 \tag{1.63}$$

with L_2 semisimple. Without loss of generality we can assume $L(z)$ is traceless ($\operatorname{tr} L$ is constant along the Lax flow, and adding a scalar constant in t to the Lax matrix L does not affect the Lax equation

$\partial_t L = [M, L]$); hence without loss of generality we set $L_2 = \frac{1}{2}\sigma_3$. As explained above, the coadjoint orbit through L has dimension 4; using the following parametrization with coordinates x_1, y_1, x_2, y_2

$$\Gamma(z) = \mathbf{1} + \begin{bmatrix} 0 & x_1 \\ y_1 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & x_2 \\ y_2 & 0 \end{bmatrix} z^{-2} + \cdots \tag{1.64}$$

we have

$$L(z) = \Gamma(z)A(z)\Gamma^{-1}(z), \qquad A(z) = \left(z^2 + a_1 z + a_0 + a_{-1}z^{-1} + \cdots\right)\frac{\sigma_3}{2}. \tag{1.65}$$

Up to the rescaling $z \mapsto z - \frac{a_1}{2}$ and renaming the a_j's, we can assume $a_1 = 0$. Hence

$$L(z) = \begin{bmatrix} \frac{z^2}{2} + \frac{a_0}{2} + x_1 y_1 & -zx_1 - x_2 \\ zy_1 + y_2 & -\frac{z^2}{2} - \frac{a_0}{2} - x_1 y_1 \end{bmatrix} \tag{1.66}$$

Note that the condition that $L = \Gamma A \Gamma^{-1}$ is a polynomial uniquely determines $a_{-1}, a_{-2}, \dots$, as well as the higher order terms in (1.64). E.g.

$$a_{-1} = -2(x_1 y_2 + x_2 y_1), \qquad a_{-2} = 2(x_1^2 y_1^2 - x_2 y_2 + a_0 x_1 y_1). \tag{1.67}$$

Moreover, we can compute the Kirillov–Konstant symplectic form (1.61) from

$$\begin{aligned} \Gamma^{-1}\mathrm{d}\Gamma &= \begin{bmatrix} 0 & \mathrm{d}x_1 \\ \mathrm{d}y_1 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} -x_1\mathrm{d}y_1 & \mathrm{d}x_2 \\ \mathrm{d}y_2 & -y_1\mathrm{d}x_1 \end{bmatrix} z^{-2} + \cdots, \\ \Gamma^{-1}\mathrm{d}\Gamma \wedge \Gamma^{-1}\mathrm{d}\Gamma &= z^{-2}\sigma_3 \mathrm{d}x_1 \wedge \mathrm{d}y_1 \\ &\quad + z^{-3}\left(\sigma_3\left(\mathrm{d}x_1 \wedge \mathrm{d}y_2 + \mathrm{d}x_2 \wedge dy_1\right) + \left(x_1\sigma_+ - y_1\sigma_-\right)\mathrm{d}x_1 \wedge \mathrm{d}y_1\right) + \cdots \end{aligned}$$

as

$$\omega = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(A(z)\Gamma^{-1}\mathrm{d}\Gamma \wedge \Gamma^{-1}\mathrm{d}\Gamma\right) = \mathrm{d}y_1 \wedge \mathrm{d}x_2 + \mathrm{d}y_2 \wedge \mathrm{d}x_1.$$

The only nonzero Poisson brackets are

$$\{y_1, x_2\} = \{y_2, x_1\} = 1. \tag{1.68}$$

As an example we shall consider the Lax flow

$$\dot{L} = [M, L], \qquad M = \operatorname*{res}_{\xi=\infty} \Gamma(\xi)\frac{\sigma_3}{2}\Gamma^{-1}(\xi)\frac{\xi \mathrm{d}\xi}{z - \xi} = \begin{bmatrix} \frac{z}{2} & -x_1 \\ y_1 & -\frac{z}{2} \end{bmatrix} \tag{1.69}$$

denoting $\dot{f} := \partial_t f$ throughout this example. This flow is written down explicitly as

$$\dot{x}_1 = x_2, \quad \dot{y}_1 = -y_2, \quad \dot{x}_2 = -a_0 x_1 - 2x_1^2 y_1, \quad \dot{y}_2 = a_0 y_1 + 2x_1 y_1^2. \tag{1.70}$$

As predicted by the general theory, (1.70) is hamiltonian with respect to the bracket (1.68), with hamiltonian

$$H = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(A(z)\frac{\sigma_3}{2}z\right)\mathrm{d}z = -\frac{a_{-2}}{2} = -x_1^2 y_1^2 + x_2 y_2 - a_0 x_1 y_1. \tag{1.71}$$

To solve these equations we perform the aforementioned symplectic reduction of (1.70) with respect to the hamiltonian action of constant diagonal conjugation. This hamiltonian action is generated by the Hamiltonian flow

$$\begin{cases} \dot{x}_i = x_i = \{F, x_i\} & (i = 1, 2) \\ \dot{y}_i = -y_i = \{F, y_i\} & (i = 1, 2) \end{cases}, \qquad F := \operatorname*{res}_{z=\infty} \operatorname{tr}\left(A(z)\frac{\sigma_3}{2}\right)\mathrm{d}z = -\frac{a_{-1}}{2} = x_1 y_2 + x_2 y_1. \tag{1.72}$$

The quotient under the conjugation action can be parametrized (away from $x_1 = 0$) with reduced variables

$$\widetilde{y}_1 = x_1 y_1, \qquad \widetilde{y}_2 = x_1 y_2, \qquad \widetilde{x}_2 = \frac{x_2}{x_1} \tag{1.73}$$

and the Poisson bracket of these reduced variables is

$$\{\widetilde{y}_1, \widetilde{x}_2\} = 1, \qquad \{\widetilde{x}_2, \widetilde{y}_2\} = \widetilde{x}_2, \qquad \{\widetilde{y}_1, \widetilde{y}_2\} = -\widetilde{y}_1. \tag{1.74}$$

F, H descend to well defined functions $\widetilde{F}, \widetilde{H}$ on the quotient

$$\widetilde{F} = \widetilde{y}_2 + \widetilde{y}_1 \widetilde{x}_2, \qquad \widetilde{H} = -\widetilde{y}_1^2 + \widetilde{x}_2 \widetilde{y}_2 - a_0 \widetilde{y}_1. \tag{1.75}$$

$\widetilde{F}$ is a Casimir of the reduced Poisson bracket (1.74); the symplectic leaf $\widetilde{F} = f$ is parametrized by Darboux coordinates $\widetilde{y}_1, \widetilde{x}_2$ and the flow (1.70) is given by the reduced Hamiltonian

$$\widetilde{H}_f := -\widetilde{y}_1^2 - a_0 \widetilde{y}_1 + f \widetilde{x}_2 - \widetilde{x}_2^2 \widetilde{y}_1 \tag{1.76}$$

obtained by the substitution $\widetilde{y}_2 = f - \widetilde{y}_1 \widetilde{x}_2$. Performing the canonical change of variables

$$\widetilde{y}_1 = p - \frac{a_0}{2} - \frac{q^2}{2}, \qquad \widetilde{x}_2 = -q, \qquad \{p, q\} = 1 \tag{1.77}$$

the reduced Hamiltonian and the reduced equations of motion (1.70) read

$$\widetilde{H}_f = -p^2 + \frac{q^4}{4} + \frac{a_0}{2} q^2 - f q + \frac{a_0^2}{4}, \qquad \begin{cases} \dot{q} = \frac{\partial \widetilde{H}_f}{\partial p} = -2p \\ \dot{p} = -\frac{\partial \widetilde{H}_f}{\partial q} = -q^3 - a_0 q + f. \end{cases} \tag{1.78}$$

Using the first integral $\widetilde{H}_f = E$, we obtain q up to quadratures in elliptic functions as

$$t - t_0 = \int_{q(t_0)}^{q(t)} \frac{\mathrm{d}q}{\sqrt{q^4 + 2a_0 q^2 - 4fq + a_0^2 - 4E}}. \tag{1.79}$$

Then $p = -\frac{\dot{q}}{2}$, variables $\widetilde{x}_2, \widetilde{y}_1, \widetilde{y}_2$ are found by direct substitution, and original variables x_1, y_1, x_2, y_2 are recovered by (1.73), where x_1 is found from (1.70) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \log x_1 = \frac{\dot{x}_1}{x_1} = \frac{x_2}{x_1} = \widetilde{x}_2. \tag{1.80}$$

Finally, using(1.66) we compute the spectral curve $\det(L(z) - w) = 0$

$$w^2 = \frac{z^4}{4} + \frac{a_0}{2} z^2 + \frac{a_{-1}}{2} z + \frac{a_{-2}}{2} + \frac{a_0^2}{4}. \tag{1.81}$$

It coincides with the elliptic curve $\widetilde{H}_f = E$ by $w \leftrightarrow p, z \leftrightarrow q$ (recall that $f = -\frac{a_{-1}}{2}$ and $E = -\frac{a_{-2}}{2}$). The coincidence with the elliptic curve in the solution (1.79) of the isospectral deformation is a manifestation of the general fact that isospectral deformation equations can be solved in terms of theta functions on the spectral curve; more precisely, the flow linearizes on the Jacobian of the spectral curve and this is probably one of the most crucial points in the whole theory of integrable systems.

Remark 1.4.2. *It is not possible to find Γ_ν if $L^{\nu}_{\ell_\nu}$ is not semisimple; geometrically this happens when $z = z_\nu$ is a branch point of the spectral curve $\det(L(z) - w) = 0$, considered as a ramified cover of the z-plane. One can nevertheless generalize the previous discussion; the completely general case is quite involved, so we shall focus here on an example which will be of particular interest in the following of this thesis. Namely, we restrict to the case in which $L(z)$ has only one pole at $z = 0$ (and vanishes at ∞) with a maximally non semisimple leading order, i.e.*

$$L(z) = \sum_{j=1}^{\ell} L_j z^{-j} \tag{1.82}$$

where L_ℓ is a single $N \times N$ Jordan block with eigenvalue 0

$$L_\ell = \left(\begin{array}{c|ccc} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & \cdots & 0 \end{array} \right) \tag{1.83}$$

and we work under the genericness assumption that $(L_{\ell-1})_{N,1}$ (the Lidskii pseudovalue *[Lb]) is nonzero; up to rescaling z we assume $(L_{\ell-1})_{N,1} = 1$, i.e. we assume*

$$L_{\ell-1} = \left(\begin{array}{c|c} & \\ & \\ \hline 1 & \end{array} \right). \tag{1.84}$$

We perform the shear transformation

$$\widetilde{L}(z) := z^{-\frac{1}{N}} z^{-S} L(z) z^{S}, \quad S := \frac{1}{N}\mathrm{diag}(0,1,...,N-1) \tag{1.85}$$

and now

$$\widetilde{L}(z) = \Pi z^{-\ell}(\mathbf{1} + \mathcal{O}(z^{\frac{1}{N}})), \qquad \Pi = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \tag{1.86}$$

has a semisimple leading order Π. *Therefore, up to considering the variable* $z^{1/N}$ *in place of* z *the theory of isospectral deformations proceeds similarly to the generic case treated above.*

1.4.2 Isomonodromic deformations

Preliminaries on linear matrix ODEs with rational coefficients. We consider a linear matrix ODE with rational coefficients

$$\Psi' = L\Psi \tag{1.87}$$

where $L(z)$ is, as before, a rational matrix with poles only at $z = z_1, ..., z_m, \infty$, see (1.37). Note that now it is best to regard $L(z)$ as a differential $L(z)\mathrm{d}z$, therefore some care must be paid about the point at ∞, considering that $\mathrm{d}z$ has a double pole at $z = \infty$.

As before, up to assuming semisimplicity of $L^{\infty}_{\ell_\infty}$ (however one can proceed similarly as in Rem. 1.4.2 to lift this assumption) we can perform a constant gauge transformation diagonalizing $L^{\infty}_{\ell_\infty}$.

We recall the following standard facts about (1.87); for their proof and more details we refer to the literature, e.g. [JMU; HS; FIKN].

1. Let z_0 be any regular point of the differential $L(z)\mathrm{d}z$ and Ψ_0 a constant $N \times N$ invertible matrix. There exists a unique germ at z_0 of fundamental matrix solution $\Psi(z)$ to (1.87) such that $\Psi(z_0) = \Psi_0$. As the ODE is linear, this germ $\Psi(z)$ can be analytically continued to the whole universal cover of $\mathbb{C} \setminus \{z_1, ..., z_m\}$. In particular, analytic continuation around closed loops yields the monodromy representation;

$$M : \pi_1(\mathbb{C} \setminus \{z_1, ..., z_m\}; z_0) \to \mathrm{GL}_N : [\gamma] \mapsto \Psi_0^{-1}\Psi(z_0 + [\gamma]) =: M_{[\gamma]} \tag{1.88}$$

where $\Psi(z_0 + [\gamma])$ denotes the analytic continuation of $\Psi(z)$ along the homotopy class $[\gamma]$ of the loop γ based at z_0. The monodromy representation M is a group anti-homomorphism, i.e. $M_{[\gamma_1][\gamma_2]} = M_{[\gamma_2]}M_{[\gamma_1]}$; moreover M transforms by conjugation if we change the initial value Ψ_0 and/or the base-point a_0. We call $M_\nu := M_{[\gamma_\nu]}$ for a simple loop γ_ν encircling z_ν and no other pole, in counter-clockwise direction; as we are on the Riemann sphere we have the relation in the fundamental group

$$\gamma_1 \cdots \gamma_m = \gamma_\infty \tag{1.89}$$

implying the constraint

$$M_m \cdots M_1 = M_\infty. \tag{1.90}$$

2. We can formally solve (1.87) near any singularity $z_1, ..., z_m, \infty$ by the ansatz

$$\Psi_\nu = \Gamma_\nu \mathrm{e}^{\Xi_\nu(z)} \tag{1.91}$$

where $\Gamma_\nu(z)$ as in (1.3) for $\nu = 1, ..., m, \infty$ and $\Xi_\nu(z)$ as in (1.2); here the matrices Γ_ν are constant invertible matrices diagonalizing the leading orders $L^{\nu}_{\ell_\nu}$. This formal solution exists and is unique (once the G_ν's have been fixed) under the assumption of semisimplicity of the $L^{\nu}_{\ell_\nu}$'s and of *nonresonance*:

when $L(z)$ has a simple pole at $z = z_\nu$, $L^{\nu}_{\ell_\nu} = L^{\nu}_1$ is semisimple with eigenvalues distinct modulo integers.

This means that the numbers $\lambda_{\nu,\alpha}$ and $t_{\nu,k,\alpha}$ in (1.2) are completely determined from the equation $\Psi' = L\Psi$, as well as the terms $\Gamma^{(j)}_\nu$ in (1.3) (the nonresonance condition, and in general the semisimplicity of the leading order, ensure that the $\Gamma^{(j)}_\nu$'s are computed by a well defined recursion). If $L(z)\mathrm{d}z$ has a pole of order $r_\nu + 1$ ($r_\nu \geq 0$ "Poincaré rank") at z_ν (as a differential) then $t_{\nu,k,\alpha} \neq 0$ only for $k \leq r_\nu$.

3. When $L(z)\mathrm{d}z$ has a simple pole at z_ν (i.e. $r_\nu = 0$, "regular singularity") the formal solution (1.91) actually converges, and it defines therefore a genuine (possibly multivalued) analytic solution to (1.87) in a neighborhood of z_ν. Due to linearity such solutions extend to the whole universal cover of $\mathbb{C} \setminus \{z_1, ..., z_m\}$. In particular, such solutions can be compared with any chosen solution $\Psi(z)$, i.e. (implying analytic continuation)
$$\Psi(z) = \Gamma_\nu(z)\mathrm{e}^{\Xi_\nu(z)}C_\nu \tag{1.92}$$
for some $C_\nu \in \mathrm{GL}_N$, called *connection matrices*. It is convenient to compare everything with the solution at $z = \infty$, i.e. to require $C_\infty = \mathbf{1}$. Moreover, due to the form of Ξ_ν, see (1.2), we find the following expression for the monodromy M_ν around z_ν;
$$M_\nu = C_\nu^{-1}\mathrm{e}^{2\pi\mathrm{i}\,\mathrm{diag}(\lambda_{\nu,1},...,\lambda_{\nu,N})}C_\nu. \tag{1.93}$$
Correspondingly, the numbers $\lambda_{\nu,1}, ..., \lambda_{\nu,N}$ are called *formal monodromy exponents* in this context; it can be proved that they coincide with the eigenvalues of the residue of L at the simple pole $z = z_\nu$.

4. When $L(z)\mathrm{d}z$ has a pole of order greater than 1 (i.e. $r_\nu > 0$, "irregular singularity") the formal solution (1.91) does not converge. It represents however asymptotic expansion of genuinely analytic solutions in suitable sectors. To be precise on this point, let us introduce the notation $\mathcal{R} := \mathbb{C} \setminus \{z_1, ..., z\}$ and $\widetilde{\mathcal{R}}$ for the universal cover of $\mathcal{R}$. Fix $2r_\nu + 1$ open sectors $\mathscr{S}_{\nu,1}, ..., \mathscr{S}_{\nu,2r_\nu+1}$ in $\widetilde{\mathcal{R}}$ with vertex at z_ν and opening angle slightly more than $\frac{\pi}{r_\nu}$ such that their union projected down to $\mathcal{R}$ is a punctured neighborhood of z_ν, and such that every pair of non-adjacent sectors do not intersect in $\widetilde{\mathcal{R}}$; let us also assume that $\mathscr{S}_{\nu,1}$ and $\mathscr{S}_{\nu,2r_\nu+1}$ project down to the same sector in $\mathcal{R}$. Then, for every $\ell = 1, ..., 2r_\nu + 1$ there exists an analytic solution $\Psi_{\nu,\ell}$ of $\Psi' = L\Psi$ such that $\Psi_{\nu,\ell} \sim \Psi_\nu$ as $z \to z_\nu$ in $\widetilde{\mathcal{R}}$ within $\mathscr{S}_{\nu,\ell}$, where $\Psi_\nu = \Gamma_\nu \mathrm{e}^{\Xi_\nu}$ is the formal solution (1.91). We can compare these analytic solutions in adjacent sectors as
$$\Psi_{\nu,\ell+1} = \Psi_{\nu,\ell}S_{\nu,\ell}, \qquad \ell = 1, ..., 2r_\nu \tag{1.94}$$
for some $S_{\nu,\ell} \in \mathrm{GL}_N$ ($\ell = 1, ..., 2r_\nu$), called *Stokes' matrices*.

   Let us now explain the triangularity of Stokes' matrices. Multiplication on the right by $S_{\nu,\ell}$ takes a linear combination of the columns of $\Psi_{\nu,\ell}$, and since in the overlap of adjacent sectors both $\Psi_{\nu,\ell+1}$ and $\Psi_{\nu,\ell}$ have the same asymptotic expansion Ψ_ν, it follows that $S_{\nu,\ell}$ can only add to a column α of $\Psi_{nu,\ell}$ some scalar multiple of another column β of $\Psi_{\nu,\ell}$ which is *subleading* in the overlap of the sectors, i.e. α, β must be such that $\mathrm{Re}\,\frac{t_{\nu,r_\nu,\beta}}{(z-z_\nu)^{r_\nu}} < \mathrm{Re}\,\frac{t_{\nu,r_\nu,\alpha}}{(z-z_\nu)^{r_\nu}}$ for all $z \in \mathscr{S}_{\nu,\ell} \cap \mathscr{S}_{\nu,\ell+1}$. More explicitly, if we define a total order (depending on ν, ℓ) on the set $\{1, ..., N\}$ by saying $\alpha \preceq \beta$ if and only if $\mathrm{Re}\,\frac{t_{\nu,r_\nu,\beta}}{(z-z_\nu)^{r_\nu}} < \mathrm{Re}\,\frac{t_{\nu,r_\nu,\alpha}}{(z-z_\nu)^{r_\nu}}$ for all $z \in \mathscr{S}_{\nu,\ell} \cap \mathscr{S}_{\nu,\ell+1}$, then
$$S_{\nu,\ell} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \preceq \beta. \end{cases} \tag{1.95}$$
   We can define connection matrices also in this case, by
$$\Psi_{\infty,1} = \Psi_{\nu,1}C_\nu. \tag{1.96}$$
   Finally, we note that traversing all these sectors from $\mathcal{S}_{\nu,1}$ to $\mathscr{S}_{\nu,2r_\nu+1}$ in $\widetilde{\mathcal{R}}$ we come back to the same sector in $\mathcal{R}$ and so we infer the following refined expression for the monodromy matrices M_ν, generalizing (1.93);
$$M_\nu = C_\nu^{-1}\mathrm{e}^{2\pi\mathrm{i}\,\mathrm{diag}(\lambda_{\nu,1},...,\lambda_{\nu,N})}S_{\nu,2r_\nu}^{-1}\cdots S_{\nu,1}^{-1}C_\nu. \tag{1.97}$$

Monodromy map, essential monodromy map, and isomonodromic deformations. The correct point of view on monodromy is then to look at the *refined* (usually called *generalized*) *monodromy data* suggested by (1.97), consisting of formal exponents $\lambda_{\nu,\alpha}$, connection matrices C_ν, and Stokes' matrices $S_{\nu,\ell}$. We say correct in the sense that if two ODEs share the same generalized monodromy data then they coincide. We now go for a more precise formulations of this statement.

Let $\mathscr{L} = \{L(z)\}$ be the set of all rational matrices $L(z)$ such that the differential $L(z)\mathrm{d}z$ has poles at m finite points and ∞ only, of orders $r_\nu + 1$ ($r_\nu \geq 0$, $\nu = 1, ..., m, \infty$), and such that the leading orders at these singularities are semisimple, and with eigenvalues distinct modulo integers if the pole is simple ($r_\nu = 0$). Then we have a well-defined *monodromy map* which maps $L \in \mathscr{L}$ to the collection of generalized monodromy data $(\mathbf{t}, \mathbf{S}, \mathbf{C}, \boldsymbol{\lambda})$ introduced above, where we denote

$$\mathbf{S} = (S_{\nu,1}, ..., S_{\nu,2r_\nu})_{\nu=1,...,m,\infty} \tag{1.98}$$
$$\mathbf{C} = (C_1, ..., C_m) \tag{1.99}$$
$$\boldsymbol{\lambda} = (\lambda_{\nu,1}, ..., \lambda_{\nu,N})_{\nu=1,...,m,\infty} \tag{1.100}$$

and as before

$$\mathbf{t} = (\{t_{\nu,k,\alpha}\}_{\nu=1,...,m,\infty,\ k=1,...,r_\nu,\ \alpha=1,...,N}, \{z_\nu\}_{\nu=1,...,m}) \tag{1.101}$$

where $z_1, ..., z_m \in \mathbb{C}$ are the poles of $L \in \mathscr{L}$. It can be proven that the monodromy map is injective, namely if two $L_1, L_2 \in \mathscr{L}$ have the same generalized monodromy data then $L_1 = L_2$. The proof of this fact goes roughly as follows; let L_1, L_2 have the same generalized monodromy data, then let Ψ_1, Ψ_2 be the solutions of $\Psi_i' = L_i \Psi_i$ for $i = 1, 2$ normalized at the same regular point $z_0 \in \mathbb{C}$, $\Psi_i(z_0) = \Psi_0$ for some fixed $\Psi_0 \in \mathrm{GL}_N$ (Ψ_i are holomorphic in the universal cover of $\mathbb{C} \setminus \{z_1, ..., z_m\}$); consider the matrix ratio $g := \Psi_1 \Psi_2^{-1}$. It easily follows from the definition of generalized monodromy data that $g(z)$ actually extends to an entire function on the Riemann sphere; for instance, $g(z+[\gamma_\nu]) = \Psi_1 M_\nu M_\nu^{-1} \Psi_2^{-1} = g(z)$ due to the fact that Ψ_1, Ψ_2 have the same monodromy around z_ν, and so $g(z)$ is meromorphic on the Riemann sphere with poles at worst at the $z_\nu' s$, and similarly we can actually prove that the latter are removable singularities of g. Finally, we conclude by Liouville theorem that g is a constant, $g(z) \equiv g(z_0) = \mathbf{1}$, i.e. $\Psi_1 = \Psi_2$. Therefore also $L_1 = \Psi_1' \Psi_1^{-1} = \Psi_2' \Psi_2^{-1} = L_2$ and the claim is proved. For more details the reader is referred to the literature, see e.g. [FIKN].

Define the *essential* mondromy data of $L \in \mathscr{L}$ as the set $(\mathbf{S}, \mathbf{C}, \boldsymbol{\lambda})$, i.e. we are neglecting the position of the poles of L and the parameters $t_{\nu,k,\alpha}$. The goal of isomonodromic deformations is to describe the fibers of the essential monodromy map, i.e. to describe how should $L \in \mathscr{L}$ depend on $\mathbf{t}$ in such a way that the essential monodromy stays constant in $\mathbf{t}$. In this case we say that L depends isomonodromically on $\mathbf{t}$. Note that, contrarily to the isospectral case (1.45), the finite poles $z_1, ..., z_m$ are now deformation parameters.

This problem of describing isomonodromic deformations of L has a long history, as it dates back to Riemann in the simplest cases, and was fully addressed and solved in complete generality in [JMU]. The result is that L depends isomonodromically on $\mathbf{t}$ if and only if L satisfies the following compatible system of nonlinear ODEs;

$$\delta L = [\mathcal{M}, L] + \mathcal{M}' \tag{1.102}$$

where $\mathcal{M}$ has been introduced in (1.5). The compatibility of (1.102) follows from the zero curvature equation $\delta\mathcal{M} = \mathcal{M} \wedge \mathcal{M}$ of Prop. 1.1.2; indeed

$$\begin{aligned}\delta[\mathcal{M}, L] + \delta\mathcal{M}' &= [\delta\mathcal{M}, L] - [\mathcal{M}, \delta L] + \delta\mathcal{M}' = [\delta\mathcal{M}, L] - [\mathcal{M}, [\mathcal{M}, L]] - [\mathcal{M}, \mathcal{M}'] + \delta\mathcal{M}' \\ &= [\delta\mathcal{M} - \mathcal{M} \wedge \mathcal{M}, L] + (\delta\mathcal{M} - \mathcal{M} \wedge \mathcal{M})' = 0.\end{aligned} \tag{1.103}$$

We now briefly sketch the proof of the fact that L depends isomonodromically on $\mathbf{t}$ if and only if (1.102) is satisfied. First, assume that L depends isomonodromically on $\mathbf{t}$; then for each analytic solution $\Psi_{\nu,\ell}$ defined above of $\Psi' = L\Psi$, the ratio $\delta\Psi_{\nu,\ell} \cdot \Psi_{\nu,\ell}^{-1}$ is independent of $\nu, \ell = 1, ..., 2r_\nu + 1$, because of $\delta C_\nu = 0 = \delta S_{\nu,\ell}$. In particular by Liouville theorem we must have $\delta\Psi_{\nu,\ell} = \mathcal{M}\Psi_{\nu,\ell}$ where $\mathcal{M}$ coincides with the sum of singular parts in (1.5). Then

$$\delta L = \delta(\Psi_{\nu,\ell}' \cdot \Psi_{\nu,\ell}^{-1}) = (\delta\Psi_{\nu,\ell})' \cdot \Psi_{\nu,\ell}^{-1} - \Psi_{\nu,\ell}' \cdot \Psi_{\nu,\ell}^{-1} \cdot \delta\Psi_{\nu,\ell} \cdot \Psi_{\nu,\ell}^{-1} = \mathcal{M}' + \mathcal{M}L - L\mathcal{M} \tag{1.104}$$

and (1.102) is proved. Conversely if (1.102) is satisfied then the system

$$\begin{cases} \Psi' = L\Psi \\ \delta\Psi = \mathcal{M}\Psi \end{cases} \tag{1.105}$$

is compatible. Let $\Psi_{\nu,\ell}$ be the solutions of this system, in particular solution of $\Psi' = L\Psi$, as specified above. Then, by definition of Stokes' matrices (1.94) we have

$$\mathcal{M} = \delta\Psi_{\nu,\ell+1} \cdot \Psi_{\nu,\ell+1}^{-1} = \delta\Psi_{\nu,\ell} \cdot \Psi_{\nu,\ell}^{-1} + \Psi_{\nu,\ell}\delta S_{\nu,\ell} \cdot S_{\nu,\ell}^{-1}\Psi_{\nu,\ell}^{-1} = \mathcal{M} + \Psi_{\nu,\ell}\delta S_{\nu,\ell} \cdot S_{\nu,\ell}^{-1}\Psi_{\nu,\ell}^{-1} \tag{1.106}$$

and so $\delta S_{\nu,\ell} = 0$, that means all Stokes' matrices are constant. Similarly comparing solutions at z_ν and at ∞ we have, by definition of connection matrices (1.96)

$$\mathcal{M} = \delta\Psi_{\infty,1} \cdot \Psi_{\infty,1}^{-1} = \delta\Psi_{\nu,1} \cdot \Psi_{\nu,1}^{-1} + \Psi_{\nu,1}\delta C_\nu \cdot C_\nu\Psi_{\nu,1}^{-1} = \mathcal{M} + \Psi_{\nu,1}^{-1}\delta C_\nu \cdot C_\nu^{-1}\Psi_{\nu,1}^{-1} \tag{1.107}$$

and so $\delta C_\nu = 0$, that means all connection matrices are constant as well. Finally $\boldsymbol{\lambda}$ must also be constant, as otherwise $\mathcal{M}$ would have logarithmic singularities at the z_ν's, which is not the case by assumption. For more detail we refer to [JMU].

Example 1.4.3. *Let us consider the ("Fuchsian") case where $L(z)\mathrm{d}z$ has just simple poles at points $z_1, \ldots, z_m$, and not at ∞, i.e. $L = \sum_{j=1}^m \frac{L_j}{z-z_j}$, with $L_1 + \cdots + L_m = 0$. In such case the only parameters are $z_1, \ldots, z_m$ and we have*

$$\Xi_j = \operatorname{diag}(\lambda_{j,1}, \ldots, \lambda_{j,N}) \log(z - z_j), \quad j = 1, \ldots, m \tag{1.108}$$

from (1.2). Moreover, in this case the monodromy exponents $\lambda_{j,\alpha}$ coincide with the eigenvalues of L_j, $j = 1, \ldots m$, as it can be shown by plugging the ansatz $\Psi = \Gamma_j \mathrm{e}^{\Xi_j}$ into the ODE $\Psi' = L\Psi$. Hence we can compute

$$\mathcal{M}\left(\frac{\partial}{\partial z_j}\right) = -\operatorname*{res}_{\xi = z_j} \Gamma(\xi) \frac{\operatorname{diag}(\lambda_{j,1}, \ldots, \lambda_{j,N})}{\xi - z_j} \Gamma^{-1}(\xi) \frac{\mathrm{d}\xi}{z - \xi} = -\frac{L_j}{z - z_j} \tag{1.109}$$

from (1.5). The isomonodromic deformation system (1.102) reads as

$$\frac{\partial}{\partial z_j} L = \sum_{i=1}^m \frac{1}{z - z_i} \frac{\partial L_i}{\partial z_j} + \cancel{\frac{L_j}{(z - z_j)^2}} = \left[-\frac{L_j}{z - z_j}, L\right] + \cancel{\mathcal{M}'} \tag{1.110}$$

for the dependent matrix variables $L_j = L_j(z_1, \ldots, z_m)$. This system can be rewritten more concretely matching the polar parts of both sides in the above equation as

$$\begin{cases} \frac{\partial L_i}{\partial z_j} = \frac{[L_i, L_j]}{z_i - z_j} & i \neq j \\ \frac{\partial L_j}{\partial z_j} = -\sum_{i \neq j} \frac{[L_i, L_j]}{z_i - z_j} \end{cases} \tag{1.111}$$

which is the standard form of the Schlesinger system *[Sb].*

One can always use the Möbius group to fix three of the points z_ν to $0, 1, \infty$; the first interesting case corresponds then to $m = 4$, which has a one-dimensional space of deformation parameters (the cross-ratio of the four poles) and it can be shown that the relevant nonlinear ODE which one obtains in this case is the Painlevé VI equation, see e.g. [C; FIKN].

Example 1.4.4. *This example should be compared with the isospectral version, Ex. (1.4.1). Fix $N = 2$ and consider $\Psi' = L\Psi$ for*

$$L(z) = L_0 + L_1 z + L_2 z^2 \tag{1.112}$$

with L_2 semisimple. Without loss of generality we can assume $L(z)$ is traceless (for if $w' = -\frac{1}{2}\operatorname{tr} L(z)$ then the transformation $\Psi \mapsto \mathrm{e}^w \Psi$ sends $L \mapsto L - \frac{1}{2}\operatorname{tr} L$ which is traceless); hence without loss of generality we set $L_2 = \frac{1}{2}\sigma_3$. Moreover, by a translation $z \mapsto z - \frac{(L_1)_{11}}{2}$ we can assume L_1 off-diagonal. Summarizing, we assume

$$L(z) = \begin{pmatrix} \ell_0 + \frac{z^2}{2} & b_0 + b_1 z \\ c_0 + c_1 z & -\ell_0 - \frac{z^2}{2} \end{pmatrix}. \tag{1.113}$$

Since it is interesting to compare the present example of isomonodromic deformations with the isospectral case of Ex. (1.4.1), comparison of (1.113) with (1.66) suggests to perform the change of coordinates

$$\ell_0 = \frac{a_0}{2} + x_1 y_1, \quad b_0 = -x_2, \quad b_1 = -x_1, \quad c_0 = y_2, \quad c_1 = y_1. \tag{1.114}$$

The ODE $\Psi' = L\Psi$ has an irregular singularity of Poincaré rank 3 at ∞; plugging the ansatz (1.91) into the equation we obtain the formal solution

$$\Psi = \Gamma \mathrm{e}^{\Xi}, \qquad \Gamma = \mathbf{1} + \begin{pmatrix} x_2 y_2 & x_1 \\ y_1 & -x_2 y_2 \end{pmatrix} z^{-1} + \mathcal{O}(z^{-2}), \qquad \Xi = \left(\lambda \log z + tz + \frac{z^3}{3}\right) \frac{\sigma_3}{2} \tag{1.115}$$

with

$$\lambda = -2(x_1 y_2 + x_2 y_1), \quad t = a_0. \tag{1.116}$$

Let us note that we have one isomonodromic time, t which corresponds to the Casimir a_0 of the Lie–Poisson bracket discussed in the previous section about isospectral deformations; it is in this sense that *it is usually said that isomonodromic deformations are a* de-autonomization *of isospectral deformations. Explicitly, in this case the de-autonomization consists in the identification*

$$a_0 \mapsto t \tag{1.117}$$

of a Casimir with the time of the deformation. Note that with the notations of Ex. 1.4.1 we have $\lambda = a_{-1}$*, which remains a constant of motion.*

Let us write down the isomonodromic deformation equation in t*; first compute directly from the definition* (1.5) *of* $\mathcal{M}$ *as the singular part of* $\Gamma\delta\Xi\Gamma^{-1}$ *at* $z = \infty$ *(including the constant term)*

$$M := \mathcal{M}\left(\frac{\partial}{\partial t}\right) = \operatorname*{res}_{\xi=\infty} \Gamma(\xi)\frac{\sigma_3}{2}\Gamma^{-1}(\xi)\frac{\xi \mathrm{d}\xi}{z-\xi} = \begin{pmatrix} \frac{z}{2} & -x_1 \\ y_1 & -\frac{z}{2} \end{pmatrix} \tag{1.118}$$

which obviously coincides with (1.69)*. The isomonodromic deformation is described by* (1.102) *which reads in this case as*

$$\dot{L} = [M, L] + M' = [M, L] + \frac{\sigma_3}{2} \tag{1.119}$$

where we denote $\dot{f} := \partial_t f$ *throughout this example. This flow is written down explicitly exacly as in* (1.70)*, provided the idenfication* (1.117)*;*

$$\dot{x}_1 = x_2, \quad \dot{y}_1 = -y_2, \quad \dot{x}_2 = -tx_1 - 2x_1^2 y_1, \quad \dot{y}_2 = ty_1 + 2x_1 y_1^2. \tag{1.120}$$

The equations are hamiltonian with respect to the bracket (1.68) *with time-dependent hamiltonian*

$$H = -x_1^2 y_1^2 + x_2 y_2 - t x_1 y_1 \tag{1.121}$$

obtained from (1.71) *by the identification* (1.117)*.*

Let us reduce the isomonodromic equations (1.120) *to the canonical form of the* Painlevé II *equation. To this end perform the symplectic reduction of* (1.120) *with respect to the hamiltonian action of constant diagonal conjugation, as in Ex. 1.4.1. This hamiltonian action is generated by the Hamiltonian flow*

$$\begin{cases} \dot{x}_i = x_i = \{F, x_i\} & (i = 1, 2) \\ \dot{y}_i = -y_i = \{F, y_i\} & (i = 1, 2) \end{cases}, \qquad F := -\frac{\lambda}{2} = x_1 y_2 + x_2 y_1 \tag{1.122}$$

and indeed the formal monodromy exponent λ *is a constant of motion. The quotient under the conjugation action can be parametrized (away from* $x_1 = 0$*) with reduced variables* (1.73) *whose Poisson bracket is given in* (1.74)*.* F, H *descend to well defined functions* $\widetilde{F}, \widetilde{H}$ *on the quotient*

$$\widetilde{F} = \widetilde{y}_2 + \widetilde{y}_1 \widetilde{x}_2, \qquad \widetilde{H} = -\widetilde{y}_1^2 + \widetilde{x}_2 \widetilde{y}_2 - t\widetilde{y}_1 \tag{1.123}$$

and $\widetilde{F}$ *is a Casimir of the reduced Poisson bracket* (1.74)*; the symplectic leaf* $\widetilde{F} = f$ *is parametrized by Darboux coordinates* $\widetilde{y}_1, \widetilde{x}_2$ *and the flow* (1.120) *is given by the reduced Hamiltonian*

$$\widetilde{H}_f := -\widetilde{y}_1^2 - t\widetilde{y}_1 + f\widetilde{x}_2 - \widetilde{x}_2^2 \widetilde{y}_1 \tag{1.124}$$

obtained by the substitution $\widetilde{y}_2 = f - \widetilde{y}_1 \widetilde{x}_2$*. We continue in the analogy with Ex. 1.4.1, however now the canonical change of coordinates* (1.77) *is a time-dependent canonical change of coordinates*

$$\widetilde{x}_2 = -q = \frac{\partial S}{\partial p}, \quad \widetilde{y}_1 = p - \frac{t}{2} - \frac{q^2}{2} = \frac{\partial S}{\partial \widetilde{x}_2}, \quad S = S(p, \widetilde{x}_2, t) := \left(p - \frac{t}{2} - \frac{\widetilde{x}_2^2}{6}\right)\widetilde{x}_2, \quad \{p, q\} = 1 \tag{1.125}$$

hence the hamiltonian $\widetilde{H}_f$ *needs to be corrected to*

$$H_f^{PII} := \widetilde{H}_f + \dot{S} = -p^2 + \frac{q^4}{4} + \frac{t}{2}q^2 + \left(\frac{1}{2} - f\right)q + \frac{t^2}{4}. \tag{1.126}$$

Hence we have finally obtained the Painlevé II equation

$$\ddot{q} = 2q^3 + 2tq + 1 - 2f \tag{1.127}$$

directly in its hamiltonian form [Oa; JM]

$$\begin{cases} \dot{q} = \frac{\partial H_f^{PII}}{\partial p} = -2p \\ \dot{p} = -\frac{\partial H_f^{PII}}{\partial q} = -q^3 - tq - \frac{1}{2} + f. \end{cases} \tag{1.128}$$

The Painlevé II equation appears in various applications, ranging from random matrix theory [TW; CKV] to nonlinear optics [GJ].

Let us mention that all Painlevé equations arise from isomonodromic deformations of a 2×2 linear ODE with rational coefficients [JM].

The isomonodromic tau function. Painlevé property. The isomonodromic deformation equations (1.102) imply that the formal solutions $\Psi_\nu = \Gamma_\nu \mathrm{e}^{\Xi_\nu}$ satisfy (1.4); hence we can introduce the *isomonodromic* tau function as in Def. 1.1.5.

Solutions of the isomonodromic deformation equations enjoy the *Painlevé property*, namely they have only movable poles [I] off the critical locus where $z_{\nu_1} = z_{\nu_2}$ for some $1 \leq \nu_1 \neq \nu_2 \leq m$ or $t_{\nu,r_\nu,\alpha} = t_{\nu,r_\nu,\beta}$ for some $1 \leq \alpha \neq \beta \leq N$. Correspondingly, the isomonodromic tau function is holomorphic in the universal cover of the complement of this critical locus, and its zeros correpond to poles of the solutions of the isomonodromic deformation equations. This was soon conjectured [JMU] and goes under the name of Malgrange–Miwa–Palmer theorem [Mf; Mb; Pa]. We will review this point in the next chapter in the general framework of Riemann–Hilbert problems.

Let us mention that the solutions of these isomonodromic deformations provide very interesting meromorphic functions, due to their Painlevé property. These functions occur in a wide range of applications. Just to name a few instances, the six Painlevé transcendents [I] appear:

- in the study of correlation functions of impenetrable Bose gases [JMMS] and of the 2D Ising model [DIK], and more generally in Conformal Field Theories [GIL];
- in 2D Quantum Gravity [DS], and more generally 2D Topological Field Theories [De];
- in Random Matrix Theory [TW].

1.4.3 Gelfand–Dickey tau functions

We connect here with some more classical notion of tau functions for integrable hierarchies. The results of this thesis are derived independently of the general results presented below, which we include however to give more context to our discussion.

Here we consider for simplicity a strictly formal setting for Kadomtsev–Petviashvili (KP) tau functions. Let us first recall the definition of KP tau functions from the *Sato Grassmannian* perspective. For a short introduction to the KP hierarchy and its tau functions we refer to App. A.

Consider an infinite set of variables $\mathbf{t} = (t_1, t_2, ...)$, and define a grading $\deg t_j := j$. The algebra $\mathbb{C}[\![\mathbf{t}]\!]$ of formal series is the completion of the algebra $\mathbb{C}[\mathbf{t}]$ of polynomials with respect to the filtration

$$\mathbb{C}[\mathbf{t}] \supset \mathscr{I}_1 \supset \mathscr{I}_2 \supset \cdots \tag{1.129}$$

where $\mathscr{I}_d$ is the ideal of polynomials in $\mathbf{t}$ of degree at least d, i.e. it is the inverse limit

$$\mathbb{C}[\![\mathbf{t}]\!] := \varprojlim_{d\in\mathbb{N}} \frac{\mathbb{C}[\mathbf{t}]}{\mathscr{I}_d}. \tag{1.130}$$

More concretely, we have to think of a formal series in $\mathbb{C}[\![\mathbf{t}]\!]$ as a well defined rule to give the complex number which is the coefficient of any (finite) monomial in $\mathbf{t}$.

Introduce the algebra $\mathbb{C}((z^{-1}))$ of formal Laurent series at $z = \infty$, i.e. its elements are expressions $\sum_{n\in\mathbb{Z}} a_n z^n$ for which there exists n_* such that $a_n = 0$ as soon as $n > n_*$. Denote $\mathcal{R}$ the $\mathbb{C}$-algebra

$$\mathcal{R} := \varprojlim_{d\in\mathbb{N}} \mathbb{C}((z^{-1})) \otimes \frac{\mathbb{C}[\mathbf{t}]}{\mathscr{I}_d}. \tag{1.131}$$

More concretely, an element of $\mathcal{R}$ assigns to every (finite) monomial in $\mathbf{t}$ a formal Laurent series.

Denote $\xi := \sum_{s\geq 1} z^s t_s \in \mathcal{R}$, and recall the elementary Schur polynomials [Ma] $p_j(\mathbf{t})$, $j = 0, 1, 2, ...$ defined by

$$\mathrm{e}^{\xi} = \sum_{j\geq 0} p_j(\mathbf{t}) z^j \tag{1.132}$$

e.g.

$$p_0(\mathbf{t}) = 1, \quad p_1(\mathbf{t}) = t_1, \quad p_2(\mathbf{t}) = t_2 + \frac{1}{2}t_1^2, \quad p_3(\mathbf{t}) = t_3 + t_1 t_2 + \frac{1}{6}t_1^3, \cdots . \tag{1.133}$$

Noting that $p_j(\mathbf{t})$ is homogeneous of degree j in $\mathbf{t}$, we conclude from (1.132) that $\mathrm{e}^{\xi} \in \mathcal{R}$. More generally:

Lemma 1.4.5. *If $f \in \mathbb{C}((z^{-1}))$ the element $\mathrm{e}^{\xi} f$ is well defined in $\mathcal{R}$.*

This is evident, as up to terms of degree d in $\mathbf{t}$ this amounts to the well defined multiplication of the polynomial $1+p_1(\mathbf{t})z+\cdots+p_d(\mathbf{t})z^d$ with the formal Laurent series f.

Let us now introduce the Sato grassmannian [SS; SWa].

Definition 1.4.6. *A* Sato subspace *of* $\mathbb{C}((z^{-1}))$ *is a subspace spanned by* $f_j\in\mathbb{C}((z^{-1}))$ *for* $j=1,2,\dots$ *such that* $f_j=z^{j-1}(1+\mathcal{O}(z^{-1}))$ *for* j *large enough. The* Sato grassmannian $\mathrm{Gr}^{\frac{\infty}{2}}$ *is the set of all Sato subspaces of* $C((z^{-1}))$. *The* big cell $\mathrm{Gr}_0^{\frac{\infty}{2}}$ *of the Sato grassmannian consists of Sato subspaces where* $f_j=z^{j-1}(1+\mathcal{O}(z^{-1}))$ *for* all $j\geq 1$.

We will always restrict to the big cell of the Sato grassmannian.

For $f=(f_j)_{j\geq1}\in\mathrm{Gr}_0^{\frac{\infty}{2}}$ one defines the *tau function* as the formal expression

$$\tau^f(\mathbf{t})=\frac{\cdots\wedge \mathrm{e}^{\xi}f_3\wedge \mathrm{e}^{\xi}f_2\wedge \mathrm{e}^{\xi}f_1\wedge z^{-1}\wedge z^{-2}\wedge\cdots}{\cdots\wedge z^2\wedge z\wedge 1\wedge z^{-1}\wedge z^{-2}\wedge\cdots} \tag{1.134}$$

to be computed using skew-symmetry and multi-linearity of the wedge product $\wedge$. We now contend that $\tau(\mathbf{t})$ is an honest element of $\mathbb{C}[\![\mathbf{t}]\!]$. Incidentally, notice that expression (1.134) depends only on the linear subspace generated by $f_1,f_2,\dots$ in $\mathbb{C}((z^{-1}))$, and this explains the *grassmannian* terminology.

To this end introduce the Schur polynomials [Ma] $s_\lambda(\mathbf{t})$ for each partition $\lambda\in\mathbb{Y}$

$$s_\lambda(\mathbf{t}):=\det\left(p_{\lambda_k+j-k}(\mathbf{t})\right)_{j,k=1}^{\ell(\lambda)} \tag{1.135}$$

where $\ell(\lambda)$ denotes the *length* of λ. For every d, Schur polynomials s_λ with $|\lambda|=d$ form a basis of the homogeneous part of degree d of $\mathbb{C}[\![\mathbf{t}]\!]$.

Lemma 1.4.7. *Formula* (1.134) *defines an element in* $\mathbb{C}[\![\mathbf{t}]\!]$; *it is more explicitly expressed as*

$$\tau(\mathbf{t})=\sum_{\lambda\in\mathbb{Y}}F_\lambda s_\lambda(\mathbf{t}) \tag{1.136}$$

where F_λ *("Plücker coordinates") are given as*

$$F_\lambda=\det\left(f_{j,\lambda_k+j-k}\right)_{j,k=1}^{|\lambda|} \tag{1.137}$$

where $f_j=z^{j-1}(1+\sum_{\ell\geq1}f_{j,\ell}z^{-\ell})$.

The Schur polynomials $s_\lambda(\mathbf{t})$ are closely related to characters of GL_N

$$\chi_\lambda(x_1,\dots,x_N):=\frac{\det\left(x_j^{\lambda_k+N-k}\right)_{j,k=1}^N}{\Delta(x_1,\dots,x_N)}=\frac{\det\left(x_j^{\lambda_k+N-k}\right)_{j,k=1}^N}{\det\left(x_j^{N-k}\right)_{j,k=1}^N} \tag{1.138}$$

see App. B, in particular (B.12). The relation goes as follows; we have

$$s_\lambda(\mathbf{t}(x_1,\dots,x_N))=\chi_\lambda(x_1,\dots,x_N) \tag{1.139}$$

for all $\lambda\in\mathbb{Y}$ of lenght $\ell(\lambda)\leq N$, where the *Miwa times* (or *Newton polynomials*) are defined as

$$\mathbf{t}(x_1,\dots,x_N)=(t_1(x_1,\dots,x_N),t_2(x_1,\dots,x_N),\dots),\quad t_j(x_1,\dots,x_N):=\frac{1}{j}(x_1^j+\cdots+x_N^j). \tag{1.140}$$

(This explains the grading of the variables $\mathbf{t}$.)

We can exploit this relation as follows. For a Sato subspace $f=(f_j)_{j\geq1}$ and all $N\geq1$ set

$$\tau_N^f(x_1,\dots,x_N):=\frac{\det\left(f_j(x_k)\right)_{j,k=1}^N}{\det\left(x_k^{j-1}\right)_{j,k=1}^N}=\frac{\det\left(f_j(x_k)\right)_{j,k=1}^N}{\prod_{1\leq j<k\leq N}(x_k-x_j)}. \tag{1.141}$$

As τ_N^f is the ratio of two alternating polynomials in $x_1,\dots,x_N$ it is a symmetric function. As such it can be expressed in terms of the Miwa times (1.140). Let us call $\tau_N^f(\mathbf{t})$ the result of this substitution.

Proposition 1.4.8 ([Kb; IZb]). *Terms of degree d in $\tau_N^f(\mathbf{t})$ do not depend on N as soon as $N \geq d$, and moreover under the same assumption these terms coincide with the same terms in $\tau^f(\mathbf{t})$.*

First we establish the following lemma, consequence of the *Binet–Cauchy formula*. We formulate it separately as it will be useful later on.

Lemma 1.4.9. *Let $g_j(x) = \sum_{\ell \geq 0} g_{j,\ell} x^\ell$. Then*

$$\frac{\det\left(g_j(x_k)\right)_{j,k=1}^N}{\Delta(x_1, ..., x_N)} = \sum_{\ell(\lambda) \leq N} \det\left(g_{j,\lambda_k+N-k}\right)_{j,k=1}^N \chi_\lambda(x_1, ..., x_N) \tag{1.142}$$

where the sum is over partitions λ of length $\ell(\lambda) \leq N$. , and $\chi_\lambda(x_1, ..., x_N)$ are the characters (1.138).

Proof of Lemma 1.4.9. It follows directly by the Binet–Cauchy formula, as

$$\det\left(g_j(x_k)\right)_{j,k=1}^N = \det\left(\begin{pmatrix} g_{1,1} & g_{1,2} & \cdots \\ \vdots & \vdots & \\ g_{1,1} & g_{1,2} & \cdots \end{pmatrix} \cdot \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \\ x_1^2 & \cdots & x_N^2 \\ \vdots & & \vdots \end{pmatrix}\right)$$

and we have to take the determinant of the product of an $N \times \infty$ times an $\infty \times N$ matrix. ∎

Proof of Prop. 1.4.8. We write $x_k := \zeta_k^{-1}$ and $g_j(x) := x^{N-1} f_j(x^{-1})$, hence

$$(-1)^{\frac{N(N-1)}{2}} \frac{\det(f_j(\zeta_k))_{j,k=1}^N}{\Delta(\zeta_1, ..., \zeta_N)} = \frac{\det(g_j(x_k))_{j,k=1}^N}{\Delta(x_1, ..., x_N)} = \sum_{\ell(\lambda) \leq N} \det\left(g_{j,\lambda_k+N-k}\right)_{j,k=1}^N \chi_\lambda(x_1, ..., x_N)$$

where the last step is due to Lemma 1.4.9 with $g_j(x) = \sum_{\ell \geq 0} g_{j,\ell} x^\ell$. Since $f_j(\zeta) = \zeta^{j-1}\left(1 + \sum_{\ell \geq 1} f_{j,\ell} \zeta^{-\ell}\right)$ we have

$$g_{j,\ell} = f_{j,\ell-N+j} \tag{1.143}$$

with the understanding that $f_{j,n} := 0$ for $n < 0$ and $f_{j,0} := 1$. Directly from Lemma 1.143 and (1.139) we can write

$$\tau_N^f(\mathbf{t}) = \sum_{\ell(\lambda) \leq N} \det\left(f_{j,\lambda_k+j-k}\right)_{j,k=1}^N s_\lambda(\mathbf{t})$$

and the proof is complete by the observation that the coefficients $\det\left(f_{j,\lambda_k+j-k}\right)_{j,k=1}^N$ do not depend on N as soon as $N \geq d$. This follows because the matrix $\left(f_{j,\lambda_k+j-k}\right)_{j,k=1}^N$ has the following block structure, provided $N \geq d$ so that $\lambda_i = 0$ for $i > d$;

$$\left(\begin{array}{ccc|ccc} f_{1,\lambda_1} & \cdots & f_{1,\lambda_d-d+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{d,\lambda_1+d-1} & \cdots & f_{d,\lambda_d} & 0 & \cdots & 0 \\ \hline f_{d+1,\lambda_1+d} & \cdots & f_{d+1,\lambda_d+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{N,\lambda_1+N-1} & \cdots & f_{N,\lambda_d+N-d-1} & f_{N,\lambda_d+N-d} & \cdots & 1 \end{array}\right)$$

hence

$$\det\left(f_{j,\lambda_k+j-k}\right)_{j,k=1}^N = \det\left(f_{j,\lambda_k+j-k}\right)_{j,k=1}^d$$

and this completes the proof. ∎

Let us now consider Gelfand–Dickey (GD) tau functions; by definition, a tau function of the rth GD hierarchy ($r = 2, 3, ...$) is a KP tau function associated with $f = (f_j)_{j \geq 1} \in \mathrm{Gr}_0^{\frac{\infty}{2}}$ of the form

$$f_j(z) = z^{j-1}\left(1 + \sum_{\ell \geq 1} f_{j,\ell} z^{-\ell}\right) \tag{1.144}$$

satisfying the *periodicity* property

$$f_{j+r,\ell} = f_{j,\ell}. \tag{1.145}$$

Lemma 1.4.10. *Tau functions of the* r*th GD hierarchy do not depend on variables* $t_r, t_{2r}, t_{3r}, \cdots$.

This lemma can be guessed directly by looking at (1.134). One can give a more explicit proof following [IZb].

The connection with the tau functions considered in this chapter can be expressed as follows. We consider for simplicity the KdV case, which is the $r = 2$ case of GD hierarchies. So, let us consider $f = (f_j)_{j\geq 1} \in \mathrm{Gr}_0^{\frac{\infty}{2}}$ satisfying the periodicity property $f_{j+2,\ell} = f_{j,\ell}$. Introduce the 2×2 matrix-valued formal series

$$\Gamma(z;\mathbf{t}) := \begin{pmatrix} \frac{\tau(\mathbf{t}-[z^{-1}])}{\tau(\mathbf{t})} & \frac{\tau(\mathbf{t}+[z^{-1}])}{\tau(\mathbf{t})} \\ -\left(\frac{\partial}{\partial t_1}+z\right)\frac{\tau(\mathbf{t}-[z^{-1}])}{\tau(\mathbf{t})} & -\left(\frac{\partial}{\partial t_1}-z\right)\frac{\tau(\mathbf{t}+[z^{-1}])}{\tau(\mathbf{t})} \end{pmatrix}, \qquad \mathbf{t} = (t_1, t_3, t_5, \ldots) \tag{1.146}$$

and

$$\Xi(z;\mathbf{t}) := \frac{\sigma_3}{2}\sum_{j\geq 0} t_{2j+1} z^{2j+1}. \tag{1.147}$$

Theorem 1.4.11 ([BDYa])**.** *The KdV tau function* (1.134) *(for a point in the big cell of the Sato grassmannian satisfying the KdV periodicity) satisfies*

$$\delta \log \tau(\mathbf{t}) = -\operatorname*{res}_{z=\infty}\left(\Gamma^{-1}\Gamma'\delta\Xi\right)\mathrm{d}z \tag{1.148}$$

These methods can be applied to Gelfand–Dickey and Drinfeld–Sokolov tau functions as well, we refer to [CW; BDYb] for further informations.

CHAPTER 2

Malgrange differential

A Riemann–Hilbert problem (RHP) is the problem of analytic factorization of a matrix J defined on a contour Σ. RHPs are intimately related with singuar integral equations and appear in a large number of diverse problems of Mathematical Physics. In this chapter we follow [Bc] and, given a RHP depending analytically on some parameters, associate a differential in the space of parameters, termed Malgrange differential. When the Malgrange differential (or some simple modification of it) is closed, it can be used to introduce a tau function as a logarithmic potential. This approach to tau functions unveils their meaning as regularized determinants of the associated singular integral equations; namely, the tau function is a section of a line bundle whose zero locus coincides with the set of points in the parameter space for which the RHP is not solvable. Finally, some formal aspects are related with the content of the previous chapter.

The material for this chapter is mainly extracted from [Bc; Bd; BCc; Mb].

2.1 Introduction

A complete and precise discussion of the general theory of Riemann–Hilbert problems (RHPs) goes far beyond the scope of this thesis. In this chapter we consider a fairly simple setting for RHPs with very mild analytic assumptions, so that much of the machinery of Operator Theory involved in the general theory of RHPs will not be needed; this setting is however enough for the rest of this thesis.

We refer to the monographs [CG; Ga; Mi] for the general development of the theory of RHPs, or [Da] and [AF, Chap. 7] for more introductory discussions.

In this introduction we overview some general facts about RHPs and outline the content of the rest of this chapter.

Riemann–Hilbert problems. Suppose we are given an oriented contour Σ in the complex z-plane, which we assume sufficiently smooth (precise formulations below). Then at each point of Σ the orientation defines two sides of Σ which will be denoted with a $+$ sign (to the left of Σ) and with a $-$ sign (to the right of Σ). Suppose we are also given a *jump* matrix $J : \Sigma \to \mathrm{GL}_N$. The RHP amounts to finding an $N \times N$ matrix valued function $\Gamma = \Gamma(z)$ such that both Γ, Γ^{-1} are analytic and bounded in $\mathbb{C} \setminus \Sigma$ and satisfy the *jump condition* $\Gamma_+ = \Gamma_- J$ at Σ, where $\Gamma_\pm$ denote the boundary values of Γ from the $\pm$ sides of Σ.

Below we will specify better the analytic details of the RHPs we are going to work with; we anticipate that we always consider jump matrices which admit (piecewise) analytic continuation to a tubular neighborhood of Σ. Under this assumption the jump condition $\Gamma_+ = \Gamma_- J$ has an obvious meaning, as $\Gamma_\pm$ analytically extend slightly across Σ. For the general case we refer to the mentioned literature.

The transformation $\Gamma(z) \mapsto C\Gamma(z)$ with $C \in \mathrm{GL}_N$ constant invertible matrix sends solutions of the RHP to solutions. It is thus convenient to normalize the solution by $\Gamma(z_0) = \mathbf{1}$, usually $z_0 = \infty$. Under this normalization the solution of the RHP, as stated above, is unique. Indeed for two solutions Γ_1, Γ_2 consider the ratio $R := \Gamma_1 \Gamma_2^{-1}$ (by assumptions Γ_i are bounded and invertible in $\mathbb{C} \setminus \Sigma$ so we can consider this ratio). This ratio R has no jump on Σ, for we have

$$R_+ = (\Gamma_1)_+ (\Gamma_2^{-1})_+ = (\Gamma_1)_- J J^{-1} (\Gamma_2^{-1})_- = (\Gamma_1)_- (\Gamma_2^{-1})_- = R_-. \tag{2.1}$$

Hence R is an entire bounded matrix function of z, therefore by Liouville theorem and our normalization $\Gamma_i(z_0) = \mathbf{1}$ $(i = 1, 2)$ we have $R \equiv \mathbf{1}$, i.e. $\Gamma_1 \equiv \Gamma_2$ and the solution is unique.

We will see below that when Σ has endpoints then the solution does not necessarily exists, as stated; in such a situation we can only require Γ, Γ^{-1} to be bounded far from the endpoints, and to ensure uniqueness of the solution we have to fix suitable growth conditions at the endpoints. General analysis of this problem is not needed in the following.

Sokhotski–Plemelj formulæ, scalar RHPs, index of matrix RHPs. Let Σ be a contour in the z-plane (in this section it may either be a loop or an arc, in any case it is assumed to be smooth and non self-intersecting) and f a function defined on Σ which is *Hölder continuous*, i.e. we have

$$|f(\xi_1) - f(\xi_2)| \leq C|\xi_1 - \xi_2|^\alpha \tag{2.2}$$

for some $0 < \alpha \leq 1$ and for all $\xi_1, \xi_2 \in \Sigma$. In this situation, define the *Cauchy principal value integral*

$$\text{P.V.} \int_\Sigma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi := \lim_{\epsilon \to 0} \int_{\Sigma \cap \{|\xi - z| > \epsilon\}} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi. \tag{2.3}$$

Of course, when $z \notin \Sigma$, the Cauchy principal value integral (2.3) coincides with the standard Cauchy integral $F(z) := \int_\Sigma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi$. However (2.3) is well defined also for $z \in \Sigma$. It is a classical fact, see e.g. [AF, Chap. 7], that, under such circumstances, the Cauchy integral $F(z)$ is a *sectionally analytic* function of $z \in \mathbb{C} \setminus \Sigma$. By this we mean that $F(z)$ is analytic off Σ and admits limiting values $F_\pm(z)$ for $z \in \Sigma^o$ ($\Sigma^o := \Sigma \setminus \{\text{endpoints}\}$) from the left $(+)$ and the right $(-)$ of Σ, where the limit is taken along any path lying entirely in the $\pm$ side of Σ. Moreover we have the following expressions for the limiting values $F_\pm$, that go under the name of *Sokhotski–Plemelj* formulæ;

$$F_\pm(z) = \pm \frac{f(z)}{2} + \frac{1}{2\pi \mathrm{i}} \text{P.V.} \int_\Sigma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \qquad z \in \Sigma^o. \tag{2.4}$$

Clearly the two identities in (2.4) are equivalent to

$$F_+(z) - F_-(z) = f(z), \qquad F_+(z) + F_-(z) = \frac{1}{\mathrm{i}\pi} \text{P.V.} \int_\Sigma \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \qquad z \in \Sigma^o. \tag{2.5}$$

With the aid of the Sokhotski–Plemelj formulæ we are able to determine the condition for solvability and to provide the solution to scalar ($N = 1$) RHPs. Indeed suppose we are given the contour Σ and the jump $J(z) : \Sigma \to \mathbb{C} \setminus \{0\}$. Taking the logarithm we have

$$\log \Gamma_+ - \log \Gamma_- = \log J \tag{2.6}$$

and by the first identity in (2.5) we would like to find the solution in the form

$$\Gamma(z) = \exp\left(\frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log J(\xi)}{\xi - z} \mathrm{d}\xi \right). \tag{2.7}$$

However we run into the following problems.

1. When Σ is a loop in $\mathbb{P}^1$ then $\log J(z)$ must be Hölder continuous, see (2.2). This can only happen if the *index*
$$\frac{1}{2\pi \mathrm{i}} \oint_\Sigma \mathrm{d} \log J(\xi) \tag{2.8}$$
which in principle is an arbitrary integer, vanishes (otherwise $\log J(z)$ cannot possibly be even continuous).

2. When Σ has endpoints, in general $\int_\Sigma \frac{\log J(\xi)}{\xi - z} \mathrm{d}\xi$ has logarithmic singularities as $z \to$endpoints [Ga; Mi; AF], hence the solution Γ cannot possibly be bounded analytic with bounded inverse near the endpoints of Σ. Moreover, if we do not require boundedness, we loose uniqueness of the solution; indeed for any solution $\Gamma(z)$ then $g(z)\Gamma(z)$, where $g(z)$ is analytic in $\mathbb{P}^1$ but for isolated singularities at the endpoints of Σ, is again a solution.

To overcome the second issue one has to complement the RHP with suitable boundary conditions at the endpoints, if any.

Regarding the first, we note that actually the *index condition*

$$\mathrm{ind}_\Sigma \log J := \frac{1}{2\pi \mathrm{i}} \oint_\Sigma \mathrm{d} \log J(\xi) = 0 \tag{2.9}$$

when Σ is a loop is a necessary condition for the existence of a solution Γ; indeed if Γ is a solution

$$\mathrm{ind}_\Sigma \log J = \frac{1}{2\pi \mathrm{i}} \oint_\Sigma \mathrm{d} \log J(\xi) = \frac{1}{2\pi \mathrm{i}} \oint_\Sigma \mathrm{d} \log \Gamma_+ - \frac{1}{2\pi \mathrm{i}} \oint_\Sigma \mathrm{d} \log \Gamma_- \tag{2.10}$$

and by Cauchy's argument principle each of the last two terms vanishes, as it is equal to the number of zeros minus the number of poles of $\Gamma_\pm$ in the respective domains of definition; as $\Gamma_\pm$ are by assumption bounded with bounded inverse, they have nor zeros nor poles and the index vanishes.

We content ourselves with this simplified overview of these general features of scalar RHPs and refer for more details to the aforementioned literature.

For *matrix* RHPs, the index condition (2.9) should be replaced by

$$\mathrm{ind}_\Sigma \log \det J := \frac{1}{2\pi \mathrm{i}} \oint_\Sigma \mathrm{d} \log \det J = 0 \tag{2.11}$$

(for instance note that the determinant solves a scalar RHP and apply the considerations above). Similar considerations about the endpoints apply to this case too.

However, when Σ is a loop in $\mathbb{P}^1$ the index condition (2.9) is not enough to guarantee in general the existence of a solution (compare with Thm. 2.2.1 below). In the following we will always consider case by case the issue of solvability of the RHPs we will consider.

Outline. For this chapter we have the following main goals.

1. Introduce the *Malgrange differential* for a RHP depending on parameters, following [Bc]. The Malgrange differential is not closed in general; there is however a general formula for its exterior differential [Bc; Bd], see Prop. 2.2.5 and Prop. 2.3.2. However in several interesting cases the Malgrange differential is either closed or closed up to a simple explicit modification. In the latter cases one can introduce the tau function as a logarithmic potential for the Malgrange differential. We also hint at the connection between zeros of the tau function and non-solvability of the RHP. We will first consider the case when Σ is the unitary circle (or any finite union of disjoint circles) (Sec. 2.2), and then the case where we allow much more general contours (Sec. 2.3).

2. Study *Schlesinger transformations* of RHP, which namely consist in looking for sectionally *meromorphic* matrix function Γ satisfying the usual jump condition $\Gamma_+ = \Gamma_- J$ on Σ with poles and zeros at some fixed points away from Σ. Locations of these points play the role of new parameters in the RHP; this procedure is related to the *dressing method* [ZS] in the theory of integrable systems. We then study the effect of Schlesinger transformations on the Malgrange differential; the main result, which was dates back in some less general and detailed form to [JM], is reported in Sec. 2.4 following the analysis of [BCc].

3. Connect some computational aspects of tau functions from RHPs to the formal aspects examined in the previous chapter, see Sec. 2.5. E.g. this will permit to use the formulæ of Thm. 1.2.2 for logarithmic derivatives of the tau function, and this will be used extensively in the sequel of this work.

2.2 RHPs on the circle, Töplitz operators and Malgrange differential

RHPs on the circle and Bikhoff theorem. We start with the case $\Sigma = S^1 := \{|z| = 1\}$. We consider jump matrices $J : S^1 \to \mathrm{GL}_N$ which are actually analytic for z in a annulus $1 - \epsilon < |z| < 1 + \epsilon$.

The solution Γ of the RHP consists of a sectionally analytic matrix function satisfying $\Gamma_+ = \Gamma_- J$; by definition, this means a pair of $N \times N$ matrix valued functions $\Gamma_\pm : \mathbf{D}_\pm \to \mathrm{GL}_N$ analytic for $z \in \mathbf{D}_+ := \{|z| < 1\}$, $\mathbf{D}_- := \{|z| > 1\}$ admitting boundary values $\Gamma_\pm$ at S^1 related as $\Gamma_+ = \Gamma_- J$.

Due to our assumption regarding analyticity of Γ we infer that $\Gamma_\pm$ actually extend to analytic functions in wider domains $\Gamma_\pm : \mathbf{D}_\pm^{(\epsilon)} \to \mathrm{GL}_N$, $\mathbf{D}_+^{(\epsilon)} := \{|z| < 1 + \epsilon\}$, $\mathbf{D}_-^{(\epsilon)} := \{|z| > 1 - \epsilon\}$, and then the identity of boundary values $\Gamma_+ = \Gamma_- J$ has the meaning of identity of functions in the annulus $1 - \epsilon < |z| < 1 + \epsilon$.

Note that more or less by definition this situation corresponds to finding the space of holomorphic sections of a rank N vector bundle on $\mathbb{P}^1$ defined by the transition function J.

One can address very explicitly the problem of solvability of this type of RHP, by the following fundamental theorem of Birkhoff [Be], later generalized and put into context of vector bundles by Grothendieck [Gb].

Theorem 2.2.1 (Birkhoff [Be]). *For every $J(z)$ as above, i.e. J a GL_N-valued function analytic for $1-\epsilon < |z| < 1+\epsilon$, there exist a unique set of integers $k_1 \geq \cdots \geq k_N$ and a unique sectionally analytic matrix function $\Gamma = \Gamma(z)$ such that*

$$J(z) = \Gamma_-^{-1}(z) \begin{pmatrix} z^{k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z^{k_N} \end{pmatrix} \Gamma_+(z). \tag{2.12}$$

We omit the proof of this classical result, which can be found e.g. in [CG].

The numbers $k_1 \geq \cdots \geq k_N$ are called (*right*[1]) *partial indexes*, and clearly the total index is the sum of the partial indexes, $\mathrm{ind}_\Sigma \log J = k_1 + \cdots + k_N$. Therefore, necessary and sufficient condition for the existence of a solution to the RHP on S^1 is the vanishing of *all* partial indexes (in particular the index condition (2.11) is not sufficient to guarantee the existence of the solution).

Töplitz operators. Introduce the Hilbert space $H := L^2(S^1, \mathrm{d}z) = H_+ \oplus H_-$ where H_+ consists of functions with only nonnegative Fourier modes (i.e. functions admitting analytic continuation to the interior of S^1) and H_- consists of functions with only negative Fourier modes (i.e. functions admitting analytic continuation to the exterior of S^1 and vanishing at infinity). H_+ and H_- are mutually orthogonal, and the associated orthogonal projectors can be represented by the Cauchy integrals

$$C_\pm : H \to H_\pm : f(z) \mapsto (C_\pm f)(z) := \frac{1}{2\pi\mathrm{i}} \oint_{S^1} \frac{f(\xi)}{\xi - z_\pm} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{2.13}$$

where $z_\pm$ denotes the corresponding boundary value. Indeed by the Sokhotski–Plemelj formula and its consequence (2.5) we have $C_+ - C_- = id_H$. Moreover $C_\pm^2 = \pm C_\pm$, hence $\pm C_\pm$ are the orthogonal projectors on their range which is $H_\pm$ (we refer for more details to the literature, see e.g. [Da]).

In the following we denote by $\vec{H}_\pm$ the spaces $H_\pm \otimes \mathbb{C}^N$, intended as the space of *row*-vector valued L^2-functions on S^1 with only nonnegative $(+)$ or negative $(-)$ Fourier modes; similarly let $\vec{H} := \vec{H}_+ \oplus \vec{H}_-$. In the interest of lighter notations, we denote by the same symbol $C_\pm$ the extension $C_\pm \otimes \mathbf{1} : \vec{H} \to \vec{H}_+$.

Introduce the *Töplitz* operator $T_\Phi : \vec{H}_+ \to \vec{H}_+$ with *(matrix) symbol* $\Phi \in H \otimes \mathrm{Mat}_N$;

$$(T_\Phi \vec{f})(z) := (C_+(\vec{f}\Phi))(z) = \oint_{S^1} \frac{\vec{f}(\xi)\Phi(\xi)}{\xi - z_+} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}. \tag{2.14}$$

Proposition 2.2.2. *The Riemann–Hilbert problem on the unit circle S^1 with jump matrix J admits a solution if and only if $T_{J^{-1}}$ is invertible; in this case the inverse is given as*

$$\left(T_{J^{-1}}^{-1} \vec{f}\right)(z) = (C_+(\vec{f}\Gamma_-^{-1}))(z)\Gamma_+(z) = \oint_{S^1} \frac{\vec{f}(\xi)\Gamma_-(\xi)\Gamma_+(z)}{\xi - z} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}. \tag{2.15}$$

For the proof see e.g. [CG].

Let us also mention the following important fact, for which we refer to [CG]. Under our assumption of J analytic in a tubular neighborhood of S^1, then T_J is a Fredholm operator (by definition, it has finite dimensional kernel and cokernel); moreover, its Fredholm index (by definition, dimension of kernel

[1] There is of course a *dual* result stating that there exists a factorization of the form $J(z) = \widehat{\Gamma}_+(z) \begin{pmatrix} z^{\widehat{k}_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z^{\widehat{k}_N} \end{pmatrix} \widehat{\Gamma}_-^{-1}(z)$, the numbers $\widehat{k}_1 \geq \cdots \geq \widehat{k}_N$ termed *left* partial indices.

minus dimension of cokernel) coincides to the winding index $\mathrm{ind}_\Sigma \log J$. As we are working under the assumption of vanishing index (2.11) the Fredholm index is constantly equal to zero; however this does not automatically imply that $T_{J^{-1}}$ is invertible, as combining Thm. 2.2.1 and Prop. 2.2.2 the invertibility of $T_{J^{-1}}$ is equivalent to vanishing of *all partial* indices.

Widom constants and dual RHP. It is natural to try to find some notion of determinant for the Töplitz operator $T_{J^{-1}}$ whose invertibility governs the solvability of the RHP. As a matter of fact, there is no notion of determinant in general for Töplitz operators. In this paragraph we discuss one first natural proxy for a determinant of the Töplitz operator. This is not however the object we shall be mostly interested in the following of the thesis, however the topic is closely related and indeed relevant to us.

Widom [Wb] observed that the operator $T_{J^{-1}}T_J$ differs from the identity by a trace-class operator hence it admits a Fredholm determinant. Before explaining this point, let us stress the meaning of the Töplitz operator T_J; directly by Prop. 2.2.2 its invertibility is related to the existence of a sectionally analytic matrix $\bar\Gamma$ such that

$$\bar\Gamma_+ = \bar\Gamma_- J^{-1}. \tag{2.16}$$

The problem of finding a sectionally analytic $\bar\Gamma$ satisfying (2.16) is called *dual* RHP.

$T_{J^{-1}}T_J$ admits a Fredholm determinant because we have, using Sokhotski–Plemelj formula $C_+ = C_- + id_{\vec H}$, for all $\vec f \in \vec H_+$,

$$T_{J^{-1}}T_J\vec f = C_+((C_+(\vec f J))J^{-1}) = C_+((C_-(\vec f J)J^{-1}) + \vec f) = \vec f + C_+((C_-(\vec f J))J^{-1}) \tag{2.17}$$

and we have the following lemma.

Lemma 2.2.3. *Under our assumptions of J analytic and invertible in a tubular neighborhood of S^1 having vanishing winding index, see (2.9), the operator $C_+((C_-(\vec f J))J^{-1})$ is trace-class.*

Hence it is natural to define the *Widom tau function* (see also [CGL])

$$\tau_{Widom}(J) := \det{}_{H_+}(T_{J^{-1}}T_J) \tag{2.18}$$

where the Fredholm determinant $\det_{H_+}$ is defined thanks to Lemma 2.2.3. Let us stress again that $\tau_{Widom} = 0$ if and only if J fails to have one of the two factorizations

$$J = \Gamma_-^{-1}\Gamma_+ = \bar\Gamma_+^{-1}\bar\Gamma_- \tag{2.19}$$

i.e. if and only if either the direct of dual RHP are not solvable. This is particularly useful when J admits by construction the dual factorization $J = \bar\Gamma_+^{-1}\bar\Gamma_-$, so that T_J is always invertible and so $\tau_{Widom} = 0$ if and only if the (direct) RHP is not solvable.

For future reference it is worth pointing out the following variational formula, see [Wb; CGL].

Proposition 2.2.4. *Assume that J depends on parameters $\mathbf{t}$ and let as usual δ be the differential in these parameters. We have;*

$$\delta \log \tau_{Widom} = \oint_\Sigma \mathrm{tr}\left[\left(\widetilde\Gamma'_-\widetilde\Gamma_-^{-1} + \Gamma_+^{-1}\Gamma'_+\right)J^{-1}\delta J\right]\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.20}$$

The proof is a computation using Jacobi variational formula, for which we refer to loc. cit.

Let us make once for all the following remark. In the most general setting for RHPs one requires existence of the boundary values $\Gamma_\pm$ but nothing is said about $\Gamma'_\pm$. However in our comfortable setting, Γ extends analytically slightly across S^1 hence the boundary values of Γ' also exist.

In the following paragraph we shall consider instead a different object, the *Malgrange differential*, directly related to the solvability of the direct RHP only.

Malgrange differential. The setup is as above that of a jump matrix $J(z;\mathbf{t})$ jointly analytic for z in the annulus $1-\epsilon < |z| < 1+\epsilon$ and for $\mathbf{t} \in \mathcal{U}$ some open domain in some $\mathbb{C}^d$. We always assume the index condition $\mathrm{ind}_{S^1}\det J = 0$ identically in $\mathcal{U}$. In this section we omit the proofs and refer to the literature.

Let us recall the following facts from [Mb]. The locus in $\mathcal{U}$ where the RHP $\Gamma_+ = \Gamma_- J$ on S^1 is solvable is open; its complement Θ is called *Malgrange divisor*. In other words

$$\Theta = \{\mathbf{t} : \; T_{J^{-1}(\mathbf{t})} \text{ is not invertible}\}. \tag{2.21}$$

Denoting δ the differential in the parameters $\mathbf{t} \in \mathcal{U}$, the operator (recall the notation (2.14))

$$T_{J^{-1}}^{-1} \delta T_{J^{-1}} + T_{\delta J J^{-1}} : \vec{H}_+ \to \vec{H}_+ \tag{2.22}$$

defined for $\mathbf{t} \in \mathcal{U} \setminus \Theta$ is trace-class; its trace reads

$$\operatorname{tr}_{\vec{H}_+} \left(T_{J^{-1}}^{-1} \delta T_{J^{-1}} + T_{\delta J J^{-1}} \right) = \oint_{S^1} \operatorname{tr} \left(\Gamma_+^{-1} \Gamma_+ J^{-1} \delta J \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \tag{2.23}$$

(It is interesting to compare this expression with (2.20).) The differential (2.23) is logarithmic, namely it has simple poles along the Malgrange divisor Θ. Moreover if $\mathbf{t}_* \in \Theta$ the Poincaré residue of (2.23) at $\mathbf{t}_*$ equals $\dim \ker T_{J(\mathbf{t}_*)}$ [Mb].

Now let us follow [Bc] and consider a simple modification of the differential (2.23)

$$\Omega := \oint_{S^1} \operatorname{tr} \left(\Gamma_-^{-1} \Gamma_-' \delta J J^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{2.24}$$

and term it *Malgrange differential*. Indeed we have

$$\Omega = \oint_{S^1} \operatorname{tr} \left(\Gamma_+^{-1} \Gamma_+ J^{-1} \delta J \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} - \oint_{S^1} \operatorname{tr} \left(J' J^{-1} \delta J J^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{2.25}$$

as it is easy to check using the cyclic property of the trace and the jump condition $\Gamma_+ = \Gamma_- J$, $\Gamma_+' = \Gamma_-' J + \Gamma_- J'$.

