Scuola Internazionale Superiore di Studi Avanzati - Trieste

Doctoral Thesis

A Tale of Two Indices
or
How to Count Black Hole Microstates in AdS/CFT

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Abstract

In this thesis we investigate on the microscopic nature of the entropy of black holes in string theory in the context of AdS/CFT correspondence. We focus our attention on two particularly interesting cases in different dimensions. First, we consider static dyonic BPS black holes in AdS_4 in 4d $\mathcal{N} = 2$ gauged supergravities with vector and hyper multiplets. More precisely, we focus on the example of BPS black holes in $\mathrm{AdS}_4 \times S^6$ in massive Type IIA, whose dual three-dimensional holographic description is known and simple. To provide a microscopic counting of the black hole entropy we employ a topologically twisted index in the dual field theory, which can be computed exactly and non-perturbatively with localization techniques. We find perfect match at leading order. Second, we turn to rotating electrically-charged BPS black holes in $\mathrm{AdS}_5 \times S^5$ in Type IIB. Here, the microscopic entropy counting is done in terms of the superconformal index of the dual $\mathcal{N} = 4$ Super-Yang-Mills theory. We proceed by deriving a new formula, which expresses the index as a finite sum over the solution set of certain transcendental equations, that we dub Bethe Ansatz Equations, of a function evaluated at those solutions. Then, using the latter formula, we reconsider the large N limit of the superconformal index. We find an exponentially large contribution which exactly reproduces the Bekenstein-Hawking entropy of the black holes of Gutowski-Reall. Besides, the large N limit exhibits a complicated structure, with many competing exponential contributions and Stokes lines, hinting at new physics.

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Preface

This thesis is based on the following publications and preprints, listed in chronological order:

[1] F. Benini, H. Khachatryan, and P. Milan, "Black hole entropy in massive Type IIA," Class. Quant. Grav. **35** no. 3, (2018) 035004, `arXiv:1707.06886 [hep-th]`

[2] F. Benini and P. Milan, "A Bethe Ansatz type formula for the superconformal index," `arXiv:1811.04107 [hep-th]`. Submitted to Communications in Mathematical Physics.

[3] F. Benini and P. Milan, "Black holes in 4d $\mathcal{N} = 4$ Super-Yang-Mills," `arXiv:1812.09613 [hep-th]`

Contents

Introduction and summary

One of the fascinating aspects of the physics of black holes is its connection with the laws of thermodynamics. Of particular importance is the fact that black holes carry a macroscopic entropy [4–8], semi-classically determined in terms of the horizon area by the famous Bekenstein-Hawking formula:

$$S_{\text{BH}} = \frac{k_B\, c^3}{\hbar\, G_N} \frac{\text{Area}}{4} \,, \tag{1}$$

where c is the speed of light, k_B is the Boltzmann constant, $\hbar$ is the reduced Planck constant and G_N is the Newton constant (to simplify the notation, throughout the rest of this thesis we will work in natural units and set $c = k_B = \hbar = 1$). In this formula, the $\frac{1}{4}$ proportionality factor is very important and it has been very precisely determined by Hawking [8], using arguments involving the black hole thermal radiation—the Hawking radiation.

In the search for a theory of quantum gravity, explaining the microscopic origin of black hole thermodynamics is a fundamental but challenging test. As string theory is proposed to embed gravity in a consistent quantum system, it should in particular provide a microscopic description of (1) in terms of a degeneracy of string states. Evidences that such expectation is correct have been put forward by appealing to the nature of black holes arising in string theory. The latter can indeed be formed from systems of branes [9] which, in turn, admit a description in terms of a worldvolume gauge theory [10–15]. This provides a powerful alternative point of view, besides the gravitational description, much more amenable to a quantum treatment. This framework was first used by Strominger and Vafa [16] to show that, within string theory, one can give a microscopic statistical interpretation to the thermodynamic Bekenstein-Hawking entropy (1) of BPS black holes in flat space.

Many different setups have been analyzed since then, including quantum corrections, to an impressive precision [17–21] (see *e.g.* [22] for more references). In essentially all examples, the microscopic counting is performed in a 2d conformal field theory (CFT) that appears—close to the black hole horizon—as one plays with the moduli available in string theory (and takes advantage of various dualities). In fact, also the entropy of BPS black holes in AdS_3 is well understood, since the microstate counting can be performed in the 2d CFT related to AdS_3 by the AdS/CFT correspondence [23–25] (see *e.g.* [26, 27] for reviews and further references). It is important to stress here that many of the derivations of the black hole entropy in these contexts involve the use of the Cardy formula [28], which gives the asymptotic growth of CFT states for large charges.

In the case of asymptotically-AdS black holes in dimension $d \geq 4$ the situation is different, because in general there is no regime in which the black holes are described by a 2d CFT. On the other hand, the AdS/CFT duality constitutes a natural and wonderful framework to study their properties at the quantum level. The duality provides a non-perturbative definition of quantum gravity, in terms of a CFT living at the boundary of AdS space. Therefore, one would expect the black holes to appear as ensembles of states with exponentially large degeneracy in the dual field theory.[1] The problem of offering a microscopic account of the black hole entropy is then rephrased into that of counting particular CFT states.

Despite the very favorable setup, the problem in four or more dimensions has remained unsolved for many years, and only recently a concrete example was successfully studied in [29, 30]. There, the entropy match was obtained for static dyonic BPS black holes in $AdS_4 \times S^7$ in M-theory. The strategy of the computation is the following: First, the black holes are conveniently described by a consistent truncation of M-theory on S^7 to a 4d $\mathcal{N} = 2$ gauged supergravity with three Abelian vector multiplets—the so-called STU model [31]. Then, the microscopic counting is performed in the 3d boundary theory—the ABJM theory [32]. Being the black holes both electrically and magnetically charged under (flavor and) R-symmetry, the holographic duality identifies their microstates with ground states of the dual QFT quantized on S^2 (or a Riemann surface, depending on the black hole geometry) with a topological twist [33], necessary to account for the magnetic charges. Such states can be conveniently counted by a (refined) Witten index [34], defined in [35, 36] and called "topologically twisted index":

$$Z(\mathfrak{p}, \Delta) \coloneqq \operatorname{Tr}_{\mathcal{H}(S^2)} (-1)^F \, e^{-\beta H} \, e^{2\pi i \sum_a \Delta_a \mathfrak{q}_a} \, . \tag{2}$$

Here Δ_a are chemical potentials for the electric charges $\mathfrak{q}_a$, while H is the Hamiltonian on S^2, whose dependence on the topological twist is in terms of a set of integers $\mathfrak{p}_a$, associated to the magnetic charges of the black hole. Because of supersymmetry, only the ground states contribute.[2] Furthermore, the index can be exactly computed using supersymmetric localization [37] (see *e.g.* [38, 39] for nice introductions and [40] for more detailed reviews). In order to make contact with weakly-curved gravity, one should take a large N limit in the QFT. Assuming that at leading order there are no dangerous

[1] By contrast, for black holes in flat space one uses a different QFT description for each black hole.

[2] In absence of deformations, the ground states have $H = 0$. However, when real masses σ_a are turned on, the ground state energy is modified to $H = \sum_a \sigma_a \mathfrak{q}_a$ and what we use as chemical potentials are the complex parameters $\Delta_a + i\beta\sigma_a/2\pi$.

cancelations due to $(-1)^F$,[3] the quantum degeneracies can be extracted with a Fourier transform, which at large N is well approximated by a Legendre transform:

$$S_{\text{BH}} = \log Z(\mathfrak{p}, \widehat{\Delta}) - 2\pi i \sum\nolimits_a \widehat{\Delta}_a \mathfrak{q}_a \,, \tag{3}$$

with $\widehat{\Delta}_a$ such that the right-hand-side is extremized. This procedure has been dubbed "$\mathcal{I}$-extremization principle" in [29] and, for black holes in $\text{AdS}_4 \times S^7$, it exactly reproduces the Bekenstein-Hawking entropy (1). Notice that here the microstate computation in the index only involves the large N limit and thus it reproduces the black hole entropy for any value of the charges, as opposed to the case of asymptotically flat black holes, where the Cardy formula is used.

As a generalization, putting together the results in [29, 30, 43, 44] the entropy match is expected to work for generic static BPS black holes that can be described in consistent truncations to 4d $\mathcal{N} = 2$ gauged supergravities with only vector multiplets. More precisely, in the gravity description, the near-horizon region of the BPS black holes is controlled by attractor equations [43]. Schematically (and in a frame with purely electric gauging) the Bekenstein-Hawking entropy (1) is given by the value of the function

$$\mathcal{S} \,\propto\, i \sum\nolimits_a \frac{\mathfrak{p}_a \partial_a \mathcal{F}(X) - \mathfrak{q}_a X^a}{\sum_b X^b} \tag{4}$$

at its critical point. Here $(\mathfrak{p}_a, \mathfrak{q}_a)$ are the magnetic and electric charges of the black hole (in suitable units), X^a are the symplectic sections parametrizing the scalars in vector multiplets, and $\mathcal{F}$ is the prepotential. Identifying $\Delta_a = X^a / \sum_b X^b$ and using that the prepotential is homogeneous of degree two, one can write

$$\mathcal{S} \,\propto\, \sum\nolimits_a \Big(i\, \mathfrak{p}_a \frac{\partial \mathcal{F}(\Delta)}{\partial \Delta_a} - i\, \mathfrak{q}_a \Delta_a \Big) \,. \tag{5}$$

On the other hand, it has been shown in [44] that, for a large class of quiver gauge theories appearing in AdS/CFT pairs, the large N limit of the index is related to the large N limit of the S^3 free energy F_{S^3} by

$$\log Z = \frac{1}{4} \sum\nolimits_a \mathfrak{p}_a \frac{\partial F_{S^3}}{\partial \Delta_a} \,. \tag{6}$$

Thus, provided one verifies the proportionality between the supergravity prepotential and the S^3 free energy—which is a property of the conformal vacuum and has nothing to do with black holes—one has also matched the entropy of static dyonic BPS black holes.

Thanks to this general approach, a number of microscopic matches for dyonic BPS

[3] An argument, similar to the one in [41, 42], was given in [29] that the index at large N counts the precise number of single-center black hole states, while one expects cancelations to take place on states related to multi-center black holes and hair.

black holes/strings in diverse dimensions and with compact and non-compact horizons have been performed in [44–68] (see *e.g.* [69] for a nice and detailed review). Attempts to reproduce some subleading corrections have been made in [70–76]. Furthermore, proposals for a geometric gravitational dual to the $\mathcal{I}$-extremization principle have been put forward in [77–81].

As the large amount of recently produced literature can testify, the remarkable success of this approach has opened the way to new studies, devoted to improve our understanding of the microscopic nature of the black hole entropy. The aim of this thesis is to present some new developments in this program, which extend the original results to more general and, perhaps, more interesting scenarios in a universal framework. We summarize below the novel results and outline the structure of this essay:

Part I. In the first part we analyze the case of asymptotically-AdS$_4$ dyonic black holes arising in string/M-theory from consistent truncations to 4d $\mathcal{N}=2$ supergravity with both vector multiplets and hypermultiplets. In particular, we describe how their entropy is microscopically determined in terms of the topologically twisted index of the holographic dual 3d QFT. The structure of this part consists of a single chapter:

Chapter 1. This chapter contains the main results obtained in [1]. Here we study the case of black holes arising from consistent truncations to 4d $\mathcal{N}=2$ gauged supergravity with both vector and hyper multiplets. Whenever the latter multiplets are present, they can give mass to some of the vector multiplets, constraining the values of the vector multiplet scalars at the horizon. This affects the formulation of the $\mathcal{I}$-extremization principle in such a way that the standard argument presented above does not go through. The purpose of this chapter is then to investigate whether the entropy of AdS$_4$ black holes can be microscopically reproduced also in such more general theories. We consider a particularly interesting example: black holes in AdS$_4\times S^6$ in massive Type IIA. The AdS$_4$ vacuum has been recently constructed in [82] and the dual three-dimensional SCFT has been identified as well. It is a 3d $\mathcal{N}=2$ $SU(N)_k$ Chern-Simons gauge theory with three adjoint chiral multiplets and superpotential $W=\operatorname{Tr}X[Y,Z]$. Besides, the near horizon geometries of static dyonic BPS black holes have been identified in [83] (see also [84]). Making use of the attractor equations with hypermultiplets and dyonic gaugings [85], we are able to reproduce—at leading order—the entropy of those black holes from a microscopic counting. Notice, as a remark, that the black holes considered here are in massive Type IIA [86], as opposed to M-theory. As a result, the entropy scales as $N^{5/3}$ as opposed to $N^{3/2}$. Yet, the microstate counting works perfectly, thus providing a non-trivial check of the robustness of the proposal in [29].

Part II. In the second part of this thesis we depart from the realm of asymptotically-AdS dyons to study a very different case: purely electric and rotating black holes. We show that, differently from the dyonic case, in absence of magnetic charges the microscopic counting is done in the dual QFT in terms of a different index—the "superconformal index" [87–89]. Although in this case the boundary theory is not topologically twisted, a generalized version of the $\mathcal{I}$-extremization principle can be formulated and the entropy match perfectly works. The structure of this part is composed of two chapters:

Chapter 2. This chapter discusses the content of [2]. Here, the main purpose is to set the ground for the entropy match of electrically charged rotating black holes by discussing some important features of the superconformal index. In QFTs with superconformal invariance, the latter object counts (with sign) the number of local operators in short representations of the superconformal algebra. The counting can be done keeping track of the spin and other charges of the operators. Crucially, the superconformal index is protected by supersymmetry and it can be computed exactly and non-perturbatively. Despite its simplicity, this observable contains a lot of information about the theory, and indeed it has been studied in all possible dimensions (*i.e.* up to six) and under so many angles (see *e.g.* [40] for reviews). Having in mind the discussion on asymptotically-AdS$_5$ black holes of the next chapter, here we focus on the four-dimensional index.

Because the index does not depend on continuous deformations of the theory, and a suitable supersymmetric generalization thereof does not depend on the RG flow, it follows that in theories that are part of a conformal manifold and have a weakly-coupled point on it, and in theories that are asymptotically free, the evaluation of the index can be reduced to a weak coupling computation.[4] This amounts to counting all possible local operators in short representations one can write down, and then restricting to the gauge-invariant ones. In the language of radial quantization, one counts all multi-particle states in short representations on the sphere, and then imposes Gauss law. In the case of the 4d $\mathcal{N}=1$ superconformal (or supersymmetric) index of a gauge theory with gauge group G and chiral multiplets Φ_a in representation $\mathfrak{R}_a$, the counting is captured by the standard formula [87, 88, 90]:

$$\mathcal{I}(p,q;v) = \frac{(p;p)_\infty^{\mathrm{rk}(G)} (q;q)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \frac{\prod_a \prod_{\rho_a\in\mathfrak{R}_a} \Gamma\big((pq)^{r_a/2} z^{\rho_a} v^{\omega_a}; p,q\big)}{\prod_{\alpha\in\Delta}\Gamma(z^\alpha;p,q)} \prod_{i=1}^{\mathrm{rk}(G)} \frac{dz_i}{2\pi i z_i} \,. \tag{7}$$

Here, briefly, p, q are the (complex) fugacities associated to the angular momentum, v collectively indicates the fugacities for flavor symmetries, z indicates the fugacities for the gauge symmetry, r_a are the R-charges, and Γ is the elliptic gamma function.

Inspired by recent work of Closset, Kim and Willett [91, 92], we show that, when the fugacities for the angular momentum satisfy

$$q^a = p^b \tag{8}$$

for some coprime positive integers a, b, then one can derive an alternative, very different

[4] There is a small caveat: the IR superconformal R-symmetry must be visible in the UV, *i.e.* it should not be accidental.

formula for the 4d superconformal index.

The new formula is a finite sum over the solution set $\mathfrak{M}_{\text{BAE}}$ to certain transcendental equations—that we dub Bethe Ansatz equations (BAEs)—of a function, closely related to the integrand in (7), evaluated at those solutions. Very schematically, we prove that

$$\mathcal{I}(p,q;v) = \frac{(p;p)_\infty^{\text{rk}(G)}(q;q)_\infty^{\text{rk}(G)}}{|\mathcal{W}_G|} \sum_{z\,\in\,\mathfrak{M}_{\text{BAE}}} \sum_{\{m_i\}=1}^{ab} \mathcal{Z}\big(z\,h^{-m},p,q,v\big)\; H(z,p,q,v)^{-1}\,. \tag{9}$$

Here, the function $\mathcal{Z}$ is the integrand in the standard formula (7); h is a fugacity defined in terms of p,q by $p=h^a$ and $q=h^b$, for coprime $a,b\in\mathbb{N}$; $\mathfrak{M}_{\text{BAE}}$ is the set of solutions—on a torus of exponentiated modulus h—to the BAEs, which take the schematic form

$$Q_i(z,p,q,v)=1 \qquad\qquad \text{for } i=1,\ldots,\text{rk}(G) \tag{10}$$

in terms of some functions Q_i of the fugacities; the function H is the Jacobian

$$H(z,p,q,v) = \det_{ij} \frac{\partial Q_i(z,p,q,v)}{\partial \log z_j} \tag{11}$$

defined in terms of Q_i. A special case of this formula when $p=q$, namely $a=b=1$, was derived in [91]. All the details and definitions required for the new formula will be given throughout the chapter.

The condition (8) limits the applicability of the Bethe Ansatz (BA) formula (9) in the space of complex fugacities. Yet, the domain of the formula is rich enough to uniquely fix the index as a continuous function (with poles) of general fugacities. We offer two arguments, one that uses holomorphy of the index and one that just uses continuity. Roughly, the reason is that the set of pairs (p,q) satisfying (8) is dense in the space of general complex fugacities.

The BA formula (9) can be thought of, in some sense, as the "Higgs branch localization" partner of the standard "Coulomb branch localization" integral formula (7), using the terminology of [93, 94] (see also [95–97] for similar formulas). More precisely, the existence of a formula as (9) can be justified along the lines of [35, 36, 91, 92, 98, 99]. The superconformal index can be defined as the partition function of the Euclidean theory on $S^1\times S^3$, with suitable flat connections along S^1 and a suitable complex structure that depends on p,q, and with the Casimir energy [100, 101] stripped off.[5] The standard localization computation [102, 103] of the partition function leads to (7). However, when p,q satisfy (8), the geometry is also a Seifert torus fibration over S^2. Following [92], one expects to be able to reduce to the computation of a correlator on S^2 in an A-twisted theory coming from dimensional reduction on the torus [104], which should give an expression

[5] Notice that the superconformal index, up to a change of variables reviewed in Section 2.2.1, is a single-valued function of the fugacities, while the partition function is not [101].

as in (9). In any case, we have derived the BA formula (9) by standard manipulations of the integral expression and thus we do not rely on any such putative 2d reduction.

Chapter 3. The last chapter of this thesis contains the material discussed in [3]. Rotating, purely electric black holes are very important solutions of general relativity and generalizations thereof, including string theory. Providing a microscopic entropy computation for such black holes constitutes a very important yet challenging problem. Famously, the microstate counting for BPS black holes in AdS_5 has remained a long-standing open problem, which dates back to the work of [88, 105, 106]. In this context, BPS black holes arise as rotating electrically-charged solutions of Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ [107–111]. Their holographic description is in terms of 1/16 BPS states of the boundary 4d $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory on S^3, which can be counted (with sign) by the superconformal index. One would expect the contribution of the black hole microstates to the index to dominate the large N (*i.e.* weak curvature) expansion. However, the large N computation of the index performed in [88] showed no rapid enough growth of the number of states, and thus it could not reproduce the entropy of the dual black holes. Additionally, that result was followed by several studies of BPS operators at weak coupling [112–116] in which no sign of high degeneracy of states was found.