In the following we will invariably consider the definition above (2.24) for the Malgrange differential.

Proposition 2.2.5 ([Mb; Bc])**.** *The exterior derivative of the Malgrange differential* (2.24) *can be expressed as*

$$\delta \Omega = \oint_{S^1} \operatorname{tr} \left(\delta J J^{-1} \wedge (\delta J J^{-1})' \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \tag{2.26}$$

Therefore what is gained with respect to the previous paragraph is that the poles of Ω are only at points where the *direct* RHP is not solvable; however a feature of this construction is that $\delta\Omega \neq 0$ generally and so for the introduction of a tau function one has to pay additional care.

Let us note however the important fact that $\delta\Omega$, even if nonzero, has no pole along the Malgrange divisor Θ, and so extends to the whole parameter space $\mathcal{U}$. Moreover, in several cases of interest, non-vanishing of $\delta\Omega$ is actually a hint at the fact that the tau function should be regarded as a section of an appropriate line bundle. Somewhat more concretely, cover $\mathcal{U}$ with simply connected open sets $\mathcal{U}_a$ and by applying Poincaré lemma write $\delta\Omega = \delta\theta_a$ in $\mathcal{U}_a$ for some holomorphic differentials θ_a. In each $\mathcal{U}_a$ introduce τ_a as

$$\delta \log \tau_a = \Omega - \theta_a. \tag{2.27}$$

Assuming we can integrate $\delta \log g_{ab} = \theta_b - \theta_a$ to functions $g_{ab} : U_a \cap U_b \to \mathbb{C} \setminus \{0\}$ which satisfy the cocycle condition $g_{ab} g_{bc} = g_{ac}$, we can therefore regard the tau function $\tau := \{\tau_a\}$ as a section of the line bundle with transition functions g_{ab}, $\tau_a = \tau_b g_{ab}$.

Finally let us mention that all results of this section extend straightforwardly to the case of RHPs posed on a *finite union* of circles, just by taking direct sums (over the set of circles) of spaces and operators involved. We omit the details as we now go for a discussion of much more general contours.

2.3 General RHPs and Malgrange differential

Setting. Let us describe the general setting, which includes all RHPs considered in this thesis.

We allow the contour Σ to have transversal self intersections and endpoints; points where Σ has self-intersections or endpoints are called vertices. Let $\Sigma^o := \Sigma \setminus V$ be the complement in Σ of the set V of vertices, and let γ_j be the connected components of Σ^o. We assume that the number of vertexes and the number of connected components of Σ^o are finite. Finally, we assume that the γ_j's are smooth and oriented, hence defining $\pm$ sides at every point of Σ^o, as explained before ($+$ on the left, $-$ on the right).

Let $J : \Sigma^o \to \mathrm{GL}_N(\mathbb{C})$ be a matrix valued function defined on $\Sigma^o = \bigcup_j \gamma_j$. We shall always assume that $J|_{\gamma_j}$ is the restriction of a matrix $J_j(z)$ analytic for z in a neighborhood of γ_j.

Finally, for all γ_j extending to $z = \infty$ we assume that $J_j(z) = \mathbf{1} + \mathcal{O}(z^{-\infty})$ as $z \to \infty$.

Let us set the following notations. At a vertex $v \in V$ of Σ, denote n_v the number of components of Σ^o incident to v; we will denote $\gamma_{v,1},...,\gamma_{v,n_v}$ such components of Σ^o and assume that the cyclic order implied in this notation is the one induced by the standard orientation of the z-plane. Introduce $\sigma_{v,1},...,\sigma_{v,n_v} = \pm 1$ according to the fact that $\gamma_{v,j}$ is oriented toward v (in which case set $\sigma_{v,j} := -1$) or outward v (in which case set $\sigma_{v,J} := +1$). Denote $J_{v,1}(z),...,J_{v,n_v}(z)$ the restrictions of $J(z)$ to $\gamma_{v,1},...,\gamma_{v,n_v}$. We shall always assume that J satisfies the *no-monodromy* condition at $v \in V$, i.e. that $J_{v,1}^{\sigma_1} \cdots J_{v,n}^{\sigma_n} = \mathbf{1} + \mathcal{O}\left((z-v)^\infty\right)$ $\left(= \mathbf{1} + \mathcal{O}\left(z^{-\infty}\right)\right.$ if $\left.v = \infty\right)$. Note that this condition at an endpoint $v \in V$ (i.e. $n_v = 1$) means that $J(z) = \mathbf{1} + \mathcal{O}\left((z-v)^\infty\right)$ $\left(= \mathbf{1} + \mathcal{O}\left(z^{-\infty}\right)\right.$ if $\left.v = \infty\right)$ as $z \to v$.

The solution of the RHP defined by the data (Σ, J) is, by definition, an $N \times N$ matrix valued function $\Gamma = \Gamma(z)$ such that both Γ, Γ^{-1} are analytic and bounded in $\mathbb{C} \setminus \Sigma$, which satisfies the following two conditions;

- for all γ_j and all $P \in \gamma_j$, $\Gamma(z)$ must admit the limit $\Gamma_+(P)$ as $z \to P$ from the left of γ_j (not tangentially to Σ) and $\Gamma(z)$ must admit the limit $\Gamma_-(P)$ as $z \to P$ from the right of γ_j (not tangentially to Σ), and these limits must be related as $\Gamma_+ = \Gamma_- J$;
- We have a Poincaré asymptotic expansion $\Gamma(z) \to \mathbf{1} + \mathcal{O}(z^{-1})$ as $z \to \infty$ uniformly in every subsector of $\mathbb{C} \setminus \Sigma$.

Let us make a comment on the last (normalization) condition. Due to the assumption $J = \mathbf{1} + \mathcal{O}(z^{-\infty})$ as $z \to \infty$ then actually $\Gamma(z) = \mathbf{1} + \sum_{k \geq 1} \Gamma_k z^{-k}$ in the sense of a Poincaré asymptotic series (see e.g. [HS]), whose coefficients Γ_k do not depend on the sector of $\mathbb{C} \setminus \Sigma$ at ∞. If Σ does not extend to ∞ then this is a genuine Taylor expansion.

RHPs are very often formulated in slightly different ways; in all the cases considered in this thesis, they can be recast in this form by simple modifications.

RHPs and singular integral equations. We have a connection with singular integral equations, generalizing Prop. 2.2.2

Proposition 2.3.1. *The RHP $\Gamma_+ = \Gamma_- J$ is solvable if and only if so is the following singular integral equation in $L^2(\Sigma, \mathrm{d}z) \otimes \mathrm{Mat}_N \ni F$;*

$$F(z) - \mathbf{1} = \int_\Sigma \frac{F(\xi)(J(\xi) - \mathbf{1})}{\xi - z_-} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \tag{2.28}$$

where z_- denotes the boundary value to take in the singular integral.

For the proof see e.g. [CG].

Malgrange differential. In [Bc], inspired by the results for RHPs on a finite disjoint union of circles, the differential

$$\Omega := \int_\Sigma \mathrm{tr}\, \left(\Gamma_-^{-1}\Gamma_-' \delta J J^{-1}\right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{2.29}$$

was posited as an object of interest for general RHPs depending analytically on parameters $\mathbf{t} \in \mathcal{U}$. Again, it has poles on the (generalized) Malgrange *divisor* where the solution Γ does not exists, i.e. where the corresponding singular integral equation (2.28) is not solvable.

We have a slightly more involved formula for the exterior derivative, generalizing Prop. 2.2.5; however remarkably the two-form $\delta\Omega$ extends again to the whole parameter space $\mathcal{U}$, as it depends explicitly on J only, and not on Γ.

Proposition 2.3.2 ([Bc; Bd])**.** *The exterior derivative of the Malgrange differential* (2.29) *can be expressed as*

$$\delta\Omega = \int_\Sigma \mathrm{tr}\, \left(\delta J J^{-1} \wedge (\delta J J^{-1})'\right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} + \sum_{v \in V} \eta_v \tag{2.30}$$

where the contributions η_v at the vertices $v \in V$ are given as

$$\eta_v = \frac{1}{4\pi \mathrm{i}} \sum_{\ell=2}^{n_v} M_{\ell-1} \delta M_{\ell-1} \wedge \delta N_\ell N_\ell^{-1} \tag{2.31}$$

where we set (with the notations introduced above)

$$M_\ell := \lim_{z\to v} J^{\sigma_v}_{v,\ell}(z), \qquad N_\ell := M_\ell \cdots M_{n_v}. \tag{2.32}$$

The localized terms η_v vanish for endpoints. Note also that due to the no-monodromy condition η_v only depends on the cyclic ordering of rays of Σ meeting at v.

Therefore in general $\delta\Omega$ is not zero; the considerations exposed after Prop. 2.2.5 regarding the definition of the tau function as a section of an appropriate line bundle apply equally well here.

We limit ourselves to this brief overview of the topic and refer to the original literature for more details.

2.4 Schlesinger transformations

The notion of Schlesinger transformation goes back to Schlesinger [Sb]; roughly speaking it is a *discrete isomonodromic deformation* of the data in (1.2) in that we shift the formal monodromy exponents $\lambda_{\nu,\alpha}$ by integer multiples of $2\pi\mathrm{i}$. In more recent time it was reconsidered in [JM] and later extended and studied in depth in [BCc].

The results of this section are actually the core of the computations which we will perform in the applications of later chapters.

Elementary Schlesinger transformations. Suppose we are given a RHP $\Gamma_+ = \Gamma_- J$. Then we *twist* it as follows. Fix points $a, b \in \mathbb{C}\setminus\Sigma$ and indices $\alpha, \beta \in \{1, ..., N\}$. The elementary Schlesinger transform $\Gamma_{\left\{\begin{smallmatrix} a & b\\ \alpha & \beta\end{smallmatrix}\right\}}$, which we denote shortly as $\underline{\Gamma} = \underline{\Gamma}(z)$, is a matrix function of z such that $\underline{\Gamma}, \underline{\Gamma}^{-1}$ are bounded and analytic in the complement of small disks around a, b in $\mathbb{P}^1\setminus\Sigma$, satisfying the jump condition $\underline{\Gamma}_+ = \underline{\Gamma}_- J$ along Σ, the growth conditions

$$\underline{\Gamma}(z) = \begin{cases} \mathcal{O}(1)(z-a)^{\mathrm{E}_{\alpha\alpha}}, & z\to a\\ \mathcal{O}(1)(z-b)^{-\mathrm{E}_{\beta\beta}}, & z\to b\end{cases} \tag{2.33}$$

and also the normalization $\underline{\Gamma}(\infty) = \mathbf{1}$. We allow a to coincide with b, in which case we have to assume $\alpha \neq \beta$.

By an application of Liouville theorem, the elementary Schlesinger transform is unique if any exists (again, for any pair of solutions $\underline{\Gamma}_1, \underline{\Gamma}_2$ the ratio $\underline{\Gamma}_1\underline{\Gamma}_2^{-1}$ has no jump on Σ and is bounded as $z\to a, z\to b$ hence it is analytic and bounded everyhwere; it is equal to 1 at ∞ therefore it is $\mathbf{1}$ everywhere by Liouville theorem).

The simple key observation to study solvability of the twisted problem (2.33) is that if $\underline{\Gamma} = \Gamma_{\left\{\begin{smallmatrix} a & b\\ \alpha & \beta\end{smallmatrix}\right\}}$ then the ratio $R := \underline{\Gamma}\Gamma^{-1}$ is a rational function with a simple pole at $z = b$ only. This is easily seen again by the Liouville theorem (because R has no jump on Σ). Therefore we may consider the ansatz $\underline{\Gamma}(z) = R(z)\Gamma(z)$ where $R(z)$ is rational with a simple pole at $z = b$ and, due to the normalization $\underline{\Gamma}(\infty) = \mathbf{1}$, tends to $\mathbf{1}$ as $z\to\infty$. So it is convenient to set

$$R(z) = \mathbf{1} + \frac{U}{z-b} \tag{2.34}$$

for a yet unspecified matrix U. We claim that U is completely determined then by the growth conditions (2.33). First, the pole condition at $z = b$ in the column β implies

$$U\Gamma(b)\vec{e}_j = 0, \qquad j\in\{1, ..., N\}\setminus\{\beta\} \tag{2.35}$$

denoting $\vec{e}_j$ the standard basis of column vectors $\vec{e}_j := (0, ..., 1, ..., 0)^\top$, the 1 being in the jth position. This implies that U is actually rank one of the form

$$U = \vec{f}\vec{e}_\beta^\top\Gamma^{-1}(b) \tag{2.36}$$

where $\vec{f}$ is some column vector. Finally the zero condition at $z = a$ in the column α implies

$$\Gamma(a)\vec{e}_\alpha + \frac{\vec{f}\vec{e}_\beta^\top\Gamma^{-1}(b)\Gamma(a)\vec{e}_\alpha}{a-b} = 0 \tag{2.37}$$

which is solved as $\vec{f} = \frac{b-a}{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}}\Gamma(a)\vec{e}_\alpha$ and finally

$$R(z) = \mathbf{1} + \frac{b-a}{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}} \frac{\Gamma(a)\mathrm{E}_{\alpha\beta}\Gamma^{-1}(b)}{z-b}. \tag{2.38}$$

It is now easy to show that if $(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha} \neq 0$ then $\underline{\Gamma}(z) := \left(\mathbf{1} + \frac{b-a}{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}} \frac{\Gamma(a)\mathrm{E}_{\alpha\beta}\Gamma^{-1}(b)}{z-b}\right)\Gamma(z)$ is the Schlesinger transformation of Γ as defined above.

Note that everything remains true in the coalescing case $a = b$ by replacing the scalar $\frac{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}}{b-a}$ with $\left(\Gamma^{-1}(a)\Gamma'(a)\right)_{\beta\alpha}$ (in this case we must have $\alpha \neq \beta$ and so $\frac{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}}{b-a} = \frac{(\Gamma^{-1}(b)\Gamma(a)-\mathbf{1})_{\beta\alpha}}{b-a}$ and this claim follows simply by taking the limit $b \to a$).

The scalar $\frac{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}}{b-a}$ governs therefore the solvability of the twisted problem and it is legitimate to suppose that it is related to the tau function; we now show that this is indeed the case.

First, let us translate the elementary Schlesinger transformation into a RHP of the usual form; this can be done by augmenting the contour Σ to $\widetilde{\Sigma} := \Sigma \cup \partial\mathbf{D}_a \cup \partial\mathbf{D}_b$, where $\mathbf{D}_x := \{|z-x| < \epsilon\}$ for ϵ small enough so that $\Sigma, \partial\mathbf{D}_a, \partial\mathbf{D}_b$ are mutually disjoint, and extending $J : \Sigma \to \mathrm{GL}_N$ to $\widetilde{J} : \widetilde{\Sigma} \to \mathrm{GL}_N$ by

$$\widetilde{J}(z) := \begin{cases} J(z) & z \in \Sigma \\ (z-a)^{\mathrm{E}_{\alpha\alpha}} & z \in \partial\mathbf{D}_a \\ (z-b)^{-\mathrm{E}_{\beta\beta}} & z \in \partial\mathbf{D}_b \end{cases} \tag{2.39}$$

and then solving the RHP $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_-\widetilde{J}$ is equivalent to finding the Schlesinger transform $\underline{\Gamma} = \Gamma\left\{\begin{smallmatrix} a & b \\ \alpha & \beta \end{smallmatrix}\right\}$; indeed $\underline{\Gamma}$ exists if and only if $\widetilde{\Gamma}$ exists, and in such case they are related as

$$\underline{\Gamma}(z) = \begin{cases} \widetilde{\Gamma}(z)(z-a)^{\mathrm{E}_{\alpha\alpha}} & z \in \overline{\mathbf{D}_a} \\ \widetilde{\Gamma}(z)(z-b)^{-\mathrm{E}_{\beta\beta}} & z \in \overline{\mathbf{D}_b} \\ \widetilde{\Gamma} & \text{otherwise.} \end{cases} \tag{2.40}$$

Consider the associated Malgrange differentials;

$$\begin{aligned} \Omega :&= \int_\Sigma \mathrm{tr}\,\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\delta J(z)J^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \\ \widetilde{\Omega} :&= \int_{\widetilde{\Sigma}} \mathrm{tr}\,\left(\widetilde{\Gamma}_-^{-1}(z)\widetilde{\Gamma}_-'(z)\delta\widetilde{J}(z)\widetilde{J}^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \end{aligned} \tag{2.41}$$

Let us stress that the deformation δ now acts also on the parameters a, b; of course it is always understood that this infinitesimal variation does not move the disks $\mathbf{D}_a, \mathbf{D}_b$, but just act on the jump matrix $\widetilde{J}$.

Proposition 2.4.1. *We have*

$$\widetilde{\Omega} - \Omega = \delta \log \frac{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}}{b-a}. \tag{2.42}$$

The proof of this result is an explicit computation using $\underline{\Gamma}(z) = R(z)\Gamma(z)$ with $R(z)$ found as above. We omit it as we shall consider a more general case below, see Thm. 2.4.7, including this elementary case as a special case.

Note that as a consequence of this proposition, if $\delta\widetilde{\Omega} = 0 = \delta\Omega$ and so we have tau functions $\delta\log\tau = \Omega$ and $\delta\log\widetilde{\tau} = \widetilde{\Omega}$, then

$$\frac{\widetilde{\tau}}{\tau} = \frac{(\Gamma^{-1}(b)\Gamma(a))_{\beta\alpha}}{b-a}. \tag{2.43}$$

Before going for this general case let us stress more closely the connection with the Sato type formulæ of Sec. 1.3.

Schlesinger transformations and Sato formulæ. We can consider, similarly as above, the case where (informally speaking) we add a pole at ∞ and a zero at a. More precisely, this time the matrix $\underline{\Gamma}$ is required to be bounded with bounded inverse away from a and ∞, to satisfy the jump condition $\underline{\Gamma}_+ = \underline{\Gamma}_- J$ on Σ, and the growth conditions

$$\underline{\Gamma}(z) = \begin{cases} \mathcal{O}(1)(z-a)^{\mathrm{E}_{\alpha\alpha}}, & z \to a \\ (\mathbf{1} + \mathcal{O}(z^{-1}))z^{\mathrm{E}_{\beta\beta}}, & z \to \infty. \end{cases} \tag{2.44}$$

Note that the condition at ∞ is also a normalization condition. Similarly to above, we are looking for a rational matrix $R(z)$ with a pole at ∞ only such that $\underline{\Gamma} = R\Gamma$. It is easy to see that $R(z)$ must be in the form $R(z) = R_0 + \mathrm{E}_{\beta\beta} z$; denoting $\Gamma(z) = \mathbf{1} + \Gamma_1 z^{-1} + \mathcal{O}(z^{-2})$ as $z \to \infty$, we find from the condition at ∞ that

$$R_0 + \mathrm{E}_{\beta\beta}\Gamma_1 = \mathbf{1} + \vec{f}\vec{e}_\beta^{\top} \tag{2.45}$$

for some column vector $\vec{f}$. Finally from the condition at $z = a$ we obtain

$$\vec{f}\vec{e}_\beta^{\top}\Gamma(a)\vec{e}_\alpha = (\mathrm{E}_{\beta\beta}(\Gamma_1 - a) - \mathbf{1})\vec{e}_\alpha \tag{2.46}$$

hence the solution for R, and so for $\underline{\Gamma}$, requires inverting the matrix element $\vec{e}_\beta^{\top}\Gamma(a)\vec{e}_\alpha = \Gamma_{\beta\alpha}(a)$. Again this suggests the following formula, which can be actually verified (and follows from the general result below), expressing the difference of the Malgrange differentials (2.41);

$$\widetilde{\Omega} - \Omega = \delta \log(\Gamma(a))_{\beta\alpha} \tag{2.47}$$

implying the relation of tau functions (if they can be defined as $\delta \log \tau = \Omega, \delta \log \widetilde{\tau} = \widetilde{\Omega}$)

$$\frac{\widetilde{\tau}}{\tau} = \Gamma_{\beta\alpha}(a) \tag{2.48}$$

The analogy (for $\beta = \alpha$) with the Sato formula (1.34) is now manifest.

General Schlesinger transformations. A general Schlesinger transformation is a composition of elementary ones. It is interesting to study directly the effect of this composition on the Malgrange differential, following the same strategy used above for the elementary case. The arguments below are recalled from [BCc], to which we refer for further details.

First the definition. Let $\mathcal{A}, \mathcal{B}$ be two collection of points in $\mathbb{C}P^1 \setminus \Sigma$, not necessarily disjoint. Note that we allow the points in $\mathcal{A}, \mathcal{B}$ to be at infinity (including therefore also the case of the previous paragraph).

For each $a \in \mathcal{A}$ let $L_a = \mathrm{diag}(\ell_{a,1}, ..., \ell_{a,N})$ and for each $b \in \mathcal{B}$ let $K_b = \mathrm{diag}(k_{b,1}, ..., k_{b,N})$, matrices of *nonnegative integers*. Informally speaking we will twist a RHP by adding zeros at $\mathcal{A}$ and poles at $\mathcal{B}$, with multiplicities prescribed by the diagonal matrices L_a, K_b.

Assume the following consistency conditions;

- if $c \in \mathcal{A} \cap \mathcal{B}$, then $L_c K_c = 0$, and
- $\sum_{a\in\mathcal{A}} \mathrm{tr}\, L_a = \sum_{b\in\mathcal{B}} \mathrm{tr}\, K_b$.

Generalizing the above definition, the Schlesinger transform $\Gamma_{\left\{\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ L & K \end{smallmatrix}\right\}}$, which we denote shortly as $\underline{\Gamma} = \underline{\Gamma}(z)$, is a matrix function of z such that $\underline{\Gamma}, \underline{\Gamma}^{-1}$ are bounded and analytic in the complement of small disks around the points of $\mathcal{A}, \mathcal{B}$ in $\mathbb{P}^1 \setminus \Sigma$, satisfying the jump condition $\underline{\Gamma}_+ = \underline{\Gamma}_- J$ along Σ, the growth conditions

$$\underline{\Gamma}(z) = \begin{cases} \mathcal{O}(1)(z - c)^{L_c - K_c}, & z \to c \in \mathcal{C} \setminus \{\infty\} \\ (\mathbf{1} + \mathcal{O}(z^{-1}))z^{K_\infty - L_\infty}, & z \to \infty \end{cases} \tag{2.49}$$

where we set

$$\mathcal{C} := \mathcal{A} \cup \mathcal{B}. \tag{2.50}$$

The condition at ∞ includes a normalization condition. In the interest of shorter notations we agree that if $c \in \mathcal{A} \setminus \mathcal{B}$ then $L_c := 0$ and if $c \in \mathcal{B} \setminus \mathcal{A}$ then $K_c = 0$. Similarly for the point at infinity, i.e. if $\infty \notin \mathcal{A}$ then $L_\infty := 0$ and if $\infty \notin \mathcal{B}$ then $K_\infty := 0$

As before, the key observation to study solvability of (2.49) is that the ratio $R(z) := \underline{\Gamma}(z)\Gamma^{-1}(z)$ is a rational function of z, with poles at $\mathcal{B}$ only; this is a consequence of Liouville theorem as before, since R has no jump on Σ.

Again, we can translate the Schlesinger transformation into a RHP of the usual form; this can be done by augmenting the contour Σ to $\widetilde{\Sigma} := \Sigma \cup \partial\mathbf{D}$ where

$$\mathbf{D} = \bigcup_{c\in\mathcal{A}\cup\mathcal{B}} \mathbf{D}_c \tag{2.51}$$

where again $\mathbf{D}_x := \{|z-x| < \epsilon\}$ for ϵ small enough so that all these disks are are mutually disjoint and disjoint from Σ, and extending $J : \Sigma \to \mathrm{GL}_N$ to $\widetilde{J} : \widetilde{\Sigma} \to \mathrm{GL}_N$ by

$$\widetilde{J}(z) := \begin{cases} J(z) & z \in \Sigma \\ (z-a)^{L_a} & z \in \partial \mathbf{D}_a \\ (z-b)^{-K_b} & z \in \partial \mathbf{D}_b \end{cases} \tag{2.52}$$

and then solving the RHP $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_- \widetilde{J}$ is equivalent to finding the Schlesinger transform $\underline{\Gamma} = \Gamma_{\left\{ \begin{smallmatrix} a & b \\ \alpha & \beta \end{smallmatrix} \right\}}$; indeed $\underline{\Gamma}$ exists if and only if $\widetilde{\Gamma}$ exists, and in such case they are related as

$$\underline{\Gamma}(z) = \widetilde{\Gamma}(z) \begin{cases} (z-c)^{L_c - K_c} & z \in \mathbf{D}_c,\ c \in \mathcal{C} \\ \mathbf{1} & z \in \mathbb{C} \setminus \overline{\mathbf{D}}. \end{cases} \tag{2.53}$$

Consider the associated Malgrange differentials exactly as in (2.41).

Let us stress that the deformation δ now acts also on the parameters $\mathcal{A}, \mathcal{B}$; of course it is always understood that this infinitesimal variation does not move the disks $\mathbf{D}_a, \mathbf{D}_b$, but just act on the jump matrix $\widetilde{J}$.

We want to find an expression for the difference of the Malgrange differentials akin to that of Prop. 2.4.1. The first (easy) part is the following computation.

It is convenient to introduce $P : \partial \mathbf{D} \to \mathrm{GL}_N$ as

$$P(z) := (z-c)^{L_c - K_c} \tag{2.54}$$

for $z \in \mathbf{D}_c$, $c \in \mathcal{C}$. In this way we have

$$\widetilde{J}(z) = \begin{cases} J(z) & z \in \Sigma \\ P(z) & z \in \partial \mathbf{D}. \end{cases} \tag{2.55}$$

Proposition 2.4.2. *We have*

$$\widetilde{\Omega} - \Omega = \oint_{\partial \mathbf{D}} \mathrm{tr}\, \left(R^{-1} R' \delta(\Gamma P)(\Gamma P)^{-1} + \Gamma^{-1} \Gamma' \delta P P^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{2.56}$$

where $\mathbf{D}$ *is defined in* (2.51) *and* P *is defined in* (2.54).

Proof. We have, using (2.55),

$$\widetilde{\Omega} - \Omega = \int_\Sigma \mathrm{tr}\, \left(\widetilde{\Gamma}_-^{-1} \widetilde{\Gamma}_-' \delta J J^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} + \oint_{\partial \mathbf{D}} \mathrm{tr}\, \left(\widetilde{\Gamma}^{-1} \widetilde{\Gamma}' \delta P P^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} - \int_\Sigma \mathrm{tr}\, \left(\Gamma_-^{-1} \Gamma_-' \delta J J^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{2.57}$$

and then using that $\widetilde{\Gamma}_- = R\Gamma$ (as we are always on the $-$ side of $\partial \mathbf{D}$) the above is rewritten

$$\begin{aligned} \widetilde{\Omega} - \Omega = & \int_\Sigma \mathrm{tr}\, \left(R^{-1} R' \Gamma_- \delta J J^{-1} \Gamma_- \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \\ & + \oint_{\partial \mathbf{D}} \mathrm{tr}\, \left(R^{-1} R' \Gamma_- \delta P P^{-1} \Gamma_-^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} + \oint_{\partial \mathbf{D}} \mathrm{tr}\, \left(\Gamma_-^{-1} \Gamma_-' \delta P P^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \end{aligned} \tag{2.58}$$

The first term can be rewritten as follows; introducing the jump operator $\Delta_\Sigma[F] := F_+ - F_-$ we have $\Gamma_- \delta J J^{-1} \Gamma_-^{-1} = \Delta_\Sigma(\delta \Gamma \Gamma^{-1})$. Moreover by Cauchy theorem

$$\int_\Sigma \mathrm{tr}\, \left(R^{-1} R' \Gamma_- \delta J J^{-1} \Gamma_- \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} = \int_\Sigma \Delta_\Sigma \left[\mathrm{tr}\, \left(R^{-1} R' \delta \Gamma \Gamma^{-1} \right) \right] \frac{\mathrm{d}z}{2\pi \mathrm{i}} = \oint_{\partial \mathbf{D}} \mathrm{tr}\, \left(R^{-1} R' \delta \Gamma \Gamma^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{2.59}$$

and the proof is complete. ■

The characteristic matrix. We wish now to find conditions under which there exists a rational matrix R for which $\underline{\Gamma} = R\Gamma$ provides the solution to the general twisted problem described above.

This rational matrix R is defined, analogously to the examples examined above, by its expansions near its poles and zeros. Therefore it is natural to consider the following (more general) setting as follows; let Γ be a collection of *formal* germs of analytic functions at the points $c \in \mathcal{C}$ and at ∞ of the form $\Gamma = \mathbf{1} + \mathcal{O}(z^{-1})$.

We seek a rational matrix R such that:

R1: $R(z)$ is analytic and analytically invertible for $z \in \mathbb{C} \setminus \mathcal{C}$;

R2: $R(z)\Gamma(z)(z-c)^{-(L_c-K_c)}$ is a formal analytic germ for every $c \in \mathcal{C} \setminus \{\infty\}$;

R3: $R(z)\Gamma(z)z^{L_\infty - K_\infty} = \mathbf{1} + \mathcal{O}(z^{-1})$ as $z \to \infty$.

The solution R to such problem is unique (by an application of Liouville theorem). Let us now translate this problem into a finite dimensional linear problem.

To this end introduce the Hilbert space

$$\vec{H} := L^2(\partial \mathbf{D}, \mathrm{d}z) \otimes \mathbb{C}^N \tag{2.60}$$

where its element are *row*-vector valued L^2 functions on $\partial\mathbf{D}$; as $\mathbf{D}$ is a disjoin union of disks, $\vec{H}$ decomposes as an orthogonal direct sum of the Hilbert spaces $L^2(\partial\mathbf{D}_c, \mathrm{d}z) \otimes \mathbb{C}^N$ for $c \in \mathcal{C}$. Hence an element of $\vec{H}$ is a collection of Fourier expansions for every circle centered at $c \in \mathcal{C}$; accordingly, let us introduce the spaces $\vec{H}_+, \vec{H}_- \subset \vec{H}$ where elements of $\vec{H}_+$ (resp. $\vec{H}_-$) have only nonnegative (resp. negative) Fourier modes in $L^2(\partial\mathbf{D}_c, \mathrm{d}z)$ for all $c \in \mathcal{C}$. $\vec{H}_+, \vec{H}_-$ are mutually orthogonal, and let $\pm C_\pm$ the associated Cauchy projectors;

$$(C_\pm \vec{f})(z) := \oint_{S^1} \frac{\vec{f}(\xi)\mathrm{d}\xi}{(\xi - z_\pm)2\pi\mathrm{i}}. \tag{2.61}$$

Introduce the finite dimensional spaces

$$V := C_-\left(\vec{H}_+(\Gamma P)^{-1}\right), \qquad W := C_-\left(\vec{H}_+\Gamma P\right) \tag{2.62}$$

where P is defined in (2.54).

Remark 2.4.3. *Let us comment once for all on the following point (which will be relevant in the applications). It is convenient to allow both Σ to extend to ∞ and $\infty \in \mathcal{C}$. Recalling that we are assuming $J = \mathbf{1} + \mathcal{O}(z^{-\infty})$ as $z \to \infty$ along Σ, in this case Γ is only* formally analytic *at ∞, namely it has an asymptotic expansion in the sense of Poincaré and not a properly convergent Taylor expansion; even though strictly speaking the spaces V, W are not properly defined as $\vec{H}_+(\Gamma P)^{-1} \not\subseteq \vec{H}_+$ and similarly $\vec{H}_+\Gamma P \not\subseteq \vec{H}_+$ (i.e. the series are not necessarily square-summable), however we interpret $C_\pm$ as the projectors on the nonnegative (+) or negative (-) tails; moreover, strictly speaking, the integral representation (2.61) of $C_\pm$ cannot be used in such case. All the results below clearly extend to this case with no modification and we will omit any further remark of this kind.*

It is convenient to introduce the local parameters, for $c \in \mathcal{C}$,

$$z_c := \begin{cases} z - c & c \neq \infty, \\ \frac{1}{z} & c = \infty. \end{cases} \tag{2.63}$$

Lemma 2.4.4. *The spaces V, W have the same finite dimension $\sum_a \operatorname{tr} L_a = \sum_b \operatorname{tr} K_b$. More precisely, they admit bases as follows;*

$$V = \bigoplus_{\substack{a \in \mathcal{A} \\ 1 \leq \alpha \leq N \\ 1 \leq \ell \leq \ell_{a,\alpha}}} \mathbb{C} v_{a,\alpha,\ell}, \qquad W = \bigoplus_{\substack{b \in \mathcal{B} \\ 1 \leq \beta \leq N \\ 1 \leq k \leq \ell_{b,\beta}}} \mathbb{C} w_{b,\beta,k} \tag{2.64}$$

where

$$v_{a,\alpha,\ell} := C_-\left(\vec{e}_\alpha (z_a)^{\ell_{a,\alpha}-\ell}(\Gamma P)^{-1}\right) = C_-\left(\frac{\vec{e}_\alpha \Gamma^{-1}(z)}{(z_a)^{\ell - \delta_{a,\infty}}}\right) \tag{2.65}$$

$$w_{b,\beta,k} := C_-\left(\vec{e}_\beta (z_b)^{k_{b,\beta}-k} P\right) = C_-\left(\frac{\vec{e}_\beta}{(z_b)^{k-\delta_{b,\infty}}}\right). \tag{2.66}$$

The proof of this lemma is immediate, we refer to [BCc] for more details; let us note however that in writing the basis for W we have used the fact that $\vec{H}_+ = \vec{H}_+\Gamma^{-1}$ as Γ is a collection of formal germs of analytic functions for all $c \in \mathcal{C}$, and therefore $\vec{H}_+\Gamma P = \vec{H}_+ P$.

Introduce the linear map

$$\mathcal{G} : V \to W : \vec{v} \mapsto C_-(v\Gamma P). \tag{2.67}$$

Proposition 2.4.5. *The map $\mathcal{G}$ is well defined. It is invertible if and only if the rational matrix R satisfying R1, R2, R3 above exists; in such case the inverse $\mathcal{G}^{-1}$ is expressed in terms of R as*

$$\mathcal{G}^{-1}(w) = C_-(w(R\Gamma P)^{-1})R \tag{2.68}$$

and the rational matrix R is expressed in terms of $\mathcal{G}^{-1}$ as

$$R(z) = \mathbf{1} - \mathcal{G}^{-1}(C_-(\Gamma P)) \tag{2.69}$$

where in the last term we mean the matrix formed adjoining the rows $\mathcal{G}^{-1}(C_-(\vec{e}_\alpha \Gamma P))$ for $\alpha = 1, ..., N$.

For the proof we refer to [BCc].

Hence the existence of R is equivalent to the invertibility of the linear map $\mathcal{G}$. We shall now follow loc. cit. and prove that the determinant of the linear map $\mathcal{G}$ with respect to the bases of Lemma 2.4.4 provides the difference of the Malgrange differentials (2.41).

It is almost immediate to write down the matrix $\mathbf{G} = \left(\mathbf{G}_{(a,\alpha,\ell),(b,\beta,k)}\right)$ representing the linear map $\mathcal{G}$ with respect to the bases of Lemma 2.4.4 (up to a reordering of the basis for W)

$$\mathcal{G} v_{a,\alpha,\ell} = \sum_{\substack{b\in\mathcal{B} \\ 1\le\beta\le N \\ 1\le k\le \ell_{b,\beta}}} \mathbf{G}_{(a,\alpha,\ell),(b,\beta,k)} w_{b,\beta,k_{b,\beta}-k+1} \tag{2.70}$$

as

$$\mathbf{G}_{(a,\alpha,\ell),(b,\beta,k)} = \operatorname*{res}_{y=a} \operatorname*{res}_{w=b} \frac{\left(\Gamma^{-1}(w)\Gamma(y)\right)_{\beta\alpha}}{(y_a)^{\ell-\delta_{a,\infty}} (w_b)^{k-\delta_{b,\infty}} (w-y)} \mathrm{d}w \mathrm{d}y. \tag{2.71}$$

Proposition 2.4.6. *Denoting again δ the differential in the parameters of the original RHP for Γ (if any) and in the location of the points $c \in \mathcal{C}$, we have*

$$\delta \log\det \mathbf{G} = \oint_{\partial \mathbf{D}} \operatorname{tr}\left(R^{-1}R'\delta(\Gamma P)(\Gamma P)^{-1} + \Gamma^{-1}\Gamma'\delta P P^{-1}\right) \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \tag{2.72}$$

For the proof we refer to [BCc].

Finally, combining Prop. 2.4.2 with Prop. 2.4.6 we immediately get the following result regarding the difference of the Malgrange differentials (2.41).

Theorem 2.4.7. *We have*

$$\widetilde{\Omega} - \Omega = \delta \log\det \mathbf{G}. \tag{2.73}$$

Rational dressing. For later convenience we consider a very similar situation, where we *dress* a RHP $\Gamma_+ = \Gamma_- J$ to $\widehat{\Gamma}_+ = \widehat{\Gamma}_- \widehat{J}$ with $\widehat{J} = D^{-1}JD$ with a rational diagonal matrix D. We leave full generality at this point and shift the focus to the case which we will need to consider in the applications of the following chapters, that of a *polynomial diagonal matrix D*.

Fix n points $z_{\alpha,1}, ..., z_{\alpha,n_\alpha}$ ($\alpha = 1, ..., N$, $n := n_1 + ... + n_N$) in $\mathbb{C} \setminus \Sigma$ and consider the polynomial diagonal matrix

$$D = \operatorname{diag}\left(\prod_{j=1}^{n_1}(z - z_{1,j}), ..., \prod_{j=1}^{n_N}(z - z_{N,j})\right). \tag{2.74}$$

Let us also assume that all the points $z_{\alpha,j}$ ($\alpha = 1, ..., N$, $j = 1, ..., n_\alpha$) are all distinct; this is just to simplify the exposition, and the final result Thm. 2.4.8 below extends to the case of coalescing points in the sense of the limit.

Given a RHP $\Gamma_+ = \Gamma_- J$ posed on a contour Σ, $\Gamma(z) = \mathbf{1} + \mathcal{O}(z^{-1})$ as $z \to \infty$, consider the *dressed* RHP

$$\widehat{\Gamma}_+(z) = \widehat{\Gamma}_-(z)\widehat{J}(z), \ z \in \Sigma, \qquad \widehat{\Gamma}(z) = \mathbf{1} + \mathcal{O}(z^{-1}), \ z \to \infty \tag{2.75}$$

where $\widehat{J} := D^{-1}JD$.

The matrix $R := \widehat{G}D^{-1}\Gamma^{-1}$ has no jump at Σ, hence it is a rational matrix. Moreover, this is nothing new with respect to the general Schlesinger transformations studied above. Indeed $\underline{\Gamma} = \Gamma_{\left\{\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ L & K \end{smallmatrix}\right\}} := R\Gamma = \widehat{\Gamma}D^{-1}$ is the Schlesinger transformation corresponding to the data

$$\mathcal{A} = \{\text{zeros of } D^{-1}(z)\} = \{\infty\} \tag{2.76}$$
$$\mathcal{B} = \{\text{zeros of } D(z)\} = \{z_{\beta,j} : \ \beta = 1, ..., N, \ j = 1, ..., n_\beta\} \tag{2.77}$$

with (recall the simplifying assumption of distinct points $z_{\alpha,j}$)

$$\ell_{\infty,\alpha} = n_\alpha, \qquad \alpha = 1, ..., N \tag{2.78}$$
$$k_{z_{\beta,j},\beta} = 1, \qquad \beta = 1, ..., N, \ j = 1, ..., n_\beta. \tag{2.79}$$

We can therefore use the previous results to compute the difference of Malgrange differentials (note the difference with (2.41))

$$\begin{aligned} \Omega :&= \int_\Sigma \operatorname{tr}\left(\Gamma_-^{-1}(z)\Gamma_-'(z)\delta J(z)J^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \\ \widehat{\Omega} :&= \int_\Sigma \operatorname{tr}\left(\widehat{\Gamma}_-^{-1}(z)\widehat{\Gamma}_-'(z)\delta \widehat{J}(z)\widehat{J}^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \end{aligned} \tag{2.80}$$

in terms of the $n \times n$ characteristic matrix (2.71) associated with this type of Schlesinger transformation, which reads

$$\mathbf{G}_{(\alpha,\ell),(\beta,j)} = \operatorname*{res}_{y=\infty}\operatorname*{res}_{w=z_{\beta,j}} \frac{y^{\ell-1}\left(\Gamma^{-1}(w)\Gamma(y)\right)_{\beta\alpha}}{(w-z_{\beta,j})(w-y)}\mathrm{d}w\mathrm{d}y = -\operatorname*{res}_{y=\infty}\frac{y^{\ell-1}\left(\Gamma^{-1}(z_{\beta,j})\Gamma(y)\right)_{\beta\alpha}}{y-z_{\beta,j}}\mathrm{d}y \tag{2.81}$$

where the indexes are $\alpha, \beta = 1, ..., N$, $\ell = 1, ..., n_\alpha$, $j = 1, ..., n_\beta$.

Theorem 2.4.8. *We have*

$$\begin{aligned} \widehat{\Omega} - \Omega = &\ \delta \log\left(\frac{\det \mathbf{G}}{\prod_{\beta=1}^N \Delta(z_{\beta,1}, ..., z_{\beta,n_\beta})}\right) \\ &+ \int_\Sigma \operatorname{tr}\left(D^{-1}D'JD^{-1}\delta D J^{-1} - D^{-1}D'D^{-1}\delta D + D^{-1}D'\delta J J^{-1} - D^{-1}\delta D J^{-1}J'\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \end{aligned} \tag{2.82}$$

where $\Delta(z_{\beta,1}, ..., z_{\beta,n_\beta}) := \prod_{1\le j<j'\le n_\beta}(z_{\beta,j} - z_{\beta,j'})$.

Proof. We start by computing, using $\widehat{\Gamma} = R\Gamma D$ and $\widehat{J} = D^{-1}JD$,

$$\widehat{\Gamma}_-^{-1}\widehat{\Gamma}_-' = D^{-1}\Gamma_-^{-1}R^{-1}R'\Gamma_- D + D^{-1}\Gamma_-^{-1}\Gamma_-' D + D^{-1}D' \tag{2.83}$$
$$\delta\widehat{J}\widehat{J}^{-1} = D^{-1}JD^{-1}\delta D J^{-1}D - D^{-1}\delta D + D^{-1}\delta J J^{-1} D \tag{2.84}$$

hence $\operatorname{tr}\left(\widehat{\Gamma}_-^{-1}\widehat{\Gamma}_-'\delta\widehat{J}\widehat{J}^{-1}\right)$ is expressed as

$$\begin{aligned} \operatorname{tr}(&J^{-1}\Gamma_-^{-1}R^{-1}R'\Gamma_- JD^{-1}\delta D - \Gamma_-^{-1}R^{-1}R'\Gamma_-\delta D D^{-1} + \Gamma_- R^{-1}R'\Gamma_-\delta J J^{-1} \\ &+J^{-1}\Gamma_-^{-1}\Gamma_-' J\delta D D^{-1} - \Gamma_-^{-1}\Gamma_-'\delta D D^{-1} + \Gamma_-^{-1}\Gamma_-'\delta J J^{-1} \\ &+D^{-1}D'JD^{-1}\delta D J^{-1} - D^{-1}D'D^{-1}\delta D + D^{-1}D'\delta J J^{-1}) \end{aligned}$$

and using again the jump operator $\Delta_\Sigma[F] := F_+ - F_-$, the identities

$$\Gamma_+' = \Gamma_-' J + \Gamma_- J', \qquad \Gamma_-\delta J J^{-1}\Gamma_-^{-1} = \Delta_\Sigma[\delta\Gamma\Gamma^{-1}] \tag{2.85}$$

and the cyclic property of the trace we compute

$$\widehat{\Omega} - \Omega = \int_\Sigma \operatorname{tr}(\Delta_\Sigma[\Gamma^{-1}R^{-1}R'\Gamma D^{-1}\delta D + R^{-1}R'\delta\Gamma\Gamma^{-1} + \Gamma^{-1}\Gamma' D^{-1}\delta D] \tag{2.86}$$
$$- J^{-1}J'D^{-1}\delta D + D^{-1}D'JD^{-1}\delta D J^{-1} - D^{-1}D'D^{-1}\delta D + D^{-1}D'\delta J J^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{2.87}$$

Note that $\Gamma^{-1}R^{-1}R'\Gamma D^{-1}\delta D + R^{-1}R'\delta\Gamma\Gamma^{-1} = R^{-1}R'\delta(\Gamma D)(\Gamma D)^{-1}$. Moreover the integral over Σ of the jump can be rewritten thanks to Cauchy theorem as

$$\int_\Sigma \mathrm{tr}\,(\Delta_\Sigma[R^{-1}R'\delta(\Gamma D)(\Gamma D)^{-1} + \Gamma^{-1}\Gamma' D^{-1}\delta D]\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \oint_{\partial\mathbf{D}} \mathrm{tr}\,(R^{-1}R'\delta(\Gamma D)(\Gamma D)^{-1} + \Gamma^{-1}\Gamma' D^{-1}\delta D)\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.88}$$

Summarizing, we have proved

$$\begin{aligned}\widehat{\Omega} - \Omega = &\oint_{\partial\mathbf{D}} \mathrm{tr}\,(R^{-1}R'\delta(\Gamma D)(\Gamma D)^{-1} + \Gamma^{-1}\Gamma' D^{-1}\delta D)\frac{\mathrm{d}z}{2\pi\mathrm{i}}\\ &+ \int_\Sigma \mathrm{tr}\,(-J^{-1}J'D^{-1}\delta D + D^{-1}D'JD^{-1}\delta D J^{-1} - D^{-1}D'D^{-1}\delta D + D^{-1}D'\delta J J^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}}\end{aligned} \tag{2.89}$$

Our goal is to compare this expression with the expression (2.72) of Prop. 2.4.6. To this end let us introduce the diagonal matrix U (piecewise defined on $\partial\mathbf{D}$) according to $D = PU$ and note that U is by construction regular in the interior of $\partial\mathbf{D}$. Therefore we analyze the two terms in (2.72).

- $\mathrm{tr}\,(\Gamma^{-1}\Gamma'\delta P P^{-1}) = \mathrm{tr}\,(\Gamma^{-1}\Gamma'\delta D D^{-1}) - \mathrm{tr}\,(\Gamma^{-1}\Gamma'\delta U U^{-1})$ and the last term is analytic in $\mathbf{D}$ therefore by Cauchy theorem

$$\oint_{\partial\mathbf{D}} \mathrm{tr}\,(\Gamma^{-1}\Gamma'\delta P P^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \oint_{\partial\mathbf{D}} \mathrm{tr}\,(\Gamma^{-1}\Gamma'\delta D D^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.90}$$

- Inserting $D = PU$ we have

$$\mathrm{tr}\,(R^{-1}R'\delta(\Gamma D)(\Gamma D)^{-1}) = \mathrm{tr}\,(R^{-1}R'\delta(\Gamma P)(\Gamma P)^{-1}) + \mathrm{tr}\,(R^{-1}R'\Gamma P\delta U U^{-1}P^{-1}\Gamma^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{2.91}$$

hence let us introduce $R_+ = R\Gamma P$, which is analytic in $\mathbf{D}$ by construction, so to write

$$\mathrm{tr}\,(R^{-1}R'\Gamma P\delta U U^{-1}P^{-1}\Gamma^{-1}) = \mathrm{tr}\,(R_+^{-1}R_+'\delta U U^{-1}) - \mathrm{tr}\,(\Gamma^{-1}\Gamma'\delta U U^{-1}) - \mathrm{tr}\,(P^{-1}P'\delta U U^{-1}) \tag{2.92}$$

and in the right side of the last identity the only term which is not analytic in $\mathbf{D}$ is the last one, and therefore we have

$$\oint_{\partial\mathbf{D}} \mathrm{tr}\,(R^{-1}R'\delta(\Gamma D)(\Gamma D)^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \oint_{\partial\mathbf{D}} \mathrm{tr}\,(R^{-1}R'\delta(\Gamma P)(\Gamma P)^{-1} - \mathrm{tr}\,(P^{-1}P'\delta U U^{-1}))\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{2.93}$$

Summarizing again, comparing with (2.72) we have proved

$$\begin{aligned}\widehat{\Omega} - \Omega = &\delta\log\det\mathbf{G} - \oint_{\partial\mathbf{D}} \mathrm{tr}\,(P^{-1}P'U^{-1}\delta U)\frac{\mathrm{d}z}{2\pi\mathrm{i}}\\ &+ \int_\Sigma \mathrm{tr}\,(-J^{-1}J'D^{-1}\delta D + D^{-1}D'JD^{-1}\delta D J^{-1} - D^{-1}D'D^{-1}\delta D + D^{-1}D'\delta J J^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}}.\end{aligned} \tag{2.94}$$

It remains to show that

$$\oint_{\partial\mathbf{D}} \mathrm{tr}\,(P^{-1}P'U^{-1}\delta U)\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \delta\log\prod_{\beta=1}^N \Delta(z_{\beta,1},\ldots,z_{\beta,n_\beta}). \tag{2.95}$$

To this end we compute

$$\begin{aligned}\oint_{\partial\mathbf{D}} \mathrm{tr}\,(P^{-1}P'U^{-1}\delta U)\frac{\mathrm{d}z}{2\pi\mathrm{i}} &= \sum_{\beta=1}^N\sum_{k=1}^{n_\beta} \operatorname*{res}_{z=z_{\beta,k}} \frac{1}{z-z_{\beta,k}} \sum_{\substack{k'\neq k\\ k'=1,\ldots,n_\beta}} \frac{\mathrm{d}z_{\beta,k'}}{z_{\beta,k'}-z}\\ &= \sum_{\beta=1}^N \sum_{\substack{k,k'=1,\ldots,n_\beta\\ k\neq k'}} \frac{\mathrm{d}z_{\beta,k'}}{z_{\beta,k'}-z_{\beta,k}}\\ &= \sum_{\beta=1}^N \sum_{\substack{k,k'=1,\ldots,n_\beta\\ k<k'}} \frac{\mathrm{d}z_{\beta,k}-\mathrm{d}z_{\beta,k'}}{z_{\beta,k}-z_{\beta,k'}}\\ &= \sum_{\beta=1}^N \delta\log(z_{\beta,k}-z_{\beta,k'}) = \delta\log\Delta.\end{aligned} \tag{2.96}$$

The proof is complete. ∎

2.5 Malgrange and tau differentials

We briefly outline how the formal situation of the first chapter arises naturally in certain situations where the jump matrix J of a RHP $\Gamma_+ = \Gamma_- J$, which depends on parameters $J = J(z;\mathbf{t})$, can be conjugated to a matrix J_0 which is constant in $\mathbf{t}$.

To simplify the discussion let us consider explicitly the case where Ξ of (1.2) has a single pole at $z = \infty$, but the following facts admit a straightforward generalization to the multi-pole case. More concretely, introduce

$$\Xi(z;\mathbf{t}) = \sum_{k\geq 1} \mathrm{diag}(t_{1,k}, ..., t_{N,k}) z^k, \qquad \mathbf{t} = (t_{\alpha,k})_{\alpha=1,\dots,N,\ k\geq 1} \tag{2.97}$$

and suppose that J has the form

$$J(z;\mathbf{t}) = \mathrm{e}^{\Xi(z;\mathbf{t})} J_0(z) \mathrm{e}^{-\Xi(z;\mathbf{t})}. \tag{2.98}$$

It follows that Ψ is analytic off Σ and satisfies the jump condition

$$\Psi_+ = \Psi_- J_0 \tag{2.99}$$

hence $\mathcal{M} = \delta\Psi\Psi^{-1}$ is a single valued in z differential in $\mathbf{t}$. By Liouville theorem it must be given by the expression (1.5). It follows also that Γ satisfies (1.4); we finally claim that the tau differential coincides up to a simple term with the Malgrange differential in this situation.

Indeed, using (2.98) we have $\delta J J^{-1} = \delta\Xi + \cancel{\mathrm{e}^{\Xi}\delta J_0 J_0^{-1}\mathrm{e}^{-\Xi}} - J\delta\Xi J^{-1}$ and so the Malgrange differential is rewritten as

$$\int_\Sigma \mathrm{tr}\left(\Gamma_-^{-1}\Gamma_-'\delta J J^{-1}\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \int_\Sigma \mathrm{tr}\left(\Gamma_-^{-1}\Gamma_-'\delta\Xi - \Gamma_-^{-1}\Gamma_-' J\delta\Xi J^{-1}\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{2.100}$$

$$= -\int_\Sigma \Delta_\Sigma\left[\mathrm{tr}\left(\Gamma^{-1}\Gamma'\delta\Xi\right)\right]\frac{\mathrm{d}z}{2\pi\mathrm{i}} + \int_\Sigma \mathrm{tr}\left(J^{-1}J'\delta\Xi\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{2.101}$$

where we use again the jump operator $\Delta_\Sigma[F] := F_+ - F_-$. By Cauchy theorem the first integral reduces to the formal residue at $z = \infty$

$$-\int_\Sigma \Delta_\Sigma\left[\mathrm{tr}\left(\Gamma_-^{-1}\Gamma_-'\delta\Xi\right)\right]\frac{\mathrm{d}z}{2\pi\mathrm{i}} = -\operatorname*{res}_{z=\infty}\mathrm{tr}\left(\Gamma^{-1}\Gamma'\delta\Xi\right)\mathrm{d}z \tag{2.102}$$

which is the tau differential. Summarizing, we have the following relation between Malgrange and tau differentials;

$$\int_\Sigma \mathrm{tr}\left(\Gamma_-^{-1}\Gamma_-'\delta J J^{-1}\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} = -\operatorname*{res}_{z=\infty}\mathrm{tr}\left(\Gamma^{-1}\Gamma'\delta\Xi\right)\mathrm{d}z + \int_\Sigma \mathrm{tr}\left(J^{-1}J'\delta\Xi\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{2.103}$$

Note that when the Malgrange and tau differentials coincide, then we can use the formulæ of Thm. 1.2.2 to compute the logarithmic derivatives of the tau function.

Finally, let us observe that if J_0 is actually constant, i.e. independent of z, then we also have an ODE

$$\Psi' = L\Psi \tag{2.104}$$

and we get an isomonodromic system in the sense of [JMU], as explained in Sec. 1.4.2. Moreover, referring for more details to [Bc], one can formulate a RHP associated with the monodromy data of the ODE for which the Malgrange and tau differentials coincide in this case (due to particular structure of the jump matrices the difference term in (2.103) vanishes). Let us stress that this isomonodromic context was actually the original motivation for the introduction of the Malgrange differential [Mb; Bc]; indeed the Malgrange differential is a generalization of the isomonodromic tau differential, as it can be used to encode the dependence on the monodromy data, which must be kept fixed in the isomonodromic setting. We refer to the original literature for more details about this point.

Part II

Tau Functions: Applications to Matrix Models and Enumerative Geometry

CHAPTER 3

Ensembles of normal matrices with semiclassical potentials

We consider unitarily diagonalizable matrices with spectrum on some contour Σ. The set of such matrices is endowed with a (complex) measure $\mathrm{e}^{\operatorname{tr} V(M)}\mathrm{d}M$, where $V'(z)$ is a rational function (V is termed semiclassical potential*). Following [BEH] we show that the partition function of this model, as a function of (the parameters entering) V, coincides with the tau function associated with the standard Fokas–Its–Kitaev RHP for the (pseudo)orthogonal polynomials associated with the measure $\mathrm{e}^{V(z)}\mathrm{d}z$. As applications, we prove classical formulæ for the expectation values of products and ratios of characteristic polynomials and we obtain formulæ for the correlators of these models; the latter are then applied to the Gaussian and Laguerre ensembles, for which these correlators have a geometric/combinatorial relevance.*

Main references for this chapter are [BEH; BHa; Da; DYb; GGR].

3.1 The partition function and orthogonal polynomials

Let $V(z)$ be a (possibly multi-valued) function of z such that $V'(z)$ is rational, let us say with poles at $z_1, ..., z_m, \infty$ of order $d_1, ..., d_m, d_\infty$ respectively. With a notation similar to (1.2), we write

$$V(z) = V(z; \mathbf{t}) = \sum_{\nu=1,\dots,m,\infty} V_\nu(z; \mathbf{t}), \qquad V_\nu = \begin{cases} \sum_{k=1}^{d_\nu} \frac{t_{k,\nu}}{(z-z_\nu)^k} + \lambda_\nu \log(z - z_\nu) & \nu = 1, ..., m \\ \sum_{k=1}^{d_\infty} t_{k,\infty} z^k & \nu = \infty. \end{cases} \tag{3.1}$$

We consider the (complex) measure $\exp\operatorname{tr}(V(M; \mathbf{t}))\mathrm{d}M$[1] on the set of normal matrices (see App. B)

$$H_N(\Sigma) = \{U \operatorname{diag}(z_1, ..., z_N)U^\dagger : \ U \in \mathrm{U}_N, \ z_i \in \Sigma\} \tag{3.2}$$

where Σ is a smooth contour in the complex plane, avoiding branch cuts, if any is needed for the logarithms in (3.1). We assume that Σ is a (finite union of) non self-intersecting smooth contour(s), possibly with endpoints. In the case an endpoint of Σ is located at a pole z_ν of $V'(z)$ then:

- if $d_\nu > 0$ we assume that Σ lies in the region $\operatorname{Re} V > 0$ in an open neighborhood of z_ν;
- if $d_\nu = 0$ we assume $\operatorname{Re} \lambda_\nu > -1$.

Under these assumptions, the *partition function* is defined as

$$\mathcal{Z}_N(\mathbf{t}) := \int_{H_N(\Sigma)} \mathrm{e}^{\operatorname{tr}(V(M))}\mathrm{d}M = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^N \ell!} \int_{\Sigma^N} \Delta^2(z_1, ..., z_N)\mathrm{e}^{V(z_1)+...+V(z_N)}\mathrm{d}z_1 \cdots \mathrm{d}z_N \tag{3.3}$$

where here and elsewhere $\Delta(z_1, ..., z_N) := \prod_{1\le a<b\le N}(z_b - z_a)$ is the Vandermonde determinant, and we refer to (B.10) in App. B for the last identity.

[1] It would be more appropriate to consider the measure $\mathrm{e}^{-\operatorname{tr} V(M)}\mathrm{d}M$, but we want to avoid tedious signs in the formulæ of Sec. 3.5

This model is intimately connected with the theory of orthogonal polynomials, as we now review[2]. These are a sequence of monic polynomials $\pi_0(z), \pi_1(z), ..., \pi_\ell(z) = \pi_\ell(z; \mathbf{t}) = z^\ell + \cdots$, satisfying

$$\int_\Sigma \pi_\ell(z)\pi_{\ell'}(z)\mathrm{e}^{V(z)}\mathrm{d}z = h_\ell \delta_{\ell,\ell'} \tag{3.4}$$

for some $h_\ell = h_\ell(\mathbf{t})$. Note that in particular $\int_\Sigma \pi_\ell(z) z^k \mathrm{e}^{V(z)}\mathrm{d}z = 0$ for all $k < \ell$, and so we also have

$$h_\ell = \int_\Sigma z^\ell \pi_\ell(z)\mathrm{e}^{V(z)}\mathrm{d}z. \tag{3.5}$$

Assuming that that $\mathrm{e}^{V(z)}\mathrm{d}z$ has finite moments

$$m_j := \int_\Sigma z^j \mathrm{e}^{V(z)}\mathrm{d}z \tag{3.6}$$

(at least for $j \leq 2\ell - 1$ for some $\ell \geq 0$), the orthogonal polynomial $\pi_\ell(z) = z^\ell + c_{\ell-1}z^{\ell-1} + \cdots + c_0$ is uniquely determined by the linear system

$$D_\ell \begin{pmatrix} c_0 \\ \vdots \\ c_{\ell-1} \end{pmatrix} = \begin{pmatrix} m_\ell \\ \vdots \\ m_{2\ell-1} \end{pmatrix}, \qquad D_\ell := \begin{pmatrix} m_0 & m_1 & \cdots & m_{\ell-1} \\ m_1 & m_2 & \cdots & m_\ell \\ \vdots & \vdots & \ddots & \vdots \\ m_{\ell-1} & m_{\ell+1} & \cdots & m_{2\ell-2} \end{pmatrix} \tag{3.7}$$

hence by Cramer rule

$$\pi_\ell = \frac{1}{D_\ell} \det \begin{pmatrix} m_0 & m_1 & \cdots & m_\ell \\ m_1 & m_2 & \cdots & m_{\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{\ell-1} & m_\ell & \cdots & m_{2\ell-1} \\ 1 & z & \cdots & z^\ell \end{pmatrix}. \tag{3.8}$$

Therefore the orthogonal polynomial $\pi_\ell(z)$ exists and is unique provided that $\det D_\ell \neq 0$[3].

In the following we assume that $\pi_0(z), ..., \pi_{2N-1}(z)$ exist; this is true in a full-measure open set in the space of parameters $\mathbf{t}$, by the above discussion.

Lemma 3.1.1. *Let $\int_\Sigma \pi_\ell^2(z)\mathrm{e}^{V(z)}\mathrm{d}z = h_\ell = h_\ell(\mathbf{t})$ as in (3.4), then the partition function (3.3) admits the expression*

$$\mathcal{Z}_N(\mathbf{t}) := \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N-1} \ell!} \prod_{\ell=0}^{N-1} h_\ell(\mathbf{t}). \tag{3.9}$$

Proof. Recall the Vandermonde determinant $\Delta(z_1, ..., z_N) = \prod_{1 \leq a < b \leq N}(z_b - z_a) = \det\left(z_i^{j-1}\right)_{i,j=1}^N$. By the properties of the determinant, since $\pi_\ell(z) = z^\ell + ...$ are monic polynomials, we may write $\Delta(z_1, ..., z_N) = \det\left(\pi_{j-1}(z_i)\right)_{i,j=1}^N$. Expanding the square of the determinant in (3.3) we have

$$\mathcal{Z}_N(\mathbf{t}) = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N} \ell!} \sum_{\sigma,\rho \in \mathfrak{S}_N} (-1)^{|\sigma||\rho|} \int_{\Sigma^N} \prod_{j=1}^N \pi_{\sigma(j)-1}(z_j)\pi_{\rho(j)-1}(z_j)\mathrm{e}^{V(z_j)}\mathrm{d}z_j. \tag{3.10}$$

Due to orthogonality, terms in which $\pi = \rho$ give the only nonzero contributions, hence

$$\mathcal{Z}_N(\mathbf{t}) = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N} \ell!} N! h_0(\mathbf{t}) \cdots h_\ell(\mathbf{t}) \tag{3.11}$$

and the proof is complete. ■

We now recall two fundamental properties of orthogonal polynomials, the *three-term recurrence* and the *Christoffel–Darboux formula*.

[2]A better name would be pseudo-orthogonal; orthogonality should be understood in the L^2 sense, i.e. $\int_\Sigma \pi_\ell \overline{\pi_{\ell'}} \mathrm{e}^V \mathrm{d}z = h_\ell \delta_{\ell,\ell'}$. Note however that when $\Sigma \subseteq \mathbb{R}$ these are really orthogonal polynomials.

[3]Note that this is always the case when $\Sigma \subseteq \mathbb{R}$ as in this case D_ℓ is a symmetric positive-definite matrix.

Lemma 3.1.2 (three-term recurrence)**.** *The monic orthogonal polynomials satisfy the following recursion, as long as they exist;*

$$z\pi_\ell(z) = \pi_{\ell+1}(z) + \beta_\ell \pi_\ell(z) + \frac{h_\ell}{h_{\ell-1}}\pi_{\ell-1}(z), \qquad \ell \geq 1. \tag{3.12}$$

Proof. As orthogonality implies linear independence, the polynomials $\pi_0(z), ..., \pi_\ell(z)$ form a basis of the space of polynomials of degree $\leq \ell$. The difference $z\pi_\ell(z) - \pi_{\ell+1}(z)$ is a polynomial of degree $\leq \ell$ hence it must be a linear combination of $\pi_0(z), ..., \pi_\ell(z)$, let us say

$$z\pi_\ell(z) - \pi_{\ell+1}(z) = \sum_{j=0}^{\ell} \beta_j \pi_j(z). \tag{3.13}$$

Then, exploiting the orthogonality property (3.4),

$$\begin{aligned}
\beta_0 &= \frac{1}{h_0}\int_\Sigma \sum_{j=0}^{\ell} \beta_j \pi_j(z) \mathrm{e}^{V(z)} \mathrm{d}z = \frac{1}{h_0}\int_\Sigma (z\pi_\ell(z) - \pi_{\ell+1}(z))\mathrm{e}^{V(z)}\mathrm{d}z = 0 \\
\beta_1 &= \frac{1}{h_1}\int_\Sigma \sum_{j=1}^{\ell} z\beta_j \pi_j(z) \mathrm{e}^{V(z)} \mathrm{d}z = \frac{1}{h_1}\int_\Sigma (z^2\pi_\ell(z) - z\pi_{\ell+1}(z))\mathrm{e}^{V(z)}\mathrm{d}z = 0 \\
&\dots \\
\beta_{\ell-2} &= \frac{1}{h_{\ell-2}}\int_\Sigma \sum_{j=\ell-2}^{\ell} z^{\ell-2}\beta_j \pi_j(z) \mathrm{e}^{V(z)} \mathrm{d}z = \frac{1}{h_{\ell-2}}\int_\Sigma (z^{\ell-1}\pi_\ell(z) - z^{\ell-2}\pi_{\ell+1}(z))\mathrm{e}^{V(z)}\mathrm{d}z = 0 \\
\beta_{\ell-1} &= \frac{1}{h_{\ell-1}}\int_\Sigma \sum_{j=\ell-1}^{\ell} z^{\ell-1}\beta_j \pi_j(z) \mathrm{e}^{V(z)} \mathrm{d}z = \frac{1}{h_{\ell-1}}\int_\Sigma (z^{\ell}\pi_\ell(z) - z^{\ell-1}\pi_{\ell+1}(z))\mathrm{e}^{V(z)}\mathrm{d}z = \frac{h_\ell}{h_{\ell-1}}.
\end{aligned}$$

where we used (3.5). The proof is complete. ∎

Lemma 3.1.3 (Christoffel–Darboux formula)**.** *For all $N > 0$ we have*

$$\sum_{\ell=0}^{N-1} \frac{\pi_\ell(z)\pi_\ell(w)}{h_\ell} = \frac{1}{h_{N-1}} \frac{\pi_N(z)\pi_{N-1}(w) - \pi_{N-1}(z)\pi_N(w)}{z-w}. \tag{3.14}$$

Proof. We exploit the three-term recurrence (3.12) as follows;

$$\begin{aligned}
(z-w)\sum_{\ell=0}^{N-1} \frac{\pi_\ell(z)\pi_\ell(w)}{h_\ell} &= \sum_{\ell=0}^{N-1} \frac{z\pi_\ell(z)\pi_\ell(w)}{h_\ell} - \sum_{\ell=0}^{N-1} \frac{\pi_\ell(z)w\pi_\ell(w)}{h_\ell} \\
&= \sum_{\ell=0}^{N-1} \frac{\pi_{\ell+1}(z)\pi_\ell(w)}{h_\ell} + \cancel{\frac{\beta_\ell\pi_\ell(z)\pi_\ell(w)}{h_\ell}} + \frac{\pi_{\ell-1}(z)\pi_\ell(w)}{h_{\ell-1}} \\
&\qquad - \frac{\pi_\ell(z)\pi_{\ell+1}(w)}{h_\ell} - \cancel{\frac{\beta_\ell\pi_\ell(z)\pi_\ell(w)}{h_\ell}} - \frac{\pi_\ell(z)\pi_{\ell-1}(w)}{h_{\ell-1}} \\
&= \sum_{\ell=0}^{N-1} \frac{\pi_{\ell+1}(z)\pi_\ell(w)}{h_\ell} - \frac{\pi_\ell(z)\pi_{\ell-1}(w)}{h_{\ell-1}} - \frac{\pi_\ell(z)\pi_{\ell+1}(w)}{h_\ell} + \frac{\pi_{\ell-1}(z)\pi_\ell(w)}{h_{\ell-1}}
\end{aligned}$$

but this is a telescopic sum, hence only the term with $\ell = N-1$ survives (observe that we must set $\pi_{-1} := 0$ to make the three-term recurrence valid for $\ell = 0$) and so

$$(z-w)\sum_{\ell=0}^{N-1} \frac{\pi_N(z)\pi_\ell(w)}{h_\ell} = \frac{\pi_N(z)\pi_{N-1}(w) - \pi_{N-1}(z)\pi_N(w)}{h_{N-1}}. \tag{3.15}$$

The proof is complete. ∎

Note the *confluent* Christoffel–Darboux formula

$$\sum_{\ell=0}^{N-1} \frac{\pi_\ell^2(z)}{h_\ell} = \frac{\pi_N(z)\pi'_{N-1}(z) - \pi_{N-1}(z)\pi'_N(z)}{h_{N-1}} \tag{3.16}$$

which is obtained from (3.14) by taking the limit $w \to z$.

Let us point out that the results exposed here and below (with the main exception of Sec. 3.5) hold true for much more general weights than $\mathrm{e}^{V(z)}$ with $V'(z)$ rational in z; our choice is dictated by the fact that under this assumption the isomonodromic method applies and the results of Chap. 1 become available. For more general informations about orthogonal polynomials we refer to the literature, e.g. [Da].

Probabilistic interpretation for real contours If Σ is a finite union of intervals in $\mathbb{R}$ and V is real, the measure $\frac{1}{\mathcal{Z}_N}\mathrm{e}^{\mathrm{tr}\, V(M)}\mathrm{d}M$ is actually a probability measure on the space H_N of hermitian matrices of size N. Using (B.10) the joint probability density of the N eigenvalues $x_1, ..., x_N$ can be expressed as[4]

$$\frac{1}{\widetilde{\mathcal{Z}}_N}\Delta^2(x_1, ..., x_N)\mathrm{e}^{V(x_1)+\cdots+V(x_N)} = \frac{1}{N!}\det\left(K_N(x_i, x_j)\right)_{i,j=1}^N$$
$$\widetilde{\mathcal{Z}}_N := \frac{1}{N!}h_0\cdots h_{N-1}$$

where $K_N(x, y)$ is called *correlation kernel* and is defined as

$$K_N(x, y) := \sum_{\ell=0}^{N-1}\frac{\pi_\ell(x)\pi_\ell(y)}{h_\ell}\mathrm{e}^{\frac{1}{2}(V(x)+V(y))} = \frac{1}{h_{N-1}}\frac{\pi_N(x)\pi_{N-1}(y) - \pi_{N-1}(x)\pi_N(y)}{x - y}\mathrm{e}^{\frac{1}{2}(V(x)+V(y))} \tag{3.17}$$

where we used (3.14) in the last equality. This is the integral kernel of the orthogonal projector of $L^2(\Sigma, \mathrm{d}x)$ onto the span of $\pi_0(x)\mathrm{e}^{\frac{1}{2}V(x)}, ..., \pi_{N-1}(x)\mathrm{e}^{\frac{1}{2}V(x)}$. The square of a projector concides with the projector itself, hence we have the *self-reproducing property*

$$\int_\Sigma K_N(x, t)K_N(t, y)\mathrm{d}t = K_N(x, y) \tag{3.18}$$

which in turn allows to write all *correlation functions* of the eigenvalues in a determinantal form

$$\rho_k(x_1, ..., x_k) := \frac{1}{N!}\int_{\mathbb{R}^{N-k}}\det\left(K_N(x_i, x_j)\right)_{i,j=1}^N \mathrm{d}x_{k+1}\cdots\mathrm{d}x_N = \frac{\det\left(K_N(x_i, x_j)\right)_{i,j=1}^k}{(N-k+1)_k} \tag{3.19}$$

in terms of the same kernel, whence the name correlation kernel.

Let us mention that (3.19) implies that this statistical model for the eigenvalues is a *determinantal point field*, which is by definition a random point field whose correlation functions admit the expression (3.19) for some kernel, for all $k \geq 2$ [Sc]. For the same reason, appropriate scaling limits as $N \to \infty$ of this model give rise to determinantal point fields as well. Finally we point out that all such examples that we shall encounter belong to the *integrable type* of Its–Izergin–Slavnov–Korepin [IIKS].

3.2 The standard RHP for orthogonal polynomials

We now define the standard RHP for orthogonal polynomials of Its–Fokas–Kitaev [IKF]. First off, introduce the *Cauchy–Hilbert transforms*

$$\widehat{\pi}_\ell(z) := \frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\pi_\ell(w)\mathrm{e}^{V(w)}}{w - z}\mathrm{d}w. \tag{3.20}$$

Lemma 3.2.1. *We have the Poincaré asymptotic expansion*

$$\widehat{\pi}_\ell(z) \sim -\frac{1}{2\pi\mathrm{i}}\frac{1}{z^{\ell+1}}\sum_{j\geq 0}\frac{1}{z^j}\int_\Sigma w^{j+\ell}\pi_\ell(w)\mathrm{e}^{V(w)}\mathrm{d}w \tag{3.21}$$

as $z \to \infty$, uniformly within any sector of $\mathbb{C}\setminus\Sigma$.

The above with the understanding that if Σ does not extend to $z = \infty$ then the expansion is valid in an open neighborhood of $z = \infty$, and is in particular a Taylor expansion.

[4] According to the fact that in this paragraph $\Sigma \subseteq \mathbb{R}$ we use the variable x instead of z.

Proof. For any $J \geq 0$ we have the identity

$$\frac{1}{w-z} + \frac{1}{z}\sum_{j=0}^{J-1}\frac{w^j}{z^j} = \frac{1}{z^J}\frac{w^J}{w-z} \tag{3.22}$$

and so

$$\widehat{\pi}_\ell(z) + \frac{1}{2\pi \mathrm{i}}\sum_{j=0}^{J-1}\frac{1}{z^j}\int_\Sigma w^j \pi_\ell(w)\mathrm{e}^{V(w)}\mathrm{d}w = \frac{1}{2\pi \mathrm{i} z^J}\int_\Sigma \frac{w^J \mathrm{e}^{V(w)}}{w-z}\mathrm{d}w \tag{3.23}$$

In any open subsector at $z=\infty$ of $\mathbb{C}\setminus\Sigma$ the above remainder can be estimated as

$$\left|\frac{1}{2\pi \mathrm{i} z^J}\int_\Sigma \frac{w^J \mathrm{e}^{V(w)}}{w-z}\mathrm{d}w\right| = \frac{1}{z^{J+1}}\left|\frac{1}{2\pi}\int_\Sigma \frac{w^J \mathrm{e}^{V(w)}}{1-\frac{w}{z}}\mathrm{d}w\right| < Kz^{J+1} \tag{3.24}$$

for some K depending on the opening of the subsector and J only. Hence we have proven the asymptotic expansion

$$\widehat{\pi}_\ell(z) \sim \frac{1}{z^{\ell+1}}\sum_{j\geq 0}\frac{a_j}{z^j}, \qquad a_j := -\frac{1}{2\pi \mathrm{i}}\int_\Sigma w^j \pi_\ell(w)\mathrm{e}^{V(w)}\mathrm{d}w \tag{3.25}$$

as $z\to\infty$, uniformly within any open subsector of $\mathbb{C}\setminus\Sigma$. Uniformity in all sectors of $\mathbb{C}\setminus\Sigma$ is proven by slightly rotating the contour of integration Σ at $z=\infty$, thanks to Cauchy theorem. Finally, $a_j = 0$ for $j=0,\dots,\ell-1$ by orthogonality and the proof is complete. ∎

In a similar way one can analyze the behavior of the Cauchy–Hilbert transform $\widehat{\pi}_\ell(z)$ near the endpoints of Σ. When the endpoint is located at a pole z_ν of $V'(z)$ then $\widehat{\pi}_\ell(z)$ is bounded near z_ν (and has an asymptotic expansion which is computed as in Lemma 3.2.1). When the endpoint is not located at a pole of $V'(z)$ then $\widehat{\pi}_\ell(z)$ has a logarithmic singularity.

Next, by the Sokhotski–Plemelj formulæ (2.5) we also note that the Cauchy–Hilbert transforms $\widehat{\pi}_\ell(z)$ admit boundary values $\widehat{\pi}_\ell(z_\pm)$ for $z\in\Sigma$ from the two sides of Σ, and the latter are related as

$$\widehat{\pi}_\ell(z_+) = \widehat{\pi}_\ell(z_-) + \pi_\ell(z)\mathrm{e}^{V(z)}, \qquad z\in\Sigma. \tag{3.26}$$

Introduce the matrix

$$\Gamma(z) := \begin{pmatrix} \pi_N(z) & \widehat{\pi}_N(z) \\ -\frac{2\pi \mathrm{i}}{h_{N-1}}\pi_{N-1}(z) & -\frac{2\pi \mathrm{i}}{h_{N-1}}\widehat{\pi}_{N-1}(z)\end{pmatrix} \tag{3.27}$$

omitting the dependence on N. A direct corollary of Lemma 3.2.1 and (3.26) is that $\Gamma(z)$ is the unique solution to the following

RHP 3.2.2 (Its–Fokas–Kapaev [IKF]).

$$\begin{cases}\Gamma_+(z) = \Gamma_-(z)J(z), & z\in\Sigma \\ \Gamma(z) = \left(\mathbf{1}+\mathcal{O}\left(z^{-1}\right)\right)z^{N\sigma_3}, & z\to\infty\end{cases} \tag{3.28}$$

where the jump matrix is given as

$$J(z) := \begin{pmatrix} 1 & \mathrm{e}^{V(z)} \\ 0 & 1\end{pmatrix}. \tag{3.29}$$

RHP 3.2.2 must be complemented with suitable growth conditions at the endpoints of Σ, provided by the analysis sketched after the proof of Lemma 3.2.1.

The growth condition at $z=\infty$ in RHP 3.2.2 is not exactly as in the general setting of the last chapter; the understanding here and in similar situations which will occur below, is that this type of RHP can be recast in the prototypical form (with the normalization condition $\Gamma(\infty)=\mathbf{1}$) by replacing $\Gamma(z)\mapsto\Gamma(z)z^{-N\sigma_3}$ and $J(z)\mapsto z^{N\sigma_3}J(z)z^{-N\sigma_3} = \begin{pmatrix} 1 & z^{2N}\mathrm{e}^{V(z)} \\ 0 & 1\end{pmatrix}$.

Finally, let us note that $\det\Gamma(z)$ has no jump along Σ, and goes to 1 as $z\to\infty$, therefore $\det\Gamma(z)\equiv 1$; in particular

$$\pi_N(z)\widehat{\pi}_{N-1}(z) - \pi_{N-1}(z)\widehat{\pi}_N(z) \equiv -\frac{h_{N-1}}{2\pi \mathrm{i}}. \tag{3.30}$$

Connection with a system of monodromy-preserving equations Note that the jump matrix (3.29) can be conjugated to a constant matrix

$$J(z) = \begin{pmatrix} 1 & \mathrm{e}^{V(z)} \\ 0 & 1 \end{pmatrix} = \mathrm{e}^{\frac{\sigma_3}{2}V(z)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{e}^{-\frac{\sigma_3}{2}V(z)}. \tag{3.31}$$

Define

$$\Psi = \Psi(z;\mathbf{t}) := \Gamma(z;\mathbf{t})\mathrm{e}^{\frac{\sigma_3}{2}V(z;\mathbf{t})} \tag{3.32}$$

which is a (possibly) multi-valued function, analytic in $z \in \mathbb{C} \setminus (\Sigma \bigcup \{z_1, ..., z_m\})$. The ratio

$$L := \Psi' \Psi^{-1} \tag{3.33}$$

is continuous on Σ because

$$\Psi_+ = \Psi_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \Psi'_+ = \Psi'_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.34}$$

hence

$$L_+ = \Psi'_+ \Psi_+^{-1} = \Psi'_- \Psi_-^{-1} = L_-. \tag{3.35}$$

Therefore L extends to a function which is analytic in $z \in \mathbb{C} \setminus (\partial\Sigma \cup \{z_1, ..., z_m\})$. Moreover it is easy to see that L has at worst a pole of order d_ν at $z = z_\nu$, see (3.1), and a simple pole at the endpoints of Σ which are not located at zeros of $V'(z)$. Therefore L is actually a meromorphic function of z. Similarly, the differential in the parameters $\mathcal{M} := \delta\Psi\Psi^{-1}$ depends rationally on z and by Liouville theorem

$$\mathcal{M} = \sum_{\nu=1,\dots,m,\infty} \operatorname*{res}_{w=z_\nu} \Gamma_\nu(w;\mathbf{t}) \frac{\delta V(w;\mathbf{t})\sigma_3}{2} \Gamma_\nu^{-1}(w;\mathbf{t}) \frac{\mathrm{d}w}{z-w}. \tag{3.36}$$

Hence the compatible system of linear ODEs

$$\Psi' = L\Psi, \qquad \delta\Psi = \mathcal{M}\Psi \tag{3.37}$$

is an isomonodromic system in the sense of [JMU] (compare with Sec. 1.4.2 and Sec. 2.5).