Very recently, the issue received renewed attention leading towards a different conclusion. First, the authors of [117, 118] formulated an extremization principle to extract the black hole entropy in terms of a Legendre transform, and then noticed the latter shares intriguing similarities with anomaly polynomials and Casimir energy of the dual field theory. Second, the authors of [119] related the black hole entropy to the (complexified) regularized on-shell action of the gravitational black hole solutions, and then compared the latter with the $S^1 \times S^3$ supersymmetric partition function of the field theory, finding perfect agreement at leading order in large N. Third, the authors of [120] analyzed the index in a double-scaling Cardy-like limit, finding quantitative evidence that the index does account for the entropy of large BPS black holes (whose size is much larger than the AdS radius). Lastly, in [121] it was observed that, even at finite values of the fugacities, the index exhibits a deconfinement transition before the Hawking-Page transition related to the known AdS_5 black holes, pointing towards the existence of hairy black holes.

In this chapter, we offer a resolution of the issue by revisiting the counting of 1/16 BPS states in the boundary $\mathcal{N} = 4$ SYM theory at large N. We approach the problem by using the Bethe Ansatz formula for the superconformal index, introduced in the previous chapter. This expression allows for an easier analysis of the large N limit, similar to the one performed in [29, 30]. We find that the superconformal index, *i.e.* the grand canonical partition function of 1/16 BPS states, does in fact grow very rapidly with N—as $\mathcal{O}(e^{N^2})$—for generic complex values of the fugacities. Although the BA formula can handle the general case, this is technically difficult and therefore we restrict to states and black holes with two equal angular momenta, as in [108].

The BA formulation reveals that the large N limit has a complicated structure. There are many exponentially large contributions, that somehow play the role of saddle points. As we vary the complex fugacities, those contributions compete and in different regions of the parameter space, different contributions dominate. This gives rise to Stokes lines, separating different domains of analyticity of the limit. The presence of Stokes lines also resolves the apparent tension with the computation of [88], that was performed with real fugacities. We show that when the fugacities are taken to be real, all exponentially large contributions organize into competing pairs that can conceivably cancel against each other. The fact that for real fugacities the index suffers from strong and non-generic cancelations was already stressed in [120, 121].

Our main result is to identify a particular exponential contribution, such that extracting from it the microcanonical degeneracy of states *exactly* reproduces the Bekenstein-Hawking entropy of BPS black holes in AdS_5 (whose Legendre transform was obtained in [117, 118]). This is in line with the double-scaling Cardy-like limit of [120]. Along the way, we show that the very same $\mathcal{I}$-extremization principle [29, 30] found in AdS_4, is also at work in AdS_5 guaranteeing that the index captures the total number of single-center BPS black hole states.

At the same time, we step into many other exponentially large contributions: we expect them to describe very interesting new physics, that we urge to uncover. To that purpose, we study in greater detail the case of BPS black holes with equal charges and angular momenta [107]. We find that while for large black holes their entropy dominates the superconformal index, this is not so for smaller black holes. This seems to suggest[6] that an instability, possibly towards hairy or multi-center black holes, might develop as the charges are decreased. Similar observations were made in [120, 121]. It would be extremely interesting if there were some connections with the recent works [122–126], and we leave this issue for future investigations.

For more recent advances in the study of the AdS_5 black holes, the dual index and generalizations thereof we refer to [127–134], where approaches similar to the ones discussed here are employed. See also [135, 136] for discussions on rotating black holes in dimensions other than five.

[6]We are grateful to Shiraz Minwalla and Sameer Murthy for suggesting this possibility to us.

Part I

The topologically twisted index and AdS_4 black holes

1 Black hole entropy in massive Type IIA

We study asymptotically-AdS_4 dyonic black holes in massive Type IIA on S^6, arising from a consistent truncation to 4d $\mathcal{N} = 2$ supergravity with both vector multiplets and hypermultiplets. In particular, we show that the Bekenstein-Hawking entropy of these black holes can be holographically reproduced in terms of the topologically twisted index of the dual 3d QFT.

The chapter is organized as follows. In Section 1.1 we describe the near-horizon geometries of static dyonic BPS black holes in $AdS_4 \times S^6$. We recast their entropy in the form of the solution to an extremization problem. In Section 1.2 we compute the index in the field theory, at leading order in N, and express again the microstate degeneracy as the solution to an extremization problem. In Section 1.3 we show that the two problems coincide. We conclude in Section 1.4.

Note added. When this work was under completion, we became aware of the related works [49] and [50] that overlap with ours.

1.1 Dyonic black holes in massive Type IIA

We study BPS black holes in massive Type IIA on $AdS_4 \times S^6$. The supersymmetric AdS_4 vacuum, corresponding to the near-horizon geometry of N D2-branes in the presence of k units of RR 0-form flux (the Romans mass [86]), has been constructed in [82]. The S^6 is squashed, as a squashed S^2 bundle over $\mathbb{CP}^2$, and it preserves $U(1)_R \times SU(3)$ isometry. The first factor is an R-symmetry, and the solution preserves $4 + 4$ supercharges.

We are interested in static dyonic BPS black holes in this geometry, and they are more conveniently described within a consistent truncation to 4d $\mathcal{N} = 2$ gauged supergravity. In particular, massive Type IIA on $AdS_4 \times S^6$ admits a consistent truncation to $ISO(7)$ dyonically-gauged 4d $\mathcal{N} = 8$ supergravity [82], where $ISO(7) = SO(7) \ltimes \mathbb{R}^7$ (see also [137,138]). This theory has many supersymmetric and non-supersymmetric AdS solutions, and the one we are interested in preserves $\mathcal{N} = 2$ supersymmetry and a $U(1)_R \times SU(3)$ subgroup of $ISO(7)$.[7] Dyonic black holes generically break $U(1)_R \times SU(3)$ to its maximal

[7] To the best of our knowledge, no complete classification of AdS vacua is available for this theory.

torus, and can be described by a further consistent truncation to a 4d $\mathcal{N} = 2$ gauged supergravity with vector and hyper multiplets. Such truncations are characterized by a subgroup $G_0 \subset ISO(7)$ under which all fields are neutral. We are thus interested in the case where $G_0 = U(1)^2$. Such an $\mathcal{N} = 2$ truncation contains three vector multiplets and one hypermultiplet, and what is gauged is a group $\mathbb{R} \times U(1)$ of isometries of the hypermultiplet moduli space [83].

When dealing with 4d $\mathcal{N} = 2$ supergravity, it is convenient to use the language of special geometry [139–141].[8] Let us restrict to the case with Abelian gauge fields, then the formalism is covariant with respect to symplectic $Sp(2n_V + 2, \mathbb{Z})$ electric-magnetic transformations (n_V is the number of vector multiplets). We use a notation $V^M = (V^\Lambda, V_\Lambda)$ for symplectic vectors, where $\Lambda = 0, \ldots, n_V$, and define the symplectic scalar product

$$\langle V, W\rangle = V^M \Omega_{MN} V^N = V_\Lambda W^\Lambda - V^\Lambda W_\Lambda \tag{1.1.1}$$

in terms of the symplectic form $\Omega = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$.

First, the complex scalars z^a in the vector multiplets (with $a = 1, \ldots, n_V$) describe a special Kähler manifold $\mathcal{M}_{\text{SK}}$. We can give it a (redundant) parametrization in terms of holomorphic sections X^Λ. The holomorphic sections are collected into a covariantly-holomorphic symplectic vector

$$\mathcal{V} = e^{\mathcal{K}(z^a, \bar{z}^a)/2} \begin{pmatrix} X^\Lambda(z^a) \\ \mathcal{F}_\Lambda(z^a) \end{pmatrix} \tag{1.1.2}$$

with $D_{\bar{a}}\mathcal{V} = \partial_{\bar{a}}\mathcal{V} - \frac{1}{2}(\partial_{\bar{a}}\mathcal{K})\mathcal{V} = 0$. Here $\mathcal{K}(z^a, \bar{z}^a) = -\log\left[i(\mathcal{F}_\Lambda \overline{X}^\Lambda - X^\Lambda \overline{\mathcal{F}}_\Lambda)\right]$ is the Kähler potential for the metric on $\mathcal{M}_{\text{SK}}$, namely $ds^2_{\text{SK}} = -(\partial_a \partial_{\bar{b}} \mathcal{K}) dz^a d\bar{z}^{\bar{b}}$, while $\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}$ are the derivatives of the prepotential $\mathcal{F}$. Thus the covariantly-holomorphic sections satisfy $\langle \mathcal{V}, \overline{\mathcal{V}}\rangle = -i$. In addition to z^a, the vector multiplets contain gauge fields A^a which, together with the graviphoton A^0, form a symplectic vector $\mathcal{A}^M = (A^\Lambda, \widetilde{A}_\Lambda)$ where $\widetilde{A}_\Lambda$ are dual to A^Λ under electric-magnetic duality.

In our case[9] $n_V = 3$ and the special Kähler manifold is $\mathcal{M}_{\text{SK}} = \big(SU(1,1)/U(1)\big)^3$, parametrized by $\{z^a\}_{a=1,2,3}$. The prepotential is

$$\mathcal{F} = -2\sqrt{X^0 X^1 X^2 X^3} \tag{1.1.3}$$

(as in the STU model [31]) and the holomorphic sections can be parametrized as

$$X^\Lambda = \big(-z^1 z^2 z^3,\, -z^1,\, -z^2,\, -z^3\big)\,, \qquad \mathcal{F}_\Lambda = \big(1,\, z^2 z^3,\, z^1 z^3,\, z^1 z^2\big)\,. \tag{1.1.4}$$

Indeed, the scalar potential being function of 70 scalars, finding all its critical points is a very hard task.

[8] We follow the notation of [85].

[9] More details about this gauged supergravity and its action can be found in [83].

In other words $X^1X^2X^3/X^0 = 1$. The Kähler potential is $\mathcal{K} = -\sum_{a=1}^3 \log\big(2\,\mathbb{Im}\,z^a\big)$ and the metric is

$$ds^2_{\text{SK}} = \frac{1}{4}\sum_{a=1}^{3}\frac{dz^a\,d\bar{z}^{\bar{a}}}{(\mathbb{Im}\,z^a)^2}\ . \tag{1.1.5}$$

Thus the scalars z^a live on the upper half plane.

Second, the real scalars q^u in hypermultiplets (with $u = 1,\dots,4n_H$ and n_H is the number of hypermultiplets) describe a quaternionic Kähler manifold $\mathcal{M}_{\text{QK}}$. The dyonic gauging involves an isometry of $\mathcal{M}_{\text{QK}}$ with associated commuting Killing vectors k_α (where α parametrizes the isometry generators). The specific gauging is described by an embedding tensor ${\Theta_M}^\alpha$ that contains information about the coupling of gravitini and hypermultiplets to the gauge fields. One requires the locality constraint $\langle\Theta^\alpha,\Theta^\beta\rangle = 0$ that ensures the existence of a frame where the gauging is purely electric [142]. Hence, one constructs a symplectic Killing vector $\mathcal{K}^u_M = {\Theta_M}^\alpha k^u_\alpha$ and then the covariant derivatives of the scalars q^u are given by

$$Dq^u = dq^u - \langle\mathcal{A},\mathcal{K}^u\rangle = dq^u + A^\Lambda{\Theta_\Lambda}^\alpha k^u_\alpha - \widetilde{A}_\Lambda\Theta^{\Lambda\alpha}k^u_\alpha\ . \tag{1.1.6}$$

The isometries of $\mathcal{M}_{\text{QK}}$ descend from $SU(2)$-triplets P^x_α of moment maps, where $SU(2)$ acts on the supercharges and $x = 1,2,3$. Once again, one can use the embedding tensor to construct a symplectic vector

$$\mathcal{P}^x_M = {\Theta_M}^\alpha P^x_\alpha\ . \tag{1.1.7}$$

The $SU(2)$ index x is related to an $SU(2)$ bundle over $\mathcal{M}_{\text{QK}}$, and one can thus perform local $SU(2)$ rotations.

In our case $n_H = 1$ and the hypermultiplet manifold is $\mathcal{M}_{\text{QK}} = SU(2,1)/\big(SU(2)\times U(1)\big)$. We parametrize it with $q^u = (\sigma,\phi,\zeta,\tilde{\zeta})$ and its metric is given by

$$ds^2_{\text{QK}} = h_{uv}dq^u dq^v = \frac{1}{4}\,e^{4\phi}\Big(d\sigma + \tfrac{1}{2}\big(\zeta d\tilde{\zeta} - \tilde{\zeta}d\zeta\big)\Big)^2 + d\phi^2 + \frac{1}{4}\,e^{2\phi}\big(d\zeta^2 + d\tilde{\zeta}^2\big)\ . \tag{1.1.8}$$

The dyonic gauging involves an Abelian $\mathbb{R}\times U(1)$ isometry of $\mathcal{M}_{\text{QK}}$ with Killing vectors

$$k_{\mathbb{R}} = \partial_\sigma\ , \qquad\qquad k_{U(1)} = \zeta\partial_{\tilde{\zeta}} - \tilde{\zeta}\partial_\zeta\ . \tag{1.1.9}$$

Here $\alpha = \mathbb{R}, U(1)$. They descend from moment maps

$$\begin{aligned} P^+_{\mathbb{R}} &= 0\ , & P^+_{U(1)} &= e^\phi(\tilde{\zeta} - i\zeta)\ , \\ P^3_{\mathbb{R}} &= -\frac{1}{2}e^{2\phi}\ , & P^3_{U(1)} &= 1 - \frac{1}{4}e^{2\phi}(\zeta^2 + \tilde{\zeta}^2)\ , \end{aligned} \tag{1.1.10}$$

where $P^+_\alpha = P^1_\alpha + iP^2_\alpha$. The embedding tensor is

$$\Theta^{M\alpha} = \begin{pmatrix} \Theta^{\Lambda\alpha} \\ \hline \Theta_\Lambda{}^\alpha \end{pmatrix} = \left(\begin{array}{cccc|cccc} m & 0 & 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & g & g \end{array} \right)^{\mathsf{T}} \tag{1.1.11}$$

where g, m are the electric and magnetic coupling constants, respectively, with dimension of mass. We will assume $g, m > 0$. Notice that the hypermultiplet is charged only under one linear combination of the three $U(1)$ gauge symmetries associated with the vector multiplets, namely under $\sum_{a=1}^3 A^a$. All fields are neutral under the remaining $G_0 = U(1)^2 \subset ISO(7)$. On the other hand, σ plays the role of a Stückelberg field that gives mass to the graviphoton A^0.

The magnetic gauging m is induced by the Romans mass in the massive Type IIA uplift of this theory [82]. It has the effect to mix the graviphoton A^0 with its magnetic dual $\widetilde{A}_0$, and in the Lagrangian it induces a topological term which requires the use of an auxiliary 2-form field $\mathcal{B}^0$ [142]. This produces an extra Abelian 1-form gauge symmetry with parameter ξ^0, such that:

$$\mathcal{B}^0 \to \mathcal{B}^0 - d\xi^0 \,, \qquad A^0 \to A^0 + \frac{1}{2} m\, \xi^0 \,, \qquad \widetilde{A}_0 \to \widetilde{A}_0 + \frac{1}{2} g\, \xi^0 \,. \tag{1.1.12}$$

This symmetry will be useful later when studying the BPS equations.

1.1.1 Black hole horizons

We consider static BPS black holes with dyonic charges and horizons given by a compact Riemann surface $\Sigma_\mathfrak{g}$. In particular, we can have spherical (S^2, $\mathfrak{g} = 0$), flat toroidal (T^2, $\mathfrak{g} = 1$) or hyperbolic (locally $\mathbb{H}^2$, $\mathfrak{g} > 1$) horizons. The metric ansatz takes the form

$$ds^2 = -e^{-2U(r)} dt^2 + e^{2U(r)} dr^2 + e^{2\left(\psi(r) - U(r)\right)} ds^2_{\Sigma_\mathfrak{g}} \tag{1.1.13}$$

in terms of radial functions U, ψ. Here $ds^2_{\Sigma_\mathfrak{g}}$ is the metric on $\Sigma_\mathfrak{g}$ with constant scalar curvature $R_{\Sigma_\mathfrak{g}} = 2\kappa$ and $\kappa = 1$ for $\mathfrak{g} = 0$, $\kappa = 0$ for $\mathfrak{g} = 1$, $\kappa = -1$ for $\mathfrak{g} > 1$. Locally we can take

$$ds^2_{\Sigma_\mathfrak{g}} = d\theta^2 + f^2_\mathfrak{g}(\theta)\, d\varphi^2 \,, \qquad f_\mathfrak{g}(\theta) = \begin{cases} \sin\theta & \mathfrak{g} = 0 \\ 1 & \mathfrak{g} = 1 \\ \sinh\theta & \mathfrak{g} > 1 \,. \end{cases} \tag{1.1.14}$$

The scalars are taken to have radial dependence. The ansatz for the gauge fields $\mathcal{A}^M$ is such that it fixes the electric charges e_Λ and the magnetic charges p^Λ of the black hole:

$$p^\Lambda = \frac{1}{\mathrm{Vol}(\Sigma_\mathfrak{g})} \int_{\Sigma_\mathfrak{g}} H^\Lambda \,, \qquad e_\Lambda = \frac{1}{\mathrm{Vol}(\Sigma_\mathfrak{g})} \int_{\Sigma_\mathfrak{g}} G_\Lambda \,. \tag{1.1.15}$$

Here $H^\Lambda = dA^\Lambda + \delta^{\Lambda 0}\frac{1}{2}m\mathcal{B}^0$ and $G_\Lambda = 8\pi G_N\, \delta(\mathscr{L} d\text{vol}_4)/\delta H^\Lambda$, where $\mathscr{L}$ is the Lagrangian of the model. The correction term ensures that the charges are gauge invariant, however it is always possible to choose a gauge in which the 2-form $\mathcal{B}^0$ vanishes. The volume of $\Sigma_{\mathfrak{g}}$ is

$$\text{Vol}(\Sigma_{\mathfrak{g}}) = 2\pi\eta \;, \qquad\qquad \eta = \begin{cases} 2|\mathfrak{g}-1| & \text{for } \mathfrak{g} \neq 1 \\ 1 & \text{for } \mathfrak{g} = 1 \;. \end{cases} \tag{1.1.16}$$

We collect the electric and magnetic charges into a symplectic vector (in general r dependent)

$$\mathcal{Q} = (p^\Lambda, e_\Lambda) \;. \tag{1.1.17}$$

It will be convenient to define also

$$\mathcal{Q}^x = \langle \mathcal{P}^x, \mathcal{Q} \rangle \;, \tag{1.1.18}$$

which is an $SU(2)$ triplet of scalars.

To find BPS solutions we should specify an ansatz for the Killing spinors as well. The condition such that the gauge connections cancel the spin connection in the gravitini variations boils down to (see *e.g.* [143])

$$\kappa\, \epsilon_A = -\mathcal{Q}^x (\sigma^x)_A{}^B \Gamma^{\hat{t}\hat{r}} \epsilon_B \;. \tag{1.1.19}$$

Here ϵ_A is a doublet of spinors, $A = 1, 2$ is an index in the fundamental of $SU(2)$ and hatted indices correspond to vielbein. By taking the square of this equation we obtain the constraint

$$\mathcal{Q}^x \mathcal{Q}^x = \kappa^2 \;. \tag{1.1.20}$$

For $\kappa \neq 0$, (1.1.19) halves the number of preserved supercharges. As we will see, in the near-horizon region one finds $\mathcal{Q}^\pm = 0$. Using local $SU(2)$ rotations we could always enforce this condition on the whole solution. Then, in order to solve (1.1.19), we would impose the projector[10]

$$\epsilon_A = (\sigma^3)_A{}^B \Gamma^{\hat{t}\hat{r}} \epsilon_B \;. \tag{1.1.21}$$

This gives us the stronger constraint

$$\mathcal{Q}^3 = -\kappa \;, \tag{1.1.22}$$

which will turn out to be the BPS constraint on the charges.[11] In practice we will not work with this rotated frame, both because we want to keep the moment maps in their simple form (1.1.10), and because in any case we will only consider near-horizon solutions.

[10] Such a projector corresponds to the one imposed by the topological twist in the boundary theory.

[11] This is equivalent to the BPS constraint $\langle \mathcal{G}, \mathcal{Q} \rangle = -\kappa$ in the case without hypermultiplets. In that case $\mathcal{G} = (g^\Lambda, g_\Lambda)$ is the symplectic vector of magnetic and electric gaugings, also called Fayet-Iliopoulos terms.

Had we chosen the opposite sign in (1.1.21), we would have considered anti-BPS solutions and the constraint (1.1.22) would have had the opposite sign. For $\kappa = 0$ we are led to the same constraint (1.1.22), however it seems that there is no need to impose projectors. Nevertheless, the projector (1.1.21)—or the one with opposite sign—is imposed by requiring the gaugini variations to vanish for generic charges. From a careful analysis of the BPS equations one derives another constraint [85]:

$$\mathcal{K}^u h_{uv} \langle \mathcal{K}^v, \mathcal{Q} \rangle = 0 \ . \tag{1.1.23}$$

This will be useful later.

The only full black hole solution that has been constructed in this theory to date has equal magnetic and electric charges [84]. However, near-horizon geometries are much easier to construct—thanks to the attractor equations [43, 85, 143]—and they have been explicitly constructed in [83]. Since the near horizon geometry is all we need to determine the Bekenstein-Hawking entropy of the black hole, we will restrict to that. In fact, as we will see, we do not even need to find the full near-horizon geometries explicitly in order to exhibit a match with the microscopic field theory computation.

The near-horizon geometry is $\mathrm{AdS}_2 \times \Sigma_{\mathfrak{g}}$, corresponding to the functions

$$e^{2U} = \frac{r^2}{L^2_{\mathrm{AdS}_2}} \ , \qquad\qquad e^{2(\psi - U)} = L^2_{\Sigma_{\mathfrak{g}}} \ , \tag{1.1.24}$$

while all scalars are constant. The full near-horizon solutions are fixed by attractor equations [43, 85, 143]. Let us define[12]

$$\mathcal{Z}(z^a; p^\Lambda, e_\Lambda) = \langle \mathcal{Q}, \mathcal{V} \rangle \ , \qquad\qquad \mathcal{L}(z^a, q^u) = \langle \mathcal{P}^3, \mathcal{V} \rangle \ . \tag{1.1.25}$$

Then the BPS equations imply

$$\langle \mathcal{K}^u, \mathcal{V} \rangle = 0 \tag{1.1.26}$$

as well as

$$\partial_a \frac{\mathcal{Z}}{\mathcal{L}} = 0 \ , \qquad\qquad -i \frac{\mathcal{Z}}{\mathcal{L}} = L^2_{\Sigma_{\mathfrak{g}}} \ , \tag{1.1.27}$$

supplemented by the constraints (1.1.22) and (1.1.23). In the equation above, ∂_a is a derivative with respect to the vector multiplet scalars z^a. The first equation is in fact equivalent to $D_a\big(\mathcal{Z} - i L^2_{\Sigma_{\mathfrak{g}}} \mathcal{L}\big) = 0$, when combined with the second one. Moreover the second equation computes the horizon area.

Our strategy will be to use the equations to fix the hypermultiplet scalars and enforce the constraints they impose on the vector multiplet scalars and the charges, but leave the

[12] The definition of $\mathcal{L}$ here differs from the more common one $\mathcal{L}_{\text{there}} = \langle \mathcal{Q}^x \mathcal{P}^x, \mathcal{V} \rangle$ that is used, for instance, in [85]. The one here, used *e.g.* in [143, 144], allows us to treat all cases $\kappa = \{1, 0, -1\}$ uniformly.

remaining freedom in the vector multiplet scalars unfixed. Let us begin with (1.1.26). The vector $\mathcal{K}^\phi$ is identically zero, while the other ones give

$$e^{-\frac{\mathcal{K}}{2}}\langle\mathcal{K}^\sigma,\mathcal{V}\rangle = gX^0 - m\mathcal{F}_0 \,, \qquad e^{-\frac{\mathcal{K}}{2}}\langle\mathcal{K}^\zeta,\mathcal{V}\rangle = -\tilde{\zeta}g\sum_{a=1}^3 X^a \,, \qquad e^{-\frac{\mathcal{K}}{2}}\langle\mathcal{K}^{\tilde{\zeta}},\mathcal{V}\rangle = \zeta g\sum_{a=1}^3 X^a \,. \tag{1.1.28}$$

Since σ is a Stückelberg field shifted by $\mathbb{R}$ gauge transformations, we can gauge fix it to zero. Together with (1.1.26) we obtain, at the horizon:[13]

$$\sigma = \zeta = \tilde{\zeta} = 0 \,, \qquad\qquad \prod_{a=1}^3 z^a = -\frac{m}{g} \,. \tag{1.1.29}$$

Then we consider (1.1.23). Imposing $\zeta = \tilde{\zeta} = 0$ the only non-vanishing components are with $\Lambda = 0$, either up or down. They give a constraint on the graviphoton charges:

$$m\, e_0 - g\, p^0 = 0 \,. \tag{1.1.30}$$

Finally we impose (1.1.22). When $\zeta = \tilde{\zeta} = 0$ only $\mathcal{P}^3$ is non-vanishing, while $\mathcal{P}^\pm = 0$. Using (1.1.30) we find $\mathcal{Q}^3 = g\sum_{a=1}^3 p^a$, and thus we obtain the BPS constraint on the charges

$$\sum_{a=1}^3 p^a = -\frac{\kappa}{g} \,. \tag{1.1.31}$$

Instead of trying to solve the remaining equations in (1.1.27) (explicit solutions can be found in [83]), we aim to reduce them to a simpler extremization problem. We evaluate the functions $\mathcal{L}$ and $\mathcal{Z}$ at the horizon, imposing $\zeta = \tilde{\zeta} = 0$:

$$\begin{aligned} \mathcal{L} &= e^{\mathcal{K}/2}\Big[-\frac{1}{2}\,e^{2\phi}\big(gX^0 - m\mathcal{F}_0\big) + g\big(X^1 + X^2 + X^3\big)\Big] \\ \mathcal{Z} &= e^{\mathcal{K}/2}\big(e_\Lambda X^\Lambda - p^\Lambda\mathcal{F}_\Lambda\big) \,. \end{aligned} \tag{1.1.32}$$

When imposing $D_a\big(\mathcal{Z} - iL^2_{\Sigma_\mathfrak{g}}\mathcal{L}\big) = \partial_a\big[e^{-\mathcal{K}/2}\big(\mathcal{Z} - iL^2_{\Sigma_\mathfrak{g}}\mathcal{L}\big)\big] = 0$ we are supposed to vary the functions with respect to *independent* scalars z^a. However the hypermultiplet scalar $e^{2\phi}$ plays the role of a Lagrange multiplier for the second constraint in (1.1.29), therefore we can reduce to the problem of extremizing $-i\mathcal{Z}/\mathcal{L}$ with respect to *constrained* scalars satisfying (1.1.29). Imposing the constraint we find

$$-i\frac{\mathcal{Z}}{\mathcal{L}} = -\frac{i}{g^2}\,\frac{\sum_{a=1}^3\big(g\,e_a z^a - m\,p^a/z_a\big)}{\sum_{a=1}^3 z^a} \qquad \text{with (1.1.29)} \,. \tag{1.1.33}$$

[13]Here we are using that $\sum_{a=1}^3 X^a = \sum_{a=1}^3 z^a \neq 0$ since z^a take values on the upper half plane. However, even relaxing this condition and allowing—in principle—specific values of z^a for which (1.1.26) is solved leaving $\zeta, \tilde{\zeta}$ unconstrained, for $\kappa = \pm 1$ we still find that (1.1.20) and (1.1.23) imply $\zeta = \tilde{\zeta} = 0$. We conclude that there exist no special solutions to (1.1.26) besides (1.1.29).

Although not needed here, notice that the equations $\partial_a\big[e^{-\mathcal{K}/2}\big(\mathcal{Z} - iL^2_{\Sigma_\mathfrak{g}}\mathcal{L}\big)\big] = 0$ with variations with respect to independent z^a, combined with the constraint (1.1.29), fix the value of the Lagrange multiplier $e^{2\phi}$, which is the last hypermultiplet scalar we had not fixed yet.

1.1.2 The Bekenstein-Hawking entropy

The Bekenstein-Hawking entropy S_{BH} of the black holes is given by the horizon area:

$$S_{\mathrm{BH}} = \frac{\mathrm{Area}}{4G_N} = \frac{2\pi\eta L^2_{\Sigma_\mathfrak{g}}}{4G_N} \,. \tag{1.1.34}$$

The attractor equations (1.1.27) determine the area in terms of the value of $-i\mathcal{Z}/\mathcal{L}$ at its critical point. We can then introduce a function

$$\mathcal{S}(z^a;p^a,e_a) = -i\frac{2\pi\eta}{4G_N}\,\frac{\mathcal{Z}}{\mathcal{L}} = -\frac{2\pi i}{g^2}\,\frac{\eta}{4G_N}\,\frac{\sum_{a=1}^3\big(g\,e_a z^a - m\,p^a/z_a\big)}{\sum_{a=1}^3 z^a} \tag{1.1.35}$$

of two complex variables, in which the three scalars satisfy $\prod_{a=1}^3 z^a = -m/g$ and the charges satisfy $\sum_{a=1}^3 p^a = -\kappa/g$. The entropy is equal to the extremal value of this function:

$$S_{\mathrm{BH}} = \mathcal{S}(\widehat{z}^a;p^a,e_a) \qquad \text{with } \widehat{z}^a \text{ such that} \qquad \partial_{z^a}\mathcal{S}(z^a;p^a,e_a)\Big|_{z^a=\widehat{z}^a} = 0\,. \tag{1.1.36}$$

We should now comment on the existence of the BPS black hole horizons we have studied above. A generic solutions of the BPS equations satisfying the ansatze (1.1.13)–(1.1.15) does not give rise to a well-defined large smooth black hole (with finite horizon area). For this to be the case further requirements need to be satisfied by the charges (p^a,e_a). First of all we should impose all the metric components to be real (or, equivalently that $L^2_{\mathrm{AdS}_2}$ and $L^2_{\Sigma_\mathfrak{g}}$ be real positive) as well as that $\widehat{z}^a$ live on the upper half-plane. These inequalities can drastically reduce the domain of charges leading to a large smooth BPS horizon, although they cannot modify its dimensionality. Solving these inequalities is a very important but hard task that, to the best of our knowledge, has not been exhaustively addressed for the $ISO(7)$ dyonically-gauged supergravity or its truncations. We will not address it here, but we leave it for future investigations. Instead, we focus on a more tractable necessary condition to have a good near-horizon geometry, which is that $\mathcal{S}(\widehat{z}^a;p^a,e_a)$ be real positive. This imposes a further polynomial constraint on the charges. We also note that for every choice of charges (p^a,e_a)—satisfying (1.1.31)—it is always possible to perform a common shift of e_a such that $\mathcal{S}(\widehat{z}^a;p^a,e_a)$ becomes real (not necessarily positive, though). This is a shift of the R-charge of the black hole. Such a shift does not affect the extremization problem, therefore it does not change $\widehat{z}^a$, but it shifts

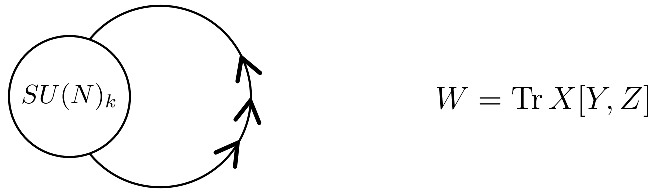

Figure 1.1: Quiver diagram and superpotential of the 3d dual to massive Type IIA on S^6.

$\mathcal{S}$ by an imaginary amount. We conclude that (before applying quantization conditions) the domain of charges (p^a, e_a) leading to large smooth BPS horizons has dimension 4.

We can describe the procedure in a slightly different way. First we fix magnetic charges that satisfy the BPS constraint (1.1.31), and flavor charges $e_a - e_3$ for $a = 1, 2$. Then we determine the unique value of the R-charge $e_R = e_3$ such that $\mathcal{S}(\widehat{z}^a; p^a, e_a)$ is real. In other words, for given magnetic and flavor charges, there is a unique value of the R-charge such that a large smooth black hole with those charges can possibly exist. As we will see in Section 1.3, this procedure has a direct counterpart in the field theory analysis.

1.2 Microscopic counting in field theory

The three-dimensional quantum field theory dual to massive Type IIA on S^6, whose consistent truncation we studied in the previous section, has been identified in [82]. It is an $\mathcal{N} = 2$ supersymmetric Chern-Simons-matter theory with gauge group $SU(N)$ and level k (related to the Romans mass), coupled to three chiral multiplets X, Y, Z in the adjoint representation, and with a superpotential given by

$$W = \operatorname{Tr} X[Y, Z] \; . \tag{1.2.1}$$

The corresponding quiver diagram is represented in Figure 1.1 (it coincides with the quiver of 4d $\mathcal{N} = 4$ SYM, which we will study in Chapter 3). The global symmetry of the theory is $SU(3) \times U(1)_R$, where the latter is the R-symmetry. We find it convenient to adopt two different bases for the maximal torus $U(1)^2 \times U(1)_R$ of the global symmetry:

	R_1	R_2	R_3	$\mathfrak{q}_1$	$\mathfrak{q}_2$	r
X	2	0	0	1	0	0
Y	0	2	0	0	1	0
Z	0	0	2	-1	-1	2

(1.2.2)

Here the generators $R_{1,2,3}$ of the first basis are all R-charges, as they give charge 2 to the superpotential (1.2.1). On the other hand, in the second basis we have two flavor charges $\mathfrak{q}_a = \frac{1}{2}(R_a - R_3)$, with $a = 1, 2$. Moreover, we have chosen an R-symmetry generator $r = R_3$ that gives integer charges to all fields.

The regime in which the bulk gravitational theory is weakly coupled corresponds to the large N limit with k fixed (or at least $N \gg k$). The BPS dyonic black hole solutions in AdS_4 induce, via the rules of AdS/CFT [23–25], relevant deformations of the boundary theory. First of all the 3d CS theory is placed on $\Sigma_\mathfrak{g} \times \mathbb{R}$, where $\Sigma_\mathfrak{g}$ is a Riemann surface with the same genus $\mathfrak{g}$ as the black hole horizon. Second, the theory is topologically twisted on $\Sigma_\mathfrak{g}$ [145] in such a way that one complex supercharge is preserved. In other words, there is a background gauge field V on $\Sigma_\mathfrak{g}$, coupled to an R-symmetry, equal and opposite to the spin connection and therefore such that $\frac{1}{2\pi}\int_{\Sigma_\mathfrak{g}} dV = \mathfrak{g} - 1$.[14] In the presence of flavor symmetries there are multiple choices one can make for the R-symmetry used in the twist. We can parametrize those choices by fixing r as the R-symmetry and introducing Abelian background gauge fields A_a coupled to the symmetry currents associated with the flavor charges $\mathfrak{q}_a$. Doing so (in the Cartan subalgebra, without loss of generality) turns on magnetic fluxes on $\Sigma_\mathfrak{g}$:

$$\mathfrak{p}_a = \frac{1}{2\pi}\int_{\Sigma_\mathfrak{g}} dA_a \in \Gamma^\vee_{SU(3)} \,. \tag{1.2.3}$$

The numbers $\mathfrak{p}_a$ are GNO quantized [146] in the coroot lattice $\Gamma^\vee_{SU(3)}$ of the flavor symmetry, and effectively parametrize the twist. It turns out to be convenient to introduce an auxiliary flux parameter, formally associated to the R-symmetry, that is defined linearly in terms of the other ones. Then the fluxes $\mathfrak{p}_a$ in field theory correspond to the magnetic charges of the black hole (the precise normalization will be fixed in Section 1.3). In our case we introduce $\mathfrak{p}_3$, besides $\mathfrak{p}_{1,2}$, such that $\sum_{a=1}^3 \mathfrak{p}_a = 2(\mathfrak{g}-1)$. This description is directly associated to the first basis in (1.2.2) and it is convenient because the Weyl group of $SU(3)$ acts as permutations of the indices $a = 1, 2, 3$. Finally, the Hamiltonian of the theory is subject to the so-called "real mass deformation". Viewing the Abelian gauge fields A_a as components (in a Cartan basis) of a background vector multiplet $\mathcal{V}$ associated with the flavor symmetry group, this relevant deformation can be parametrized by non-zero constant configurations σ_a (again in a Cartan basis) for the adjoint scalars in the bottom component of $\mathcal{V}$. The latter preserve the original supersymmetries and they introduce mass terms which lift the spectrum of the Hamiltonian to a gapped one. Additionally, they modify the supersymmetry algebra in the following way:

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = H - \sum\nolimits_a \sigma_a \mathfrak{q}_a \,, \tag{1.2.4}$$

where $\mathcal{Q}$ is the preserved supercharge, H is the Hamiltonian on $\Sigma_\mathfrak{g}$ and $\mathfrak{q}_a$ are the flavor charges defined as in (1.2.2).

The black hole microstates correspond to ground states of this system, therefore in order to give a microscopic account of the black hole entropy we should count them [29,30]. This is a non-trivial problem because the theory is strongly coupled in the IR. However we can have a good estimate, in the large N limit, of the number of ground states by

[14]We can turn on a background flux because all gauge-invariant operators have integer R-charge.

computing an index, dubbed "topologically twisted index":

$$Z(\mathfrak{p},\Delta) \coloneqq \operatorname{Tr}_{\mathcal{H}(\Sigma_\mathfrak{g})} (-1)^F \, e^{-\beta H} \, e^{2\pi i \sum_a \alpha_a \mathfrak{q}_a} \, . \tag{1.2.5}$$

Here, F is the fermion number and α_a are real chemical potentials associated to the flavor charges $\mathfrak{q}_a$. Furthermore, notice that the Hamiltonian H on $\Sigma_\mathfrak{g}$ explicitly depends on the flavor fluxes $\mathfrak{p}_a$ and on the real masses σ_a. This object is a Witten index [34]: it only receives contributions from ground states $H = \sum_a \sigma_a \mathfrak{q}_a$, such that

$$Z(\mathfrak{p},\Delta) = \operatorname{Tr}_{\mathcal{Q}=0} (-1)^F \, e^{2\pi i \sum_a \Delta_a \mathfrak{q}_a} \tag{1.2.6}$$

is a meromorphic function of $\Delta_a \coloneqq \alpha_a + i\beta\sigma_a/2\pi$, which play the role of complexified chemical potentials. As a remark, let us stress that, in the case where the real masses σ_a are switched off, the index is singular as the spectrum of H becomes continuous and so the trace is ill-defined. Nevertheless, a well-defined formula for the index in the limit $\sigma_a \to 0$ can be obtained via analytic continuation.

The index (1.2.6) is protected by supersymmetry and it can alternatively be thought of as the supersymmetric partition function of the Euclidean theory on the topologically twisted $S^1 \times \Sigma_\mathfrak{g}$ background. As such it can be computed exactly with localization techniques [35, 36, 98], and it takes the following form:

$$Z(\mathfrak{p},\Delta) = \frac{1}{|\mathcal{W}_G|} \sum_{\mathfrak{m} \,\in\, \Gamma_G^\vee} \oint_{\text{JK}} Z_{\text{int}}(x,\mathfrak{m};y,\mathfrak{p}) \, . \tag{1.2.7}$$

Here $\mathcal{W}_G$ is the gauge Weyl group, $\Gamma_G^\vee$ is the co-root lattice of the gauge group G, and the sum is over gauge fluxes $\mathfrak{m}$ on $\Sigma_\mathfrak{g}$. Then Z_{int} is a meromorphic rk(G)-form on the space of complexified flat gauge connections on S^1, which can be parametrized by gauge fugacities x. Finally $y_a = e^{2\pi i \Delta_a}$ are (complex) fugacities for the flavor symmetries. The integral is a contour integral along a particular contour called the Jeffrey-Kirwan residue [147]. We refer to [35, 36, 98] for details.