Here the isomonodromic times $\mathbf{t}$ include $t_{k,\nu}, z_\nu$ and the endpoints of Σ. The latter are fuchsian singularities of $\Psi' = L\Psi$, whose motion is governed by the relative isomonodromic deformation equations. This can be used for applications in the context illustrated in 3.1 in connection with *gap probabilities* [TW].

3.3 The partition function as a tau function

The following theorem was originally proven in [BEH].

Theorem 3.3.1. *The partition function $\mathcal{Z}_N$ defined in (3.3) coincides with the isomonodromic tau function of the monodromy-preserving deformation system of orthogonal polynomials (3.37), i.e. the following relation holds true*

$$\delta \log \mathcal{Z}_N = \Omega \tag{3.38}$$

where Ω denotes the Malgrange differential of the RHP for orthogonal polynomials (3.2.2).

Proof. We start by differentiating the orthogonality relation (3.4) as

$$\delta h_\ell = \int_\Sigma \pi_\ell^2(z)\delta V(z)\mathrm{e}^{V(z)}\mathrm{d}z \tag{3.39}$$

where we note that the term $2\int_\Sigma \pi_\ell(z)\delta\pi_\ell(z)\mathrm{e}^{V(z)}\mathrm{d}z$ vanishes due to orthogonality; indeed the degree of $\delta\pi_\ell(z)$ as a polynomial in z is strictly less than ℓ, as $\pi_\ell(z)$ is monic. Hence, using Lemma 3.1.1, we have

$$\delta \log Z_N = \sum_{\ell=0}^{N-1} \frac{\delta h_\ell}{h_\ell} = \sum_{\ell=0}^{N-1} \int_\Sigma \frac{\pi_\ell^2}{h_\ell} \delta V \mathrm{e}^V \mathrm{d}z = \int_\Sigma \frac{\pi_N \pi'_{N-1} - \pi'_N \pi_{N-1}}{h_{N-1}} \delta V \mathrm{e}^V \mathrm{d}z$$

where we have used the confluent Christoffel–Darboux formula (3.16). Finally we see that, using (3.30),

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\mathrm{tr}\,\left(\Gamma^{-1}\Gamma'\delta J J^{-1}\right)\\
&= \frac{1}{2\pi \mathrm{i}}\mathrm{tr}\,\left(\begin{pmatrix} -\frac{2\pi\mathrm{i}}{h_{N-1}}\widehat{\pi}_{N-1} & -\widehat{\pi}_N \\ \frac{2\pi\mathrm{i}}{h_{N-1}}\pi_{N-1} & \pi_N\end{pmatrix}\begin{pmatrix} \pi'_N & \widehat{\pi}'_N \\ -\frac{2\pi\mathrm{i}}{h_{N-1}}\pi'_{N-1} & -\frac{2\pi\mathrm{i}}{h_{N-1}}\widehat{\pi}'_{N-1}\end{pmatrix}\begin{pmatrix} 0 & \delta V \mathrm{e}^V \\ 0 & 0\end{pmatrix}\begin{pmatrix} 1 & -\mathrm{e}^V \\ 0 & 1\end{pmatrix}\right)\\
&= \frac{\pi_N\pi'_{N-1} - \pi'_N\pi_{N-1}}{h_{N-1}}\delta V \mathrm{e}^V
\end{aligned}$$

and the proof is complete. ■

3.4 Expectation values of products and ratios of characteristic polynomials

We can provide a first application of Thm. 3.3.1 and of the theory of Schlesinger transformations (Sec. 2.4) giving an alternative proof of the following formula for expectations of products and ratios of characteristic polynomials, originally proven in [BHa; BDS].

Introduce the notation

$$\langle f(M)\rangle := \frac{1}{\mathcal{Z}_N(\mathbf{t})}\int_{\mathrm{H}_N(\Sigma)} f(M)\mathrm{e}^{\mathrm{tr}\,V(M;\mathbf{t})}\mathrm{d}M \tag{3.40}$$

for any scalar function $f \in L^1(\mathrm{H}_N(\Sigma), \mathrm{e}^{\mathrm{tr}\,V(M)}\mathrm{d}M)$. Note that we are omitting the size N in the notation (3.40).

Theorem 3.4.1. *For any $a_1, ..., a_\ell, b_1, ..., b_m \in \mathbb{C}\setminus\Sigma$ with $0 \le \ell, 0 \le m \le N$ we have*

$$\begin{aligned}
&\left\langle \frac{\prod_{i=1}^{\ell}\det(a_i\mathbf{1} - M)}{\prod_{i=1}^{m}\det(b_i\mathbf{1} - M)}\right\rangle_N\\
&= \frac{(-1)^{\frac{\ell(\ell-1)}{2}+m}\prod_{j=1}^{m}\frac{2\pi\mathrm{i}}{h_{N-j}}}{\Delta(a_1, ..., a_\ell)\Delta(b_1, ..., b_m)}\det\begin{pmatrix} \pi_{N-m}(a_1) & \cdots & \pi_{N-m}(a_\ell) & \widehat{\pi}_{N-m}(b_1) & \cdots & \widehat{\pi}_{N-m}(b_m)\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi_{N+\ell-1}(a_1) & \cdots & \pi_{N+\ell-1}(a_\ell) & \widehat{\pi}_{N+\ell-1}(b_1) & \cdots & \widehat{\pi}_{N+\ell-1}(b_m)\end{pmatrix}
\end{aligned} \tag{3.41}$$

Note the case $\ell = 1, m = 0$ which recovers the formula of Heine

$$\langle \det(a\mathbf{1} - M)\rangle_N = \pi_N(a) \tag{3.42}$$

expressing the monic orthogonal polynomial as the expectation value of the characteristic polynomial.

Proof. We have

$$\left\langle \frac{\prod_{i=1}^{\ell}\det(M - a_i\mathbf{1})}{\prod_{i=1}^{m}\det(M - b_i\mathbf{1})}\right\rangle_N = \frac{\widehat{\mathcal{Z}}_N}{\mathcal{Z}_N} \tag{3.43}$$

where

$$\widehat{\mathcal{Z}}_N := \int_{\mathrm{H}_N(\Sigma)} \frac{\prod_{i=1}^{\ell}\det(M - a_i\mathbf{1})}{\prod_{i=1}^{m}\det(M - b_i\mathbf{1})}\mathrm{e}^{\mathrm{tr}\,V(M)}\mathrm{d}M = \int_{\mathrm{H}_N(\Sigma)} \mathrm{e}^{\mathrm{tr}\,\widehat{V}(M)}\mathrm{d}M \tag{3.44}$$

the potential $\widehat{V}(z)$ being defined by the relation

$$\mathrm{e}^{\widehat{V}(z) - V(z)} := \frac{\prod_{i=1}^{\ell}(z - a_i)}{\prod_{i=1}^{m}(z - b_i)}. \tag{3.45}$$

Note that $\widehat{V}'(z)$ is rational again, hence Thm. 3.3.1 applies to $\mathcal{Z}_N$ and $\widehat{\mathcal{Z}}_N$. More precisely, let $J, \widehat{J}$ be the jump matrices of the RHPs 3.2.2 associated with the measure $\mathrm{e}^{V(z)}\mathrm{d}z, \mathrm{e}^{\widehat{V}(z)}\mathrm{d}z$ on Σ, respectively. Then

$$J = \begin{pmatrix} 1 & \mathrm{e}^{V(z)} \\ 0 & 1 \end{pmatrix}, \quad \widehat{J} = \begin{pmatrix} 1 & \mathrm{e}^{\widehat{V}(z)} \\ 0 & 1 \end{pmatrix} = D^{-1} J D, \quad D(z) := \begin{pmatrix} \prod_{i=1}^{m}(z-b_i) & 0 \\ 0 & \prod_{i=1}^{\ell}(z-a_i) \end{pmatrix} \tag{3.46}$$

so that denoting $\Gamma, \widehat{\Gamma}$ the solutions of the respective RHPs, Thm. 3.3.1 implies that

$$\delta \log \frac{\widehat{\mathcal{Z}}_N}{\mathcal{Z}_N} = \widehat{\Omega} - \Omega \tag{3.47}$$

where

$$\Omega := \int_\Sigma \mathrm{tr}\left(\Gamma^{-1}(z)\Gamma'(z)\delta J(z) J^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi \mathrm{i}}, \qquad \widehat{\Omega} := \int_\Sigma \mathrm{tr}\left(\widehat{\Gamma}^{-1}(z)\widehat{\Gamma}'(z)\delta \widehat{J}(z) \widehat{J}^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{3.48}$$

and δ is the differential with respect to all parameters $\mathbf{t}, a_1, ..., a_\ell, b_1, ..., b_m$. This is precisely the setting of Thm. 2.4.8, hence translating to the present situation we have

$$\delta \log \frac{\widehat{\mathcal{Z}}_N}{\mathcal{Z}_N} = \widehat{\Omega} - \Omega = \delta \log \left(\frac{\det \mathbf{G}}{\Delta(a_1, ..., a_\ell)\Delta(b_1, ..., b_m)} \right) \tag{3.49}$$

(all the other terms in the statement of 2.4.8 vanish due to the structure $\mathbf{1}$+strictly upper triangular of the jump matrix J). Here the matrix $\mathbf{G}$ is found directly from the general form (2.81)[5]; it has the structure

$$\mathbf{G} = (\mathbf{A}|\mathbf{B}) \tag{3.50}$$

where $\mathbf{A} = (\mathbf{A}_{k,j})$ $(k = 1, ..., \ell+m, j = 1, ..., \ell)$ is a $(\ell+m)\times \ell$ rectangular matrix with entries

$$\mathbf{A}_{k,j} = -\operatorname*{res}_{y=\infty} \frac{y^{N+k-1}\left(\Gamma^{-1}(a_j)\Gamma(y)\right)_{22}}{y-a_j}\mathrm{d}y \tag{3.51}$$

while $\mathbf{B} = (\mathbf{B}_{k,j})$ $(k = 1, ..., \ell+m, j = 1, ..., m)$ is a $(\ell+m)\times m$ rectangular matrix with entries

$$\mathbf{B}_{k,j} = -\operatorname*{res}_{y=\infty} \frac{y^{-N+k-1}\left(\Gamma^{-1}(b_j)\Gamma(y)\right)_{11}}{y-b_j}\mathrm{d}y. \tag{3.52}$$

Therefore the entries of $\mathbf{A}$ are found from the expansion as $y \to \infty$ of

$$\frac{\left(\Gamma^{-1}(a_j)\Gamma(y)\right)_{22}}{y-a_j} = -\frac{2\pi \mathrm{i}}{h_{N-1}} \frac{\pi_N(a_j)\widehat{\pi}_{N-1}(y) - \pi_{N-1}(a_j)\widehat{\pi}_N(y)}{y-a_j} \tag{3.53}$$

and those of $\mathbf{B}$ from that of

$$\frac{\left(\Gamma^{-1}(b_j)\Gamma(y)\right)_{11}}{y-b_j} = -\frac{2\pi \mathrm{i}}{h_{N-1}} \frac{\pi_N(y)\widehat{\pi}_{N-1}(b_j) - \pi_{N-1}(y)\widehat{\pi}_N(b_j)}{y-b_j}. \tag{3.54}$$

It is convenient at this point to interrupt the proof and to state and prove the following consequence of the Christoffel–Darboux identity.

Lemma 3.4.2. *For all $N > 0$ we have*

$$\sum_{k=0}^{N-1} \frac{\pi_k(z)\widehat{\pi}_k(w)}{h_k} = \frac{1}{h_{N-1}} \frac{\pi_N(z)\widehat{\pi}_{N-1}(w) - \pi_{N-1}(z)\widehat{\pi}_N(w)}{z-w}. \tag{3.55}$$

[5] With a minor modification due to the expansion $\Gamma(y) \sim y^{N\sigma_3}$ as $y \to \infty$.

Proof of lemma. Let us start from the right side of (3.55) and apply first the definition of Cauchy–Hilbert transform and then the standard Christoffel–Darboux identity (3.14);

$$\frac{1}{h_{N-1}}\frac{\pi_N(z)\widehat{\pi}_{N-1}(w)-\pi_{N-1}(z)\widehat{\pi}_N(w)}{z-w} \tag{3.56}$$

$$=\frac{1}{2\pi \mathrm{i}}\int_\Sigma \frac{1}{h_{N-1}}\frac{\pi_N(z)\pi_{N-1}(w')-\pi_{N-1}(z)\pi_N(w')}{(z-w)(w'-w)}\mathrm{e}^{V(w')}\mathrm{d}w' \tag{3.57}$$

$$=\frac{1}{2\pi \mathrm{i}}\int_\Sigma \frac{1}{h_{N-1}}\frac{\pi_N(z)\pi_{N-1}(w')-\pi_{N-1}(z)\pi_N(w')}{z-w'}\left(\frac{1}{w'-w}-\frac{1}{z-w}\right)\mathrm{e}^{V(w')}\mathrm{d}w' \tag{3.58}$$

$$=\frac{1}{2\pi \mathrm{i}}\sum_{k=0}^{N-1}\int_\Sigma \left(\frac{1}{w'-w}-\frac{1}{z-w}\right)\frac{\pi_k(z)\pi_k(w')}{h_k}\mathrm{e}^{V(w')}\mathrm{d}w' = \sum_{k=0}^{N-1}\frac{\pi_k(z)\widehat{\pi}_k(w)}{h_k} \tag{3.59}$$

where in the last step we have used that $\int_\Sigma \pi(w')\mathrm{e}^{V(w')}\mathrm{d}w'=0$. ■

Let us return to the proof of Thm. 3.4.1. Due to (3.53)-(3.54) and to the lemma, the entries of $\mathbf{A}$ are to be found from the expansion at $y=\infty$ of

$$-\frac{2\pi \mathrm{i}}{h_{N-1}}\frac{\pi_N(a_j)\widehat{\pi}_{N-1}(y)-\pi_{N-1}(a_j)\widehat{\pi}_N(y)}{y-a_j}=2\pi \mathrm{i}\sum_{k=0}^{N-1}\frac{\pi_k(a_j)\widehat{\pi}_k(y)}{h_k} \tag{3.60}$$

and those of $\mathbf{B}$ from that of

$$-\frac{2\pi \mathrm{i}}{h_{N-1}}\frac{\pi_N(y)\widehat{\pi}_{N-1}(b_j)-\pi_{N-1}(y)\widehat{\pi}_N(b_j)}{y-b_j}=-2\pi \mathrm{i}\sum_{k=0}^{N-1}\frac{\pi_k(y)\widehat{\pi}(b_j)}{h_k}. \tag{3.61}$$

Therefore, using $\widehat{\pi}_k(y)=-\frac{h_k}{2\pi \mathrm{i}}y^{-k-1}(1+\mathcal{O}(y^{-1}))$ (see Lemma 3.2.1) and $\pi_k(y)=y^k(1+\mathcal{O}(y^{-1}))$, the formula of the statement follows, up to an integration constant independent of the a_j's and b_j's; this constant is uniquely fixed analyzing the behavior for $a_j,b_j\to\infty$, again using the same asymptotic expansions for $\pi_k,\widehat{\pi}_k$ of large argument. More explicitly, for $a_j,b_j\to\infty$ we have

$$\left\langle \frac{\prod_{i=1}^{\ell}\det(a_i\mathbf{1}-M)}{\prod_{i=1}^{m}\det(b_i\mathbf{1}-M)}\right\rangle_N \sim \left(\frac{\prod_{j=1}^{\ell}a_j}{\prod_{j=1}^{m}b_j}\right)^N \tag{3.62}$$

and

$$\det\begin{pmatrix}\pi_{N-m}(a_1) & \cdots & \pi_{N-m}(a_\ell) & \widehat{\pi}_{N-m}(b_1) & \cdots & \widehat{\pi}_{N-m}(b_m)\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \pi_{N+\ell-1}(a_1) & \cdots & \pi_{N+\ell-1}(a_\ell) & \widehat{\pi}_{N+\ell-1}(b_1) & \cdots & \widehat{\pi}_{N+\ell-1}(b_m)\end{pmatrix} \tag{3.63}$$

$$\sim\det\begin{pmatrix}a_1^{N-m} & \cdots & a_\ell^{N-m} & -\frac{h_{N-m}}{2\pi \mathrm{i}}b_1^{-N+m-1} & \cdots & -\frac{h_{N-m}}{2\pi \mathrm{i}}b_m^{-N+m-1}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ a_1^{N+\ell-1} & \cdots & a_\ell^{N+\ell-1} & -\frac{h_{N+\ell-1}}{2\pi \mathrm{i}}b_1^{-N-\ell} & \cdots & -\frac{h_{N+\ell-1}}{2\pi \mathrm{i}}b_m^{-N-\ell}\end{pmatrix} \tag{3.64}$$

$$\sim\prod_{j=1}^{m}\left(-\frac{h_{N-j}}{2\pi \mathrm{i}}\right)\det\begin{pmatrix}a_1^N & \cdots & a_\ell^N\\ \vdots & \ddots & \vdots\\ a_1^{N+\ell-1} & \cdots & a_\ell^{N+\ell-1}\end{pmatrix}\det\begin{pmatrix}b_1^{-N+m-1} & \cdots & b_m^{-N+m-1}\\ \vdots & \ddots & \vdots\\ b_1^{-N} & \cdots & b_m^{-N}\end{pmatrix} \tag{3.65}$$

$$=\left(\frac{\prod_{j=1}^{\ell}a_j}{\prod_{j=1}^{m}b_j}\right)^N\prod_{j=1}^{m}\left(-\frac{h_{N-j}}{2\pi \mathrm{i}}\right)(-1)^{\frac{\ell(\ell-1)}{2}}\Delta(a_1,...,a_\ell)\Delta(b_1,...,b_m). \tag{3.66}$$

The proof is complete. ■

3.5 Connected correlators

Using the formulæ of Thm. 1.2.2 we can provide expressions for the logarithmic derivatives of the partition function (3.3). Let us first comment on the interpretation of these derivatives; indeed

$$\frac{\partial^r \mathcal{Z}_N(\mathbf{t})}{\partial t_{k_1,\nu_1}\cdots\partial t_{k_r,\nu_r}}=\int_{H_N(\Sigma)}\mathrm{tr}\,(M-z_{\nu_1})^{-k_1}\cdots\mathrm{tr}\,(M-z_{\nu_r})^{-k_r}\mathrm{e}^{\mathrm{tr}\,V(M;t)}\mathrm{d}M \tag{3.67}$$

where we agree that

$$(M - z_\nu)^{-k} := M^k \text{ when } \nu = \infty. \tag{3.68}$$

Therefore, using the notation (3.40),

$$\frac{1}{\mathcal{Z}_N(\mathbf{t})} \frac{\partial^r \mathcal{Z}_N(\mathbf{t})}{\partial t_{k_1,\nu_1} \cdots \partial t_{k_r,\nu_r}} = \left\langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \cdots \mathrm{tr}\,(M - z_{\nu_r})^{-k_r} \right\rangle. \tag{3.69}$$

Recalling the formula

$$\frac{\partial^r \log Z}{\partial \lambda_1 \cdots \partial \lambda_r} = \sum_{\mathcal{P} \text{ partition of } \{1,\dots,r\}} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)! \prod_{I \in \mathcal{P}} \frac{\left(\prod_{i\in I} \frac{\partial}{\partial \lambda_i}\right) Z}{Z}, \tag{3.70}$$

for logarithmic derivatives and introducing the *connected expectation values*

$$\langle f_1, \dots, f_r \rangle_c := \sum_{\mathcal{P} \text{ partition of } \{1,\dots,r\}} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)! \prod_{I\in\mathcal{P}} \left\langle \prod_{i \in I} f_i \right\rangle, \tag{3.71}$$

we have

$$\frac{\partial^r \log \mathcal{Z}_N(\mathbf{t})}{\partial t_{k_1,\nu_1} \cdots \partial t_{k_r,\nu_r}} = \left\langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1}, \dots, \mathrm{tr}\,(M - z_{\nu_r})^{-k_r} \right\rangle_c. \tag{3.72}$$

We call $\left\langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \cdots \mathrm{tr}\,(M - z_{\nu_r})^{-k_r} \right\rangle_c$ *connected correlators.* E.g.

$$\begin{aligned}
\frac{\partial \log \mathcal{Z}_N(\mathbf{t})}{\partial t_{k_1,\nu_1}} &= \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \rangle_c = \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \rangle, \\
\frac{\partial^2 \log \mathcal{Z}_N(\mathbf{t})}{\partial t_{k_1,\nu_1} \partial t_{k_2,\nu_2}} &= \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1}, \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \rangle_c \\
&= \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \rangle - \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \rangle \langle \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \rangle, \\
\frac{\partial^3 \log \mathcal{Z}_N(\mathbf{t})}{\partial t_{k_1,\nu_1} \partial t_{k_2,\nu_2} \partial t_{k_3,\nu_3}} &= \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1}, \mathrm{tr}\,(M - z_{\nu_2})^{-k_2}, \mathrm{tr}\,(M - z_{\nu_3})^{-k_3} \rangle_c \\
&= \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \mathrm{tr}\,(M - z_{\nu_3})^{-k_3} \rangle \\
&\quad - \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \rangle \langle \mathrm{tr}\,(M - z_{\nu_3})^{-k_3} \rangle \\
&\quad - \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \mathrm{tr}\,(M - z_{\nu_3})^{-k_3} \rangle \langle \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \rangle \\
&\quad - \langle \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \mathrm{tr}\,(M - z_{\nu_3})^{-k_3} \rangle \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \rangle \\
&\quad + 2 \langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1} \rangle \langle \mathrm{tr}\,(M - z_{\nu_2})^{-k_2} \rangle \langle \mathrm{tr}\,(M - z_{\nu_3})^{-k_3} \rangle.
\end{aligned}$$

From Thm. 1.2.2 we obtain the following formulæ for the connected correlators. Let Γ be as in (3.27) the solution of the RHP (3.2.2), and introduce the matrix $R = R(z;\mathbf{t})$ as[6]

$$R := \mathrm{diag}\left(1, -\frac{h_{N-1}}{2\pi \mathrm{i}}\right) \Gamma \frac{\sigma_3}{2} \Gamma^{-1} \,\mathrm{diag}\left(1, -\frac{2\pi \mathrm{i}}{h_{N-1}}\right) = \frac{\sigma_3}{2} - \frac{2\pi \mathrm{i}}{h_{N-1}} \begin{pmatrix} \pi_{N-1}\widehat{\pi}_N & -\pi_N \widehat{\pi}_N \\ \pi_{N-1}\widehat{\pi}_{N-1} & -\pi_{N-1}\widehat{\pi}_N \end{pmatrix} \tag{3.73}$$

where we use (3.30), and the functions

$$S_1(z;\mathbf{t}) := \mathrm{tr}\left(\Gamma^{-1}(z;\mathbf{t})\Gamma'(z;\mathbf{t})\frac{\sigma_3}{2}\right), \tag{3.74}$$

$$S_r(z_1, \dots, z_r; \mathbf{t}) : = -\frac{1}{r} \sum_{\sigma \in \mathfrak{S}_r} \frac{\mathrm{tr}\,\left(R(z_{\sigma(1)};\mathbf{t}) \cdots R(z_{\sigma(r)};\mathbf{t})\right)}{(z_{\sigma(1)} - z_{\sigma(2)}) \cdots (z_{\sigma(r-1)} - z_{\sigma(r)})(z_{\sigma(r)} - z_{\sigma(1)})} - \frac{1}{2} \frac{\delta_{r,2}}{(z_1 - z_2)^2}. \tag{3.75}$$

Theorem 3.5.1. *We have*

$$\left\langle \mathrm{tr}\,(M - z_{\nu_1})^{-k_1}, \dots, \mathrm{tr}\,(M - z_{\nu_r})^{-k_r} \right\rangle_c = \operatorname*{res}_{z_1 = z_{\nu_1}} \cdots \operatorname*{res}_{z_1 = z_{\nu_1}} S_r(z_1, \dots, z_r) \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_r}{(z_1 - z_{\nu_1})^{k_1} \cdots (z_r - z_{\nu_r})^{k_r}} \tag{3.76}$$

where S_r is defined in (3.74)-(3.75) and, as above, we agree that $\frac{1}{w - z_\nu} := w$ for $\nu = \infty$.

[6] It is convenient to get rid of some constant (in z) factors in the matrix $\Gamma\frac{\sigma_3}{2}\Gamma^{-1}$, by conjugation with a constant diagonal matrix; this transformation does not spoil the formulæ of Thm. 3.5.1, as it is follows by the Ad-invariance of the trace.

Applications of this Theorem to the Gaussian and Laguerre Unitary Ensembles are considered below.

3.5.1 Gaussian Unitary Ensemble

The *Gaussian Unitary Ensemble* (GUE) is the statistical model of a random hermitian matrix of size N distributed according to the probability measure

$$\frac{1}{\mathcal{Z}_N^{\mathsf{GUE}}(0)} \exp \operatorname{tr}\left(-\frac{M^2}{2}\right) \mathrm{d}M. \tag{3.77}$$

The normalization $\mathcal{Z}_N^{\mathsf{GUE}}(0)$ is a gaussian integral;

$$\mathcal{Z}_N^{\mathsf{GUE}}(0) = \int_{\mathrm{H}_N} \exp \operatorname{tr}\left(-\frac{M^2}{2}\right) \mathrm{d}M = \int_{\mathbb{R}^{N^2}} \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^N M_{ii}^2 - \sum_{1\le a<b\le N}|M_{ab}|^2} \mathrm{d}M = \sqrt{2}^N \sqrt{\pi}^{N^2}. \tag{3.78}$$

The monic orthogonal polynomials are given in terms of the *Hermite polynomials*[7]

$$\pi_\ell(z) := 2^{-\frac{\ell}{2}} H_\ell\left(\frac{z}{\sqrt{2}}\right) = (-1)^\ell \mathrm{e}^{\frac{z^2}{2}}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}\mathrm{e}^{-\frac{z^2}{2}}\right). \tag{3.79}$$

The last identity is the *Rodrigues formula*; using it and integrating by parts we obtain

$$\int_{-\infty}^{+\infty} z^k \pi_\ell(z) \mathrm{e}^{-\frac{z^2}{2}} \mathrm{d}z = \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{z^2}{2}} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} z^k\right) \mathrm{d}z = \begin{cases} 0 & k<\ell \\ \sqrt{2\pi}\ell! & k=\ell. \end{cases} \tag{3.80}$$

Hence we have proven the orthogonality property (3.4) in the form

$$\int_{-\infty}^{+\infty} \pi_\ell(z)\pi_{\ell'}(z)\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z = \sqrt{2\pi}\ell!\delta_{\ell,\ell'} \tag{3.81}$$

i.e.

$$h_\ell = \sqrt{2\pi}\ell!. \tag{3.82}$$

We have a linear ODE (compare with (3.37), the parameters are absent for the time being)

$$\Psi'(z) = L(z)\Psi(z), \qquad L(z) = \begin{pmatrix} -\frac{z}{2} & \frac{\mathrm{i}N!}{\sqrt{2\pi}} \\ \frac{\mathrm{i}\sqrt{2\pi}}{(N-1)!} & \frac{z}{2} \end{pmatrix} \tag{3.83}$$

for the matrix

$$\Psi(z) := \Gamma(z)\mathrm{e}^{-\frac{z^2}{4}\sigma_3}, \qquad \Gamma(z) := \begin{pmatrix} \pi_N(z) & \widehat{\pi}_{N-1}(z) \\ -\frac{\mathrm{i}\sqrt{2\pi}}{(N-1)!}\pi_{N-1}(z) & -\frac{\mathrm{i}\sqrt{2\pi}}{(N-1)!}\widehat{\pi}_{N-1}(z) \end{pmatrix} \tag{3.84}$$

compare with (3.27) and (3.32). The linear ODE (3.83) has an irregular singularity of Poincaré rank 2 at $z=\infty$ and no other singularity; the irregular singularity is generic in the sense of Sec. 1.4.2, as the leading order at $z=\infty$ of L is $\frac{\sigma_3}{2}$ which has distinct eigenvalues.

Proposition 3.5.2. *1. We have the explicit formal expansion*

$$\Gamma(z) \sim \sum_{j\ge 0} \frac{1}{z^{2j}} \begin{pmatrix} \frac{(-1)^j N!}{2^j j!(N-2j)!} & \frac{\mathrm{i}}{\sqrt{2\pi}z}(2j-1)!!(2j+1)_N \\ \frac{-\mathrm{i}\sqrt{2\pi}}{z}\frac{(-1)^j}{2^j j!(N-2j-1)!} & \frac{(2j-1)!!(2j+1)_{N-1}}{(N-1)!} \end{pmatrix} z^{N\sigma_3} \tag{3.85}$$

as $z\to\infty$ within any of the two sectors in $\mathbb{C}\setminus(-\infty,+\infty)$.

2. We have the formal expansion

$$R(z) \sim \frac{\sigma_3}{2} + \sum_{\ell\ge 0}\frac{1}{z^{2\ell+2}}\begin{pmatrix} N\mathcal{A}_{\ell,N} & -zN\mathcal{B}_{\ell,N+1} \\ z\mathcal{B}_{\ell,N} & -N\mathcal{A}_{\ell,N} \end{pmatrix} \tag{3.86}$$

[7] The notation $\mathrm{He}_\ell(z)$ is also used in the literature for the monic orthogonal polynomials $\pi_\ell(z)$.

where

$$\mathcal{A}_{\ell,N} := (2\ell+1)!! \sum_{j=0}^{\ell} 2^j \binom{\ell}{j}\binom{N}{j+1} = N(2\ell+1)!!\,{}_2F_1\left({-\ell, 1-N \atop 2} \,\middle|\, 2\right)$$

$$\mathcal{B}_{\ell,N} := N(2\ell-1)!! \sum_{j=0}^{\ell} 2^j \binom{\ell}{j}\binom{N-1}{j} = N(2\ell-1)!!\,{}_2F_1\left({-\ell, 1-N \atop 1} \,\middle|\, 2\right).$$

as $z \to \infty$ within any of the two sectors in $\mathbb{C} \setminus (-\infty, +\infty)$.

Proof.

1. The first column follows from the formula

$$\pi_\ell(z) = z^\ell \sum_{j\geq 0} \frac{(-1)^j \ell!}{2^j j!(\ell-2j)!} \frac{1}{z^{2j}} \tag{3.87}$$

for monic Hermite polynomials. For the second column let us apply Lemma 3.2.1, Rodrigues formula (3.79) and integration by parts to compute

$$\begin{aligned}
\widehat{\pi}_\ell(z) &\sim -\frac{1}{2\pi \mathrm{i}} \frac{1}{z^{\ell+1}} \sum_{j\geq 0} \frac{1}{z^j} \int_{-\infty}^{+\infty} w^{j+\ell} \pi_\ell(w) \mathrm{e}^{-\frac{w^2}{2}} \mathrm{d}w \\
&= -\frac{1}{2\pi \mathrm{i}} \frac{1}{z^{\ell+1}} \sum_{j\geq 0} \frac{1}{z^j} \int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}w^\ell} w^{j+\ell} \right) \mathrm{e}^{-\frac{w^2}{2}} \mathrm{d}w \\
&= -\frac{1}{2\pi \mathrm{i}} \frac{1}{z^{\ell+1}} \sum_{j\geq 0} \frac{(j+1)_\ell}{z^j} \int_{-\infty}^{+\infty} w^j \mathrm{e}^{-\frac{w^2}{2}} \mathrm{d}w.
\end{aligned}$$

In the last expression we see that j must be even; redenoting $j \mapsto 2j$ and using the gaussian integral

$$\int_{-\infty}^{+\infty} w^{2j} \mathrm{e}^{-\frac{w^2}{2}} = \sqrt{2\pi}(2j-1)!! \tag{3.88}$$

we have

$$\widehat{\pi}_\ell(z) \sim \frac{\mathrm{i}}{\sqrt{2\pi}} \frac{1}{z^{\ell+1}} \sum_{j\geq 0} \frac{(2j-1)!!(2j+1)_\ell}{z^{2j}} \tag{3.89}$$

from which the expansion of the second column of Γ follows too.

2. The statement follows from (3.87) and (3.89). For instance for the entry R_{11} we have to compute

$$\begin{aligned}
-\frac{2\pi \mathrm{i}}{h_{N-1}} \pi_{N-1} \widehat{\pi}_N &= -\frac{2\pi \mathrm{i}}{h_{N-1}} \frac{\mathrm{i}}{\sqrt{2\pi}} \sum_{j,k\geq 0} \frac{1}{z^{2(j+k)+2}} \frac{(-1)^j}{2^j j!} \frac{(2k-1)!!(2k+1)_N}{(N-1-2j)!} \\
&= \sum_{\ell \geq 0} \frac{1}{z^{2\ell+2}} \sum_{j=0}^{\ell} \frac{(-1)^j}{2^j j!} \frac{(2(\ell-j)-1)!!(2(\ell-j)+1)_N}{(N-1-2j)!} \\
&= (2\ell+1)!! \sum_{\ell\geq 0} \frac{1}{z^{2\ell+2}} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \binom{N+2j}{2\ell+1}
\end{aligned}$$

where in the last step we used trivial simplifications, e.g.

$$(2(\ell-j)-1)!! = \frac{(2(\ell-j))!}{2^{\ell-j}(\ell-j!)}, \qquad \frac{(2(\ell-j))!(2(\ell-j)+1)_N}{(N-1-2j)!} = \binom{N+2(\ell-j)}{2\ell+1} \tag{3.90}$$

and then replaced summation index $j \mapsto \ell - j$. Now in principle

$$\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \binom{N+2j}{2\ell+1} = (2\ell+1)!! \binom{N}{2\ell+1} {}_3F_2\left({-\ell, \frac{1}{2}+\frac{N}{2}, 1+\frac{N}{2} \atop \frac{N}{2}-\ell, \frac{N+1}{2}-\ell} \,\middle|\, 1\right) \tag{3.91}$$

but we have a nice simplification to a *Gauss* hypergeometric function;

$$\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\binom{N+2j}{2\ell+1} = \sum_{j=0}^{\ell}2^j\binom{\ell}{j}\binom{N}{j+1} = N\,{}_2F_1\left(\,{}^{-\ell,1-N}_{\;\;2}\,\middle|\,2\right). \tag{3.92}$$

To prove (3.92) we note the identity

$$(\beta)_k = \beta(\beta+1)\cdots(\beta+k-1) = \frac{\partial}{\partial x}x^{\beta+k-1}\bigg|_{x=1} \tag{3.93}$$

so that the left side of (3.92) is

$$\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j}\binom{N+2j}{2\ell+1} = (-1)^{\ell}\frac{\mathrm{d}^{2\ell+1}}{\mathrm{d}x^{2\ell+1}}\frac{x^N(1-x^2)^{\ell}}{(2\ell+1)!}\bigg|_{x=1} \tag{3.94}$$

hence it must be equal to (changing variable $x = 1+y$)

$$\begin{aligned}\frac{\mathrm{d}^{2\ell+1}}{\mathrm{d}y^{2\ell+1}}\frac{(1+y)^N(2y+y^2)^{\ell}}{(2\ell+1)!}\bigg|_{y=0} &= \frac{1}{(2\ell+1)!}\frac{\mathrm{d}^{2\ell+1}}{\mathrm{d}y^{2\ell+1}}\sum_{s\geq 0}\sum_{j=0}^{\ell}2^j\binom{\ell}{j}\binom{N}{s}y^{s+2\ell-j}\bigg|_{y=0}\\ &= \sum_{j=0}^{\ell}2^j\binom{\ell}{j}\binom{N}{j+1} = (-1)^{\ell}N\,{}_2F_1\left(\,{}^{-\ell,1-N}_{\;\;2}\,\middle|\,2\right).\end{aligned}$$

This proves the statement for the entry R_{11}. The other entries are computed likewise.

■

Consider now the deformed GUE partition function

$$\mathcal{Z}_N^{\mathsf{GUE}}(\mathbf{t}) := \int_{\mathrm{H}_N}\mathrm{e}^{\mathrm{tr}\left(-\frac{M^2}{2}+\sum_{j\geq 1}t_jM^j\right)}\mathrm{d}M, \qquad \mathbf{t} = (t_1,t_2,...) \tag{3.95}$$

assuming that $t_j = 0$ for $j > 2K$ for some $K \geq 1$ and $\mathrm{Re}\,t_{2K} < 0$ so that the integral in (3.95) is well defined; the results below are unaffected by this arbitrary truncation of the times. The deformed GUE partition function serves as a generating functional of connected correlators of the GUE. The following result, originally proven in [DYb], follows directly by the results of this chapter, and it provides an effective way to compute the generating function (3.95).

The interest of this result is that the deformed GUE partition function (3.95) is known to be related to the enumeration of *ribbon graphs on surfaces* [Hd; BIPZ], see also [DYb, App. A.3]. This connection is expressed, for $k_1,...,k_r \geq 3$ as

$$\frac{N^{r-\frac{|k|}{2}}}{r!}\langle \mathrm{tr}\,M^{k_1},...,\mathrm{tr}\,M^{k_r}\rangle_c^{\mathsf{GUE}} = \sum_{g\geq 0}N^{2-2g}a_g(k_1,...,k_r), \qquad |k| := k_1+\cdots+k_r \tag{3.96}$$

where

$$\langle \mathrm{tr}\,M^{k_1},...,\mathrm{tr}\,M^{k_r}\rangle_c^{\mathsf{GUE}} := \frac{\partial^r\log\mathcal{Z}_N^{\mathsf{GUE}}(\mathbf{t})}{\partial t_{k_1}\cdots\partial t_{k_r}}\bigg|_{\mathbf{t}=0} \tag{3.97}$$

and

$$a_g(k_1,...,k_r) := \sum_{\substack{\text{connected oriented ribbon graphs } G \text{ of genus } g\\ \text{with } r \text{ vertices of valencies } k_1,...,k_r}}\frac{1}{|\mathrm{Aut}G|} \tag{3.98}$$

where $|\,\mathrm{Aut}\,G|$ is the order of the automorphism group of the ribbon graph G.

Introduce the (formal) generating functions (for $r = 1,2,...$)

$$\mathcal{F}_r^{\mathsf{GUE}}(z_1,...,z_r) = \sum_{k_1,...,k_r\geq 1}\frac{\langle \mathrm{tr}\,M^{k_1},...,\mathrm{tr}\,M^{k_r}\rangle_c^{\mathsf{GUE}}}{z_1^{k_1+1}\cdots z_r^{k_r+1}}. \tag{3.99}$$

Theorem 3.5.3 ([DYb]). *We have*

$$\mathcal{F}_r^{\mathsf{GUE}}(z_1,...,z_r)=\begin{cases}\int_z^\infty\left(R_{11}(w)-\frac{1}{2}\right)\mathrm{d}w & r=1\\ \frac{\mathrm{tr}\,(R(z_1)R(z_2))-\frac12}{(z_1-z_2)^2} & r=2\\ -\frac{1}{r}\sum_{\pi\in\mathfrak{S}_r}\frac{\mathrm{tr}\,\left(R(z_{\pi(1)})\cdots R(z_{\pi(r)})\right)}{(z_{\pi(1)}-z_{\pi(2)})\cdots(z_{\pi(r)}-z_{\pi(1)})} & r\geq 3\end{cases}$$

where $R(z)$ is identified with the formal series in the expansion (3.86).

Equivalent formulæ for the cases $r=1,2$ were given in [HZ; MS]. Note that the case $r=1$ boils down to the explicit expression

$$\left\langle \mathrm{tr}\,M^{2\ell}\right\rangle^{\mathsf{GUE}}=(2\ell-1)!!\sum_{j=0}^{\ell}2^j\binom{\ell}{j}\binom{N}{j+1}=(2\ell-1)!!N\,{}_2F_1\left(\begin{smallmatrix}-\ell,1-N\\2\end{smallmatrix}\Big|\,2\right). \tag{3.100}$$

Let us make the trivial observation that this reduces for $N=1$ to the usual scalar gaussian integral (3.88), where ribbon graphs are not weighted by their genus. We also note that we have the well-known *planar limit*

$$\lim_{N\to\infty}\frac{\left\langle \mathrm{tr}\,M^{2\ell}\right\rangle^{\mathsf{GUE}}}{N^{\ell+1}}=\frac{1}{\ell+1}\binom{2\ell}{\ell} \tag{3.101}$$

to the *Catalan numbers*, moments of the celebrated *Wigner semicircle law* [F]; this follows by the trivial estimate $\binom{N}{j+1}\sim\frac{N^{j+1}}{(j+1)!}$ for large N.

Proof. Let us first consider the case $r=1$. By Thm. 3.5.1 and definition (3.74) we have

$$\frac{\partial}{\partial t_k}\log\mathcal{Z}_N^{\mathsf{GUE}}(\mathbf{t})\Big|_{\mathbf{t}=0}=-\operatorname*{res}_{z=\infty}\mathrm{tr}\,\left(\Gamma^{-1}(z)\Gamma'(z)\frac{\sigma_3}{2}\right)z^k\mathrm{d}z=-\operatorname*{res}_{z=\infty}\mathrm{tr}\,\left(L(z)R(z)\frac{\sigma_3}{2}\right)z^k\mathrm{d}z \tag{3.102}$$

where L is given in (3.83), $R=\Gamma\frac{\sigma_3}{2}\Gamma^{-1}$ as in (3.73), and we use the identity

$$\Gamma'=L\Gamma-\frac{z}{2}\Gamma\sigma_3. \tag{3.103}$$

Note therefore that, denoting $\mathcal{F}_1^{\mathsf{GUE}}=:\mathcal{F}_1$ for short,

$$\mathcal{F}_1=\mathrm{tr}\,(LR)+\frac{z}{2}-(\mathrm{tr}\,(LR))_0 \tag{3.104}$$

where we have to subtract the singular part at $z=\infty$ of $\mathrm{tr}\,(LR)=-\frac{z}{2}+(\mathrm{tr}\,(LR))_0+\mathcal{O}(z^{-1})$, denoting $(\mathrm{tr}\,(LR))_0$ the constant term. Here we are identifying R with its formal expansion at $z=\infty$. Taking one derivative in z we obtain

$$\mathcal{F}_1'=\mathrm{tr}\,(L'R)+\frac12 \tag{3.105}$$

as, due to $R'=[L,R]$, $\mathrm{tr}\,(LR')=\mathrm{tr}\,(L[L,R])=\frac12\mathrm{tr}\,([L^2,R])=0$. Finally, we have $L'=-\frac{\sigma_3}{2}$, see (3.83), hence

$$\mathcal{F}_1'=-\frac12\mathrm{tr}\,(\sigma_3R)+\frac12=-\left(R_{11}-\frac12\right) \tag{3.106}$$

where we use $\mathrm{tr}\,R=0$, which integrates to the claimed expression for $\mathcal{F}_1^{\mathsf{GUE}}$.

The cases $r\geq 2$ follow from Thm. 3.5.1 by the following computation;

$$\mathcal{F}_r^{\mathsf{GUE}}=(z_1,...,z_r)=\sum_{k_1,...,k_r\geq1}\frac{\langle\mathrm{tr}\,M^{k_1},...,\mathrm{tr}\,M^{k_r}\rangle_c^{\mathsf{GUE}}}{z_1^{k_1+1}\cdots z_r^{k_r+1}} \tag{3.107}$$

$$=\sum_{k_1,...,k_r\geq1}\frac{\operatorname*{res}_{\xi_1=\infty}\cdots\operatorname*{res}_{\xi_r=\infty}S_r(\xi_1,...,\xi_r)\xi_1^{k_1}\mathrm{d}\xi_1\cdots\xi_r^{k_r}\mathrm{d}\xi_r}{z_1^{k_1+1}\cdots z_r^{k_r+1}} \tag{3.108}$$

$$=S_r(z_1,...,z_r) \tag{3.109}$$

which is the desired formula, by definition (3.75). ■

E.g. we have

$$\langle \operatorname{tr} M^6, \operatorname{tr} M^6, \operatorname{tr} M^6 \rangle_c^{\mathsf{GUE}} = 3600(3421N^2 + 4803N^4 + 1160N^6 + 60N^8) = 3!N^6 \sum_{g=0}^{3} N^{2-2g} a_g(6,6,6) \tag{3.110}$$

computing the (weighted) numbers of connected oriented ribbon graphs of genus $g = 0, ..., 3$ with three 6-valent vertices. For more applications and examples see [DYb].

3.5.2 Laguerre Unitary Unsemble

In this section we review the results obtained in collaboration with Massimo Gisonni and Tamara Grava.

The *Laguerre Unitary Ensemble* (LUE) is the statistical model of a random positive definite hermitian matrix of size N distributed according to the probability measure

$$\frac{1}{\mathcal{Z}_N^{\mathsf{LUE}}(0)} \det{}^\alpha M \exp \operatorname{tr}\,(-M)\,\mathrm{d}M. \tag{3.111}$$

The normalization $\mathcal{Z}_N^{\mathsf{LUE}}(0)$ is computed below, see (3.116).

The parameter α will be left implicit. For the time being it is enough to assume $\operatorname{Re}\alpha > -1$; in the following discussion we can either assume that α lies in a suitable right half-plane of the complex plane or that $\alpha \in \mathbb{C} \setminus \mathbb{Z}$; see Rem. 3.5.5 on this point.

The monic orthogonal polynomials are given in terms of the *generalized Laguerre polynomials*

$$\pi_\ell(z) := (-1)^\ell \ell! L_\ell^{(\alpha)}(z) = (-1)^\ell z^{-\alpha} \mathrm{e}^z \left(\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} (z^{\alpha+\ell}\mathrm{e}^{-z}) \right). \tag{3.112}$$

The last identity is the *Rodrigues formula*; using it and integrating by parts we obtain

$$\int_0^{+\infty} z^k \pi_\ell(z) z^\alpha \mathrm{e}^{-z} \mathrm{d}z = \int_0^{+\infty} z^{\alpha+\ell}\mathrm{e}^{-z} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} z^k \right) \mathrm{d}z = \begin{cases} 0 & k < \ell \\ \ell!\Gamma(\alpha+\ell+1) & k = \ell. \end{cases} \tag{3.113}$$

Hence we have proven the orthogonality property (3.4) in the form

$$\int_0^{+\infty} \pi_\ell(z)\pi_{\ell'}(z) z^\alpha \mathrm{e}^{-z} \mathrm{d}z = \ell!\Gamma(\alpha+\ell+1)\delta_{\ell,\ell'} \tag{3.114}$$

i.e.

$$h_\ell = \ell!\Gamma(\alpha+\ell+1). \tag{3.115}$$

This allows to compute the normalization $\mathcal{Z}_N^{\mathsf{LUE}}(0)$ from Lemma 3.1.1 as

$$\mathcal{Z}_N^{\mathsf{LUE}}(0) = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N-1} \ell!} \prod_{\ell=0}^{N-1} h_\ell = \pi^{\frac{N(N-1)}{2}} \prod_{\ell=0}^{N-1} \Gamma(\alpha+\ell+1). \tag{3.116}$$

We have a linear ODE (compare with (3.37), the parameters are absent for the time being)

$$\Psi'(z) = L(z)\Psi(z), \qquad L(z) = -\frac{\sigma_3}{2} + \frac{1}{z} \begin{pmatrix} N + \frac{\alpha}{2} & -\frac{N!\Gamma(N+\alpha+1)}{2\pi\mathrm{i}} \\ \frac{2\pi\mathrm{i}}{(N-1)!\Gamma(N+\alpha)} & -N - \frac{\alpha}{2} \end{pmatrix}. \tag{3.117}$$

for the matrix

$$\Psi(z) := \Gamma(z) z^{\frac{\alpha}{2}\sigma_3} \mathrm{e}^{-\frac{z}{2}\sigma_3}, \qquad \Gamma(z) := \begin{pmatrix} \pi_N(z) & \widehat{\pi}_{N-1}(z) \\ -\frac{2\pi\mathrm{i}}{(N-1)!\Gamma(N+\alpha)}\pi_{N-1}(z) & -\frac{2\pi\mathrm{i}}{(N-1)!\Gamma(N+\alpha)}\widehat{\pi}_{N-1}(z) \end{pmatrix} \tag{3.118}$$

compare with (3.27) and (3.32). The linear ODE (3.117) has an irregular singularity of Poincaré rank 1 at $z = \infty$ and a regular singularity at $z = 0$, with Frobenius indices $\pm\frac{\alpha}{2}$; the irregular singularity at $z = \infty$ is generic in the sense of Sec. 1.4.2, as the leading order at $z = \infty$ of L is $\frac{\sigma_3}{2}$ which has distinct eigenvalues. The regular singularity at $z = 0$ is generic ("nonresonant") in the sense of Sec. 1.4.2 if and only if α is not an integer (compare with Rem. 3.5.5).

Proposition 3.5.4. *1. We have the explicit formal expansions*

$$\Gamma(z) \sim \sum_{j\geq 0} \frac{1}{z^j} \begin{pmatrix} (-1)^j \binom{N}{j} (N+\alpha-j+1)_j & -\frac{1}{z}\frac{1}{2\pi \mathrm{i}}\Gamma(N+\alpha+j+1)(j+1)_N \\ -\frac{1}{z}\frac{2\pi\mathrm{i}}{(N-1)!\Gamma(N+\alpha-j)}(-1)^j\binom{N-1}{j} & \frac{(j+1)_{N-1}}{(N-1)!}(N+\alpha)_j \end{pmatrix} z^{N\sigma_3} \tag{3.119}$$

as $z \to \infty$ within $\mathbb{C}\setminus[0,+\infty)$, and

$$\Gamma(z) \sim (-1)^N \sum_{j\geq 0} z^j \begin{pmatrix} (-1)^j\binom{N}{j}(\alpha+j+1)_{N-j} & \frac{\Gamma(\alpha-j)(j+1)_N}{2\pi\mathrm{i}} \\ \frac{2\pi\mathrm{i}}{(N-1)!\Gamma(\alpha+j+1)}(-1)^j\binom{N-1}{j} & \frac{\Gamma(\alpha-j)(j+1)_{N-1}}{(N-1)!\Gamma(N+\alpha)} \end{pmatrix} \tag{3.120}$$

as $z\to 0$ within $\mathbb{C}\setminus[0,+\infty)$.

2. *We have the formal expansions*

$$R(z) \sim \frac{\sigma_3}{2} + \sum_{\ell\geq 0}\frac{1}{z^{\ell+2}}\begin{pmatrix} \mathcal{A}^{(\infty)}_{\ell,N} & -zN(N+\alpha)\mathcal{B}^{(\infty)}_{\ell,N+1} \\ z\mathcal{B}^{(\infty)}_{\ell,N} & -\mathcal{A}^{(\infty)}_{\ell,N}\end{pmatrix} \tag{3.121}$$

where

$$\mathcal{A}^{(\infty)}_{\ell,N} := \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}(N-j)_{\ell+1}(N+\alpha-j)_{\ell+1} = \frac{1}{\ell!}{}_3F_2\left(\left.{-\ell,1-N,1-\alpha-N \atop -\ell-N,-\ell-\alpha-N}\right| 1\right)$$

$$\mathcal{B}^{(\infty)}_{\ell,N} := \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}(N-j)_{\ell}(N+\alpha-j)_{\ell} = \frac{1}{\ell!}{}_3F_2\left(\left.{-\ell,1-N,1-\alpha-N \atop 1-\ell-N,1-\ell-\alpha-N}\right| 1\right)$$

as $z\to\infty$ within $\mathbb{C}\setminus[0,+\infty)$, and

$$R(z) \sim \frac{\sigma_3}{2} + \sum_{\ell\geq 0} z^\ell \begin{pmatrix} \mathcal{A}^{(0)}_{\ell,N} & -N(N+\alpha)\mathcal{B}^{(0)}_{\ell,N+1} \\ \mathcal{B}^{(0)}_{\ell,N} & -\mathcal{A}^{(0)}_{\ell,N}\end{pmatrix} \tag{3.122}$$

where

$$\mathcal{A}^{(0)}_{\ell,N} := \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\frac{(N-j)_{\ell+1}}{(\alpha-\ell+j)_{\ell+1}} = \frac{(N)_{\ell+1}}{\ell!(\alpha-\ell)_{\ell+1}}{}_3F_2\left(\left.{-\ell,1-N,\alpha-\ell \atop -\ell-N,\alpha+1}\right| 1\right)$$

$$\mathcal{B}^{(0)}_{\ell,N} := -\frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\frac{(N-j)_{\ell}}{(\alpha-\ell+j)_{\ell+1}} = -\frac{(N)_{\ell}}{\ell!(\alpha-\ell)_{\ell+1}}{}_3F_2\left(\left.{-\ell,1-N,\alpha-\ell \atop 1-\ell-N,\alpha+1}\right| 1\right)$$

as $z\to 0$ within $\mathbb{C}\setminus[0,+\infty)$.

Remark 3.5.5. *We have to comment on the nature of the expansions as $z\to 0$. If α is an integer then the expansions for $\widehat{\pi}_\ell$ contain divergent coefficients; this can be seen in particular from (3.125). However, if we retain only the non-divergent coefficients we obtain a valid asymptotic relation, up to the order of the first divergent coefficient. If α is not an integer, then the expansions are valid in their entirety.*

Proof.

1. The first column follows from the formula

$$\pi_\ell(z) = z^\ell \sum_{j\geq 0}\frac{(-1)^j}{z^j}\binom{\ell}{j}(\alpha+\ell+1-j)_j = (-1)^\ell\sum_{j=0}^{\ell} z^j(-1)^j\binom{\ell}{j}(\alpha+j+1)_{\ell-j} \tag{3.123}$$

for monic Laguerre polynomials. For the second column let us apply Lemma 3.2.1, Rodrigues formula (3.112) and integration by parts to compute

$$\begin{aligned}\widehat{\pi}_\ell(z) &\sim -\frac{1}{2\pi\mathrm{i}}\frac{1}{z^{\ell+1}}\sum_{j\geq 0}\frac{1}{z^j}\int_0^{+\infty} w^{j+\ell}\pi_\ell(w)w^\alpha \mathrm{e}^{-w}\mathrm{d}w \\ &= -\frac{1}{2\pi\mathrm{i}}\frac{1}{z^{\ell+1}}\sum_{j\geq 0}\frac{1}{z^j}\int_0^{+\infty}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}w^\ell}w^{j+\ell}\right)w^{\alpha+\ell}\mathrm{e}^{-w}\mathrm{d}w \\ &= -\frac{1}{2\pi\mathrm{i}}\frac{1}{z^{\ell+1}}\sum_{j\geq 0}\frac{(j+1)_\ell}{z^j}\int_0^{+\infty} w^{\alpha+j+\ell}\mathrm{e}^{-w}\mathrm{d}w.\end{aligned}$$

Hence

$$\widehat{\pi}_\ell(z) \sim= -\frac{1}{2\pi\mathrm{i}}\frac{1}{z^{\ell+1}}\sum_{j\geq 0}\frac{\Gamma(\alpha+j+\ell+1)(j+1)_\ell}{z^j} \tag{3.124}$$

from which the expansion of the second column of Γ at $z=\infty$ follows. For the expansion at $z=0$ of the Cauchy–Hilbert transform we reason as in Lemma 3.2.1 and compute, using again Rodrigues formula (3.112) and integration by parts,

$$\begin{aligned}\int_0^{+\infty}\frac{\pi_\ell(w)w^\alpha\mathrm{e}^{-w}}{w-z}\mathrm{d}w &\sim \sum_{j\geq 0} z^j\int_0^{+\infty}\frac{\pi_\ell(w)w^\alpha\mathrm{e}^{-w}}{w^{j+1}}\mathrm{d}w\\ &= \sum_{j\geq 0} z^j\int_0^{+\infty} w^{\alpha+\ell}\mathrm{e}^{-w}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}w^\ell}\frac{1}{w^{j+1}}\right)\mathrm{d}w\\ &= \sum_{j\geq 0} z^j(-1)^\ell(j+1)_\ell\int_0^{+\infty} w^{\alpha-j-1}\mathrm{e}^{-w}\mathrm{d}w\end{aligned}$$

hence

$$\widehat{\pi}_\ell(z) \sim \frac{(-1)^\ell}{2\pi\mathrm{i}}\sum_{j\geq 0} z^j\Gamma(\alpha-j)(j+1)_\ell \tag{3.125}$$

and the expansion at $z=0$ is proven.

2. The statement follows from (3.123) and (3.124) and (3.125). For instance for the entry R_{11} at $z=\infty$ we have to compute

$$\begin{aligned}-\frac{2\pi\mathrm{i}}{h_{N-1}}\pi_{N-1}\widehat{\pi}_N &= \sum_{j,k\geq 0}\frac{1}{z^{j+k+2}}\frac{(-1)^j\binom{N-1}{j}\Gamma(N+\alpha+k+1)(k+1)_N}{(N-1)!\Gamma(N+\alpha-j)}\\ &= \sum_{\ell\geq 0}\frac{1}{z^{2\ell+2}}\sum_{j=0}^{\ell}(-1)^j\frac{\Gamma(N+\alpha+\ell-j+1)}{\Gamma(N+\alpha-j)}\frac{(\ell-j+1)_N\binom{N-1}{j}}{(N-1)!}\\ &= \sum_{\ell\geq 0}\frac{1}{z^{2\ell+2}}\sum_{j=0}^{\ell}(-1)^j\frac{(N+\alpha-j)_{\ell+1}(N-j)_{\ell+1}}{j!(\ell-j)!}\end{aligned}$$

where in the last step we have used the following elementary identities

$$\frac{\Gamma(N+\alpha+\ell-j+1)}{\Gamma(N+\alpha-j)} = (N+\alpha-j)_{\ell+1}, \qquad \frac{(\ell-j+1)_N\binom{N-1}{j}}{(N-1)!} = \frac{(N-j)_{\ell+1}}{j!(\ell-j)!}. \tag{3.126}$$

This proves the statement about the expansion at $z=\infty$ for the entry R_{11}. The expansions for the other entries of R are computed likewise. The expansion at $z=0$ is derived in a completely similar manner.

■

Let us observe a *reciprocity phenomenon* between the expansions at $z=0,\infty$.

Lemma 3.5.6. *We have the identities;*

$$\frac{\mathcal{A}^{(0)}_{\ell,N}}{\ell+1} = \frac{1}{c_\ell(\alpha)}\frac{\mathcal{A}^{(\infty)}_{\ell-1,N}}{\ell}, \qquad \ell\geq 1$$

$$\mathcal{B}^{(0)}_{\ell,N} = -\frac{\mathcal{B}^{(\infty)}_{\ell,N}}{c_\ell(\alpha)}, \qquad \ell\geq 0$$

involving the coefficients defined right after (3.121) *and* (3.122)*. Here*

$$c_\ell(\alpha) := (\alpha-\ell)\cdots(\alpha-1)\alpha(\alpha+1)\cdots(\alpha+\ell). \tag{3.127}$$

Consider now the deformed LUE partition function

$$\mathcal{Z}_N^{\mathsf{LUE}}(\mathbf{t}) := \int_{\mathrm{H}_N^+} \det{}^\alpha M \exp \mathrm{tr} \left(-M + \sum_{j\neq 0} t_j M^j \right) \mathrm{d}M, \qquad \mathbf{t} = (..., t_{-2}, t_{-1}, t_1, t_2, ...) \tag{3.128}$$

denoting H_N^+ the cone of positive definite hermitian matrices of size N. Here we are assuming that $t_j = 0$ for $j > K_+$ and $j < K_-$ for some $K_+, K_- \geq 1$ and $\mathrm{Re}\, t_{K_+} < 0$, so that the model is well defined; the results below are unaffected by this arbitrary truncation of the times. The deformed LUE partition function serves as a generating functional of connected correlators of the LUE. The following result has been derived in [GGR] and it provides an effective way to compute the generating function (3.128).

The interest of this result is that the connected correlators of the LUE are known to be related to *weighted double monotone Hurwitz numbers* [CDO]. This connection is expressed as follows. Let $\mu_1 \geq \cdots \geq \mu_r \geq 1$ be a fixed partition of length r and weight $|\mu| := \mu_1 + \cdots + \mu_r$. Substitute the parameter α by $C := 1 + \frac{\alpha}{N}$; then

$$\frac{N^{r-|\mu|}|\mu|!}{z_\mu} \langle \mathrm{tr}\, M^{\mu_1}, ..., \mathrm{tr}\, M^{\mu_r} \rangle_c^{\mathsf{LUE}} = \sum_{g\geq 0} N^{2-2g} C^{|\mu|+2-2g-r} H_g^{>}(\mu; C)$$
$$\frac{N^{r-|\mu|}|\mu|!}{z_\mu} \langle \mathrm{tr}\, M^{-\mu_1}, ..., \mathrm{tr}\, M^{-\mu_r} \rangle_c^{\mathsf{LUE}} = \sum_{g\geq 0} \frac{N^{2-2g}}{(C-1)^{|\mu|+2g-2+r}} H_g^{\geq}(\mu; C)$$

where $z_\mu := \prod_{i\geq 1} m_i! i^{m_i}$ (m_i :=multiplicity of i in μ),

$$\langle \mathrm{tr}\, M^{k_1}, ..., \mathrm{tr}\, M^{k_r} \rangle_c^{\mathsf{LUE}} := \left. \frac{\partial^r \log \mathcal{Z}_N^{\mathsf{LUE}}(\mathbf{t})}{\partial t_{k_1} \cdots \partial t_{k_r}} \right|_{\mathbf{t}=0} \tag{3.129}$$

and $H_g^{>}(\mu; C)$ (resp. $H_g^{\geq}(\mu; C)$) are the *strictly* (resp. *weakly*) *weigthed double monotone Hurwitz numbers*; their definition goes as follows.

First, for $\mu = (\mu_1, ..., \mu_r), \nu = (\nu_1, ..., \nu_s)$ partitions of the same integer $|\mu| = |\nu|$, define the *strictly* (resp. *weakly*) *monotone double Hurwitz numbers* $h_g^{>}(\mu; \nu)$ (resp. $h_g^{\geq}(\mu; \nu)$) as the number of $(m+2)$-tuples $(\alpha, \beta, \tau_1, ..., \tau_m)$ of permutations in $\mathfrak{S}_{|\mu|}$ such that:

(i) $m = r + s + 2g - 2$;

(ii) α, β have cycle type μ, ν respectively, $\tau_1, ..., \tau_m$ are transpositions;

(iii) we have $\alpha\tau_1 \cdots \tau_m = \beta$;

(iv) $\alpha, \tau_1, ..., \tau_m$ generate a transitive subgroup of $S_{|\mu|}$;

(v) writing $\tau_j = (a_j, b_j)$ with $a_j < b_j$ we have

$$b_1 < \cdots < b_m \qquad (\text{resp. } b_1 \leq \cdots \leq b_m). \tag{3.130}$$

Then the *weighted* double monotone Hurwitz numbers are finally defined as

$$H_g^{>}(\mu; C) := \sum_{s\geq 1} \sum_{\substack{\text{partitions } \nu \text{ of length } \ell(\nu)=s \\ \text{and weight } |\nu|=\nu_1+\cdots+\nu_s=|\mu|}} h_g^{>}(\mu; \nu) C^s$$
$$H_g^{\geq}(\mu; C) := \sum_{s\geq 1} \sum_{\substack{\text{partitions } \nu \text{ of length } \ell(\nu)=s \\ \text{and weight } |\nu|=\nu_1+\cdots+\nu_s=|\mu|}} h_g^{\geq}(\mu; \nu) \frac{1}{(C-1)^s}.$$

It would be very interesting to understand whether *mixed* type correlators $\langle \mathrm{tr}\, M^{k_1}, ..., \mathrm{tr}\, M^{k_r} \rangle_c^{\mathsf{LUE}}$, where the k_j's do not necessarily have the same sign, have an analogous combinatorial interpretation in terms of factorizations in some symmetric group.

To formulate concisely the result below let us introduce the following (formal) generating functions

$$\mathcal{F}_{1,0}^{\mathsf{LUE}}(z) := \sum_{k\geq 1} \frac{1}{z^{k+1}} \left\langle \mathrm{tr}\, M^k \right\rangle^{\mathsf{LUE}}, \qquad \mathcal{F}_{0,1}^{\mathsf{LUE}}(z) := \sum_{k\geq 1} z^{k-1} \left\langle \mathrm{tr}\, M^{-k} \right\rangle^{\mathsf{LUE}},$$

$$\mathcal{F}_{2,0}^{\mathsf{LUE}}(z_1,z_2) := \sum_{k_1,k_2\geq 1} \frac{1}{z_1^{k_1+1} z_2^{k_2+1}} \left\langle \mathrm{tr}\, M^{k_1} \mathrm{tr}\, M^{k_2} \right\rangle_c^{\mathsf{LUE}},$$

$$\mathcal{F}_{1,1}^{\mathsf{LUE}}(z_1,z_2) := \sum_{k_1,k_2,\geq 1} \frac{z_2^{k_2-1}}{z_1^{k_1+1}} \left\langle \mathrm{tr}\, M^{k_1} \mathrm{tr}\, M^{-k_2} \right\rangle_c^{\mathsf{LUE}},$$

$$\mathcal{F}_{0,2}^{\mathsf{LUE}}(z_1,z_2) := \sum_{k_1,k_2\geq 1} z_1^{k_1-1} z_2^{k_2-1} \left\langle \mathrm{tr}\, M^{-k_1} \mathrm{tr}\, M^{-k_2} \right\rangle_c^{\mathsf{LUE}},$$

and in general for all $r = r_+ + r_- \geq 1$

$$\mathcal{F}_{r_+,r_-}^{\mathsf{LUE}}(z_1, ..., z_r) := \sum_{k_1,...,k_r\geq 1} \frac{1}{z_1^{\sigma_1 k_1+1} \cdots z_r^{\sigma_r k_r+1}} \left\langle \mathrm{tr}\, M^{\sigma_1 k_1} \cdots \mathrm{tr}\, M^{\sigma_r k_r} \right\rangle_c^{\mathsf{LUE}} \tag{3.131}$$

where

$$\sigma_1 = ... = \sigma_{r_+} = +, \qquad \sigma_{r_++1} = ... = \sigma_r = -. \tag{3.132}$$

Theorem 3.5.7. *([GGR]) The generating functions (3.131) can be expressed as*

$$\mathcal{F}_{1,0}^{\mathsf{LUE}}(z) = \frac{1}{z}\int_z^\infty \left((R_+(w))_{11} - \frac{1}{2}\right) \mathrm{d}w, \qquad \mathcal{F}_{0,1}^{\mathsf{LUE}}(z) = \frac{1}{z}\int_0^z (R_-(w))_{11}\, \mathrm{d}w - \frac{1}{2},$$

$$\mathcal{F}_{2,0}^{\mathsf{LUE}}(z_1,z_2) = \frac{\mathrm{tr}\,(R_+(x_1)R_+(x_2)) - \frac{1}{2}}{(z_1-z_2)^2}, \qquad \mathcal{F}_{1,1}^{\mathsf{LUE}}(z_1,z_2) = -\frac{\mathrm{tr}\,(R_+(z_1)R_-(z_2)) - \frac{1}{2}}{(z_1-z_2)^2},$$

$$\mathcal{F}_{0,2}^{\mathsf{LUE}}(z_1,z_2) = \frac{\mathrm{tr}\,(R_-(z_1)R_-(z_2)) - \frac{1}{2}}{(z_1-z_2)^2},$$

and in general

$$\mathcal{F}_{r_+,r_-}^{\mathsf{LUE}}(z_1,...,z_r) = -\left(\frac{(-1)^{r_-}}{r} \sum_{\pi\in\mathfrak{S}_r} \frac{\mathrm{tr}\,\left(R_{\sigma_{\pi(1)}}(z_{\pi(1)}) \cdots R_{\sigma_{\pi(r)}}(z_{\pi(r)})\right)}{(z_{\pi(1)}-z_{\pi(2)})\cdots(z_{\pi(r-1)}-z_{\pi(r)})(z_{\pi(r)}-z_{\pi(1)})} - \frac{\delta_{r,2}}{2(z_1-z_2)^2}\right) \tag{3.133}$$

where $r = r_+ + r_- \geq 2$, $\mathfrak{S}_r$ is the group of permutations of $\{1,...,r\}$, and we use the signs $\sigma_1, ..., \sigma_r$ of (3.132). Here $R_\pm(z)$ are the formal series

$$R_+(z) := \frac{\sigma_3}{2} + \sum_{\ell\geq 0} \frac{1}{z^{\ell+2}} \begin{pmatrix} \mathcal{A}_{\ell,N}^{(\infty)} & -zN(N+\alpha)\mathcal{B}_{\ell,N+1}^{(\infty)} \\ z\mathcal{B}_{\ell,N}^{(\infty)} & -\mathcal{A}_{\ell,N}^{(\infty)} \end{pmatrix}$$

$$R_-(z) := \frac{\sigma_3}{2} + \sum_{\ell\geq 0} z^\ell \begin{pmatrix} \mathcal{A}_{\ell,N}^{(0)} & -N(N+\alpha)\mathcal{B}_{\ell,N+1}^{(0)} \\ \mathcal{B}_{\ell,N}^{(0)} & -\mathcal{A}_{\ell,N}^{(0)} \end{pmatrix}$$

compare with (3.121) and (3.122).

The proof is below; let us first make a few comments about one-point correlators.

Thm. 3.5.7 generalizes formulæ for one-point correlators which are already known in the literature, see [HSS] for the case $k \geq 1$ and [CMOS] for the general case (i.e. k also negative). Indeed, our formulæ for the generating series $\mathcal{F}_{1,0}^{\mathsf{LUE}}$ and $\mathcal{F}_{0,1}^{\mathsf{LUE}}$ boil down to the following identities, for $\ell \geq 1$;

$$\left\langle \mathrm{tr}\, M^\ell \right\rangle^{\mathsf{LUE}} = \frac{\mathcal{A}_{\ell-1,N}^{(\infty)}}{\ell} = \frac{1}{\ell!}\sum_{j=0}^{\ell-1} (-1)^j \binom{\ell-1}{j} (N-j)_\ell (N+\alpha-j)_\ell \tag{3.134}$$

$$\left\langle \mathrm{tr}\, M^{-\ell} \right\rangle^{\mathsf{LUE}} = \frac{\mathcal{A}_{\ell-1,N}^{(0)}}{\ell} = \frac{1}{\ell!}\sum_{j=0}^{\ell-1} (-1)^j \binom{\ell-1}{j} \frac{(N-j)_\ell}{(\alpha-\ell+1+j)_\ell}. \tag{3.135}$$

In particular, for $\ell \geq 1$ we have from Lemma 3.5.6

$$\left\langle \mathrm{tr}\, M^{-\ell-1} \right\rangle^{\mathsf{LUE}} = \frac{\left\langle \mathrm{tr}\, M^{\ell} \right\rangle^{\mathsf{LUE}}}{c_\ell(\alpha)}, \qquad c_\ell(\alpha) := (\alpha-\ell)\cdots(\alpha-1)\alpha(\alpha+1)\cdots(\alpha+\ell). \tag{3.136}$$

Let us note an equivalent formula to (3.134) for $\langle \mathrm{tr}\, M^\ell \rangle^{\mathsf{LUE}}$ which is more suited to take the planar limit $N \to \infty$ in the regime $C := 1 + \frac{\alpha}{N}$ fixed. Using the identity (3.93) we rewrite

$$\left\langle \mathrm{tr}\, M^{\ell} \right\rangle^{\mathsf{LUE}} = \frac{1}{\ell!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell-1}{j} (N-j)_\ell (N+\alpha-j)_\ell = \frac{\partial^\ell}{\partial x^\ell} \frac{\partial^\ell}{\partial y^\ell} \frac{x^N y^{N+\alpha}(xy-1)^{\ell-1}}{\ell!} \bigg|_{x=1,y=1} \tag{3.137}$$

and then changing variable $1+\xi = x, 1+\eta = y$ the last expression is equal to

$$\frac{\partial^\ell}{\partial x^\ell} \frac{\partial^\ell}{\partial y^\ell} \frac{(1+\xi)^N (1+\eta)^{N+\alpha} (\xi\eta + \xi + \eta)^{\ell-1}}{\ell!} \bigg|_{\xi=0,\eta=0} = \frac{1}{\ell} \sum_{\substack{a,b\geq 0 \\ a+b\leq \ell-1}} \binom{N}{a+1} \binom{N+\alpha}{b+1} \frac{\ell!^2}{a!b!(\ell-1-a-b)!}. \tag{3.138}$$

Then the leading order, in the aforementioned regime $N+\alpha = CN$, is easily found using the trivial asymptotics $\binom{N}{k} \sim \frac{N^k}{k!}$ for $N \to \infty$; it occurs for the terms for which $a+b = \ell-1$ as

$$\frac{N^{\ell+1}}{\ell} \sum_{b=0}^{\ell-1} \frac{C^{b+1}\ell!^2}{(\ell-b)!(b+1)!(\ell-b-1)!b!} = N^{\ell+1} \sum_{b=0}^{\ell-1} \frac{C^{b+1}}{\ell} \binom{\ell}{b} \binom{\ell}{b+1} = N^{\ell+1} \sum_{s=1}^{\ell} \mathcal{N}_{\ell,s} C^s \tag{3.139}$$

involving the *Narayana numbers* $\mathcal{N}_{\ell,s} := \frac{1}{\ell}\binom{\ell}{s-1}\binom{\ell}{s}$ ($\ell > 0$, $s = 1, ..., \ell$), reproducing a result of Wigner [F]. This is related to the positive moments of the equilibrium measure

$$\frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi C x}, \qquad x_\pm := (1 \pm \sqrt{C})^2. \tag{3.140}$$

The large N limit of negative one-point moments then follows from (3.136); these are related to the negative moments of the equilibrium measure (3.140).

Proof of Thm. 3.5.7. Again, it follows from 3.5.1, with the definition (3.74)-(3.75). We begin with the case $r = 1$. Let us first consider the case $k \geq 1$, for which by Thm. 3.3.1 and the discussion of formal residue expression for the Malgrange differential of Chap. 2 we have

$$\frac{\partial}{\partial t_k} \log \mathcal{Z}_N^{\mathsf{LUE}}(\mathbf{t}) \bigg|_{\mathbf{t}=0} = -\operatorname*{res}_{z=\infty} \mathrm{tr} \left(\Gamma^{-1}(z)\Gamma'(z)\frac{\sigma_3}{2} \right) z^k \mathrm{d}z = -\operatorname*{res}_{z=\infty} \mathrm{tr} \left(L(z)R(z)\frac{\sigma_3}{2} \right) z^k \mathrm{d}z \tag{3.141}$$

where L is given in (3.117), $R = \Gamma \frac{\sigma_3}{2} \Gamma^{-1}$ as in (3.73), and we use the identity

$$\Gamma' = L\Gamma - \left(\frac{1}{2} - \frac{\alpha}{2z} \right) \Gamma \sigma_3. \tag{3.142}$$

Note therefore that

$$\mathcal{F}_{1,0}^{\mathsf{LUE}} = \mathrm{tr}\, (LR) + \frac{1}{2} \tag{3.143}$$

where we have to subtract the constant term $\mathrm{tr} \left(-\left(\frac{\sigma_3}{2}\right)^2 \right) = -\frac{1}{2}$ at $z = \infty$ of $\mathrm{tr}\,(LR)$. Multiplying by z and taking one derivative in z we obtain

$$(z\mathcal{F}_{1,0}^{\mathsf{LUE}})' = \mathrm{tr}\, ((zL)'R) + \frac{1}{2} \tag{3.144}$$

as, due to $R' = [L, R]$, $\mathrm{tr}\,(LR') = \mathrm{tr}\,(L[L,R]) = \frac{1}{2}\mathrm{tr}\,([L^2, R]) = 0$. Here we are identifying R with its expansion R_+ at $z = \infty$. Finally, we have $(zL)' = -\frac{\sigma_3}{2}$, see (3.83), hence

$$(z\mathcal{F}_{1,0}^{\mathsf{LUE}})' = -\frac{1}{2}\mathrm{tr}\, (\sigma_3 R) + \frac{1}{2} = -\left(R_{11} - \frac{1}{2} \right) \tag{3.145}$$

where we use $\operatorname{tr} R = 0$, and the proof is complete integrating this expression.

Similarly, for $k \leq -1$ let us rename $k \mapsto -k \geq 1$ and consider

$$\frac{\partial}{\partial t_{-k}} \log \mathcal{Z}_N^{\mathsf{LUE}}(\mathbf{t})\Big|_{\mathbf{t}=0} = -\operatorname*{res}_{z=0} \operatorname{tr}\left(\Gamma^{-1}(z)\Gamma'(z)\frac{\sigma_3}{2}\right) z^{-k}\mathrm{d}z = -\operatorname*{res}_{z=\infty} \operatorname{tr}\left(L(z)R(z)\frac{\sigma_3}{2}\right) z^{-k}\mathrm{d}z + \frac{\delta_{k,1}}{2} \tag{3.146}$$

where we used again (3.142). Note therefore that

$$\mathcal{F}_{0,1}^{\mathsf{LUE}} = \operatorname{tr}(LR) + \frac{1}{2} \tag{3.147}$$

where this time we are identifying R with its expansion R_- at $z = 0$. This is formally identical to (3.143) and so from this point on the proof proceeds exactly as in the previous case.

Finally, for $r \geq 2$ we have

$$\mathcal{F}_{r_+,r_-}^{\mathsf{LUE}}(z_1,...,z_r) = \sum_{k_1,...,k_r \geq 1} \frac{\langle \operatorname{tr} M^{\sigma_1 k_1}, ..., \operatorname{tr} M^{\sigma_r k_r}\rangle_c^{\mathsf{GUE}}}{z_1^{\sigma_1 k_1 + 1} \cdots z_r^{\sigma_r k_r + 1}} \tag{3.148}$$

$$= \sum_{k_1,...,k_r \geq 1} \frac{\operatorname*{res}_{\xi_1=\infty} \cdots \operatorname*{res}_{\xi_{r_+}=\infty} \operatorname*{res}_{\xi_{r_+ +1}=0} \cdots \operatorname*{res}_{\xi_r=0} S_r(\xi_1,...,\xi_r)\xi_1^{\sigma_1 k_1}\mathrm{d}\xi_1 \cdots \xi_r^{\sigma_r k_r}\mathrm{d}\xi_r}{z_1^{\sigma_1 k_1 + 1} \cdots z_r^{\sigma_r k_r + 1}} \tag{3.149}$$

$$= -\left(\frac{(-1)^{r_-}}{r} \sum_{\pi \in \mathfrak{S}_r} \frac{\operatorname{tr}\left(R_{\sigma_{\pi(1)}}(z_{\pi(1)}) \cdots R_{\sigma_{\pi(r)}}(z_{\pi(r)})\right)}{(z_{\pi(1)} - z_{\pi(2)}) \cdots (z_{\pi(r-1)} - z_{\pi(r)})(z_{\pi(r)} - z_{\pi(1)})} - \frac{\delta_{r,2}}{2(z_1 - z_2)^2}\right) \tag{3.150}$$

which is the desired formula, where in the last step we replaced the analytic function S_r with its formal expansion for $x_1, ..., x_{r+} \to \infty$ and $x_{r_+ +1}, ..., x_r \to 0$, compare with (3.75). ■

For more applications and examples see [GGR].

CHAPTER 4

Kontsevich–Witten tau function

The Kontsevich matrix integral was introduced by Kontsevich [Kb] as a tool to prove Witten conjecture, as it provides a bridge between combinatorics of intersection numbers over the moduli spaces of curves and the KdV hierarchy. In this chapter we review Witten conjecture and and then focus on the interpretation of the Kontsevich matrix integral as an isomonodromic tau function, following [BCa]; applications of the isomonodromic approach are effective formulæ for the intersection numbers first obtained in [BDYa], which we recall.

Main references for this chapter are [Wc; Kb; BDYa; BCa].

4.1 Witten Conjecture

Inspired by physical intuition about 2D quantum gravity, in 1991 Witten [Wc] proposed his celebrated conjecture, establishing a very prolific connection between algebraic geometry and integrable systems. This conjecture was first proven by Kontsevich [Kb], by the use of the matrix model that now bears his name; other proofs were later given by Okounkov and Pandharipande [Ob; OP], Kazarian and Lando [KL], Mirzakhani [Md; Me].

We first introduce the moduli spaces of curves. As there are many excellent reviews [Ha; Lc; Zc; LZ], and more extended references [ACG; HM] on the topic, we content ourselves with the following brief overview.

Moduli spaces of curves. Peculiar to algebraic geometry is the study of families [Mh]. Indeed it is often the case that a parameter space for certain varieties it is naturally a variety (e.g. grassmannians, conics in the plane). Already Riemann [R] noted that algebraic curves over $\mathbb{C}$ (what we call Riemann surfaces[1]) of a fixed genus g can be parametrized, up to isomorphism, by $3g-3$ complex numbers, as soon as $g \geq 2$. It is natural to expect that the set of isomorphism classes of Riemann surfaces of genus g carries a variety-like structure of complex dimension $3g-3$.

More precisely, the modern rigorous statement is the following (more precisely, see Thm. 4.1.2 below). Fix $g, n \geq 0$ with $2g-2+n > 0$ (*stability condition*). Then the set $\mathcal{M}_{g,n}$ of isomorphism classes of compact and connected Riemann surfaces of genus g, with the additional structure of n *distinct* ordered marked points on it (the isomorphisms are then required to fix these marked points), carries the structure of a complex *orbifold* of dimension $3g-3+n$.

For the general definition of orbifold we refer to the aforementioned literature, let us just say that an orbifold is a natural generalization of manifold in the following sense. Recall that the quotient of a manifold by a free smooth proper group action is a manifold itself (properness ensures the Hausdorff property). The notion of (basic) orbifold arises lifting the freeness property; somewhat more precisely, a basic orbifold is a pair (X, G) where X is a connected and simply connected manifold and G a finite group acting properly, smoothly and with *finite* stabilizers on X. Then one can define more general orbifolds by gluing in a suitable way basic orbifolds.

The precise definition is a bit more technical and goes beyond the scope of this basic explanation, so let us see one example that illustrates the situation.

Example 4.1.1 (elliptic curves)**.** *An elliptic curve is a Riemann surface of genus* 1 *with the choice of a marked point on it. It is well known that an elliptic curve* $\mathcal{E}$ *is represented as* $\mathcal{E}_\tau = \frac{\mathbb{C}}{\Lambda_\tau}$ *where* $\Lambda_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$

[1]For us a Riemann surface will be a one-dimensional compact connected complex manifold.

is a lattice in $\mathbb{C}$*, with* $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$*; this is proven by the fact that the Abel map is an isomorphism in this case. The marked point is given by the class of* $0 \in \mathbb{C}$*.*

Moreover, it is well known that any isomorphism $\mathcal{E}_\tau \to \mathcal{E}_{\tau'} : z \mapsto z'$ *must be of the form*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad z' = \frac{z}{c\tau + d} = z(a - c\tau'), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \tag{4.1}$$

(i.e. $a, b, c, d \in \mathbb{Z}$ *and* $ad - bc = 1$*). Therefore we can identify*

$$\mathcal{M}_{1,1} = \frac{\mathbb{H}}{\mathrm{SL}_2(\mathbb{Z})}. \tag{4.2}$$

It carries naturally the structure of (basic) orbifold, as the action of $\mathrm{SL}_2(\mathbb{Z})$ *is proper, smooth, and with finite stabilizers. It is really an orbifold and not just a manifold, as generically the stabilizer is* $\{\pm\mathbf{1}\}$*, but for the square lattice* $\tau = \mathrm{i}$ *and the hexagonal lattice* $\tau = \mathrm{e}^{\mathrm{i}\pi/3}$ *the stabilizers are bigger, respectively cyclic of order 4 generated by* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *for the square lattice and cyclic of order 6 generated by* $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ *for the hexagonal lattice.*

Note that we always have to assume $2g - 2 + n > 0$; this stability condition ensures that the automorphism group of the Riemann surfaces of genus g with n marked points is finite (compare with Thm. 4.1.2). This rules out the *unstable* cases $g = 0$, $n = 0, 1, 2$ (with continuous families of Möbius transformations) and the case $g = 1$, $n = 0$ (with the translation as a continuous family of automorphisms). From the point of view of *uniformization*, $2g - 2 + n > 0$ is equivalently expressed as the fact that the universal cover of a Riemann surface of genus g with n points removed is the Poincaré disk.

There are (at least) two approaches to define the orbifold complex structure on general moduli spaces $\mathcal{M}_{g,n}$ with $2g - 2 + n > 0$. One is via *Teichmüller theory* [He], the other via Mumford's *Geometric Invariant Theory* [Mg].

The first one is more explicit; basically it realizes $\mathcal{M}_g$ (usually in this theory $n = 0$) as the quotient of the *Teichmüller space* (the space of all conformal structures up to isotopy on a fixed orientable topological surface Σ_g of genus g; such space is homeomorphic to a ball in $\mathbb{C}^{3g-3}$ by Bers theorem [Bb]) with respect to the *mapping class group* (the group of orientation preserving diffeomorphisms up to isotopy of Σ_g). This naturally displays a real orbifold structure on $\mathcal{M}_g$ (stabilizers are automorphism groups of the curves), and various types of explicit coordinates; with more work one also obtains the the complex structure.

The second one is more direct, although more abstract, and has the advantage of directly providing the structure of a quasi-projective algebraic variety to and a natural compactification of $\mathcal{M}_{g,n}$, which we now turn our attention to.

Deligne–Mumford compactification The orbifolds $\mathcal{M}_{g,n}$ are not compact (compare with Ex. (4.1.1), about $\mathcal{M}_{1,1}$). However they admit a convenient compactification, by adding nodal curves. A nice heuristic explanation of the appearance of nodal curves is obtained considering the simplest case $\mathcal{M}_{0,4} \ni [(C; p_1, p_2, p_3, p_4)]$ where C is a rational curve and $p_i \in C$ are distinct point on it and [] denotes the equivalence class of the equivalence relation $(C; p_i) \sim (C', p_i')$ if and only if $\varphi : C \to C'$ is a biholomorphism such that $\varphi(p_i) = p_i'$.

Fixing $p_1 = 0, p_2 = 1, p_3 = \infty$ by a Möbius transformation, $\mathcal{M}_{0,4}$ is parametrized by $p_4 = t \in \mathbb{C} \setminus \{0, 1\}$, and taking the limit $t \to 0$ (which must be defined for any compactification) we obtain $p_1 = p_4$ and $p_1 \neq p_2$. However for any nonzero t we have, by definition of equivalence class,

$$[(\mathbb{C}P^1; 0, 1, \infty, t)] = [(\mathbb{C}P^1; 0, \frac{1}{t}, \infty, 1)] \tag{4.3}$$

and in the same limit $t \to 0$ we now have $p_2 = p_3$ and $p_1 \neq p_4$. This suggests that for any reasonable compactification, the limit $t \to 0$ should include both configurations; this is achieved declaring the limit to be a singular rational curve with one separating node, and p_1, p_4 in one component and p_2, p_3 in the other one. For more details about this example see [Zc; LZ].

We now give the precise definition and statement of existence.

A *stable curve with* n *marked points* ($n \geq 0$) is a tuple $(C; p_1, ..., p_n)$ consisting of a connected curve C and n distinct points $p_i \in C$, $p_i \neq p_j$, such that

- C is smooth but for a finite set of nodes[2] away from $p_1, ..., p_n$, and
- C has no nontrivial infinitesimal automorphisms, or (equivalently) has a finite automorphism group.

The second condition admits a more effective formulation; recall that any curve C whose only singularities are nodes can be *normalized* by detaching all pairs of disks attached at a node. By this procedure one obtains a smooth (disconnected in general) curve, which is called *normalization* of C, and denoted $\hat{C}$. Then C has a finite automorphism group if and only if any connected component of the normalization $\hat{C}$ has genus $\hat{g}$ and number $\hat{n}$ of marked points (i.e. preimages of the marked points p_i and of the nodes of C) satisfying $2\hat{g} - 2 + \hat{n} > 0$.

Then we recall the following fundamental result, see the original work [DM] or the more recent reference [HM, Chap.4].

Theorem 4.1.2 (Deligne and Mumford, 1969)**.** *For each $g \geq 0, n \geq 0$ fulfilling $2g-2+n > 0$ there exists a compact complex orbifold $\overline{\mathcal{M}}_{g,n}$ of complex dimension $3g-3+n$ such that $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open dense Zariski subset. Moreover, there exists a compact complex orbifold $\overline{\mathcal{C}}_{g,n}$ and a map $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that*

- *the fibers of π are stable curves of genus g with n marked points,*
- *every stable curve of genus g with n marked points is isomorphic to one and only one fiber of π, and*
- *the stabilizer at $p \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of the stable curve $\pi^{-1}(p)$.*

In the language of algebraic geometry, the map $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is called *universal curve*, and it realizes $\overline{\mathcal{M}}_{g,n}$ as a *fine* moduli space in the category of complex orbifolds [HM].

Psi-classes. An orbifold *vector bundle* $\pi : E \to X$ of rank r over a basic orbifold (X, G) (we recall that this means that X is a connected and simply connected complex manifold and G is a finite group acting properly, smoothly, and with finite stabilizers on X) is a vector bundle $\pi : E \to X$ where E is also endowed with an action of G on E, lifting that on X in the sense that π is G-equivariant. For more general orbifolds one makes out an orbifold vector bundle by gluing several of these pieces.

Over $\mathcal{M}_{g,n}$ there are natural orbifold line bundles $\mathcal{L}_1, ..., \mathcal{L}_n$, the fiber of $\mathcal{L}_i$ over (the equivalence class of) a stable curve $(C; p_1, ..., p_n)$ being the cotangent line $T^*_{p_i}C$. There is no problem in extending these orbifold line bundles to the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, as the marked points p_i are away from the nodes, and let us denote $\mathcal{L}_i$ also these line bundles.

The orbifold (co)homology ring are defined as the (co)homology ring of the underlying topological spaces. The main important feature of orbifold (co)homology is the use rational coefficients instead of integral coefficients. E.g. the homology class of an irreducible sub-orbifold Y of X is defined as $\frac{1}{k}[\widehat{Y}] \in H_*(\widehat{X}; \mathbb{Q})$ where $\widehat{Y}$ and $\widehat{X}$ are the underlying topological spaces of the orbifolds Y and X, respectively, and k is the order of the stabilizer of a generic point in Y. Accordingly, characteristic classes are rational and not integral; for instance, the Chern class of an orbifold line bundle is a *weighted* count of zeros of a generic section, the weight being the inverse of the order of the stabilizer at the zero. For more details we refer to the literature (see also Ex. 4.1.3).

Let $\psi_i := c_1(\mathcal{L}_i)$ the Chern class of the line bundles over $\overline{\mathcal{M}}_{g,n}$ introduced above. The Witten conjecture concerns the *Witten intersection numbers*

$$\langle \tau_{r_1} \cdots \tau_{r_n} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{r_1} \cdots \psi_n^{r_n} \in \mathbb{Q}. \tag{4.4}$$

In (4.4) the integration over $\overline{\mathcal{M}}_{g,n}$ on the right side denotes pairing with the orbifold fundamental class $[\overline{\mathcal{M}}_{g,n}]$, however the notation as an integral is normal practice in the literature. We agree that in the right side of (4.4) $g := \frac{1}{3}(r_1 + ... + r_n - n) + 1$ (so that $\psi_1^{r_1} \cdots \psi_n^{r_n}$ is a cohomology class in the top-dimensional cohomology space $H^{6g-6+2n}\left(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}\right)$ and can be paired with the orbifold fundamental class) with the

[2] A point of C is called a node if and only if it is locally analytically isomorphic to a neighborhood of $(0,0)$ in $\{(x,y) \in \mathbb{C}^2 : xy = 0\}$. Locally, a node is diffeomorphic to the disjoint union of two disks glued at the respective origins.

implicit assumption that $\langle \tau_{r_1} \cdots \tau_{r_n} \rangle := 0$ whenever g is not a nonnegative integer or in the unstable cases $g = 0, n = 1, 2$.

Example 4.1.3 (elliptic curves and modular forms)**.** *The orbifold line bundle $\mathcal{L}_1 \to \mathcal{M}_{1,1}$ is easily described. Indeed, in view of Ex. 4.1.1 it is obtained from the trivial vector bundle*

$$\mathbb{C}\mathrm{d}z \times \mathbb{H} \to \mathbb{H} \tag{4.5}$$

by the following lift of the action of $\mathrm{SL}_2(\mathbb{Z})$ *of* (4.1)*;*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .(\mathrm{d}z, \tau) = (\mathrm{d}z', \tau') = \left(\frac{\mathrm{d}z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \tag{4.6}$$

Sections of $\mathcal{L}_1$ are sections of the trivial bundle (4.5) *equivariant with respect to the action* (4.6)*, i.e.*

$$f(\tau)\mathrm{d}z = f(\tau')\mathrm{d}z' = f\left(\frac{a\tau + b}{c\tau + d}\right) \frac{\mathrm{d}z}{c\tau + d} \Rightarrow f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)f(\tau). \tag{4.7}$$

More generally, section of tensor powers $\mathcal{L}_1^{\otimes k}$ should satisfy

$$f(\tau)\mathrm{d}z^{\otimes k} = f(\tau')\mathrm{d}z'^{\otimes k} = f\left(\frac{a\tau + b}{c\tau + d}\right) \frac{\mathrm{d}z^{\otimes k}}{(c\tau + d)^k} \Rightarrow f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau). \tag{4.8}$$

These are precisely modular forms of weight k, with respect to the full modular group $\mathrm{SL}_2(\mathbb{Z})$ *[Za].*

*This picture can be extended so to include the point (*cusp*) at $\tau = \infty$ (which is not an orbifold point, i.e. it has stabilizer equal to the stabilizer at a generic point); holomorphic sections of $\mathcal{L}_1^{\otimes k}$ over the compactification $\overline{\mathcal{M}}_{1,1} = \mathbb{H} \cup \{\infty\}$ are precisely modular forms of weight k holomorphic also at ∞ (in the sense that the Fourier series*[3] *$f(\tau) = \sum_{n \geq 0} a_n \mathrm{e}^{2\mathrm{i}\pi n\tau}$ satisfies $a_n = 0$ for all $n < 0$).*

It is well known [Za, Prop. 2 on page 9] that for a holomorphic modular form of weight k, which is holomorphic at ∞ too, we have

$$\sum_{\tau \in \frac{\mathbb{H}}{\mathrm{SL}_2(\mathbb{Z})}} \frac{1}{n_\tau} \mathrm{ord}_\tau(f) + \mathrm{ord}_\infty(f) = \frac{k}{12} \tag{4.9}$$

where $\mathrm{ord}_\tau(f)$ *is the order of vanishing of f at τ (the modular transformation property* (4.8) *it is well defined for $\tau \in \frac{\mathbb{H}}{\mathrm{SL}_2(\mathbb{Z})}$, as it only depends on the* $\mathrm{SL}_2(\mathbb{Z})$*-orbit of τ) and*

$$n_\tau := \begin{cases} 2 & \tau = \mathrm{i} \\ 3 & \tau = \mathrm{e}^{\mathrm{i}\pi/3} \\ 1 & \textit{otherwise.} \end{cases} \tag{4.10}$$

The proof is just an integration of the logarithmic form $\mathrm{d}\log f$ *along the boundary of the fundamental domain of the* $\mathrm{SL}_2(\mathbb{Z})$ *action on $\mathbb{H}$, paying attention to poles at the boundary of the fundamental domain and at the orbifold points $\tau = \mathrm{i}, \mathrm{e}^{\mathrm{i}\pi/3}$; see loc. cit.*

As holomorphic modular forms exist (for $k \geq 4$) we can apply this argument to compute the Chern number $\int_{\overline{\mathcal{M}}_{1,1}} c_1(\mathcal{L}_1^{\otimes k}) = k\langle \tau_1 \rangle$, compare with the definition (4.4)*. Indeed this number is precisely the orbifold weighted count of zeros of an holomorphic section of $\mathcal{L}_1^{\otimes k}$, as in* (4.9)*; recalling that the generic stabilizer of $\mathcal{M}_{1,1}$ is*[4] *$\{\pm\mathbf{1}\}$ of order 2, pairing with the orbifold fundamental class of $\overline{\mathcal{M}}_{1,1}$ gives an extra factor of 2, and so we obtain the identity*

$$k\langle \tau_1 \rangle = \frac{1}{2} \cdot \frac{k}{12} \Rightarrow \langle \tau_1 \rangle = \frac{1}{24}. \tag{4.11}$$

For an alternative derivation of (4.11) *using a pencil of elliptic curves see [LZ, Ex. 4.6.6].*

Example 4.1.4 (genus zero)**.** *It can be verified explicitly for low n and proven in general (see e.g. [Wc; Zc; LZ; ACG]) that we have the following formula for genus zero Witten intersection numbers in terms of multinomial coefficients;*

$$\langle \tau_{r_1} \cdots \tau_{r_n} \rangle = \binom{n-3}{r_1, \ldots, r_n} = \frac{(n-3)!}{r_1! \cdots r_n!}, \qquad r_1 + \cdots + r_n = n - 3. \tag{4.12}$$

[3] As a consequence of the modular transformation property $f(\tau + 1) = f(\tau)$.

[4] Every elliptic curve has the elliptic involution as a nontrivial automorphism.

The proof is a simple induction based on the string equation*, see below. It is worth noting that rational curves with $n \geq 3$ marked points are strongly rigid, i.e. they have trivial automorphism group; correspondingly the spaces $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ are really smooth projective varieties (they admit very nice description and combinatorial structure in terms of iterated blow-ups of $\mathbb{P}^{n-3}$ [Ka]). Accordingly, the genus zero Witten intersection numbers (4.12) are integer (rather than just rational) numbers.*

Witten conjecture. In principle the problem of computing Witten intersection numbers (4.4), beyond the examples in genus 0 and 1 which we have considered above, is general very hard.

However, in 1990 Witten surprised the mathematical community with the following conjecture [Wc], now called Kontsevich–Witten theorem. To formulate it, collect the Witten intersection numbers into the generating function ("free energy")

$$F(\mathbf{T}) := \sum_{n\geq 1}\sum_{r_1,\dots,r_n\geq 0}\langle \tau_{r_1}\cdots\tau_{r_n}\rangle \frac{T_{r_1}\cdots T_{r_n}}{n!} = \frac{T_0^3}{6} + \frac{T_1}{24} + \frac{T_0^3T_1}{24} + \frac{T_0T_2}{24} + \frac{T_1^2}{24} + \cdots \tag{4.13}$$

where $\mathbf{T} = (T_0, T_1, \dots)$.

Theorem 4.1.5 (Witten conjecture)**.** *The exponential $\tau(\mathbf{T}) := \exp F(\mathbf{T})$ is a tau function of the KdV hierarchy*[5] *in the variables $\mathbf{t} = (t_1, t_3, t_5, \dots)$ defined by*

$$t_{2k+1} := -\frac{\sqrt[3]{2^{2k+1}}}{(2k+1)!!}T_k. \tag{4.14}$$

Let us briefly comment on the origin of this conjecture, which connected for the first time two seemingly very far subjects (infinite dimensional integrable systems, the KdV world, with the world of algebraic geometry). The bridge is the theory of 2D quantum gravity.

2D quantum gravity is to be regarded as a theory of random metrics on a topological surface, the *path-integral* extending also to a summation over all possible topologies of the surface (i.e. to a summation over the genus of the surface). One approach to such a theory considers a discretization of the surfaces and related combinatorics, and it was well known in the 80s that the correlators for such a theory were intimately related (via matrix models) with integrable hierarchies of PDEs, in particular with the KdV hierarchy (see for instance [DGZ] or the more mathematics oriented review in [LZ, Sec. 3.6], and references therein). Yet another approach to such a theory considers a supersymmetric localization of the path-integral to a finite-dimensional integral over the moduli space of metrics over a topological curve, equivalent to an integration over complex structures.

Witten [Wc], motivated by the fact that the same *string equation* (see below) appeared in both approaches to 2D quantum gravity, conjectured the equivalence thereof, expressed as an identity of the correlators of each theory.

Virasoro constraints. The fact that the exponential of the generating function (4.13) is a tau function of the KdV hierarchy implies highly nontrivial recursion relations at the level of Witten intersection numbers (4.4). It was soon realized [DVV] that such recursions can be best expressed in terms of the so called *Virasoro constraints*.

Introduce the family of differential operators L_n for $n = -1, 0, 1, 2, \dots$ as

$$\begin{aligned} L_n &= \frac{T_0^2}{2}\delta_{n,-1} + \frac{\delta_{n,0}}{8} + \sum_{i\geq 0}\frac{(2(i+n)+1)!!}{(2i-1)!!}(T_i - \delta_{i,1})\frac{\partial}{\partial T_{i+n}} \\ &\quad + \frac{1}{2}\sum_{i=0}^{n-1}(2i+1)!!(2(n-i)-1)!!\frac{\partial^2}{\partial T_i \partial T_{n-1-i}} \\ &= \sqrt[3]{4^n}\left(\frac{t_1^2}{2}\delta_{n,-1} + \frac{\delta_{n,0}}{8} + \sum_{i\geq 0}(2i+1)\left(t_{2i+1} + \frac{2}{3}\delta_{i,1}\right)\frac{\partial}{\partial t_{2(i+n)+1}} + \frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2}{\partial t_{2i+1}\partial t_{2(n-i)-1}}\right) \end{aligned} \tag{4.15}$$

called *Virasoro operators*. We have used the variables $\mathbf{T} = (T_0, T_1, \dots), \mathbf{t} = (t_1, t_3, \dots)$ related as (4.14).

Theorem 4.1.6 ([DVV]). *The following two facts about the formal series $\tau(\mathbf{t}) := \exp F(\mathbf{t})$, with the generating function defined in (4.13), are equivalent.*

- $\tau(\mathbf{t})$ *is a tau function of the KdV hierarchy, satisfying the string equation $L_{-1}\tau = 0$.*
- $\tau(\mathbf{t})$ *satisfies all* Virasoro constraints, *in the sense that $L_j\tau = 0$ for all $j \geq -1$.*

For the proof we refer to the original work [DVV] or for a review to [ACG, Thm. 3.2, Chap. XX]. It can be readily checked that the operators (4.15) satisfy the *Virasoro*[6] commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{4.16}$$

Actually this is just half of the Virasoro algebra, and the *central charge* is invisible therefore the commutation relations are the same of half of the Witt algebra. Incidentally, let us recall that the Witt algebra is generated by the infinitesimal holomorphic transformations $\mathbb{L}_j := -z^{j+1}\frac{\partial}{\partial z}$, which commute as $[\mathbb{L}_m, \mathbb{L}_n] = (m-n)\mathbb{L}_{m+n}$. Below we shall connect directly the Virasoro constraints $L_j\tau = 0$ for the Kontsevich–Witten tau function with the Witt algebra of infinitesimal holomorphic transformations.

The first Virasoro constraint $L_{-1}\tau = 0$ reads as

$$\left(\frac{T_0^2}{2} + \sum_{i\geq 1}(T_i - \delta_{i,1})\frac{\partial}{\partial T_{i-1}}\right)\tau(\mathbf{T}) = \sqrt[3]{2}\left(\frac{t_1^2}{4} + \sum_{i\geq 1}\frac{2i+1}{2}\left(t_{2i+1} + \frac{2}{3}\delta_{i,1}\right)\frac{\partial}{\partial t_{2i-1}}\right)\tau(\mathbf{t}) = 0, \tag{4.17}$$

and is called, following physical terminology, *string equation*; it is related to the vector field $\frac{\partial}{\partial z}$ of translations in the z-plane (see below).

The second constraint $L_0\tau = 0$ reads as

$$\left(\frac{1}{8} + \sum_{i\geq 0}(2i+1)(T_i - \delta_{i,1})\frac{\partial}{\partial T_i}\right)\tau(\mathbf{T}) = 2\left(\frac{1}{16} + \sum_{i\geq 1}\frac{2i+1}{2}\left(t_{2i+1} + \frac{2}{3}\delta_{i,1}\right)\frac{\partial}{\partial t_{2i+1}}\right)\tau(\mathbf{t}) = 0, \tag{4.18}$$

and is called, again following physical terminology *dilaton equation*; it is related to the vector field $z\frac{\partial}{\partial z}$ of dilations in the z plane (see below).

Witten [Wc] proved the string equation $L_{-1}\tau = 0$ for the generating function $\tau = \exp F$, with geometric methods. This point was one of the first strong motivations of his conjecture.

Without going much more into the details, let us note how the Virasoro constraints express recursion relations between Witten intersection numbers (4.4).

For instance, the string equation $L_{-1}\tau = 0$ is equivalent to the following relation

$$\langle \tau_0\tau_{r_1}\cdots\tau_{r_n}\rangle = \sum_{\substack{j=1,\dots,n\\ r_j\geq 1}}\langle \tau_{r_1}\cdots\tau_{r_j-1}\cdots\tau_{r_n}\rangle \tag{4.19}$$

from which, together with the trivial case $\langle\tau_0^3\rangle = \int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$, it is easy to establish the formula (4.12).

The dilaton equation $L_0\tau = 0$ is equivalent to the following relation

$$\langle \tau_1\tau_{r_1}\cdots\tau_{r_n}\rangle = (2g-2+n)\langle \tau_{r_1}\cdots\tau_{r_n}\rangle. \tag{4.20}$$

More generally the Virasoro constraints $L_n\tau = 0$ for $n \geq -1$ allow in principle to compute recursively all Witten intersection numbers (4.4) from the *initial datum* $\langle\tau_0^3\rangle = 1$. For more details see e.g. [ACG, Lemma 2.10, Chap. XX]. Below we use our formalism of tau functions to derive explicit generating functions for the same intersection numbers; such formulæ were first considered in [BDYa] and are very effective for computations of the Witten intersection numbers (4.4).

[6] The Virasoro algebra is generated by elements L_n for $n \in \mathbb{Z}$ commuting as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}.$$

The parameter $c \in \mathbb{C}$ is called central charge and it gives the one-parameter family of all possible central extentions of the Witt algebra, recovered for $c = 0$.

4.2 Kontsevich matrix integral

Kontsevich matrix integral. In his proof of the Witten conjecture [Kb], Kontsevich introduces the following matrix integral which now bears his name;

$$\mathcal{Z}_N^{\mathsf{K}}(Y) := \frac{\int_{\mathrm{H}_N} \exp \operatorname{tr}\left(\mathrm{i}\frac{M^3}{3} - YM^2\right) \mathrm{d}M}{\int_{\mathrm{H}_N} \exp \operatorname{tr}\left(-YM^2\right) \mathrm{d}M} \tag{4.21}$$

where $Y = (y_1, ..., y_N)$ ("external source") is a diagonal matrix with positive entries, $y_i > 0$, so that the integrals in (4.21) are absolutely convergent.

The Kontsevich matrix integral can be regarded as a matrix version of the Airy function[7]

$$\mathrm{Ai}(z) := \frac{1}{2\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon} \exp\left(\mathrm{i}\frac{x^3}{3} + \mathrm{i}xz\right) \mathrm{d}x \tag{4.22}$$

where the integral is independent of $\epsilon > 0$, which is added to make the integral absolutely convergent.

Indeed it can be easily seen from this integral representation of the Airy function that $\mathcal{Z}_{N=1}^{\mathsf{K}}(y) = 2\sqrt{\pi y}\mathrm{e}^{\frac{2}{3}y^3}\mathrm{Ai}(y^2)$. More generally we have the following determinantal expression.

Lemma 4.2.1. *The Kontsevich matrix integral* (4.21) *can be expressed as follows:*

$$\mathcal{Z}_N^{\mathsf{K}}(Y) = \left(2\sqrt{\pi}\right)^N \frac{\det\sqrt{Y} \exp \operatorname{tr}\left(\frac{2}{3}Y^3\right)}{\Delta(Y)} \det\left(\mathrm{Ai}^{(j-1)}(y_k^2)\right)_{j,k=1}^N \tag{4.23}$$

where we denote $\mathrm{Ai}^{(j-1)}(z) := \frac{\mathrm{d}^{j-1}\mathrm{Ai}(z)}{\mathrm{d}z^{j-1}}$ *derivatives of the Airy function* (4.22).

Proof. Let us consider first the gaussian integral in the denominator of (4.21). We have

$$\operatorname{tr}(YM^2) = \sum_{i,j=1}^N y_i M_{ij} M_{ji} = \sum_{i,j=1}^N y_i |M_{ij}|^2 = \sum_{i<j}^N (y_i + y_j)|M_{ij}|^2 + \sum_{i=1}^N y_i |M_{ii}|^2 \tag{4.24}$$

where we use the identity $M_{ij} = \overline{M_{ji}}$ as M is hermitian, therefore writing $M_{ij} = M'_{ij} + \mathrm{i}M''_{ij}$ we have

$$\begin{aligned}
&\int_{\mathrm{H}_N} \exp \operatorname{tr}(-YM^2)\mathrm{d}M \\
&= \prod_{1\le i<j\le N} \int_{\mathbb{R}} \mathrm{e}^{(y_i+y_j)M'^{\,2}_{ij}} \mathrm{d}M'_{ij} \prod_{1\le i<j\le N} \int_{\mathbb{R}} \mathrm{e}^{(y_i+y_j)M''^{\,2}_{ij}} \mathrm{d}M''_{ij} \prod_{i=1}^N \int_{\mathbb{R}} \mathrm{e}^{y_i M_{ii}^2} \mathrm{d}M_{ii} \\
&= \prod_{1\le i<j\le N} \frac{\pi}{y_i + y_j} \prod_{i=1}^N \sqrt{\frac{\pi}{y_i}} = \frac{\sqrt{\pi^{N^2}}}{\det\sqrt{Y}} \frac{\Delta(Y)}{\Delta(Y^2)}.
\end{aligned} \tag{4.25}$$

For the numerator of (4.21) instead we have the following chain of equalities;

$$\int_{\mathrm{H}_N} \exp \operatorname{tr}\left(\mathrm{i}\frac{M^3}{3} - YM^2\right) \mathrm{d}M \tag{4.26}$$

$$\stackrel{(1)}{=} \exp\left(\frac{2}{3}\operatorname{tr} Y^3\right) \int_{H_N} \exp \operatorname{tr}\left(\mathrm{i}\frac{M'^3}{3} + \mathrm{i}M'Y^2\right) \mathrm{d}M' \tag{4.27}$$

$$\stackrel{(2)}{=} \frac{1}{N!} \exp\left(\frac{2}{3}\operatorname{tr} Y^3\right) \int_{\mathbb{R}^N} \Delta^2(X) \prod_{j=1}^N \mathrm{e}^{\frac{\mathrm{i}x_j^3}{3}} \mathrm{d}x_j \int_{\mathrm{U}_N/(\mathrm{U}_1^N)} \mathrm{d}U \exp \operatorname{tr}\left(\mathrm{i}Y^2 U X U^\dagger\right) \tag{4.28}$$

$$\stackrel{(3)}{=} \frac{\pi^{\frac{N(N-1)}{2}}}{N!} \exp\left(\frac{2}{3}\operatorname{tr} Y^3\right) \int_{\mathbb{R}^N} \frac{\Delta(X) \det\left(\mathrm{e}^{\mathrm{i}x_j y_k^2}\right)_{j,k=1}^N}{\Delta(\mathrm{i}Y^2)} \prod_{j=1}^N \mathrm{e}^{\frac{\mathrm{i}x_j^3}{3}} \mathrm{d}x_j \tag{4.29}$$

$$\stackrel{(4)}{=} \pi^{\frac{N(N-1)}{2}} \frac{\exp\left(\frac{2}{3}\operatorname{tr} Y^3\right)}{\Delta(\mathrm{i}Y^2)} \det\left(\int_{\mathbb{R}} x^{N-j} \exp\left(\frac{\mathrm{i}x^3}{3} + \mathrm{i}xy_k^2\right) \mathrm{d}x\right)_{j,k=1}^N. \tag{4.30}$$

[7] We use this particular integral representation (4.22) for the Airy function as it is the most convenient for our purposes.

In (1) we perform a shift $M' := M + \mathrm{i}Y$ and an analytic continuation: the integral is now only conditionally convergent, it is absolutely convergent only when understood as integration over $H_n + \mathrm{i}\epsilon\mathbf{1}$ for any $\epsilon > 0$. In (2) we apply Weyl integration formula (Prop. B.1.1) and we use the notation $X = \mathrm{diag}(x_1, ..., x_n)$. In (3) we apply Harish-Chandra formula (B.11) and in (4) Andreief identity (Lemma B.3.1). The proof is completed by the identity

$$\int_{\mathbb{R}} x^\ell \exp\left(\frac{\mathrm{i}x^3}{3} + \mathrm{i}xz\right) \mathrm{d}x = \frac{2\pi}{\mathrm{i}^\ell} \mathrm{Ai}^{(\ell)}(z) \tag{4.31}$$

which directly follows from (4.22). ■

Asymptotic expansion and intersection numbers Recall [AS] that the Airy function has the asymptotic expansion

$$\mathrm{Ai}(z^2) \sim \frac{\mathrm{e}^{-\frac{2}{3}z^3}}{\sqrt{4\pi z}} \sum_{j\geq 0} \frac{(6j-1)!!}{(2j)!72^j} \frac{(-1)^j}{z^{3j}} \tag{4.32}$$

for $z \to \infty$, uniformly in closed subsectors of $|\arg z| < \frac{\pi}{2}$. From the expression of Lemma 4.2.1 we obtain that we have an asymptotic expansion

$$\mathcal{Z}_N^{\mathsf{K}}(Y) \sim \tau_N^f \tag{4.33}$$

where, with the notation of Sec. 1.4.3, we set

$$\tau_N^f(z_1, ..., z_N) := \frac{\det\left(f_j(z_k)\right)_{j,k=1}^N}{\det\left(z_k^{j-1}\right)} = \frac{\det\left(f_j(z_k)\right)_{j,k=1}^N}{\prod_{1\leq j<k\leq N}(z_k - z_j)} \tag{4.34}$$

(compare with (1.141)) where $f = (f_j)_{j\geq 1} \in \mathrm{Gr}_0^{\frac{\infty}{2}}$ with the formal series $f_j(z) = z^{j-1}(1 + \mathcal{O}(z^{-1}))$ defined by

$$\mathrm{Ai}^{(j-1)}(z^2) \sim (-1)^{j-1} \frac{\mathrm{e}^{-\frac{2}{3}z^3}}{\sqrt{4\pi z}} f_j(z) \tag{4.35}$$

as $z \to \infty$ within $|\arg z| < \frac{\pi}{2}$. We recall from Sec. 1.4.3 that $\tau_N^f(z_1, ..., z_N)$ gives a well defined limit $\tau^f(\mathbf{t})$. Moreover it follows from the Airy differential equation

$$\mathrm{Ai}''(z) = z\mathrm{Ai}(z) \tag{4.36}$$

that writing $f_j = z^{j-1}(1 + \sum_{\ell\geq 1} f_{j,\ell} z^{-\ell})$

$$f_{j+2,\ell} = f_{j,\ell} \tag{4.37}$$

and therefore $\tau^f(\mathbf{t})$ is a KdV tau function. This is called *Kontsevich–Witten tau function*.

We are finally ready to state the main result of Kontsevich, in particular implying Thm. 4.1.5.

Theorem 4.2.2 (Kontsevich, 1991 [Kb])**.** *The series $\tau^f(\mathbf{t})$ coincides with $\tau(\mathbf{T}) = \exp F(\mathbf{T})$, where $F(\mathbf{T})$ is the generating function (4.13) of Witten intersection numbers, and the variables $\mathbf{t} = (t_1, t_3, ...)$ and $\mathbf{T} = (T_0, T_1, ...)$ are related as in (4.14).*

The proof in [Kb] uses a combinatorial description of (a top-dimensional stratum in) the moduli spaces $\overline{\mathcal{M}}_{g,n}$ based on a theorem of Strebel [Sd] about decomposition of Riemann surfaces by horizontal trajectories of suitable quadratic differentials.

4.3 Isomonodromic method

The bare system. Fix three angles $\beta_\pm, \beta_0$ such that

$$-\pi < \beta_- < -\frac{\pi}{3}, \quad -\frac{\pi}{3} < \beta_0 < \frac{\pi}{3}, \quad \frac{\pi}{3} < \beta_+ < \pi \tag{4.38}$$

and define four sectors I, II, III, IV in the complex z-plane, with $-\pi < \arg z < \pi$, as follows

$$\begin{aligned} z \in I &\iff -\pi < \arg z < \beta_-, & z \in II &\iff \beta_- < \arg z < \beta_0, \\ z \in III &\iff \beta_0 < \arg z < \beta_+, & z \in IV &\iff \beta_+ < \arg z < \pi. \end{aligned} \tag{4.39}$$

Let $\Sigma := \mathbb{R}_- \sqcup \left(\bigsqcup_{j\in\{0,\pm\}} \mathrm{e}^{\mathrm{i}\beta_j}\mathbb{R}_+\right)$ be the oriented contour delimiting the sectors $I, ..., IV$, as in figure 6.1. Let $\omega := \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$ and define

$$\Psi(z) := \begin{cases} \begin{pmatrix} \omega^2\mathrm{Ai}(\omega^{-1}z) & \mathrm{i}\omega^{-\frac{1}{2}}\mathrm{Ai}(\omega z) \\ \omega\mathrm{Ai}'(\omega^{-1}z) & \mathrm{i}\omega^{\frac{1}{2}}\mathrm{Ai}'(\omega z) \end{pmatrix} & z\in I \\ \\ \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{i}\omega^{-\frac{1}{2}}\mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & \mathrm{i}\omega^{\frac{1}{2}}\mathrm{Ai}'(\omega z) \end{pmatrix} & z\in II \\ \\ \begin{pmatrix} \mathrm{Ai}(z) & -\mathrm{i}\omega^{\frac{1}{2}}\mathrm{Ai}(\omega^{-1}z) \\ \mathrm{Ai}'(z) & -\mathrm{i}\omega^{-\frac{1}{2}}\mathrm{Ai}'(\omega^{-1}z) \end{pmatrix} & z\in III \\ \\ \begin{pmatrix} \omega\mathrm{Ai}(\omega z) & -\mathrm{i}\omega^{\frac{1}{2}}\mathrm{Ai}(\omega^{-1}z) \\ \omega^2\mathrm{Ai}'(\omega z) & -\mathrm{i}\omega^{-\frac{1}{2}}\mathrm{Ai}'(\omega^{-1}z) \end{pmatrix} & z\in IV. \end{cases} \tag{4.40}$$

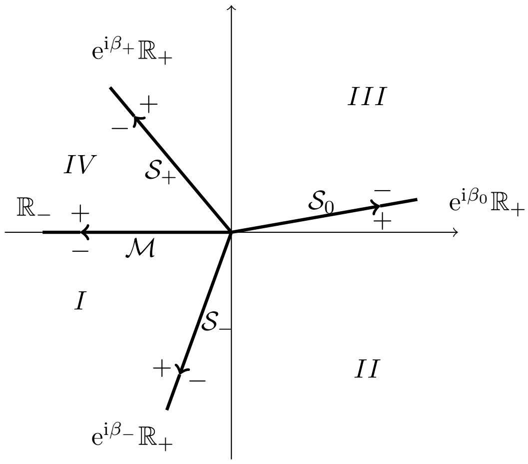


Figure 4.1: Jump $\widetilde{M}$ of Ψ along Σ: $\Psi_+ = \Psi_-\widetilde{M}$.

Consider the matrix form of the Airy ODE

$$\Psi'(z) = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}\Psi(z). \tag{4.41}$$

Proposition 4.3.1. *1. $\Psi(z)$ solves (4.41) in all sectors $I, ..., IV$.*

2. $\Psi(z)$ has the same asymptotic expansion in all sectors $I, ..., IV$

$$\Psi(z) \sim z^S G\left(\mathbf{1} + \mathcal{O}\left(z^{-\frac{1}{2}}\right)\right)\mathrm{e}^{\Xi(z)} \tag{4.42}$$

where S, G, Ξ are defined as

$$S := \mathrm{diag}\left(-\frac{1}{4}, \frac{1}{4}\right), \quad G := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \Xi(z) := \mathrm{diag}\left(-\frac{2}{3}z^{\frac{3}{2}}, \frac{2}{3}z^{\frac{3}{2}}\right). \tag{4.43}$$

3. $\Psi(z)$ satisfies a jump condition along Σ

$$\Psi_+(z) = \Psi_-(z)\widetilde{M}, \qquad z\in\Sigma \tag{4.44}$$

where boundary values are taken with respect to the orientation of Σ shown in figure 6.1 and $\widetilde{M} : \Sigma \to \mathrm{SL}(2,\mathbb{C})$ is piecewise defined as

$$\widetilde{M} := \begin{cases} \mathcal{S}_{0,\pm}, & z\in \mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+ \\ \mathcal{M}, & z\in\mathbb{R}_- \end{cases} \tag{4.45}$$

where

$$\mathcal{S}_- := \begin{pmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{pmatrix}, \quad \mathcal{S}_0 := \begin{pmatrix} 1 & \mathrm{i} \\ 0 & 1 \end{pmatrix}, , \quad \mathcal{S}_+ := \begin{pmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}. \tag{4.46}$$

4. *The identity* $\det \Psi(z) \equiv 1$ *holds identically in all sectors.*

We omit the elementary proof.

In the terminology of linear complex ordinary differential equations (reviewed in Sec. 1.4.2) $\mathcal{S}_{\pm,0}$ are the Stokes matrices (note their triangular structure) and $\mathcal{M}$ the formal monodromy of the singularity $z = \infty$ of (4.41). Notice the *no-monodromy condition* $\mathcal{M}\mathcal{S}_+\mathcal{S}_0\mathcal{S}_- = \mathbf{1}$. Note also that we are in a non-generic case in the sense explained in Sec. 1.4.2, as the eigenvalues of the leading order at ∞ of the connection matrix in (4.41) are all 0; this explains the appearance of non-integer powers in the asymptotic expansions of solutions, compare with Rem. 1.4.2.

Remark 4.3.2. *The RHP associated with the Stokes' phenomenon of the Airy equation appears in random matrix theory, in the context of* universality *as the* local paramatrix at the edge of the spectrum. *The connection of intersection theory on the moduli spaces of curves with the edge of the spectrum matrix model was first proposed by Okounkov [Ob], and lead to an independent proof of Witten conjecture. In loc. cit. the GUE is considered, but the Kontsevich model is actually universal in the sense explained in [BCb].*

Extension of the Kontsevich matrix integral to all sectors. We have seen that $\mathcal{Z}_N^{\mathsf{K}}(Y)$ admits a regular asymptotic expansion for large Y when $\mathrm{Re}\, y_j > 0$. As $\mathrm{Ai}(z)$ are entire functions we could try to analytically continue $\mathcal{Z}_N^{\mathsf{K}}(Y)$ to the region $\mathrm{Re}\, y_j < 0$ via the right side of (4.23). However, this would result in the fact that $\mathcal{Z}_N^{\mathsf{K}}(Y)$ does not admit a regular asymptotic expansion in the region where some $\mathrm{Re}\, y_j < 0$.

It is convenient for our purposes to have a regular expansion near infinity also in the sector $\mathrm{Re}\, y_j < 0$ (and, in fact, the *same* expansion), therefore we need to consider the following extension of $\mathcal{Z}_N^{\mathsf{K}}(Y)$. To this end we start from the representation of Lemma 4.2.1 in terms of the function $\varphi(z, N)$ defined in (6.2); in the left plane we replace them by other solution to the ODE (6.3) in appropriate way so as to preserve the regularity of the asymptotic expansion. The logic is completely parallel to the one used in [BCa] (and reviewed in the previous chapter) and is forced on us by the Stokes' phenomenon of the solutions to the ODE (4.41), which is closely related to the Airy differential equation of the previous chapter.

Definition 4.3.3. *We order the variables* y_j *so that* $\mathrm{Re}\, y_j > 0$ *for* $j = 1, ..., n_1$ *and* $\mathrm{Re}\, y_j < 0$ *for* $j = n_1 + 1, ..., n_1 + n_2 = N$. *We denote* $\vec{\lambda} = (\lambda_1, ..., \lambda_{n_1})$ *and* $\vec{\mu} = (\mu_1, ..., \mu_{n_2})$ *with* $y_j = \sqrt{\lambda_j}$ *for* $j = 1, ..., n_1$ *and* $y_{n_1+j} = -\sqrt{\mu_j}$ *for* $j = 1, ..., n_2$, *all roots being principal. We define the extended Kontsevich partition function by the expression*

$$\mathcal{Z}_N^{\mathsf{K}}(\vec{\lambda}, \vec{\mu}) := (2\sqrt{\pi})^N \mathrm{e}^{U(\vec{\lambda}, \vec{\mu})}\, \Delta(\vec{\lambda}, \vec{\mu}) \det \left(\begin{array}{c} \left(\omega^{\frac{1}{2}} \mathrm{Ai}^{(j-1)}(\omega^{-1}\lambda_k) \right)_{1 \le k \le n_1,\ \lambda_k \in I} \\ \hline \left(\mathrm{Ai}^{(j-1)}(\lambda_k) \right)_{1 \le k \le n_1,\ \lambda_k \in II \cup III} \\ \hline \left(\omega^{-\frac{1}{2}} \mathrm{Ai}^{(j-1)}(\omega\lambda_k) \right)_{1 \le k \le n_1,\ \lambda_k \in IV} \\ \hline \left(\omega^{\frac{1}{4}} \mathrm{Ai}^{(j-1)}(\omega\mu_k) \right)_{n_1+1 \le k \le n,\ \mu_k \in I \cup II} \\ \hline \left(\omega^{-\frac{1}{4}} \mathrm{Ai}^{(j-1)}(\omega^{-1}\mu_k) \right)_{n_1+1 \le k \le n,\ \mu_k \in III \cup IV} \end{array} \right)_{1 \le j \le n} \tag{4.47}$$

where

$$U(\vec{\lambda}; \vec{\mu}) := \frac{2}{3} \sum_{j=1}^{n_1} \lambda_j^{\frac{3}{2}} - \frac{2}{3} \sum_{j=1}^{n_2} \mu_j^{\frac{3}{2}} \tag{4.48}$$

and

$$\Delta(\vec{\lambda}, \vec{\mu}) := \frac{\prod_{j=1}^{n_1} \lambda_j^{\frac{1}{4}} \prod_{j=1}^{n_2} (-\mu_j)^{\frac{1}{4}}}{\prod_{1 \le j < k \le n_1} \left(\sqrt{\lambda_k} - \sqrt{\lambda_j} \right) \prod_{1 \le j < k \le n_2} \left(\sqrt{\mu_j} - \sqrt{\mu_k} \right) \prod_{j=1}^{n_1} \prod_{k=1}^{n_2} \left(\sqrt{\lambda_j} + \sqrt{\mu_k} \right)}. \tag{4.49}$$

We deduce that $\mathcal{Z}_N^{\mathsf{K}}(\vec{\lambda},\vec{\mu})$ as defined in (4.47) has a regular asymptotic expansion when $\lambda_j,\mu_j\to\infty$ in the indicated sectors. This regular asymptotic expansion coincides with the already discussed regular asymptotic expansion of $\mathcal{Z}_N^{\mathsf{K}}(Y)$ for $\operatorname{Re} y_k=\operatorname{Re}\sqrt{\lambda_k}\geq 0$. As analytic functions, $\mathcal{Z}_N^{\mathsf{K}}(\vec{\lambda},\vec{\mu})=\mathcal{Z}_N^{\mathsf{K}}(Y)$ provided that $n_2=0$, $\lambda_k\in II\cup III$ and $y_k=\sqrt{\lambda_k}$ for all $k=1,..,N$.

We point out that the definition (4.47) depends not only on the belonging of y_j to the left/right half-planes but also on the placement of the boundaries between the sectors I–IV, i.e. on the angles $\beta_0,\beta_\pm$ in (4.38). If we move the boundaries within the bounds of (4.38) then this yields different functions $\mathcal{Z}_N^{\mathsf{K}}(\vec{\lambda},\vec{\mu})$ but all admitting the same asymptotic expansion as $\vec{\lambda},\vec{\mu}$ tend to infinity within the respective sectors. We opted to leave this dependence on the sectors understood.

Rational dressing. We fix points (compare with the paragraph above) $\vec{\lambda}=(\lambda_1,...,\lambda_{n_1})$ and $\vec{\mu}=(\mu_1,...,\mu_{n_2})$ and the matrix

$$D(z;\vec{\lambda},\vec{\mu}):=\operatorname{diag}(\pi_+,\pi_-)\,,\qquad \pi_\pm:=\prod_{j=1}^{n_1}\left(\sqrt{\lambda_j}\pm\sqrt{z}\right)\prod_{j=1}^{n_2}\left(\sqrt{\mu_j}\mp\sqrt{z}\right) \tag{4.50}$$

and $J:\Sigma\to\mathrm{SL}(2,\mathbb{C})$

$$J:=\left(D^{-1}\,\mathrm{e}^{\Xi}\right)_-\,\widetilde{M}\,\left(\mathrm{e}^{-\Xi}\,D\right)_+ \tag{4.51}$$

$\widetilde{M}$ and the notation $\pm$ for boundary values being as in (4.45).

The boundary value specifications $\pm$ in (4.51) give different values along the cut $\mathbb{R}_-$ only. In particular it is easy to check that $J|_{\mathbb{R}_-}$ does not depend on $\vec{\lambda},\vec{\mu}$. The angles $\beta_{0,\pm}$ can be chosen so that none of zeros of D occur along the three rays $\mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+$.

The construction is such that along the three rays $\mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+$ the jump matrix J is exponentially close to the identity matrix; $J(z)=\mathbf{1}+\mathcal{O}\left(z^{-\infty}\right)$ as $z\to\infty$.

We now formulate the dressed RHP.

RHP 4.3.4. *Find a* $\mathrm{Mat}(2,\mathbb{C})$*-valued function* $\Gamma=\Gamma(z;\vec{\lambda},\vec{\mu})$ *analytic in* $z\in\mathbb{C}\setminus\Sigma$*, admitting boundary values* $\Gamma_\pm$ *at* Σ *(as in figure 4.1) such that*

$$\begin{cases}\Gamma_+(z)=\Gamma_-(z)J(z) & z\in\Sigma\\ \Gamma(z)\sim z^S GY(z) & z\to\infty\end{cases} \tag{4.52}$$

where S,G *are as in (4.43),* J *as in (4.51) and* $Y(z)$ *a formal power series in* $z^{-\frac{1}{2}}$ *satisfying the normalization (we explain it below)*

$$Y(z)=\mathbf{1}+\begin{pmatrix}a & a\\ -a & -a\end{pmatrix}z^{-\frac{1}{2}}+\mathcal{O}(z^{-1}). \tag{4.53}$$

We will see that the existence of the solution to the RHP 4.3.4 depends on the non-vanishing of a function of $\vec{\lambda},\vec{\mu}$ which is (restriction of an) entire function. Hence the *Malgrange divisor* (see Chap. 2), i.e. the locus in the parameter space where the problem is unsolvable, is really a divisor and the problem is generically solvable.

Remark 4.3.5. *We observe that we can analytically continue* $\Gamma|_{IV}$ *beyond* $\arg z=\pi$ *so that the asymptotic expansion* $\Gamma\sim z^S GY$ *remains valid in a sector up to* $\arg z=\pi+\epsilon$*. Similarly said for* $\Gamma|_I$*, in a sector from* $\arg z=-\pi-\epsilon$*. By matching the expansions in the overlap sector, we obtain*

$$\mathrm{e}^{2\pi\mathrm{i}S}z^S GY(z\mathrm{e}^{2\pi\mathrm{i}})=z^S GY(z)\mathcal{M}. \tag{4.54}$$

By trivial algebra (4.54) implies the following symmetry relation for the formal power series $Y_n(z)$

$$Y(z\mathrm{e}^{2\pi\mathrm{i}})=\sigma_1 Y(z)\sigma_1. \tag{4.55}$$

In terms of the coefficients of the expansion of Y*, we find that the coefficients of the fractional powers must be odd under the conjugation (4.55), while those of the integer powers must be even. In particular this implies the following form for* Y

$$Y(z)=\mathbf{1}+\begin{pmatrix}a & b\\ -b & -a\end{pmatrix}\frac{1}{\sqrt{z}}+\mathcal{O}(z^{-1}). \tag{4.56}$$

Remark 4.3.6. *The normalization condition* (4.53) *is necessary to ensure the uniqueness of the solution to the RHP 4.3.4. To explain this, consider the identity*

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} z^S G = z^S G \left(\mathbf{1} + \frac{1}{2} \begin{pmatrix} -\alpha & -\alpha \\ \alpha & \alpha \end{pmatrix} z^{-\frac{1}{2}} \right). \tag{4.57}$$

This identity shows that the simple requirement $Y(z) = \mathbf{1} + \mathcal{O}\left(z^{-\frac{1}{2}}\right)$ *leaves the freedom of multiplying on the left by the one-parameter family of matrices indicated in* (4.57). *The normalization* $a = b$ *of* (4.53) *fixes uniquely the gauge arbitrariness implied by* (4.57). *It is chosen because of certain later convenience in the computations (as it will be explained in Chap. 5), but this or any other particular choice is otherwise irrelevant (more precisely, it does not affect the expression* $\Gamma^{-1}\Gamma'$ *and hence it does not affect the tau function).*

The extended Kontsevich partition function as the isomonodromic tau function. We can interpret the RHP 4.3.4 as an isomonodromic deformation problem. Indeed by construction it amounts to consider the rational connection on the Riemann sphere with an irregular singularity at ∞ with the same Stokes' phenomenon as the bare system, and N Fuchsian singularities with trivial monodromies. This connection is unique if any. The dependence on the parameters $\vec{\lambda}, \vec{\mu}$ is contrained by the isomonodromic equations reviewed in Sec. 1.4.2.

We explain this point a bit more in detail. The matrix $\Psi := \Gamma D^{-1} \mathrm{e}^{\Xi}$ satisfies a jump condition on Σ which is independent of z and of the parameters $\vec{\lambda}, \vec{\mu}$. Hence the ratios $\Psi' \Psi^{-1} =: L$ and $\delta\Psi\Psi^{-1} = \mathcal{M}$ have no discontinuities along Σ and are rational functions by Liouville theorem; then the system $\Psi' = L\Psi$ and $\delta\Psi = \mathcal{M}\Psi$ is an isomonodromic system in the sense explained in Sec. 1.4.2; it has a fixed Stokes' phenomenon at ∞ and N Fuchsian singularities of trivial monodromy at the points $\vec{\lambda}, \vec{\mu}$.

Following the considerations of Chap. 2 we define the tau function of this isomonodromic system as

$$\delta \log \tau = \Omega, \qquad \delta := \sum_{i=1}^{n_1} \mathrm{d}\lambda_i \frac{\partial}{\partial \lambda_i} + \sum_{i=1}^{n_2} \mathrm{d}\mu_i \frac{\partial}{\partial \mu_i} \tag{4.58}$$

in terms of the Malgrange differential

$$\Omega := \int_\Sigma \operatorname{tr} \left(\Gamma^{-1} \Gamma' \delta J J^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{4.59}$$

for the RHP 4.3.4.

Due to the construction of this RHP by a dressing of the jump matrices (rational in $\sqrt{z}$) the considerations of Thm. 2.4.8 can be applied. In particular we have the following result.

Theorem 4.3.7 ([BCa])**.** *The isomonodromic tau function* (4.58) *coincides with the extended Kontsevich partition function, i.e.*

$$\delta \mathcal{Z}_N^{\mathsf{K}}(\vec{\lambda}; \vec{\mu}) = \Omega. \tag{4.60}$$

There are however some not entirely obvious modifications to Thm. 2.4.8 to be considered in order to derive this result; they are due to the different normalization at ∞ and to the formulation of the RHP in terms of the square root variable $\sqrt{z}$. It is remarkable that the additional pieces in the Malgrange differential for this RHP (that appear because of this modifications) are precisely designed to reproduce the (extended) Kontsevich matrix integral.

For the details of the proof we refer to [BCa] and to Chap. 6 where we shall consider a more general model and the corresponding version of this result.

4.4 Applications

Limiting RHP. The ratio of the products $\pi_\pm$ in (4.50) can be rewritten as

$$\frac{\pi_+}{\pi_-} = \exp \sum_{k \geq 0} t_{2k+1} \sqrt{z^{2k+1}} \tag{4.61}$$

where we have introduced *Miwa variables* $\mathbf{t} = (t_1, t_2, ...)$

$$t_k(\vec{\lambda}, \vec{\mu}) := \frac{1}{k}\sum_{j=1}^{n_1}\left(\frac{1}{\sqrt{\lambda_j}}\right)^k + \frac{1}{k}\sum_{j=1}^{n_2}\left(\frac{1}{-\sqrt{\mu_j}}\right)^k = \frac{1}{k}\sum_{j=1}^{n}\frac{1}{y_j^k} = \frac{1}{k}\mathrm{tr}\, Y^{-k}. \tag{4.62}$$

More precisely, the expression above is actually convergent for $|z| < \min\{|\lambda_j|, |\mu_j|\}$.

Consequently, the matrix D_n can be rewritten formally as

$$D_n^{-1} \propto \exp\left(\frac{\sigma_3}{2}\sum_{k\geq 0} t_{2k+1}\sqrt{z^{2k+1}}\right) \tag{4.63}$$

up to a scalar factor constant in z; since the jump matrices of RHP 4.3.4 are obtained by conjugation via D this constant factor is irrelevant. This suggests to consider in the limit $N \to \infty$ (formally regarding variables $t_1, t_2, ...$ as independent) a new RHP as follows.

RHP 4.4.1. *Let* $\mathbf{t}$ *denote the infinite set of variables* $\mathbf{t} = (t_1, t_2, ...)$. *The formal RHP amounts to finding a* 2×2 *analytic matrix-valued function* $\Gamma = \Gamma(z; \mathbf{t})$ *in* $z \in \mathbb{C}\setminus\Sigma$ *admitting boundary values* $\Gamma_\pm$ *at* Σ *such that*

$$\begin{cases}\Gamma_+(z;\mathbf{t}) = \Gamma_-(z;\mathbf{t})J(z;\mathbf{t}) & z\in\Sigma\\ \Gamma(z;\mathbf{t}) \sim z^S G Y(z;\mathbf{t}) & z\to\infty\end{cases} \tag{4.64}$$

where $J(z;\mathbf{t}) := \mathrm{e}^{\Xi(z;\mathbf{t})_-}\widetilde{M}\mathrm{e}^{-\Xi(z;\mathbf{t})_+}$, $\widetilde{M}$ *as in* (4.45), *and* $Y(z;\mathbf{t})$ *is a formal power series in* $z^{-\frac{1}{2}}$ *satisfying the normalization*

$$Y(z;\mathbf{t}) = \mathbf{1} + \begin{pmatrix} c & -c\\ c & -c\end{pmatrix} z^{-\frac{1}{2}} + \mathcal{O}\left(z^{-1}\right) \tag{4.65}$$

for some function $c = c(\mathbf{t})$.

Remark 4.4.2. *Remark 4.3.6 applies here as well for the uniqueness of the solution to the RHP 4.4.1. Moreover, the symmetry relation* (4.55) *holds true similarly here, namely*

$$Y(z\mathrm{e}^{2\pi\mathrm{i}};\mathbf{t}) = \sigma_1 Y(z;\mathbf{t})\sigma_1. \tag{4.66}$$

We now explain a meaningful setup where the RHP 4.4.1 can be given a completely rigorous analytic meaning. The driving idea is that of truncating the time variables to some finite (odd) number.

Fix now $K \in \mathbb{N}$ and assume that $t_\ell = 0$ for all $\ell \geq 2K+2$. Set $\mathbf{t} = (t_1, \dots, t_{2K+1}, 0, \dots)$ with $t_{2K+1} \neq 0$. In addition, the angles $\beta_{0,\pm}$ (satisfying (4.38)) and the argument of t_{2K+1} must satisfy the following condition:

$$\begin{cases}\mathrm{Re}\left(\sqrt{z^{2K+1}}t_{2K+1}\right) < 0, & z\in \mathrm{e}^{\mathrm{i}\beta_\pm}\mathbb{R}_+\\ \mathrm{Re}\left(\sqrt{z^{2K+1}}t_{2K+1}\right) > 0, & z\in \mathrm{e}^{\mathrm{i}\beta_0}\mathbb{R}_+.\end{cases} \tag{4.67}$$

Under this assumption, given the particular triangular structure of the Stokes matrices $\mathcal{S}_{0,\pm}$, the jumps $M = \mathrm{e}^{\Xi_-}\widetilde{M}\mathrm{e}^{-\Xi_+}$ are exponentially close to the identity matrix along the rays $\mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+$.

Formulæ for Witten intersection numbers. The Malgrange differential of the limiting RHP described above can be expressed as a(n isomonodromic) tau differential. This follows directly from the considerations of Sec. 2.5. Proceeding exactly as in the proof of Thm. 1.2.2 we can obtain the following formulæ for Witten intersection numbers, first derived in [BDYa].

To formulate them introduce the matrix

$$R(x) = \sum_{j\geq 0}\frac{(6j+1)!!}{24^j j! x^{3j}}\begin{pmatrix} -\frac{1}{2x} & -\frac{1}{6j+1}\\ \frac{x}{6j-1} & \frac{1}{2x}\end{pmatrix}. \tag{4.68}$$

Theorem 4.4.3 ([BDYa]). *The following formula for a generating function of n-point open intersection numbers holds true for $n \geq 2$;*

$$\sum_{r_1,\cdots,r_n\geq 0} \frac{(2r_1+1)!!\cdots(2r_n+1)!!\,\langle\tau_{r_1}\cdots\tau_{r_n}\rangle}{x_1^{r_1+1}\cdots x_n^{r_n+1}} = -\frac{1}{n}\sum_{\pi\in\mathfrak{S}_n}\frac{\operatorname{tr}\left(R(x_{\pi(1)})\cdots R(x_{\pi(n)})\right)}{(x_{\pi(1)}-x_{\pi(2)})\cdots(x_{\pi(n)}-x_{\pi(1)})} - \frac{\delta_{n,2}(x_1+x_2)}{(x_1-x_2)^2}. \tag{4.69}$$

They follow directly by the arguments used in the proof of Thm. 1.2.2. We omit the details of the computation; we refer to [BDYa] and to Chap. 6 where we consider a more general model.

Explicit formulæ for Witten intersection numbers have been previously considered in [Oc; BBE; Zb]. Of course, the simple expression for one-point Witten intersection numbers [IZb]

$$\langle\tau_r\rangle = \begin{cases}\frac{1}{24^g g!} & r=3g-2\\ 0 & \text{otherwise}\end{cases} \tag{4.70}$$

can be recovered by this method as well.

Virasoro constraints. The Virasoro constraints for this model can also be deduced purely by this isomonodromic approach. The proof is completely similar to the one for the Brézin–Gross–Witten model, which we report in the next chapter following [BRb].

KdV and Painlevé I hierarchies. It was originally observed in [DS] that Kontsevich–Witten KdV tau function provides a solution to the Painlevé I hierarchy. We now review this point, as it was also one of the main motivations of [BCa] to find the analytic properties of this solution (for details about this point we refer to loc. cit.).

More precisely, let us call $x := t_1$ and introduce

$$u(x,\widetilde{\mathbf{t}}) := \frac{\partial^2}{\partial x^2}\log\tau(x,\widetilde{\mathbf{t}}), \quad \widetilde{\mathbf{t}} := (t_3,t_5,\ldots) \tag{4.71}$$

which is a solution to the KdV hierarchy

$$\frac{\partial u}{\partial t_{2\ell+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{\ell+1}[u], \quad \ell\geq 1. \tag{4.72}$$

In (4.72) we denote $\mathcal{L}_\ell[u]$ the *Lenard–Magri* differential polynomials, defined as

$$\mathcal{L}_0[u]=1, \quad \begin{cases}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{\ell+1} = \left(\frac14\frac{\mathrm{d}^3}{\mathrm{d}x^3}+2u\frac{\mathrm{d}}{\mathrm{d}x}+u_x\right)\mathcal{L}_\ell[u]\\ \mathcal{L}_{\ell+1}[u=0]=0\end{cases} \quad \text{for } \ell\geq 0. \tag{4.73}$$

Incidentally, let us compute the initial datum of the KdV hierarchy (4.72) corresponding to the Kontsevich–Witten solution.

Lemma 4.4.4. *The solution u in (4.71) of the KdV hierarchy (4.72) satisfies the initial condition*

$$u(x,\widetilde{\mathbf{t}}=0) = -\frac{x}{2}. \tag{4.74}$$

Proof. We only sketch the proof, for more details see [BCa]; the idea is that $\Gamma(z;(x,0,\ldots)) = \Psi(z+x)\mathrm{e}^{-\frac23\sqrt{z^3}-x\sqrt{z}}$ is the solution to RHP 4.4.1 for the times $\mathbf{t}=(x,0,\ldots)$, where Ψ is the matrix solution to the Airy equation introduced in (4.40). From this and the explicit expansion of $\Psi(z)$ it is easy to compute

$$\partial_x\log\tau(x,0,\ldots) = -\frac12\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\Gamma^{-1}\Gamma'\sigma_3\right)z\mathrm{d}z = -\frac{x^2}{4} \tag{4.75}$$

and the proof is complete. ■

Remark 4.4.5. *The above computation also shows that $\tau(x, 0, ...) = \mathrm{e}^{-\frac{x^3}{12}}$; this is always different from zero, as indeed RHP 4.4.1 is always solvable for any $\mathbf{t} = (x, 0, ...)$.*

Let us now write the string equation $L_{-1}\tau = 0$, see (4.17),

$$\frac{x^2}{2} + \sum_{i\geq 1}(2i+1)\left(t_{2i+1} + \frac{2}{3}\delta_{i,1}\right)\frac{\partial F}{\partial t_{2i-1}} = 0 \tag{4.76}$$

where $F = \log\tau$ is the free energy (4.13) as before, and differentiate it once in x to get

$$x + \sum_{i\geq 1}(2i+1)(t_{2i+1} + \frac{2}{3}\delta_{i,1})\frac{\partial^2 F}{\partial x \partial t_{2i-1}} = 0 \tag{4.77}$$

and substituting the integrated form $\frac{\partial F}{\partial t_{2i-1}} = \mathcal{L}_i$ of (4.72) we obtain

$$x + \sum_{i\geq 1}(2i+1)(t_{2i+1} + \frac{2}{3}\delta_{i,1})\frac{\partial \mathcal{L}_i}{\partial x} = 0 = 0. \tag{4.78}$$

Hence we have proven the following

Proposition 4.4.6. *If we set $t_\ell = 0$ for $\ell \geq 2K+3$ as above, then $u(x; t_3, ..., t_{2K+1}, 0, ...; \nu)$ solves the Kth member of the PI hierarchy;*

$$x + \sum_{i=1}^{K}(2i+1)(t_{2i+1} + \frac{2}{3}\delta_{i,1})\frac{\partial \mathcal{L}_i}{\partial x} = 0 \tag{4.79}$$

which is an ODE in x, where $t_3, ..., t_{2K+1}$ are regarded as parameters.

The case $K = 2$ gives (up to simple scalings) the Painlevé I equation.

CHAPTER 5

Brézin–Gross–Witten tau function

The Brezin–Gross–Witten (BGW) model was introduced by physicists in the 80s, in the context of QCD. Under a simple scaling, its partition function gives in the weak coupling regime a KdV tau function, termed BGW tau function; it is in many ways a close cousin of the Kontsevich–Witten tau function of last chapter. Recently, Norbury has discovered a remarkable analogue of the Witten conjecture, expressing certain intersection numbers on the moduli spaces of curves as coefficients of the BGW tau functions.

In this chapter we apply the methods developed so far to the BGW tau function, and in particular we deduce explicit formulæ for Norbury intersection numbers.

Main references for this chapter are [Ab; N; BRb].

5.1 The Brezin–Gross–Witten partition function

The *Brezin–Gross–Witten* (BGW) partition function is defined by the following unitary matrix integral [BG; GW]

$$\mathcal{Z}_N^{\mathsf{BGW}}(J;\nu) := \int_{\mathrm{U}_N} \det{}^{\nu}(UJ^\dagger)\exp\mathrm{tr}\,\left(UJ^\dagger + U^\dagger J\right)\mathrm{d}_*U. \tag{5.1}$$

Here d_*U is the normalized bi-invariant measure on the unitary group U_N, $\int_{\mathrm{U}_N}\mathrm{d}_*U = 1$. The partition function depends on an external $N\times N$ matrix J; however it depends only on the eigenvalues of $\sqrt{J^\dagger J}$, as clarified in the next proposition.

The integer parameter $\nu\in\mathbb{Z}$ in (5.1) was absent in the original formulation of the model and is added here to match with the generalization introduced in [MMS; Ab]. Interestingly, this type of generalization had appeared also in the Physics literature on QCD, see e.g. [LS; JŞV; AW].

The unitary integral (5.1) can be regarded as a matrix Bessel function; indeed for $N=1$ we have

$$\mathcal{Z}_{N=1}^{\mathsf{BGW}}(J;\nu) = \oint_{|U|=1}(UJ^*)^{\nu}\mathrm{e}^{UJ^*+\frac{J}{U}}\frac{\mathrm{d}U}{2\pi\mathrm{i}U} = |J|^{\nu}\mathrm{I}_{\nu}(2|J|) \tag{5.2}$$

where we have used an integral representation for is the modified Bessel function of the first kind $\mathrm{I}_\nu(x)$ [AS]. More generally we have the following.

Proposition 5.1.1. *Denote $\lambda_1,\dots,\lambda_N$ be the eigenvalues of $\sqrt{J^\dagger J}$. Then the partition function (5.1) is a function of $\lambda_1,\dots,\lambda_N$ only, and it can be explicitly expressed for $\nu\geq 0$ as*

$$\mathcal{Z}_N^{\mathsf{BGW}}(J;\nu) = \left(\prod_{k=1}^{N-1}k!\right)\frac{\det\left(\lambda_k^{N-j+\nu}\mathrm{I}_{N-j+\nu}(2\lambda_k)\right)_{j,k=1}^N}{\Delta(\lambda_1^2,\dots,\lambda_N^2)} = \left(\prod_{k=1}^{N-1}k!\right)\frac{\det\left(\lambda_k^{j-1+\nu}\mathrm{I}_{j-1+\nu}(2\lambda_k)\right)_{j,k=1}^N}{\prod_{1\leq j<k\leq N}(\lambda_k^2-\lambda_j^2)}. \tag{5.3}$$

The proof is based on the character expansion method [Ba], reviewed in App. B; in the same appendix we also sketch a slightly different proof of this proposition, by seeing it as confluent version of a generalization of the Harish-Chandra–Itzykson–Zuber matrix integral.

Proof. We use the formula (B.17) with $\phi(t)=t^\nu\mathrm{e}^t$ to obtain (here we use that $\nu\geq 0$)

$$\det{}^{\nu}T\exp\mathrm{tr}\,T = \sum_{\alpha\in\mathbb{Y},\,\ell(\alpha)\leq N}\det\left(\frac{1}{(\alpha_k+j-k-\nu)!}\right)_{j,k=1}^N\chi_\alpha(T) \tag{5.4}$$

and the same formula with $\phi(t) = \mathrm{e}^t$ to obtain

$$\exp \operatorname{tr} T = \sum_{\alpha \in \mathbb{Y},\, \ell(\alpha) \leq N} \det \left(\frac{1}{(\alpha_k + j - k)!} \right)_{j,k=1}^{N} \chi_\alpha(T). \tag{5.5}$$

In both expressions the sum on the right runs over partitions $\alpha = (\alpha_1, ..., \alpha_\ell)$ with length $\ell \leq N$, and $\chi_\alpha(T)$ are the characters, compare with (B.12),

$$\chi_\alpha(T) := \frac{\det \left(t_j^{\alpha_k + N - k} \right)_{j,k=1}^{N}}{\Delta(t_1, ..., t_N)} \tag{5.6}$$

where $t_1, ..., t_N$ are the eigenvalues of T. Hence, using the orthogonality property (B.16)

$$\int_{\mathrm{U}_N} \chi_\alpha(UJ^\dagger)\chi_{\alpha'}(U^\dagger J) \mathrm{d}_* U = \frac{\delta_{\alpha\alpha'}}{d_{\alpha,N}} \chi_\alpha(J^\dagger J) \tag{5.7}$$

where, see (B.13),

$$d_{\alpha,N} := \chi_\alpha(\mathbf{1}_N) = \left(\prod_{k=1}^{N} \frac{(\alpha_k + N - k)!}{(k-1)!} \right) \det \left(\frac{1}{(\alpha_k + j - k)!} \right)_{j,k=1}^{N}, \tag{5.8}$$

we integrate over the unitary group to obtain the following expression

$$\begin{aligned} \mathcal{Z}_N^{\mathsf{BGW}}(J;\nu) &= \sum_{\alpha \in \mathbb{Y},\, \ell(\alpha) \leq N} \left(\prod_{k=1}^{N} \frac{(k-1)!}{(\alpha_k + N - k)!} \right) \det \left(\frac{1}{(\alpha_k + j - k - \nu)!} \right)_{j,k=1}^{N} \chi_\alpha(J^\dagger J) \\ &= \left(\prod_{k=0}^{N-1} k! \right) \sum_{\alpha \in \mathbb{Y},\, \ell(\alpha) \leq N} \det \left(\frac{1}{(\alpha_k + N - k)!(\alpha_k + j - k - \nu)!} \right)_{j,k=1}^{N} \chi_\alpha(J^\dagger J). \end{aligned}$$

Finally an application of Lemma 1.4.9 with

$$g_j(\lambda) = \sum_{\ell \geq 0} \frac{\lambda^{2\ell}}{\ell!(\ell - (N - j + \nu))!} = \lambda^{N-j+\nu} \mathrm{I}_{-N+j-\nu}(2\lambda) \tag{5.9}$$

concludes the proof (note that $\mathrm{I}_\nu(x) = \mathrm{I}_{-\nu}(x)$ for $\nu \in \mathbb{Z}$). ■

The representation (5.2) holds true for $\nu \in \mathbb{Z}$ only; it is well known that for arbitrary complex ν one has to consider suitable contour integrals to analytically continue Bessel functions. Incidentally, let us note that there exists a similar analytic continuation for the matrix case $N \geq 1$ as well, in the form of a *generalized Kontsevich integral* [KMMMZ] of the form

$$\frac{\int_{\mathrm{H}_N(\gamma)} \exp \operatorname{tr} \left(\Lambda^2 M + M^{-1} - (\nu + N) \log M \right) \mathrm{d}M}{\int_{\mathrm{H}_N(\gamma)} \exp \operatorname{tr} \left(M^{-1} - (\nu + N) \log M \right) \mathrm{d}M} \tag{5.10}$$

where γ is a contour from $-\infty$ encircling zero counterclockwise once and going back to $-\infty$ and $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_N)$ where $\lambda_1, ..., \lambda_N$ are the eigenvalues of $\sqrt{J^\dagger J}$; for more details see [MMS; Ab].

BGW tau function and Norbury theorem. Recall [AS] the asymptotic expansion of Bessel functions

$$2\sqrt{\pi z} \mathrm{e}^{-2z} \mathrm{I}_\alpha(2z) \sim 1 + \mathcal{O}(z^{-1}) \tag{5.11}$$

as $z \to \infty$ in a suitable sector (more details below). Introducing, with the notations of Sec. 1.4.3, $f = (f_j)_{j \geq 1}$, $f_j(z) = z^{j-1}(1 + \mathcal{O}(z^{-1}))$ according to

$$2\sqrt{\pi z} z^{j-1} \mathrm{e}^{-2z} \mathrm{I}_{j-1-\nu}(2z) \sim f_j(z), \qquad j = 1, 2, ... \tag{5.12}$$

we see that the following modification of the BGW partition function ($\vec{\lambda} = (\lambda_1, ..., \lambda_N)$ with $\lambda_1, ..., \lambda_N$ the eigenvalues of $\sqrt{J^\dagger J}$), where we also replace $\nu \mapsto -\nu$ to match with the convention of [Ab],

$$\mathcal{Z}_N^{\mathsf{BGW}}(\vec{\lambda};\nu) := \frac{\det \left(2\sqrt{\pi \lambda_k} \mathrm{e}^{-2\lambda_k} \lambda_k^{j-1} I_{j-1-\nu}(2\lambda_k) \right)_{j,k=1}^{N}}{\Delta(\lambda_1, ..., \lambda_N)} \tag{5.13}$$

has the asymptotic expansion

$$\mathcal{Z}_N^{\mathsf{BGW}}(\vec{\lambda};\nu) \sim \tau_N^f(\lambda_1,...,\lambda_N) \tag{5.14}$$

where as in the previous cases, see (1.141), we have defined

$$\tau_N^f(\lambda_1,...,\lambda_N) := \frac{\det\left(f_{j-1}(\lambda_k)\right)_{j,k=1}^N}{\det\left(\lambda_k^{j-1}\right)_{j,k=1}^N} = \tag{5.15}$$

Recall that $\tau_N^f(\lambda_1,...,\lambda_N)$ has a formal limit for $N\to\infty$ in the Miwa times

$$\mathbf{t} = (t_1,t_2,....), \qquad t_k := \frac{1}{k}\mathrm{tr}\,\Lambda^{-k} \tag{5.16}$$

which we call $\tau(\mathbf{t})$.

As a consequence of the Bessel differential equation, $\tau(\mathbf{t})$ is a tau function of the KdV hierarchy. As such it does not depend on the even variables $t_2, t_4, \cdots$. It is called *BGW tau function*.

Norbury [N], starting from previous study about topological recursion on the Bessel curve [DN], has found the following beautiful result, paralleling the Witten–Kontsevich theorem (Thm. 4.1.5) for the KDV tau function just introduced, for $\nu = 0$.

To formulate the result let us review a natural web of maps between moduli spaces of curves.

1. *Forgetful map.* $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ sends a stable curve with $n+1$ marked points to the same curve with only the first n marked points (i.e. π *forgets* the last marked point).

2. *Gluing map* (I). $\rho : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ sends a stable curve with marked points $p_1,...,p_{n+2}$ to the curve with the points p_{n+1}, p_{n+2} identified, and the first n marked points.

3. *Gluing map* (II). For any choice of indexes $0 \le n' \le n$ and any $0 \le g' \le g$ the map $\phi_{g',n'}$: $\overline{\mathcal{M}}_{g',n'+1} \times \overline{\mathcal{M}}_{g-g',n-n'+1} \to \overline{\mathcal{M}}_{g,n}$ sends a pair of curves C, C' with marked points $p_1,...,p_{n'+1}$ and $p'_1,...,p'_{n-n'+1}$ to the curve $(C \sqcup C')/\sim$ where $\sim$ is the smallest equivalence relation identifying $p_{n'+1} \sim p'_{n-n'+1}$, with the n marked points $p_1,...,p_s,p'_1,...,p'_{n-n'}$ (in this order).