Taking an alternative approach [36, 98, 99] (see also [104, 148]), one can obtain an equivalent expression for the topologically twisted index, very different from (1.2.7). This formula expresses the index as a finite sum over the set of solutions $\mathfrak{M}_{\text{BAE}}$ of a system of rk(G) algebraic equations—dubbed "Bethe Ansatz Equations" (BAEs)—of a function directly obtained from the integrand Z_{int}. More precisely, the BAEs are of the form

$$e^{2\pi i B_i(x)} = 1 \, , \qquad\qquad i = 1, \ldots, r \, , \tag{1.2.8}$$

with

$$B_i(x) = \frac{1}{2\pi i} \frac{\partial \log Z_{\text{int}}(x,\mathfrak{m};y,\mathfrak{p})}{\partial \mathfrak{m}_i}\bigg|_{\mathfrak{m}=0} \, . \tag{1.2.9}$$

The new formula is given by

$$Z(\mathfrak{p},\Delta) = \frac{(-1)^{\text{rk}(G)}}{|\mathcal{W}_G|} \sum_{\hat{x}\in\mathfrak{M}_{\text{BAEs}}} Z_{\text{int}}(\hat{x},0;y,\mathfrak{p})\, H(\hat{x};y,\mathfrak{p})^{\mathfrak{g}-1} \,, \tag{1.2.10}$$

where H is the determinant of the Jacobian matrix

$$H(x;y,\mathfrak{p}) = \det_{ij} \frac{\partial e^{2\pi i B_i(x)}}{\partial \log x_j} \tag{1.2.11}$$

and

$$\mathfrak{M}_{\text{BAE}} = \left\{ \hat{x}_i\,,\ i=1,\dots,r \;\middle|\; e^{2\pi i B_i(\hat{x},y,\mathfrak{p})} = 1\,, \quad w\cdot\hat{x}\neq\hat{x} \quad \forall\, w\in\mathcal{W}_G \right\} \tag{1.2.12}$$

identifies the only solutions to (1.2.8) which are not fixed by any Weyl group element.[15] The strategy employed to obtain such a formula consists in resumming the series over $\mathfrak{m}\in\Gamma_G^\vee$ in (1.2.7) making use of some manipulations to recast it as a geometric series. Upon carefully dealing with convergence issues, this produces a sum of residues at the solutions of the BAEs. Assuming that all the acceptable solutions are simple roots, the sum of residues precisely evaluates to (1.2.10).[16] This Bethe Ansatz formula for the topologically twisted index can be given a physical interpretation along the lines of [36,98, 99,104]. Indeed, the index can be thought of as a correlator on $\Sigma_\mathfrak{g}$ in the dimensionally reduced theory on S^1. The latter is an effective two-dimensional A-twisted Landau-Ginzburg (LG) model of rk(G) Abelian twisted chiral multiplets, governed by a twisted superpotential $\widetilde{\mathcal{W}}$. In terms of the 3d degrees of freedom, the bottom components of these multiplets correspond to the gauge fugacities x and $\widetilde{\mathcal{W}}$ has a holomorphic dependence on them. Using the standard formula [104, 149], the correlator is expressed as a sum over the supersymmetric vacua of the LG model, precisely giving (1.2.10). In this language, the BAEs consist in the 2d vacuum equations and they are determined in terms of the twisted superpotential by the relation

$$B_i(x) = 2\pi i\, \frac{\partial\widetilde{\mathcal{W}}(x)}{\partial \log x_i} \,. \tag{1.2.13}$$

Moreover, the contribution of $Z_{\text{int}}|_{\mathfrak{m}=0}$ in (1.2.10) can be interpreted as the insertion in the correlator on $\Sigma_\mathfrak{g}$ of line operators wrapped on S^1, whose effect is to turn on the flavor magnetic fluxes $\mathfrak{p}_a$.[17] For further details, we refer to [36, 98, 99, 104].

[15] An equivalent requirement is that the Vandermonde determinant $\prod_{\alpha\in\Delta_G}(1-x^\alpha)$, playing the role of the vector multiplet 1-loop determinant in Z_{int}, evaluated on the solutions to the BAEs be non-vanishing [36].

[16] If the roots are not simple (1.2.10) does not hold. In this case one has to use a more general prescription, in terms of the Jeffrey-Kirwan residues.

[17] To be precise, $Z_{\text{int}}|_{\mathfrak{m}=0}$ also accounts for the contribution of an effective dilaton operator [104], which governs the coupling to curvature of the 2d LG model.

To extract the Bekenstein-Hawking entropy of the black holes, we should compute the large N limit of the topologically twisted index. Since the index (1.2.5) is in the grand canonical ensemble with respect to the electric charges, the microcanonical degeneracies (with sign) of states with fixed electric and magnetic charges $(\mathfrak{p}, \mathfrak{q})$ have to be extracted from a Fourier transform with respect to the chemical potentials:

$$Z(\mathfrak{p}, \mathfrak{q}) = \int d^d\Delta \, Z(\mathfrak{p}, \Delta) \, e^{-2\pi i \sum_a \mathfrak{q}_a \Delta_a} \,, \tag{1.2.14}$$

where d is the total rank of the flavor symmetry. Here a complication arises [30]: since the index $Z(\mathfrak{p}, \Delta)$ depends on flavor fugacities but it cannot have a fugacity for the R-symmetry (it would spoil supersymmetry), what we have on the left-hand-side is the sum of contributions from all states with fixed flavor charges but arbitrary R-charge. However in the large N limit we can assume that one R-charge sector is dominant, and we will see that it precisely coincides with the single-center black hole sector. Moreover, assuming that at large N $\log Z$ grows with an appropriate power of N, the integral (1.2.14) can be computed in the saddle-point approximation. Therefore one defines the function

$$\mathcal{I}(\Delta; \mathfrak{p}, \mathfrak{q}) = \log Z(\mathfrak{p}, \Delta) - 2\pi i \sum\nolimits_a \mathfrak{q}_a \Delta_a \tag{1.2.15}$$

such that the logarithm of the degeneracy d_{micro} of states is given by

$$\log d_{\text{micro}}(\mathfrak{p}, \mathfrak{q}) = \mathcal{I}\big(\widehat{\Delta}; \mathfrak{p}, \mathfrak{q}\big) \qquad \text{with } \widehat{\Delta}_a \text{ such that } \quad \frac{\partial \mathcal{I}}{\partial \Delta_a}\bigg|_{\widehat{\Delta}} = 0 \,, \tag{1.2.16}$$

which is the Legendre transform of $\log Z$. This is the object we expect to reproduce the entropy of dual black holes.

Here, a very important subtlety to keep in mind is that Z in (1.2.5) is an index, thus it counts states with sign $(-1)^F$ and so $\mathcal{I}(\widehat{\Delta})$ may fail to reproduce the black hole entropy due to strong fermion/boson cancelations. However, it has been argued in [29,30] that the states associated to the pure single-center BPS black holes[18] all contribute with the same sign and thus the index precisely counts their number. The argument can be summarized as follows: the holographic 3d QFT on $\Sigma_\mathfrak{g}$ can be thought of undergoing a RG flow across dimensions to a 1d Super Quantum Mechanics (SQM) in the IR, describing the BPS black holes near-horizon dynamics. Because of the AdS_2 factor in the latter geometry, the SQM must develop an $\mathfrak{su}(1,1|1)$ superconformal symmetry. The BPS black holes being supersymmetric, their microstates must be invariant under such AdS_2 (super-)isometry. Therefore, the dual QFT states—which are the ground states—transform trivially under $\mathfrak{su}(1,1|1)$. This implies in particular that they have vanishing superconformal R-charge

[18] By "pure single-center black hole" we mean the near-horizon $AdS_2 \times \Sigma_\mathfrak{g}$ solution with boundary conditions that fix the microcanonical ensemble with respect to both magnetic and electric charges [150].

$R_{\rm sc} = 0$. At this point one has to rewrite the index in (1.2.6) as

$$Z(\mathfrak{p}, \Delta) = \mathrm{Tr}_{\,\mathrm{SQM}}\, e^{\pi i R_{\rm trial}(\Delta)}\, e^{-2\pi \sum_a \mathfrak{q}_a \,\mathbb{I}\mathrm{m}\, \Delta_a}\, e^{-\beta \{\mathcal{Q}, \mathcal{Q}^\dagger\}} \,, \tag{1.2.17}$$

where the trace is over the quantum mechanical states on $\Sigma_{\mathfrak{g}}$ and $R_{\rm trial}$ is a trial R-symmetry given by

$$R_{\rm trial}(\Delta) = r + 2 \sum\nolimits_a \mathfrak{q}_a \,\mathbb{R}\mathrm{e}\, \Delta_a \,, \tag{1.2.18}$$

with $r = R_3$ defined in (1.2.2) and such that $e^{\pi i r} = (-1)^F$. It is now clear that, when $R_{\rm trial}$ is tuned to $R_{\rm sc}$, then (1.2.17) counts all the ground states with the same sign. The non-trivial observation is then that the critical points $\widehat{\Delta}_a$ of the function $\mathcal{I}$ in (1.2.15) precisely select the superconformal R-charge $R_{\rm sc} \equiv R_{\rm trial}(\widehat{\Delta})$ out of all the possible trial R-charges. Although a satisfactory physical justification is not yet available, this principle—dubbed "$\mathcal{I}$-extremization principle"—has been shown to work in various examples, guaranteeing that (1.2.16) reproduces at leading order in N the degeneracy of the quantum mechanical ground states, and so of the holographically dual pure single-center BPS black hole microstates. On the other hand, the same conclusion does not apply for all the other states in the IR SQM, including those dual to multi-center black holes and hair, whose number however we might expect to be subleading. As a remark, let us stress that this argument is very similar to the one given in [41, 42] (and nicely summarized *e.g.* in [151]) for BPS black holes in flat space.

Interestingly, it has been noticed in [30] in one example that if we introduce one auxiliary chemical potential Δ_{d+1}, defined in terms of the other ones—as we did for the flavor fluxes $\mathfrak{p}_a$—such that $\sum_a \Delta_a = 1$, we can extract the dominant R-charge from (1.2.16) by requiring $\mathcal{I}$ to be real. This is done by first explicitly expressing $\mathcal{I}$ in terms of the redundant chemical potentials Δ_a, now associated to the R-charges R_a (defined from $\mathfrak{q}_a$ similarly to (1.2.2)), and then by choosing the last R-charge R_{d+1} in such a way that $\mathbb{I}\mathrm{m}\, \mathcal{I} = 0$. As we will see, the same applies also in the example considered here.

1.2.1 The topologically twisted index

Let us go back to the specific $SU(N)_k$ theory we are interested in. We denote by $y_{1,2}$ the fugacities associated with the flavor charge $\mathfrak{q}_{1,2}$, with

$$y_a = e^{2\pi i \Delta_a} \,, \tag{1.2.19}$$

and by $\mathfrak{p}_{1,2}$ the corresponding magnetic fluxes on $\Sigma_{\mathfrak{g}}$. In order to restore the symmetry under the Weyl group of $SU(3)$, it is convenient to introduce also the auxiliary variables y_3 and $\mathfrak{p}_3$ fixed by

$$\sum\nolimits_{a=1}^{3} \mathfrak{p}_a = 2(\mathfrak{g} - 1) \,, \qquad\qquad \prod\nolimits_{a=1}^{3} y_a = 1 \,, \tag{1.2.20}$$

in such a way that, formally, each y_a is now associated to the R-charge R_a defined in (1.2.2). In order to avoid the technicality arising from the structure of the Cartan subalgebra of $\mathfrak{su}(N)$, we consider the theory with gauge group $U(N)_k$ instead. The computation of the $U(N)$ partition function is simpler, and in our case it provides the same result as the $SU(N)$ theory. In fact it has been proven in [35] that the index of a $U(N)_k$ CS theory with no topological flux is exactly equal to the index of the corresponding $SU(N)_k$ CS theory whenever the matter is neutral under the central $U(1)$ in $U(N)$. Following the rules in [35, 36, 98] and after some manipulations, the index takes the form

$$Z(\mathfrak{p},\Delta) = \frac{(-1)^N}{N!} \prod_{a=1}^{3} \frac{y_a^{N^2(1+\mathfrak{p}_a-\mathfrak{g})/2}}{(1-y_a)^{N(1+\mathfrak{p}_a-\mathfrak{g})}} \sum_{\mathfrak{m}\in\mathbb{Z}^N} \oint_{\text{JK}} \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i}\, x_i^{k\mathfrak{m}_i} \times \\ \times \prod_{j(\neq i)}^{N} \prod_{a=1}^{3} \Big(\frac{x_i - y_a x_j}{x_j - y_a x_i} \Big)^{\mathfrak{m}_i} \prod_{i\neq j}^{N} \Big(1 - \frac{x_i}{x_j} \Big)^{1-\mathfrak{g}} \prod_{a=1}^{3} \Big(1 - y_a \frac{x_i}{x_j} \Big)^{\mathfrak{g}-1-\mathfrak{p}_a} . \tag{1.2.21}$$

The integrand has poles at $x_i = 0$ and ∞ (for generic values of y_a). Assuming $k > 0$, the JK prescription selects an integration contour around $x_i = 0$ and thus the integral computes minus the sum of the residues there. Since there are poles at $x_i = 0$ only for $\mathfrak{m}_i \leq M - 1$ for some large positive M, we can restrict the sum to those values and resum the geometric series before picking the residues. This leads to the Bethe Ansatz formula, given by:

$$Z = \frac{1}{N!} \prod_{a=1}^{3} \frac{y_a^{N^2(1+\mathfrak{p}_a-\mathfrak{g})/2}}{(1-y_a)^{N(1+\mathfrak{p}_a-\mathfrak{g})}} \sum_{\hat{x}\,\in\,\mathfrak{M}_{\text{BAE}}} H(\hat{x})^{\mathfrak{g}-1} \prod_{i\neq j}^{N} \Big(1 - \frac{\hat{x}_i}{\hat{x}_j} \Big)^{1-\mathfrak{g}} \prod_{a=1}^{3} \Big(1 - y_a \frac{\hat{x}_i}{\hat{x}_j} \Big)^{\mathfrak{g}-1-\mathfrak{p}_a} . \tag{1.2.22}$$

Here the BAEs are

$$1 = e^{2\pi i B_i(x)} = x_i^k \prod_{j(\neq i)}^{N} \prod_{a=1}^{3} \frac{x_i - y_a x_j}{x_j - y_a x_i} \qquad \forall i = 1, \dots, N , \tag{1.2.23}$$

while the Jacobian term is given by

$$H(x) = \det \mathbb{B}(x) , \qquad \text{with} \quad \mathbb{B}_{ij} = \frac{\partial\, e^{2\pi i B_i(x)}}{\partial \log x_j} . \tag{1.2.24}$$

This matrix can be written in a more explicit form as

$$\mathbb{B}_{ij} = e^{2\pi i B_i(x)} \Bigg[\Big(k + \sum_{l=1}^{N} D_{il} \Big) \delta_{ij} - D_{ij} \Bigg] , \quad D_{ij} = z \frac{\partial}{\partial z} \log \Big(\frac{z-y_1}{1-y_1 z} \, \frac{z-y_2}{1-y_2 z} \, \frac{z-y_3}{1-y_3 z} \Big) \Big|_{z=\frac{x_i}{x_j}} . \tag{1.2.25}$$

We stress that (1.2.22) is an exact expression for the index, valid at finite N.

The BAEs (1.2.23) are N algebraic equations in N complex variables x_i: in general they have a large number of solutions and cannot be analytically solved. However for any solution $\{x_i\}$ we can generate other ones $\{\omega x_i\}$ where ω is a k-th root of unity. Each of the k solutions in the orbit gives the same contribution to (1.2.22).

It is convenient to perform the change of variables

$$x_i = e^{2\pi i u_i} \;, \qquad\qquad y_a = e^{2\pi i \Delta_a} \;, \tag{1.2.26}$$

where Δ_a are chemical potentials for the flavor symmetries. The angular variables are defined modulo 1, and the constraint on y_a becomes $\sum_{a=1}^{3} \Delta_a \in \mathbb{Z}$. The BAEs in the new variables take the form

$$k u_i - \frac{1}{2\pi i} \sum_{j=1}^{N} \sum_{a=1}^{3} \Big[\mathrm{Li}_1 \left(e^{2\pi i (u_j - u_i + \Delta_a)} \right) - \mathrm{Li}_1 \left(e^{2\pi i (u_j - u_i - \Delta_a)} \right) \Big] - n_i + \frac{N}{2} = 0 \;, \tag{1.2.27}$$

where the integers n_i express the angular ambiguity, while

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \tag{1.2.28}$$

are the polylogarithm functions and $\mathrm{Li}_1(z) = -\log(1-z)$. Following the discussion above (1.2.13), the BAEs can be obtained as the critical point equations (up to integer shifts, already accounted for in (1.2.27)) of the twisted superpotential $\widetilde{\mathcal{W}}$ of the dimensionally reduced theory on S^1:

$$\widetilde{\mathcal{W}} = \sum_{i=1}^{N} \frac{k}{2} u_i^2 + \frac{1}{2(2\pi i)^2} \sum_{i,j=1}^{N} \sum_{a=1}^{3} \Big[\mathrm{Li}_2 \left(e^{2\pi i (u_j - u_i + \Delta_a)} \right) - \mathrm{Li}_2 \left(e^{2\pi i (u_j - u_i - \Delta_a)} \right) \Big] - \sum_{i=1}^{N} \frac{m_i}{2} u_i \;. \tag{1.2.29}$$

Here the integers m_i incorporate the various angular ambiguities of (1.2.27). In the language of Bethe/gauge correspondence [152–154], the latter superpotential plays the role of the "Yang-Yang functional" [155] of the dual integrable system.

1.2.2 The large N limit

We proceed by computing (1.2.22) in the large N limit at fixed k. The computation is essentially the same as the one in [29], and turns out to be very similar to the computation of the large N limit of the S^3 partition function in [156, 157]. More examples have been considered in [46] and a rather general analysis have been performed in [44], therefore here we will be brief.

First of all, we assume that there is one k-fold orbit of solutions to the BAEs that dominates Z. To determine it, we consider a continuous distribution of points $u(t)$, where t is the continuous version of the discrete index $i = 1, \ldots, N$, and a density distribution

$\rho(t)$ defined by

$$\rho(t) = \frac{1}{N}\frac{di}{dt} \ . \tag{1.2.30}$$

In the continuum approximation, sums over i are turned into integrals $\sum_{i=1}^N \mapsto N\int dt\,\rho(t)$, and the density distribution is normalized as $\int dt\,\rho(t) = 1$. From numerical solutions to (1.2.23), and as suggested by [156, 157], we consider the following ansatz for the behavior of the dominant solution:

$$u(t) = N^\alpha\big(it + v(t)\big) \ , \tag{1.2.31}$$

where $v(t)$ is real and α is an exponent to be determined. Then we compute the large N limit of the twisted superpotential $\widetilde{\mathcal{W}}$, as a functional of $u(t)$ and $\rho(t)$. On general grounds, we know that the index Z is analytic in Δ_a [35], therefore it is convenient to perform all computations with $\Delta_a \in \mathbb{R}$ and analytically continue the result at the end. Only for a specific set of values of the integers m_i there is a cancelation of "long range forces" in (1.2.29) and the large N functional becomes local:

$$m_i = \Big(2\sum\nolimits_a \Delta_a - 3\Big)\sum_{j=1}^N \Big[\Theta\big(\,\mathbb{I}\mathrm{m}(u_i - u_j)\big) - \Theta\big(\,\mathbb{I}\mathrm{m}(u_j - u_i)\big)\Big] \ . \tag{1.2.32}$$

Here Θ is the Heaviside function.

The functional $\widetilde{\mathcal{W}}(v,\rho;\mu)$ is, at leading order in N:

$$\begin{aligned}\widetilde{\mathcal{W}}(v,\rho;\mu) = N^{1+2\alpha}\int dt\Big[ik\,t\,\rho(t)\,v(t) + \frac{k}{2}\rho(t)\big(v(t)^2 - t^2\big)\Big] - \\ - N^{2-\alpha}\Big[iG(\Delta)\int dt\,\frac{\rho(t)^2}{1 - i\,\dot{v}(t)} - i\mu\Big(\int dt\,\rho(t) - 1\Big)\Big] \ .\end{aligned} \tag{1.2.33}$$

The function $G(\Delta)$ is defined as

$$G(\Delta) = \sum_{a=1}^3 g_+(\Delta_a) \ , \qquad\qquad g_+(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12} \ . \tag{1.2.34}$$

The polynomial terms in the first line of the expression for $\widetilde{\mathcal{W}}$ directly descend from the first term in (1.2.29). The first term in the second line, instead, comes from the sum of polylogarithms. The derivation consists in splitting the sum as

$$\begin{aligned}\sum_{i,j=1}^N \mathrm{Li}_2\left(e^{2\pi i(u_j - u_i + \Delta_a)}\right) = N\,\mathrm{Li}_2\left(e^{2\pi i\Delta_a}\right) - (2\pi i)^2\sum_{i<j} g'_+(u_j - u_i + \Delta_a) + \\ + \sum_{i<j}^N\Big(\mathrm{Li}_2\left(e^{2\pi i(u_j - u_i + \Delta_a)}\right) - \mathrm{Li}_2\left(e^{2\pi i(u_j - u_i - \Delta_a)}\right)\Big) \ ,\end{aligned} \tag{1.2.35}$$

where we have used that $\mathrm{Li}_2(e^{2\pi iu}) + \mathrm{Li}_2(e^{-2\pi iu}) = -(2\pi i)^2 g'_+(u)$ for $0 < \mathbb{R}\mathrm{e}\,u < 1$. The first term in the expansion is obviously subleading, therefore we neglect it. Moreover, the contribution of the polynomial terms in g'_+, once plugged in (1.2.29), can be shown to properly cancel with the angular ambiguities chosen in (1.2.32). Making use of the definition of Li_2, the contribution of the last line can be turned into an integral of the form:

$$\begin{aligned}\sum_{i<j}^{N} \mathrm{Li}_2\left(e^{2\pi i(u_j-u_i+\Delta_a)}\right) &= \sum_{k=1}^{\infty}\frac{N^2}{k^2}\int_{t_-}^{t_+} dt_1\,\rho(t_1)\int_{t_1}^{t_+} dt_2\, e^{-2\pi k N^\alpha(t_2-t_1)}\times \\ &\quad\times\sum_{j=0}^{\infty}\frac{(t_2-t_1)^j}{j!}\partial_x^j\left[\rho(x)e^{2\pi i k(N^\alpha(v(x)-v(t_1))+\Delta_a)}\right]_{x=t_1} \\ &= \frac{N^{2-\alpha}}{2\pi}\,\mathrm{Li}_3\left(e^{2\pi i\Delta_a}\right)\int dt\,\frac{\rho(t)^2}{1-i\dot{v}(t)} + \text{subleading}\,. \end{aligned} \tag{1.2.36}$$

Plugging this equality in (1.2.29) and using that $\mathrm{Li}_3(e^{2\pi i u}) + \mathrm{Li}_3(e^{-2\pi i u}) = -(2\pi i)^3 g_+(u)$ for $0 < \mathbb{R}\mathrm{e}\, u < 1$, we obtain the first term in the second line of (1.2.33). Finally, we have enforced the normalization condition $\int\rho = 1$ with a Lagrange multiplier μ, and we have chosen its scaling with N for convenience. The dominant solution to the BAEs in the large N limit is obtained by extremizing $\widetilde{\mathcal{W}}$. Only for $\alpha = \frac{1}{3}$ there is a competition between the various terms and a well-behaved saddle point is found.[19] In this case the twisted superpotential scales as $N^{5/3}$.

The BAEs correspond to the system $\delta\widetilde{\mathcal{W}}/\delta v(t) = \delta\widetilde{\mathcal{W}}/\delta\rho(t) = 0$, together with the normalization condition $\partial\widetilde{\mathcal{W}}/\partial\mu = 0$. After some manipulations, the first two equations reduce to

$$\mu = -k\,t\,v(t) + \frac{ik}{2}\big(v(t)^2 - t^2\big) + \frac{2G(\Delta)\,\rho(t)}{1 - i\,\dot{v}(t)}\,. \tag{1.2.37}$$

We solve this equation taking $k > 0$ as well as

$$0 < \Delta_a < 1 \qquad \text{and} \qquad \sum\nolimits_{a=1}^{3}\Delta_a = 1\,, \tag{1.2.38}$$

which implies also $G(\Delta) > 0$. We look for solutions in which $\rho(t)$ is positive, bounded, and either integrable or with compact support between two zeros. It turns out that there exists only one solution satisfying these requirements, and it has compact support. After fixing the normalization $\int\rho = 1$, the solution is

$$\begin{aligned} v(t) &= -\frac{t}{\sqrt{3}} & \mu &= \frac{3^{7/6}k^{1/3}G(\Delta)^{2/3}}{4}\big(1 - i/\sqrt{3}\big) \\ \rho(t) &= \frac{3^{1/6}k^{1/3}}{2G(\Delta)^{1/3}} - \frac{2kt^2}{3\sqrt{3}\,G(\Delta)} & t_\pm &= \pm\frac{3^{5/6}G(\Delta)^{1/3}}{2k^{1/3}} \end{aligned} \tag{1.2.39}$$

with support on the interval $\mathcal{D} = [t_-, t_+]$. The density $\rho(t)$ vanishes at $t_\pm$.

[19]Moreover, in this scaling argument we are assuming that k does not scale with N.

We notice that the k-fold degeneracy of the solutions is invisible in the large N limit: the k solutions in the orbit are related by shifts of $v(t)$ by $1/kN^{1/3}$. The solution for $\sum_{a=1}^{3} \Delta_a = 2$ is similar to (1.2.39): just map $v(t) \to -v(t)$, $G(\Delta) \to -G(\Delta)$ and $\mu \to \mu^*$. The density $\rho(t)$ is well-defined because $G(\Delta) < 0$ in this range. The cases $\sum_a \Delta_a = 0, 3$ imply $\Delta_a = 0, 1$ respectively (since $0 \leq \Delta_a \leq 1$) and do not lead to solutions to the BAEs.

It was proven in [44] that, for a large class of quiver gauge theories including the one we are studying here, the following relation holds:

$$\widetilde{\mathcal{W}}(\Delta)\Big|_{\text{BAEs}} = -i\frac{3}{5}\, N^{5/3}\mu(\Delta_a)\ . \tag{1.2.40}$$

On the left-hand-side is the twisted superpotential (1.2.33) evaluated on the solution (1.2.39). The relation is indeed satisfied in our case. If we restrict to $\sum_a \Delta_a = 1$ there is also a connection with the S^3 partition function F_{S^3} [158–160] of the gauge theory, in the large N limit:

$$2\pi i\, \widetilde{\mathcal{W}}(\Delta)\Big|_{\text{BAEs}} = \frac{1}{4}\, F_{S^3}\big(R_a = 2\Delta_a\big)\ , \tag{1.2.41}$$

where R_a are the R-charges. The S^3 partition function of the gauge theory we are studying here has been considered *e.g.* in [161].

The last step is to compute the large N limit of the expression (1.2.22) for Z, as a functional of the solutions (v, ρ) to the BAEs, and then to plug in the dominant solution (1.2.39) we found. Once again, the computation is essentially as the one in [29].[20] It turns out that at large N the logarithm of the index grows as $N^{5/3}$. In particular the k-fold degeneracy of the solutions is irrelevant at leading order in N. As a functional of the solutions to the BAEs and at leading order in N, the index is given by:

$$\log Z(\mathfrak{p}, \Delta; v, \rho) = -2\pi N^{5/3} f_+(\mathfrak{p}, \Delta, \mathfrak{g}) \int_{\mathcal{D}} dt\, \frac{\rho(t)^2}{1 - i\,\dot{v}(t)} \tag{1.2.42}$$

with

$$f_+(\mathfrak{p}, \Delta, \mathfrak{g}) = \sum_{a=1}^{3} \big(\mathfrak{g} - 1 - \mathfrak{p}_a\big)\, g_+'(\Delta_a) + \frac{(1-\mathfrak{g})}{12}\ . \tag{1.2.43}$$

This expression has been obtained using the same technique as for the twisted superpotential $\widetilde{\mathcal{W}}$ in (1.2.33). Notice that, only the last two products in (1.2.22) give contribution to the leading order in N. Plugging in the solution (1.2.39) we find

$$\log Z(\mathfrak{p}, \Delta) = -\frac{3^{7/6}\,\pi}{5}\, \frac{f_+(\mathfrak{p}, \Delta, \mathfrak{g})}{G(\Delta)^{1/3}}\, k^{1/3} N^{5/3}\big(1 - i/\sqrt{3}\big)\ . \tag{1.2.44}$$

This expression can be further simplified recalling that $\mathfrak{p}_a$ and Δ_a are constrained by

[20] In [29] it was crucial to keep into account "exponential tails". In the cases where $\log Z$ scales like $N^{5/3}$ such tails do not play a role [44].

(1.2.20). Specializing to the case in which[21] $\sum_{a=1}^{3} \Delta_a = 1$, one finds that

$$f_+(\mathfrak{p}, \Delta, \mathfrak{g}) = -\frac{1}{2}\Delta_1\Delta_2\Delta_3 \sum_{a=1}^{3} \frac{\mathfrak{p}_a}{\Delta_a} \,, \qquad G(\Delta) = \frac{1}{2}\Delta_1\Delta_2\Delta_3 \,. \tag{1.2.45}$$

We are thus led to the simple expression:

$$\log Z(\mathfrak{p}, \Delta) = \frac{3^{7/6}\,\pi}{2^{2/3}5}\big(1 - i/\sqrt{3}\big)\; k^{1/3} N^{5/3} \big(\Delta_1\Delta_2\Delta_3\big)^{2/3} \sum_{a=1}^{3} \frac{\mathfrak{p}_a}{\Delta_a} \,. \tag{1.2.46}$$

This expression seems not to depend on $\mathfrak{g}$, however recall that the fluxes $\mathfrak{p}_a$ are constrained as in (1.2.20) and that introduces the dependence on $\mathfrak{g}$.

In fact, the general analysis of [44] gives a compact way to compute the index once the dominant solution to the BAEs is found:

$$\log Z = 2\pi i \sum_a \mathfrak{p}_a \, \frac{\partial \widetilde{\mathcal{W}}(\Delta)}{\partial \Delta_a}\bigg|_{\text{BAEs}} \,. \tag{1.2.47}$$

This expression agrees with (1.2.46) and determines a relation between the topologically twisted index and the S^3 free energy as stated in (6).

1.3 Entropy matching through attractor equations

We compare the Bekenstein-Hawking entropy computed from supergravity in Section 1.1 with the microstate counting from field theory in Section 1.2.

First of all we need a dictionary between the charges. In field theory there are three electric and magnetic charges $(\mathfrak{p}_a, \mathfrak{q}_a)$ that are integer, and $\mathfrak{p}_a$ satisfy the BPS constraint (1.2.20). To understand the quantization condition of (p^a, e_a) in supergravity (see a similar discussion in [30]) we recall that the Yang-Mills action is normalized in the same way as the Einstein-Hilbert term. Rescaling to canonical normalization we find

$$\mathfrak{p}_a = \eta g \, p^a \in \mathbb{Z} \,, \qquad \mathfrak{q}_a = \frac{\eta}{4G_N g}\, e_a \in \mathbb{Z} \,. \tag{1.3.1}$$

This is compatible with (1.1.31).

Then we need a dictionary between the field theory chemical potentials Δ_a, constrained by (1.2.38), and the supergravity vector multiplet scalars z^a, constrained by (1.1.29). We propose

$$\Delta_a = \frac{z^a}{\sum_{a=1}^{3} z^a} \,. \tag{1.3.2}$$

[21]The solution for the other case, in which $\sum_a \Delta_a = 2$, can be obtained from this one simply mapping $\Delta_a \to 1 - \Delta_a$.

This automatically guarantees $\sum_a \Delta_a = 1$. The map (1.3.2) is three-to-one (before taking into account that the scalars z^a take values in the upper half-plane), not invertible: a common rotation of z^a by $e^{2\pi i/3}$ leaves the Δ_a's invariant. This resonates with the fact that the large N index (1.2.46) is not a single-valued function in the complex Δ_a-plane. The inverse of (1.3.2) is

$$z^a = e^{\frac{i\pi}{3}} \left(\frac{m}{g}\right)^{1/3} \frac{\Delta_a}{(\Delta_1\Delta_2\Delta_3)^{1/3}} \tag{1.3.3}$$

which has in fact three sheets[22] and automatically guarantees $\prod_{a=1}^{3} z_a = -m/g$. One also obtains the relation $\big(\prod_a \Delta_a\big)^{1/3}\big(\sum_a z_a\big) = e^{i\pi/3}\,(m/g)^{1/3}$.

Finally we need a dictionary between the field theory dimensionless parameters N and k and the supergravity dimensionful parameters g, m and G_N [82]:

$$\frac{m^{1/3}g^{-7/3}}{4G_N} = \frac{3^{2/3}}{2^{2/3}\,5}\, k^{1/3}N^{5/3}\,, \qquad\qquad \frac{16\pi^3}{3}\left(\frac{m}{g}\right)^5 = Nk^5\,. \tag{1.3.4}$$

Although not needed here, the relation with the Type IIA mass parameter is $k = 2\pi\ell_s m$.

Consider now the index function $\mathcal{I}(\Delta;\mathfrak{p},\mathfrak{q}) = \log Z(\mathfrak{p},\Delta) - 2\pi i \sum_a \Delta_a \mathfrak{q}_a$ whose value at the critical point computes the large N ground state degeneracy:

$$\mathcal{I} = \frac{2^{-1/3}\,3^{2/3}\,\pi\, e^{-i\pi/6}}{5}\, k^{1/3}N^{5/3}\,\big(\Delta_1\Delta_2\Delta_3\big)^{2/3} \sum_{a=1}^{3} \frac{\mathfrak{p}_a}{\Delta_a} - 2\pi i \sum_{a=1}^{3} \mathfrak{q}_a\Delta_a\,. \tag{1.3.5}$$

Using the dictionaries for the various quantities we can rewrite it as

$$\mathcal{I} = \mathcal{S} = -\frac{2\pi i}{g^2}\,\frac{\eta}{4G_N}\,\frac{\sum_{a=1}^{3}\big(g\, e_a z^a - m\, p^a/z_a\big)}{\sum_{a=1}^{3} z^a}\,, \tag{1.3.6}$$

exactly matching the entropy function $\mathcal{S}$ in (1.1.35) we found in supergravity. Notice in particular that the supergravity variables z^a provide a global description of the parameter space, on which the function $\mathcal{I} = \mathcal{S}$ is single valued.

Summarizing, we have reduced the classical supergravity computation of the horizon area and the quantum field theory computation of the ground state degeneracy—more precisely, of its index—to the same extremization problem: finding the value of a complex function at its critical point. Since the two functions $\mathcal{S}$ and $\mathcal{I}$ coincide (as functions of variables with the same constraint), the result is guaranteed to be the same: the black hole entropy exactly equals the ground state degeneracy at leading order.

Notice that from the field theory index we can also reproduce the R-charge of the black holes, along the lines of [30]. In field theory the flavor charges are $\mathfrak{q}_a = \frac{1}{2}(R_a - R_3)$. Keeping them fixed, we perform a common shift of the R_a's (which does not affect the extremization problem since $\sum_{a=1}^{3}\Delta_a = 1$) in such a way that the value of $\mathcal{I}$ at the critical

[22] It is important that one uses the same branch of the root for the three values $a = 1, 2, 3$.

point becomes real. Then we read off the R-charge $r = R_3$. Exactly the same procedure fixes the black hole R-charge $e_R = e_3$ in supergravity, as we commented upon at the end of Section 1.1.

1.4 Conclusions

In this chapter we have studied the entropy of static dyonic BPS black holes in 4d $\mathcal{N} = 2$ gauged supergravities with vector and hyper multiplets. We have focused on a specific example: BPS black holes in $\mathrm{AdS}_4 \times S^6$ in massive Type IIA. We have shown that, similarly to the case with no hypermultiplets, the entropy can be expressed as the value of a function $\mathcal{S}$ at its critical point. Moreover we have shown that the entropy can be reproduced with a microscopic computation in the dual (via AdS/CFT) 3d QFT: there the logarithm of the number of states can be reduced to the very same extremization problem, thus giving evidence that the $\mathcal{I}$-extremization principle applies also in this case.

It would be interesting to understand the case with hypermultiplets more in general. The hypermultiplets can give mass to some of the vector multiplets, thus effectively reducing the extremization problem to a submanifold of the vector multiplet scalar manifold $\mathcal{M}_{\mathrm{SK}}$. Only this submanifold seems to be visible to the QFT index. Presumably, a general matching argument (similar to the one presented in the introduction for the cases with no hypermultiplets) would involve not only the prepotential on $\mathcal{M}_{\mathrm{SK}}$, but also the Killing vector fields on the hypermultiplet scalar manifold $\mathcal{M}_{\mathrm{QK}}$ that are gauged, and the embedding tensor. How these quantities appear on the QFT side is unclear to us.

Part II

The superconformal index and AdS_5 black holes

2 A Bethe Ansatz formula for the superconformal index

We derive a new formula—dubbed "Bethe Ansatz formula"—for the superconformal index of 4d $\mathcal{N}=1$ supersymmetric theories and we carefully discuss its properties. The chapter is organized as follows. In Section 2.1 we review the standard formula for the 4d superconformal index, carefully stressing its regime of applicability. In Section 2.2 we present our new Bethe Ansatz formula in great detail, and then we derive it in Section 2.2.2. We give some conclusions in Section 2.3.

2.1 The 4d superconformal index

Let us review the standard formulation of the superconformal index [87, 88], thus fixing our notation. This object counts local operators in short representations of the 4d $\mathcal{N}=1$ superconformal algebra (SCA) $\mathfrak{su}(2,2|1)$. Going to radial quantization, this is the same as counting (with sign) $\frac{1}{4}$-BPS states of the theory on S^3.

The bosonic part of the superconformal algebra is $\mathfrak{su}(2,2)\oplus\mathfrak{u}(1)_R$, where the first factor is the 4d conformal algebra and the second one is the R-symmetry. We pick on S^3 one Poincaré supercharge, specifically $\mathcal{Q}=\overline{Q}_-$, and its conjugate conformal supercharge $\mathcal{Q}^\dagger=S_+$. Together with $\Delta=\frac{1}{2}\{\mathcal{Q},\mathcal{Q}^\dagger\}$ they form an $\mathfrak{su}(1|1)$ superalgebra. The superconformal index is then equal to the Witten index

$$\mathcal{I}(t) \coloneqq \operatorname{Tr}_{\mathcal{H}(S^3)}(-1)^F e^{-\beta\Delta}\prod\nolimits_k t_k^{J_k}\,, \tag{2.1.1}$$

where J_k are Cartan generators of the commutant of $\mathfrak{su}(1|1)$ in the full SCA and t_k are the associated complex fugacities. By standard arguments [34], $\mathcal{I}(t)$ counts only states with $\Delta=0$, *i.e.* annihilated by both $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, and thus it does not depend on β. On the other hand, it is holomorphic in the fugacities t_k, which serve both as regulators and as refinement parameters.