There is a small caveat to complete the above definitions; the results of the forgetting and gluing procedures outlined above are not necessarily *stable* curves (in the sense of their automorphism group being finite). In such a case one further shrinks to a point the unstable components. For more details see the literature mentioned in Chap. 4. Note that the image of gluing maps $\rho, \phi_{h,I}$ are contained in the *boundary divisor* in $\overline{\mathcal{M}}_{g,n}$, consisting of stable curves with at least one node.

Theorem 5.1.2 ([N]). *1. There exists a family of cohomology classes $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ ("Norbury classes"), for all pair of indexes $g,n \ge 0$ satisfying the stability property $2g-2+n>0$, such that*

(a) $\Theta_{g,n}$ is of homogeneous, i.e. $\Theta(g,n) \in H^{d(g,n)}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ for some $d(g,n)\in\mathbb{N}$,

(b) the family is closed with respect to pullbacks via gluing maps, i.e.

$$\rho^*\Theta_{g,n} = \Theta_{g-1,n+2}, \qquad \phi^*_{g',n'}\Theta_{g,n} = \left(\pi^*\Theta_{g',n'}\right)\left(\pi^*\Theta_{g-g',n-n'}\right) \tag{5.17}$$

where the gluing maps $\rho, \phi_{g',n'}$ are recalled above, and

(c) the family behaves as follows under pullback via the forgetful map π

$$\Theta_{g,n+1} = \psi_{n+1}\left(\pi^*\Theta_{g,n}\right) \tag{5.18}$$

where the psi-classes $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ have been reviewed in Chap. 4.

Moreover, for any such family we have $d(g,n) = 4g-4+2n$ and the family is unique provided we normalize $\Theta_{1,1} = 3\psi_1 \in H^2(\overline{\mathcal{M}}_{1,1},\mathbb{Q})$.

2. *Define the following generating function*

$$F(\mathbf{T}) := \sum_{n\geq 1}\sum_{r_1,\dots,r_n\geq 0}\langle \Theta, \tau_{r_1}\cdots\tau_{r_n}\rangle \frac{T_{r_1}\cdots T_{r_n}}{n!} \tag{5.19}$$

introducing the notation

$$\langle \Theta, \tau_{r_1}\cdots\tau_{r_n}\rangle := \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}\psi_1^{r_1}\cdots\psi_n^{r_n} \tag{5.20}$$

for "Norbury intersection numbers", normalized as above as $\Theta_{1,1} = 3\psi_1$ and with the understanding that they vanish unless the dimensional constraint $r_1+\cdots+r_n = g-1+n$ is met. Then defining the scaling

$$T_k = \frac{(2k+1)!!}{2^{2k+1}} t_{2k+1} \tag{5.21}$$

between the sets of variables $\mathbf{T} = (T_0, T_1, \dots)$ and $\mathbf{t} = (t_1, t_3, \dots)$, we have $\tau(\mathbf{t}) = \exp F(\mathbf{t})$, where $\tau(\mathbf{t})$ is the BGW tau function introduced above.

It is an open problem to understand whether the deformation switching on ν also has a similar algebro-geometric interpretation.

5.2 Isomonodromic method

The bare system. Fix two angles β_1, β_2 in the range

$$-\pi < \beta_1 < \beta_2 < \pi \tag{5.22}$$

and define Σ to be the contour in the z–plane consisting of the three rays $z<0$, $\arg z = \beta_1$, $\arg z = \beta_2$, see Fig. 5.1. Introduce the following 2×2 matrix $\Psi(z)$, analytic for $z \in \mathbb{C}\setminus\Sigma$:

$$\Psi(z) := \sqrt{\frac{2}{\pi}} \times \begin{cases} \begin{pmatrix} \pi I_{-\nu}(2\sqrt{z}) + \mathrm{i}\mathrm{e}^{\mathrm{i}\nu\pi}K_{-\nu}(2\sqrt{z}) & -K_{-\nu}(2\sqrt{z}) \\ \pi\sqrt{z}I_{1-\nu}(2\sqrt{z}) - \mathrm{i}\mathrm{e}^{\mathrm{i}\nu\pi}\sqrt{z}K_{1-\nu}(2\sqrt{z}) & \sqrt{z}K_{1-\nu}(2\sqrt{z}) \end{pmatrix} & -\pi < \arg z < \beta_1 \\[2ex] \begin{pmatrix} \pi I_{-\nu}(2\sqrt{z}) & -K_{-\nu}(2\sqrt{z}) \\ \pi\sqrt{z}I_{1-\nu}(2\sqrt{z}) & \sqrt{z}K_{1-\nu}(2\sqrt{z}) \end{pmatrix} & \beta_1 < \arg z < \beta_2 \\[2ex] \begin{pmatrix} \pi I_{-\nu}(2\sqrt{z}) - \mathrm{i}\mathrm{e}^{-\mathrm{i}\nu\pi}K_{-\nu}(2\sqrt{z}) & -K_{-\nu}(2\sqrt{z}) \\ \pi\sqrt{z}I_{1-\nu}(2\sqrt{z}) + \mathrm{i}\mathrm{e}^{-\mathrm{i}\nu\pi}\sqrt{z}K_{1-\nu}(2\sqrt{z}) & \sqrt{z}K_{1-\nu}(2\sqrt{z}) \end{pmatrix} & \beta_2 < \arg z < \pi \end{cases} \tag{5.23}$$

where $I_\alpha(x)$, $K_\alpha(x)$ are the modified Bessel functions of order α of the first and second kind respectively [AS] and we stipulate henceforth that all the roots are principal. Note that we are implying the dependence on ν.

The following proposition is elementary and the proof is omitted.

Proposition 5.2.1. *In every sector of $\mathbb{C}\setminus\Sigma$ the following statements hold true.*

1. *The following ODE is satisfied;*

$$\Psi'(z) = \begin{pmatrix} -\frac{\nu}{2z} & \frac{1}{z} \\ 1 & \frac{\nu}{2z} \end{pmatrix}\Psi(z). \tag{5.24}$$

2. *We have the asymptotic expansion below;*[1]

$$\Psi(z) \sim z^{-\frac{\sigma_3}{4}} G\left(\mathbf{1} + \frac{1}{16\sqrt{z}}\begin{pmatrix} -(1-2\nu)^2 & 2-4\nu \\ -2+4\nu & (1-2\nu)^2 \end{pmatrix} + \mathcal{O}\left(z^{-1}\right)\right)\mathrm{e}^{2\sqrt{z}\sigma_3}, \quad z\to\infty \tag{5.25}$$

[1] We use the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

where

$$G := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{5.26}$$

3. *We have* $\det \Psi(z) \equiv 1$.

Moreover, the matrix $\Psi(z)$ satisfies the following jump condition along Σ;

$$\Psi(z_+) = \Psi(z_-)S(z), \quad z \in \Sigma \tag{5.27}$$

where $\pm$ denote boundary values as in Fig. 5.1 and $S(z)$ is the following piecewise constant matrix defined on Σ;

$$S(z) := \begin{cases} \mathrm{i}\sigma_1 & z < 0 \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}e^{\mathrm{i}\nu\pi} & 1 \end{pmatrix} & \arg z = \beta_1 \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}e^{-\mathrm{i}\nu\pi} & 1 \end{pmatrix} & \arg z = \beta_2. \end{cases} \tag{5.28}$$

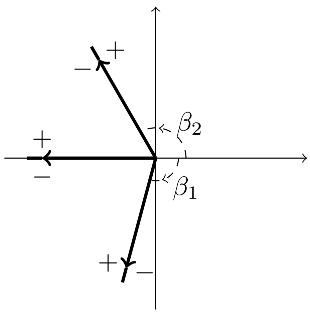


Figure 5.1: Contour Σ, and notation for the boundary values.

In the terminology of linear ODEs with rational coefficients (see Sec. 1.4.2) this piecewise analytic matrix Ψ displays explicitly the Stokes' phenomenon at ∞ of the matrix ODE (5.24). The jump along the rays $e^{\mathrm{i}\beta_{1,2}}$ are the Stokes' matrices (note their triangular property) while the jump along $\mathbb{R}_-$ is the formal monodromy matrix.

Note that the no-monodromy condition is not satisfied, and there is an additional pole at $z = 0$ of the ODE (5.24), which is a Fuchsian singularity.

Extended BGW partition function. For later convenience we introduce an extension of $\widehat{\mathcal{Z}}_N^{\mathsf{BGW}}(\vec{\lambda};\nu)$, having the same regular asymptotic expansion when the λ_j's go to infinity within arbitrary sectors of the λ-plane, not only uniformly within a sector $|\arg \lambda_j| < \frac{\pi}{2}$. The strategy is parallel to that in Chap. 4.

We introduce, for $-\pi < \arg\lambda < \pi$ and $k \geq 1$, the functions

$$\xi_k(\lambda) := \sqrt{\frac{2}{\pi}}\lambda^{k-1} \times \begin{cases} \mathrm{i}K_{k-\nu-1}(2e^{\mathrm{i}\pi}\lambda) & \text{if } -\pi < \arg\lambda < -\frac{\pi}{2} \\ \pi I_{k-\nu-1}(2\lambda) - \mathrm{i}e^{\mathrm{i}(k-\nu)\pi}K_{k-\nu-1}(2\lambda) & \text{if } -\frac{\pi}{2} < \arg\lambda < \frac{\beta_1}{2} \\ \pi I_{k-\nu-1}(2\lambda) & \text{if } \frac{\alpha_1}{2} < \arg\lambda < \frac{\alpha_2}{2} \\ \pi I_{k-\nu-1}(2\lambda) + \mathrm{i}e^{\mathrm{i}(k+\nu)\pi}K_{k-\nu-1}(2\lambda) & \text{if } \frac{\beta_2}{2} < \arg\lambda < \frac{\pi}{2} \\ -\mathrm{i}K_{k-\nu-1}(2e^{-\mathrm{i}\pi}\lambda) & \text{if } \frac{\pi}{2} < \arg\lambda < \pi. \end{cases} \tag{5.29}$$

The motivation behind this convoluted definition is that the above functions have the *same* asymptotic expansion

$$\xi_k(\lambda) \sim \frac{1}{\sqrt{2\lambda}}e^{2\lambda}\lambda^{k-1}(1 + \mathcal{O}(\lambda^{-1})), \quad \lambda \to \infty \tag{5.30}$$

in every sector of $-\pi < \arg\lambda < \pi$ appearing in the definition (5.29).

Remark 5.2.2. *Note that*

$$\xi_1(\lambda) = \begin{cases} \Psi_{11}(\lambda^2) & \text{if } -\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2} \\ \pm \mathrm{i}\Psi_{12}(\lambda^2 \mathrm{e}^{\mp 2\pi \mathrm{i}}) & \text{if } \frac{\pi}{2} < \pm\arg\lambda < \pi \end{cases}, \qquad \xi_2(\lambda) = \begin{cases} \Psi_{21}(\lambda^2) & \text{if } -\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2} \\ \mp \mathrm{i}\Psi_{22}(\lambda^2 \mathrm{e}^{\mp 2\pi \mathrm{i}}) & \text{if } \frac{\pi}{2} < \pm\arg\lambda < \pi. \end{cases} \tag{5.31}$$

For arbitrary $\lambda_1, ..., \lambda_N$ in $\mathbb{C}\setminus\Sigma$, we define

$$\widehat{\mathcal{Z}}_N(\vec{\lambda};\nu) := \frac{\det\left(\sqrt{2\lambda_j}\mathrm{e}^{-2\lambda_j}\xi_k(\lambda_j)\right)_{j,k=1}^N}{\Delta(\lambda_1,...,\lambda_N)} \tag{5.32}$$

and call it *extended BGW partition function.*

By construction the extended BGW partition function has the same regular asymptotic expansion, $\widehat{\mathcal{Z}}_N(\vec{\lambda};\nu) \sim \tau_N^f(\lambda_1,...,\lambda_N)$, when the λ_j's go to ∞ in every sector of the complex plane, see (5.30). Notice that $\widehat{\mathcal{Z}}_N(\vec{\lambda};\nu) = \mathcal{Z}_N(\vec{\lambda};\nu)$ provided that $\frac{\beta_1}{2} < \arg\lambda_j < \frac{\beta_2}{2}$ for all $j = 1,...,N$.

Rational dressing. Following the strategy already discussed in Chap. 4 for the Kontsevich–Witten tau function, we consider the following dressing of the RHP associated with the Stokes' phenomenon of the bare ODE (5.24).
Fix $\lambda_1,...,\lambda_N \in \mathbb{C}\setminus\Sigma$; from now on we imply dependence on $\vec{\lambda}$. Introduce

$$D(z) := \prod_{j=1}^N \begin{pmatrix} \lambda_j + \sqrt{z} & 0 \\ 0 & \lambda_j - \sqrt{z} \end{pmatrix} \tag{5.33}$$

$$J(z) := D^{-1}(z_+)\mathrm{e}^{2\sigma_3\sqrt{z_+}}S(z)\mathrm{e}^{-2\sigma_3\sqrt{z_-}}D(z_-) \tag{5.34}$$

where the notation $\pm$ refers to the boundary values as in Fig. 5.1; the distinction between boundary values is only important along $z < 0$. The matrix J reads more explicitly

$$J(z) = \begin{cases} \mathrm{i}\sigma_1 & z<0 \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\nu\pi}\mathrm{e}^{-4\sqrt{z}}\prod_{j=1}^N \frac{\lambda_j+\sqrt{z}}{\lambda_j-\sqrt{z}} & 1 \end{pmatrix} & \arg z = \beta_1 \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\nu\pi}\mathrm{e}^{-4\sqrt{z}}\prod_{j=1}^N \frac{\lambda_j+\sqrt{z}}{\lambda_j-\sqrt{z}} & 1 \end{pmatrix} & \arg z = \beta_2. \end{cases} \tag{5.35}$$

Notice that $J(z) = \mathbf{1} + \mathcal{O}(z^{-\infty})$ as $z\to\infty$ along the rays $\arg z = \beta_1, \beta_2$.

RHP 5.2.3. *Find a 2×2 matrix $\Gamma(z) = \Gamma(z;\vec{\lambda})$, analytic for $z \in \mathbb{C}\setminus\Sigma$ satisfying the following jump condition for $z\in\Sigma$*

$$\Gamma_+(z) = \Gamma_-(z)J(z), \tag{5.36}$$

the growth condition at zero

$$\Gamma(z) \sim \mathcal{O}(1)\Psi(z), \quad z \to 0, \tag{5.37}$$

where $\Psi(z)$ was introduced above in (5.23), and the normalization condition at infinity

$$\Gamma(z) \sim z^{-\frac{\sigma_3}{4}}GY(z), \quad z\to\infty, \tag{5.38}$$

where

$$G = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad Y(z) = \mathbf{1} + \begin{pmatrix} a & a \\ -a & -a \end{pmatrix}\frac{1}{\sqrt{z}} + \mathcal{O}\left(\frac{1}{z}\right) \in \mathrm{GL}\left(2, \mathbb{C}\left[\!\left[\frac{1}{\sqrt{z}}\right]\!\right]\right), \tag{5.39}$$

for some $a = a(\vec{\lambda})$ independent of z.

Remark 5.2.4. *The jump on the negative semi-axis $z<0$ in RHP 5.2.3 is due to the multi–valuedness of $\sqrt{z}$. The position of this cut is completely arbitrary. By considering the analytic continuation beyond this cut we find that*

$$(z\mathrm{e}^{2\pi\mathrm{i}})^{-\frac{\sigma_3}{4}} G Y(z\mathrm{e}^{2\pi\mathrm{i}}) = z^{-\frac{\sigma_3}{4}} G Y(z)\mathrm{i}\sigma_1 \tag{5.40}$$

which in turn implies the following symmetry property

$$Y(z\mathrm{e}^{2\pi\mathrm{i}}) = \sigma_1 Y(z)\sigma_1. \tag{5.41}$$

Hence the coefficients in front of even, resp. odd, powers of $\sqrt{z}$ have the form $\begin{pmatrix} u & v \\ v & u\end{pmatrix}$, resp. $\begin{pmatrix} u & v \\ -v & -u\end{pmatrix}$.

Remark 5.2.5. *The conditions (5.37) and (5.38) are required to ensure uniqueness of the solution to the RHP (5.2.3). The growth condition (5.37) is necessary as the product of the jump matrices at $z=0$ is not the identity matrix (the no-monodromy condition is not fulfilled in this case). The necessity of the normalization condition (5.38) is explained as follows; indeed one may require the simpler boundary behavior $\Gamma(z)\sim z^{-\frac{\sigma_3}{4}}G\left(\mathbf{1}+\mathcal{O}\left(z^{-1/2}\right)\right)$. However this would not uniquely fix the solution as follows from the identity*

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1\end{pmatrix} z^{-\frac{\sigma_3}{4}} G = z^{-\frac{\sigma_3}{4}} G\left(\mathbf{1}+\frac{1}{2}\begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha\end{pmatrix} z^{-1/2}\right) \tag{5.42}$$

which would leave us with a one–parameter family of solutions, obtained one from the other by left multiplication by a matrix $\begin{pmatrix} 1 & 0 \\ \alpha & 1\end{pmatrix}$, $\alpha\in\mathbb{C}$. This is completely analogous to the case considered in Chap. 4. It follows from the same identity (5.42) that the condition (5.38) removes this ambiguity. This gauge fixing is chosen purely because of certain later convenience (see Lemma 5.3.7) and is otherwise entirely arbitrary. Indeed the tau function to be defined shortly is invariant under any transformation multiplying Γ on the left by an arbitrary constant (in z) matrix.

The matrix $\Psi(z)\mathrm{e}^{-2\sqrt{z}\sigma_3}$ satisfies the jump condition (5.36) and the growth condition (5.37) for $N=0$ but the asymptotic expansion (5.25) does not meet the requirement (5.38). However we have

$$\begin{pmatrix} 1 & 0 \\ \frac{3-8\nu+4\nu^2}{16} & 1\end{pmatrix}\Psi(z)\mathrm{e}^{-2\sqrt{z}\sigma_3}\sim z^{-\frac{\sigma_3}{4}}G\left(\mathbf{1}+\frac{1-4\nu^2}{32\sqrt{z}}\begin{pmatrix} 1 & 1 \\ -1 & -1\end{pmatrix}+\mathcal{O}(z^{-1})\right),\quad z\to\infty \tag{5.43}$$

which does fulfill (5.38), with $a=\frac{1-4\nu^2}{32}$. This provides the solution to RHP 5.2.3 for $N=0$.

The extended BGW partition function as the isomonodromic tau function. We can interpret the RHP 5.2.3 as an isomonodromic deformation problem. Indeed by construction it amounts to consider the rational connection on the Riemann sphere with an irregular singularity at ∞ with the same Stokes' phenomenon as the bare system, and N Fuchsian singularities with trivial monodromies. This connection is unique if any. The dependence on the parameters $\vec{\lambda}$ is contrained by the isomonodromic equations reviewed in Sec. 1.4.2.

We explain this point a bit more in detail. The matrix $\Psi:=\Gamma D^{-1}\mathrm{e}^{2\sqrt{z}\sigma_3}$ satisfies a jump condition along Σ which is independent of z and of the parameters $\vec{\lambda}$. Hence the ratios $\Psi'\Psi^{-1}=:L$ and $\delta\Psi\Psi^{-1}=\mathcal{M}$ have no discontinuities along Σ and are rational functions by Liouville theorem; then the system $\Psi'=L\Psi$ and $\delta\Psi=\mathcal{M}\Psi$ is an isomonodromic system in the sense explained in Sec. 1.4.2; it has a fixed Stokes' phenomenon at ∞ and N Fuchsian singularities of trivial monodromy at the points $\vec{\lambda}$.

Following the considerations of Chap. 2 we define the tau function of this isomonodromic system as

$$\delta\log\tau=\Omega,\qquad \delta:=\sum_{i=1}^{n_1}\mathrm{d}\lambda_i\frac{\partial}{\partial\lambda_i}+\sum_{i=1}^{n_2}\mathrm{d}\mu_i\frac{\partial}{\partial\mu_i} \tag{5.44}$$

in terms of the Malgrange differential

$$\Omega:=\int_\Sigma \mathrm{tr}\,\left(\Gamma^{-1}\Gamma'\delta J J^{-1}\right)\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{5.45}$$

for the RHP 5.2.3.

Due to the construction of this RHP by a dressing of the jump matrices (rational in $\sqrt{z}$) the considerations of Thm. 2.4.8 can be applied. In particular we have obtained the following result.

Theorem 5.2.6 ([BRb]). *The isomonodromic tau function (4.58) coincides with the extended BGW partition function, i.e.*

$$\delta \widehat{\mathcal{Z}}_N^{\mathsf{BGW}}(\vec{\lambda};\nu) = \Omega. \tag{5.46}$$

We postpone the proof to Sec. 5.4.

5.3 Applications

Limiting RHP. Consider the (2,1)-entry of the jump matrix (5.35); the following identity

$$\mathrm{e}^{-4\sqrt{z}} \prod_{j=1}^{N} \frac{\lambda_j + \sqrt{z}}{\lambda_j - \sqrt{z}} = \exp\left(2\sum_{\ell\geq 0}\left[\left(\frac{1}{\lambda_1^{2\ell+1}} + \cdots + \frac{1}{\lambda_N^{2\ell+1}}\right) - 2\delta_{\ell,0}\right]\frac{\sqrt{z}^{2\ell+1}}{2\ell+1}\right) \tag{5.47}$$

holds uniformly over compact sets in $|z| < \min_j |\lambda_j|^2$. Together with the definition of the Miwa times

$$t_{2\ell+1} := \frac{1}{2\ell+1}\left(\frac{1}{\lambda_1^{2\ell+1}} + \cdots + \frac{1}{\lambda_N^{2\ell+1}}\right) \tag{5.48}$$

it suggests to consider the *phase function*

$$\vartheta(z;\mathbf{t}) := \sum_{\ell\geq 0}(t_{2\ell+1} - 2\delta_{\ell,0})\sqrt{z^{2\ell+1}}, \qquad \mathbf{t} = (t_1, t_3, \dots) \tag{5.49}$$

and to formulate the following limiting RHP. Set, for some fixed but arbitrary $K > 0$,

$$\vartheta(z;\mathbf{t}) := \sum_{\ell=0}^{K}(t_{2\ell+1} - 2\delta_{\ell,0})\sqrt{z^{2\ell+1}}, \tag{5.50}$$

$$J(z;\mathbf{t}) := \mathrm{e}^{-\vartheta(z_-;\mathbf{t})\sigma_3} S(z) \mathrm{e}^{\vartheta(z_+;\mathbf{t})\sigma_3} = \begin{cases} \mathrm{i}\sigma_1 & z<0 \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\nu\pi}\mathrm{e}^{2\vartheta(z;\mathbf{t})} & 1\end{pmatrix} & \arg z = \beta_1 \\ \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\nu\pi}\mathrm{e}^{2\vartheta(z;\mathbf{t})} & 1\end{pmatrix} & \arg z = \beta_2. \end{cases} \tag{5.51}$$

To give strictly non-formal sense to this discussion, we agree that $\mathbf{t} := (t_1, t_3, \dots, t_{2K+1}, 0, 0, \dots)$ for some fixed (but arbitrary) K. We also assume that $t_{2K+1} \neq 0$ satisfies

$$\mathrm{Re}\left(\sqrt{z^{2K+1}}\, t_{2K+1}\right) < 0, \text{ for } \arg z = \alpha_{1,2} \tag{5.52}$$

so that $J(z;\mathbf{t}) \sim \mathbf{1} + \mathcal{O}(z^{-\infty})$ along $\arg z = \beta_{1,2}$.

RHP 5.3.1. *Find a 2×2 matrix $\Gamma(z;\mathbf{t})$, analytic for $z \in \mathbb{C}\setminus\Sigma$ satisfying the following jump condition along Σ*

$$\Gamma(z_+;\mathbf{t}) = \Gamma(z_-;\mathbf{t}) J(z;\mathbf{t}), \tag{5.53}$$

the growth condition at zero

$$\Gamma(z;\mathbf{t}) \sim \mathcal{O}(1)\Psi(z), \qquad z \to 0 \tag{5.54}$$

where Ψ is defined in (5.23), and the normalization condition at infinity

$$\Gamma(z;\mathbf{t}) \sim z^{-\frac{\sigma_3}{4}} G Y(z;\mathbf{t}), \qquad z\to\infty, \tag{5.55}$$

where

$$G = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \quad Y(z;\mathbf{t}) = \mathbf{1} + \begin{pmatrix}a(\mathbf{t}) & a(\mathbf{t})\\ -a(\mathbf{t}) & -a(\mathbf{t})\end{pmatrix}\frac{1}{\sqrt{z}} + \mathcal{O}\left(\frac{1}{z}\right) \in \mathrm{GL}\left(2, \mathbb{C}\left[\!\left[\frac{1}{\sqrt{z}}\right]\!\right]\right), \tag{5.56}$$

for some $a = a(\mathbf{t})$ of $\mathbf{t}$ independent of z.

The considerations regarding the uniqueness exposed in Rem. 5.2.5 apply equally well here.

Formulæ for logarithmic correlators. Directly from Thm. 1.2.2 we have obtained the following result. Introduce the matrix $R(z;\nu)$

$$R(z;\nu) := \sum_{k\geq 0}\frac{(2k-1)!!}{k!(8z)^k}\begin{pmatrix}\frac{1}{2}\left(\frac{1}{2}-\nu\right)_{k+1}\left(\frac{1}{2}+\nu\right)_k & \left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k\\ -z\left(\frac{1}{2}-\nu\right)_{k+1}\left(\frac{1}{2}+\nu\right)_{k-1} & -\frac{1}{2}\left(\frac{1}{2}-\nu\right)_{k+1}\left(\frac{1}{2}+\nu\right)_k\end{pmatrix}. \tag{5.57}$$

Theorem 5.3.2 ([BRb])**.** *For all $\ell \geq 0$ we have*

$$\left.\frac{\partial \tau(\mathbf{t};\nu)}{\partial t_\ell}\right|_{\mathbf{t}=0} = \frac{(2\ell-1)!!}{2^{3\ell+2}(\ell+1)!}\left(\frac{1}{2}-\nu\right)_{\ell+1}\left(\frac{1}{2}+\nu\right)_{\ell+1} \tag{5.58}$$

and for all $n \geq 2$ we have

$$\sum_{\ell_1,\dots,\ell_n\geq 0}\frac{1}{z_1^{1+\ell_1}\cdots z_n^{1+\ell_n}}\left.\frac{\partial^n\tau(\mathbf{t};\nu)}{\partial t_{\ell_1}\cdots\partial t_{\ell_n}}\right|_{\mathbf{t}=0}$$
$$= \frac{(-1)^{n-1}}{n}\sum_{\pi\in\mathfrak{S}_n}\frac{\mathrm{tr}\,\left(R(z_{\pi_1};\nu)\cdots R(z_{\pi_n};\nu)\right)}{(z_{\pi_1}-z_{\pi_2})\cdots(z_{\pi_{n-1}}-z_{\pi_n})(z_{\pi_n}-z_{\pi_1})} - \frac{z_1+z_2}{(z_1-z_2)^2}\delta_{n,2}. \tag{5.59}$$

Remark 5.3.3. *The same formulæ have been derived independently in [DYZb].*

Note that $R(z;\nu)$ is a power series in z whose coefficients are polynomials in ν. Moreover, $R(z;\nu)$ satisfies the following identity

$$R(z;-\nu) = \begin{pmatrix}1 & 0\\ -\nu & 1\end{pmatrix}R(z;\nu)\begin{pmatrix}1 & 0\\ \nu & 1\end{pmatrix} \tag{5.60}$$

from which we conclude that the BGW tau function is invariant under $\nu \mapsto -\nu$, namely all the coefficients in the expansion of the BGW tau function are even polynomials in ν.

In particular when ν is a half-integer, $R(z;\nu)$ is actually a Laurent polynomial in z which reflects the fact that the BGW tau function is a polynomial in this case; see [Ab] for a description of these polynomials in terms of Schur polynomials.

The proof is a simple application (for the precise details of this case see [BRb]) of the formulæ derived in Thm. 1.2.2, evaluated at $\mathbf{t} = 0$ by means of the following lemma, simplifying the products of Bessel functions appearing in the relevant expansion at ∞ which we need for R.

Lemma 5.3.4. *We have, at the level of asymptotic expansions,*

$$\sqrt{z}\Psi(z)\sigma_3\Psi^{-1}(z) = R(z;\nu) \tag{5.61}$$

where $R(z;\nu)$ is defined in (5.57).

Proof. We compute $R(z;\nu)$ in the sector $\alpha_1 < \arg z < \alpha_2$, the result holds in every sector due to the fact that $\Psi(z)$ has the same asymptotic expansion in every sector by construction. Hence we compute

$$\sqrt{z}\Psi(z)\sigma_3\Psi^{-1}(z) = \sqrt{z}\begin{pmatrix}R_{11} & R_{12}\\ R_{21} & -R_{11}\end{pmatrix}, \quad \begin{cases}R_{11} := 2\sqrt{z}\left(I_{-\nu}(2\sqrt{z})K_{1-\nu}(2\sqrt{z}) - I_{1-\nu}(2\sqrt{z})K_{-\nu}(2\sqrt{z})\right)\\ R_{12}(z) := 4I_{-\nu}(2\sqrt{z})K_{-\nu}(2\sqrt{z})\\ R_{21} := 4zI_{1-\nu}(2\sqrt{z})K_{1-\nu}(2\sqrt{z}).\end{cases} \tag{5.62}$$

From the ODE (5.24) we deduce

$$\left(\frac{R}{\sqrt{z}}\right)' = \left[A, \frac{R}{\sqrt{z}}\right], \qquad A = \begin{pmatrix} -\frac{\nu}{2z} & \frac{1}{z} \\ 1 & \frac{\nu}{2z} \end{pmatrix} \tag{5.63}$$

from which we obtain the system of ODEs

$$\begin{cases} 2zR_{11}' = -2zR_{12} + 2R_{21} \\ 2zR_{12}' = -4R_{11} - 2\nu R_{12} \\ 2zR_{21}' = 4zR_{11} + 2\nu R_{21}. \end{cases} \tag{5.64}$$

Consider, at the formal level, the following integral transform

$$f(z) = \sum_{k\geq 0} f_k z^{-k-\frac{1}{2}} \mapsto \widehat{f}(t) := \sum_{k\geq 0} \frac{f_k}{(2k)!} t^{2k}, \qquad f(z) = \int \widehat{f}(t) \mathrm{e}^{-t\sqrt{z}} \mathrm{d}t \tag{5.65}$$

for which

$$2\widehat{zf'(z)} = -\frac{\mathrm{d}}{\mathrm{d}t}(t\widehat{f}(t)), \quad \widehat{zf(z)} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{f}(t). \tag{5.66}$$

Hence, by (5.64) and (5.66), the formal series $\widehat{R_{11}}(t), \widehat{R_{12}}(t), \widehat{R_{21}}(t)$ satisfy the system

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}\left(t\widehat{R_{11}}(t)\right) = -2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{R_{12}}(t) + 2\widehat{R_{21}}(t) \\ -\frac{\mathrm{d}}{\mathrm{d}t}\left(t\widehat{R_{12}}(t)\right) = -4\widehat{R_{11}}(t) - 2\nu\widehat{R_{12}}(t) \\ -\frac{\mathrm{d}}{\mathrm{d}t}\left(t\widehat{R_{21}}(t)\right) = 4\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{R_{11}}(t) + 2\nu\widehat{R_{21}}(t). \end{cases} \tag{5.67}$$

Solving for $\widehat{R_{11}}(t)$ and $\widehat{R_{21}}(t)$ from the first two equations in (5.67) we obtain

$$\begin{aligned} \widehat{R_{11}}(t) &= \frac{1-2\nu}{4}\widehat{R_{12}}(t) + \frac{t}{4}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{R_{12}}(t), \\ \widehat{R_{21}}(t) &= \frac{2\nu-1}{8}\widehat{R_{12}}(t) + \frac{2\nu-3}{8}t\frac{\mathrm{d}}{\mathrm{d}t}\widehat{R_{12}}(t) + \left(1 - \frac{t^2}{8}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{R_{12}}(t) \end{aligned} \tag{5.68}$$

and inserting this in the third equation in (5.67) we obtain ODE

$$t\left(16-t^2\right)\frac{\mathrm{d}^3}{\mathrm{d}t^3}\widehat{R_{12}}(t) + 2\left(16-3t^2\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{R_{12}}(t) + \left(4\nu^2-7\right)t\frac{\mathrm{d}}{\mathrm{d}t}\widehat{R_{12}}(t) + \left(4\nu^2-1\right)\widehat{R_{12}}(t) = 0. \tag{5.69}$$

Now, from the expansions [AS]

$$I_\alpha(x) \sim \frac{1}{\sqrt{2\pi x}}\mathrm{e}^x \sum_{k\geq 0} \frac{\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k}{k!(2x)^k}, \quad K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\sum_{k\geq 0}\frac{(-1)^k\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k}{k!(2x)^k} \tag{5.70}$$

we see that

$$R_{12}(z) = 4I_{-\nu}(2\sqrt{z})K_{-\nu}(2\sqrt{z}) = \frac{1}{\sqrt{z}}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \tag{5.71}$$

is a power series containing only negative odd powers of $\sqrt{z}$ and so, from (5.65),

$$\widehat{R_{12}}(t) = 1 + \mathcal{O}\left(t^2\right) \tag{5.72}$$

is a power series containing only positive even powers of t. Hence we are interested in even power series solutions $\widehat{R_{12}}(t) = 1 + \mathcal{O}(t^2)$ of the ODE (5.69); by the Frobenius method it is possible to conclude that there exists exactly one such solution, which can be written in closed form in terms of the Gauss hypergeometric function

$$\widehat{R_{12}}(t) = {}_2F_1\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; 1; \frac{t^2}{16}\right) = \sum_{k\geq 0}\frac{\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k}{(k!)^2}\frac{t^{2k}}{16^k}. \tag{5.73}$$

Finally, recalling transformation (5.65) we have

$$\sqrt{z}R_{12}(z) = \sum_{k\geq 0}\frac{\left(\frac{1}{2}-\nu\right)_k\left(\frac{1}{2}+\nu\right)_k(2k)!}{(k!)^2}\frac{z^{-k}}{16^k} \tag{5.74}$$

which simplifies to the (1,2)–entry in (5.57) by the identity $(2k)! = 2^k k!(2k-1)!!$. The other entries of (5.57) are obtained by substituting (5.74) into (5.68). ■

Formulæ for Norbury intersection numbers. Just setting $\nu = 0$ in the previous result we obtain the following formulæ for Norbury intersection numbers. Indeed, by Norbury's theorem we have

$$\langle \Theta, \tau_{r_1} \cdots \tau_{r_n} \rangle = \frac{2^{2r_1+1} \cdots 2^{2r_n+1}}{(2r_1+1)!! \cdots (2r_n+1)!!} \left. \frac{\partial^n \tau(\mathbf{t})\big|_{\nu=0}}{\partial t_{r_1} \cdots \partial t_{r_n}} \right|_{\mathbf{t}=0}. \tag{5.75}$$

Note that $R(z; \nu = 0)$ has the following simple expression

$$R(z; \nu = 0) = \sum_{k \geq 0} \frac{(2k-1)!!^3}{k!(2z)^k} \begin{pmatrix} \frac{2k+1}{4} & 1 \\ -z\frac{2k+1}{2k-1} & -\frac{2k+1}{4} \end{pmatrix}. \tag{5.76}$$

Corollary 5.3.5. *For all $g \geq 1$ we have*

$$\langle \Theta, \tau_{g-1} \rangle = \frac{(2g-1)!!(2g-3)!!}{8^g g!} \tag{5.77}$$

and for all $n \geq 2$ we have

$$\begin{aligned} &\sum_{r_1,\dots,r_n \geq 0} \frac{(2r_1+1)!! \cdots (2r_n+1)!!}{2^{2r_1+1} \cdots 2^{2r_n+1} z_1^{1+r_1} \cdots z_n^{1+r_n}} \langle \Theta, \tau_{r_1} \cdots \tau_{r_n} \rangle \\ &\quad = \frac{(-1)^{n-1}}{n} \sum_{\pi \in \mathfrak{S}_n} \frac{\operatorname{tr}\left(R(z_{\pi_1}; \nu = 0) \cdots R(z_{\pi_n}; \nu = 0)\right)}{(z_{\pi_1} - z_{\pi_2}) \cdots (z_{\pi_{n-1}} - z_{\pi_n})(z_{\pi_n} - z_{\pi_1})} - \frac{z_1 + z_2}{(z_1 - z_2)^2} \delta_{n,2}. \end{aligned} \tag{5.78}$$

Virasoro constraints. The identification of the Brezin-Gross-Witten tau function as an appropriate isomonodromic tau function allows us also to derive independently the Virasoro constraints for this model, already known in the case $\nu = 0$ from [GN; MMS; DN] and in the general case from [Ab] by other methods. In concrete terms, we introduce the following differential operators;

$$L_m := \sum_{\ell \geq 0} \frac{2\ell+1}{2} \left(t_{2\ell+1} - 2\delta_{\ell,0}\right) \frac{\partial}{\partial t_{2\ell+2m+1}} + \frac{1}{4} \sum_{\ell=0}^{m-1} \frac{\partial^2}{\partial t_{2\ell+1} \partial t_{2m-2\ell-1}} + \left(\frac{1-4\nu^2}{16}\right) \delta_{m,0}, \quad m \geq 0. \tag{5.79}$$

They satisfy the Virasoro commutation relations

$$[L_m, L_n] = (m-n) L_{m+n}, \qquad m, n \geq 0. \tag{5.80}$$

Note that, contrarily to the Kontsevich–Witten case considered in Chap. 4 there is no $L_{-1} = 0$ constraint. This can be seen as the isomonodromic system now does not possess a translation symmetry $z \mapsto z + \epsilon$, as now the Fuchsian singularity $z = 0$ must be fixed.

Theorem 5.3.6 ([Ab]). *The Virasoro operators annihilate the BGW tau function;*

$$L_m \tau(\mathbf{t}; \nu) = 0, \qquad m \geq 0. \tag{5.81}$$

We now give a proof of this theorem, solely by means of our approach. We stress that the same computations, with very little modifications, can be used to prove the Virasoro constraints for the Kontsevich–Witten tau function considered in Chap. 4.

First, we need to study more in detail the (limiting) isomonodromic system. Introduce for convenience

$$\Xi(z; \mathbf{t}) := -\vartheta(z; \mathbf{t}) \sigma_3. \tag{5.82}$$

Repeating similar arguments as before, we find the following compatible system of ODEs for the matrix $\Psi(z; \mathbf{t}) := \Gamma(z; \mathbf{t}) \mathrm{e}^{\Xi(z; \mathbf{t})}$

$$\frac{\partial \Psi(z; \mathbf{t})}{\partial z} = L(z; \mathbf{t}) \Psi(z; \mathbf{t}), \qquad \frac{\partial \Psi(z; \mathbf{t})}{\partial t_{2\ell+1}} = M_\ell(z; \mathbf{t}) \Psi(z; \mathbf{t}), \ \ \ell = 0, \dots, K \tag{5.83}$$

Lemma 5.3.7. *The matrices $M_\ell(z;\mathbf{t})$ are polynomials in z of degree $\ell+1$ which can be written as*

$$M_\ell(z;\mathbf{t}) = -\left(\Psi(z;\mathbf{t})\sigma_3\Psi^{-1}(z;\mathbf{t})\sqrt{z}^{2\ell+1}\right)_+ \tag{5.84}$$

where $()_+$ denotes the polynomial part[2] *of a Laurent expansion in z around $z=\infty$. The matrix $L(z;\mathbf{t})$ is a rational matrix with a simple pole at $z=0$ which can be written as*

$$L(z;\mathbf{t}) = \frac{1}{z}\left(-\frac{\sigma_3}{4} + \sum_{\ell\geq 0}\frac{2\ell+1}{2}(t_{2\ell+1}-2\delta_{\ell,0})M_\ell(z;\mathbf{t})\right). \tag{5.85}$$

Proof. In this proof we omit the dependence on $(z;\mathbf{t})$. The matrix $M_\ell = \frac{\partial\Psi}{\partial t_{2\ell+1}}\Psi^{-1}$ has no jumps along Σ. In principle it may have an isolated singularity at $z=0$ (a pole or worse); however this cannot happen because of condition (5.54). Therefore M_ℓ has a removable singularity at $z=0$ and thus extends to an entire function. From inspection of the asymptotic behaviour of Ψ at ∞, it follows that M_ℓ is an entire function of z with polynomial growth at $z=\infty$. By the Liouville Theorem M_ℓ is a polynomial of z, which coincides then with the polynomial part of its asymptotic expansion;

$$M_\ell = \left(\frac{\partial\Psi}{\partial t_{2\ell+1}}\Psi^{-1}\right)_+ = \underbrace{\left(z^{-\frac{\sigma_3}{4}}G\frac{\partial Y}{\partial t_{2\ell+1}}Y^{-1}G^{-1}z^{\frac{\sigma_3}{4}}\right)_+}_{=0} - \left(\Psi\sigma_3\Psi^{-1}\frac{\partial\vartheta}{\partial t_{2\ell+1}}\right)_+ = -\left(\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2\ell+1}\right)_+ \tag{5.86}$$

where the first term vanishes thank to our choice of normalization in (5.55).

The same reasoning applies to $L=\Psi'\Psi^{-1}$, with the only exception that, in view of growth condition at $z=0$ (5.54), L has a simple pole at $z=0$. It follows by the Liouville Theorem that L is a rational function of z, which coincides then with the Laurent expansion at ∞ truncated at the term in z^{-1}; namely

$$\begin{aligned} L &= \frac{1}{z}\left(z\Psi'\Psi^{-1}\right)_+ = -\frac{\sigma_3}{4z} + \underbrace{\frac{1}{z}\left(zz^{-\frac{\sigma_3}{4}}GY'Y^{-1}G^{-1}z^{\frac{\sigma_3}{4}}\right)_+}_{=0} - \frac{1}{z}\left(z\Psi\sigma_3\Psi^{-1}\vartheta'\right)_+ \\ &= -\frac{\sigma_3}{4z} - \sum_{\ell\geq 0}\frac{2\ell+1}{2z}(t_\ell-2\delta_{\ell,0})\left(z\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2\ell-1}\right)_+ = \frac{1}{z}\left(-\frac{\sigma_3}{4} + \sum_{\ell\geq 0}\frac{2\ell+1}{2}(t_\ell-2\delta_{\ell,0})M_\ell\right) \end{aligned} \tag{5.87}$$

where again the term indicated vanishes thank to our choice of normalization in (5.55). ■

The compatible system (5.83) is the isomonodromic system whose tau function, defined as

$$\frac{\partial\tau(\mathbf{t})}{\partial t_{2\ell+1}} = \operatorname*{res}_{z=\infty}\operatorname{tr}\left(\Gamma^{-1}(z;\mathbf{t})\Gamma'(z;\mathbf{t})\sigma_3\sqrt{z}^{2\ell+1}\right)\mathrm{d}z, \qquad \ell=1,...,K, \tag{5.88}$$

reproduces the BGW tau function, up to truncating the times as $\mathbf{t}=(t_1,t_3,...,t_{2K+1},0,...)$ (hence there is no confusion in denoting both $\tau(\mathbf{t})$).

Hereafter we drop the explicit notation of dependence on $z,\mathbf{t}$ and denote

$$\widetilde{t}_\ell := t_{2\ell+1}-2\delta_{\ell,0}, \quad \partial_\ell := \frac{\partial}{\partial\widetilde{t}_\ell} = \frac{\partial}{\partial t_{2\ell+1}}. \tag{5.89}$$

We collect below some simple preliminary results that will be needed below for the proof of the Virasoro constraints.

Lemma 5.3.8. *The following identity holds true for all $k\geq 0$;*

$$\operatorname*{res}_{z=\infty}\operatorname{tr}\left(zL'\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z + \frac{2k+3}{2}\partial_k\log\tau = 0. \tag{5.90}$$

Proof. The (formal or not) residue of a total differential vanishes, hence

$$\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\Psi'\sigma_3\Psi^{-1}\sqrt{z}^{2k+3}\right)'\mathrm{d}z = 0 \tag{5.91}$$

[2]Note that by (5.41) the expression $\Psi(z;\mathbf{t})\sigma_3\Psi^{-1}(z;\mathbf{t})\sqrt{z}^{2\ell+1}$ has an expansion in integer powers of z only.

and computing the left hand side using $\Psi' = L\Psi$ we have

$$\begin{aligned}&\operatorname*{res}_{z=\infty} \operatorname{tr}\left((L\Psi)'\sigma_3\Psi^{-1}\sqrt{z}^{2k+3} - L\Psi\sigma_3\Psi^{-1}\Psi'\Psi^{-1}\sqrt{z}^{2k+3} + \frac{2k+3}{2}\Psi'\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z \\ &= \operatorname*{res}_{z=\infty} \operatorname{tr}\left(L'\Psi\sigma_3\Psi^{-1} + \cancel{L^2\Psi\sigma_3\Psi^{-1}} - \cancel{L\Psi\sigma_3\Psi^{-1}L}\right)\sqrt{z}^{2k+3}\mathrm{d}z + \frac{2k+3}{2}\partial_k \log\tau\end{aligned} \tag{5.92}$$

where the two terms indicated cancel out thanks to the cyclic property of the trace. ■

Lemma 5.3.9. *The following formulæ hold true, for all $a, b, c \geq 0$;*

$$\partial_b\partial_c \log\tau = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(M_b'\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2c+1}\right)\mathrm{d}z, \tag{5.93}$$

$$\partial_a\partial_b\partial_c \log\tau = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\left(\partial_a M_b' + [M_b', M_a]\right)\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2c+1}\right)\mathrm{d}z. \tag{5.94}$$

Proof. We start from (5.88)

$$\partial_c \log\tau = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\Psi'\sigma_3\Psi^{-1}\sqrt{z^{2c+1}}\right)\mathrm{d}z \tag{5.95}$$

and applying ∂_b using $\partial_b\Psi = M_b\Psi$ we get

$$\begin{aligned}\partial_b\partial_c \log\tau &= \operatorname*{res}_{z=\infty} \operatorname{tr}\left((M_b\Psi)'\sigma_3\Psi^{-1}\sqrt{z}^{2c+1} - \Psi'\sigma_3\Psi^{-1}M_b\sqrt{z}^{2c+1}\right)\mathrm{d}z \\ &= \operatorname*{res}_{z=\infty} \operatorname{tr}\left(M_b'\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2c+1} + \cancel{M_b\Psi'\sigma_3\Psi^{-1}\sqrt{z}^{2c+1}} - \cancel{\Psi'\sigma_3\Psi^{-1}M_b\sqrt{z}^{2c+1}}\right)\mathrm{d}z\end{aligned} \tag{5.96}$$

where the two terms cancel due to the cyclic property of the trace; (5.93) is proven. Now apply ∂_a to (5.93) to obtain

$$\partial_a\partial_b\partial_c \log\tau = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\left((\partial_a M_b')\Psi\sigma_3\Psi^{-1} + M_b'M_a\Psi\sigma_3\Psi^{-1} - M_b'\Psi\sigma_3\Psi^{-1}M_a\right)\sqrt{z}^{2c+1}\right)\mathrm{d}z \tag{5.97}$$

which simplifies to (5.94), once again thanks to the cyclic property of the trace. ■

As a last preliminary, let us use the expansion

$$Y(z;\mathbf{t}) = \mathbf{1} + \begin{bmatrix} a & a \\ -a & -a\end{bmatrix} z^{-\frac{1}{2}} + \begin{bmatrix} b & c \\ c & b\end{bmatrix} z^{-1} + \begin{bmatrix} d & e \\ -e & -d\end{bmatrix} z^{-\frac{3}{2}} + \begin{bmatrix} f & g \\ g & f\end{bmatrix} z^{-2} + \mathcal{O}\left(z^{-\frac{5}{2}}\right) \tag{5.98}$$

with $a = a(\mathbf{t})$,..., $g = g(\mathbf{t})$, to compute

$$M_0 = \begin{bmatrix} -2a & -1 \\ -z-2c & 2a\end{bmatrix}, \tag{5.99}$$

$$M_1 = \begin{bmatrix} 2(ab-ac-e)-2az & 4a^2+2c-z \\ 2(ae-ad-c^2+bc-g)-2zc-z^2 & 2(-ab+ac+e)+2az\end{bmatrix}, \tag{5.100}$$

and, by direct use of (5.88) we also find

$$\partial_0 \log\tau = 2a, \tag{5.101}$$

$$\partial_1 \log\tau = -4ab + 3d + e. \tag{5.102}$$

Proof of $L_0\tau = 0$. We compute from (5.85)

$$zL' = z\left(\frac{\sigma_3}{4z^2} + \frac{1}{z}\sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_\ell' - \frac{1}{z^2}\sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_\ell\right) = \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_\ell' - L. \tag{5.103}$$

Substitution in (5.90) shows that for all $k \geq 0$ we have

$$\begin{aligned}0 &= \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell \operatorname*{res}_{z=\infty}\operatorname{tr}(M_\ell'\Psi\sigma_3\Psi\sqrt{z}^{2k+1})\mathrm{d}z - \operatorname*{res}_{z=\infty}\operatorname{tr}\left(L\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right) + \frac{2k+3}{2}\partial_k\log\tau \\ &= \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell\partial_\ell\partial_k\log\tau + \frac{2k+1}{2}\partial_k\log\tau = \partial_k\left(\frac{L_0\tau}{\tau}\right)\end{aligned} \tag{5.104}$$

where we use (5.93) and the fact that $L\Psi = \Psi'$; the last identity implies $\frac{L_0\tau}{\tau} = C$ for some constant C; evaluation at $t_\ell = 0$, i.e. $\widetilde{t}_\ell = -2\delta_{\ell,0}$, using the definition of L_0 in (5.79) shows that

$$C = \left.\frac{L_0\tau}{\tau}\right|_{\mathbf{t}=0} = -\partial_0 \log\tau|_{\mathbf{t}=0} + \frac{1-4\nu^2}{16} = -\frac{1-4\nu^2}{16} + \frac{1-4\nu^2}{16} = 0 \tag{5.105}$$

where we use $\partial_0 \log\tau|_{\mathbf{t}=0} = \frac{1-4\nu^2}{16}$, which follows from the explicit formula (5.58).

Remark 5.3.10. *The constraint $L_0\tau = 0$ follows also from the dilation covariance of the RHP 5.3.1. Concretely, the matrix $\Psi(\mathrm{e}^u z;\mathbf{t})$ ($u \in \mathbb{R}$) satisfies the same jump condition as $\Psi(z;\mathbf{t})$, as the latter satisfies a jump condition with matrices independent of $z,\mathbf{t}$; further we have the boundary behaviour*

$$\Psi(\mathrm{e}^u z) \sim \mathrm{e}^{-\frac{u}{4}\sigma_3} z^{-\frac{\sigma_3}{4}} G\left(\mathbf{1} + \begin{bmatrix} a(\mathbf{t}) & a(\mathbf{t}) \\ -a(\mathbf{t}) & -a(\mathbf{t})\end{bmatrix}\mathrm{e}^{-\frac{u}{2}} z^{-\frac{1}{2}} + \mathcal{O}(z^{-1})\right)\mathrm{e}^{-\vartheta(z;\mathbf{t}(u))\sigma_3}, \quad z\to\infty \tag{5.106}$$

where $t_\ell(u) := \mathrm{e}^{\frac{2\ell+1}{2}u} t_\ell$. It follows that $\mathrm{e}^{\frac{u}{4}\sigma_3}\Gamma(\mathrm{e}^u z;\mathbf{t}(-u))$ solves RHP 5.3.1, the solution of which is unique, hence

$$\Gamma(z;\mathbf{t}) = \mathrm{e}^{\frac{u}{4}\sigma_3}\Gamma(\mathrm{e}^u z;\mathbf{t}(-u)). \tag{5.107}$$

Therefore, for all $k\geq 0$ we have

$$\operatorname*{res}_{z=\infty} \operatorname{tr}\left(\Gamma^{-1}(z;\mathbf{t})\Gamma'(z;\mathbf{t})\sigma_3\sqrt{z}^{2k+1}\right) = \operatorname*{res}_{z=\infty}\operatorname{tr}\left(\Gamma^{-1}(\mathrm{e}^u z;\mathbf{t}(-u))\Gamma'(\mathrm{e}^u z;\mathbf{t}(-u))\sigma_3\sqrt{z}^{2k+1}\right) \tag{5.108}$$

and the last expression does not depend on u by construction; setting the first variation in u equal to zero we recover $\partial_k\left(\frac{L_0\tau}{\tau}\right) = 0$ for all $k\geq 0$, from which we can derive $L_0\tau = 0$ as above.

Note that due to the special point $z = 0$, RHP 5.3.1 does not have a translation covariance property. Therefore it lacks a Virasoro constraint of the form L_{-1} as in the case of the Kontsevich–Witten tau function.

Proof of $L_1\tau = 0$ As a consequence of the recursion

$$zM_\ell = M_{\ell+1} - (M_{\ell+1})_0 \Rightarrow zM'_\ell = M'_{\ell+1} - M_\ell \tag{5.109}$$

where $()_0$ denotes the constant term in z, we multiply (5.103) by z to get

$$\begin{aligned} z^2L' &= \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell zM'_\ell - zL = \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M'_{\ell+1} - \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_\ell - zL \\ &= \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_{\ell'+1} - \underbrace{\left(\sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_\ell - \frac{\sigma_3}{4}\right)}_{=zL} - \frac{\sigma_3}{4} - zL = \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell M'_{\ell+1} - 2zL - \frac{\sigma_3}{4} \end{aligned} \tag{5.110}$$

and we use (5.90) with $k\mapsto k+1$:

$$\begin{aligned} 0 &= \operatorname*{res}_{z=\infty}\operatorname{tr}\left(z^2L'\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z + \frac{2k+5}{2}\partial_k\log\tau \\ &= \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell \operatorname*{res}_{z=\infty}\operatorname{tr}(M'_{\ell+1}\Psi\sigma_3\Psi\sqrt{z}^{2k+1})\mathrm{d}z - \operatorname*{res}_{z=\infty}\operatorname{tr}\left(2zL\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right) \\ &\quad + \frac{2k+5}{2}\partial_{k+1}\log\tau - \frac{1}{4}\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\sigma_3\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z \\ &= \sum_{\ell\geq 0}\frac{2\ell+1}{2}\widetilde{t}_\ell\partial_{\ell+1}\partial_k\log\tau + \frac{2k+1}{2}\partial_{k+1}\log\tau - \frac{1}{4}\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\sigma_3\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z \end{aligned} \tag{5.111}$$

where we have used (5.93) and $L\Psi = \Psi'$.

Lemma 5.3.11. *We have*

$$-\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\sigma_3\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z = \partial_k\left(\frac{\partial_0^2\tau}{\tau}\right). \tag{5.112}$$

Proof of Lemma 5.3.11. Note that

$$\partial_k\left(\frac{\partial_0^2\tau}{\tau}\right)=\partial_k\left(\partial_0^2\log\tau+(\partial_0\log\tau)^2\right)$$
$$=\partial_k\partial_0^2\log\tau+2(\partial_0\log\tau)(\partial_k\partial_0\log\tau)=\partial_k\partial_0^2\log\tau+4a\partial_k\partial_0\log\tau \tag{5.113}$$

where we have used (5.101) in the last step. Using Lemma 5.3.9 we obtain

$$\partial_k\partial_0^2\log\tau+4a\partial_k\partial_0\log\tau=\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\left(\partial_0M_0'+[M_0',M_0]+4aM_0'\right)\Psi\sigma_3\Psi^{-1}\sqrt{z}^{2k+1}\right)\mathrm{d}z \tag{5.114}$$

and the statement (5.112) boils down to the identity

$$\cancel{\partial_0M_0'}+[M_0',M_0]+4aM_0'=-\sigma_3 \tag{5.115}$$

which is easily checked using (5.99). ∎

Back to the proof of $L_1\tau=0$, we see from the last line of (5.111) together with Lemma 5.3.11 that we have proven $\partial_k\left(\frac{L_1\tau}{\tau}\right)=0$ for all $k\geq 0$. Hence $L_1\tau=C\tau$ for some constant C; evaluation at $\mathbf{t}=(0,0,...)$ shows that $C=0$ (e.g. using the explicit formulæ we have obtained for logarithmic derivatives of the tau function) and so $L_1\tau=0$.

Proof of $L_2\tau=0$. Using the recursion (5.109) we see that

$$zM_{\ell+1}'=M_{\ell+2}'-M_{\ell+1}=M_{\ell+2}'-zM_\ell-(M_{\ell+1})_0 \tag{5.116}$$

where again we denote $()_0$ the constant term in z; we then compute from (5.110)

$$z^3L'=\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell zM_{\ell+1}'-2z^2L-\frac{\sigma_3}{4}z$$
$$=\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_{\ell+2}'-z\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_\ell-\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell\left(M_{\ell+1}\right)_0-2z^2L-\frac{\sigma_3}{4}z$$
$$=\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_{\ell+2}'-\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell\left(M_{\ell+1}\right)_0-3z^2L-\frac{\sigma_3}{2}z. \tag{5.117}$$

Lemma 5.3.12. *We have the identity*

$$-\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell\left(M_{\ell+1}\right)_0=\begin{bmatrix}-b+c & -a\\ \frac{3}{2}(d-e) & -b-c\end{bmatrix} \tag{5.118}$$

where $a=a(\mathbf{t}),...,e=e(\mathbf{t})$ *are as in* (5.98).

Proof of Lemma 5.3.12. Since $(z^2\Psi)'$ satisfies the same jump condition as Ψ along Σ, it follows that the ratio $(z^2\Psi)'\Psi^{-1}$ is an entire matrix–valued function; indeed from (5.54) we see that this ratio is analytic also at $z=0$. Since this ratio has polynomial growth at $z=\infty$, see (5.55), we conclude that $(z^2\Psi)'\Psi^{-1}$ is actually a polynomial, which coincides with the polynomial part of its expansion at $z=\infty$;

$$(z^2\Psi)'\Psi^{-1}=\left(2z\mathbf{1}-z^2\frac{\sigma_3}{4z}+z^2z^{-\frac{\sigma_3}{4}}GY'Y^{-1}G^{-1}z^{\frac{\sigma_3}{4}}+z^2\Psi\vartheta'\sigma_3\Psi^{-1}\right)_+$$
$$=2z\mathbf{1}-z\frac{\sigma_3}{4}+\left(z^2z^{-\frac{\sigma_3}{4}}GY'Y^{-1}G^{-1}z^{\frac{\sigma_3}{4}}\right)_++\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_{\ell+1}. \tag{5.119}$$

However, it is trivial to compute $(z^2\Psi)'\Psi^{-1}=2z\mathbf{1}+z^2L$, which has no constant term in z. Therefore also the constant term in z in (5.119) vanishes and hence

$$-\left(\sum_{\ell\geq0}\frac{2\ell+1}{2}\widetilde{t}_\ell M_{\ell+1}\right)_0=\left(z^2z^{-\frac{\sigma_3}{4}}GY'Y^{-1}G^{-1}z^{\frac{\sigma_3}{4}}\right)_0=\begin{bmatrix}-b+c & -a\\ \frac{3}{2}(d-e) & -b-c\end{bmatrix} \tag{5.120}$$

and the Lemma is proven. ∎

Back to the proof of $L_2\tau = 0$, we obtain from (5.117) together with Lemma 5.3.12

$$z^3 L' = \sum_{\ell\geq 0} \frac{2\ell+1}{2}\widetilde{t}_\ell M'_{\ell+2} - 3z^2 L + \begin{bmatrix} -\frac{z}{2}-b+c & -a \\ \frac{3}{2}(d-e) & \frac{z}{2}-b-c \end{bmatrix} \tag{5.121}$$

and inserting this expression in (5.90) with $k \mapsto k+2$ we have

$$\begin{aligned} 0 &= \operatorname*{res}_{z=\infty} \operatorname{tr}\left(z^3 L' \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z + \frac{2k+7}{2} \partial_{k+2} \log \tau \\ &= \sum_{\ell \geq 0} \frac{2\ell+1}{2} \widetilde{t}_\ell \operatorname*{res}_{z=\infty} \operatorname{tr}\left(M'_{\ell+2} \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z - 3 \operatorname*{res}_{z=\infty} \left(z^3 L \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z \\ &\quad + \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\begin{bmatrix} -\frac{z}{2}-b+c & -a \\ \frac{3}{2}(d-e) & \frac{z}{2}-b-c \end{bmatrix} \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z + \frac{2k+7}{2} \partial_{k+2} \log \tau \\ &= \sum_{\ell \geq 0} \frac{2\ell+1}{2} \widetilde{t}_\ell \partial_k \partial_{\ell+2} \log \tau + \frac{2k+1}{2} \partial_{k+2} \log \tau \\ &\quad + \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\begin{bmatrix} -\frac{z}{2}-b+c & -a \\ \frac{3}{2}(d-e) & \frac{z}{2}-b-c \end{bmatrix} \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z. \end{aligned} \tag{5.122}$$

The final part is the computation of the last term in the above equation. This is done in the following Lemma.

Lemma 5.3.13. *We have*

$$\operatorname*{res}_{z=\infty} \operatorname{tr}\left(\begin{bmatrix} -\frac{z}{2}-b+c & -a \\ \frac{3}{2}(d-e) & \frac{z}{2}-b-c \end{bmatrix} \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z = \partial_k \left(\frac{\partial_0 \partial_1 \tau}{2\tau} \right). \tag{5.123}$$

Proof of Lemma 5.3.13. Note that

$$\begin{aligned} \partial_k \left(\frac{\partial_0 \partial_1 \tau}{\tau} \right) &= \partial_k \left(\partial_0 \partial_1 \log \tau + (\partial_0 \log \tau)(\partial_1 \log \tau) \right) \\ &= \partial_k \partial_0 \partial_1 \log \tau + (\partial_k \partial_0 \log \tau)(\partial_1 \log \tau) + (\partial_0 \log \tau)(\partial_k \partial_1 \log \tau) \\ &= \partial_k \partial_0 \partial_1 \log \tau + (-4ab+3d+e)(\partial_k \partial_0 \log \tau) + 2a(\partial_k \partial_1 \log \tau) \end{aligned} \tag{5.124}$$

where we have used (5.101) and (5.102). Using Lemma 5.3.9 and the explicit expressions (5.99) and (5.100) we obtain

$$\begin{aligned} \partial_k \left(\frac{\partial_0 \partial_1 \tau}{2\tau} \right) &= \frac{1}{2} \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\left(\partial_1 M'_0 + [M'_0, M_1] + (-4ab+3d+e) M'_0 + 2a M'_1 \right) \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z \\ &= \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\begin{bmatrix} -\frac{z}{2}+c & -a \\ \frac{3}{2}(-d+e) & \frac{z}{2}-c \end{bmatrix} \Psi \sigma_3 \Psi^{-1} \sqrt{z}^{2k+1} \right) \mathrm{d}z \end{aligned} \tag{5.125}$$

and the proof is complete, as $\operatorname{tr}\left(\begin{bmatrix} -b & 0 \\ 0 & -b \end{bmatrix} \Psi \sigma_3 \Psi^{-1} \right) = -b \operatorname{tr}\left(\Psi \sigma_3 \Psi^{-1} \right) = -b \operatorname{tr}(\sigma_3) = 0.$ ■

From the last line of (5.122) combined with Lemma 5.3.13 we obtain $\partial_k \left(\frac{L_2 \tau}{\tau} \right) = 0$, for all $k \geq 0$. It follows that $L_2\tau = C\tau$ for some integration constant C; evaluation at $\mathbf{t} = (0, 0, \ldots)$ shows that $C = 0$ (e.g. using the explicit formulæ we have obtained for logarithmic derivatives of the tau function) and so $L_2 \tau = 0$.

Proof of Thm. 5.3.6. We have proven $L_n \tau = 0$ for $n = 0, 1, 2$. It remains to show that $L_{n+1}\tau = 0$ for $n \geq 2$. The proof is given by induction on $n \geq 2$: assume that $L_n\tau = 0$ for some $n \geq 2$, then exploiting the Virasoro commutation relation (5.80) we have

$$L_{n+1}\tau = \frac{1}{n-1} \left(L_n L_1 \tau - L_1 L_n \tau \right) = 0 \tag{5.126}$$

and the proof of Thm. 5.3.6 is complete.

KdV and Painlevé XXXIV hierarchies. As we briefly reviewed in Chap. 4, the Kontsevich–Witten KdV tau function provides a solution to the Painlevé I hierarchy. We now observe that the BGW tau function is instead related with the PXXXIV hierarchy.

More precisely, let us call $x := t_1$ and introduce

$$u(x, \widetilde{\mathbf{t}}) := \frac{\partial^2}{\partial x^2} \log \tau(x, \widetilde{\mathbf{t}}), \quad \widetilde{\mathbf{t}} := (t_3, t_5, \ldots) \tag{5.127}$$

which is a solution to the KdV hierarchy

$$\frac{\partial u}{\partial t_{2\ell+1}} = \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{L}_{\ell+1}[u], \quad \ell \geq 1. \tag{5.128}$$

In (4.72) we denote $\mathcal{L}_\ell[u]$ the *Lenard–Magri* differential polynomials, defined as

$$\mathcal{L}_0[u] = 1, \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{\ell+1} = \left(\frac{1}{4}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + u_x\right)\mathcal{L}_\ell[u] \\ \mathcal{L}_{\ell+1}[u=0] = 0 \end{cases} \quad \text{for } \ell \geq 0. \tag{5.129}$$

Lemma 5.3.14. *The solution u in (4.71) of the KdV hierarchy (4.72) satisfies the initial condition*

$$u(x, \widetilde{\mathbf{t}} = 0) = \frac{1 - 4\nu^2}{8(2-x)^2}. \tag{5.130}$$

Proof. The time $x = t_1$ is related to shifts of the variable z in the RHP 5.3.1; more precisely, restricting to real values of x for simplicity, we easily see that

$$\Gamma(z; (x, 0, \ldots)) = \left(1 - \frac{x}{2}\right)^{\frac{\sigma_3}{2}} \Gamma_0\left(\left(1 - \frac{x}{2}\right)^2 z\right) \tag{5.131}$$

is the solution to (5.3.1) for this choice of times $t_1 = x$, $\widetilde{\mathbf{t}} = 0$. Here we assume $-2 < x < 2$ and take the principal branch of the square roots. At the level of asymptotic expansions, we are replacing $\sqrt{z} \to \left(1 - \frac{x}{2}\right)\sqrt{z}$ in the asymptotic expansion of $\Gamma_0(z)$; from (5.43) we see that

$$\Gamma(z; (x, 0, \ldots)) \sim z^{-\frac{\sigma_3}{4}} G\left(\mathbf{1} + \frac{1 - 4\nu^2}{32\left(1 - \frac{x}{2}\right)\sqrt{z}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathcal{O}(z^{-1})\right), \quad z \to \infty. \tag{5.132}$$

Using (5.132) a direct computation shows that

$$\partial_x \log \tau(x, 0, \ldots) = \operatorname*{res}_{z=\infty} \operatorname{tr}\left(\Gamma^{-1}(z; (x, 0, \ldots))\Gamma'(z; (x, 0, \ldots))\sigma_3\sqrt{z}\right) \mathrm{d}z = \frac{1 - 4\nu^2}{8(2-x)} \tag{5.133}$$

which completes the proof. ∎

Remark 5.3.15. *Note that (5.133) implies that*

$$\tau(x, 0, \ldots) = C(2-x)^{\frac{4\nu^2 - 1}{8}} \tag{5.134}$$

for some nonvanishing integration constant $C \neq 0$, which indicates that RHP 5.3.1 for $\mathbf{t} = (x, 0, \ldots)$ is solvable for all values of $x \neq 2$.

Let us now write the Virasoro constraint $L_0\tau = 0$, see (5.79), as

$$(x-2)\frac{\partial \log \tau}{\partial x} + \sum_{\ell \geq 1}(2\ell+1)t_{2\ell+1}\frac{\partial \log \tau}{\partial t_{2\ell+1}} + \frac{1 - 4\nu^2}{8} = 0 \tag{5.135}$$

and taking two derivatives in x we have

$$(x-2)\frac{\partial^3 \log \tau}{\partial x^3} + 2\frac{\partial^2 \log \tau}{\partial x^2} + \sum_{\ell \geq 1}(2\ell+1)t_{2\ell+1}\frac{\partial^3 \log \tau}{\partial_x^2 \partial t_{2\ell+1}} = 0. \tag{5.136}$$

The following proposition then follows from the definition (4.71) of u and the KdV hierarchy equations (4.72).

Proposition 5.3.16. *If we set $t_\ell = 0$ for $\ell \geq K+1$ as above, then $u(x; t_3, ..., t_{2K+1}, 0, ...)$ solves the Kth member of the PXXXIV hierarchy;*

$$2u + (x-2)u_x + \sum_{\ell=1}^{K} (2\ell+1) t_{2\ell+1} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{L}_{\ell+1}[u] = 0 \tag{5.137}$$

which is an ODE in x, where $t_3, ..., t_{2K+1}$ are regarded as parameters.

The Painlevé XXXIV hierarchy has been considered in [CJP] and it is related by a *Miura transformation* to the Painlevé II hierarchy, first introduced in [FN].

For example, the case $K = 1$, denoting $t_3 =: t$, in (5.137) is

$$\frac{3}{4} t u_{xxx} + 9tuu_x + (x-2)u_x + 2u = 0. \tag{5.138}$$

By the simple scaling

$$x = 2 - \left(\frac{3t}{4}\right)^{\frac{1}{3}} y, \qquad u(x) = \left(\frac{2}{9t^2}\right)^{\frac{1}{3}} v(y) \tag{5.139}$$

(5.138) reads

$$v_{yyy} + 6vv_y - yv_y - 2v = 0 \tag{5.140}$$

which we call, following the literature, see e.g. [CJP], the Painlevé XXXIV equation.

It is known [I; FA] that (5.140) is equivalent to the Painlevé II equation

$$w_{yy} = w^3 + yw + \alpha, \tag{5.141}$$

in the sense that the *Miura transformation*

$$v = -w^2 - w_y, \qquad w = \frac{v_y + \alpha}{2v - y} \tag{5.142}$$

is a one–to–one map between solutions to (5.140) and to (5.141).

Using (5.98), (5.99) and (5.100) we can write down explicitly the Lax pair for (5.138) as

$$L(z) = L_1 z + L_0 + \frac{L_{-1}}{z}, \qquad M := \mathcal{M}\left(\frac{\partial}{\partial x}\right) = \begin{pmatrix} -2a & -1 \\ -z - 2a_x + 4a^2 & 2a \end{pmatrix} \tag{5.143}$$

where

$$L_1 = \begin{pmatrix} 0 & 0 \\ -\frac{3t_3}{2} & 0 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} -3ta & -\frac{3t}{2} \\ 6ta^2 + 3ta_x - \frac{x}{2} + 1 & 3ta \end{pmatrix}, \tag{5.144}$$
$$L_{-1} = \begin{pmatrix} -(x-2)a - 6ta_x a - \frac{3}{2} ta_{xx} - \frac{1}{4} & -\frac{x-2}{2} - 3ta_x \\ 2(x-2)a^2 + 12ta_x a^2 + 6ta_{xx} a + a + 12ta_x^2 + 2(x-2)a_x + \frac{3}{2} ta_{xxx} & (x-2)a + 6ta_x a + \frac{3}{2} ta_{xx} + \frac{1}{4} \end{pmatrix}.$$

Indeed, the compatibility of $\Psi' = L\Psi$ and $\Psi_x = M\Psi$ implies the zero curvature condition

$$L_x - M' - [M, L] = \frac{1}{z} \begin{pmatrix} 0 & 0 \\ \frac{3}{2} ta_{xxxx} + 36ta_{xx}a_x + 2(x-2)a_{xx} + 4a_x & 0 \end{pmatrix} = 0 \tag{5.145}$$

which, identifying $u = 2a_x$ from (5.101), gives (5.138). Setting $t = -\frac{4}{3}$, $x - 2 = y$ and $4a(x) = \alpha(y)$ we obtain the following Lax pair for (5.140);

$$L = \begin{pmatrix} \alpha + \frac{2\alpha_{yy} - y\alpha + 2\alpha\alpha_y - 1}{4z} & 2 + \frac{2\alpha_y - y}{2z} \\ 2z - \frac{y}{2} - \frac{\alpha^2}{2} - \alpha_y + \frac{2\alpha + y\alpha^2 + 4y\alpha_y - 2\alpha^2\alpha_y - 8\alpha_y^2 - 4\alpha\alpha_{yy} - 4\alpha_{yyy}}{8z} & -\alpha - \frac{2\alpha_{yy} - y\alpha + 2\alpha\alpha_y - 1}{4z} \end{pmatrix},$$
$$M = \begin{pmatrix} -\frac{\alpha}{2} & -1 \\ -z + \frac{\alpha^2}{4} + \frac{\alpha_y}{2} & \frac{\alpha}{2} \end{pmatrix}, \qquad \begin{cases} \Psi' = L\Psi \\ \Psi_y = M\Psi \end{cases} \Rightarrow \quad \alpha_{yyyy} + 6\alpha_y\alpha_{yy} - y\alpha_{yy} - 2\alpha_y = 0 \tag{5.146}$$

which is (5.140) for $v := \alpha_y$. Finally we note that after a gauge transformation on (5.146) of the form $\widehat{L} = GLG^{-1}$, $\widehat{M} = GMG^{-1} + G_yG^{-1}$ with $G = \begin{pmatrix} 1 & 0 \\ \frac{\alpha}{2} & 1 \end{pmatrix}$ we obtain a Lax pair

$$\widehat{L} = \begin{pmatrix} \frac{2v_y-1}{4z} & \frac{2v-y}{2z}+2 \\ 2z - v - \frac{y}{2} + \frac{-2v^2+yv-v_{yy}}{2z} & \frac{1-2v_y}{4z} \end{pmatrix}, \qquad \widehat{M} = \begin{pmatrix} 0 & -1 \\ v-z & 0 \end{pmatrix} \tag{5.147}$$

for (5.140) in v directly.

5.4 Proof of Thm. 5.2.6

Schlesinger transform matrix R and characteristic matrix. Consider the bare matrix Ψ defined in (5.23) and introduce

$$\Gamma_0 := \Psi(z)\mathrm{e}^{-2\sqrt{z}\sigma_3}. \tag{5.148}$$

Γ_0 is, up to a simple gauge fact in front, the solution of RHP 5.2.3 for $N = 0$ (and consequently no parameters $\vec{\lambda}$).

Suppose RHP 5.2.3 has a solution $\Gamma(z) = \Gamma(z;\vec{\lambda})$. Then there exists a rational matrix $R(z) = R(z;\vec{\lambda})$ with simple poles at $z = \lambda_1^2, ..., \lambda_N^2$ only such that

$$\Gamma(z) = R(z)\Gamma_0(z)D(z). \tag{5.149}$$

This is easily seen by Liouville theorem as $R(z) := \Gamma(z)D^{-1}(z)\Gamma_0^{-1}(z)$ is continuous along Σ, while having at worse simple poles at $z = \lambda_1^2, ..., \lambda_N^2$. This falls within the theory developed in Chap. 2.

Hereafter we employ the short notation $\partial_j := \frac{\partial}{\partial \lambda_j}$ and we consider the case $\operatorname{Re}\lambda_j \geq 0$ only for clarity's sake; the general case is a straightforward generalization.

Following the strategy outlined in Chap. 2 one can seek a characterization in terms of a finite-dimensional linear map for the existence of a rational function R such that $\Gamma = R\Gamma_0 D$ is the solution of RHP 5.2.3. Referring for more details about the modification of the general Thm. 2.4.8 for the present situation to [BCa, App. B], let us state that such a rational functions exists if and only if the following *characteristic matrix* is nondegenerate;

$$\mathbf{G} = (\mathbf{G}_{j,k})_{j,k=1}^N, \quad \mathbf{G}_{j,k} = \begin{cases} -\operatorname*{res}_{z=\infty} \frac{z^k}{z-\lambda_j^2}\mathbf{e}_2^\top \Gamma_0^{-1}(\lambda_j^2)\Gamma_0(z)G^{-1}z^{\frac{\sigma_3}{4}}\mathbf{e}_{1+k} & \text{if } -\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2} \\ -\operatorname*{res}_{z=\infty} \frac{z^k}{z-\lambda_j^2}\mathbf{e}_1^\top \Gamma_0^{-1}(\lambda_j^2\mathrm{e}^{\mp 2\pi\mathrm{i}})\Gamma_0(z)G^{-1}z^{\frac{\sigma_3}{4}}\mathbf{e}_{1+k} & \text{if } \frac{\pi}{2} < \pm\arg\lambda < \pi \end{cases} \tag{5.150}$$

where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the index in $\mathbf{e}_{1+k}$ is understood $\mod 2$ (e.g. $\mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_4 = \mathbf{e}_2$); $\Gamma_0(z)$ is as in (5.148), and note that the gauge factor of (5.148) is irrelevant here, as $\mathbf{G}_{j,k}$ is invariant under $\Gamma_0 \mapsto B\Gamma_0$ for any $B \in \mathrm{GL}(2,\mathbb{C})$.

The residue in (5.150) is by definition a formal residue, i.e. we regard

$$\Gamma_0(z)G^{-1}z^{\frac{\sigma_3}{4}} = z^{-\frac{\sigma_3}{4}}GY(z)G^{-1}z^{\frac{\sigma_3}{4}} = \mathbf{1} + \mathcal{O}(z^{-1}) \in \mathrm{GL}\left(2, \mathbb{C}\left[\!\left[z^{-1}\right]\!\right]\right) \tag{5.151}$$

as a formal power series and the formal residue is simply the coefficient of z^{-1}. It can be checked that thanks to the property (5.41) the expression (5.151) contains integer powers of z only.

The following variational formula has been proven in [BCc, App. B], and is a direct corollary of Prop. 2.4.6;

$$\partial_j \log\det\mathbf{G} = \sum_{k=1}^N \operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left(R^{-1}R'\partial_j J_k J_k^{-1}\right) + \operatorname*{res}_{z=\infty} \operatorname{tr}\left(R^{-1}R'\partial_j J_\infty J_\infty^{-1}\right) + \sum_{k=1}^N \operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left(\Gamma_0^{-1}\Gamma_0'\partial_j U_k U_k^{-1}\right) \tag{5.152}$$

where

$$J_k := \Gamma_0(z)\begin{bmatrix} 1 & 0 \\ 0 & \lambda_k^2 - z \end{bmatrix}, \quad J_\infty := \Gamma_0(z)D(z)G^{-1}z^{\frac{\sigma_3}{4}}, \quad U_k := \begin{bmatrix} 1 & 0 \\ 0 & z-\lambda_k^2 \end{bmatrix}, \qquad k = 1, ..., n. \tag{5.153}$$

Malgrange differential and extended Kontsevich–Penner partition function. The following manipulation of the determinant of the characteristic matrix is the main step in order to prove Thm. 5.2.6.

Proposition 5.4.1. *We have*

$$\det \mathbf{G} = C \det \left(\mathrm{e}^{-2\lambda_j}\xi_k(\lambda_j)\right)_{j,k=1}^n \tag{5.154}$$

where the proportionality constant C (irrelevant in the following) is

$$C := (-1)^{\lfloor \frac{a}{2} \rfloor}(-\mathrm{i})^{b_+}\mathrm{i}^{b_-}, \quad a := \#\left\{j:\ -\frac{\pi}{2}\arg\lambda_j < \frac{\pi}{2}\right\}, \quad b_\pm := \#\left\{j:\ \frac{\pi}{2} < \pm\arg\lambda_j < \pi\right\}. \tag{5.155}$$

Proof. Let us consider the case $-\frac{\pi}{2} < \arg\lambda_j < \frac{\pi}{2}$ first; by the definition (5.150) and simple algebra using (5.31), we see that the $(2m+1)$th, resp. $(2m+2)$th, column of $\mathbf{G}$ is the second, resp. the first, entry in the row vector coefficient of z^{-m} in

$$\frac{\mathrm{e}^{-2\lambda_j}}{1-\frac{\lambda_j^2}{z}}[-\xi_2(\lambda_j),\xi_1(\lambda_j)](\mathbf{1}+\mathcal{O}(z^{-1})) = \sum_{m\geq 0}\frac{\mathrm{e}^{-2\lambda_j}\lambda_j^{2m}}{z^m}[-\xi_2(\lambda_j),\xi_1(\lambda_j)](\mathbf{1}+\mathcal{O}(z^{-1})), \tag{5.156}$$

where j is the row index of the columns of $\mathbf{G}$. Hence we note that the first column of $\mathbf{G}$ is given by $[\mathrm{e}^{-2\lambda_j}\xi_1(\lambda_j)]_{j=1}^N$ and the second one by $[-\mathrm{e}^{-2\lambda_j}\xi_2(\lambda_j)]_{j=1}^N$.

For the next columns we proceed by induction. Indeed, as the $\mathcal{O}(z^{-1})$ term in (5.156) does not depend on the row index j, it follows that the $(2m+1)$th column is $[\mathrm{e}^{-2\lambda_j}\lambda_j^{2m}\xi_1(\lambda_j)]_{j=1}^N$ up to a linear combination of the previous (odd) column. Similarly the $(2m+2)$th column is $[-\mathrm{e}^{-2\lambda_j}\lambda_j^{2m}\xi_2(\lambda_j)]_{j=1}^N$ up to a linear combination of the previous (even) columns. Now we recall [AS]

$$I_{\alpha+1}(2\lambda) = I_{\alpha-1}(2\lambda) - \frac{\alpha}{\lambda}I_\alpha(2\lambda), \quad K_{\alpha+1}(2\lambda) = K_{\alpha-1}(2\lambda) + \frac{\alpha}{\lambda}K_\alpha(2\lambda) \tag{5.157}$$

which implies

$$\xi_{k+2}(\lambda) = \lambda^2\xi_k(\lambda) - (k-\nu)\xi_{k+1}(\lambda) \text{ when } -\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2} \tag{5.158}$$

and so

$$\begin{aligned} \lambda^{2m}\xi_1(\lambda) &\equiv \xi_{2m+1}(\lambda) \quad \bmod\ (\xi_1(\lambda),...,\xi_{2m}(\lambda)) \\ \lambda^{2m}\xi_2(\lambda) &\equiv \xi_{2m+2}(\lambda) \quad \bmod\ (\xi_1(\lambda),...,\xi_{2m+1}(\lambda)) \quad (m\geq 1). \end{aligned} \tag{5.159}$$

It follows that the matrices $\mathbf{G}$ and $[(-1)^{k-1}\mathrm{e}^{-2\lambda_j}\xi_k(\lambda_j)]_{j,k=1}^m$ differ by multiplication by a unimodular matrix, more precisely by a triangular matrix with 1's along the diagonal; in particular they have the same determinant and Proposition is proven when $-\frac{\pi}{2} < \arg\lambda_j < \frac{\pi}{2}$.

The case when $\frac{\pi}{2} < \pm\arg\lambda_j < \pi$ is completely analogous so we just briefly comment on the differences; expression (5.156), in view of (5.150) and (5.31), must be replaced by

$$\frac{\mathrm{e}^{-2\lambda_j}}{1-\frac{\lambda_j^2}{z}}[\pm\mathrm{i}\xi_2(\lambda_j),\pm\mathrm{i}\xi_1(\lambda_j)](\mathbf{1}+\mathcal{O}(z^{-1})) = \sum_{m\geq 0}\frac{\mathrm{e}^{-2\lambda_j}\lambda_j^{2m}}{z^m}[\pm\mathrm{i}\xi_2(\lambda_j),\pm\mathrm{i}\xi_1(\lambda_j)](\mathbf{1}+\mathcal{O}(z^{-1})) \tag{5.160}$$

while the recursion (5.158) must be replaced by

$$\xi_{k+1}(\lambda) = \lambda^2\xi_{k-1}(\lambda) + (k-\nu)\xi_k(\lambda), \qquad \frac{\pi}{2} < \pm\arg\lambda < \pi \tag{5.161}$$

which is again a consequence of (5.157). Hence (5.159) holds true in the case $\frac{\pi}{2} < \pm\arg\lambda_j < \pi$ as well and as above, taking care of the $\pm$'s and $\pm$i's, we have the thesis. ∎

We are finally ready to complete the proof of Thm. 5.2.6. Let us compute the Malgrange form

$$\Omega(\partial_j) := \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{tr}\ \left(\Gamma(z_-)^{-1}\Gamma'(z_-)\partial_j J(z)J^{-1}(z)\right)\mathrm{d}z \tag{5.162}$$

by using $\Gamma = R\Gamma_0 D^{-1}$ and $J = D^{-1}J_0 D$ where $J_0(z) := \mathrm{e}^{2\sqrt{z_-}\sigma_3}S(z)\mathrm{e}^{-2\sqrt{z_+}\sigma_3}$, with S defined in (5.28). After some elementary algebra we obtain

$$\Omega(\partial_j) = \sum_{z_*\in\left\{\lambda_1^2,\dots,\lambda_N^2,\infty\right\}} \operatorname*{res}_{z=z_*} \mathrm{tr}\ \left(R^{-1}R'\Gamma_0\partial_j DD^{-1}\Gamma_0^{-1} + \Gamma_0^{-1}\Gamma_0'\partial_j DD^{-1}\right) \tag{5.163}$$

and by using the identities

$$\partial_j J_\infty J_\infty^{-1} = \Gamma_0 \partial_j D D^{-1} \Gamma_0^{-1} \tag{5.164}$$

we obtain (comparing with (5.152))

$$\Omega(\partial_j) = \partial_j \log\det \mathbf{G} + \sum_{k=1}^{N} \operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left(\Gamma_0^{-1} R^{-1} R' \Gamma_0 (\partial_j D D^{-1} - \partial_j U_k U_k^{-1})\right) \tag{5.165}$$

as $\operatorname*{res}_{z=\infty} \operatorname{tr}\left(\Gamma_0^{-1}\Gamma_0' \partial_j D D\right) = 0$. Introducing now the matrices

$$T_k := D U_k^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\prod_{k'\neq k}(\lambda_{k'} - \sqrt{z})}{\lambda_k + \sqrt{z}} \end{bmatrix}, \quad R_k^+ := R\Gamma_0 U_k, \quad k = 1, ..., n \tag{5.166}$$

which are analytic at $z = \lambda_k^2$ and satisfy $\partial_j D D^{-1} - \partial_j U_k U_k^{-1} = \partial_j T_k T_k^{-1}$ we compute each summand in the right–hand side of (5.165) as

$$\begin{aligned}
&\operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left(R^{-1}R'\Gamma_0 \partial_j T_k T_k^{-1} \Gamma_0^{-1}\right) = \operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left((U_k^{-1}\Gamma_0^{-1}R^{-1})(R'\Gamma_0 U_k)\partial_j T_k T_k^{-1}\right) \\
&\quad = \operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left((U_k^{-1}\Gamma_0^{-1}R^{-1})((R\Gamma_0 U_k)' - R\Gamma_0' U_k - R\Gamma_0 U_k')\partial_j T_k T_k^{-1}\right) \\
&\quad = \underbrace{\operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left((R_k^+)^{-1}(R_k^+)'\partial_j T_k T_k^{-1}\right)}_{=0} - \underbrace{\operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left(\Gamma_0^{-1}\Gamma_0' \delta T_k T_k^{-1}\right)}_{=0} - \operatorname*{res}_{z=\lambda_k^2} \operatorname{tr}\left(U_k^{-1} U_k' \partial_j T_k T_k^{-1}\right) \\
&\quad = - \operatorname*{res}_{z=\lambda_k^2} \frac{1}{z-\lambda_k^2} \left(\partial_j \left(\frac{\prod_{k'\neq k}(\lambda_{k'} - \sqrt{z})}{\lambda_k + \sqrt{z}}\right) \frac{\lambda_k + \sqrt{z}}{\prod_{k'\neq k}(\lambda_{k'} - \sqrt{z})}\right) \\
&\quad = - \operatorname*{res}_{z=\lambda_k^2} \frac{1}{z-\lambda_k^2} \times \begin{cases} \frac{1}{\lambda_j - \sqrt{z}} & \text{if } j \neq k \\ \frac{1}{\lambda_j - \sqrt{z}} & \text{if } j = k \end{cases} = \begin{cases} \frac{1}{\lambda_k - \lambda_j} & \text{if } j \neq k \\ \frac{1}{2\lambda_k} & \text{if } j = k. \end{cases}
\end{aligned} \tag{5.167}$$

From (5.165) we get, after a simple integration,

$$\Omega(\partial_j) = \partial_j \log\left(\frac{\prod_{j=1}^{N} \sqrt{\lambda_j}}{\Delta(\lambda_1, ..., \lambda_N)} \det \mathbf{G}\right). \tag{5.168}$$

In view of (5.154) and (5.32) the proof of Thm. 5.2.6 is complete.

CHAPTER 6

Kontsevich–Penner tau function

The Kontsevich matrix integral can be generalized with addition of a logarithmic term to the cubic potential. The resulting model is called Kontsevich–Penner model, and is a tau function of the modified KP hierarchy. The interest in this model is motivated by a conjecture of Alexandrov, Buryak and Tessler relating it to open intersection numbers. In this chapter we apply the methods developed so far to this model, by identifying the partition function with the isomonodromic tau function of an appropriate 3×3 *system.*

Main references for this chaper are [ABT; BRc].

6.1 The Kontsevich–Penner model and open intersection numbers

The *Kontsevich–Penner matrix integral* is

$$\mathcal{Z}_N^{\mathsf{KP}}(Y;Q) := \frac{\det(\mathrm{i}Y)^Q}{\int_{\mathrm{H}_N} \exp\mathrm{tr}\,(-YM^2)} \int_{\mathrm{H}_N} \frac{\exp\mathrm{tr}\,\left(\frac{\mathrm{i}}{3}M^3 - YM^2\right)}{\det(M+\mathrm{i}Y)^Q}\mathrm{d}M. \tag{6.1}$$

It is a function of the diagonal matrix $Y = \mathrm{diag}(y_1, ..., y_N)$ satisfying $y_j > 0$, so that the integrals are absolutely convergent. We assume for simplicity that Q is integer, but it can be regarded as an arbitrary complex parameter. For $Q = 0$ it reduces to Kontsevich matrix integral (4.21).

It is a deformation of the Kontsevich matrix integral by addition of a logarithmic term in the potential; purely logarithmic potential was first considered by Penner [Pc], whence the name.

It has attracted some interest [Aa; Ac; BHb] due to Conjecture 6.1.5 below, formulated in [ABT], relating this matrix integral to intersection numbers over the moduli spaces of *open* Riemann surfaces in a similar way as the Kontsevich matrix integral (4.21) is related to Witten intersection numbers, see Chap. 4.

Similarly to Lemma 4.2.1 we have the following determinantal expression for the Kontsevich–Penner matrix integral (6.1).

First, introduce the sequence of functions ($Q \in \mathbb{Z}$)

$$\phi(z;Q) := \frac{\mathrm{i}^Q}{\sqrt{2\pi}} \int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\exp\left(\frac{\mathrm{i}x^3}{3} + \mathrm{i}xz\right)}{x^Q}\mathrm{d}x \tag{6.2}$$

The integral is absolutely convergent for any $\epsilon > 0$ and it defines an entire function of z (independent of ϵ).

The functions (6.2) are close relatives of the Airy function (4.22); note that $\phi(z; Q = 0) = \sqrt{2\pi}\mathrm{Ai}(z)$. Moreover, for arbitrary Q they satisfy a third order linear differential equation

$$\phi'''(z;Q) - z\phi'(z;Q) + (Q-1)\phi(z;Q) = 0. \tag{6.3}$$

Combining it with the trivial identity

$$\phi'(z;Q) = -\phi(z;Q-1) \tag{6.4}$$

we obtain

$$\phi(z;Q-3) - z\phi(z;Q-1) - (Q-1)\phi(z;Q) = 0. \tag{6.5}$$

We shall return to the properties of these functions below.

Lemma 6.1.1. *The Kontsevich–Penner matrix integral* (4.21) *can be expressed as follows:*

$$\mathcal{Z}_N^{\mathsf{KP}}(Y;Q) = \sqrt{2^N}\frac{\det\sqrt{Y^{2Q+1}}\exp\mathrm{tr}\,\left(\frac{2}{3}Y^3\right)}{\Delta(Y)}\det\left(\phi^{(j-1)}(y_k^2;Q)\right)_{j,k=1}^N \tag{6.6}$$

where the functions $\phi(z;Q)$ *are defined in* (6.2).

Proof. The gaussian integral in the denominator of (6.1) has already been computed in the proof of Lemma 4.2.1, hence we only report report the result;

$$\int_{\mathrm{H}_N}\exp\mathrm{tr}\,(-YM^2)\mathrm{d}M = \frac{\sqrt{\pi^{N^2}}}{\det\sqrt{Y}}\frac{\Delta(Y)}{\Delta(Y^2)}. \tag{6.7}$$

For the numerator we also proceed analogously to the proof of Lemma 4.2.1; we have

$$\int_{\mathrm{H}_N}\exp\mathrm{tr}\,\left(\frac{\mathrm{i}}{3}M^3 - YM^2 - Q\log(M+\mathrm{i}Y)\right)\mathrm{d}M \tag{6.8}$$

$$\overset{(1)}{=}\exp\left(\frac{2}{3}\mathrm{tr}\,Y^3\right)\int_{H_N}\exp\mathrm{tr}\,\left(\mathrm{i}\frac{M'^3}{3} + \mathrm{i}M'Y^2 - Q\log M'\right)\mathrm{d}M' \tag{6.9}$$

$$\overset{(2)}{=}\frac{1}{N!}\exp\left(\frac{2}{3}\mathrm{tr}\,Y^3\right)\int_{\mathbb{R}^N}\Delta^2(X)\prod_{j=1}^N\frac{\mathrm{e}^{\frac{\mathrm{i}x_j^3}{3}}}{x_j^Q}\mathrm{d}x_j\int_{\mathrm{U}_N/(\mathrm{U}_1^N)}\mathrm{d}U\exp\mathrm{tr}\,\left(\mathrm{i}Y^2UXU^\dagger\right) \tag{6.10}$$

$$\overset{(3)}{=}\frac{\pi^{\frac{N(N-1)}{2}}}{N!}\exp\left(\frac{2}{3}\mathrm{tr}\,Y^3\right)\int_{\mathbb{R}^N}\frac{\Delta(X)\det\left(\mathrm{e}^{\mathrm{i}x_jy_k^2}\right)_{j,k=1}^N}{\Delta(\mathrm{i}Y^2)}\prod_{j=1}^N\frac{\mathrm{e}^{\frac{\mathrm{i}x_j^3}{3}}}{x_j^Q}\mathrm{d}x_j \tag{6.11}$$

$$\overset{(4)}{=}\pi^{\frac{N(N-1)}{2}}\frac{\exp\left(\frac{2}{3}\mathrm{tr}\,Y^3\right)}{\Delta(\mathrm{i}Y^2)}\det\left(\int_{\mathbb{R}}x^{N-j-Q}\exp\left(\frac{\mathrm{i}x^3}{3}+\mathrm{i}xy_k^2\right)\mathrm{d}x\right)_{j,k=1}^N. \tag{6.12}$$

In (1) we perform a shift $M' := M+\mathrm{i}Y$ and an analytic continuation: the integral is now only conditionally convergent, it is absolutely convergent only when understood as integration over $H_n+\mathrm{i}\epsilon\mathbf{1}$ for any $\epsilon>0$. In (2) we apply Weyl integration formula (Prop. B.1.1) and we use the notation $X=\mathrm{diag}(x_1,...,x_n)$. In (3) we apply Harish-Chandra formula (B.11) and in (4) Andreief identity (Lemma B.3.1). The proof is completed by the identity

$$\frac{\mathrm{i}^Q}{\sqrt{2\pi}}\int_{\mathbb{R}}x^{\ell-Q}\exp\left(\frac{\mathrm{i}x^3}{3}+\mathrm{i}xz\right)\mathrm{d}x = \frac{1}{\mathrm{i}^\ell}\phi^{(\ell)}(z) \tag{6.13}$$

which directly follows from (6.2). ■

Asymptotic expansions and open intersection numbers. Similarly to Airy functions, (6.2) admit explicit asymptotic expansions as $z\to\infty$ within $|\arg z|<\pi$

Proposition 6.1.2. *When* $z\to\infty$ *within the sector* $|\arg z|<\pi$ *we have*

$$\phi(z;Q)\sim\frac{\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)}{\sqrt{2}z^{\frac{Q}{2}+\frac{1}{4}}}F_-(z;Q) \tag{6.14}$$

where

$$F_-(z;Q) = 1+\sum_{j\geq1}(-1)^jC_j(Q)z^{-\frac{3j}{2}},\qquad C_j(Q) := \sum_{b=0}^{2j}\frac{(-1)^b}{3^b b!}\binom{-Q}{2j-b}\frac{\Gamma\left(\frac{1}{2}+j+b\right)}{\sqrt{\pi}}. \tag{6.15}$$

Proof. Let us first consider $z\in\mathbb{R}_+$, $z\to+\infty$. According to *Laplace's method* the main contributions to $\phi(z;Q)$ for large z come from the saddles of the exponent $\mathrm{i}\frac{x^3}{3}+\mathrm{i}xz$, provided that the contour of integration can be deformed into the curve of steepest descent through some of the saddles in a neighbourhood of the saddle points. In the present case there are two saddles, $\pm\mathrm{i}\sqrt{z}$. The identity

$$\mathrm{i}\frac{x^3}{3}+\mathrm{i}xz = \mp\frac{2}{3}z^{\frac{3}{2}}\mp\sqrt{z}\left(x\mp\mathrm{i}\sqrt{z}\right)^2+\frac{\mathrm{i}}{3}\left(x\mp\mathrm{i}\sqrt{z}\right)^3 \tag{6.16}$$

shows that the direction of steepest descent at $\mathrm{i}\sqrt{z}$ is horizontal, while at $-\mathrm{i}\sqrt{z}$ is vertical. Hence we deform the contour of integration passing through $\mathrm{i}\sqrt{z}$ along the direction of steepest descent. Using the expansion

$$x^{-Q}\exp\left(\frac{\mathrm{i}}{3}(x-\mathrm{i}\sqrt{z})^3\right)=\sum_{a,b\geq 0}\binom{-Q}{a}\frac{\mathrm{i}^{b-a-Q}}{b!3^b}z^{-\frac{a+Q}{2}}(x-\mathrm{i}\sqrt{z})^{a+3b} \tag{6.17}$$

we find

$$\begin{aligned}\phi(z;Q)&=\frac{\mathrm{i}^Q}{\sqrt{2\pi}}\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)\int_{\mathbb{R}+\mathrm{i}\epsilon}x^{-Q}\exp\left(-\sqrt{z}\left(x-\mathrm{i}\sqrt{z}\right)^2+\frac{\mathrm{i}}{3}\left(x-\mathrm{i}\sqrt{z}\right)^3\right)\mathrm{d}x\sim\\&\sim\frac{\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)}{\sqrt{2\pi}}\sum_{a,b\geq 0}\binom{-Q}{a}\frac{\mathrm{i}^{b-a}}{b!3^b}z^{-\frac{a+Q}{2}}\int_{-\infty}^{+\infty}\exp\left(-\sqrt{z}\xi^2\right)\xi^{a+3b}\mathrm{d}\xi=\\&=\frac{\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)}{\sqrt{2\pi}z^{\frac{Q}{2}+\frac{1}{4}}}\sum_{a,b\geq 0,\ a+b\text{ even}}\binom{-Q}{a}\frac{\mathrm{i}^{b-a}}{b!3^b}\Gamma\left(\frac{1+a+3b}{2}\right)z^{-\frac{3}{4}(a+b)}=\\&=\frac{\exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right)}{\sqrt{2}z^{\frac{Q}{2}+\frac{1}{4}}}\sum_{j\geq 0}(-1)^jC_j(Q)z^{-\frac{3}{2}j}\end{aligned} \tag{6.18}$$

where, in the second line, $\xi=x-\mathrm{i}\sqrt{z}$, in the last step $a+b=2j$ and $C_j(Q)$ is as in (6.15).

The asymptotic expansion holds in the whole sector $|\arg z|<\pi$ by standard arguments that are completely parallel to the well–known case of the Airy functions. Roughly speaking, this follows from the general asymptotic theory of solutions to a linear ODE with rational coefficients, see [Wa]; indeed, since $\phi(z;Q)$ is a solution to such an ODE, see (6.3), the asymptotic expansion is valid in a sector including the positive real semiaxis. In principle, this sector is [1] $|\arg z|<\frac{\pi}{6}$, but as that $\phi(z;Q)$ is subleading with respect to all other solutions, its asymptotic expansion must be valid also across the Stokes's lines $\arg z=\pm\frac{\pi}{6}$. For more details see loc. cit. ∎

Remark 6.1.3. *In different sectors (e.g. in $\pi<\arg z<3\pi$) a formal analytic continuation of the expression in the right-hand side of (6.14) is needed, so we shall consider also the power series*

$$F_+(z;Q)=1+\sum_{j\geq 1}C_j(Q)z^{-\frac{3j}{2}}. \tag{6.19}$$

Remark 6.1.4. *As a corollary of the recurrence relation (6.5) we obtain the following recurrence relation for the formal series $F_\pm(z;Q)$:*

$$F_\pm(z;Q-2)-F_\pm(z;Q)\pm Qz^{-\frac{3}{2}}F_\pm(z;Q+1)=0. \tag{6.20}$$

From the expression of Lemma 6.1.1 we obtain that we have an asymptotic expansion

$$\mathcal{Z}_N^{\mathsf{KP}}(Y)\sim\tau_N^f \tag{6.21}$$

where, with the notation of Sec. 1.4.3, we set

$$\tau_N^f(z_1,...,z_N):=\frac{\det\left(f_j(z_k)\right)_{j,k=1}^N}{\det\left(z_k^{j-1}\right)_{j,k=1}^N} \tag{6.22}$$

(compare with (1.141)) where $f=(f_j)_{j\geq 1}\in\mathrm{Gr}_0^{\frac{\infty}{2}}$ with the formal series $f_j(z)=z^{j-1}(1+\mathcal{O}(z^{-1}))$ defined by

$$\phi^{(j-1)}(z^2)\sim(-1)^{j-1}\frac{\mathrm{e}^{-\frac{2}{3}z^3}}{\sqrt{2z^{2Q+1}}}f_j(z) \tag{6.23}$$

as $z\to\infty$ within $|\arg z|<\frac{\pi}{2}$. We recall from Sec. 1.4.3 that $\tau_N^f(z_1,...,z_N)$ gives a well defined limit $\tau^f(\mathbf{t})$. It is called *Kontsevich–Penner tau function.*

[1]Since, up to a shearing transformation, as in Rem. 1.4.2, the Poincaré rank at ∞ in the square root variable $\sqrt{z}$ is 3.

We now briefly introduce the conjecture of [ABT], which has been proposed as an analogue of the Witten–Kontsevich theorem, Thm. 4.1.5, to explain the algebro-geometric and combinatorial meaning of the coefficients of the Kontsevich–Penner tau function. To formulate it, consider rescaled times $\mathbf{T} = (T_1, T_2, ...)$ defined by

$$T_k := (-1)^k\, k!!\, 2^{-\frac{k}{3}}\, t_k \tag{6.24}$$

where we have used the double factorial $k!! := \prod_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j)$ for any nonnegative integer k. Then the expansion

$$\log \tau^f(\mathbf{T}) = \frac{T_1^3}{6} + \left(\frac{1}{24} + \frac{Q^2}{2}\right) T_3 + QT_1T_2 + \frac{Q^2}{2} T_2T_4 + \left(\frac{1}{24} + \frac{Q^2}{2}\right) T_3^2 + QT_1^2T_4 + \cdots \tag{6.25}$$

is a deformation of the generating function (4.13) of Witten intersection numbers (4.4), up to a change of variables; the T_{2k+1}'s of this chapter are the T_k's of Chap. 4 ($k = 0, 1, 2, ...$), the even variables $T_2, T_4, ...$ of the present chapter have no analogue in Chap. 4. Define the polynomials in Q with rational coefficients

$$\langle \tau_{\frac{d_1}{2}} \cdots \tau_{\frac{d_n}{2}} \rangle_{\mathsf{open}} \in \mathbb{Q}[Q] \tag{6.26}$$

for all $n \geq 1$ and $d_j \geq 0$ by the expansion

$$\log \tau^f(\mathbf{T}) = \sum_{n\geq 1} \sum_{d_1, \cdots, d_n \geq 0} \langle \tau_{\frac{d_1}{2}} \cdots \tau_{\frac{d_n}{2}} \rangle_{\mathsf{open}} \frac{T_{d_1+1} \cdots T_{d_n+1}}{n!}. \tag{6.27}$$

Conjecture 6.1.5. *The polynomials* (6.26) *are the* open intersection numbers*.*

The open intersection numbers [PST; T; BT] are a generalization of Witten intersection numbers(4.4); the generalization consists in considering moduli spaces of *open* Riemann surfaces (by definition, compact connected Riemann surfaces with a finite number of holomorphically embedded open disks removed) with n marked points, which may belong to the boundary.

Rigorous definition of open intersection numbers is a challenging topic, whose grounds were laid in [PST]. The main difficulties arise because such moduli spaces are non-compact *real* orbifolds, hence their compactifications produce real boundaries and then integration of cohomology classes is in general an ill-defined problem as it stands. Moreover, such moduli spaces are in general not orientable.

A more detailed introduction to the topic of open intersection numbers, which is a very recent and active field of research, goes beyond the possibilities of this thesis. Let us just comment on the fact that they are polynomials in Q defined in terms of a combinatorial formula [T], the coefficient in front of Q^b representing the contribution from the moduli space of open Riemann surfaces with b boundary components.

Remark 6.1.6. *In [Sa] the author provides an alternative construction of the open intersection numbers and proves that their generating function is precisely* $\log \mathcal{Z}_N^{\mathsf{KP}}(\mathbf{t}; Q)$*. The relation between the two definitions is still not clear.*

Let us also mention that the conjecture, which reduces to Witten–Kontsevich theorem (Thm. 4.1.5) for $Q = 0$, has been proven by Alexandrov [Aa] in the case $Q = 1$ (where the intersection numbers are not weighted by the number of boundary components); the proof exploits a formula of Buryak for open intersection numbers [Bg].

6.2 Isomonodromic method

The bare system. For a reason that will be clear below, we shall start from the *formal adjoint*[2] of the ODE (6.3), namely

$$u'''(z) - zu(z) - Qu(z) = 0. \tag{6.28}$$

Note that $u(z) = \phi(z; 1-Q)$ solves (6.28).

[2]Let us remind that the (formal) adjoint of a linear differential operator $\sum_{k=0}^{K} a_k(z)\partial_\lambda^k$ is $\sum_{k=0}^{K} (-\partial_z)^k a_k(z)$.

We need to introduce another linearly independent solution of (6.28). To this end we consider the case $Q \geq 0$ only; the other case $Q < 0$ is completely similar. Let us define

$$g(z;0) := 1, \quad g(z;Q) := \frac{(-\mathrm{i})^Q}{\Gamma(Q)} \int_0^{+\infty \mathrm{e}^{\mathrm{i}\epsilon}} x^{Q-1} \exp\left(\frac{\mathrm{i}x^3}{3} + \mathrm{i}xz\right) \mathrm{d}x, \qquad Q = 1, 2, \dots. \tag{6.29}$$

(Note that $g(z;0)$ is also the limit as $Q \to 0$ of $g(z;Q)$). The integral is absolutely convergent for any $0 < \epsilon < \frac{\pi}{3}$ so it defines an entire function of z (independent of ϵ). Using integration by parts it is easy to check that $u(z) = g(z;Q)$ satisfies the ODE (6.28).

Proposition 6.2.1. *When $z \to \infty$ within the sector $-\frac{\pi}{3} < \arg z < \pi$*

$$g(z;Q) \sim z^{-Q}\left(1 + \mathcal{O}\left(z^{-3}\right)\right). \tag{6.30}$$

Proof. Use the Cauchy theorem to rotate the contour

$$g(z;Q) = \frac{(-\mathrm{i})^Q}{(Q-1)!} \int_0^{+\infty} x^{Q-1}\, \mathrm{e}^{\frac{\mathrm{i}x^3}{3}}\, \mathrm{e}^{\mathrm{i}zx}\, \mathrm{d}x \tag{6.31}$$

(now the integral is only conditionally convergent). The series expansion $x^{Q-1} \exp \frac{\mathrm{i}x^3}{3} = \sum_{a \geq 0} \frac{\mathrm{i}^a}{3^a a!} x^{3a+Q-1}$ together with Watson's lemma (see e.g. [Od]) gives

$$g(z;Q) \sim \frac{(-\mathrm{i})^Q}{(Q-1)!} \sum_{a \geq 0} \frac{i^a}{3^a a!} \Gamma(3a+Q)(-\mathrm{i}z)^{-3a-Q}, \quad -\frac{\pi}{2} < \arg(-\mathrm{i}z) < \frac{\pi}{2} \tag{6.32}$$

Rotating the contour of integration within the sector $0 < \arg x < \frac{\pi}{3}$ we infer that the above asymptotic expansion holds in the bigger sector $-\frac{\pi}{3} < \arg z < \pi$. ∎

Fix three angles $\beta_\pm, \beta_0$ such that

$$-\pi < \beta_- < -\frac{\pi}{3}, \quad -\frac{\pi}{3} < \beta_0 < \frac{\pi}{3}, \quad \frac{\pi}{3} < \beta_+ < \pi \tag{6.33}$$

and define four sectors I, II, III, IV in the complex z-plane, with $-\pi < \arg z < \pi$, as follows

$$\begin{gathered} z \in I \iff -\pi < \arg z < \beta_-, \quad z \in II \iff \beta_- < \arg z < \beta_0, \\ z \in III \iff \beta_0 < \arg z < \beta_+, \quad z \in IV \iff \beta_+ < \arg z < \pi. \end{gathered} \tag{6.34}$$

Let $\Sigma := \mathbb{R}_- \sqcup \left(\bigsqcup_{j \in \{0,\pm\}} \mathrm{e}^{\mathrm{i}\beta_j}\mathbb{R}_+\right)$ be the oriented contour delimiting the sectors $I, ..., IV$, as in figure 6.1. Let $\omega := \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$, $\nabla := \begin{pmatrix} 1 & \partial_z & \partial_z^2 \end{pmatrix}^\top$, and define

$$\Psi(z) := \begin{cases} \left(\omega^{-Q}\, \nabla g(\omega^{-1}z;Q) \,\middle|\, \omega^{-Q}\, \nabla\phi(\omega^{-1}z; 1-Q) \,\middle|\, \mathrm{i}\omega^{-\frac{Q}{2}}\, \nabla\phi(\omega z; 1-Q) \right) & z \in I \\ \left(\omega^{Q}\, \nabla g(\omega z;Q) \,\middle|\, -\nabla\phi(\lambda; 1-Q) \,\middle|\, \mathrm{i}\omega^{-\frac{Q}{2}}\, \nabla\phi(\omega\lambda; 1-Q) \right) & z \in II \\ \left(\nabla g(z;Q) \,\middle|\, -\nabla\phi(\lambda; 1-Q) \,\middle|\, -\mathrm{i}\omega^{\frac{Q}{2}}\, \nabla\phi(\omega^{-1}\lambda; 1-Q) \right) & z \in III \\ \left(\omega^{-Q}\, \nabla g(\omega^{-1}z;Q) \,\middle|\, \omega^{Q}\, \nabla\phi(\omega\lambda; 1-Q) \,\middle|\, -\mathrm{i}\omega^{\frac{Q}{2}}\, \nabla\phi(\omega^{-1}\lambda; 1-Q) \right) & z \in IV. \end{cases} \tag{6.35}$$

Consider the matrix form

$$\Psi'(z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ Q & z & 0 \end{pmatrix} \Psi(z). \tag{6.36}$$

Proposition 6.2.2. *1. $\Psi(z)$ solves (6.36) in all sectors $I, ..., IV$.*

2. $\Psi(z)$ has the same asymptotic expansion in all sectors $I, ..., IV$

$$\Psi(z) \sim z^S G \left(\mathbf{1} + \mathcal{O}\left(z^{-\frac{1}{2}}\right)\right) z^L \mathrm{e}^{\vartheta(z)} \tag{6.37}$$

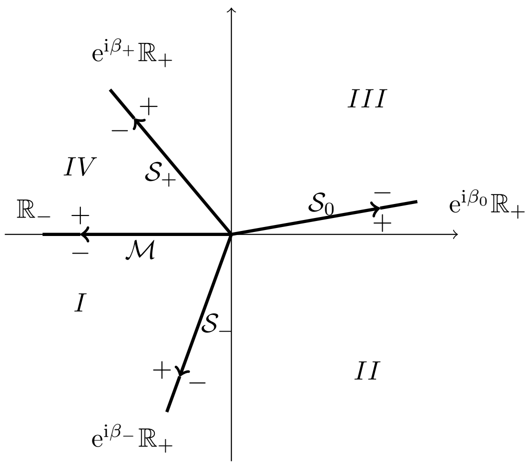


Figure 6.1: Jump $\widetilde{M}$ of Ψ along Σ: $\Psi_+ = \Psi_- \widetilde{M}$.

where S, G, L, ϑ are defined as

$$\begin{gathered} S := \operatorname{diag}\left(-\frac{1}{2}, -\frac{1}{2}, 0\right), \quad G := \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \\ L := \operatorname{diag}\left(-Q + \frac{1}{2}, \frac{Q}{2} + \frac{1}{4}, \frac{Q}{2} + \frac{1}{4}\right), \quad \vartheta(z) := \operatorname{diag}\left(0, -\frac{2}{3}z^{\frac{3}{2}}, \frac{2}{3}z^{\frac{3}{2}}\right). \end{gathered} \tag{6.38}$$

3. $\Psi(z)$ *satisfies a jump condition along* Σ

$$\Psi_+(z) = \Psi_-(z)\widetilde{M}, \qquad z \in \Sigma \tag{6.39}$$

where boundary values are taken with respect to the orientation of Σ *shown in figure 6.1 and* $\widetilde{M} : \Sigma \to \mathrm{SL}(3, \mathbb{C})$ *is piecewise defined as*

$$\begin{gathered} \widetilde{M} := \begin{cases} \mathcal{S}_{0,\pm}, & z \in \mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+ \\ \mathcal{M}, & z \in \mathbb{R}_- \end{cases} \\ \mathcal{S}_- := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{(-1)^Q\sqrt{2\pi}}{\Gamma(Q)} & \mathrm{i}(-1)^Q & 1 \end{pmatrix}, \quad \mathcal{S}_0 := \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\mathrm{i}\sqrt{2\pi}}{\Gamma(Q)} & 1 & \mathrm{i}(-1)^Q \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathcal{S}_+ := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{(-1)^Q\sqrt{2\pi}}{\Gamma(Q)} & \mathrm{i}(-1)^Q & 1 \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\mathrm{i}(-1)^Q \\ 0 & -\mathrm{i}(-1)^Q & 0 \end{pmatrix}. \end{gathered} \tag{6.40}$$

4. The identity $\det \Psi(z) \equiv 1$ *holds identically in all sectors.*

Proof. The differential equation follows from the discussion above. The asymptotic expansion (6.37) follows by analytic continuation of the expansions (6.14) and (6.30). For the jump use the following identities, consequence of the Cauchy Theorem

$$\phi(z; 1-Q) + \omega^Q \phi(\omega z; 1-Q) + \omega^{-Q}\phi(\omega^{-1}z; 1-Q) = 0, \tag{6.41}$$

$$g(z; Q) - \omega^Q g(\omega z; Q) = -\frac{\mathrm{i}\sqrt{2\pi}}{\Gamma(Q)}\phi(z; 1-Q). \tag{6.42}$$

Finally, $\det \Psi(z)$ is constant in z as it follows from the fact that the connection (6.36) is traceless; moreover it tends to 1 at $z = \infty$. ■

In the terminology of linear complex ordinary differential equations (reviewed in Sec. 1.4.2) $\mathcal{S}_{\pm,0}$ are the Stokes matrices (note their triangular structure) and $\mathcal{M}$ the formal monodromy of the singularity

$z = \infty$ of (6.36). Notice the *no-monodromy condition* $\mathcal{M}\mathcal{S}_+\mathcal{S}_0\mathcal{S}_- = \mathbf{1}$. Note also that we are in a non-generic case in the sense explained in Sec. 1.4.2, as the eigenvalues of the leading order at ∞ of the connection matrix in (6.36) are all 0; this explains the appearance of non-integer powers in the asymptotic expansions of solutions, compare with Rem. 1.4.2.

Remark 6.2.3. *Note that for $Q = 0$ the jump matrices in (6.40) are block-diagonal of the form* $\left(\begin{array}{c|c} 1 & \\ \hline & \widetilde{M}_0 \end{array}\right)$ *where $\widetilde{M}_0$ are the jump matrices (4.45). This illustrates the reduction to the Kontsevich–Witten case of Chap. 4 from the RHP point of view.*

Extension of the Kontsevich–Penner matrix integral to all sectors. We have seen that $\mathcal{Z}_N^{\mathsf{KP}}(Y;Q)$ admits a regular asymptotic expansion for large Y when $\operatorname{Re} y_j > 0$. As $\varphi(z;Q)$ are entire functions we could try to analytically continue $\mathcal{Z}_N^{\mathsf{KP}}(Y;Q)$ to the region $\operatorname{Re} y_j < 0$ via the right-hand side of (6.6). However, this would result in the fact that $\mathcal{Z}_N^{\mathsf{KP}}(Y;Q)$ does not admit a regular asymptotic expansion in the region where some $\operatorname{Re} y_j < 0$.

It is convenient for our purposes to have a regular expansion near infinity also in the sector $\operatorname{Re} y_j < 0$ (and, in fact, the *same* expansion), therefore we need to consider the following extension of $\mathcal{Z}_N^{\mathsf{KP}}(Y;Q)$. To this end we start from the representation of Lemma 6.1.1 in terms of the function $\varphi(z,N)$ defined in (6.2); in the left plane we replace them by other solution to the ODE (6.3) in appropriate way so as to preserve the regularity of the asymptotic expansion. The logic is completely parallel to the one used in [BCa] (and reviewed in the previous chapter) and is forced on us by the Stokes' phenomenon of the solutions to the ODE (6.36), which is closely related to the Airy differential equation of the previous chapter.

Definition 6.2.4. *We order the variables y_j so that $\operatorname{Re} y_j > 0$ for $j = 1,...,n_1$ and $\operatorname{Re} y_j < 0$ for $j = n_1+1,...,n_1+n_2 = N$. We denote $\vec{\lambda} = (\lambda_1,...,\lambda_{n_1})$ and $\vec{\mu} = (\mu_1,...,\mu_{n_2})$ with $y_j = \sqrt{\lambda_j}$ for $j = 1,...,n_1$ and $y_{n_1+j} = -\sqrt{\mu_j}$ for $j = 1,....,n_2$, all roots being principal. We define the extended Kontsevich–Penner partition function by the expression*

$$\mathcal{Z}_N^{\mathsf{KP}}(\vec{\lambda},\vec{\mu};Q) := 2^{\frac{N}{2}} \mathrm{e}^{U(\vec{\lambda},\vec{\mu})}\, \Delta(\vec{\lambda},\vec{\mu};Q)\, \det \left(\begin{array}{c} \left(\omega^{Q+\frac{1}{2}} \phi^{(j-1)}(\omega^{-1}\lambda_k;Q)\right)_{1\le k\le n_1,\ \lambda_k\in I} \\ \hline \left(\phi^{(j-1)}(\lambda_k;Q)\right)_{1\le k\le n_1,\ \lambda_k\in II\cup III} \\ \hline \left(\omega^{-Q-\frac{1}{2}} \phi^{(j-1)}(\omega\lambda_k;Q)\right)_{1\le k\le n_1,\ \lambda_k\in IV} \\ \hline \left(\omega^{\frac{Q}{2}+\frac{1}{4}} \phi^{(j-1)}(\omega\mu_k;Q)\right)_{n_1+1\le k\le n,\ \mu_k\in I\cup II} \\ \hline \left(\omega^{-\frac{Q}{2}-\frac{1}{4}} \phi^{(j-1)}(\omega^{-1}\mu_k;Q)\right)_{n_1+1\le k\le n,\ \mu_k\in III\cup IV} \end{array} \right)_{1\le j\le n} \tag{6.43}$$

where

$$U(\vec{\lambda};\vec{\mu}) := \frac{2}{3}\sum_{j=1}^{n_1} \lambda_j^{\frac{3}{2}} - \frac{2}{3}\sum_{j=1}^{n_2} \mu_j^{\frac{3}{2}} \tag{6.44}$$

and

$$\Delta(\vec{\lambda},\vec{\mu};Q) := \frac{\prod_{j=1}^{n_1} \lambda_j^{\frac{1}{4}+\frac{Q}{2}} \prod_{j=1}^{n_2} (-\mu_j)^{\frac{1}{4}+\frac{Q}{2}}}{\prod_{1\le j<k\le n_1} \left(\sqrt{\lambda_k}-\sqrt{\lambda_j}\right) \prod_{1\le j<k\le n_2} \left(\sqrt{\mu_j}-\sqrt{\mu_k}\right) \prod_{j=1}^{n_1}\prod_{k=1}^{n_2} \left(\sqrt{\lambda_j}+\sqrt{\mu_k}\right)}. \tag{6.45}$$

We deduce that $\mathcal{Z}_N^{\mathsf{KP}}(\vec{\lambda},\vec{\mu};Q)$ as defined in (6.43) has a regular asymptotic expansion when $\lambda_j,\mu_j \to \infty$ in the indicated sectors. This regular asymptotic expansion coincides with the already discussed regular asymptotic expansion of $\mathcal{Z}_N^{\mathsf{KP}}(Y;Q)$ for $\operatorname{Re} y_k = \operatorname{Re}\sqrt{\lambda_k} \ge 0$. As analytic functions, $\mathcal{Z}_N^{\mathsf{KP}}(\vec{\lambda},\vec{\mu};Q) = \mathcal{Z}_N^{\mathsf{KP}}(Y;Q)$ provided that $n_2 = 0$, $\lambda_k \in II\cup III$ and $y_k = \sqrt{\lambda_k}$ for all $k = 1,..,N$.

We point out that the definition (6.43) depends not only on the belonging of y_j to the left/right half-planes but also on the placement of the boundaries between the sectors I–IV, i.e. on the angles $\beta_0, \beta_\pm$ in (6.33). If we move the boundaries within the bounds of (6.33) then this yields different functions $\mathcal{Z}_N^{\mathsf{KP}}(\vec{\lambda},\vec{\mu};Q)$ but all admitting the same asymptotic expansion as $\vec{\lambda},\vec{\mu}$ tend to infinity within the respective sectors. We opted to leave this dependence on the sectors understood.

Rational dressing. Similarly as in Chap. 4 we fix points (compare with the paragraph above) $\vec{\lambda} = (\lambda_1, ..., \lambda_{n_1})$ and $\vec{\mu} = (\mu_1, ..., \mu_{n_2})$ and the matrix

$$D(z; \vec{\lambda}, \vec{\mu}) := \operatorname{diag}(\alpha, \pi_+, \pi_-)$$
$$\alpha := \prod_{j=1}^{n_1} \sqrt{\lambda_j} \prod_{j=1}^{n_2} \sqrt{\mu_j}, \qquad \pi_\pm := \prod_{j=1}^{n_1} \left(\sqrt{\lambda_j} \pm \sqrt{z}\right) \prod_{j=1}^{n_2} \left(\sqrt{\mu_j} \mp \sqrt{z}\right) \tag{6.46}$$

and $J : \Sigma \to \mathrm{SL}(3, \mathbb{C})$

$$J := \left(D^{-1} \,\mathrm{e}^{\vartheta}\right)_- \, \widetilde{M} \, \left(\mathrm{e}^{-\vartheta} \, D\right)_+ \tag{6.47}$$

$\widetilde{M}$ and the notation $\pm$ for boundary values being as in (6.40).

The boundary value specifications $\pm$ in (6.47) give different values along the cut $\mathbb{R}_-$ only. In particular it is easy to check that $J|_{\mathbb{R}_-}$ does not depend on $\vec{\lambda}, \vec{\mu}$. The angles $\beta_{0,\pm}$ can be chosen so that none of zeros of D occur along the three rays $\mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+$.

The construction is such that along the three rays $\mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+$ the jump matrix J is exponentially close to the identity matrix; $J(z) = \mathbf{1} + \mathcal{O}\left(z^{-\infty}\right)$ as $z \to \infty$.

We now formulate the dressed RHP. In the interest of simpler notations, we drop the dependence on N below.

RHP 6.2.5. *Find a* $\mathrm{Mat}(3, \mathbb{C})$*-valued function* $\Gamma = \Gamma(z; \vec{\lambda}, \vec{\mu})$ *analytic in* $z \in \mathbb{C} \setminus \Sigma$*, admitting non-tangential boundary values* $\Gamma_\pm$ *at* Σ *(as in figure 6.1) such that*

$$\begin{cases} \Gamma_+(z) = \Gamma_-(z) J(z) & z \in \Sigma \\ \Gamma(z) \sim z^S \, G \, Y(z) \, z^L & \lambda \to \infty \end{cases} \tag{6.48}$$

where S, G, L *are as in* (6.38)*,* J *as in* (6.47) *and* $Y(z)$ *a formal power series in* $z^{-\frac{1}{2}}$ *satisfying the normalization*

$$Y(z) = \mathbf{1} + \begin{pmatrix} 0 & a & -a \\ 0 & c & c \\ 0 & -c & -c \end{pmatrix} z^{-\frac{1}{2}} + \mathcal{O}(z^{-1}). \tag{6.49}$$

We will see that the existence of the solution to the RHP 6.2.5 depends on the non-vanishing of a function of $\vec{\lambda}, \vec{\mu}$ which is (restriction of an) entire function. Hence the *Malgrange divisor* (see Chap. 2), i.e. the locus in the parameter space where the problem is unsolvable, is really a divisor and the problem is generically solvable.

Remark 6.2.6. *We observe that we can analytically continue* $\Gamma|_{IV}$ *beyond* $\arg z = \pi$ *so that the asymptotic expansion* $\Gamma \sim z^S G Y z^L$ *remains valid in a sector up to* $\arg z = \pi + \epsilon$*. Similarly said for* $\Gamma|_I$*, in a sector from* $\arg z = -\pi - \epsilon$*. By matching the expansions in the overlap sector, we obtain*

$$z^S \,\mathrm{e}^{2\pi \mathrm{i} S} \, G \, Y(\lambda \mathrm{e}^{2\pi \mathrm{i}}) \, z^L \,\mathrm{e}^{2\pi \mathrm{i} L} = z^S \, G \, Y(z) \, z^L \, \mathcal{M}. \tag{6.50}$$

By trivial algebra (6.50) *implies the following symmetry relation for the formal power series* $Y_n(z)$

$$Y(z\mathrm{e}^{2\pi \mathrm{i}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.51}$$

In terms of the coefficients of the expansion of Y*, we find that the coefficients of the fractional powers must be odd under the conjugation* (6.51)*, while those of the integer powers must be even. In particular this implies the following form for* Y

$$Y(z) = \mathbf{1} + \begin{pmatrix} 0 & a & -a \\ b & c & d \\ -b & -d & -c \end{pmatrix} z^{-\frac{1}{2}} + \mathcal{O}(z^{-1}) \,. \tag{6.52}$$

Remark 6.2.7. *The normalization condition (6.49) is necessary to ensure the uniqueness of the solution to RHP 6.2.5. To explain this, consider the identity*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{pmatrix} z^S G = z^S G \left(\mathbf{1} + \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\alpha}{\sqrt{2}} & -\frac{\beta}{2} & -\frac{\beta}{2} \\ \frac{\alpha}{\sqrt{2}} & \frac{\beta}{2} & \frac{\beta}{2} \end{pmatrix} \frac{1}{\sqrt{z}} \right). \tag{6.53}$$

This identity shows that the simple requirement $Y(z) = \mathbf{1} + \mathcal{O}\left(z^{-\frac{1}{2}}\right)$ leaves the freedom of multiplying on the left by the two-parameter family of matrices indicated in (6.53). The normalization (6.49) fixes uniquely the gauge arbitrariness implied by (6.53).

The extended Kontsevich–Penner partition function as the isomonodromic tau function. We can interpret the RHP 6.2.5 as an isomonodromic deformation problem. Indeed by construction it amounts to consider the rational connection on the Riemann sphere with an irregular singularity at ∞ with the same Stokes' phenomenon as the bare system, and N Fuchsian singularities with trivial monodromies. This connection is unique if any. The dependence on the parameters $\vec{\lambda}, \vec{\mu}$ is contrained by the isomonodromic equations reviewed in Sec. 1.4.2.

We explain this point a bit more in detail. The matrix $\Psi := \Gamma D^{-1} \mathrm{e}^{\vartheta}$ satisfies a jump condition on Σ which is independent of z and of the parameters $\vec{\lambda}, \vec{\mu}$. Hence the ratios $\Psi' \Psi^{-1} =: L$ and $\delta\Psi\Psi^{-1} = \mathcal{M}$ have no discontinuities along Σ and are rational functions by Liouville theorem; then the system $\Psi' = L\Psi$ and $\delta\Psi = \mathcal{M}\Psi$ is an isomonodromic system in the sense explained in Sec. 1.4.2; it has a fixed Stokes' phenomenon at ∞ and N Fuchsian singularities of trivial monodromy at the points $\vec{\lambda}, \vec{\mu}$.

Following the considerations of Chap. 2 we define the tau function of this isomonodromic system as

$$\delta \log \tau = \Omega, \qquad \delta := \sum_{i=1}^{n_1} \mathrm{d}\lambda_i \frac{\partial}{\partial \lambda_i} + \sum_{i=1}^{n_2} \mathrm{d}\mu_i \frac{\partial}{\partial \mu_i} \tag{6.54}$$

in terms of the Malgrange differential

$$\Omega := \int_\Sigma \mathrm{tr}\, \left(\Gamma^{-1} \Gamma' \delta J J^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{6.55}$$

for the RHP 6.2.5.

Due to the construction of this RHP it is clear that the considerations of Thm. 2.4.8 can be applied. In particular we have obtained the following result.

Theorem 6.2.8 ([BRc]). *The isomonodromic tau function (6.54) coincides with the extended Kontsevich–Penner partition function, i.e.*

$$\delta \mathcal{Z}_N^{\mathsf{KP}}(\vec{\lambda}; \vec{\mu}; Q) = \Omega. \tag{6.56}$$

The proof is postponed to Sec. 6.4. It is reported as again there are some substantial modifications with respect to the general Thm. 2.4.8, due to the different normalization at $z = \infty$ and the formulation of the RHP in terms of the square root variable $\sqrt{z}$.

6.3 Applications

Limiting RHP. The products $\pi_\pm$ in (6.46) can be rewritten formally as

$$\frac{\alpha}{\pi_+} = \exp \sum_{k \geq 1} (-1)^k t_k z^{\frac{k}{2}}, \qquad \frac{\alpha}{\pi_-} = \exp \sum_{k \geq 1} t_k z^{\frac{k}{2}} \tag{6.57}$$

where α is as in (6.46) and we have introduced *Miwa variables* $\mathbf{t} = (t_1, t_2, \ldots)$

$$t_k(\vec{\lambda}, \vec{\mu}) := \frac{1}{k} \sum_{j=1}^{n_1} \left(\frac{1}{\sqrt{\lambda_j}} \right)^k + \frac{1}{k} \sum_{j=1}^{n_2} \left(\frac{1}{-\sqrt{\mu_j}} \right)^k = \frac{1}{k} \sum_{j=1}^{n} \frac{1}{y_j^k} = \frac{1}{k} \mathrm{tr}\, Y^{-k}. \tag{6.58}$$

Consequently, the matrix D_n can be rewritten formally as

$$D_n^{-1} = \alpha^{-1} \exp\left(\sum_{k\geq 1} t_k \sqrt{z^k}\theta_k\right), \qquad \theta_k := \mathrm{diag}(0, (-1)^k, 1). \tag{6.59}$$

More precisely, the expression above is actually convergent for $|z| < \min\{|\lambda_j|, |\mu_j|\}$.

Note that D_n acts by conjugation on the jumps of Γ_n and hence the scalar constant α in (6.59) is irrelevant. In the limit $n \to \infty$ we can formally consider the variables $t_1, t_2, ...$ as independent. We then arrive at a (formal) limit of the RHP (dropping α as explained above) for the matrix

$$\Psi(z;\mathbf{t}) = \Gamma(z;\mathbf{t})\mathrm{e}^{\Xi(z;\mathbf{t})}, \qquad \Xi(z;\mathbf{t}) := \sum_{k\geq 1}\left(t_k + \frac{2}{3}\delta_{k,3}\right)\sqrt{z^k}\theta_k. \tag{6.60}$$

Consequently, the matrix $\Gamma(z;\mathbf{t})$ solves a new RHP as follows:

RHP 6.3.1. *Let $\mathbf{t}$ denote the infinite set of variables $\mathbf{t} = (t_1, t_2, ...)$. The formal RHP amounts to finding a 3×3 analytic matrix-valued function $\Gamma = \Gamma(z;\mathbf{t})$ in $z \in \mathbb{C}\setminus\Sigma$ admitting non-tangential boundary values $\Gamma_\pm$ at Σ such that*

$$\begin{cases} \Gamma_+(z;\mathbf{t}) = \Gamma_-(z;\mathbf{t})J(z;\mathbf{t}) & z \in \Sigma \\ \Gamma(z;\mathbf{t}) \sim z^S G Y(z;\mathbf{t}) z^L & z \to \infty \end{cases} \tag{6.61}$$

where $J(z;\mathbf{t}) := \mathrm{e}^{\Xi(z;\mathbf{t})_-}\widetilde{M}\mathrm{e}^{-\Xi(z;\mathbf{t})_+}$, $\widetilde{M}$ as in (6.40), and $Y(z;\mathbf{t})$ is a formal power series in $z^{-\frac{1}{2}}$ satisfying the normalization

$$Y(z;\mathbf{t}) = \mathbf{1} + \begin{bmatrix} 0 & a & -a \\ 0 & c & -c \\ 0 & c & -c \end{bmatrix} z^{-\frac{1}{2}} + \mathcal{O}\left(z^{-1}\right) \tag{6.62}$$

for some functions $a = a(\mathbf{t}), c = c(\mathbf{t})$.

Remark 6.3.2. *Remark 6.2.7 applies here as well for the uniqueness of the solution to the RHP 6.3.1. Moreover, the symmetry relation (6.51) holds true similarly here, namely*

$$Y(z\mathrm{e}^{2\pi\mathrm{i}};\mathbf{t}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} Y(z;\mathbf{t}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{6.63}$$

We now explain a meaningful setup where the RHP 6.3.1 can be given a completely rigorous analytic meaning. The driving idea is that of truncating the time variables to some finite (odd) number.

Fix now $K \in \mathbb{N}$ and assume that $t_\ell = 0$ for all $\ell \geq 2K+2$. Set $\mathbf{t} = (t_1, \dots, t_{2K+1}, 0, \dots)$ with $t_{2K+1} \neq 0$. In addition, the angles $\beta_{0,\pm}$ (satisfying (6.33)) and the argument of t_{2K+1} must satisfy the following condition:

$$\begin{cases} \mathrm{Re}\left(z^{\frac{2K+1}{2}} t_{2K+1}\right) < 0, & z \in \mathrm{e}^{\mathrm{i}\beta_\pm}\mathbb{R}_+ \\ \mathrm{Re}\left(z^{\frac{2K+1}{2}} t_{2K+1}\right) > 0, & z \in \mathrm{e}^{\mathrm{i}\beta_0}\mathbb{R}_+. \end{cases} \tag{6.64}$$

Under this assumption, given the particular triangular structure of the Stokes matrices $\mathcal{S}_{0,\pm}$, the jumps $M = \mathrm{e}^{\Xi_-}\widetilde{M}\mathrm{e}^{-\Xi_+}$ are exponentially close to the identity matrix along the rays $\mathrm{e}^{\mathrm{i}\beta_{0,\pm}}\mathbb{R}_+$.

Formulæ for open intersection numbers. The Malgrange differential of the limiting RHP described above can be expressed as a(n isomonodromic) tau differential. This follows directly from the considerations of Sec. 2.5. Proceeding exactly as in the proof of Thm. 1.2.2 we have obtained the following formulæ for the (conjectural) *open intersection numbers*.

To formulate them introduce $P^j_{a,b}(Q)$ (polynomials in Q, $a, b = 0, \pm 1$, $j = 0, 1, 2, ...$) by the generating functions[3]

$$\sum_{m\geq 0} P_{a,b}^{2m}(Q)\frac{Z^m}{\left(\frac{1+a-b}{2}\right)_{3m}} = \mathrm{e}^{\frac{Z}{3}}\,{}_2F_2\left(\begin{matrix}\frac{1-a-b-2Q}{2} & \frac{1+a+b+2Q}{2}\\ \frac{1}{2} & \frac{1+a-b}{2}\end{matrix}\middle| -\frac{Z}{4}\right)$$

$$\sum_{m\geq 0} P_{a,b}^{2m+1}(Q)\frac{Z^m}{\left(\frac{2+a-b}{2}\right)_{3m+1}} = -\frac{2Q+a+b}{2}\mathrm{e}^{\frac{Z}{3}}\,{}_2F_2\left(\begin{matrix}\frac{2-a-b-2Q}{2} & \frac{2+a+b+2Q}{2}\\ \frac{3}{2} & \frac{2+a-b}{2}\end{matrix}\middle| -\frac{Z}{4}\right) \tag{6.65}$$

and the matrix

$$R(x) := \sum_{j\geq 0}\begin{bmatrix} Q\,P_{1,-1}^{j}(Q)x^{-\frac{3j+2}{2}} & P_{-1,-1}^{j}(Q)x^{-\frac{3j}{2}} & P_{0,-1}^{j}(Q)x^{-\frac{3j+1}{2}}\\ Q\,P_{1,0}^{j}(Q)x^{-\frac{3j+1}{2}} & P_{-1,0}^{j}(Q)x^{-\frac{3j-1}{2}} & P_{0,0}^{j}(Q)x^{-\frac{3j}{2}}\\ Q\,P_{1,1}^{j}(Q)x^{-\frac{3j}{2}} & P_{-1,1}^{j}(Q)x^{-\frac{3j-2}{2}} & P_{0,1}^{j}(Q)x^{-\frac{3j-1}{2}}\end{bmatrix} \tag{6.66}$$

Theorem 6.3.3 ([BRc]). *The following formula for a generating function of one-point open intersection numbers holds true;*

$$\sum_{\ell\geq 0}\left\langle\tau_{\frac{\ell}{2}-2}\right\rangle_{\mathsf{open}} x^{\frac{\ell}{2}} = \mathrm{e}^{\frac{x^3}{6}}\left({}_2F_2\left(\begin{matrix}\frac{1}{2}-Q & \frac{1}{2}+Q\\ \frac{1}{2} & \frac{1}{2}\end{matrix}\middle| -\frac{x^3}{8}\right) + Q\,x^{\frac{3}{2}}\,{}_2F_2\left(\begin{matrix}1-Q & 1+Q\\ 1 & \frac{3}{2}\end{matrix}\middle| -\frac{x^3}{8}\right)\right). \tag{6.67}$$

The following formula for a generating function of n-point open intersection numbers holds true for $n\geq 2$;

$$\sum_{\ell_1,\cdots,\ell_n\geq 0}\frac{(-1)^{\ell_1+\cdots+\ell_n+n}(\ell_1+1)!!\cdots(\ell_n+1)!!}{2^{\frac{\ell_1+\cdots+\ell_n+n}{3}}}\frac{\left\langle\tau_{\frac{\ell_1}{2}}\cdots\tau_{\frac{\ell_n}{2}}\right\rangle_{\mathsf{open}}}{x_1^{\frac{\ell_1}{2}+1}\cdots x_s^{\frac{\ell_n}{2}+1}} \tag{6.68}$$

$$= -\frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n}\frac{\operatorname{tr}\left(R(x_{\sigma(1)})\cdots R(x_{\sigma(s)})\right)}{(x_{\sigma(1)}-x_{\sigma(2)})\cdots(x_{\sigma(s)}-x_{\sigma(1)})} - \frac{\delta_{n,2}}{(\sqrt{x_1}-\sqrt{x_2})^2}. \tag{6.69}$$

Remark 6.3.4. *It is possible to prove that the generating function (6.67) is equivalent to the following formula;*

$$\sum_{\ell\geq 0}\left\langle\tau_{\frac{\ell}{2}-2}\right\rangle_{\mathsf{open}} x^{\frac{\ell}{2}} = \mathrm{e}^{\frac{x^3}{24}}\sum_{j\geq 0}\frac{A_j(Q)}{(j-1)!!}x^{\frac{3j}{2}} \tag{6.70}$$

where the coefficients A_j are defined by

$$\left(\frac{2+x}{2-x}\right)^Q = \sum_{j\geq 0}A_j(Q)x^j. \tag{6.71}$$

Let us remark that $\mathrm{e}^{\frac{x^3}{24}}$ is the generating function of Witten (closed) intersection numbers, compare with (4.70).

The above theorem follows directly by the arguments used in the proof of Thm. 1.2.2 and by the identification with an isomonodromic tau function, and consequent possibility of using the explicit formula provided by the tau differential (as the bare solution is explicitly known in terms of the functions ϕ, g).

For more details on the application of the arguments of Thm. 1.2.2 to this situation we refer to [BRc], where it is proven that the relevant matrix is $R = \Psi\mathrm{E}_{3,3}\Psi^{-1}$. However, to give more explanations about the final formulæ, let us report the following lemma from loc. cit., which provides a simplification of certain products of asymptotic expansions of the functions $\phi(z;Q)$, appearing in the matrix $\Psi\mathrm{E}_{3,3}\Psi^{-1}$.

[3] Note that for $a-b+1=0$ or $a-b+2=0$ both sides of (6.65) have simple poles, and then the meaning of the identity is that of the residue.

Lemma 6.3.5. *For $a, b \in \{0, \pm 1\}$, let*

$$F_-(z; Q+a)\, F_+(Z; -Q-b) = \sum_{k\geq 0} P^k_{a,b}(Q) z^{-\frac{3k}{2}}. \tag{6.72}$$

Then the polynomials $P^k_{a,b}(Q)$ in the indeterminate Q coincide with those in (6.65).

Proof. The expression $z^{-\frac{a-b+1}{2}} F_-(z; Q+a) F_+(z; -Q-b)$ is the formal expansion of the product of a solution to $\partial_z^3 - z\partial_z + Q + a - 1$ and of a solution to $\partial_z^3 - z\partial_z - Q - b + 1$. As such it is annihilated by the following ninth-order differential operator;

$$\begin{gathered}(a-b-2)(a-b+1)(a-b+4) + \left(3(a-b)^2 - 3(a-b) - 46\right) z\partial_z - 30\, z^2 \partial_z^2 + \\ + \left(102 - 3(a-b) + 3a^2 + 21ab + 3b^2 + 27Q\,(a+b+Q)\,Q - 4z^3\right)\partial_z^3 + 3\,(23 - a + b)\, z\partial_z^4 \\ + 9\, z^2 \partial_z^5 + 3\,(-11 + a - b)\,\partial_z^6 - 6\, z\partial_z^7 + \partial_z^9.\end{gathered} \tag{6.73}$$

Introduce the power series (a formal Laplace transform)

$$G(x) := \sum_{k\geq 0} P^k_{a,b}(Q) \frac{x^{\frac{3k+a-b-1}{2}}}{\Gamma\left(\frac{3k+a-b+1}{2}\right)}. \tag{6.74}$$

Then $G(x)$ is annihilated by the third-order differential operator

$$\begin{gathered}\partial_x^3 - \frac{3\left(3x^3 - 2\right)}{4x}\partial_x^2 + \frac{3(2x^6 - (7+a-b)\,x^3 - ((a-b)^2 - a + b + 2))}{4x^2}\partial_x + \\ + \frac{-x^9 + 3(a-b+3)x^6 - (3(a+b)^2 + 15ab + 9(a-b) + 6 + 27Q(a+b+Q))x^3 + (a-b+2)(a-b-2)}{4x^3}\end{gathered} \tag{6.75}$$

obtained from (6.73) by replacing z with ∂_x and ∂_z with $-x$. We are therefore interested in power-series solutions around the Fuchsian singularity $x = 0$ of (6.75). It is easily checked that for $a, b \in \{0, \pm 1\}$ the equation (6.75) is resonant and the Frobenius solutions at $x = 0$ span a two-dimensional space[4] generated by the two series below;

$$\begin{aligned} G_1(x) &:= x^{\frac{a-b-1}{2}} \mathrm{e}^{\frac{x^3}{3}} {}_2F_2\left(\left.\begin{matrix}\frac{1-a-b-2Q}{2} & \frac{1+a+b+2Q}{2} \\ \frac{1}{2} & \frac{1+a-b}{2}\end{matrix}\right| -\frac{x^3}{4}\right) \\ G_2(x) &:= x^{\frac{a-b+2}{2}} \mathrm{e}^{\frac{x^3}{3}} {}_2F_2\left(\left.\begin{matrix}\frac{2-a-b-2Q}{2} & \frac{2+a+b+2Q}{2} \\ \frac{3}{2} & \frac{2+a-b}{2}\end{matrix}\right| -\frac{x^3}{4}\right).\end{aligned} \tag{6.76}$$

By matching with $F_-(z; Q+a)F_+(z; -Q-b) = 1 - \frac{(a-b+2)(2Q+a+b)}{4} z^{-\frac{3}{2}} + \mathcal{O}\left(z^{-3}\right)$ we obtain

$$G(x) = \frac{G_1(x)}{\Gamma\left(\frac{a-b+1}{2}\right)} - \frac{2Q+a+b}{2}\frac{G_2(x)}{\Gamma\left(\frac{a-b+2}{2}\right)}. \tag{6.77}$$

The proof is complete. ∎

Virasoro constraints. For simplicity we derive only the first two Virasoro constraints, again going under the name of *string* and *dilaton* equations; they are a deformation of the Virasoro constraint for the Kontsevich–Witten tau function discussed in Chap. 4.

They read as

$$\left(\sum_{k\geq 3} \frac{k}{2} t_k \frac{\partial}{\partial t_{k-2}} + \frac{\partial}{\partial t_1} + \frac{t_1^2}{4} + Qt_2\right)\tau(\mathbf{t}) = 0 \tag{6.78}$$

$$\left(\sum_{k\geq 1} \frac{k}{2} t_k \frac{\partial}{\partial t_k} + \frac{\partial}{\partial t_3} + \frac{1}{16} + \frac{3Q^2}{4}\right)\tau(\mathbf{t}) = 0. \tag{6.79}$$

There is actually a family of Virasoro constraints [Aa]; it is possible to prove higher Virasoro constraints also along the lines of the proof in Chap. 5.

[4] The expansion of the third solution involves logarithms.

Let us introduce the following action of the shifts and dilations on the times $\mathbf{t}$, $\mathbf{t}_S(x;\mathbf{t})$, $\mathbf{t}_D(x;\mathbf{t})$ according to

$$\Xi(z;\mathbf{t}_S(x,\mathbf{t})) = \Xi(z+x;\mathbf{t})_+, \qquad \Xi(z;\mathbf{t}_D(x,\mathbf{t})) = \Xi(\mathrm{e}^x z;\mathbf{t})_+ \tag{6.80}$$

where $+$ denotes the polar part at $z=\infty$, i.e., we keep only *strictly* positive powers of $z^{\frac{1}{2}}$ in the Puiseux expansion at infinity. At first order in x we have

$$\mathbf{t}_S(x,\mathbf{t}) = \mathbf{t} + x\mathbb{L}_{-1}\mathbf{t} + \mathcal{O}(x^2), \quad \mathbf{t}_D(x,\mathbf{t}) = \mathbf{t} + x\mathbb{L}_0\mathbf{t} + \mathcal{O}(x^2) \tag{6.81}$$

where the vector fields $\mathbb{L}_{-1}$ and $\mathbb{L}_0$ are

$$\mathbb{L}_{-1} := \sum_{k\geq 3}\frac{k}{2}t_k\frac{\partial}{\partial t_{k-2}} + \frac{\partial}{\partial t_1}, \quad \mathbb{L}_0 := \sum_{k\geq 1}\frac{k}{2}t_k\frac{\partial}{\partial t_k} + \frac{\partial}{\partial t_3} \tag{6.82}$$

Lemma 6.3.6. *The following identities hold true*

$$\Psi(z+x;\mathbf{t}) = \mathrm{diag}(1,\mathrm{e}^\eta,\mathrm{e}^\eta)\Psi(z;\mathbf{t}_S(x;\mathbf{t})), \quad \eta := \sum_{k\geq 1}x^k t_{2k}, \tag{6.83}$$

$$\Psi(\mathrm{e}^x z;\mathbf{t}) = \mathrm{e}^{x(S+L)}\Psi(z;\mathbf{t}_D(x;\mathbf{t})). \tag{6.84}$$

Proof. Consider the sectionally analytic matrix $\widehat{\Psi}(z;\mathbf{t}) := \Psi(z+x;\mathbf{t})$; it has constant jumps on the sectors translated by $-x$. In each of these sectors, the restriction admits entire analytic continuation under the assumption that $\mathbf{t} = (t_1,...,t_{2K+1},0,...)$ and the condition (6.64) on $\beta_{0,\pm}$. We denote by the same symbol $\widehat{\Psi}(z;\mathbf{t})$ the piecewise analytic matrix function with the same sectors as $\Psi(z;\mathbf{t})$. Now, the jumps of $\widehat{\Psi}(z;\mathbf{t})$ are the same as those of $\Psi(z;\mathbf{t})$. Hence the matrix $\widehat{\Gamma}(z;\mathbf{t}) := \widehat{\Psi}(z;\mathbf{t})\mathrm{e}^{-\Xi(z;\mathbf{t}_S)}$ (with $\mathbf{t}_S = \mathbf{t}_S(x,\mathbf{t})$ for brevity) necessarily solves a RHP with jumps equal to those of $\Gamma(z;\mathbf{t}_S)$ but with a different normalization at $z=\infty$;

$$\widehat{\Gamma}(z;\mathbf{t}) \sim (z+x)^S\, G\, Y(z+x;\mathbf{t})\,(z+x)^L\,\mathrm{e}^{\Xi(z+x;\mathbf{t})-\Xi(z;\mathbf{t}_S)}. \tag{6.85}$$

The trailing factor has the form:

$$\exp(\Xi(z+x;\mathbf{t}) - \Xi(z;\mathbf{t}_S)) = \mathrm{diag}(1,\mathrm{e}^\eta,\mathrm{e}^\eta)(\mathbf{1}+\mathcal{O}(z^{-1})), \quad \eta = \sum_{k\geq 1}x^k t_{2k}. \tag{6.86}$$

The prefactor $\mathrm{diag}(1,\mathrm{e}^\eta,\mathrm{e}^\eta)$ in the right side of (6.86) commutes with G, hence it follows from the uniqueness of the solution to the RHP 6.3.1 that $\widehat{\Gamma}(z;\mathbf{t}) = \mathrm{diag}(1,\mathrm{e}^\eta,\mathrm{e}^\eta)\Gamma(z;\mathbf{t}_S)$ and (6.83) is proved. The proof for the dilations follows along the same lines; the sectionally analytic matrix $\widetilde{\Gamma}(z;\mathbf{t}) := \Psi(\mathrm{e}^x z;\mathbf{t})\mathrm{e}^{-\Xi(z;\mathbf{t}_D)}$ (with $\mathbf{t}_D = \mathbf{t}_D(x;\mathbf{t})$) solves a RHP with jumps equal to those of $\Gamma(z;\mathbf{t}_D)$ but with a different normalization at $z=\infty$;

$$\widetilde{\Gamma}(z;\mathbf{t}) \sim \mathrm{e}^{xS}\, z^S\, G\, Y(\mathrm{e}^x z;\mathbf{t})\,\mathrm{e}^{xL}\, z^L\,\mathrm{e}^{\Xi(\mathrm{e}^x z;\mathbf{t})-\Xi(z;\mathbf{t}_D)} \tag{6.87}$$

and taking e^{xL} on the left (it commutes with G) one obtains $\widetilde{\Gamma}(z;\mathbf{t}) = \mathrm{e}^{x(S+L)}\Gamma(z;\mathbf{t}_D)$. ■

Now we are in position to derive (6.78) and (6.79). For the String equation we apply (6.83) of the Lemma, writing $\mathbf{t}_S = \mathbf{t}_S(x,\mathbf{t})$ for short,

$$\begin{aligned} -\frac{\partial}{\partial t_j}\log\tau(\mathbf{t}) &= \operatorname*{res}_{z=\infty}\mathrm{tr}\left(z^{\frac{j}{2}}\Gamma^{-1}(z;\mathbf{t})\Gamma'(z;\mathbf{t})\theta_j\right) = \operatorname*{res}_{z=\infty}\mathrm{tr}\left(z^{\frac{j}{2}}\Psi^{-1}(z;\mathbf{t})\Psi'(z;\mathbf{t})\theta_j\right) = \\ &= \operatorname*{res}_{z=\infty}\mathrm{tr}\left(z^{\frac{j}{2}}\Psi^{-1}(z-x;\mathbf{t}_S)\Psi'(z-x;\mathbf{t}_S)\theta_j\right) = \operatorname*{res}_{z=\infty}\mathrm{tr}\left((z+x)^{\frac{j}{2}}\Psi^{-1}(z;\mathbf{t}_S)\Psi'(z;\mathbf{t}_S)\theta_j\right) \end{aligned} \tag{6.88}$$

The last expression does not depend on x by construction, so its first variation in x vanishes:

$$\operatorname*{res}_{z=\infty}\mathrm{tr}\left(\frac{j}{2}z^{\frac{j}{2}-1}\Psi^{-1}(z;\mathbf{t})\Psi'(z;\mathbf{t})\theta_j\right) + \mathbb{L}_{-1}\operatorname*{res}_{z=\infty}\mathrm{tr}\left(z^{\frac{j}{2}}\Psi^{-1}(z;\mathbf{t})\Psi'(z;\mathbf{t})\theta_j\right) = 0 \tag{6.89}$$

In terms of the tau function

$$\frac{j}{2}\frac{\partial}{\partial t_{j-2}}\log\tau(\mathbf{t}) + \frac{1}{2}\delta_{j,1}t_1 + Q\delta_{j,2} + \mathbb{L}_{-1}\frac{\partial}{\partial t_j}\log\tau(\mathbf{t}) = 0 \tag{6.90}$$

which gives

$$\frac{\partial}{\partial t_j}\left(\mathbb{L}_{-1}\log\tau(\mathbf{t})+\frac{t_1^2}{4}+Qt_2\right)=0 \tag{6.91}$$

for all $j=1,2,\dots$. Therefore we conclude that $\mathbb{L}_{-1}\log\tau(\mathbf{t})+\frac{t_1^2}{4}+Qt_2=const$ and the integration constant is easily seen to be 0 by evaluation at $\mathbf{t}=0$ (we use the identity $\langle\tau_0\rangle=0$ which implies $\frac{\partial}{\partial t_1}\log\tau(\mathbf{t})\Big|_{\mathbf{t}=0}=0$). The String equation (6.78) is established.

The dilaton equation follows by very similar computations. Write $\mathbf{t}_D=\mathbf{t}_D(x;\mathbf{t})$ and use (6.84):

$$-\frac{\partial}{\partial t_j}\log\tau(\mathbf{t})=\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\mathrm{e}^{\frac{j}{2}x}z^{\frac{j}{2}}\Psi^{-1}(z;\mathbf{t}_D)\Psi'(z;\mathbf{t}_D)\right) \tag{6.92}$$

The first variation in x of the above vanishes:

$$\operatorname*{res}_{z=\infty}\operatorname{tr}\left(\frac{j}{2}z^{\frac{j}{2}}\Psi^{-1}(z;\mathbf{t})\Psi'(z;\mathbf{t})\theta_j\right)+\mathbb{L}_0\operatorname*{res}_{z=\infty}\operatorname{tr}\left(z^{\frac{j}{2}}\Psi^{-1}(z;\mathbf{t})\Psi'(z;\mathbf{t})\theta_j\right)=0. \tag{6.93}$$

In terms of the tau function:

$$\left(\frac{j}{2}\frac{\partial}{\partial t_j}+\mathbb{L}_0\frac{\partial}{\partial t_j}\right)\log\tau(\mathbf{t})=\frac{\partial}{\partial t_j}\mathbb{L}_0\log\tau(\mathbf{t})=0 \tag{6.94}$$

Therefore $\mathbb{L}_0\log\tau(\mathbf{t})=const$, and the constant is easily evaluated at $\mathbf{t}=0$ as

$$\mathbb{L}_0\log\tau(\mathbf{t})|_{\mathbf{t}=0}=\frac{\partial}{\partial t_3}\log\tau(\mathbf{t})\Big|_{\mathbf{t}=0}=-\frac{3}{2}\langle\tau_1\rangle=-\frac{1+12Q^2}{16} \tag{6.95}$$

and the dilaton equation (6.79) is established as well.

6.4 Proof of Thm. 6.2.8

Schlesinger transform matrix R and characteristic matrix. Let us call $\Gamma_0:=\Psi\mathrm{e}^{-\vartheta}$ where Ψ is the solution to the bare system given in (6.35). This is the solution of RHP 6.2.5 for $N=0$.

Introduce the matrix $R:=\Gamma D^{-1}\Gamma_0^{-1}$. By the arguments already reviewed several times, the matrix $R(z)$ is a rational function of z, with simple poles at $z\in\vec{\lambda},\vec{\mu}$ only.

Again, existence of such a rational matrix R is equivalent to find the inverse of a finite-dimensional linear map. Referring to [BRc] for more details, let us consider the following construction, analogous to the general one in Chap. 2.