To be more precise, the states counted by (2.1.1) have $\Delta=E-2J_+-\frac{3}{2}r=0$, where E is the conformal Hamiltonian or dimension, $J_\pm$ are the Cartan generators of the angular momentum $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \subset \mathfrak{su}(2,2)$, and r is the superconformal $U(1)_R$ charge. Moreover, the subalgebra of $\mathfrak{su}(2,2|1)$ which commutes with $\mathfrak{su}(1|1)$ has Cartan generators $E + J_+$ and J_-. Therefore we write

$$\mathcal{I}(p,q) = \mathrm{Tr}_{\,\Delta=0}\,(-1)^F\, p^{\frac{1}{3}(E+J_+)+J_-} q^{\frac{1}{3}(E+J_+)-J_-} = \mathrm{Tr}_{\,\Delta=0}\,(-1)^F\, p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} \,, \qquad (2.1.2)$$

where $J_{1,2} = J_+ \pm J_-$ parametrize the rotated frame $\mathfrak{u}(1)_1 \oplus \mathfrak{u}(1)_2 \subset \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ and p, q are the associated fugacities (up to a shift by $r/2$). Whenever the theory enjoys flavor symmetries, one can introduce fugacities v_α for the Cartan generators of the flavor group. Then, the index will depend holomorphically also on v_α.

The trace formula (2.1.2) can be exactly evaluated at all regimes in the couplings. Indeed, since $\mathcal{I}$ is invariant under any continuous deformation of the theory, one can explicitly account for the contribution of every gauge-invariant state with $\Delta = 0$ in the free regime [88,105,106]. In particular, the contributions of all the multi-particle states are simply encoded in the plethystic exponential [162] of the "single-letter partition functions", whereas the restriction to the gauge-invariant sector is done by integrating the latter contributions over the gauge group. This procedure yields a finite-dimensional integral formula for the superconformal index, which can be expressed as an elliptic hypergeometric integral [90].[23]

For concreteness, we consider a generic $\mathcal{N} = 1$ gauge theory with semi-simple gauge group G, flavor symmetry group G_F and non-anomalous $U(1)_R$ R-symmetry. We assume that the theory flows in the IR to a non-trivial fixed point and we parametrize $U(1)_R$ with the superconformal R-charge sitting in the SCA of the IR CFT (assuming this is visible in the UV). Furthermore, the matter content consists of n_χ chiral multiplets Φ_a in representations $\mathfrak{R}_a$ of G, carrying flavor weights ω_a in some representations $\mathfrak{R}_F$ of G_F and with superconformal R-charges r_a. Additionally, we turn on flavor fugacities v_α, with $\alpha = 1, \ldots, \mathrm{rk}(G_F)$, parametrizing the maximal torus of G_F. The integral representation of the superconformal index is given by

$$\mathcal{I}(p,q;v) = \frac{(p;p)_\infty^{\mathrm{rk}(G)}(q;q)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \frac{\prod_{a=1}^{n_\chi}\prod_{\rho_a\in\mathfrak{R}_a}\Gamma\big((pq)^{r_a/2} z^{\rho_a} v^{\omega_a};p,q\big)}{\prod_{\alpha\in\Delta}\Gamma(z^\alpha;p,q)} \prod_{i=1}^{\mathrm{rk}(G)} \frac{dz_i}{2\pi i z_i} \,. \qquad (2.1.3)$$

The integration variables z_i parametrize the maximal torus of G, and the integration contour is the product of $\mathrm{rk}(G)$ unit circles. Then ρ_a are the weights of the representation $\mathfrak{R}_a$, α parametrizes the roots of G and $|\mathcal{W}_G|$ is the order of the Weyl group. Moreover,

[23] An alternative way to obtain the integral formula is to use supersymmetric localization [37]. Indeed, the supersymmetric partition function Z of the theory on a primary Hopf surface $\mathcal{H}_{p,q} \simeq S^1 \times S^3$ can be computed with localization [102,103] and it is related to the superconformal index through $Z = e^{-E_{\mathrm{SUSY}}}\mathcal{I}$, where E_{SUSY} is the supersymmetric Casimir energy [100,101].

we have introduced the notation $z^{\rho_a} = \prod_{i=1}^{\mathrm{rk}(G)} z_i^{\rho_a^i}$ and $v^{\omega_a} = \prod_{\alpha=1}^{\mathrm{rk}(G_F)} v_\alpha^{\omega_a^\alpha}$, whereas

$$\Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z} \,, \qquad |p|<1 \,, \quad |q|<1 \tag{2.1.4}$$

is the elliptic gamma function [163] and

$$(z;q)_\infty = \prod_{n=0}^{\infty} (1-zq^n) \,, \qquad |q|<1 \tag{2.1.5}$$

is the q-Pochhammer symbol (see Appendix A for details).

This representation makes manifest the holomorphic dependence of the index on p, q, v_α. It is important to stress that the expression (2.1.3), which is a contour integral along $\mathrm{rk}(G)$ unit circles, is only valid as long as the fugacities stay within the following domain:

$$|p|, |q| < 1 \,, \qquad |pq| < \left|(pq)^{r_a/2} v^{\omega_a}\right| < 1 \,, \qquad \forall\, a \,. \tag{2.1.6}$$

These conditions descend from the requirement of convergence of the plethystic representation of the index, from which (2.1.3) is derived. The plethystic expansion of the elliptic gamma function,

$$\Gamma(z;p,q) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} \frac{z^m - (pq)^m z^{-m}}{(1-p^m)(1-q^m)}\right] \,, \tag{2.1.7}$$

converges for

$$|pq| < |z| < 1 \qquad \text{and} \qquad |p|, |q| < 1 \,. \tag{2.1.8}$$

The domain (2.1.6) then follows from requiring the integrand of (2.1.3) to have a convergent expansion. Indeed, within the domain of convergence (2.1.8), the elliptic gamma function is a single-valued analytic function with no zeros, poles nor branch cuts. Both $\Gamma(z;p,q)$ and $(z;q)_\infty$ can be analytically continued to $z \in \mathbb{C}$. However, when we analytically continue the integral (2.1.3) outside the domain (2.1.6), the integration contour must be continuously deformed in order to take into account the movement of the various poles of the integrand in the complex plane, in such a way that the poles do not cross the contour. As a result, for generic fugacities the integration contour is not as simple as a product of unit circles. To avoid this complication, throughout this chapter we will always work within (2.1.6)—and perform analytic continuation only at the end, if needed.

It will be useful to set some new notation. We define a set of chemical potentials

$$p = e^{2\pi i\tau} \,, \qquad q = e^{2\pi i\sigma} \,, \qquad v_\alpha = e^{2\pi i\xi_\alpha} \,, \qquad z_i = e^{2\pi i u_i} \,, \tag{2.1.9}$$

as well as a fictitious chemical potential ν_R for the R-symmetry, whose value is fixed to

$$\nu_R = \frac{1}{2}(\tau+\sigma) \tag{2.1.10}$$

by supersymmetry. Moreover, we redefine the elliptic gamma function as a (periodic) function of the chemical potentials:

$$\widetilde{\Gamma}(u,\tau,\sigma) = \Gamma\big(e^{2\pi i u}; e^{2\pi i\tau}, e^{2\pi i\sigma}\big) \;, \tag{2.1.11}$$

so that the integrand of (2.1.3) can be expressed as

$$\mathcal{Z}(u;\xi,\nu_R,\tau,\sigma) = \frac{\prod_{a=1}^{n_\chi}\prod_{\rho_a\in\mathfrak{R}_a}\widetilde{\Gamma}\big(\rho_a(u)+\omega_a(\xi)+r_a\nu_R;\tau,\sigma\big)}{\prod_{\alpha\in\Delta}\widetilde{\Gamma}\big(\alpha(u);\tau,\sigma\big)} \;. \tag{2.1.12}$$

At last, we define

$$\kappa_G = \frac{(p;p)_\infty^{\mathrm{rk}(G)}(q;q)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \;. \tag{2.1.13}$$

The integral representation of the index takes then the following compact form:

$$\mathcal{I}(p,q;v) = \kappa_G \int_{\mathbb{T}^{\mathrm{rk}(G)}} \mathcal{Z}(u;\xi,\nu_R,\tau,\sigma)\; d^{\mathrm{rk}(G)}u \;. \tag{2.1.14}$$

The integration contour $\mathbb{T}^{\mathrm{rk}(G)}$ is represented on the u-plane by a product of straight segments of length one on the real axes. In terms of the chemical potentials, the domain (2.1.6) can be rewritten as:

$$\mathrm{Im}\,\tau,\, \mathrm{Im}\,\sigma > 0 \;, \qquad 0 < \mathrm{Im}\,\omega_a(\xi) < \mathrm{Im}(\tau+\sigma) \;, \qquad \forall\, a \;. \tag{2.1.15}$$

The integral formula (2.1.14) is the starting point of our analysis. In the next section we will focus our attention to the case where τ/σ is a rational number to derive—from (2.1.14)—a new formula that expresses the index as a finite sum.

2.2 A new Bethe Ansatz formula

The integral representation (2.1.14) of the superconformal index is valid for generic complex values of the chemical potentials within the domain (2.1.15). However, if we restrict to a case where

$$\tau/\sigma \in \mathbb{Q}_+ \;, \tag{2.2.1}$$

we can prove an alternative formula, very similar to the expression (1.2.10) for the topologically twisted index in 3d, describing the superconformal index as a finite sum over the set of solutions to certain transcendental equations, which again we call "Bethe Ansatz Equations" (BAEs). We will first present the formula in detail, and then provide a proof. In Section 2.2.1 we will also discuss the properties of the set of pairs (τ,σ) satisfying (2.2.1).

Let us take

$$\tau = a\omega \;, \qquad \sigma = b\omega \qquad \text{with } a, b \in \mathbb{N} \text{ such that } \gcd(a,b) = 1 \tag{2.2.2}$$

and $\operatorname{Im}\omega > 0$. This implies $q^a = p^b$, as in (8). We can set $p = h^a$ and $q = h^b$ with $h = e^{2\pi i \omega}$, although we will mostly work with chemical potentials. We introduce the BAEs as the set of equations

$$Q_i(u;\xi,\nu_R,\omega) = 1 \;, \qquad \forall\, i = 1,\ldots,\mathrm{rk}(G) \;, \tag{2.2.3}$$

written in terms of "BA operators" defined as

$$Q_i(u;\xi,\nu_R,\omega) = \prod_{a=1}^{n_\chi} \prod_{\rho_a \in \mathfrak{R}_a} P\big(\rho_a(u) + \omega_a(\xi) + r_a \nu_R; \omega\big)^{\rho_a^i} \;. \tag{2.2.4}$$

The basic BA operator is

$$P(u;\omega) = \frac{e^{-\pi i \frac{u^2}{\omega} + \pi i u}}{\theta_0(u;\omega)} \;, \tag{2.2.5}$$

where $\theta_0(u;\omega) = (z;h)_\infty (z^{-1}h;h)_\infty$ with $z = e^{2\pi i u}$ and $h = e^{2\pi i \omega}$.

The BA operators satisfy three important properties. First, they are doubly-periodic in the gauge chemical potentials:

$$Q_i(u + n + m\omega;\xi,\nu_R,\omega) = Q_i(u;\xi,\nu_R,\omega) \;, \qquad \forall\, n_i,\, m_i \in \mathbb{Z} \;, \quad i = 1,\ldots \mathrm{rk}(G) \;. \tag{2.2.6}$$

Second, they are invariant under $SL(2,\mathbb{Z})$ modular transformations of ω:

$$Q_i(u;\xi,\nu_R,\omega) = Q_i(u;\xi,\nu_R,\omega+1) = Q_i\left(\frac{u}{\omega};\frac{\xi}{\omega},\frac{\nu_R}{\omega},-\frac{1}{\omega}\right) = Q_i(-u;-\xi,-\nu_R,\omega) \;. \tag{2.2.7}$$

The last equality represents invariance under the center of $SL(2,\mathbb{Z})$. Third, they capture the quasi-periodicity of the index integrand:

$$Q_i(u;\xi,\nu_R,\omega)\, \mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega) = \mathcal{Z}(u - \delta_i ab\omega;\xi,\nu_R,a\omega,b\omega) \;, \tag{2.2.8}$$

valid $\forall\, i$ and where $\delta_i = (\delta_{ij})_{j=1}^{\mathrm{rk}(G)}$ so that $(u - \delta_i ab\omega)_j = u_j - \delta_{ij} ab\omega$. Notice that, thanks to (2.2.7), one might prefer to rewrite Q_i in terms of the function $\theta(u;\omega) = e^{-\pi i u + \pi i \omega/6}\, \theta_0(u;\omega)$ that has simpler modular properties [91] (see Appendix A). Nevertheless, here we will refrain from doing so.

Because of the double-periodicity of Q_i, the actual number of solutions $\hat{u}_i$ to the system of BAEs (2.2.3) is infinite. However, the solutions can be grouped into a finite number of equivalence classes $[\hat{u}_i]$ such that $\hat{u}_i \sim \hat{u}_i + 1 \sim \hat{u}_i + \omega$. In other words, the equations and their solutions are well-defined on a torus $\mathbb{T}^{2\mathrm{rk}(G)}$ which is the product of $\mathrm{rk}(G)$ identical complex tori of modular parameter ω, and the number of solutions on the torus is finite. The modular invariance (2.2.7) confirms that the equations are well-defined on the torus. We define

$$\mathfrak{M}_{\text{BAE}} = \left\{ [\hat{u}_i]\,,\ i = 1, \ldots, \text{rk}(G) \;\middle|\; Q_i\big([\hat{u}]; \xi, \nu_R, \omega\big) = 1\,, \quad w \cdot [\hat{u}] \neq [\hat{u}] \quad \forall\, w \in \mathcal{W}_G \right\} \tag{2.2.9}$$

as the set of solutions (on the torus) that are not fixed by non-trivial elements of the Weyl group. For definiteness we can choose, as representatives, the elements living in a fundamental domain of the torus with modulus ω, *i.e.* with $0 \leq \text{Re}\,\hat{u}_i < 1$ and $0 \leq \text{Im}\,\hat{u}_i < \text{Im}\,\omega$. Notice that, because of (2.2.7), the solutions must organize into representations of $SL(2, \mathbb{Z})$.

As we prove below, thanks to the properties of the BA operators, we can rewrite the superconformal index as a sum over solutions to the BAEs in the following way:

$$\mathcal{I}(p, q; v) = \kappa_G \sum_{\hat{u}\, \in\, \mathfrak{M}_{\text{BAE}}} \mathcal{Z}_{\text{tot}}(\hat{u}; \xi, \nu_R, a\omega, b\omega)\ H(\hat{u}; \xi, \nu_R, \omega)^{-1}\ . \tag{2.2.10}$$

Here

$$\mathcal{Z}_{\text{tot}}(u; \xi, \nu_R, a\omega, b\omega) = \sum_{\{m_i\}=1}^{ab} \mathcal{Z}(u - m\omega; \xi, \nu_R, a\omega, b\omega)\ , \tag{2.2.11}$$

where $\mathcal{Z}$ is precisely the integrand defined in (2.1.12) and

$$H(u; \xi, \nu_R, \omega) = \det_{ij} \left[\frac{1}{2\pi i} \frac{\partial Q_i(u; \xi, \nu_R, \omega)}{\partial u_j} \right] \tag{2.2.12}$$

is the contribution from the Jacobian of the change of variables $u_i \mapsto Q_i(u)$. Notice that both the function H, and the function $\mathcal{Z}_{\text{tot}}$ evaluated on the solutions to the BAEs, are doubly-periodic on the product of complex tori of modulus ω. On the other hand, because $SL(2, \mathbb{Z})$ is not a symmetry of the superconformal index, the summand $\kappa_G\, \mathcal{Z}_{\text{tot}}\, H^{-1}$ is not invariant under modular transformations of ω.

A specialization of this formula to the case $\tau = \sigma$ was derived in [91], while a three-dimensional analog was derived in [92, 99]. In the next section we will spell out in detail how the BA formula uniquely fixes the index for all values of the complex fugacities, using either holomorphy or continuity. In Section 2.2.2 we will derive the final formula (2.2.10), starting from the integral representation (2.1.14). The proof is rather technical and it does not give new physical insights on the main result. Therefore, uninterested readers may directly jump to Chapter 3.

2.2.1 Continuation to generic fugacities

Our BA formula (2.2.10) can only be applied for special values of the angular fugacities that satisfy $q^a = p^b$. We will offer two arguments, one based on holomorphy and the other based on just continuity, that this is enough to completely determine the index for all values of the complex fugacities.

Using the standard definition (2.1.2), the index is not a single-valued function of the angular fugacities p, q—unless the R-charges of chiral multiplets are all even. This is also apparent from the integral formula (2.1.3). On the other hand, regarded as a function of chemical potentials τ, σ each living on the upper half-plane $\mathbb{H}$, the index is single-valued and holomorphic. Keeping the flavor fugacities fixed in the argument that follows, the BA formula applies to points $(\tau, \sigma) \in \mathbb{H}^2$ such that $\tau/\sigma \in \mathbb{Q}_+$. Such a set is dense in a hyperplane $\mathcal{J} \cong \mathbb{R}^3$ of real codimension one in $\mathbb{H}^2$ defined as $\mathcal{J} = \big\{(\tau, \sigma) \,\big|\, \tau/\sigma \in \mathbb{R}_+\big\}$. Thus, the BA formula determines the index on $\mathcal{J}$ by continuity. On the other hand, we know that the index is a holomorphic function on $\mathbb{H}^2$, therefore its restriction to $\mathcal{J}$ completely fixes the function on $\mathbb{H}^2$ by analytic continuation.

It turns out that we can refine the argument in such a way that we only use continuity, and not holomorphy, of the index. This is because if we think in terms of angular fugacities p, q each living in the open unit disk $\mathbb{D}$, then the set of points $(p, q) \in \mathbb{D}^2$ such that $q^a = p^b$ for coprime $a, b \in \mathbb{N}$ is dense in $\mathbb{D}^2$. This fact is not completely obvious, and we show it in Appendix C.

Unfortunately, the index (2.1.3) is not a single-valued function of p, q if we keep the flavor fugacities v_α fixed, unless the R-charges are all even. However, it is always possible to find a change of variables which expresses $\mathcal{I}$ as a single-valued function of a set of new fugacities. The latter is defined by

$$\Delta_a = \omega_a(\xi) + r_a \nu_R \quad \Longrightarrow \quad y_a = e^{2\pi i \Delta_a} = v^{\omega_a} (pq)^{\frac{r_a}{2}} \;, \qquad \forall\, a = 1, \dots, n_\chi \;. \tag{2.2.13}$$

This gives us a set of (redundant) chemical potentials Δ_a, one for each chiral multiplet present in the theory, which must satisfy some linear constraint, following the requirement of invariance of the theory under flavor and R-symmetry. Suppose, indeed, the theory has a superpotential given by

$$W(\Phi) = \sum\nolimits_A W_A(\Phi) \;, \tag{2.2.14}$$

where each $W_A(\Phi)$ is a gauge-invariant homogeneous polynomial of degree n_A. Then, for each term in (2.2.14), the following linear constraints must be satisfied:

$$\sum_{a \in A} r_a = 2 \;, \qquad \sum_{a \in A} \omega_a^\alpha = 0 \;, \qquad \forall\, \alpha = 1, \dots, \mathrm{rk}(G) \;. \tag{2.2.15}$$

Here we used $a \in A$ to indicate the chiral components Φ_a which are present in W_A. The first equation imposes that the superpotential has R-charge 2. The second equation constrains W to be invariant under G_F. Indeed, $\omega_a = (\omega_a^\alpha)_{\alpha=1}^{\mathrm{rk}(G_F)}$ are the flavor weights carried by Φ_a. A similar role is played by ABJ anomalies.

Translating (2.2.15) to the definition of Δ_a, we obtain

$$\sum_{a \in A} \Delta_a = 2\nu_R = \tau + \sigma \qquad \forall\, A\,. \tag{2.2.16}$$

In such a new set of variables we have

$$\mathcal{Z}(u; \Delta, \tau, \sigma) = \frac{\prod_{a=1}^{n_\chi} \prod_{\rho_a \in \mathfrak{R}_a} \widetilde{\Gamma}(\rho_a(u) + \Delta_a; \tau, \sigma)}{\prod_{\alpha \in \Delta} \widetilde{\Gamma}(\alpha(u); \tau, \sigma)}\,, \tag{2.2.17}$$

showing that the index is now a well-defined, single-valued and continuous function (in fact, also holomorphic) of the fugacities p, q, y_a. Indeed, recall that the elliptic gamma function is a single-valued function of its arguments, and notice that the constraints (2.2.16) always involve integer combinations of τ, σ, thus never introducing non-trivial monodromies under integer shifts. Once again, the BA formula can be applied whenever $q^a = p^b$ and for generic values of y_a. Since such a set of points is dense in the space of generic fugacities, we conclude that the BA formula fixes the index completely.

2.2.2 Proof of the formula

We prove the formula (2.2.10) in three steps. First we verify the properties (2.2.6) and (2.2.8) of the BA operators. Then we use them to modify the contour of the integral (2.1.14) and to reduce it to a sum of simple residues. Finally we prove that the only poles that contribute to the residue formula are determined by the BAEs, thus obtaining (2.2.10).

Properties of the BA operators

First, we prove the identities (2.2.6) and (2.2.8). For later convenience, let us briefly recall the anomaly cancellation conditions that are required to have a well-defined four-dimensional theory. These requirements can be expressed in terms of the anomaly coefficients. In particular, let $\mathbf{i} = (i, \alpha)$ collectively denote the Cartan indices of the gauge $\times$ flavor group, where $i = 1, \ldots, \mathrm{rk}(G)$ are the gauge indices and $\alpha = 1, \ldots, \mathrm{rk}(G_F)$ are the flavor indices. Moreover, define $\mathbf{a} = (a, \rho_a)$ as running over all chiral multiplets components, where ρ_a are the weights of the gauge representation $\mathfrak{R}_a$. Then the anomaly coefficients for gauge/flavor symmetries are defined by

$$\mathcal{A}^{\mathbf{ijk}} = \sum_{\mathbf{a}} Q^{\mathbf{i}}_{\mathbf{a}} Q^{\mathbf{j}}_{\mathbf{a}} Q^{\mathbf{k}}_{\mathbf{a}} \,, \qquad \mathcal{A}^{\mathbf{ij}} = \sum_{\mathbf{a}} Q^{\mathbf{i}}_{\mathbf{a}} Q^{\mathbf{j}}_{\mathbf{a}} \,, \qquad \mathcal{A}^{\mathbf{i}} = \sum_{\mathbf{a}} Q^{\mathbf{i}}_{\mathbf{a}} \,, \tag{2.2.18}$$

where $Q^{\mathbf{i}}_{\mathbf{a}} = Q^{\mathbf{i}}_{(a,\rho_a)} = (\rho^i_a, \omega^\alpha_a)$ are the components of the gauge $\times$ flavor weights carried by the chiral multiplets. The first and the last coefficient in (2.2.18) are associated with the gauge^3 and mixed gauge-gravitational^2 perturbative anomalies. The second term—sometimes called pseudo-anomaly coefficient—describes the non-perturbative or global anomaly [164–166] when the corresponding perturbative anomaly vanishes.

Similarly, the perturbative anomaly coefficients involving the R-symmetry are defined by

$$\begin{aligned} \mathcal{A}^{\mathbf{ij}R} &= \sum_{\mathbf{a}} Q^{\mathbf{i}}_{\mathbf{a}} Q^{\mathbf{j}}_{\mathbf{a}} (r_{\mathbf{a}} - 1) + \delta^{\mathbf{ij},ij} \sum_{\alpha \in \Delta} \alpha^i \alpha^j & \qquad \mathcal{A}^{\mathbf{i}RR} &= \sum_{\mathbf{a}} Q^{\mathbf{i}}_{\mathbf{a}} (r_{\mathbf{a}} - 1)^2 \\ \mathcal{A}^{RRR} &= \sum_{\mathbf{a}} (r_{\mathbf{a}} - 1)^3 + \dim G & \qquad \mathcal{A}^{R} &= \sum_{\mathbf{a}} (r_{\mathbf{a}} - 1) + \dim G \,, \end{aligned} \tag{2.2.19}$$

whereas the pseudo R-anomaly coefficients are

$$\mathcal{A}^{\mathbf{i}R} = \sum_{\mathbf{a}} Q^{\mathbf{i}}_{\mathbf{a}} (r_{\mathbf{a}} - 1) \qquad \mathcal{A}^{RR} = \sum_{\mathbf{a}} (r_{\mathbf{a}} - 1)^2 + \dim G \,. \tag{2.2.20}$$

Anomaly cancellation is realized by a set of conditions on the coefficients defined above, that a well-defined quantum gauge theory must satisfy. We will also restrict to the case that the gauge group G is semi-simple. The conditions for the cancellation of the gauge and gravitational anomaly are

$$\mathcal{A}^{ijk} = \mathcal{A}^{i} = 0 \qquad \text{and} \qquad \mathcal{A}^{ij} \in 4\mathbb{Z} \qquad \text{for } G \text{ semi-simple} \,. \tag{2.2.21}$$

The conditions for the cancellation of the ABJ anomalies of G_F and $U(1)_R$, namely that those are global symmetries of the quantum theory, are

$$\mathcal{A}^{ij\alpha} = \mathcal{A}^{ijR} = 0 \,. \tag{2.2.22}$$

Finally,

$$\mathcal{A}^{i\alpha\beta} = \mathcal{A}^{i\alpha R} = \mathcal{A}^{iRR} = 0 \qquad \text{and} \qquad \mathcal{A}^{i\alpha} = \mathcal{A}^{iR} = 0 \tag{2.2.23}$$

simply follow from the restriction to semi-simple gauge group G.

We now focus on describing some properties of the basic BA operator

$$P(u;\omega) = \frac{e^{-\pi i \frac{u^2}{\omega} + \pi i u}}{\theta_0(u;\omega)} \,. \tag{2.2.24}$$

First, consider the function

$$\theta_0(u;\omega) = (z;h)_\infty (z^{-1}h;h)_\infty = \prod_{k=0}^{\infty}(1-zh^k)(1-z^{-1}h^{k+1}) \,, \qquad z = e^{2\pi i u} \,,\ h = e^{2\pi i \omega} \tag{2.2.25}$$

which is holomorphic in z and h, and satisfies the following properties (see Appendix A for details):

$$\begin{aligned} &\theta_0(u+n+m\omega;\omega) = (-1)^m \, e^{-2\pi i m u - \pi i m(m-1)\omega} \, \theta_0(u;\omega) \qquad \forall\, n,m \in \mathbb{Z} \\ &\theta_0(-u;\omega) = \theta_0(u+\omega;\omega) = -e^{-2\pi i u} \, \theta_0(u;\omega) \,. \end{aligned} \tag{2.2.26}$$

They immediately imply

$$\begin{aligned} &P(-u;\omega) = -P(u;\omega) \\ &P(u+n+m\omega;\omega) = (-1)^{n+m} \, e^{-\frac{\pi i}{\omega}(2nu+n^2)} \, P(u;\omega) \qquad \forall\, n,m \in \mathbb{Z} \,. \end{aligned} \tag{2.2.27}$$

It turns out that the basic BA operator has also nice modular transformation properties:

$$P(u;\omega+1) = e^{\pi i \frac{u^2}{\omega(\omega+1)}} \, P(u;\omega) \,, \qquad P\left(\frac{u}{\omega}; -\frac{1}{\omega}\right) = e^{\pi i \left(\frac{u^2}{\omega} - \frac{\omega}{6} - \frac{1}{6\omega} + \frac{1}{2}\right)} \, P(u;\omega) \,. \tag{2.2.28}$$

In order to prove (2.2.8), we also need to show that

$$\begin{aligned} P(u+r\nu_R;\omega)^m \, \widetilde{\Gamma}(u+r\nu_R;a\omega,b\omega) &= (-1)^{\frac{abm^2}{2} + \frac{m(a+b-1)}{2}} \, e^{-\frac{\pi i m u^2}{\omega} + \pi i a b m^2 u - \pi i m(a+b)(r-1)u} \times \\ &\times h^{-\frac{m^3 ab}{6} + \frac{ab(a+b)m^2(r-1)}{4} - \frac{m(a+b)^2(r-1)^2}{8} + \frac{m(a^2+b^2+2)}{24}} \, \widetilde{\Gamma}(u+r\nu_R - mab\omega;a\omega,b\omega) \,. \end{aligned} \tag{2.2.29}$$

Here $r \in \mathbb{R}$ mimics the contribution from the R-charge of a generic multiplet in the theory. Notice that all factors in front of $\widetilde{\Gamma}$ in the r.h.s. of (2.2.29) explicitly depend on the fermion R-charge $r-1$. This will be crucial to ensure anomaly cancellation in the full BA operator.

Proof. The identity (2.2.29) follows from the properties of the elliptic gamma function. Indeed, for generic τ and σ, we have that

$$\widetilde{\Gamma}(u+\tau;\tau,\sigma) = \theta_0(u;\sigma)\,\widetilde{\Gamma}(u;\tau,\sigma) \,, \qquad \widetilde{\Gamma}(u+\sigma;\tau,\sigma) = \theta_0(u;\tau)\,\widetilde{\Gamma}(u;\tau,\sigma) \,. \tag{2.2.30}$$

Moreover, there exists a factorization property (see Theorem 5.4 of [163]) which expresses $\widetilde{\Gamma}(u;a\omega,b\omega)$ as a product of elliptic gamma functions with equal periods:

$$\widetilde{\Gamma}(u;a\omega,b\omega) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\widetilde{\Gamma}(u+(as+br)\omega;ab\omega,ab\omega) \,, \tag{2.2.31}$$

valid for $a, b \in \mathbb{Z}$ (not necessarily coprime). Using both (2.2.30) and (2.2.31) we obtain the identity

$$\widetilde{\Gamma}(u+ab\omega;a\omega,b\omega)=\left[\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\theta_0\big(u+(as+br)\omega;ab\omega\big)\right]\times\widetilde{\Gamma}(u;a\omega,b\omega) \tag{2.2.32}$$

and its generalizations to $m\in\mathbb{Z}$, given by

$$\begin{aligned}\widetilde{\Gamma}(u+mab\omega;a\omega,b\omega)=(-z)^{-\frac{abm(m-1)}{2}}\,h^{-\frac{m(m-1)}{2}\frac{ab(2ab-a-b)}{2}-\frac{m(m-1)(m-2)a^2b^2}{6}}\times\\ \times\left[\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\theta_0\big(u+(as+br)\omega;ab\omega\big)^m\right]\times\widetilde{\Gamma}(u;a\omega,b\omega)\ .\end{aligned} \tag{2.2.33}$$

Now, by enforcing the assumption that $\gcd(a,b)=1$, we can use the properties of numerical semigroups (see Appendix B for a review) to reduce the periods of the theta functions from $ab\omega$ to ω. In order to do so, let us introduce some notation. We call $\mathcal{R}(a,b)$ the set of non-negative integer linear combinations of a,b:

$$\mathcal{R}(a,b)=\{am+bn\,|\,m,n\in\mathbb{Z}_{\geq0}\}. \tag{2.2.34}$$

Then $\mathcal{R}(a,b)$ forms a numerical semigroup, which can be thought of as a subset of $\mathbb{Z}_{\geq0}$, closed under addition, with only a finite number of excluded non-vanishing elements. The latter elements form the so-called set of gaps $\overline{\mathcal{R}}(a,b)=\mathbb{N}\setminus\mathcal{R}(a,b)$. The highest element of $\overline{\mathcal{R}}(a,b)$ is the Frobenius number $F(a,b)=ab-a-b$, whereas the order of $\overline{\mathcal{R}}(a,b)$ is called the genus $\chi(a,b)$ and the sum of all its elements is the weight $w(a,b)$. It is a classic result in mathematics that, in terms of a,b, the latter read

$$\chi(a,b)=\frac{(a-1)(b-1)}{2}\ ,\qquad\qquad w(a,b)=\frac{(a-1)(b-1)(2ab-a-b-1)}{12}\ . \tag{2.2.35}$$

Thanks to the properties of these objects, we can use the following identities (proved in Appendix B):

$$\begin{aligned}\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(zh^{as+br};h^{ab})_\infty&=\frac{(z;h)_\infty}{\prod_{k\in\overline{\mathcal{R}}(a,b)}(1-zh^k)}\\ \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(z^{-1}h^{ab-as-br};h^{ab})_\infty&=(z^{-1}h;h)_\infty\prod_{k\in\overline{\mathcal{R}}(a,b)}(1-z^{-1}h^{-k})\ ,\end{aligned} \tag{2.2.36}$$

which lead to

$$\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\theta_0\big(u+(as+br)\omega;\omega\big)=(-z)^{-\chi(a,b)}\,h^{-w(a,b)}\,\theta_0(u;\omega)\ . \tag{2.2.37}$$

Substituting into (2.2.33) we obtain

$$\widetilde{\Gamma}(u+mab\omega;a\omega,b\omega)=(-z)^{-\frac{abm^2}{2}+\frac{m(a+b-1)}{2}}\times \\ \times h^{-\frac{abm^3}{6}+\frac{ab(a+b)m^2}{4}-\frac{(a^2+b^2+3ab-1)m}{12}}\,\theta_0(u;\omega)^m\,\widetilde{\Gamma}(u;a\omega,b\omega)\ . \quad (2.2.38)$$

Finally, applying (2.2.38) to the l.h.s. of (2.2.29) proves the latter identity. □

We now turn to analyzing the full BA operators. Notice that, in the definition (2.2.4), Q_i receive contribution only from the chiral multiplets of the theory. The vector multiplets do not appear in (2.2.4) because their contribution simply amounts to

$$\prod_{\alpha\in\Delta}P\big(\alpha(u);\omega\big)^{-\alpha^i}=\prod_{\alpha>0}\left[\frac{P\big(-\alpha(u);\omega\big)}{P\big(\alpha(u);\omega\big)}\right]^{\alpha^i}=(-1)^{\sum_{\alpha>0}\alpha^i}=1\ , \quad (2.2.39)$$

which holds true if G is semi-simple, as in this case the sum of positive roots is always an even integer. Despite this fact, as far as the proof of (2.2.10) is concerned, we write

$$Q_i(u;\xi,\nu_R,\omega)=\prod_{a=1}^{n_\chi}\prod_{\rho_a\in\mathfrak{R}_a}P\big(\rho_a(u)+\omega_a(\xi)+r_a\nu_R;\omega\big)^{\rho_a^i}\times\prod_{\alpha\in\Delta}P\big(\alpha(u);\omega\big)^{-\alpha^i} \quad (2.2.40)$$

without simplifying the vector multiplet contribution.

At this point, using (2.2.27) we can show that Q_i satisfies:

$$\begin{aligned}Q_i(u+n;\xi,\nu_R,\omega)&=(-1)^{\mathcal{A}^{ij}n_j}\,e^{-\frac{\pi i}{\omega}\left(\mathcal{A}^{ijk}n_j(2u_k+n_k)+2\mathcal{A}^{ij\alpha}n_j\xi_\alpha+2\mathcal{A}^{ijR}n_j\omega\right)}\,Q_i(u;\xi,\nu_R,\omega)\\ Q_i(u+m\omega;\xi,\nu_R,\omega)&=(-1)^{\mathcal{A}^{ij}m_j}\,Q_i(u;\xi,\nu_R,\omega)\ ,\end{aligned} \quad (2.2.41)$$

which, thanks to (2.2.21)–(2.2.23), reduce to (2.2.6) $\forall\, n_i,m_i\in\mathbb{Z}$ in an anomaly-free theory. Similarly, (2.2.27) and (2.2.28) together with the anomaly cancelation conditions (2.2.21)–(2.2.23) imply (2.2.7). Moreover, using (2.2.40), we can write

$$Q_i(u;\xi,\nu_R,\omega)\,\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega)= \\ =\frac{\prod_{a,\rho_a}P\big(\rho_a(u)+\omega_a(\xi)+r_a\nu_R;\omega\big)^{\rho_a^i}\,\widetilde{\Gamma}\big(\rho_a(u)+\Delta_a;a\omega,b\omega\big)}{\prod_{\alpha\in\Delta}P\big(\alpha(u);\omega\big)^{\alpha^i}\,\widetilde{\Gamma}\big(\alpha(u);a\omega,b\omega\big)}\ . \quad (2.2.42)$$

Applying (2.2.29), the latter equation can be written as

$$Q_i(u;\xi,\nu_R,\omega)\,\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega)=(-1)^{\frac{ab}{2}\mathcal{A}^{ii}+\frac{a+b-1}{2}\mathcal{A}^i}\,e^{\pi iab\left(\mathcal{A}^{iij}u_j+\mathcal{A}^{ii\alpha}\xi_\alpha\right)}\times \\ \times e^{-\frac{\pi i}{\omega}\left(\mathcal{A}^{ijk}u_ju_k+\mathcal{A}^{i\alpha\beta}\xi_\alpha\xi_\beta+2\mathcal{A}^{ij\alpha}u_j\xi_\alpha\right)}\,e^{-\pi i(a+b)\left(\mathcal{A}^{ijR}u_j+\mathcal{A}^{i\alpha R}\xi_\alpha\right)+\frac{\pi iab(a+b)}{2}\mathcal{A}^{iiR}\omega}\times \\ \times e^{-\frac{\pi i(a+b)^2}{4}\mathcal{A}^{iRR}\omega-\frac{\pi ia^2b^2}{3}\mathcal{A}^{iii}\omega+\frac{\pi i(a^2+b^2)}{12}\mathcal{A}^i\omega}\,\mathcal{Z}(u-\delta_iab\omega;\xi,\nu_R,\omega)\ , \quad (2.2.43)$$

which, by anomaly cancellation, reduces to (2.2.8).

Residue formula

We now use the BA operators and their properties to modify the contour of integration of the index in (2.1.14). For our purposes, it is sufficient to implement the following trivial relation:

$$\begin{aligned}\mathcal{I}(p,q;v) &= \kappa_G \oint \mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega)\, d^{\mathrm{rk}(G)}u \\ &= \kappa_G \oint \frac{\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)}{\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)}\, \mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega)\, d^{\mathrm{rk}(G)}u\ .\end{aligned} \tag{2.2.44}$$

The numerator of the integrand can be expanded as

$$\begin{aligned}&\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)\times\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega) = \\ &\quad = \sum_{n=0}^{\mathrm{rk}(G)}\frac{(-1)^n}{n!}\sum_{i_1\neq\dots\neq i_n}^{\mathrm{rk}(G)} Q_{i_1}(u;\xi,\nu_R,\omega)\dots Q_{i_n}(u;\xi,\nu_R,\omega)\,\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega) \\ &\quad = \sum_{n=0}^{\mathrm{rk}(G)}\frac{(-1)^n}{n!}\sum_{i_1\neq\dots\neq i_n}^{\mathrm{rk}(G)} \mathcal{Z}\big(u-(\delta_{i_1}+\ldots+\delta_{i_n})ab\omega;\xi,\nu_R,a\omega,b\omega\big)\ ,\end{aligned} \tag{2.2.45}$$

where, in the last line, we have used the shift property (2.2.8). Plugging the last equation back in (2.2.44) gives:

$$\mathcal{I}(p,q;v) = \kappa_G \sum_{n=0}^{\mathrm{rk}(G)}\frac{(-1)^n}{n!}\sum_{i_1\neq\dots\neq i_n}^{\mathrm{rk}(G)} I_{i_1\dots i_n}(p,q;v)\ , \tag{2.2.46}$$

with

$$\begin{aligned}I_{i_1\dots i_n}(p,q;v) &= \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \frac{\mathcal{Z}\big(u-(\delta_{i_1}+\ldots+\delta_{i_n})ab\omega;\xi,\nu_R,a\omega,b\omega\big)}{\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)}\, d^{\mathrm{rk}(G)}u \\ &= \oint_{\mathcal{C}_{i_1\dots i_n}} \frac{\mathcal{Z}\big(u;\xi,\nu_R,a\omega,b\omega\big)}{\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)}\, d^{\mathrm{rk}(G)}u\end{aligned} \tag{2.2.47}$$

and where

$$\mathcal{C}_{i_1\dots i_n} = \mathbb{T}^{\mathrm{rk}(G)-n}\times\bigcup_{k=1}^{n}\big\{|z_{i_k}|=|h|^{-ab};\circlearrowleft\big\}\ . \tag{2.2.48}$$

This is a contour where $z_{i_1},\dots,z_{i_n}$ live on circles of radius $|h|^{-ab}$, whereas the other variables z_j parametrize the unit circles in $\mathbb{T}^{\mathrm{rk}(G)-n}$. The second line in (2.2.47) has been obtained by implementing the change of variables $u_{i_k}\mapsto u_{i_k}+ab\omega$ for $k=1,\dots,n$ and using the periodicity (2.2.6).