Let us introduce $\mathcal{H}:=L^2(\partial\mathbf{D}_+,\mathrm{d}z)\otimes\mathbb{C}^3$, where $\mathbb{C}^3$ are row-vectors. The space $\mathcal{H}$ is isomorphic to the direct sum of $n+1$ copies of $L^2(S^1)\otimes\mathbb{C}^3$, i.e. $\mathcal{H}$ has a basis given by

$$(z-\zeta)^r\,\chi_{\partial\mathbf{D}_\zeta}(z)\,\mathbf{e}_j^\top,\quad z^{-r-1}\,\chi_{\partial\mathbf{D}_\infty}(z)\,\mathbf{e}_j^\top,\qquad r\in\mathbb{Z},\ j\in\{1,2,3\},\ \zeta\in\vec{\lambda},\vec{\mu} \tag{6.96}$$

where $\mathbf{e}_j$ is the standard basis of column-vectors in $\mathbb{C}^3$ and χ_X the indicator function of the set X. Consider the subspace $\mathcal{H}_+$ consisting of row-vectors which are analytic in $\mathbf{D}_+$ and vanish at $z=\infty$; equivalently, $\mathcal{H}_+$ has a basis given by (6.96) restricted to $r\geq 0$. Let $C_\pm:\mathcal{H}\to\mathcal{H}$ the projectors defined by the Cauchy integrals

$$C_\pm[f](z):=\oint_{\partial\mathbf{D}_+}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{f(w)}{w-z_\pm}. \tag{6.97}$$

The range of C_+ is $\mathcal{H}_+$ and we denote by $\mathcal{H}_-$ the range of C_-, namely, functions that admit analytic extension to $\mathbf{D}_-$; from the Sokhotski–Plemelj formula $C_++\mathcal{C}_-=\mathrm{Id}$, it follows that $\pm C_\pm:\mathcal{H}\to\mathcal{H}_\pm$ are complementary projectors. Introduce the following subspaces of $\mathcal{H}_-$

$$V:=C_-[\mathcal{H}_+\mathbf{J}^{-1}],\qquad W:=C_-[\mathcal{H}_+\mathbf{J}] \tag{6.98}$$

where

$$\mathbf{J}(z)=\begin{cases}\mathbf{J}_\zeta(z):=\Gamma_0(\lambda)(\zeta-z)^{\mathrm{E}_{33}}, & z\in\partial\mathbf{D}_\zeta,\ \zeta\in\vec{\lambda}\\ \mathbf{J}_\zeta(z):=\Gamma_0(z)(\zeta-z)^{\mathrm{E}_{22}}, & z\in\partial\mathbf{D}_\zeta,\ \zeta\in\vec{\mu}\\ \mathbf{J}_\infty(z):=\Gamma_0(z)D(z)z^{-L}G^{-1}z^{-S}, & z\in\partial\mathbf{D}_\infty.\end{cases} \tag{6.99}$$

Note that $\mathbf{J}_\infty(z) = G_\infty(z) H_\infty(z)$ where

$$H_\infty := z^S G \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^N z^{\frac{N}{2}} & 0 \\ 0 & 0 & \lambda^{\frac{N}{2}} \end{bmatrix} G^{-1} z^{-S} = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & z^{\frac{N}{2}} & 0 \\ 0 & 0 & z^{\frac{N}{2}} \end{bmatrix} & N \text{ even} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^{\frac{N-1}{2}} \\ 0 & \lambda^{\frac{N+1}{2}} & 0 \end{bmatrix} & N \text{ odd} \end{cases} \tag{6.100}$$

and $G_\infty := \mathbf{J}_\infty H_\infty^{-1}$ is formally analytic at $\lambda = \infty$. Actually, due to symmetry (6.51), one can check that G_∞

$$G_\infty = \mathbf{J}_\infty H_\infty^{-1} = \Gamma_0 z^{-L} \widetilde{D} G^{-1} z^{-S} \sim z^S G Y_0 \widetilde{D} G^{-1} z^{-S} \tag{6.101}$$

$$\widetilde{D} := D \operatorname{diag}(1, (-1)^N z^{-\frac{N}{2}}, z^{-\frac{N}{2}}) \tag{6.102}$$

has an expansion in integer powers of z only.

Then $\{v_\zeta\}_{\zeta \in \vec{\lambda}, \vec{\mu}}$ and $\{w_\ell\}_{\ell=1}^N$ defined as

$$v_\zeta := \begin{cases} \mathbf{e}_3^\top \frac{\Gamma_0^{-1}(\zeta)}{z-\zeta}, & \zeta \in \vec{\lambda} \\ \mathbf{e}_2^\top \frac{\Gamma_0^{-1}(\zeta)}{z-\zeta}, & \zeta \in \vec{\mu} \end{cases} \qquad \begin{cases} w_{2m+1} := z^m \mathbf{e}_2^\top, & m = 0, \dots, \lfloor \frac{N-1}{2} \rfloor \\ w_{2m+2} := z^m \mathbf{e}_3^\top, & m = 0, \dots, \lfloor \frac{N-2}{2} \rfloor \end{cases} \tag{6.103}$$

are bases of V and W respectively. To prove that $\{w_\ell\}_{\ell=1}^N$ is a basis of W we use that $G_\infty(z)$ is formally analytic with formally analytic inverse at $\lambda = \infty$ so that $W = C_-[\mathcal{H}_+ H_\infty]$, where $H_\infty(z)$ is as in (6.100).

Proceeding along the same logical steps as in Chap. 2, the linear operator

$$\mathbb{G} : V \to W : v \mapsto C_-[v\mathbf{J}] \tag{6.104}$$

is well defined. Moreover, its invertibility is equivalent to the existence of the Schlesinger transform matrix R; in fact the inverse is given in such case by

$$\mathbb{G}^{-1} : V \to W : w \mapsto C_-[w\mathbf{J}^{-1}R^{-1}]R. \tag{6.105}$$

This is a rephrasing of Prop. 2.4.5.

By expressing the operator $\mathbb{G}$ in the bases (6.103) we obtain the *characteristic matrix* $\mathbf{G} = (\mathbf{G}_{k,\ell})_{k,\ell=1}^N$

$$\mathbf{G}_{k,\ell} := \left[\frac{\left[\operatorname*{res}_{z=\infty} \frac{z^{\lfloor \frac{\ell-1}{2} \rfloor}}{z-\lambda_k} \mathbf{e}_3^\top \Gamma_0^{-1}(\lambda_k) G_\infty(z) \mathbf{e}_{2+(\ell \bmod 2)} \right]_{k=1,\dots,n_1}}{\left[\operatorname*{res}_{z=\infty} \frac{z^{\lfloor \frac{\ell-1}{2} \rfloor}}{z-\mu_k} \mathbf{e}_2^\top \Gamma_0^{-1}(\mu_k) G_\infty(z) \mathbf{e}_{2+(\ell \bmod 2)} \right]_{k=1,\dots,n_2}} \right]_{\ell=1,\dots,n}. \tag{6.106}$$

As a consequence of Prop. 2.4.6 (see also [BCc, Theorem B.1]), the following variational formula holds true;

$$\delta \log\det \mathbf{G} = \int_{\partial \mathbf{D}_-} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \operatorname{tr}\left(R^{-1}R'\delta\mathbf{J}\mathbf{J}^{-1}\right) + \sum_{\zeta \in \vec{\lambda}, \vec{\mu}} \operatorname*{res}_{z=\zeta} \operatorname{tr}\left(\Gamma_0^{-1}\Gamma_0' \delta U_\zeta U_\zeta^{-1} \,\mathrm{d}z\right) \tag{6.107}$$

where δ is the differential with respect to the parameters as in (6.54), and

$$U_\zeta := \begin{cases} (z-\zeta)^{\mathrm{E}_{33}}, & \zeta \in \vec{\lambda} \\ (z-\zeta)^{\mathrm{E}_{22}}, & \zeta \in \vec{\mu} \end{cases} \tag{6.108}$$

and $\int_{\partial \mathbf{D}_-} \frac{\mathrm{d}z}{2\pi \mathrm{i}}$ is understood as the sum over the (formal) residues at $z \in \{\vec{\lambda}, \vec{\mu}, \infty\}$.

Then we have the following manipulation of the determinant of the characteristic matrix.

Proposition 6.4.1. *The following formula holds*

$$\det \mathbf{G} = \pm \mathrm{e}^{U(\vec{\lambda},\vec{\mu})} \det \begin{bmatrix} \left[\omega^{Q+\frac{1}{2}}\phi^{(j-1)}(\omega^{-1}\lambda_k;Q)\right]_{1\le k\le n_1,\ \lambda_k\in I} \\ \hline \left[\phi^{(j-1)}(\lambda_k;Q)\right]_{1\le k\le n_1,\ \lambda_k\in II\cup III} \\ \hline \left[\omega^{-Q-\frac{1}{2}}\phi^{(j-1)}(\omega\lambda_k;Q)\right]_{1\le k\le n_1,\ \lambda_k\in IV} \\ \hline \left[\omega^{\frac{Q}{2}+\frac{1}{4}}\phi^{(j-1)}(\omega\mu_k;Q)\right]_{n_1+1\le k\le n,\ \mu_k\in I\cup II} \\ \hline \left[\omega^{-\frac{Q}{2}-\frac{1}{4}}\phi^{(j-1)}(\omega^{-1}\mu_k;Q)\right]_{n_1+1\le k\le n,\ \mu_k\in III\cup IV} \end{bmatrix}_{1\le j\le n} \tag{6.109}$$

where $U(\vec{\lambda},\vec{\mu})$ has been defined in (6.44) and the explicit sign is irrelevant to our purposes.

The proof of this proposition is a rather lengthy manipulation of the characteristic matrix. We report its proof after completing the proof of Thm. 6.2.8, which uses the result of this proposition.

Malgrange differential and extended Kontsevich–Penner partition function. The following are computations similar to those of the previous chapter. From $\Gamma = R\Gamma_0 D$ and $J = D^{-1}J_0 D$ where $J_0(z) := \mathrm{e}^{\vartheta(z_-)}\widetilde{M}\mathrm{e}^{-\vartheta(z_+)}$, with $\widetilde{M}$ defined in (6.40), one obtains

$$\begin{aligned} \Gamma_-^{-1}\Gamma_-' &= D^{-1}\Gamma_{0-}^{-1}R^{-1}R'\Gamma_{0-}D + D^{-1}\Gamma_{0-}^{-1}\Gamma_{0-}'D + D^{-1}D' \\ \delta J J^{-1} &= D^{-1}J_0\delta D D^{-1}J_0^{-1}D - D^{-1}\delta D \end{aligned} \tag{6.110}$$

so that using (6.110) and the cyclicity of the trace,

$$\begin{aligned} \operatorname{tr}(\Gamma_-\Gamma_-'\delta J J^{-1}) &= \operatorname{tr}(\Gamma_{0-}^{-1}R^{-1}R'\Gamma_{0-}J_0\delta D D^{-1}J_0^{-1} - D^{-1}\Gamma_{0-}^{-1}R^{-1}R'\Gamma_{0-}\delta D + \\ &+\Gamma_{0-}^{-1}\Gamma_{0-}'J_0\delta D D^{-1}J_0^{-1} - D^{-1}\Gamma_{0-}^{-1}\Gamma_{0-}'\delta D + D'D^{-1}J_0\delta D D^{-1}J_0^{-1} - D^{-1}D'D^{-1}\delta D) \end{aligned} \tag{6.111}$$

It is easy to check, thanks to the block–triangular structure of M_0 in (6.47), that the last two terms above are traceless and thus drop out. The remaining terms can be rewritten, using $\Gamma_{0+} = \Gamma_{0-}J_0$, $\Gamma_{0+}' = \Gamma_{0-}'J_0 + \Gamma_{0-}J_0'$, as

$$\begin{aligned} &\operatorname{tr}(R^{-1}R'\Gamma_{0+}\delta D D^{-1}\Gamma_{0+}^{-1} - R^{-1}R'\Gamma_{0-}\delta D D^{-1}\Gamma_{0-}^{-1} + \Gamma_{0+}^{-1}\Gamma_{0+}'\delta D D^{-1} - J_0^{-1}J_0'\delta D D^{-1} + \\ &-\Gamma_{0-}^{-1}\Gamma_{0-}'\delta D D^{-1}) = \Delta_\Sigma\left[\operatorname{tr}(R^{-1}R'\Gamma_0\delta D D^{-1}\Gamma_0^{-1} + \Gamma_0^{-1}\Gamma_0'\delta D D^{-1})\right] \end{aligned} \tag{6.112}$$

where Δ_Σ is the jump operator $\Delta_\Sigma[f] = f_+ - f_-$ and we have used $\operatorname{tr}(J_0^{-1}J_0'\delta D D^{-1}) = 0$. Let us call $\Sigma' := \Sigma \setminus \mathbb{R}_-$ and let $\widetilde{\Sigma}$ be the contour depicted in figure 6.2, which has the property that $\int_{\Sigma'}\Delta[f]\mathrm{d}z = \int_{\widetilde{\Sigma}} f(z)\mathrm{d}z$, so that

$$\begin{aligned} \Omega &= \int_{\Sigma'}\Delta\left[\operatorname{tr}(R^{-1}R'\Gamma_0\delta D D^{-1}\Gamma_0^{-1} + \Gamma_0^{-1}\Gamma_0'\delta D D^{-1})\right]\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \\ &= \int_{\widetilde{\Sigma}}\operatorname{tr}(R^{-1}R'\Gamma_0\delta D D^{-1}\Gamma_0^{-1} + \Gamma_0^{-1}\Gamma_0'\delta D D^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \end{aligned} \tag{6.113}$$

Applying Cauchy's Theorem we can deform $\widetilde{\Sigma}$ as in figure 6.2 so that finally

$$\Omega = \int_{\partial\mathbf{D}_-}\operatorname{tr}(R^{-1}R'\Gamma_0\delta D D^{-1}\Gamma_0^{-1} + \Gamma_0^{-1}\Gamma_0'\delta D D^{-1})\frac{\mathrm{d}z}{2\pi\mathrm{i}} \tag{6.114}$$

with the understanding that $\int_{\partial\mathbf{D}_-}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}$ is the sum over the (formal) residues at $\lambda \in \{\vec{\lambda},\vec{\mu},\infty\}$. We want to compare now the last expression (6.114) for Ω with (6.107). To this end we note the identities

$$\delta\mathbf{J}_\infty\mathbf{J}_\infty^{-1} = \Gamma_0\delta D D^{-1}\Gamma_0^{-1} \qquad \delta\mathbf{J}_\zeta\mathbf{J}_\zeta^{-1} = \Gamma_0\delta U_\zeta U_\zeta^{-1}\Gamma_0^{-1} \tag{6.115}$$

where U_ζ has been defined in (6.108), and the identities

$$\operatorname*{res}_{z=\zeta}\Gamma_0^{-1}\Gamma_0'\delta D D^{-1} = \operatorname*{res}_{z=\zeta}\Gamma_0^{-1}\Gamma_0'\delta U_\zeta U_\zeta^{-1}, \quad \zeta \in \vec{\lambda},\vec{\mu} \tag{6.116}$$

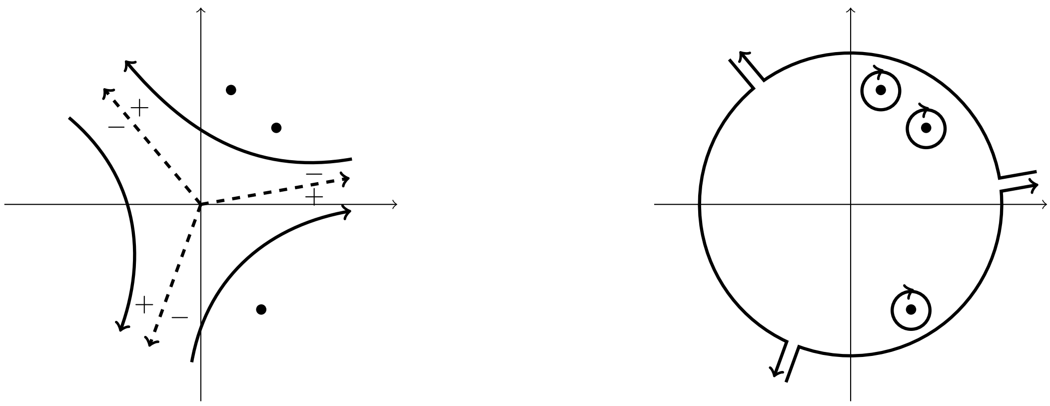


Figure 6.2: On the left, contour Σ' (dashed) and $\widetilde{\Sigma}$ such that $\int_{\Sigma'}(f_+ - f_-)\mathrm{d}z = \int_{\widetilde{\Sigma}} f(z)\mathrm{d}z$; $\widetilde{\Sigma}$ must leave all poles of f (the dots in the picture) on the right. On the right, the deformation of $\widetilde{\Sigma}$ using the Cauchy Theorem.

which follow from $\frac{\mathrm{d}\sqrt{\zeta}}{\sqrt{\zeta}-\sqrt{z}} = \frac{\mathrm{d}\zeta}{\zeta - z} + \mathcal{O}(1)$ as $z \to \zeta$. We finally obtain

$$\Omega - \delta \log \det \mathbf{G} = - \sum_{\lambda \in \vec{\lambda}, \vec{\mu}} \operatorname*{res}_{z=\zeta} \operatorname{tr} \left(R^{-1} R' \Gamma_0 \left(\delta D D^{-1} - \delta U_\zeta U_\zeta^{-1} \right) \Gamma_0^{-1} \right) + \operatorname*{res}_{z=\infty} \operatorname{tr} \left(\Gamma_0^{-1} \Gamma_0' \delta D D^{-1} \right). \tag{6.117}$$

A simple computation for the last term in (6.117) shows that

$$\operatorname*{res}_{z=\infty} \operatorname{tr} \left(\Gamma_0^{-1} \Gamma_0' \delta D D^{-1} \right) \mathrm{d}z = -\operatorname{tr}\left((S+L) \mathrm{E}_{11} \right) \delta\alpha\, \alpha^{-1} = N \sum_{\zeta \in \vec{\lambda}, \vec{\mu}} \frac{\mathrm{d}\sqrt{\zeta}}{\sqrt{\zeta}}. \tag{6.118}$$

Define $T_\zeta := U_\zeta^{-1} D$ and $R_+ := R \Gamma_0 U_\zeta$, for $\zeta \in \vec{\lambda}, \vec{\mu}$. Notice that T_ζ, R_+ are analytic at $z \in \vec{\lambda}, \vec{\mu}$ and that $\delta D D^{-1} - \delta U_\zeta U_\zeta^{-1} = \delta T_\zeta T_\zeta^{-1}$ and so for all $\zeta \in \vec{\lambda}, \vec{\mu}$

$$\begin{aligned}
\operatorname*{res}_{z=\zeta} \operatorname{tr} \left(R^{-1} R' \Gamma_0 \delta T_\zeta T_\zeta^{-1} \Gamma_0^{-1} \right) &= \operatorname*{res}_{\lambda=\zeta} \operatorname{tr} \left((U_\zeta^{-1} \Gamma_0^{-1} R^{-1})(R' \Gamma_0 U_\zeta) \delta T_\zeta T_\zeta^{-1} \right) \mathrm{d}z = \\
&= \operatorname*{res}_{z=\zeta} \operatorname{tr} \left((U_\zeta^{-1} \Gamma_0^{-1} R^{-1})((R\Gamma_0 U_\zeta)' - R\Gamma_0' U_\zeta - R\Gamma_0 U_\zeta') \delta T_\zeta T_\zeta^{-1} \right) \mathrm{d}z = \\
&= \underbrace{\operatorname*{res}_{z=\zeta} \operatorname{tr} \left(R_+^{-1} R_+' \delta T_\zeta T_\zeta^{-1} \right) \mathrm{d}z}_{=0} - \underbrace{\operatorname*{res}_{z=\zeta} \operatorname{tr} \left(\Gamma_0^{-1} \Gamma_0' \delta T_\zeta T_\zeta^{-1} \right) \mathrm{d}z}_{=0} - \operatorname*{res}_{z=\zeta} \operatorname{tr} \left(U_\zeta^{-1} U_\zeta' \delta T_\zeta T_\zeta^{-1} \right) \mathrm{d}z = \\
&= \operatorname*{res}_{z=\zeta} \frac{1}{z-\zeta} \sum_{\zeta' \in \vec{\lambda}, \vec{\mu}} \frac{\mathrm{d}\sqrt{\zeta'}}{\sqrt{\zeta'} - \sqrt{z}} \mathrm{d}z.
\end{aligned} \tag{6.119}$$

To summarize, we have proved

$$\Omega = \delta \log \det \mathbf{G} + \sum_{\zeta \in \vec{\lambda}, \vec{\mu}} \operatorname*{res}_{z=\zeta} \frac{1}{z-\zeta} \sum_{\zeta' \in \vec{\lambda}, \vec{\mu}} \frac{\mathrm{d}\sqrt{\zeta'}}{\sqrt{\zeta'} - \sqrt{z}} \mathrm{d}z + N \sum_{\zeta \in \vec{\lambda}, \vec{\mu}} \frac{\mathrm{d}\sqrt{\zeta}}{\sqrt{\zeta}} = \delta \log \det \mathbf{G} + \delta \log \Delta \tag{6.120}$$

which completes the proof of Thm. 6.2.8, in view of Prop. 6.4.1.

Manipulation of the characteristic determinant: proof of Prop. 6.4.1. Denote

$$\zeta_k := \begin{cases} \lambda_k & 1 \le k \le n_1 \\ \mu_{k-n_1} & n_1 + 1 \le k \le n \end{cases} \qquad A_k := \begin{cases} \mathbf{e}_3^\top \Gamma^{-1}(\zeta_k) & 1 \le k \le n_1 \\ \mathbf{e}_2^\top \Gamma^{-1}(\zeta_k) & n_1 + 1 \le k \le n \end{cases} \tag{6.121}$$

so that we rewrite the characteristic matrix (6.106) as

$$\mathbf{G}_{k,\ell} = \operatorname*{res}_{z=\infty} \frac{z^{\left\lfloor \frac{\ell-1}{2} \right\rfloor}}{\lambda - \zeta_k} A_k G_\infty(\lambda) \mathbf{e}_{2+(\ell \bmod 2)}, \qquad k, \ell = 1, \ldots, N. \tag{6.122}$$

First we compute A_k. Consider the pair of mutually adjoint (in the classical sense) differential operators $L,\ \widehat{L}$ given by

$$L := \partial_z^3 - z\partial_z - Q, \quad \widehat{L} := -\partial_z^3 + z\partial_z - Q + 1. \tag{6.123}$$

According to the general theory (see e.g. [I]) there exists a non-degenerate bilinear pairing between the kernels of $L, \widehat{L}$ that uses the *bilinear concomitant identity*; to express such identity we introduce the matrix bilinear concomitant

$$\mathcal{B}(z) := \begin{bmatrix} -z & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{6.124}$$

Given any solution u of $Lu = 0$ and any solution $\widehat{u}$ of $\widehat{L}\widehat{u} = 0$ we define their *bilinear concomitant* as the bilinear expression

$$\mathscr{B}[u, \widehat{u}] := \begin{bmatrix} \widehat{u} & \widehat{u}' & \widehat{u}'' \end{bmatrix} \mathcal{B}(z) \begin{bmatrix} u \\ u' \\ u'' \end{bmatrix} = \widehat{u}u'' + \widehat{u}''u - \widehat{u}'u' - z\widehat{u}u \tag{6.125}$$

The above expression is, in fact, independent of z and we have:

Proposition 6.4.2 (Lagrange identity)**.** *The bilinear concomitant (6.125) is independent of z and gives a non-degenerate pairing between the solution spaces of the operators $L, \widehat{L}$.*

Proof. The independence of z follows from the identity

$$0 = \widehat{u}Lu - u\widehat{L}\widehat{u} = \widehat{u}u''' + \widehat{u}'''u - z\left(\widehat{u}u' + \widehat{u}'u\right) - \widehat{u}u = \left(\widehat{u}u'' + \widehat{u}''u - \widehat{u}'u' - z\widehat{u}u\right)' = \left(\mathscr{B}[u, \widehat{u}]\right)'. \tag{6.126}$$

The nondegeneracy of the pairing follows from $\det \mathcal{B} = 1$. ■

Proposition 6.4.3. *Denote*

$$\phi_k := \begin{cases} \omega^{Q+\frac{1}{2}}\phi(\omega^{-1}\lambda_k; Q) & 1 \le k \le n_1,\ \lambda_k \in I \\ \phi(\lambda_k; Q) & 1 \le k \le n_1,\ \lambda_k \in II \cup III \\ \omega^{-Q-\frac{1}{2}}\phi(\omega\lambda_k; Q) & 1 \le k \le n_1,\ \lambda_k \in IV \\ \omega^{\frac{Q}{2}+\frac{1}{4}}\phi(\omega\mu_k; Q) & n_1 + 1 \le k \le n_1,\ \mu_k \in I \cup II \\ \omega^{-\frac{Q}{2}-\frac{1}{4}}\phi(\omega^{-1}\mu_k; Q) & n_1 + 1 \le k \le n_1,\ \mu_k \in III \cup IV \end{cases} \qquad Q_k := \begin{cases} \frac{2}{3}\lambda_k^{\frac{3}{2}} & 1 \le k \le n_1 \\ -\frac{2}{3}\mu_k^{\frac{3}{2}} & n_1 + 1 \le k \le n \end{cases} \tag{6.127}$$

Then the row-vectors A_k defined in (6.121) can be expressed as follows;

$$A_k = \mathrm{e}^{Q_k}[\phi_k, \phi_k', \phi_k'']\mathcal{B}(\zeta_k) \tag{6.128}$$

Proof. Let us consider the case $k = 1, ..., n_1$ with $\lambda_k \in II \cup III$, the other cases are completely analogous. The Proposition follows from the following identity in which we set $z = \lambda_k$:

$$[f(z; Q), -f(z; Q-1), f(z; Q-2)]\mathcal{B}(z)\Psi(z) = \mathbf{e}_3^\top. \tag{6.129}$$

The equation (6.129) follows from the fact that the left side is a constant row vector, because of Prop. 6.4.2, which tends to $\mathbf{e}_3^\top$ when $z \to +\infty$. ■

Therefore we can use the expansion (6.101) and write the characteristic matrix (6.106) as,

$$\mathbf{G}_{k,\ell} = \operatorname*{res}_{z=\infty} \frac{z^{\left\lfloor \frac{\ell-1}{2} \right\rfloor}}{z - \zeta_k} \mathrm{e}^{Q_k}[\phi_k, \phi_k', \phi_k''] \, \mathcal{B}(\zeta_k) z^S G Y_0 \widetilde{D} G^{-1} z^{-S} \mathbf{e}_{2+(\ell \bmod 2)} \quad (k, \ell = 1, ..., n). \tag{6.130}$$

(Here Y_0 is found from the expansion at $z = \infty$ of Γ_0.)

From now on we denote $F_r^\pm := \frac{1}{\sqrt{2}}F_\pm(z; r)$ for short.

Lemma 6.4.4. *Let $\phi := \phi_k$ as in (6.127) and $\zeta := \zeta_k$ as in (6.121). For any integer $J \ge 0$ the following identities of formal expansions hold true:*

$$\begin{aligned} \frac{[\phi, \phi', \phi'']\mathcal{B}(z) z^S G Y_0(z)}{z - \zeta}\mathbf{e}_2 &= -\sum_{r=1}^{J} z^{-1-\frac{r}{2}}\phi^{(r)}F^-_{-Q+r+1} \\ -\sum_{m \ge 0}\frac{\zeta^m}{z^m}\Big(z^{-\frac{J+3}{2}}\phi^{(J+1)}F^-_{-Q+J} &+ z^{-\frac{J+4}{2}}\phi^{(J+2)}F^-_{-Q+J+1} - z^{-\frac{J+5}{2}}(Q - J - 1)\phi^{(J)}F^-_{-Q+J+2}\Big) \end{aligned} \tag{6.131}$$

$$
\begin{aligned}
&\frac{[\phi,\phi',\phi'']\mathcal{B}(z)z^S GY_0(z)}{z-\zeta}\mathbf{e}_3 = \sum_{r=1}^{J}(-1)^r\lambda^{-1-\frac{r}{2}}\phi^{(r)}F^+_{-Q+r+1}\\
&+\sum_{m\geq 0}\frac{\zeta^m}{z^m}\Big((-1)^{J+1}\lambda^{-\frac{J+3}{2}}\phi^{(J+1)}F^+_{-Q+J}+(-1)^J z^{-\frac{J+4}{2}}\phi^{(J+2)}F^+_{-Q+J+1}+\\
&(-1)^J z^{-\frac{J+5}{2}}(Q-J-1)\Phi^{(J)}F^+_{-Q+J+2}\Big).
\end{aligned}
\tag{6.132}
$$

Proof. The proof is inductive with respect to J. First compute (we are only interested in the second and third columns)

$$
z^S GY_0(z)\sim\begin{bmatrix}\star & -z^{-1}F^-_{-N+1} & z^{-1}F^+_{-N+1}\\ \star & z^{-\frac{1}{2}}F^-_{-N} & z^{-\frac{1}{2}}F^+_{-N}\\ \star & -F^-_{-N-1} & F^+_{-N-1}\end{bmatrix}
\tag{6.133}
$$

where $F^\pm_r := F_\pm(z;r) = \sum_{j\geq 0}(\pm 1)^j C_j(r)\lambda^{-\frac{3j}{2}}$ and $C_n(r)$ are introduced in Prop. 6.1.2 in formulas (6.19), (6.15). Now we use the recursions (6.20) to write

$$
\mathcal{B}(z)z^S GY_0(z)\sim\begin{bmatrix}\star & z^{-\frac{3}{2}}(Q-1)F^-_{-Q+2} & z^{-\frac{3}{2}}(Q-1)F^+_{-Q+2}\\ \star & -z^{-\frac{1}{2}}F^-_{-Q} & -z^{-\frac{1}{2}}F^+_{-Q}\\ \star & -z^{-1}F^-_{-Q+1} & z^{-1}F^+_{-Q+1}\end{bmatrix}
\tag{6.134}
$$

Inserting the last expression into $\frac{[\phi,\phi',\phi'']}{z-\zeta}\mathcal{B}(z)z^S GY_0(z)\mathbf{e}_{2,3}$ and expanding $\frac{1}{z-\zeta}=\sum\limits_{m\geq 0}\frac{\zeta^m}{z^{m+1}}$ gives (6.131) and (6.132) with $J=0$.

We now proceed with the inductive step: we verify (6.131) only, (6.132) being completely analogous. Assume that (6.131) holds true for an integer $J\geq 0$ and substitute

$$
(Q-J-1)\phi^{(J)}=\zeta\phi^{(J+1)}-\phi^{(J+3)}
\tag{6.135}
$$

(obtained by taking J derivatives of $\phi'''-\zeta\phi'+(Q-1)\phi=0$) into (6.131) to get:

$$
\begin{aligned}
&-\sum_{r=1}^{J}z^{-1-\frac{r}{2}}\phi^{(r)}F^-_{-Q+r+1}\\
&-\sum_{m\geq 0}\frac{\zeta^m}{z^m}\Big(\lambda^{-\frac{J+3}{2}}\phi^{(J+1)}F^-_{-Q+J}+z^{-\frac{J+4}{2}}\phi^{(J+2)}F^-_{-Q+J+1}+z^{-\frac{J+5}{2}}(\phi^{(J+3)}-\zeta\phi^{(J+1)})F^-_{-Q+J+2}\Big).
\end{aligned}
\tag{6.136}
$$

We now re-organize the second summation

$$
\begin{aligned}
&-\left(\sum_{r=1}^{J}z^{-1-\frac{r}{2}}\phi^{(r)}F^-_{-Q+r+1}\right)-z^{-\frac{J+3}{2}}\phi^{(J+1)}F^-_{-Q+J+2}+\\
&-\sum_{m\geq 0}\frac{\zeta^m}{z^m}\Big(z^{-\frac{J+3}{2}}\phi^{(J+1)}(F^-_{-Q+J}-F^-_{-Q+J+2})+z^{-\frac{J+4}{2}}\phi^{(J+2)}F^-_{-Q+J+1}+\lambda^{-\frac{J+5}{2}}\phi^{(J+3)}F^-_{-Q+J+2}\Big).
\end{aligned}
\tag{6.137}
$$

Finally we substitute the identity $F^-_{-Q+J}-F^-_{-Q+J+2}=-z^{-\frac{3}{2}}(Q-J-2)F^-_{-N+J+3}$ obtained from (6.20) with the replacement $Q\to -Q+J+2$. This yields

$$
\begin{aligned}
&-\sum_{r=1}^{J+1}z^{-1-\frac{r}{2}}\phi^{(r)}F^-_{-Q+r+1}+\\
&-\sum_{m\geq 0}\frac{\zeta^m}{z^m}\Big(-z^{-\frac{J+6}{2}}\phi^{(J+1)}(Q-J-2)F^-_{-Q+J+3}+\lambda^{-\frac{J+4}{2}}\phi^{(J+2)}F^-_{-Q+J+1}+z^{-\frac{J+5}{2}}\phi^{(J+3)}F^-_{-Q+J+2}\Big).
\end{aligned}
\tag{6.138}
$$

This is the identity (6.131) under the substitution $J\mapsto J+1$. The proof is complete. ∎

In particular we shall use the following corollary of Lemma 6.4.4: for any $J\geq 0$ we have

$$
\begin{aligned}
\frac{[\phi_k,\phi_k',\phi_k'']\mathcal{B}(z)z^S GY_0(z)}{z-\zeta_k}\mathbf{e}_2 &= -\sum_{r=1}^{J}\lambda^{-1-\frac{r}{2}}\phi_k^{(r)}F^-_{-Q+r+1}+\mathcal{O}\left(z^{-\frac{J+3}{2}}\right)\\
\frac{[\phi_k,\phi_k',\phi_k'']\mathcal{B}(z)z^S GY_0(z)}{z-\zeta_k}\mathbf{e}_3 &= \sum_{r=1}^{J}(-1)^r z^{-1-\frac{r}{2}}\phi_k^{(r)}F^+_{-Q+r+1}+\mathcal{O}\left(z^{-\frac{J+3}{2}}\right)
\end{aligned}
\tag{6.139}
$$

By construction, the columns of the characteristic matrix are obtained as follows: the $(2K-1)$–th and $2K$–th columns of $\mathbf{G}$ correspond to, respectively, the second and first entries of the coefficient in front of z^{-K} in the 2-dimensional row-vector power-series (at $z=\infty$) below (k is the row index of $\mathbf{G}$)

$$\frac{1}{z-\zeta_k}\mathrm{e}^{Q_k}\,[\phi_k,\phi_k',\phi_k'']\,\mathcal{B}(\zeta_k)\,\lambda^S\,G\,Y_0\,\widetilde{D}\,G^{-1}\,z^{-S}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix} \tag{6.140}$$

Let us simplify the last expression: first compute

$$\widetilde{D}G^{-1}z^{-S}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0\\\beta_+z^{\frac12}&\beta_+\\-\beta_-z^{\frac12}&\beta_-\end{bmatrix} \tag{6.141}$$

where $\beta_\pm=z^{-\frac{N}{2}}\pi_\pm$. The power series (6.140) can be rewritten using the identity

$$\frac{\mathcal{B}(\zeta_k)}{z-\zeta_k}=\mathrm{E}_{11}+\frac{\mathcal{B}(z)}{z-\zeta_k} \tag{6.142}$$

(where E_{11} is the elementary unit matrix). This gives the equation

$$\begin{aligned}
&\mathrm{e}^{Q_k}[\phi_k,\phi_k',\phi_k'']\left(\mathrm{E}_{11}+\frac{\mathcal{B}(z)}{z-\zeta_k}\right)z^S\,G\,Y_0\frac{1}{\sqrt{2}}\begin{bmatrix}0&0\\\beta_+z^{\frac12}&\beta_+\\-\beta_-z^{\frac12}&\beta_-\end{bmatrix}=\\
&=\frac{\mathrm{e}^{Q_k}}{\sqrt{2}z}\left[-\sum_{r=0}^{N-1}z^{-\frac{r}{2}}\phi_k^{(r)}F^-_{-Q+r+1}+\mathcal{O}\left(z^{-\frac{N}{2}}\right),\ \sum_{r=0}^{N-1}(-1)^r z^{-\frac{r}{2}}\phi_k^{(r)}F^+_{-Q+r+1}+\mathcal{O}\left(z^{-\frac{N}{2}}\right)\right]\begin{bmatrix}\beta_+z^{\frac12}&\beta_+\\-\beta_-z^{\frac12}&\beta_-\end{bmatrix}=\\
&=\mathrm{e}^{Q_k}\left[-\sum_{r\ \text{odd}}z^{-\frac{r+1}{2}}\phi_k^{(r)}\left(1+\mathcal{O}\left(z^{-1}\right)\right)+\sum_{r\ \text{even}}\star z^{-\frac{r+2}{2}}\phi_k^{(r)}\left(1+\mathcal{O}\left(z^{-1}\right)\right)+\mathcal{O}\left(z^{-\frac{N+2}{2}}\right),\right.\\
&\qquad\left.\sum_{r\ \text{odd}}\star z^{-\frac{r+1}{2}}\phi_k^{(r)}\left(1+\mathcal{O}\left(z^{-1}\right)\right)+\sum_{r\ \text{even}}z^{-\frac{r+2}{2}}\phi_k^{(r)}\left(1+\mathcal{O}\left(z^{-1}\right)\right)+\mathcal{O}\left(z^{-\frac{N+2}{2}}\right)\right]
\end{aligned} \tag{6.143}$$

where we have used (6.139) with $J=N-1$ and then the monodromy properties $\beta_\pm(z\mathrm{e}^{2\pi\mathrm{i}})=\beta_\mp(z)$, $F_\ell^\pm(z\mathrm{e}^{2\pi\mathrm{i}})=F_\ell^\mp(z)$; the expansions in the last expression contain only integer powers of z. The $\star$ denotes an expression independent of z and of the index k and irrelevant to the discussion (we are interested in the determinant). The $\mathcal{O}$ expressions are also independent of k and hence irrelevant.

From the last expression we obtain that the wedge of the columns in $\mathbf{G}$ is, performing triangular transformations on $\mathbf{G}$ and up to an irrelevant sign,

$$\mathrm{e}^{\sum_{\ell=1}^N Q_\ell}\begin{bmatrix}\phi_1\\\vdots\\\phi_N\end{bmatrix}\wedge\begin{bmatrix}\phi_1'\\\vdots\\\phi_N'\end{bmatrix}\wedge\cdots\wedge\begin{bmatrix}\phi_1^{(N-1)}\\\vdots\\\phi_N^{(N-1)}\end{bmatrix} \tag{6.144}$$

For example, if we look at the $2K$–th column of $\mathbf{G}$ we need to extract the coefficient of z^{-K} from the first component of (6.143): the main term comes from the term $r=2K-1$ in the first sum and then there are other terms with $r<2K-1$ coming from both sums. These additional terms correspond to a linear combination of the previous columns of $\mathbf{G}$ and hence do not affect the determinant. Using $U=\sum_{\ell=1}^n Q_\ell$, the proof of Proposition 6.4.1 is complete.

CHAPTER 7

Stationary Gromov–Witten theory of the Riemann sphere

In this chapter we consider stationary Gromov–Witten invariants of $\mathbb{P}^1$. In particular, inspired by recent formulæ of Dubrovin, Yang and Zagier, we make precise connections with matrix models; we obtain results which are slightly different from those in the literature. Moreover, we make a connection with a suitable scaling of the Charlier ensemble.

Main references for this chapter are [DYZa; BRa].

7.1 Stationary Gromov–Witten invariants of $\mathbb{P}^1$ and Dubrovin–Yang–Zagier formulæ

One extremely important and far-reaching generalization of Witten–Kontsevich theorem, Thm. 4.1.5, is Gromov-Witten (GW) theory [KM; BM]. A complete introduction to the topic can be found e.g. in [FP].

In the stationary GW theory of $\mathbb{P}^1$ one is interested in the generating function

$$\begin{aligned} F_{\mathbb{P}^1}(T_0, T_1, T_2, ...; \epsilon) &:= \sum_{n\geq 1} \sum_{k_1,\dots,k_n\geq 0} \frac{T_{k_1}\cdots T_{k_n}}{n!} \langle \tau_{k_1}\cdots\tau_{k_n}\rangle_{\mathbb{P}^1,d} \\ &= \left(\frac{1}{\epsilon^2} - \frac{1}{24}\right) T_0 + \frac{T_0^2}{2\epsilon^2} + \frac{T_0^3}{6\epsilon^2} + \left(\frac{1}{4\epsilon^2} + \frac{1}{24} + \frac{7\epsilon^2}{5760}\right) T_2 + \cdots \end{aligned} \tag{7.1}$$

of stationary GW invariants of $\mathbb{P}^1$

$$\langle \tau_{k_1}\cdots\tau_{k_n}\rangle_{\mathbb{P}^1,d} := \sum_{g\geq 0} \epsilon^{2g-2} \int_{\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1;d)\right]} \psi_1^{k_1}\cdots\psi_n^{k_n} \mathrm{ev}_1^*\omega\cdots\mathrm{ev}_n^*\omega. \tag{7.2}$$

Here $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1;d)$ denotes the moduli stack of degree d stable maps from Riemann surfaces of genus g with n marked points to $\mathbb{P}^1$; $\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1;d)\right]$ is the virtual fundamental class [BF], which allows integration of characteristic classes, in this case the psi-classes ψ_i as above (pulled back via the forgetful map $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1;d) \to \overline{\mathcal{M}}_{g,n}$) and the classes $\mathrm{ev}_i^*\omega$ (pullback of the normalized Kähler class $\omega \in H^2(\mathbb{P}^1;\mathbb{Z})$, $\int_{\mathbb{P}^1}\omega = 1$, via the evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1;d) \to \mathbb{P}^1$ at the ith marked point).

The dimensional constraint $k_1+...+k_n = 2(g-1+d)$ allows to recover the degree d for every coefficient of the generating function (7.1). The exponential $\exp F_{\mathbb{P}^1}$ is a tau function of the Toda hierarchy [OP; DZb].

In this chapter we connect formulæ discovered by Dubrovin and Yang [DYa] for the generating function (7.1) to matrix models, following (in reverse) the isomonodromic method of the previous chapters. The motivation is to connect with certain matrix models that have been proposed in the Physics literature.

Dubrovin, Yang and Zagier formulæ. Let us review explicit formulæ for stationary GW invariants of $\mathbb{P}^1$, conjectured by Dubrovin and Yang in [DYa] and proven together with Zagier in [DYZa] (and also proven independently by Marchal in [Mc] within the framework of topological recursion). This result can be summarized as follows.

Introduce the 2×2 matrix valued formal series

$$R(z;\epsilon) := \frac{\pi}{\epsilon\cos(\pi z)} \begin{pmatrix} \mathrm{J}_{z-\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \\ \mathrm{J}_{z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \end{pmatrix} \begin{pmatrix} \mathrm{J}_{-z-\frac{1}{2}}\left(\frac{2}{\epsilon}\right) & \mathrm{J}_{-z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(z^{-1}) \tag{7.3}$$

where $\mathrm{J}_\nu(z)$ are the Bessel functions of the first kind, identified with their formal expansions as $z \to +\infty$ [AS]. Introduce also the expressions[1]

$$S_1 = \frac{1}{\epsilon}\left(\frac{\pi}{\epsilon\cos(\pi z)} \begin{pmatrix} \mathrm{J}_{-z-\frac{1}{2}}\left(\frac{2}{\epsilon}\right) & \mathrm{J}_{-z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \end{pmatrix} \begin{pmatrix} \partial_z \mathrm{J}_{z-\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \\ \partial_z \mathrm{J}_{z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \end{pmatrix} + \log(\epsilon z)\right), \tag{7.4}$$

$$S_n = -\frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \frac{\mathrm{tr}\,\left(R(z_{\sigma(1)};\epsilon)\cdots R(z_{\sigma(n)};\epsilon)\right)}{(z_{\sigma(1)}-z_{\sigma(2)})\cdots(z_{\sigma(n-1)}-z_{\sigma(n)})(z_{\sigma(n)}-z_{\sigma(1)})} - \frac{\delta_{n,2}}{(z_1-z_2)^2} \tag{7.5}$$

understood as formal series in $z_1^{-1}, \ldots, z_n^{-1}$; note that (7.5) is well defined in this sense, as it is regular along the diagonals $z_i = z_j$.

The main result conjectured in [DYa] and proven in [DYZa] is that for the stationary GW invariants of $\mathbb{P}^1$ (7.2) entering the generating function (7.1), we have an expression in terms of formal residues, namely for all $n \geq 1$, $k_1, \ldots, k_n \geq 0$ the following identity holds true;

$$\langle \tau_{k_1}\cdots\tau_{k_n}\rangle_{\mathbb{P}^1,d} = (-1)^n \operatorname*{res}_{z_1=\infty}\cdots\operatorname*{res}_{z_n=\infty} S_n(z_1,\ldots,z_n)\prod_{j=1}^n \frac{\epsilon^{k_j+1}z_j^{k_j+1}\mathrm{d}z_j}{(k_j+1)!}. \tag{7.6}$$

In the case $n = 1$, (7.6) reproduces the explicit formula for one-point stationary GW invariants of $\mathbb{P}^1$ due to Pandharipande [Pb].

We shall now recognize these formulæ as the logarithmic derivatives of a tau function (of what we called *limiting* RHP in the previous chapters). The motivation is to make connections with matrix integrals, see Sec. 7.3

7.2 Isomonodromic method

Bare system. We shall consider the difference equation

$$f(z+1;\epsilon) + f(z-1;\epsilon) = \epsilon\left(z+\frac{1}{2}\right)f(z;\epsilon). \tag{7.7}$$

or more conveniently its 2×2 matrix form

$$\Psi(z+1) = A(z)\Psi(z), \qquad A(z) = \begin{pmatrix} \epsilon\left(z+\frac{1}{2}\right) & -1 \\ 1 & 0 \end{pmatrix} \tag{7.8}$$

which has a unique formal solution in the form[2]

$$(\mathbf{1} + \mathcal{O}(z^{-1}))\left(\frac{\epsilon z}{\mathrm{e}}\right)^{z\sigma_3} \tag{7.9}$$

as it can be easily shown by induction.

In this section we study asymptotics of solutions to the difference equation (7.7) so to encode its general solution in a 2×2 matrix solution of (7.8), piecewise analytic in suitable sectors, and having the same asymptotic expansion (7.9) in every sector. In other terms, we study the *Stokes' phenomenon* of the *difference* equation (7.8).

From now on we omit the dependence on the parameter $\epsilon > 0$, in the interest of clarity.

[1] We denote $\mathfrak{S}_n$ the symmetric group over $\{1, 2, \ldots, n\}$.
[2] We use the Pauli matrix $\sigma_3 = \mathrm{diag}(1, -1)$.

Solutions to the difference equation (7.7) can be expressed by Mellin contour integrals; in particular we choose

$$
\begin{aligned}
f(z) &:= \frac{1}{\sqrt{2\pi\epsilon}} \int_{C_1} \exp\left(\frac{1}{\epsilon}\left(x - \frac{1}{x}\right) - \left(z + \frac{3}{2}\right)\log x\right) \mathrm{d}x, \\
g(z) &:= \frac{1}{\mathrm{i}\sqrt{2\pi\epsilon}} \int_{C_2} \exp\left(\frac{1}{\epsilon}\left(x - \frac{1}{x}\right) - \left(z + \frac{3}{2}\right)\log x\right) \mathrm{d}x,
\end{aligned} \tag{7.10}
$$

where C_1, C_2 are contours in the x-plane with a branch cut along $x < 0$, $|\arg x| < \pi$, for the definition of $\log x$. More precisely

- C_1 starts from 0 with $|\arg x| < \frac{\pi}{2}$ and arrives at ∞ with $\frac{\pi}{2} < \arg x < \pi$, and
- C_2 starts from ∞ with $-\pi < \arg x < -\frac{\pi}{2}$ and arrives at ∞ with $\frac{\pi}{2} < \arg x < \pi$.

These contours are depicted in Fig. 7.1.

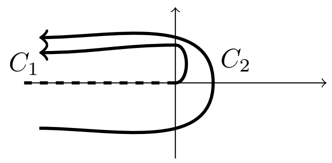


Figure 7.1: Contours C_1, C_2 in the x-plane; the dashed line represents the branch cut along $x < 0$ for the definition of $\log x$ in the integrand of(7.10).

Remark 7.2.1. *g can be expressed in terms of the Bessel function of first kind [AS]*

$$
g(z) = \sqrt{\frac{2\pi}{\epsilon}} \mathrm{J}_{z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right) \tag{7.11}
$$

while f can be expressed in terms of the Bessel function of first and second kind, or equivalently in terms of the Hankel function $\mathrm{H}^{(1)}$

$$
f(z) = \sqrt{\frac{\pi}{2\epsilon}}\left(\mathrm{i}\mathrm{J}_{z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right) - \mathrm{Y}_{z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right)\right) = \mathrm{i}\sqrt{\frac{\pi}{2\epsilon}}\mathrm{H}^{(1)}_{z+\frac{1}{2}}\left(\frac{2}{\epsilon}\right). \tag{7.12}
$$

Note that the z-dependence is in the order of the Bessel functions.

Lemma 7.2.2. *The following asymptotic relations hold true.*

1. *$f(z) \sim \left(\frac{\epsilon z}{\mathrm{e}}\right)^z (1 + \mathcal{O}(z^{-1}))$, as $z \to \infty$ within $|\arg z| < \frac{\pi}{2} - \delta$, for all $\delta > 0$.*
2. *$g(z-1) \sim \left(\frac{\epsilon z}{\mathrm{e}}\right)^{-z} (1 + \mathcal{O}(z^{-1}))$, as $z \to \infty$ within $|\arg z| \leq \pi - \delta$, for all $\delta > 0$.*

The proof is based on the steepest descent method; we defer it to Sec. 7.4.
Let us fix angles $\alpha_1, ..., \alpha_4$ satisfying

$$
-\pi < \alpha_1 < -\frac{\pi}{2} < \alpha_2 < 0 < \alpha_3 < \frac{\pi}{2} < \alpha_4 < \pi \tag{7.13}
$$

and corresponding sectors in the z-plane, with a branch cut along $z < 0$, $|\arg z| < \pi$;

$$
\mathcal{S}_1 := \{-\pi < \arg z < \alpha_1\}, \quad \mathcal{S}_j := \{\alpha_{j-1} < \arg z < \alpha_j\}\ (j = 2,3,4), \quad \mathcal{S}_5 := \{\alpha_4 < \arg z < \pi\}. \tag{7.14}
$$

Define a piecewise analytic 2×2 matrix $\Psi_0 = \Psi_0(z)$ as

$$\Psi_0(z) := \begin{cases} \begin{pmatrix} e^{-i\pi z} g(-z-1) & -e^{i\pi z} f(-z-1) \\ -e^{-i\pi z} g(-z) & e^{i\pi z} f(-z) \end{pmatrix}, & z \in \mathcal{S}_1 \\ \begin{pmatrix} \frac{1}{2\cos(\pi z)} g(-z-1) & g(z) \\ -\frac{1}{2\cos(\pi z)} g(-z) & g(z-1) \end{pmatrix}, & z \in \mathcal{S}_2 \\ \begin{pmatrix} f(z) & g(z) \\ f(z-1) & g(z-1) \end{pmatrix}, & z \in \mathcal{S}_3 \\ \begin{pmatrix} \frac{1}{2\cos(\pi z)} g(-z-1) & g(z) \\ -\frac{1}{2\cos(\pi z)} g(-z) & g(z-1) \end{pmatrix}, & z \in \mathcal{S}_4 \\ \begin{pmatrix} e^{i\pi z} g(-z-1) & -e^{-i\pi z} f(-z-1) \\ -e^{i\pi z} g(-z) & e^{-i\pi z} f(-z) \end{pmatrix}, & z \in \mathcal{S}_5 \end{cases} \tag{7.15}$$

and define also

$$\Gamma_0(z) := \Psi_0(z) \left(\frac{\epsilon z}{e}\right)^{-z\sigma_3}. \tag{7.16}$$

Proposition 7.2.3. *The following statements hold in all sectors $\mathcal{S}_1, ..., \mathcal{S}_5$;*

1. *The matrix $\Psi_0(z)$ solves the matrix difference equation (7.8), and*
2. *The matrix $\Gamma_0(z)$ admits an asymptotic expansion $\Gamma_0(z) \sim \mathbf{1} + \mathcal{O}(z^{-1})$.*

Proof.

1. Integrating by parts, we have $(i = 1, 2)$

$$0 = \int_{C_i} \partial_x \left(e^{\frac{1}{\epsilon}(x-\frac{1}{x}) - (z+\frac{1}{2})\log x} \right) dx = \int_{C_i} \left(1 + \frac{1}{x^2} - \frac{z+\frac{1}{2}}{x}\right) e^{\frac{1}{\epsilon}(x-\frac{1}{x}) - (z+\frac{1}{2})\log x} dx$$
$$= \int_{C_i} \left(e^{\frac{1}{\epsilon}(x-\frac{1}{x}) - (z+\frac{1}{2})\log x} + e^{\frac{1}{\epsilon}(x-\frac{1}{x}) - (z+2+\frac{1}{2})\log x} - \epsilon\left(z+\frac{1}{2}\right) e^{\frac{1}{\epsilon}(x-\frac{1}{x}) - (z+1+\frac{1}{2})\log x} \right) dx$$

which implies

$$f(z-1) + f(z+1) - \epsilon\left(z+\frac{1}{2}\right) f(z) = 0 = g(z-1) + g(z+1) - \epsilon\left(z+\frac{1}{2}\right) g(z). \tag{7.17}$$

Therefore the statement is true for the sector $\mathcal{S}_3$. The statement in the remaining sectors is obtained noting that if $p(z)$ is any anti-periodic function $p(z+1) = -p(z)$, then $\widetilde{f}(z) := p(z) f(-z-1)$ and $\widetilde{g}(z) := p(z) g(-z-1)$ solve the same difference equation;

$$\widetilde{f}(z-1) + \widetilde{f}(z+1) - \epsilon\left(z+\frac{1}{2}\right) \widetilde{f}(z) = 0 = \widetilde{g}(z-1) + \widetilde{g}(z+1) - \epsilon\left(z+\frac{1}{2}\right) \widetilde{g}(z). \tag{7.18}$$

2. In the sector $\mathcal{S}_3$ the statement follows directly from Lemma 7.2.2. For the sector $\mathcal{S}_1$ we exploit the fact that f, g are entire function and note that $0 < \arg(e^{i\pi} z) < \frac{\pi}{2}$, due to (7.13), so that can apply Lemma 7.2.2 as

$$e^{i\pi z} f(-z) = e^{i\pi z} f(e^{i\pi} z) \sim e^{i\pi z} \left(\frac{\epsilon e^{i\pi} z}{e}\right)^{-z} (1 + \mathcal{O}(z^{-1})) = \left(\frac{\epsilon z}{e}\right)^{-z} (1 + \mathcal{O}(z^{-1}))$$
$$e^{-i\pi z} g(-z-1) = e^{-i\pi z} g(e^{i\pi} z - 1) \sim e^{-i\pi z} \left(\frac{\epsilon e^{i\pi} z}{e}\right)^{z} (1 + \mathcal{O}(z^{-1})) = \left(\frac{\epsilon z}{e}\right)^{z} (1 + \mathcal{O}(z^{-1})).$$

The statement is proven likewise in the sectors $\mathcal{S}_2, \mathcal{S}_4, \mathcal{S}_5$. ■

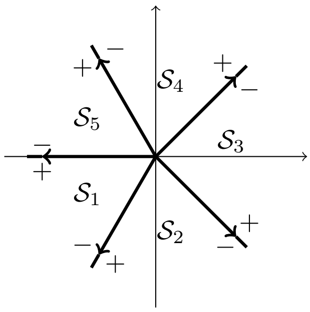


Figure 7.2: Contour Σ, sectors $\mathcal{S}_1, ..., \mathcal{S}_5$, and notation for the boundary values.

Denote

$$\Sigma := \mathrm{e}^{\mathrm{i}\alpha_1}\mathbb{R}_+ \cup \cdots \cup \mathrm{e}^{\mathrm{i}\alpha_4}\mathbb{R}_+ \cup \mathbb{R}_- \tag{7.19}$$

(rays oriented outwards) so that Γ, Ψ are analytic for $z \in \mathbb{C} \setminus \Sigma = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_5$

Lemma 7.2.4. $\Psi_0(z)$ *satisfies the jump condition*

$$\Psi_{0+}(z) = \Psi_{0-}(z)\widehat{J}_0(z) \tag{7.20}$$

where the boundary values are taken according to the orientation of Σ *(see Fig. 7.2) and the matrix* $\widehat{J}_0(z)$ *is defined on* Σ *by*

$$\widehat{J}_0(z) = \begin{cases} \widehat{J}_0^{(1)}(z) = \begin{pmatrix} \frac{1}{1+q^{-1}} & \mathrm{i}q \\ 0 & 1+q^{-1} \end{pmatrix}, & z \in \mathrm{e}^{\mathrm{i}\alpha_1}\mathbb{R}_+ \\ \widehat{J}_0^{(2)}(z) = \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{i}}{1+q} & 1 \end{pmatrix}, & z \in \mathrm{e}^{\mathrm{i}\alpha_2}\mathbb{R}_+ \\ \widehat{J}_0^{(3)}(z) = \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}}{1+q} & 1 \end{pmatrix}, & z \in \mathrm{e}^{\mathrm{i}\alpha_3}\mathbb{R}_+ \\ \widehat{J}_0^{(4)}(z) = \begin{pmatrix} \frac{1}{1+q} & \mathrm{i} \\ 0 & 1+q \end{pmatrix}, & z \in \mathrm{e}^{\mathrm{i}\alpha_4}\mathbb{R}_+ \\ \widehat{J}_0^{(5)}(z) = q^{-\sigma_3}, & z \in \mathbb{R}_- \end{cases} \tag{7.21}$$

where we denote

$$q := \mathrm{e}^{2\pi\mathrm{i}z}. \tag{7.22}$$

Proof. It is a computation based on the identity

$$g(-z-1) = 2\cos(\pi z)f(z) - \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi z}g(z), \tag{7.23}$$

which can be proven by performing the change of variable $x \mapsto -\frac{1}{x}$ in the integral defining (7.10) and applying the Cauchy theorem. Alternatively, in view of Rem. 7.2.1, this identity follows from the known relation

$$\mathrm{J}_{-\nu}(\zeta) = \mathrm{i}\sin(\pi\nu)\mathrm{H}_\nu^{(1)}(\zeta) + \mathrm{e}^{-\mathrm{i}\pi\nu}\mathrm{J}_\nu(\zeta) \tag{7.24}$$

of Hankel and Bessel functions [AS]. ∎

It follows that

$$\Gamma_{0+}(z) = \Gamma_{0-}(z)J_0(z), \qquad J_0(z) := \left(\frac{\epsilon z_-}{\mathrm{e}}\right)^{z\sigma_3} \widehat{J}_0(z) \left(\frac{\epsilon z_+}{\mathrm{e}}\right)^{-z\sigma_3} \tag{7.25}$$

where the notation for the boundary values in the definition of J_0 is relevant only along $z < 0$.

The jump matrices $\widehat{J}_0(z), J_0(z)$ satisfy the following properties.

1. $J_0^{(5)}(z) \equiv \mathbf{1}$, hence Γ_0 extends analytically across $y < 0$.

2. J_0 is exponentially close to the identity as $z \to \infty$, i.e. $J_0(z) = \mathbf{1} + \mathcal{O}(z^{-\infty})$ as z approaches ∞ along any of the rays $\mathrm{e}^{\mathrm{i}\alpha_j}\mathbb{R}_+$, $j = 1, 2, 3, 4$.

3. The no-monodromy condition $\widehat{J}_0^{(1)}(z) \cdots \widehat{J}^{(5)}(z) = \mathbf{1}$ holds true.

4. The jump matrices have unit determinant, $\det \widehat{J}_0^{(i)}(z) \equiv 1$, $i = 1, ..., 5$, $\det J^{(i)}(z) \equiv 1$, $i = 1, ..., 4$.

Lemma 7.2.5. *We have* $\det \Psi_0(z) \equiv 1 \equiv \det \Gamma_0(z)$ *identically in* z, ϵ.

Proof. As $\det \widehat{J}_0(z)$ is identically 1 on Σ, we infer that $u(z) := \det \Psi_0(z)$ is an entire function of z. Moreover, i is periodic, $u(z+1) = u(z)$, as it follows from (7.8). Hence, $u(z) \equiv 1$ everywhere. ■

Remark 7.2.6. *The results of Birkhoff [Bf] cannot be applied directly to the difference equation* (7.8), *which is a* non-generic *case with respect to his usual assumptions.*

Rational dressing. Let us denote $\Sigma' := \mathrm{e}^{\mathrm{i}\alpha_1}\mathbb{R}_+ \cup \cdots \cup \mathrm{e}^{\mathrm{i}\alpha_4}\mathbb{R}_+$.

Fix $N \geq 0$, points $\mathbf{z} = (z_1, ..., z_N)$ in the complex plane, $|\arg z_j| < \pi$; by the freedom in the choice of the angles α_i, compare with (7.13), we can assume that $z_1, ..., z_N \in \mathbb{C} \setminus \Sigma'$. Associated with this data, introduce the jump matrix $J_N(z; \mathbf{z}) : \Sigma' \to \mathrm{SL}_2(\mathbb{C})$ by

$$J_N(z; \mathbf{z}) := D_N^{-1}(z; \mathbf{z}) J_0(z) D_N(z; \mathbf{z}), \qquad D_N(z; \mathbf{z}) := \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j=1}^N (1 - \frac{z}{z_j}) \end{pmatrix}. \tag{7.26}$$

RHP 7.2.7. *Find a* 2×2 *matrix* $\Gamma_N(z; \mathbf{z})$, *analytic in every sector of* $\mathbb{C} \setminus \Sigma'$, *satisfying the following jump condition along* Σ'

$$\Gamma_{N+}(z; \mathbf{z}) = \Gamma_{N-}(z; \mathbf{z}) J_N(z; \mathbf{z}), \tag{7.27}$$

and the following boundary condition at infinity

$$\Gamma_N(z; \mathbf{z}) \sim \mathbf{1} + \mathcal{O}(z^{-1}). \tag{7.28}$$

Remark 7.2.8. *As in Lemma* (7.2.5) *it can be shown that* $\det \Gamma_N(z; \mathbf{z}) \equiv 1$ *identically in* y, *whenever* $\Gamma_N(z; \mathbf{z})$ *exists. Hence, the solution to the RHP* 7.2.7 *is unique, if it exists.*

The tau differential

$$\Omega_N = \sum_{j=1}^N \Omega_{N,j} \mathrm{d}z_j, \qquad \Omega_{N,j} := \operatorname*{res}_{z=z_j} \operatorname{tr} \left(\Gamma_N^{-1} \Gamma_N' \frac{\partial D_N}{\partial z_j} D_N^{-1} \right) \mathrm{d}z \tag{7.29}$$

and the Malgrange differential

$$\widehat{\Omega}_N = \sum_{j=1}^N \widehat{\Omega}_{N,j} \mathrm{d}z_j, \qquad \widehat{\Omega}_{N,j} := \int_{\Sigma'} \operatorname{tr} \left(\Gamma_{N-}^{-1} \Gamma_{N-}' \frac{\partial J_N}{\partial z_j} J_N^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{7.30}$$

are related as

$$\Omega_N - \widehat{\Omega}_N = \eta_N \tag{7.31}$$

where

$$\eta_N = \sum_{j=1}^N \eta_{N,j} \mathrm{d}z_j, \qquad \eta_{N,j} := \int_{\Sigma'} \operatorname{tr} \left(J_N^{-1} J_N' \frac{\partial D_N}{\partial z_j} D_N^{-1} \right) \frac{\mathrm{d}z}{2\pi \mathrm{i}}. \tag{7.32}$$

as it is easily shown. (Compare with Sec. 2.5.)

As the tau differential is closed (see Chap. 1) we introduce the tau function $\tau_N(\mathbf{z})$ as

$$\Omega_{N,j} = \frac{\partial}{\partial z_j} \log \tau_N(\mathbf{z}). \tag{7.33}$$

From the theory of *Schlesinger transformations* which we have recalled in Chap. 2 we know that a tau function related to a rational dressing of jump matrices like (7.26), admits an explicit expression in terms of the finite size determinant of the *characteristic matrix* $G_N(\mathbf{z})$, with entries

$$(G_N(\mathbf{z}))_{j,k} = -\operatorname*{res}_{z=\infty}\left(\Gamma_0^{-1}(z_j)\Gamma_0(z)\right)_{2,2}\frac{z^{k-1}\mathrm{d}z}{z-z_j}, \qquad 1\le j,k\le N \tag{7.34}$$

compare with (2.81).

The following proposition is crucial in establishing the relations with matrix models.

Proposition 7.2.9. *We have*

$$\det G_N(\mathbf{z}) = \det\left(\frac{1}{\epsilon^{k-1}}(\Gamma_0(z_j+k-1))_{1,1}\right)_{j,k=1}^{N}. \tag{7.35}$$

Proof. Introduce functions $a(z), b(z)$, analytic in every sector $\mathcal{S}_1,...,\mathcal{S}_5$, according to

$$\Psi_0(z) = \begin{pmatrix} a(z) & b(z) \\ a(z-1) & b(z-1)\end{pmatrix} \tag{7.36}$$

so that the entries (2.71) of the characteristic matrix are found as

$$\left(\frac{\epsilon z_j}{\mathrm{e}}\right)^{-z_j}\left(\frac{\epsilon z}{\mathrm{e}}\right)^{z}\frac{\begin{pmatrix} -a(z_j-1) & a(z_j)\end{pmatrix}\begin{pmatrix} b(z) \\ b(z-1)\end{pmatrix}}{z-z_j} = \sum_{k=1}^{N}(G_N(\mathbf{z}))_{j,k}z^{-k} + \mathcal{O}(z^{-N-1}) \tag{7.37}$$

where we use $\det\Gamma_0(z)\equiv 1$ from Lemma 7.2.5. Introducing the matrix

$$H(z;z_j) = \begin{pmatrix} a(z_j) & b(z) \\ a(z_j-1) & b(z-1)\end{pmatrix} \tag{7.38}$$

we rewrite (7.37) as

$$\left(\frac{\epsilon z_j}{\mathrm{e}}\right)^{-z_j}\left(\frac{\epsilon z}{\mathrm{e}}\right)^{z}\frac{\det H(z;z_j)}{z-z_j} = \sum_{k=1}^{N}(G_N(\mathbf{z}))_{j,k}z^{-k} + \mathcal{O}(z^{-N-1}). \tag{7.39}$$

Recalling the difference equation (7.8) we have

$$\begin{pmatrix} a(z_j+1) & b(z+1)+\epsilon(z_j-z)b(z) \\ a(z_j-1) & b(z-1)\end{pmatrix} = \begin{pmatrix} \epsilon\left(z_j+\frac{1}{2}\right) & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a(z_j) & b(z) \\ a(z_j-1) & b(z_j)\end{pmatrix} \tag{7.40}$$

hence we get

$$\det H(z+1;z_j+1) + \epsilon(z-z_j)a(z_j-1)b(z) = \det H(z;z_j) \tag{7.41}$$

from which we obtain

$$\det H(z+N;z_j+N) + \epsilon(z-z_j)\sum_{\ell=0}^{N} b(z+\ell)a(z_j+\ell-1) = \det H(z;z_j). \tag{7.42}$$

Finally, from $\left(\frac{\epsilon z}{\mathrm{e}}\right)^{z} b(z+\ell) = \frac{1}{(\epsilon z)^{\ell}}\widehat{b}_\ell(z)$, $\widehat{b}_\ell(z) = 1+\mathcal{O}(z^{-1})$ we rewrite (7.39) as

$$\left(\frac{\epsilon z_j}{\mathrm{e}}\right)^{-z_j}\sum_{\ell=1}^{N}\frac{a(z_j+\ell-1)\widehat{b}_\ell(z)}{\epsilon^{\ell-1}z^{\ell}} = \sum_{k=1}^{N}(G_N(\mathbf{z}))_{j,k}z^{-k} + \mathcal{O}(z^{-N-1}) \tag{7.43}$$

and so

$$G_N(\mathbf{z}) = \widehat{G}_N(\mathbf{z})B_N \tag{7.44}$$

where we write $\widehat{b}_\ell(z) = 1+\sum_{j\ge 1}\widehat{b}_\ell^{j}z^{-j}$ and

$$B := \begin{pmatrix} 1 & \widehat{b}_1^1 & \cdots & \widehat{b}_1^{N-2} \\ 0 & 1 & \cdots & \widehat{b}_2^{N-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\end{pmatrix}, \qquad (\widehat{G}_N(\mathbf{z}))_{j,k} = \left(\frac{\epsilon z_j}{\mathrm{e}}\right)^{-z_j}\frac{a(z_j+k-1)}{\epsilon^{k-1}} = \frac{(\Gamma_0(z_j+k-1))_{1,1}}{\epsilon^{k-1}} \tag{7.45}$$

and the proof is complete by taking the determinant of identity (7.44), as $\det B\equiv 1$. ■

The limiting Riemann-Hilbert-Birkhoff problem For all $N \geq 0$, we have the identity

$$D_N^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\sum_{\ell \geq 1} t_\ell(\mathbf{z}) z^\ell} \end{pmatrix}, \qquad t_\ell(\mathbf{z}) := \frac{1}{\ell} \sum_{j=1}^{N} z_j^{-\ell}. \tag{7.46}$$

This identity is non-formal provided $\min_{j=1,...,N} |z_j| > |z|$.

This prompts to introduce an independent set of times $t_1, t_2, ...,$ and[3]

$$J(z; \mathbf{t}) := \mathrm{e}^{\vartheta(z;\mathbf{t})\mathrm{E}_{22}} J_0(z) \mathrm{e}^{-\vartheta(z;\mathbf{t})\mathrm{E}_{22}}, \qquad \vartheta(z; \mathbf{t}) := \sum_{\ell \geq 1} t_\ell z^\ell \tag{7.47}$$

and to consider the following RHP.

RHP 7.2.10. *Find a 2×2 matrix $\Gamma(z; \mathbf{t})$, analytic in every sector of $\mathbb{C} \setminus \Sigma'$, satisfying the following jump condition along Σ'*

$$\Gamma_+(z; \mathbf{t}) = \Gamma_-(z; \mathbf{t}) J(z; \mathbf{t}), \tag{7.48}$$

and the following normalization infinity

$$\Gamma(z; \mathbf{t}) \sim \mathbf{1} + \mathcal{O}(z^{-1}). \tag{7.49}$$

For the sake of definiteness, in the RHP 7.2.10 one must first assume that for some $K \geq 1$ we have $t_\ell = 0$ whenever $\ell > K$. Therefore, from now on let us fix an arbitrary $K \geq 1$ and assume $t_\ell = 0$ whenever $\ell > K$. Under the assumption that

$$\mathrm{Re}\, t_K \mathrm{e}^{\mathrm{i}K\alpha_j} < 0, \quad j = 2, 3, \qquad \mathrm{Re}\, t_K \mathrm{e}^{\mathrm{i}K\alpha_j} > 0, \quad j = 1, 4 \tag{7.50}$$

we conclude that $J(z; \mathbf{t}) = \mathbf{1} + \mathcal{O}(z^{-\infty})$ as $z \to \infty$ along any ray of Σ. Hence, the solution to the RHP 7.2.10 exists and is unique for $t_1, ..., t_K$ in an open neighborhood of $\mathbf{t} = 0$, with the argument of t_K further restricted by (7.50); it defines a matrix function $\Gamma(z; \mathbf{t})$, its specifications to each sector of the z-plane being holomorphic in $t_1, ..., t_K$. Note that $\Gamma(\mathbf{t} = 0) = \Gamma_0$ by construction.

In particular, this allows to introduce the tau and the Malgrange differentials as above, see (7.29)-(7.30);

$$\Omega = \sum_{\ell=1}^{K} \Omega_\ell \mathrm{d}t_\ell, \qquad \Omega_\ell := -\operatorname*{res}_{z=\infty} \mathrm{tr}\, \left(\Gamma^{-1}\Gamma' \mathrm{E}_{22}\right) z^\ell \mathrm{d}z, \tag{7.51}$$

$$\widehat{\Omega} = \sum_{\ell=1}^{K} \widehat{\Omega}_\ell \mathrm{d}t_\ell, \qquad \widehat{\Omega}_\ell := \int_{\Sigma'} \mathrm{tr}\, \left(\Gamma_-^{-1}\Gamma_-' \frac{\partial J}{\partial t_\ell} J^{-1}\right) \frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{7.52}$$

Exactly as in (7.31), we have the relation

$$\Omega = \widehat{\Omega} + \eta, \qquad \eta = \sum_{\ell=1}^{K} \eta_\ell \mathrm{d}t_\ell, \quad \eta_\ell := -\int_{\Sigma'} \mathrm{tr}\, \left(J^{-1}J'\mathrm{E}_{22}\right) \frac{z^\ell \mathrm{d}z}{2\pi\mathrm{i}}. \tag{7.53}$$

Moreover, the tau differential is closed (see Chap. 1)

$$\frac{\partial}{\partial t_j} \Omega_k = \frac{\partial}{\partial t_k} \Omega_j \tag{7.54}$$

and so we can introduce the tau function $\tau(\mathbf{t})$ as

$$\Omega_\ell = \frac{\partial}{\partial t_\ell} \log \tau(\mathbf{t}). \tag{7.55}$$

[3] Here E_{22} is the elementary matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Identification of the limiting tau function with stationary GW invariants of $\mathbb{P}^1$. The similarity of (7.6) with the general formulæ of Thm. 1.2.2 suggests the following result.

Proposition 7.2.11. *Logarithmic derivatives of the tau function $\tau(\mathbf{t})$, defined in (7.55), coincide, up to a simple scaling, with the stationary GW invariants of $\mathbb{P}^1$ (7.2);*

$$\frac{\partial^n \log\tau(\mathbf{t})}{\partial t_{\ell_1}\cdots\partial t_{\ell_n}} = \frac{\ell_1!\cdots\ell_n!}{\epsilon^{\ell_1+\cdots+\ell_n-n}}\left\langle\tau_{k_1}\cdots\tau_{k_n}\right\rangle_{\mathbb{P}^1,d}. \tag{7.56}$$

Proof. We first consider one-point intersection numbers, $n=1$. To this end, applying definition (7.55), using the notation of (7.36) and denoting $' := \partial_z$, we compute

$$\frac{\partial}{\partial t_\ell}\log\tau(\mathbf{t})\Big|_{\mathbf{t}=0} = -\operatorname*{res}_{z=\infty}\left(\begin{pmatrix} -a(z-1) & a(z)\end{pmatrix}\begin{pmatrix} b'(z)+b(z)\log(\epsilon z)\\ b'(z-1)+b(z-1)\log(\epsilon z)\end{pmatrix}\right)z^\ell \mathrm{d}z \tag{7.57}$$

where we use the identity $\left(\frac{\epsilon z}{\mathrm{e}}\right)^{-z}\left(\left(\frac{\epsilon z}{\mathrm{e}}\right)^z\right)' = \log(\epsilon z)$. Since

$$\det\Psi_0(z) = a(z)b(z-1)-a(z-1)b(z)\equiv 1 \tag{7.58}$$

we can write

$$\frac{\partial}{\partial t_\ell}\log\tau(\mathbf{t})\Big|_{\mathbf{t}=0} = -\operatorname*{res}_{z=\infty}\left(\begin{pmatrix} -a(z-1) & a(z)\end{pmatrix}\begin{pmatrix} b'(z)\\ b'(z-1)\end{pmatrix}+\log(\epsilon z)\right)z^\ell\mathrm{d}z. \tag{7.59}$$

The formal residue is independent of the sector in which we let $z\to\infty$ by construction, as $\Gamma_0(z)$ has the same asymptotic expansion in every sector. E.g. we can assume, using the definition of $\Gamma_0(z)$ in the sector $\mathcal{S}_3$, compare with (7.16), that

$$a(z) = f(z) = \mathrm{i}\sqrt{\frac{\pi}{2\epsilon}}\mathrm{H}^{(1)}_{z+\frac12}\left(\frac{2}{\epsilon}\right)\sim\sqrt{\frac{\pi}{2\epsilon}}\frac{1}{\cos(\pi z)}J_{-z-\frac12}\left(\frac{2}{\epsilon}\right),\qquad b(z)=g(z)=\sqrt{\frac{2\pi}{\epsilon}}\mathrm{J}_{z+\frac12}\left(\frac{2}{\epsilon}\right) \tag{7.60}$$

where we use the Hankel function $\mathrm{H}^{(1)}_\nu(\zeta) = \mathrm{J}_\nu(\zeta)+\mathrm{i}\mathrm{Y}_\nu(\zeta)$, the identity

$$\mathrm{H}^{(1)}_\nu(\zeta) = \frac{\mathrm{i}}{\sin(\nu\pi)}\left(\mathrm{e}^{-\nu\pi\mathrm{i}}\mathrm{J}_\nu(\zeta)-\mathrm{J}_{-\nu}(\zeta)\right) \tag{7.61}$$

compare with (7.24) [AS], and the fact that the term involving $J_{z+\frac12}\left(\frac{2}{\epsilon}\right)$ is sub-leading as $z\to+\infty$, hence inconsequential for the computation of the formal residue (7.59). Inserting (7.60) in (7.59) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t_\ell}\log\tau(\mathbf{t})\Big|_{\mathbf{t}=0} && (7.62)\\
&= -\operatorname*{res}_{z=\infty}\left(\frac{\pi}{\epsilon\cos(\pi z)}\begin{pmatrix}\mathrm{J}_{-z+\frac12}\left(\frac{2}{\epsilon}\right) & \mathrm{J}_{-z-\frac12}\left(\frac{2}{\epsilon}\right)\end{pmatrix}\begin{pmatrix}\partial_z\mathrm{J}_{z+\frac12}\left(\frac{2}{\epsilon}\right)\\ \partial_z\mathrm{J}_{z-\frac12}\left(\frac{2}{\epsilon}\right)\end{pmatrix}+\log(\epsilon z)\right)z^\ell\mathrm{d}z && (7.63)\\
&= -\operatorname*{res}_{z=\infty}\epsilon S_1(z)z^\ell\mathrm{d}z = \frac{\ell!}{\epsilon^{\ell-1}}\left\langle\tau_{\ell-1}\right\rangle_{\mathbb{P}^1,d} && (7.64)
\end{aligned}$$

where we used (7.4) and (7.6). This proves Prop. 7.2.11 for $n=1$.

In order to proceed with higher order derivatives, we first note that we have a compatible system of ODEs of the form (compare with (1.4))

$$\frac{\partial\Gamma}{\partial t_\ell} = M_\ell\Gamma - z^\ell\Gamma\mathrm{E}_{22},\qquad \frac{\partial M_m}{\partial t_\ell}-\frac{\partial M_\ell}{\partial t_m} = [M_\ell, M_m] \tag{7.65}$$

where $M_\ell = M_\ell(z;\mathbf{t})$ is a polynomial of degree ℓ in z;

$$M_\ell(z) := \operatorname*{res}_{w=\infty}\frac{\Gamma(w;\mathbf{t})\mathrm{E}_{22}\Gamma^{-1}(w;\mathbf{t})}{w-z}w^\ell\mathrm{d}w = \operatorname*{res}_{w=\infty}\frac{U(w;\mathbf{t})}{w-z}w^\ell\mathrm{d}w,\quad \ell\geq 1 \tag{7.66}$$

where

$$U(z;\mathbf{t}) := \Gamma(z;\mathbf{t})\mathrm{E}_{22}\Gamma^{-1}(z;\mathbf{t}). \tag{7.67}$$

This fact follows by a standard application of the Liouville theorem. The matrix $\Gamma\mathrm{e}^{\vartheta\mathrm{E}_{2,2}}$ is piecewise analytic in the complex z-plane and satisfies jump conditions independent of $\mathbf{T}$ along Σ'. Hence the ratio

$\frac{\partial}{\partial t_\ell}\left(\Gamma \mathrm{e}^{\vartheta \mathrm{E}_{2,2}}\right)\left(\Gamma \mathrm{e}^{\vartheta \mathrm{E}_{2,2}}\right)^{-1} =: M_\ell$ is analytic in z everywhere and grows like a polynomial of degree ℓ at $z=\infty$. It follows that M_ℓ can be found as the polynomial part of the expansion at $z=\infty$, as in (7.66).

Then we compute second derivatives of $\log \tau(\mathbf{t})$, using the cyclic property of the trace and denoting $' := \partial_z$;

$$\begin{aligned}\frac{\partial}{\partial t_{\ell_2}}\frac{\partial}{\partial t_{\ell_1}}\log \tau(\mathbf{t}) &= -\operatorname*{res}_{z_1=\infty} \operatorname{tr} \frac{\partial}{\partial t_{\ell_2}}\left(\Gamma^{-1}(z_1;\mathbf{t})\Gamma'(z_1;\mathbf{t})\mathrm{E}_{22}\right) z_1^{\ell_1}\mathrm{d}z_1 \\ &= -\operatorname*{res}_{z_1=\infty} \operatorname{tr}\left(\Gamma^{-1}(z_1;\mathbf{t})M_{\ell_2}'(z_1;\mathbf{t})\mathrm{E}_{22}\Gamma(z_1;\mathbf{t})\right) z_1^{\ell_1}\mathrm{d}z_1 + \operatorname*{res}_{z_1=\infty}\left(\ell_2 z_1^{\ell_1+\ell_2-1}\right)\mathrm{d}z_1 \\ &= \operatorname*{res}_{z_1=\infty}\operatorname*{res}_{z_2=\infty} \frac{\operatorname{tr}\left(U(z_1;\mathbf{t})U(z_2;\mathbf{t})\right)-1}{(z_1-z_2)^2} z_1^{\ell_1} z^{\ell_2}\mathrm{d}z_1\mathrm{d}z_2.\end{aligned}$$

Lemma 7.2.12. *In the sense of asymptotic expansions at $z=\infty$, we have*

$$U(z;\mathbf{t}=0) = \sigma_1 R(z)\sigma_1 \tag{7.68}$$

where $R(z)=R(z;\epsilon)$ is given in (7.3).

Proof. Using the notation of (7.36) we compute

$$U(z;\mathbf{t}=0) = \Gamma_0(z)\mathrm{E}_{22}\Gamma_0^{-1}(z) = \begin{pmatrix} b(z) \\ b(z-1)\end{pmatrix}\begin{pmatrix} -a(z-1) & a(z)\end{pmatrix} \tag{7.69}$$

and so the proof is complete by comparing with (7.60). ■

Comparing with (7.5) and (7.6) for $n=2$ we conclude that Prop. 7.2.11 is also true for $n=2$.

To complete the proof of Prop. 7.2.11 one proceeds by induction exactly as in the general Thm. 1.2.2. ■

7.3 Connection with matrix models and discrete orthogonal polynomials.

We can finally deduce some consequences of this isomonodromic interpretation of Dubrovin–Yang–Zagier formulæ.

External source matrix model. Define, as in Sec. 1.4.3,

$$\tau_N(z_1,\dots,z_N) := \frac{\det\left(\frac{1}{\epsilon^{k-1}}\left(\frac{\epsilon z_j}{\mathrm{e}}\right)^{-z_j} f(z_j+k-1;\epsilon)\right)_{j,k=1}^N}{\prod_{1\le j<k\le N} z_k - z_j} \tag{7.70}$$

where f has been defined in (7.10). Note the asymptotic expansion

$$\left(\frac{\epsilon z}{\mathrm{e}}\right)^{-z} f(z;\epsilon) \sim 1 + \frac{24-\epsilon^2}{24\epsilon^2 z} + \frac{\epsilon^4+528\epsilon^2+576}{1152\epsilon^4 z^2} + \frac{1003\epsilon^6+95400\epsilon^4+406080\epsilon^2+69120}{414720\epsilon^6 z^3} + \dots \tag{7.71}$$

where the coefficients can be computed either by a steepest descent analysis or by the difference equation (7.7). Therefore, within the same sector we also have

$$\left(\frac{\epsilon z}{\mathrm{e}}\right)^{-z} f(z+k;\epsilon) \sim (\epsilon z)^k (1+\mathcal{O}(z^{-1})). \tag{7.72}$$

for all $k=0,1,2,\dots$. Thus we fall within the class of KP tau functions considered in Sec. 1.4.3. In particular there is a well defined limit as $N\to\infty$.

For connections of (7.70) with a matrix model with external source, see (7.79) below.

Directly by the isomonodromic interpretation of Dubrovin–Yang–Zagier formulæ and by Prop. 7.2.9 we obtain the following.

Proposition 7.3.1 ([BRa]). *The limiting expansion as $N \to \infty$ of $\log \tau_N(z_1, ..., z_N)$ in the scaled Miwa variables*

$$T_k := \frac{k!}{\epsilon^k}\left(\frac{1}{z_1^{k+1}} + \cdots + \frac{1}{z_N^{k+1}}\right) \tag{7.73}$$

coincides with the free energy of the stationary GW theory of $\mathbb{P}^1$ (7.1).

Example 7.3.2. *Using the first terms of the expansion in (7.71) we can compute $\tau_{N=4}(z_1, ..., z_4)$ up to terms of order 3 in $z_1^{-1}, ..., z_4^{-1}$ as*

$$\tau_{N=4}(z_1,...,z_4) = 1+\tfrac{24-\epsilon^2}{24\epsilon^2}\left(\tfrac{1}{z_1}+\cdots+\tfrac{1}{z_4}\right)+\left(\tfrac{1}{1152}+\tfrac{11}{24\epsilon^2}+\tfrac{1}{2\epsilon^4}\right)\left(\tfrac{1}{z_1^2}+\cdots+\tfrac{1}{z_4^2}+\tfrac{2}{z_1z_2}+\cdots+\tfrac{2}{z_3z_4}\right)$$
$$+\left(\tfrac{1003}{414720}+\tfrac{265}{1152\epsilon^2}+\tfrac{47}{48\epsilon^4}+\tfrac{1}{6\epsilon^6}\right)\left(\tfrac{1}{z_1^3}+\cdots+\tfrac{1}{z_4^3}\right)+\left(-\tfrac{1}{27648}+\tfrac{169}{384\epsilon^2}+\tfrac{23}{16\epsilon^4}+\tfrac{1}{2\epsilon^6}\right)\left(\tfrac{1}{z_1^2z_2}+\cdots+\tfrac{1}{z_3z_4^2}+\tfrac{2}{z_1z_2z_3}+\cdots+\tfrac{2}{z_2z_3z_4}\right)$$

and then, in view of the relations

$$T_0 = \frac{1}{z_1} + \cdots + \frac{1}{z_4}, \qquad T_0^2 = \frac{1}{z_1^2} + \cdots + \frac{1}{z_4^2} + \frac{2}{z_1z_2} + \cdots + \frac{2}{z_3z_4},$$
$$\frac{\epsilon^2 T_2}{2} = \frac{1}{z_1^3} + \cdots + \frac{1}{z_4^3}, \qquad \frac{T_0^3}{3} - \frac{\epsilon^2 T_2}{6} = \frac{1}{z_1^2z_2} + \cdots + \frac{1}{z_3z_4^2} + \frac{2}{z_1z_2z_3} + \cdots + \frac{2}{z_2z_3z_4},$$

the expansion for $\log \tau_{N=4}(z_1, ..., z_4)$ correctly reproduces the terms up to degree 3 given by example in (7.1).

In [ADKMV] the following analogue of the Kontsevich matrix integral for stationary GW invariants of $\mathbb{P}^1$ has been proposed[4];

$$\int_{H_N} \exp \operatorname{tr}\left(MZ - \frac{2}{\epsilon}\cosh M\right) \mathrm{d}M = \pi^{\frac{N(N-1)}{2}} \frac{\det\left(\int_{\mathbb{R}} x^k \mathrm{e}^{xz_j - \frac{2}{\epsilon}\cosh x} \mathrm{d}x\right)_{j,k=1}^N}{\prod_{1\le j<k\le N}(z_k - z_j)}. \tag{7.74}$$

The equality above can be derived as follows (see App. B). First we decompose integration in eigenvalues and angular variables

$$\int_{H_N} \exp \operatorname{tr}\left(MZ - \frac{2}{\epsilon}\cosh M\right) \mathrm{d}M = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{i=1}^N i!} \int_{\mathbb{R}^N}\left(\int_{\mathrm{U}_N} \mathrm{e}^{\operatorname{tr}\left(UXU^\dagger Z\right)}\mathrm{d}U\right)\Delta^2(X)\mathrm{e}^{-\frac{2}{\epsilon}\operatorname{tr}\cosh(X)}\mathrm{d}X \tag{7.75}$$

then we use Harish-Chandra-Itzykson-Zuber formula (B.11) to rewrite the previous expression as

$$\frac{\pi^{\frac{N(N-1)}{2}}}{N!}\frac{1}{\Delta(Z)}\int_{\mathbb{R}^N}\Delta(X)\det\left(\mathrm{e}^{x_iz_j - \frac{2}{\epsilon}\cosh x_i}\right)_{i,j=1}^N \tag{7.76}$$

and finally the equality in (7.74) is found by Andreief identity.

Noting that

$$\int_{\mathbb{R}} \mathrm{e}^{xz - \frac{2}{\epsilon}\cosh x}\mathrm{d}x = \mathrm{K}_{-z}\left(\frac{2}{\epsilon}\right) \tag{7.77}$$

where $\mathrm{K}_\nu(\zeta)$ is the modified Bessel function of second kind of order ν and argument ζ [AS], the matrix integral (7.74) can be alternatively expressed as

$$\pi^{\frac{N(N-1)}{2}}\frac{\det\left(\partial_{z_j}^{k-1}\mathrm{K}_{-z_j}\left(\frac{2}{\epsilon}\right)\right)_{j,k=1}^N}{\prod_{1\le j<k\le N}(z_k - z_j)}. \tag{7.78}$$

[4] Actually, the partition function $\int_{H_N} \exp \frac{1}{g}\operatorname{tr}\left(M\Lambda - \mathrm{e}^M - q\mathrm{e}^{-M}\right)\mathrm{d}M$ is considered. Up to minor modifications, the parameters g, q can be combined into a single parameter $\epsilon = q^{-\frac{1}{2}}g$; then (7.74) is recovered by the identification $\Lambda = \epsilon Z$.

The main difference with the model (7.70) considered in this work is the presence of derivatives instead of integral shifts. We observe that the following modification of (7.74)

$$\int_{\mathrm{H}_N} \exp \operatorname{tr}\left(ZM - \frac{2}{\epsilon}\cosh M\right)\frac{\Delta\left(\mathrm{e}^M\right)\mathrm{d}M}{\Delta(M)} = \frac{\det\left(\mathrm{K}_{-(z_j+k-1)}\left(\frac{2}{\epsilon}\right)\right)_{j,k=1}^N}{\prod_{1\le j<k\le N}(z_k - z_j)} \tag{7.79}$$

(which coincides with (7.74) for $N=1$ only) produces a result which is closer to the model (7.70)[5]; as above, $\Delta(A)$ denotes the discriminant of the characteristic polynomial of the matrix A. The equality in (7.79) is proven by the same arguments above, noting that after the angular integration using the Harish-Chandra-Itzykson-Zuber formula the left side is written as

$$\frac{\pi^{\frac{N(N-1)}{2}}}{N!}\frac{1}{\Delta(Z)}\int_{\mathbb{R}^N}\frac{\Delta\left(\mathrm{e}^X\right)}{\cancel{\Delta(X)}}\cancel{\Delta(X)}\det\left(\mathrm{e}^{x_i z_j - \frac{2}{\epsilon}\cosh x_i}\right)_{i,j=1}^N \tag{7.80}$$

and the equality follows again by Andreief identity.

Connection with the Charlier ensemble. Introduce a discrete measure

$$\mu_a := \sum_{n\ge 0} w(n;a)\delta_n, \qquad w(x;a) := \frac{\mathrm{e}^{-a}a^x}{\Gamma(x+1)} \tag{7.81}$$

supported on the nonnegative integers; here δ_x is the Dirac delta measure supported at $x\in\mathbb{R}$ and $a>0$ is a parameter. The monic discrete orthogonal polynomials $\pi_\ell(x;a) = x^\ell + \cdots$ relative to the measure (7.81) are known to be the (suitably scaled) *Charlier polynomials*;

$$\pi_\ell(x;a) : = (-a)^\ell {}_2F_0\left(-\ell, -x;\,; -\frac{1}{a}\right),$$
$$\int_{\mathbb{R}} \pi_\ell(x;a)\pi_{\ell'}(x,a)\mathrm{d}\mu_a(x) = \sum_{n\ge 0}\pi_\ell(n;a)\pi_{\ell'}(n;a)w(n;a) = a^\ell \ell!\delta_{\ell,\ell'}.$$

The following result concerning a scaling limit of these orthogonal polynomials has been communicated to us by P. Lazag.

Lemma 7.3.3 ([La])**.** *For all $\zeta\in\mathbb{R}$ and $\ell\in\mathbb{Z}$ we have*

$$\lim_{L\to+\infty}\frac{\pi_{L+\ell}\left(L+\zeta;\frac{1}{L\epsilon^2}\right)}{\Gamma(L+\zeta+1)} = \epsilon^{\zeta-\ell}\mathrm{J}_{\zeta-\ell}\left(\frac{2}{\epsilon}\right) \tag{7.82}$$

where $w(x;a)$ has been introduced in (7.81).

Consider now a matrix model of $L\times L$ hermitian matrices with spectrum distributed according to the discrete measure (7.81) (*Charlier ensemble*). In particular, the probability distribution of the eigenvalues is given by

$$\frac{1}{Z_{L,a}}\Delta^2(x_1,...,x_L)\mathrm{d}\mu_a^{\otimes L}(x_1,...,x_L), \qquad Z_{L,a} := \int_{\mathbb{R}^L}\Delta^2(x_1,...,x_L)\mathrm{d}\mu_a(x_1)\cdots\mathrm{d}\mu_a(x_L). \tag{7.83}$$

According to Thm. 3.4.1 the expectation value of a product of characteristic polynomials admits the following expression

$$\left\langle\prod_{i=1}^N \det\left(u_i\mathbf{1} - M\right)\right\rangle_{L,a} = \frac{\det\left(\pi_{L+k-1}(u_j)\right)_{j,k=1}^N}{\prod_{1\le j<k\le N}(u_k-u_j)} \tag{7.84}$$

in terms of the monic orthogonal polynomials $\pi_0,\pi_1,...$; here the expectation value is taken according to the distribution (7.83).

Combining (7.84) with Lemma 7.3.3 we obtain the following interpretation of the model (7.70).

[5]Since $\mathrm{K}_\nu(\zeta) = \frac{\pi}{2}\mathrm{i}^{\nu+1}\mathrm{H}_\nu^{(1)}(\mathrm{i}\zeta)$ one concludes that the Wick rotation $\epsilon\mapsto\mathrm{i}\epsilon$ essentially converts (7.70) to (7.79)

Proposition 7.3.4. *For all $N \geq 1$, $z_1, ..., z_N \in \mathbb{R}$, and all $\epsilon > 0$, we have the following scaling limit of the expectation value of the product of N characteristic polynomials in the Charlier ensemble, as the size L diverges;*

$$\lim_{L\to+\infty} \frac{\left\langle \prod_{i=1}^N \det\left(\left(L - z_i - \frac{1}{2}\right)\mathbf{1} - M\right)\right\rangle_{L, a=\frac{1}{L\epsilon^2}}}{\prod_{i=1}^N \Gamma(L - z_i + \frac{1}{2})} = \frac{\det\left(\epsilon^{\frac{1}{2}-z_j-k} \mathrm{J}_{\frac{1}{2}-z_j-k}\left(\frac{2}{\epsilon}\right)\right)_{j,k=1}^N}{\prod_{1\leq j<k\leq N}(z_k - z_j)} \tag{7.85}$$

where the expectation value in the left side is taken according to the distribution (7.83), with the parameter a being set to $a = \frac{1}{L\epsilon^2}$.

We recognize the model (7.70), up to minor modifications (see also the arguments after (7.60)), in the right side of (7.85).

This should provide a link to the matrix model proposed by Eguchi and Yang [EY].

7.4 Asymptotic expansions: proof of Lemma 7.2.2

It is convenient to introduce

$$\widehat{f}(z) := \int_{C_1} \exp\left(\frac{1}{\epsilon}\left(x - \frac{1}{x}\right) - \left(z + \frac{3}{2}\right)\log x\right) \mathrm{d}x,$$
$$\widehat{g}(z) := \int_{C_2} \exp\left(\frac{1}{\epsilon}\left(x - \frac{1}{x}\right) - \left(z + \frac{3}{2}\right)\log x\right) \mathrm{d}x.$$

Asymptotics for $\widehat{g}$. Let us write $\xi := \epsilon\left(z + \frac{1}{2}\right)$ so that

$$\widehat{g}(z-1) = \int_{C_2} \mathrm{e}^{\frac{1}{\epsilon}\left(x - \frac{1}{x} - \xi\log x\right)} \mathrm{d}x = |\xi| \mathrm{e}^{-\frac{\xi}{\epsilon}\log|\xi|} \int_{C_2} \mathrm{e}^{\frac{|\xi|}{\epsilon}\left(x - \mathrm{e}^{\mathrm{i}\theta}\log x\right)} \mathrm{e}^{-\frac{1}{\epsilon|\xi|x}} \mathrm{d}x \tag{7.86}$$

where $\xi = |\xi|\mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \pi$; in the second equality we performed the change of variable $x \mapsto x|\xi|$ and applied Cauchy theorem to deform the contour $|\xi|^{-1}C_0$ back to C_0. Since C_0 stays at a bounded distance from $x = 0$, we can apply Fubini theorem and write

$$\int_{C_0} \mathrm{e}^{\frac{|\xi|}{\epsilon}\left(x - \mathrm{e}^{\mathrm{i}\theta}\log x\right)} \mathrm{e}^{-\frac{1}{\epsilon|\xi|x}} \mathrm{d}x = \sum_{j\geq 0} \frac{(-1)^j}{j!\epsilon^j|\xi|^j} \int_{C_0} \frac{1}{x^j} \mathrm{e}^{\frac{|\xi|}{\epsilon}\left(x - \mathrm{e}^{\mathrm{i}\theta}\log x\right)} \mathrm{d}x. \tag{7.87}$$

We study each integral in the series in right hand side of (7.87) by the steepest descent method. The phase is $\varphi(x) := x - \mathrm{e}^{\mathrm{i}\theta}\log x$, which has one saddle point at $x = \mathrm{e}^{\mathrm{i}\theta}$. Expanding $\varphi(x) = \mathrm{e}^{\mathrm{i}\theta}(1 - \mathrm{i}\theta) + \frac{\mathrm{e}^{-\mathrm{i}\theta}}{2}(x - \mathrm{e}^{\mathrm{i}\theta})^2 + \mathcal{O}((x - \mathrm{e}^{\mathrm{i}\theta})^3)$ we see that the steepest descent direction is $\frac{\pi+\theta}{2}$.

For all $|\theta| < \pi$, the contour C_0 can be deformed to the steepest descent contour $\operatorname{Im}\varphi(x) = \operatorname{Im}\varphi(\mathrm{e}^{\mathrm{i}\theta})$ in the vicinity of $x = \mathrm{e}^{\mathrm{i}\theta}$ in such a way that the main contribution to the integral for large $|\xi|$ comes from the neighborhood of the saddle point (see Fig. 7.3), and is computed by the gaussian integral;

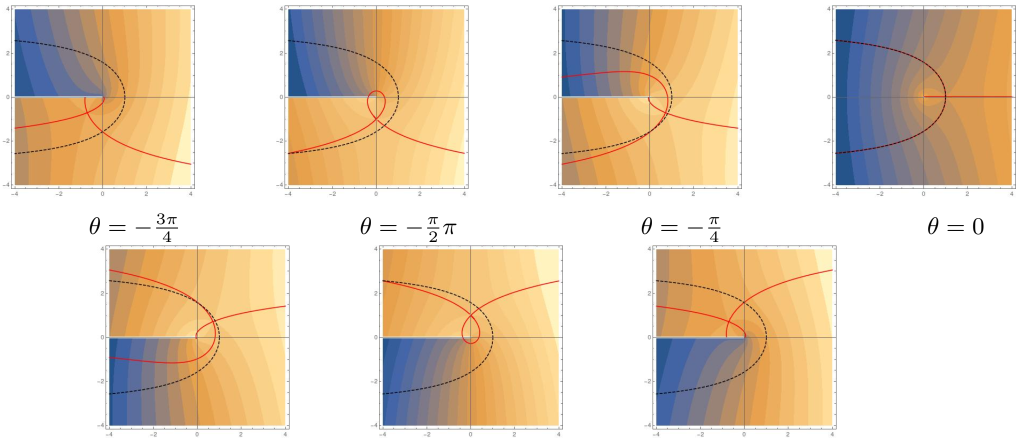
$$\begin{aligned}\widehat{g}(z-1) &\sim |\xi| \mathrm{e}^{-\frac{\xi}{\epsilon}(\log|\xi| - 1 + \mathrm{i}\theta)} \sum_{j\geq 0} \frac{(-1)^j}{j!(\epsilon|\xi|\mathrm{e}^{\mathrm{i}\theta})^j} \int_{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\frac{\pi+\theta}{2}}\mathbb{R}} \mathrm{e}^{\frac{|\xi|}{\epsilon}\frac{\mathrm{e}^{-\mathrm{i}\theta}}{2}(x - \mathrm{e}^{\mathrm{i}\theta})^2} \mathrm{d}x \\ &= \mathrm{i}\sqrt{2\pi\epsilon|\xi|} \mathrm{e}^{-\frac{\xi}{\epsilon}(\log\xi - 1)}(1 + \mathcal{O}(|\xi|^{-1})).\end{aligned}$$

Finally, we recall $\xi = \epsilon\left(z + \frac{1}{2}\right)$ and so $\sqrt{|\xi|}\mathrm{e}^{-\frac{\xi}{\epsilon}(\log\xi - 1)} \sim \left(\frac{\epsilon z}{\mathrm{e}}\right)^{-z}$.

This completes the proof of the asymptotic for $g(z-1)$.

Asymptotics for $\widehat{f}$. Let us write $\xi := \epsilon\left(z + \frac{3}{2}\right)$ and divide the contour C_1 in $C_1^{in} := C_1 \cap \{|x| \leq 1\}$ and $C_1^{out} := C_1 \cap \{|x| \geq 1\}$. Performing two different scalings $x \mapsto x|\xi|^{\pm 1}$ we have

$$\widehat{f}(z) = \frac{\mathrm{e}^{\frac{\xi}{\epsilon}\log|\xi|}}{|\xi|} \int_{|\xi|C_1^{in}} \mathrm{e}^{-\frac{|\xi|}{\epsilon}\left(\frac{1}{x} + \mathrm{e}^{\mathrm{i}\theta}\log x\right)} \mathrm{e}^{\frac{x}{\epsilon|\xi|}} \mathrm{d}x + |\xi|\mathrm{e}^{-\frac{\xi}{\epsilon}\log|\xi|} \int_{|\xi|^{-1}C_1^{out}} \mathrm{e}^{\frac{|\xi|}{\epsilon}\left(x - \mathrm{e}^{\mathrm{i}\theta}\log x\right)} \mathrm{e}^{-\frac{1}{\epsilon|\xi|x}} \mathrm{d}x \tag{7.88}$$



$\theta = -\frac{3\pi}{4}$ $\theta = -\frac{\pi}{2}\pi$ $\theta = -\frac{\pi}{4}$ $\theta = 0$

$\theta = \frac{\pi}{4}$ $\theta = \frac{\pi}{2}$ $\theta = \frac{3\pi}{4}$

Figure 7.3: Steepest descent and ascent contours $\mathrm{Im}\,\varphi(x) = \mathrm{Im}\,\varphi(\mathrm{e}^{\mathrm{i}\theta})$ for the phase $\varphi(x) = x - \mathrm{e}^{\mathrm{i}\theta}\log x$ (red), and contour C_0 (black, dashed), for $\theta = i\frac{\pi}{4}$, $i = -3, ..., 3$. Level lines of $\mathrm{Re}\,\varphi$ are also shown. In all cases it is clear how to deform C_0 to the steepest descent contour in the vicinity of the saddle point, so that the contributions from the tails at infinity are exponentially smaller than the saddle point approximation.

where $\xi = |\xi|\mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \pi$. Applying Fubini theorem, the first integral is

$$\int_{|\xi|C_1^{in}} \mathrm{e}^{-\frac{|\xi|}{\epsilon}\left(\frac{1}{x}+\mathrm{e}^{\mathrm{i}\theta}\log x\right)} \mathrm{e}^{\frac{x}{\epsilon|\xi|}}\,\mathrm{d}x = \sum_{j\geq 0} \frac{1}{j!\epsilon^j|\xi|^j} \int_{|\xi|C_1^{in}} x^j \mathrm{e}^{-\frac{|\xi|}{\epsilon}\left(\frac{1}{x}+\mathrm{e}^{\mathrm{i}\theta}\log x\right)}\mathrm{d}x \tag{7.89}$$

and the second one is also written similarly as in (7.87).

We study each integral in the series in the right hand side of (7.89) by the steepest descend method. The phase is $\varphi(x) = \frac{1}{x} - \mathrm{e}^{\mathrm{i}\theta}\log x$, which has one saddle point at $x = \mathrm{e}^{-\mathrm{i}\theta}$. Expanding $\varphi(x) = \mathrm{e}^{\mathrm{i}\theta}(1 - \mathrm{i}\theta) + \frac{\mathrm{e}^{3\mathrm{i}\theta}}{2}(x - \mathrm{e}^{-\mathrm{i}\theta})^2 + \mathcal{O}((x - \mathrm{e}^{-\mathrm{i}\theta})^3)$ we see that the steepest descent direction is $-\frac{3\theta}{2}$.

Let us restrict attention to

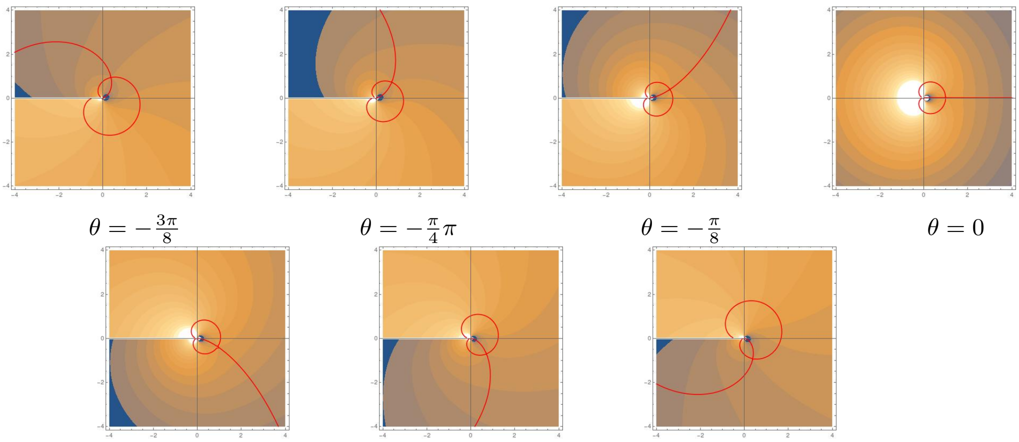
$$|\theta| < \frac{\pi}{2}. \tag{7.90}$$

The contour C_1 can be deformed so that $|\xi|C_1^{in}$ coincides with the steepest descent path in the vicinity of the saddle point $\mathrm{e}^{-\mathrm{i}\theta}$ (see Fig. 7.4), therefore giving the contribution

$$\frac{\mathrm{e}^{\frac{\xi}{\epsilon}(\log|\xi|-1+\mathrm{i}\theta)}}{|\xi|} \sum_{j\geq 0} \frac{(\mathrm{e}^{-\mathrm{i}\theta})^j}{j!\epsilon^j|\xi|^j} \int_{\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\frac{3\theta}{2}}\mathbb{R}} \mathrm{e}^{-\frac{|\xi|}{\epsilon}\frac{\mathrm{e}^{3\mathrm{i}\theta}}{2}(x-\mathrm{e}^{-\mathrm{i}\theta})^2}\mathrm{d}x = \frac{\sqrt{2\pi\epsilon}}{\xi^{\frac{3}{2}}}\mathrm{e}^{\frac{\xi}{\epsilon}(\log\xi-1)}(1+\mathcal{O}(|\xi|^{-1})) \tag{7.91}$$

where we recall that $\xi = \epsilon\left(z+\frac{3}{2}\right)$ so that $\xi^{-\frac{3}{2}}\mathrm{e}^{\frac{\xi}{\epsilon}(\log\xi-1)} \sim \left(\frac{\epsilon z}{\mathrm{e}}\right)^z$. The contribution from the other term, relative to the contour $|\xi|^{-1}C_1^{out}$, is computed similarly as above for g and is subleading with respect to (7.91), as long as we restrict to the range (7.90).

This completes the proof of the asymptotics for f.



$\theta = -\frac{3\pi}{8}$ $\theta = -\frac{\pi}{4}\pi$ $\theta = -\frac{\pi}{8}$ $\theta = 0$

$\theta = \frac{\pi}{8}$ $\theta = \frac{\pi}{4}$ $\theta = \frac{3\pi}{8}$

Figure 7.4: Steepest descent and ascent contours for the phase $\frac{1}{x} + \mathrm{e}^{\mathrm{i}\theta} \log x$ (red) for $\theta = i\frac{\pi}{4}$, $i = -3, ..., 2$. When $\theta = -\frac{3}{4}\pi$ it is convenient to move the branch cut of log. Level lines of the real part of the phase are also shown. In all cases it is clear how to deform $|\xi|C_1^{in}$ to the steepest descent contour in the vicinity of the saddle point, so that the contributions from the tails at zero and infinity are exponentially smaller than the saddle point approximation.

Part III

Appendices

APPENDIX A

KP hierarchy and its tau functions

An integrable hierarchy is a family of pairwise commuting flows; the notion generalizes finite-dimensional integrable hamiltonian systems to infinite dimensions, so to apply to "integrable" PDEs, e.g. the Korteweg-de Vries equation. A rather ubiquitous, and indeed universal, hierarchy in this context is the Kadomtsev-Petvhiashvili (KP) hierarchy. In this appendix we review the KP hierarchy and its tau functions.

A.1 Kadomtsev–Petviashvili hierarchy

Pseudo-differential operators and KP hierarchy. Let (R, ∂_R) be a differential $\mathbb{C}$-algebra, i.e. R is a $\mathbb{C}$-algebra and $\partial_R : R \to R$ a $\mathbb{C}$-linear map satisfying the Leibniz rule $\partial_R(rs) = (\partial_R r)s + r(\partial_R s)$. The $\mathbb{C}$-algebra $\Psi DO(R, \partial_R)$ of pseudo-differential operators over (R, ∂_R) consists of elements[1]

$$A = \sum_{j \ll \infty} r_j \partial^j \tag{A.1}$$

which are called pseudo-differential operators, with product defined by enforcing the generalized Leibniz rule

$$\partial^j r = \sum_{n \geq 0} \frac{j(j-1)\cdots(j-n+1)}{n!} (\partial_R^n r) \partial^{-n}, \qquad j \in \mathbb{Z}. \tag{A.2}$$

Let us denote $A = A_+ + A_-$ for all $A = \sum_{j \ll \infty} r_j \partial^j \in \Psi DO(R, \partial_R)$, where $A_+ := \sum_{0 \leq j \ll \infty} r_j \partial^j$ is the purely differential part of A.

In the following we mostly consider

$$\Psi DO := \Psi DO(\mathbb{C}[\![x, t_1, t_2, ...]\!], \partial_x) \tag{A.3}$$

for an infinite set of times $t_1, t_2, ...$

On the subspace

$$\left\{ L = \partial + \sum_{j \geq 1} u_j \partial^{-j} \right\} \subset \Psi DO \tag{A.4}$$

we consider the compatible system of nonlinear ODEs[2]

$$\frac{\partial}{\partial t_n} L = [(L^n)_+, L], \qquad n \geq 1 \tag{A.5}$$

called Kadomtsev-Petviashvili (KP) hierarchy. For the name, see (A.8) below.

Note that from the identity $0 = [L^n, L] = [(L^n)_+ + (L^n)_-, L]$ we may write the KP hierarchy as

$$\frac{\partial}{\partial t_n} L = [L, (L^n)_-], \qquad n \geq 1. \tag{A.6}$$

We postpone the proof of compatibility, as it becomes more transparent after Prop. A.1.1 (although it could be checked directly). Note however that (A.5) are well defined, for if L is in (A.4) then both $\frac{\partial}{\partial t_n} L$ and $[L, (L^n)_-]$ are in the form $\sum_{j \geq 1} \ell_j \partial^{-j}$.

[1] We use the notation $\sum_{j \ll \infty} f_j$ to denote $\sum_{j=-\infty}^{+\infty} f_j$ and that there exists some $K \in \mathbb{Z}$ such that $f_j = 0$ whenever $j \geq K$.

[2] $[A, B] = AB - BA$ denotes the commutator in the algebra ΨDO.

A solution L of the KP hierarchy satisfies in particular the $n = 1$ equation

$$\frac{\partial}{\partial t_1} L = \frac{\partial}{\partial x} L \tag{A.7}$$

which implies that L depends only on the combination $t_1 + x$; it is convenient and customary to set $x = 0$, the full dependence being restored by $t_1 \mapsto t_1 + x$. Accordingly, we restrict R to $\mathbb{C}[\![t_1, t_2, ...]\!]$.

Moreover, after some computations, one derives the following equation from $n = 2, 3$ in (A.5);

$$\frac{3}{4}\frac{\partial^2 u_1}{\partial t_2^2} = \frac{\partial}{\partial t_1}\left(\frac{\partial u_1}{\partial t_3} - 3u_1\frac{\partial u_1}{\partial t_1} - \frac{1}{4}\frac{\partial u_1}{\partial t_1^3}\right) \tag{A.8}$$

where $L = \partial + u_1\partial^{-1} + \cdots$. The nonlinear equation (A.8) is the KP equation, whence the name.

Gelfand–Dickey hierarchies. It can be shown that the condition $(L^r)_- = 0$ is invariant under the KP hierarchy flows (A.5), see e.g. [Db]. The corresponding reduced hierarchies are called r-Gelfand–Dickey (GD) hierarchies. From (A.6) it is clear that the flows $t_r, t_{2r}, t_{3r}, ...$ are trivial; consequently we can say that the solutions to the GD hierarchy do not depend on these times.

The case $r = 2$ it corresponds to the *KdV hierarchy*, and the case $r = 3$ to the *Boussinesq hierarchy*.

Undressing the KP hierarchy. As a fact, for any $\mathbb{C}$-algebra of pseudo-differential operators, the subset

$$\mathcal{G} = \left\{1 + \sum_{j\geq 1} g_j\partial^{-j}\right\} \subset \Psi DO(R, \partial_R) \tag{A.9}$$

is a multiplicative subgroup, assuming R has an identity element 1. Indeed one can easily show that the coefficients of $1 + \sum_{j\geq 1} f_j\partial^{-j} = \left(1 + \sum_{j\geq 1} g_j\partial^{-j}\right)^{-1}$ are found by a well defined recursion;

$$g_1 + f_1 = 0, \quad g_2 + f_1g_1 + f_2 = 0, \quad g_3 + f_1g_2 + f_2g_1 + f_3 - f_1\partial_R g_1 = 0, \cdots. \tag{A.10}$$

Then every pseudo-differential operator L of the form (A.4) admits a representation as

$$L = M\partial M^{-1} \tag{A.11}$$

for some $M = 1 + \sum_{j\geq 1} m_j\partial^{-j} \in \mathcal{G}$, whose coefficients m_j are again found by a well defined recursion;

$$u_1 = -\partial_R m_1, \quad u_2 = -\partial_R\left(m_2 - \frac{1}{2}m_1^2\right), \cdots. \tag{A.12}$$

The operator $M \in \mathcal{G}$ in (A.11) is unique up to the gauge arbitrariness $M \mapsto MC$ for some constant $C \in \mathcal{G}$, i.e. $C = 1 + \sum_{j\geq 1} c_j\partial^{-j}$ with $\partial_R c_1 = \partial_R c_2 = ... = 0$.

Let us go back to the setting (A.3) for the KP hierarchy.

Proposition A.1.1. *$L = M\partial M^{-1}$, with $M \in \mathcal{G}$, solves the KP hierarchy (A.5) if and only if M solves the compatible system*

$$\frac{\partial M}{\partial t_n} + (M\partial^n M^{-1})_- M = 0, \qquad n \geq 1. \tag{A.13}$$

We skip the easy proof.

Proposition A.1.2. *The KP hierarchy (A.5) is compatible;*

$$\frac{\partial}{\partial t_m}\frac{\partial}{\partial t_n}L = \frac{\partial}{\partial t_n}\frac{\partial}{\partial t_m}L. \tag{A.14}$$

Sketch of proof. Writing $X_n = -(M\partial^n M^{-1})_- = -(L^n)_-$, the proposition follows from the zero curvature identity

$$\frac{\partial X_n}{\partial t_m} - \frac{\partial X_m}{\partial t_n} = [X_m, X_n] \tag{A.15}$$

which can be easily checked. ■

Wave functions and KP hierarchy. Introduce notation $\mathbf{t} = (t_1, t_2, ...)$ and

$$\xi(z; \mathbf{t}) := \sum_{j\geq 1} t_j z^j. \tag{A.16}$$

The vector space $\mathcal{W}$, of (formal) wave functions, consists of symbols $\psi \in \mathcal{W}$

$$\psi = \psi(z; \mathbf{t}) = \left(\sum_{j\ll\infty} \xi_j(\mathbf{t}) z^j\right) e^{\xi(z;\mathbf{t})} \tag{A.17}$$

where $\xi_j \in \mathbb{C}[\![\mathbf{t}]\!]$, endowed with the natural linear structure (sum the coefficients ξ_j). There is a (left) action of ΨDO on $\mathcal{W}$, defined by setting

$$\partial^n e^{\xi(z;\mathbf{t})} = z^n e^{\xi(z;\mathbf{t})}, \qquad n \in \mathbb{Z} \tag{A.18}$$

and requiring that this action commutes with multiplication in ΨDO, i.e. $(AB)\psi = A(B\psi)$. It can be checked that this action is free and transitive, and therefore $\mathcal{W}$ is a free rank 1 (left) ΨDO-module. Therefore $\mathcal{W}$ can be identified with the vector space underlying ΨDO, where $\psi \in \mathcal{W}$ is identified with the unique $A \in \Psi DO$ such that $\psi = Ae^{\xi(z;\mathbf{t})}$. Concretely, this is the trivial identification

$$\mathcal{W} \ni \psi = \left(\sum_{j\ll\infty} \xi_j(\mathbf{t}) z^j\right) e^{\xi(z;\mathbf{t})} \leftrightarrow A = \left(\sum_{j\ll\infty} \xi_j(\mathbf{t}) \partial^j\right) \in \Psi DO. \tag{A.19}$$

Operators $\frac{\partial}{\partial t_n} : \mathcal{W} \to \mathcal{W}$ $(n \geq 1)$ are defined in a natural way; for $\psi = \left(\sum_{j\ll\infty} \xi_j(\mathbf{t}) z^j\right) e^{\xi(z;\mathbf{t})} \in \mathcal{W}$

$$\frac{\partial}{\partial t_n}\psi := \left(\sum_{j\ll\infty} \frac{\partial \xi_j(\mathbf{t})}{\partial t_n} z^j\right) e^{\xi(z;\mathbf{t})} + z^n \psi. \tag{A.20}$$

Proposition A.1.3. *$L = M\partial M^{-1}$ in (A.4), with $M \in \mathcal{G}$, solves the KP hierarchy if and only if $\psi = Me^{\xi(z;\mathbf{t})} \in \mathcal{W}$ satisfies*

$$\begin{cases} L\psi = z\psi \\ \frac{\partial}{\partial t_n}\psi = (L^n)_+\psi. \end{cases} \tag{A.21}$$

We omit the easy proof. In case (A.21) holds true, we call $\psi \in \mathcal{W}$ a KP wave function.

A.2 KP tau functions

For a KP wave function ψ we introduce $h_{n,j}$ by

$$\frac{\partial}{\partial t_n}\psi = \left(z^n + \sum_{j\geq 1} h_{n,j} z^{-j}\right)\psi. \tag{A.22}$$

E.g. $h_{n,1} = \frac{\partial}{\partial t_n}\xi_1$.

The following central result is due to Sato. To state let us first recall the elementary Schur polynomials $p_j(\mathbf{t})$, which are symmetric homogeneous degree j polynomials (with respect to $\deg t_k := k$) defined for $j = 0, 1, 2, ...$ by the generating function

$$\exp \sum_{s\geq 1} t_s \beta^s = \sum_{j\geq 0} \beta^j p_j(\mathbf{t}) \tag{A.23}$$

e.g.

$$p_0(\mathbf{t}) = 1, \quad p_1(\mathbf{t}) = t_1, \quad p_2(\mathbf{t}) = t_2 + \frac{1}{2}t_1^2, \quad p_3(\mathbf{t}) = t_3 + t_1 t_2 + \frac{1}{6}t_1^3, \cdots. \tag{A.24}$$

Theorem A.2.1. *For any KP wave function ψ introduce the $h_{n,j}$ as in (A.22). There exists a function $\tau(\mathbf{t})$, called the KP tau function, such that all the $h_{n,j}$ are expressed through derivatives of $\log\tau(\mathbf{t})$;*

$$h_{n,1} = -\frac{\partial}{\partial t_n}\frac{\partial}{\partial t_1}\log\tau(\mathbf{t})$$
$$h_{n,2} = \frac{1}{2}\frac{\partial}{\partial t_n}\left(\frac{\partial^2}{\partial t_1^2} - \frac{\partial}{\partial t_2}\right)\log\tau(\mathbf{t})$$

and in general

$$h_{n,j} = \frac{\partial}{\partial t_n}p_j\left(-\frac{\partial}{\partial t_1}, -\frac{1}{2}\frac{\partial}{\partial t_2}, -\frac{1}{3}\frac{\partial}{\partial t_3}, \cdots\right)\log\tau(\mathbf{t}) \tag{A.25}$$

where $n, j \geq 1$ and p_j are the elementary Schur polynomials (A.23). The KP tau function is defined uniquely by a KP wave function up to the gauge freedom

$$\tau(\mathbf{t}) \mapsto \mathrm{e}^{f(\mathbf{t})}\tau(\mathbf{t}) \tag{A.26}$$

(not affecting (A.25)) for some $f(\mathbf{t}) = c_1t_1 + c_2t_2 + \ldots$ linear in the times.

We omit the proof of this theorem (see e.g. [Dc]). Let us state also the following fundamental corollary instead (for its proof we we refer again to [Dc]).

Corollary A.2.2. *1. The KP wave function $\psi = \psi(z;\mathbf{t})$ is expressed in terms of the KP tau function as*

$$\psi(z;\mathbf{t}) = \frac{\tau\left(\mathbf{t} - \left[z^{-1}\right]\right)}{\tau(\mathbf{t})}\mathrm{e}^{\xi(z;\mathbf{t})} \tag{A.27}$$

where we employ the standard notation $[z^{-1}] := \left(\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \ldots\right)$.

2. A wave function $\psi \in \mathcal{W}$ expressed as (A.27) is a KP wave function if and only if τ satisfies the Hirota bilinear equations defined by the following generating function;

$$\sum_{j\geq 0} p_j(-2y_1, -2y_2, \ldots)\mathrm{e}^{y_1D_1+y_2D_2+\cdots}p_{j+1}\left(D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots\right)\tau\cdot\tau = 0. \tag{A.28}$$

These equations can also be compactly written as

$$\operatorname*{res}_{z=\infty}\tau\left(\mathbf{t} + \left[z^{-1}\right]\right)\tau\left(\mathbf{t}' - \left[z^{-1}\right]\right)\mathrm{e}^{\xi(z;\mathbf{t}-\mathbf{t}')} = 0. \tag{A.29}$$

We remind that the Hirota derivatives $D_i := D_{t_i}$ are bilinear operators[3] whose action on a pair of functions $f = f(\mathbf{t}), g = g(\mathbf{t})$ is denoted $D_i f\cdot g$ and is defined by the generating function in the parameters $\mathbf{y} = (y_1, y_2, \ldots)$

$$\left(\mathrm{e}^{\mathbf{y}\cdot\mathbf{D}}f\cdot g\right)(\mathbf{t}) = f(\mathbf{t}+\mathbf{y})g(\mathbf{t}-\mathbf{y}) \tag{A.30}$$

where $\mathbf{y}\cdot\mathbf{D} := \sum_{j\geq 1} y_jD_j$.

Sato grassmannian. In [SS] the authors proposed an interesting description of the space of solutions as an infinite-dimensional grassmannian. Indeed the bilinear form of the KP hierarchy (A.29) can be interpreted naturally as an infinite-dimensional generalization of the *Plücker relations* [MJD].

This is the point of view considered in Sec. 1.4.3. For more details we refer to the literature, e.g. [SS; SWb; MJD].

[3] Contrarily to the potentially misleading notation $D_if\cdot g$, D_i does not act on the product of f and g.

APPENDIX B

Basics of matrix integration

We review some standard techniques for integrals over matrix spaces, namely Weyl integration formula, Andreief identity, and applications of character expansions to evaluation of certain integrals over unitary groups.

B.1 Hermitian matrices. Weyl integration formula

Let H_N be the set of $N \times N$ hermitian matrices. It is a real vector space of dimension N^2, e.g. we have real coordinates

$$\mathrm{H}_N \ni M \mapsto (M_{ii}, \mathrm{Re}\, M_{ab}, \mathrm{Im}\, M_{ab}) \in \mathbb{R}^{N^2}$$

with $i = 1, ..., N$, $1 \leq a < b \leq N$.

Given this identification we can endow $\mathrm{H}_N = \mathbb{R}^{N^2}$ with the euclidean volume form

$$\mathrm{d}M := \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{1 \leq a < b \leq N} \mathrm{dRe}\, M_{ab} \prod_{1 \leq a < b \leq N} \mathrm{dIm}\, M_{ab}. \tag{B.1}$$

By the spectral theorem, each $M \in \mathrm{H}_N$ can be diagonalized by a unitary matrix $U \in \mathrm{U}_N$ and has real spectrum, i.e. for every $M \in \mathrm{H}_N$ there exists $U \in \mathrm{U}_N$ and real numbers $x_1, ..., x_N$ such that

$$M = U \,\mathrm{diag}(x_1, ..., x_N) U^{\dagger}. \tag{B.2}$$

Here and elsewhere, U_N denotes the group of $N \times N$ unitary matrices, which is a real Lie group of dimension N^2. The matrix U in (B.2) can be specified only up to a permutation action (permuting the eigenvalues x_i) and to a torus action of $(\mathrm{U}_1)^N$ (diagonal unitary matrices).

Introduce the subset $\mathrm{H}_N^{ss} \subset \mathrm{H}_N$ of hermitian matrices with distinct eigenvalues. H_N^{ss} is dense, open (in the Zariski and hence in the euclidean topology), and it has full Lebesgue measure. To see this it is enough to note that $\mathrm{H}_N \setminus \mathrm{H}_N^{ss}$ is a Zariski closed set in $\mathrm{H}_N = \mathbb{R}^{N^2}$, whose components have codimension greater or equal to 3. Indeed, $\mathrm{H}_N \setminus \mathrm{H}_N^{ss} \ni M$ is cut out by an equation which is polynomial in the entries of M, namely the discriminant of the characteristic polynomial of M should vanish. As for the codimension, observe that

$$\Delta := \{M \in \mathrm{H}_N : \; M \text{ has eigenvalues } x_1 < x_2 < ... < x_{N-1} \text{ where } x_1 \text{ has multiplicity } 2\}$$

is a smooth submanifold of H_N and we have a diffeomorphism

$$\begin{aligned} &\left\{(x_1, ..., x_{N-1}) \in \mathbb{R}^{N-1} : \; x_1 < ... < x_{N-1}\right\} \times \frac{\mathrm{U}_N}{\mathrm{U}_2 \times (\mathrm{U}_1)^{N-2}} \to \Delta : \\ &(x_1, ..., x_{N-1}, [U]) \mapsto U \,\mathrm{diag}(x_1, x_1, x_2, ..., x_{N-1}) U^{\dagger} \end{aligned} \tag{B.3}$$

where we consider the diagonal embedding $U_2 \times (\mathrm{U}_1)^{N-2} < \mathrm{U}_N$. The codomain in (B.3) is a real smooth manifold of dimension

$$N - 1 + \dim \mathrm{U}_N - (\dim \mathrm{U}_2 + (N-2) \dim \mathrm{U}_1) = N - 1 + N^2 - (4 + N - 2) = N^2 - 3.$$

Therefore, any integral over H_N with respect to a Lebesgue-absolutely continuous measure can be restricted to H_N^{ss}.

As in (B.3), we have a diffeomorphism

$$\phi : \left\{(x_1, ..., x_N) \in \mathbb{R}^N : \; x_1 < ... < x_N\right\} \times \frac{\mathrm{U}_N}{(\mathrm{U}_1)^N} \to \mathrm{H}_N^{ss}$$
$$(x_1, ..., x_N, [U]) \mapsto U \,\mathrm{diag}(x_1, ..., x_N) U^\dagger \tag{B.4}$$

where again we consider the diagonal embedding $(\mathrm{U}_1)^N < \mathrm{U}_N$.

We now compute the Jacobian determinant of ϕ, in order to derive useful formulæ reducing the integration over H_N to integration over eigenvalues and U_N. To this end, let us recall that the unitary group U_N admits a bi-invariant volume form $\mathrm{d}U$ (*Haar measure*), any other bi-invariant volume form being a nonzero scalar multiple of it. For us it is convenient to introduce $\mathrm{d}U$ as follows. Put a bi-invariant riemannian metric on $\mathrm{U}_N \subset \mathrm{Mat}_N(\mathbb{C})$ by restricting the standard bi-invariant euclidean metric

$$\|X\| := \mathrm{Re}\,\mathrm{tr}\,(X^\dagger X) = \sum_{j,k=1}^{N} |X_{jk}|^2 \tag{B.5}$$

on $\mathrm{Mat}_N(\mathbb{C}) \ni X$. Then let $\mathrm{d}U$ be the riemannian volume form on U_N associated with this riemannian metric[1]. $\mathrm{d}U$ is bi-invariant by construction.

Due to bi-invariance, $\mathrm{d}U$ descends to a measure on the homogeneous manifold $\mathrm{U}_N/(\mathrm{U}_1)^N$, which by abuse of notation we denote by the same symbol.

Proposition B.1.1. *The pullback via the diffeomorphism ϕ in (B.4) of the restriction to H_N^{ss} of the Lebesgue measure $\mathrm{d}M$ in (B.1) is*

$$\phi^* \mathrm{d}M = \Delta^2(x_1, ..., x_N) \mathrm{d}x_1 \cdots \mathrm{d}x_N \mathrm{d}U$$

where $\Delta(x_1, ..., x_N)$ is the Vandermonde determinant and $\mathrm{d}U$ is the measure on $\mathrm{U}^N/(\mathrm{U}_1)^N$ introduced above.

Proof. We first prove tha statement at points of the form $(x_1, ..., x_N, [\mathbf{1}])$. In a neighborhood of the identity $\mathbf{1} \in \mathrm{U}_N$ we can use coordinates $U = \mathbf{1} + \mathrm{i}H$ where H is hermitian, and so in a neighborhood of the point $[1] \in \mathrm{U}_N/(\mathrm{U}_1)^N$ we can use the off-diagonal entries of an hermitian matrix H as coordinates. To compute the differential of ϕ in this chart at the points $(x_1, ..., x_N, [\mathbf{1}])$, we introduce a small parameter ϵ and compute the linear part of the variation

$$\begin{aligned} &\phi(x_1 + \epsilon \mathrm{d}x_1, ..., x_N + \epsilon \mathrm{d}x_N, [\mathbf{1} + \mathrm{i}\epsilon \mathrm{d}H]) - \phi(x_1, ..., x_N, [\mathbf{1}]) \\ &\quad = \epsilon\,(\mathrm{diag}(\mathrm{d}x_1, ..., \mathrm{d}x_N) + \mathrm{i}[\mathrm{d}H, X]) + \mathcal{O}(\epsilon^2). \end{aligned}$$

This implies

$$\mathrm{d}\phi|_{(x_1,...,x_N,[\mathbf{1}])} = \mathrm{diag}(\mathrm{d}x_1, ..., \mathrm{d}x_N) + \mathrm{i}[\mathrm{d}H, X]$$

or, more explicitly, with $M = \phi(x_1, ..., x_N, [U])$ and dropping the point $(x_1, ..., x_N, [\mathbf{1}])$ from the notation,

$$\mathrm{d}M_{ii} = \mathrm{d}x_i, \qquad \mathrm{d}M_{ab} = \mathrm{i}(x_b - x_a)\mathrm{d}H_{ab}.$$

Therefore

$$\begin{aligned} &\prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{1 \le a < b \le N} \mathrm{dRe}\, M_{ab} \prod_{1 \le a < b \le N} \mathrm{dIm}\, M_{ab} \\ &\quad = \Delta^2(x_1, ..., x_N) \prod_{i=1}^{N} \mathrm{d}x_i \prod_{1 \le a < b \le N} \mathrm{dRe}\, H_{ab} \prod_{1 \le a < b \le N} \mathrm{dIm}\, H_{ab}. \end{aligned}$$

Note that in the chart $[U] = [\mathbf{1} + \mathrm{i}H]$ in a neighborhood of $[\mathbf{1}] \in \mathrm{U}_N/(\mathrm{U}_1)^N$, restriction of the metric (B.5) is written as $\mathrm{tr}\,(\mathrm{d}H^\dagger \mathrm{dH})$, and so its volume form $\mathrm{d}U$ takes the form (recall that H is off-diagonal)

$$\mathrm{d}U = \prod_{1 \le a < b \le N} \mathrm{dRe}\, H_{ab} \prod_{1 \le a < b \le N} \mathrm{dIm}\, H_{ab} \tag{B.6}$$

at the point $[\mathbf{1}] \in \mathrm{U}_N/(\mathrm{U}_1)^N$, and so the proof is complete at points of the form $(x_1, ..., x_N, [\mathbf{1}])$. The statement at a general point $(x_1, ..., x_N, [U_0])$ follows from the invariance $\mathrm{d}U = \mathrm{d}(UU_0)$ and from the invariance of the euclidean structure on H_N, $\mathrm{tr}\,(M^\dagger M) = \mathrm{tr}\,((U_0 M U_0^\dagger)^\dagger U_0 M U_0^\dagger)$. ∎

[1] Recall that the riemannian volume form associated with a riemannian metric $g = g_{ij}\mathrm{d}x^i \mathrm{d}x^j$ on a manifold of dimension d is defined as $\sqrt{\det g_{ij}(x_1, ..., x_d)}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d$, in any coordinate chart $(x^1, ..., x^d)$.

Lemma B.1.2. *For all $N \geq 1$ we have*

$$\int_{\mathrm{U}_N/(\mathrm{U}_1)^N} \mathrm{d}U = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N-1} \ell!}.$$

Proof. Since $\mathrm{H}_N \simeq \mathbb{R}^{N^2}$ and

$$\operatorname{tr} M^2 = \sum_{a,b=1}^{N} M_{ab} M_{ba} = \sum_{a,b=1}^{N} M_{ab} M_{ba} = \sum_{i=1}^{N} M_{ii}^2 + 2 \sum_{1 \leq a < b \leq N} |M_{ab}|^2$$

the Gaussian integral

$$\mathcal{I} = \int_{\mathrm{H}_N} \exp\left(-\frac{\operatorname{tr} M^2}{2}\right) \mathrm{d}M$$

is easily evaluated as $\mathcal{I} = \sqrt{2}^N \sqrt{\pi}^{N^2}$. However, denoting $\mathrm{Vol}_N = \int_{\mathrm{U}_N/(\mathrm{U}_1)^N} \mathrm{d}U$, we also have

$$\mathcal{I} = \frac{\mathrm{Vol}_N}{N!} \int_{\mathbb{R}^N} \Delta^2(x_1, ..., x_N) \mathrm{e}^{-\frac{x_1^2}{2} - \cdots - \frac{x_N^2}{2}} \mathrm{d}x_1 \cdots \mathrm{d}x_N \tag{B.7}$$

using Prop. B.1.1. The integral over $\mathbb{R}^N$ is evaluated by introducing the monic Hermite polynomials (compare with Sec. 3.5.1)

$$\pi_\ell(z) := 2^{-\frac{\ell}{2}} H_\ell\left(\frac{z}{\sqrt{2}}\right) = (-1)^\ell \mathrm{e}^{\frac{z^2}{2}} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} \mathrm{e}^{-\frac{z^2}{2}}\right)$$

which satisfy the orthogonality property

$$\int_{\mathbb{R}} \pi_\ell(z) \pi_{\ell'}(z) \mathrm{e}^{-\frac{z^2}{2}} \mathrm{d}z = \sqrt{2\pi} \ell! \delta_{\ell,\ell'}. \tag{B.8}$$

Noting that $\Delta(x_1, ..., x_N) = \det\left(x_i^{j-1}\right)_{i,j=1}^N = \det\left(\pi_{j-1}(x_i)\right)_{i,j=1}^N$ and using the orthogonality property (B.8) we have (compare with Lemma 3.1.1)

$$\int_{\mathbb{R}^N} \Delta^2(x_1, ..., x_N) \mathrm{e}^{-\frac{x_1^2}{2} - \cdots - \frac{x_N^2}{2}} \mathrm{d}x_1 \cdots \mathrm{d}x_N = N! \prod_{\ell=0}^{N-1} \int_{\mathbb{R}} \pi_\ell^2(z) \mathrm{e}^{-\frac{z^2}{2}} \mathrm{d}z = N! \sqrt{2}^N \sqrt{\pi}^N \prod_{\ell=0}^{N-1} \ell! \tag{B.9}$$

and the proof is complete comparing (B.7) and (B.9). ∎

Hence we have the following basic result.

Corollary B.1.3 (Weyl integration formula)**.** *Let $f(M) : \mathrm{H}_N \to \mathbb{C}$ a U_N-invariant scalar function, i.e. $f(M) = f(UMU^\dagger)$ for all $U \in \mathrm{U}_N$. Assuming $f \in L^1(\mathrm{H}_N, \mathrm{d}M)$ then*

$$\int_{\mathrm{H}_N} f(M) \mathrm{d}M = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N} \ell!} \int_{\mathbb{R}^N} \Delta^2(x_1, ..., x_N) f\left(\operatorname{diag}(x_1, ..., x_N)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

Proof. It follows from the chain of equalities, writing $\vec{x} = (x_1, ..., x_N)$ for short,

$$\begin{aligned}\int_{\mathrm{H}_N} f(M) \mathrm{d}M &= \int_{\mathrm{H}_N^{ss}} f(M) \mathrm{d}M = \int_{\mathrm{U}_N/(\mathrm{U}_1)^N} \mathrm{d}U \int_{\{x_1 < \ldots < x_N\}} \Delta^2(\vec{x}) f(\operatorname{diag} \vec{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_N \\ &= \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^{N-1} \ell!} \frac{1}{N!} \int_{\mathbb{R}^N} \Delta^2(\vec{x}) f(\operatorname{diag} \vec{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_N\end{aligned}$$

where we use that H_N^{ss} has full measure in H_N, we apply Lemma B.1.2, and we note that $f\left(\operatorname{diag} \vec{x}\right)$ is a symmetric function of $x_1, ..., x_N$ due to the U_N-invariance. ∎

B.2 Normal matrices. Weyl integration formula

Let Σ be a smooth contour in the complex plane. Define the following subset in the space of complex $N \times N$ matrices M

$$\mathrm{H}_N(\Sigma) := \{M = U\,\mathrm{diag}(z_1, ..., z_N)U^\dagger,\ U \in \mathrm{U}_N,\ z_1, ..., z_N \in \Sigma\}.$$

Note that $\mathrm{H}_N(\mathbb{R}) = \mathrm{H}_N$; moreover note that every $M \in \mathrm{H}_N(\Sigma)$ commutes with $M^\dagger$, i.e. they are *normal matrices*.

As before, let us consider the subset $H_N^{ss}(\Sigma) \subset H_N(\Sigma)$ of matrices with distinct eigenvalues; this open locus has a topological covering ϕ of degree $N!$

$$\phi : \Sigma^N \times \frac{\mathrm{U}_N}{(\mathrm{U}_1)^N} \to \mathrm{H}_N^{ss}(\Sigma) : (z_1, ..., z_N, [U]) \mapsto U\,\mathrm{diag}(z_1, ..., z_N)U^\dagger.$$

In analogy with Prop. B.1.1 it is natural to introduce the measure

$$\mathrm{d}M := \phi_* \frac{\Delta^2(z_1, ..., z_N)}{N!} \mathrm{d}z_1 \cdots \mathrm{d}z_N \mathrm{d}U$$

on $H_N(\Sigma)$, declaring $H_N^{ss}(\Sigma)$ to have full measure. In particular if $f \in L^1(\mathrm{H}_N(\Sigma), \mathrm{d}M)$ is a U_N-invariant complex valued function, i.e. $f(M) = f(UMU^\dagger)$ for all $U \in \mathrm{U}_N$, then

$$\int_{\mathrm{H}_N(\Sigma)} f(M)\mathrm{d}M := \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{\ell=1}^N \ell!} \int_{\Sigma^N} \Delta^2(z_1, ..., z_N) f\left(\mathrm{diag}(z_1, ..., z_N)\right) \mathrm{d}z_1 \cdots \mathrm{d}z_N. \tag{B.10}$$

B.3 Andreief identity

Lemma B.3.1 (Andreief identity [Ad]). *Let $f_1, g_1, ..., f_N, g_N : X \to \mathbb{C}$ and let μ be a measure on X. If $f_i(x)g_j(x) \in L^1(X, \mathrm{d}\mu(x))$ for all $i, j = 1, ..., N$ then $\det\left(f_i(x_j)\right)_{i,j=1}^N \det\left(g_i(x_j)\right)_{i,j=1}^N \in L^1(X^N, \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_N))$ and*

$$\int_{X^N} \det\left(f_i(x_j)\right)_{i,j=1}^N \det\left(g_i(x_j)\right)_{i,j=1}^N \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_N) = N! \det\left(\int_X f_i(x)g_j(x)\mathrm{d}\mu(x)\right)_{i,j=1}^N.$$

Proof. It follows from the following computation.

$$\int_{X^N} \det\left(f_i(x_j)\right)_{i,j=1}^N \det\left(g_i(x_j)\right)_{i,j=1}^N \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_N)$$

(definition of determinant)

$$= \sum_{\pi,\rho \in \mathfrak{S}_N} (-1)^{|\pi||\rho|} \int_{X^N} f_1(x_{\pi(1)}) \cdots f_N(x_{\pi(N)}) g_1(x_{\rho(1)}) \cdots g_N(x_{\rho(N)}) \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_N)$$

(change of variables $\widetilde{x}_i = x_{\pi(i)}$ in the integrals)

$$\sum_{\pi,\rho \in \mathfrak{S}_N} (-1)^{|\pi||\rho|} \int_{X^N} f_1(\widetilde{x}_1) \cdots f_N(\widetilde{x}_N) g_1(\widetilde{x}_{\pi^{-1}\rho(1)}) \cdots g_N(\widetilde{x}_{\pi^{-1}\rho(N)}) \mathrm{d}\mu(\widetilde{x}_1) \cdots \mathrm{d}\mu(\widetilde{x}_N)$$

(rename $\sigma := \pi^{-1}\rho$ and observe that the terms in the sum, now with indices $\pi, \sigma \in \mathfrak{S}_N$, do not depend on π)

$$= N! \sum_{\sigma \in \mathfrak{S}_N} (-1)^{|\sigma|} \int_{X^N} f_1(\widetilde{x}_1) \cdots f_N(\widetilde{x}_N) g_1(\widetilde{x}_{\sigma(1)}) \cdots g_N(\widetilde{x}_{\sigma(N)}) \mathrm{d}\mu(\widetilde{x}_1) \cdots \mathrm{d}\mu(\widetilde{x}_N)$$

(rename $\sigma \mapsto \sigma^{-1}$)

$$= N! \sum_{\sigma \in \mathfrak{S}_N} (-1)^{|\sigma|} \int_{X^N} f_1(\widetilde{x}_1) \cdots f_N(\widetilde{x}_N) g_{\sigma(1)}(\widetilde{x}_1) \cdots g_{\sigma(N)}(\widetilde{x}_N) \mathrm{d}\mu(\widetilde{x}_1) \cdots \mathrm{d}\mu(\widetilde{x}_N)$$

(the integrand is factorized)

$$= N! \sum_{\sigma \in \mathfrak{S}_N} (-1)^{|\sigma|} \int_X f_1(x) g_{\sigma(1)}(x) \mathrm{d}\mu(x) \cdots \int_X f_N(x) g_{\sigma(N)}(x) \mathrm{d}\mu(x)$$

(definition of determinant)

$$= N! \det \left(\int_X f_i(x) g_j(x) \mathrm{d}\mu(x) \right)_{i,j=1}^N .$$

■

B.4 Character expansions. Harish-Chandra–Itzykson–Zuber and other unitary integrals

In this section we consider the *normalized* bi-invariant measure on U_N

$$\mathrm{d}_* U := \frac{\mathrm{d}U}{\int_{\mathrm{U}_N} \mathrm{d}U}$$

where $\mathrm{d}U$ is any bi-invariant measure on U_N (for instance the one introduced above).

The following theorem (in a much more general form) is due to Harish-Chandra [Hb] and was subsequently rediscovered by Itzykson and Zuber [IZa].

Theorem B.4.1. *Let* $A = \operatorname{diag}(a_1, ..., a_N), B = \operatorname{diag}(b_1, ..., b_N)$ *be diagonal* $N \times N$ *matrices. Then*

$$\int_{\mathrm{U}_N} \exp \operatorname{tr} (AUBU^\dagger) \mathrm{d}_* U = \left(\prod_{\ell=0}^{N-1} \ell! \right) \frac{\det \left(\mathrm{e}^{a_j b_k} \right)_{j,k=1}^N}{\Delta(a_1, ..., a_N) \Delta(b_1, ..., b_N)}. \tag{B.11}$$

The *Harish-Chandra–Itzykson–Zuber* formula (B.11) has many proofs, see for instance [ZZ]; we choose to report here the one based on character expansion [Ba] so to introduce a method which is also relevant to the discussion of the Brezin–Gross–Witten model, in particular to Prop. 5.1.1.

Let us recall that the *Schur–Weyl construction* [FH] associates to each partition λ of length $\ell(\lambda) \leq N$ a finite dimensional irreducible representations of GL_N. We shall set $\lambda_i := 0$ for all $i > \ell(\lambda)$. The associated characters are given by *Weyl formula*

$$\chi_\lambda(T) := \frac{\det \left(t_j^{\lambda_k + N - k} \right)_{j,k=1}^N}{\Delta(t_1, ..., t_N)} = \frac{\det \left(t_j^{\lambda_k + N - k} \right)_{j,k=1}^N}{\det \left(t_j^{N-k} \right)_{j,k=1}^N} \tag{B.12}$$

where $t_1, ..., t_N$ are the eigenvalues (in the standard representation) of $T \in \mathrm{GL}_N$. Note the following formula for the dimension of these representations

$$d_{\lambda,N} := \chi_\lambda(\mathbf{1}_N) = \left(\prod_{k=1}^N \frac{(\lambda_k + N - k)!}{(k-1)!} \right) \det \left(\frac{1}{(\lambda_k + j - k)!} \right)_{j,k=1}^N \tag{B.13}$$

which follows directly by taking the limit $T \to \mathbf{1}$ in (B.12). There are several alternative formulæ for $d_{\lambda,N}$, but this one is already convenient for our later purposes.

The remarkable fact we are going to use is that (B.12) are the characters also of the representation restricted to $\mathrm{U}_N \subset \mathrm{GL}_N$, hence we can apply *Schur orthogonality*. In the present case it reads

$$\int_{\mathrm{U}_N} R^\lambda_{jk}(U) R^{\lambda'}_{j'k'}(U^\dagger) \mathrm{d}_* U = \frac{\delta_{\lambda\lambda'}}{d_{\lambda,N}} \delta_{jk'} \delta_{kj'} \tag{B.14}$$

where R^λ is the aforementioned representation of GL_N labeled by the partition λ. We note two useful immediate consequences

$$\int_{\mathrm{U}_N} \chi_\lambda(UA) \chi_{\lambda'}(U^\dagger B) \mathrm{d}_* U = \chi_\lambda(AB) \frac{\delta_{\lambda\lambda'}}{d_{\lambda,N}}, \tag{B.15}$$

$$\int_{\mathrm{U}_N} \chi_\lambda(UAU^\dagger B) \mathrm{d}_* U = \frac{\chi_\lambda(A) \chi_\lambda(B)}{d_{\lambda,N}}. \tag{B.16}$$

Proof of (B.15) and (B.16). Expanding the traces and using Schur orthogonality (B.14) we have

$$\begin{aligned}\int_{\mathrm{U}_N} \chi_\lambda(UA)\chi_{\lambda'}(U^\dagger B)\mathrm{d}_*U &= \int_{\mathrm{U}_N} \sum_{j,k,j',k'} R^\lambda_{jk}(U)R^\lambda_{kj}(A)R^{\lambda'}_{j'k'}(U^\dagger)R^{\lambda'}_{k'j'}(B)\mathrm{d}_*U \\ &= \sum_{j,k,j',k'} R^\lambda_{kj}(A)R^{\lambda'}_{k'j'}(B)\frac{\delta_{\lambda\lambda'}}{d_{\lambda,N}}\delta_{jk'}\delta_{kj'} \\ &= \sum_{j,k} R^\lambda_{kj}(A)R^\lambda_{jk}(B)\frac{\delta_{\lambda\lambda'}}{d_{\lambda,N}} = \chi_\lambda(AB)\frac{\delta_{\lambda\lambda'}}{d_{\lambda,N}}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathrm{U}_N} \chi_\lambda(UAU^\dagger B)\mathrm{d}_*U &= \int_{\mathrm{U}_N} \sum_{i,j,k,l} R^\lambda_{ij}(U)R^\lambda_{jk}(A)R^\lambda_{kl}(U^\dagger)R^\lambda_{li}(B)\mathrm{d}_*U \\ &= \sum_{i,j,k,l} R^\lambda_{jk}(A)R^\lambda_{li}(B)\frac{\delta_{jk}\delta_{il}}{d_{\lambda,N}} \\ &= \frac{1}{d_{\lambda,N}}\sum_{j,i} R^\lambda_{jj}(A)R^\lambda_{ii}(B) = \frac{\chi_\lambda(A)\chi_\lambda(B)}{d_{\lambda,N}}.\end{aligned}$$

■

Below we are going to use only (B.15), although (B.16) comes in handy in the derivation of the expression of Prop. 5.1.1 for the partition function of the Brezin–Gross–Witten model.

The last ingredient we need are the following consequences of the *Binet–Cauchy formula.*

Lemma B.4.2. *Let $\phi(t) = \sum_{n\geq 0}\phi_n t^n$. Then the following identities hold true.*

1. *If $t_1, ..., t_N$ are the eigenvalues of T, we have*

$$\phi(t_1)\cdots\phi(t_N) = \sum_{\ell(\lambda)\leq N} \chi_\lambda(T)\det\left(\phi_{\lambda_k+j-k}\right)_{j,k=1}^N \tag{B.17}$$

   where we set $\phi_j := 0$ for $j<0$.

2. *If $a_1, ..., a_N$ and $b_1, ..., b_N$ are the eigenvalues of A and B respectively, we have*

$$\frac{\det\left(\phi(a_jb_k)\right)_{j,k=1}^N}{\Delta(a_1,...,a_N)\Delta(b_1,...,b_N)} = \sum_{\ell(\lambda)\leq N}\left(\prod_{k=1}^N \phi_{\lambda_k+N-k}\right)\chi_\lambda(A)\chi_\lambda(B). \tag{B.18}$$

In both statements, the sum on the right is meant over partitions λ of length $\ell(\lambda)\leq N$. In all cases such sums come from an application of the Binet–Cauchy formula using the following identification

$$\begin{gathered}\{\text{partitions } \lambda \text{ of length } \ell(\lambda)\leq N\} \leftrightarrow \{\text{strictly increasing sequences } n_1 > \cdots > n_N \geq 0\} \\ \lambda \mapsto n_k := \lambda_k + N - k\end{gathered} \tag{B.19}$$

where we recall that we set $\lambda_i = 0$ whenever $i > \ell(\lambda)$.

Proof.

1. We compute

$$\begin{aligned}\Delta(t_1,...,t_N)\phi(t_1)\cdots\phi(t_N) &= \det\left(t_j^{N-k}\phi(t_j)\right)_{j,k=1}^N = \det\left(\sum_{n\geq 0}\phi_n t_j^{N-k+n}\right)_{j,k=1}^N \\ &= \det\left(\begin{pmatrix} 0 & 0 & \cdots & \phi_0 & \cdots \\ \vdots & \vdots & \iddots & \vdots & \\ 0 & \phi_0 & \cdots & \phi_{N-2} & \cdots \\ \phi_0 & \phi_1 & \cdots & \phi_{N-1} & \cdots \end{pmatrix}\cdot\begin{pmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_N \\ t_1^2 & \cdots & t_N^2 \\ \vdots & & \vdots \end{pmatrix}\right) \\ &= \sum_{n_1>\ldots>n_N\geq 0} \det(\phi_{n_k+j-N})_{j,k=1}^N \det(t_k^{n_j})_{j,k=1}^N\end{aligned}$$

where in the last step we use the Binet–Cauchy formula (for we need to take the determinant of the product of an $N\times\infty$ with an $\infty\times N$ matrix). The result then follows by the change of summation indices (B.19) and by Weyl formula (B.12).

2. We compute

$$\det\left(\phi(a_jb_k)\right)=\det\left(\begin{pmatrix}\phi_0 & \phi_1a_1 & \phi_2a_2^2 & \cdots\\ \vdots & \vdots & \vdots & \\ \phi_0 & \phi_1a_N & \phi_2a_N^2 & \cdots\end{pmatrix}\cdot\begin{pmatrix}1 & \cdots & 1\\ b_1 & \cdots & b_N\\ b_1^2 & \cdots & b_N^2\\ \vdots & & \vdots\end{pmatrix}\right)$$

hence from Binet–Cauchy formula we obtain

$$\begin{aligned}\det\left(\phi(a_jb_k)\right)&=\sum_{n_1>\ldots>n_N\geq 0}\det\left(\phi_{n_k}a_j^{n_k}\right)_{j,k=1}^N\det\left(b_j^{n_k}\right)_{j,k=1}^N\\ &=\sum_{\ell(\lambda)\leq N}\left(\prod_{k=1}^N\phi_{\lambda_k+N-k}\right)\det\left(a_j^{\lambda_k+N-k}\right)_{j,k=1}^N\det\left(b_j^{\lambda_k+N-k}\right)_{j,k=1}^N\end{aligned}$$

again using the change of summation indices (B.19). Dividing by $\Delta(a_1,\ldots,a_N)\Delta(b_1,\ldots,b_N)$ and using Weyl formula (B.12) the proof is complete. ■

We are now ready to give the proof of Thm. B.4.1.

Proof of Thm. B.4.1. Applying (B.17) with $\phi(t)=\exp t$, i.e. $\phi_j=\frac{1}{j!}$, and $T=UAU^\dagger B$ we have

$$\exp\operatorname{tr}(UAU^\dagger B)=\sum_{\ell(\lambda)\leq N}\det\left(\frac{1}{(\lambda_j+k-j)!}\right)\chi_\lambda(UAU^\dagger B)$$

hence using (B.15) we obtain

$$\begin{aligned}\int_{\mathrm{U}_N}\exp\operatorname{tr}(UAU^\dagger B)\mathrm{d}_*U&=\sum_{\ell(\lambda)\leq N}\det\left(\frac{1}{(\lambda_j+k-j)!}\right)\frac{\chi_\lambda(A)\chi_\lambda(B)}{d_{\lambda,N}}\\ &=\sum_{\ell(\lambda)\leq N}\left(\prod_{k=1}^N\frac{(k-1)!}{(N+\lambda_k-k)!}\right)\chi_\lambda(A)\chi_\lambda(B)\end{aligned}$$

where in the last step we use the explicit formula (B.13). The proof is complete by applying (B.18). ■

Identity (B.11) has several generalization, most notably to other groups. Confining ourselves instead to unitary matrix integrals, a straightforward generalization goes as follows [HO, App. A].

Fix any sequence of complex numbers $r=(r_j)_{j=1}^\infty$ and denote

$$r_\ell^\downarrow:=\prod_{j=1}^\ell r_j=r_1\cdots r_\ell.$$

For all partitions λ of length $\ell(\lambda)\leq N$ denote

$$r_{\lambda,N}:=\prod_{k=1}^{\ell(\lambda)}\frac{r^\downarrow_{N+\lambda_k-k}}{r^\downarrow_{N-k}}=\prod_{k=1}^{N}\frac{r^\downarrow_{N+\lambda_k-k}}{r^\downarrow_{N-k}}=\prod_{k=1}^N\left(r_{N+1-k}\cdots r_{N+\lambda_k-k}\right)\tag{B.20}$$

and set

$$\phi_r(x):=\sum_{\ell\geq 0}r_\ell^\downarrow x^\ell=1+r_1x+r_1r_2x^2+r_1r_2r_3x^3+\cdots.\tag{B.21}$$

Theorem B.4.3. *Let $A=\operatorname{diag}(a_1,\ldots,a_N),B=\operatorname{diag}(b_1,\ldots,b_N)$ be diagonal $N\times N$ matrices. Then*

$$\int_{\mathrm{U}_N}\sum_{\ell(\lambda)\leq N}d_{\lambda,N}r_{\lambda,N}\chi_\lambda(AUBU^\dagger)\mathrm{d}_*U=\frac{1}{\prod_{k=1}^{N-1}r_k^\downarrow}\frac{\det\left(\phi_r(a_jb_k)\right)_{j,k=1}^N}{\Delta(a_1,\ldots,a_N)\Delta(b_1,\ldots,b_N)}.\tag{B.22}$$

Proof. Using (B.15) and the definition (B.20) of $r_{\lambda,N}$, we have

$$\begin{aligned}\int_{\mathrm{U}_N} \sum_{\ell(\lambda)\leq N} \mathrm{d}_{\lambda,N} r_{\lambda,N}\chi_\lambda(AUBU^\dagger)\mathrm{d}_*U &= \sum_{\ell(\lambda)\leq N} r_{\lambda,N}\chi_\lambda(A)\chi_\lambda(B)\\ &= \sum_{\ell(\lambda)\leq N}\prod_{k=1}^N \frac{r^{\downarrow}_{N+\lambda_k-k}}{r^{\downarrow}_{N-k}}\chi_\lambda(A)\chi_\lambda(B)\end{aligned}$$

and now we conclude by using (B.18) and the definition (B.21) of ϕ_r. ■

Note the *confluent* version of (B.22)

$$\sum_{\ell(\lambda)\leq N} d_{\lambda,N} r_{\lambda,N}\chi_\lambda(A) = \frac{1}{\prod_{k=1}^{N-1}(k!r^{\downarrow}_k)}\frac{\det\left(a_j^{k-1}\phi_r^{k-1}(a_j)\right)_{j,k=1}^N}{\Delta(a_1,...,a_N)} \tag{B.23}$$

obtained by taking $B\to\mathbf{1}$ in (B.22).

We finally make two comments on this generalization. First, note that the Harish-Chandra–Itzykson–Zuber formula (B.11) corresponds to the choice $r_j := \frac{1}{j}$, for which (B.21) is $\phi_r(x) = \mathrm{e}^x$. Second, that the expression of Prop. 5.1.1 for the Brezin–Gross–Witten partition function, which we have derived there by the character expansion method using (B.16), can also be recognized as the confluent version (B.23) in the case $r_j := \frac{1}{j(j+\nu)}$. Indeed, in this case (B.21) is given in terms of the Bessel I function as

$$\phi_r(x) = \frac{\Gamma(\nu+1)}{\sqrt{x}^\nu}\mathrm{I}_\nu(2\sqrt{x})$$

and we note the identity

$$\det\left(x_j^{k-1}\mathrm{I}_\nu^{(k-1)}(2\sqrt{x_j})\right)_{j,k=1}^N = \det\left(\sqrt{x_j}^{k-1}\mathrm{I}_{\nu+k-1}(2\sqrt{x_j})\right)_{j,k=1}^N. \tag{B.24}$$

Identity (B.24) can be proved in the same spirit as in Prop. 5.4.1 by inductively using the identity

$$x\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{I}_\nu(2\sqrt{x}) = \sqrt{x}\mathrm{I}_{\nu+1}(2\sqrt{x}) + \frac{\nu}{2}\mathrm{I}_\nu(2\sqrt{x})$$

to recognize that the ratio of the two matrices in (B.24) is in the form identity plus a strictly triangular matrix.

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