The series of integrals in (2.2.46) can be resummed to a unique integral over a composite contour:

$$\mathcal{I}(p,q;v) = \kappa_G \oint_{\mathcal{C}} \frac{\mathcal{Z}\big(u;\xi,\nu_R,a\omega,b\omega\big)}{\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)}\, d^{\mathrm{rk}(G)}u \;, \tag{2.2.49}$$

where

$$\mathcal{C} = \sum_{n=0}^{\mathrm{rk}(G)} \frac{(-1)^n}{n!} \bigcup_{i_1\neq\cdots\neq i_n}^{\mathrm{rk}(G)} \mathcal{C}_{i_1\ldots i_n} \simeq \bigcup_{i=1}^{\mathrm{rk}(G)} \big\{|z_i|=1;\circlearrowleft\big\} \cup \big\{|z_i|=|h|^{-ab};\circlearrowleft\big\} \tag{2.2.50}$$

is a contour encircling the annulus $\mathcal{A} = \Big\{u_i \,\Big|\, 1<|z_i|<|h|^{-ab},\, i=1,\ldots,\mathrm{rk}(G)\Big\}$.

We now apply the residue theorem to (2.2.49). The integrand has simple poles coming from the denominator, whose positions are precisely described by the BAEs (2.2.3). Obviously, only the poles that lie inside the annulus $\mathcal{A}$ contribute to the contour integral. Moreover, as we do in Appendix D, one can show that whenever a particular solution $[\hat{u}]$ to the BAEs (2.2.3) is fixed (on the torus) by a non-trivial element of the Weyl group $\mathcal{W}_G$, namely $w\cdot[\hat{u}]=[\hat{u}]$, then the numerator $\mathcal{Z}(\hat{u};\xi,\nu_R,a\omega,b\omega)$ is such that cancelations take place and there is no contribution to the integral—more precisely, the function $\mathcal{Z}_{\mathrm{tot}}(\hat{u};\xi,\nu_R,a\omega,b\omega)$ defined in (2.2.11) vanishes.[24] Hence, we define the set of relevant poles by:

$$\mathfrak{M}_{\mathrm{index}} = \left\{\hat{u}_i \;\middle|\; [\hat{u}_i]\in\mathfrak{M}_{\mathrm{BAE}} \quad \text{and} \quad 1<|\hat{z}_i|<|h|^{-ab}\,,\; i=1,\ldots,\mathrm{rk}(G)\right\}. \tag{2.2.51}$$

This includes all points inside the annulus $\mathcal{A}$ such that their class belongs to $\mathfrak{M}_{\mathrm{BAE}}$. In particular, the same equivalence class $[\hat{u}_i]\in\mathfrak{M}_{\mathrm{BAE}}$ appears in $\mathfrak{M}_{\mathrm{index}}$ as many times as the number of its representatives living in $\mathcal{A}$. For this reason, we employ the following alternative description:

$$\mathfrak{M}_{\mathrm{index}} = \left\{\hat{u}_i^{(\vec{m})} = [\hat{u}_i]-m_i\,\omega \;\middle|\; [\hat{u}_i]\in\mathfrak{M}_{\mathrm{BAE}}\,,\; m_i=1,\ldots,ab\,,\; i=1,\ldots,\mathrm{rk}(G)\right\} \tag{2.2.52}$$

where, we some abuse of notation, we have denoted as $[\hat{u}_i]$ the representative in the fundamental domain of the torus as after (2.2.9).

In addition, the numerator $\mathcal{Z}$ has other poles coming from the elliptic gamma functions. As we show below, as long as the fugacities v_α, p, q are taken within the domain (2.1.6)—which is necessary in order for the standard contour integral representation (2.1.3) to be valid—those other poles either lie outside the annulus $\mathcal{A}$ or are not poles of the in-

[24] In particular, let us stress that the condition $w\cdot[\hat{u}]\neq[\hat{u}]$ in the definition of $\mathfrak{M}_{\mathrm{BAE}}$ could be relaxed with no harm: in that case, we would simply include more poles in the sum, whose residues however combine to zero.

tegrand (because the denominator has a pole of equal or higher degree) and thus do not contribute to the integral.

Therefore, working within the domain (2.1.6), we can rewrite the index as

$$\mathcal{I}(p,q;v) = (-2\pi i)^{\mathrm{rk}(G)}\ \kappa_G \sum_{\hat{u}^{(\vec{m})}\in\mathfrak{M}_{\mathrm{index}}} \operatorname*{Res}_{u=\hat{u}^{(\vec{m})}} \left[\frac{\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega)}{\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u;\xi,\nu_R,\omega)\big)}\, d^{\mathrm{rk}(G)}u \right] . \tag{2.2.53}$$

Computing the residues produces the final expression for the superconformal index:

$$\mathcal{I}(p,q;v) = \kappa_G \sum_{\hat{u}^{(\vec{m})}\in\mathfrak{M}_{\mathrm{index}}} \mathcal{Z}(\hat{u}^{(\vec{m})};\xi,\nu_R,a\omega,b\omega)\ H(\hat{u}^{(\vec{m})};\xi,\nu_R,\omega)^{-1} , \tag{2.2.54}$$

where H is defined in (2.2.12). The residue formula (2.2.54) can be rewritten, more elegantly, in the final form:

$$\mathcal{I}(p,q;v) = \kappa_G \sum_{\hat{u}\,\in\,\mathfrak{M}_{\mathrm{BAE}}} \mathcal{Z}_{\mathrm{tot}}(\hat{u};\xi,\nu_R,a\omega,b\omega)\ H(\hat{u};\xi,\nu_R,\omega)^{-1} , \tag{2.2.55}$$

where

$$\mathcal{Z}_{\mathrm{tot}}(u;\xi,\nu_R,a\omega,b\omega) = \sum_{\{m_i\}=1}^{ab} \mathcal{Z}(u-m\omega;\xi,\nu_R,a\omega,b\omega) . \tag{2.2.56}$$

To obtain this expression we have split the sum over the poles in $\mathfrak{M}_{\mathrm{index}}$ into a sum over the inequivalent solutions to the BAEs, described by the elements of $\mathfrak{M}_{\mathrm{BAE}}$, and the sum over the "repetitions" of these elements in the annulus $\mathcal{A}$. Moreover, we have used the double-periodicity of the Jacobian $H(u;\xi,\nu_R,\omega)$ to pull the latter sum inside the definition of $\mathcal{Z}_{\mathrm{tot}}$.

Analysis of the residues

The last step consists in showing that the only residues contributing to (2.2.49) come from zeros of the denominator. In particular we need to show that, remaining within the domain (2.1.6), all poles in (2.2.49) which are not given by the BAEs live outside the annulus $\mathcal{A}$ and thus do not contribute to the integral. We concretely do so by proving that every pole of $\mathcal{Z}$ inside $\mathcal{A}$ is also a pole of the denominator $\prod_i(1-Q_i)$ with a high enough degree that the integrand of (2.2.49) is non-singular at those points.

We begin by classifying the poles of $\mathcal{Z}$. Using (2.2.26), (2.2.30) and

$$\widetilde{\Gamma}(u;\tau,\sigma) = \frac{1}{\widetilde{\Gamma}(\tau+\sigma-u;\tau,\sigma)} , \tag{2.2.57}$$

we can rewrite $\mathcal{Z}$ as

$$\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega) = \prod_{\alpha>0} \theta_0\big(\alpha(u);a\omega\big)\,\theta_0\big(-\alpha(u);b\omega\big) \times \prod_{a,\rho_a} \widetilde{\Gamma}\big(\rho_a(u)+\omega_a(\xi)+r_a\nu_R;a\omega,b\omega\big)\;. \tag{2.2.58}$$

Since $\theta_0(u;\omega)$ has no poles for finite u, the only singularities of $\mathcal{Z}$ come from the elliptic gamma functions related to the chiral multiplets. These can be read off the product expansion:

$$\widetilde{\Gamma}(u;a\omega,b\omega) = \prod_{m=0}^{\infty}\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\left(\frac{1-h^{ab(m+2)-as-br}z^{-1}}{1-h^{abm+as+br}z}\right)^{m+1} \tag{2.2.59}$$

that follows from (2.2.31), and so they are given by

$$z^{\rho_a} = v^{-\omega_a} h^{-r_a(a+b)/2-abm-as-br} \tag{2.2.60}$$

for $0 \leq r \leq a-1$, $0 \leq s \leq b-1$ and $m \geq 0$. The multiplicity of each pole is $\mu^a_m = m+1$.[25] Notice that one could also write $z^{\rho_a} = v^{-\omega_a}h^{-r_a(a+b)/2-k}$ for $k \in \mathcal{R}(a,b)$.

We now turn to analyzing the denominator. More specifically, we need to find the singularities of $\prod_{i=1}^{\mathrm{rk}(G)}\big(1-Q_i(u)\big)$. From (2.2.4) and (2.2.5) we see that Q_i has a pole whenever $\theta_0\big(\rho_a(u)+\omega_a(\xi)+r_a\nu_R;\omega\big) = 0$ and $\rho_a^i > 0$. Therefore, the singularities of the denominator are given by

$$z^{\rho_a} = v^{-\omega_a}h^{-r_a(a+b)/2+n} \qquad \text{for } n \in \mathbb{Z}\;, \tag{2.2.61}$$

all with the same multiplicity $\nu^a = \sum_{i\in\mathfrak{D}_a^+}\rho_a^i$. Here $\mathcal{D}_a^{\pm}$ represents the set of indices such that $\rho_a^i > 0$, resp. $\rho_a^i < 0$, thus ν^a is the sum of the positive components of ρ_a. We notice that the denominator poles in (2.2.61) with $-n \in \mathcal{R}(a,b)$ coincide with the numerator poles. Therefore, the actual singularities of the integrand in (2.2.49) are only those points in (2.2.60) such that $\mu^a_m > \nu^a$, or more explicitly

$$m \geq \sum\nolimits_{i\in\mathfrak{D}_a^+}\rho_a^i\;. \tag{2.2.62}$$

We now want to show that, when the fugacities satisfy (2.1.6), the set of actual singularities is always living outside the annulus $\mathcal{A}$. Therefore, we first study the conditions for which (2.2.60) belong to the annulus $\mathcal{A}$. By imposing that $1 < |z_i| < |h|^{-ab}$, we obtain that

$$|h|^{-ab\sum_{i\in\mathfrak{D}_a^-}\rho_a^i} < |z^{\rho_a}| < |h|^{-ab\sum_{i\in\mathfrak{D}_a^+}\rho_a^i}\;, \qquad \forall\, a\;. \tag{2.2.63}$$

Then we determine the constraints imposed on (2.2.60) by requiring (2.1.6). In the rational

[25] In counting the multiplicity one may worry that there could be different choices of r,s that give the same $abm+as+br$ for fixed m. This is equivalent to finding non-trivial solutions to the equation $as+br = as'+br'$. However, it is easy to see that, as long as $0 \leq r,r' \leq a-1$ and $0 \leq s,s' \leq b-1$, such an equation has no non-trivial solution in $\mathbb{Z}$.

case, the latter conditions are expressed by $|h|^{a+b} < |v^{\omega_a} h^{r_a(a+b)/2}| < 1$, $\forall\, a$. These inequalities, together with $0 \leq as + br \leq 2ab - a - b$, imply that

$$|h|^{-abm} \leq |h|^{-abm-as-br} < |z^{\rho_a}| < |h|^{-abm-a(s+1)-b(r+1)} \leq |h|^{-ab(m+2)} \,. \tag{2.2.64}$$

Furthermore, requiring (2.2.62) to be satisfied, we obtain that

$$|z^{\rho_a}| > |h|^{-ab\sum_{i\in\mathfrak{D}_a^+} \rho_a^i} \,, \qquad \forall\, a \,, \tag{2.2.65}$$

which is satisfied by all the singularities of (2.2.49) coming from the numerator $\mathcal{Z}$.

At this point, we immediately notice that the intersection between (2.2.63) and (2.2.65) is empty. This means that, if the flavor fugacities satisfy (2.1.6), all poles of the integrand (2.2.49) that come from poles of the numerator $\mathcal{Z}$ live outside the annulus $\mathcal{A}$, and so the only residues contributing to the integral are those given by the BAEs. This completes the proof of (2.2.10).

2.3 Conclusions

In this chapter we have derived the BA formula (2.2.10) for the superconformal index of 4d $\mathcal{N} = 1$ gauge theories, expressing it as a sum over the solutions of a set of Bethe Ansatz Equations. This expression provides a powerful alternative to the integral representation (2.1.3), more suited to study the asymptotic expansion in interesting limits (see *e.g.* Chapter 3) and to perform numerical computations. Although the applicability of the BA formula is limited by the condition (2.2.2), we have formulated one holomorphy-based and one continuity-based arguments to show that the index can be uniquely fixed as a function of generic fugacities.

Finally, as a remark, we stress that the BA formula can be viewed as the 4d version of the analogous formula (1.2.10) for the topologically twisted index of 3d $\mathcal{N} = 2$ theories. Indeed, as mentioned in the introduction, when $q^a = p^b$ one expects, along the lines of [35,36,91,92,98,99], to be able to interpret (2.2.10) as computing a 2d correlator on S^2 in the A-twisted theory obtained from dimensional reduction of the original theory on the torus of modulus ω. Nevertheless, our derivation is entirely based on the manipulations discussed throughout the chapter.

In the next chapter we will apply the BA formula to solve a very important issue: the holographic counting of the entropy of electrically charged rotating black holes in Type IIB on $\mathrm{AdS}_5 \times S^5$.

3 | Black holes in 4d $\mathcal{N} = 4$ Super-Yang-Mills

We study the entropy of electrically charged rotating BPS black holes in AdS_5 [107–111] that can be embedded in Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ [167]. Via the AdS/CFT correspondence, we relate their microstate counting problem to the large N computation of the superconformal index of the boundary $SU(N)$ $\mathcal{N} = 4$ SYM theory. We perform such a calculation making use of the BA formula derived in Chapter 2 and, generalizing the $\mathcal{I}$-extremization principle to this setup, we find perfect agreement with the gravitational dual. Furthermore, we discuss in a simple example the presence of Stokes phenomena in the asymptotic expansion of the index and their possible physical interpretation.

This chapter is organized as follows. In Section 3.1 we review the BPS black hole solutions in AdS_5 and their entropy. In Section 3.2 we consider the dual $\mathcal{N} = 4$ SYM theory and we describe its superconformal index in the BA formula, whereas in Section 3.3 we compute the large N limit. Sections 3.4 and 3.5 are devoted to extracting the black hole entropy from the index. Finally, we conclude in Section 3.6.

3.1 BPS black holes in AdS_5

In order to set the stage, we briefly review the properties of BPS black holes in $\mathrm{AdS}_5 \times S^5$. These are solutions of Type IIB supergravity that preserve one complex supercharge [168], thus being 1/16 BPS. The metric interpolates between the AdS_5 boundary and a fibration of AdS_2 on S^3 at the horizon. Moreover, the black holes carry three charges $Q_{1,2,3}$ for $U(1)^3 \subset SO(6)$ acting on S^5, that appear as electric charges in AdS_5, and two angular momenta $J_{1,2}$ associated to the Cartan $U(1)^2 \subset SO(4)$ (each Cartan generator acts on an $\mathbb{R}^2$ plane inside $\mathbb{R}^4$). The black hole mass is fixed by the linear BPS constraint

$$M = g\Big(|J_1| + |J_2| + |Q_1| + |Q_2| + |Q_3|\Big) , \tag{3.1.1}$$

where $g = \ell_5^{-1}$ is the gauge coupling, determined in terms of the curvature radius ℓ_5 of AdS_5 (whereas charges are dimensionless). It turns out that regular BPS black holes with no closed time-like curves only exist when the five charges satisfy certain non-linear constraints. The first constraint relies on the fact that one parameterizes the solutions by four real parameters $\mu_{1,2,3}$, Ξ [111].[26] The second constraint is

$$g^2\mu_{1,2,3} > \Xi - 1 \geq 0 \ . \tag{3.1.3}$$

Alternatively, one can have the same constraint with Ξ substituted by Ξ^{-1} which corresponds to exchanging $J_1 \leftrightarrow J_2$. The third constraint is

$$S_{\text{BH}} \in \mathbb{R} \ , \tag{3.1.4}$$

where the entropy S_{BH} is defined in (3.1.10) below.

Charges and angular momenta of the black holes are completely determined by these four parameters μ_I, Ξ with $I = 1, 2, 3$. Defining

$$\gamma_1 = \sum_I \mu_I \ , \qquad \gamma_2 = \sum_{I<J} \mu_I\mu_J \ , \qquad \gamma_3 = \mu_1\mu_2\mu_3 \ , \tag{3.1.5}$$

the electric charges and angular momenta are

$$Q_I = \frac{\pi}{4G_N}\left[\frac{\mu_I}{g} + \frac{g}{2}\left(\gamma_2 - \frac{2\gamma_3}{\mu_I}\right)\right] \qquad \begin{aligned} J_1 &= \frac{\pi}{4G_N}\left[\frac{g\gamma_2}{2} + g^3\gamma_3 + \frac{\mathcal{J}}{g^3}\left(\Xi - 1\right)\right] \\ J_2 &= \frac{\pi}{4G_N}\left[\frac{g\gamma_2}{2} + g^3\gamma_3 + \frac{\mathcal{J}}{g^3}\left(\frac{1}{\Xi} - 1\right)\right] \end{aligned} \tag{3.1.6}$$

where G_N is the five-dimensional Newton constant and

$$\mathcal{J} = \prod_I \left(1 + g^2\mu_I\right) \ . \tag{3.1.7}$$

It is easy to see that one of the charges Q_I can be zero or negative.[27] There are some combinations, though, that we can bound above zero. For instance:

$$\begin{aligned} Q_1 + Q_2 + Q_3 &= \frac{\pi}{4G_N}\left[\frac{\gamma_1}{g} + \frac{g\,\gamma_2}{2}\right] > 0 \\ Q_I + Q_K &= \frac{\pi}{4G_N}\left[\frac{\mu_I + \mu_K}{g} + g\,\mu_I\mu_K\right] > 0 \qquad\qquad \text{for } I \neq K \ . \end{aligned} \tag{3.1.8}$$

[26] In [111] the authors use five real parameters $\mu_{1,2,3}$, a, b with $0 \leq a, b < g^{-1}$, however the black hole charges only depend on the combination $\Xi = \sqrt{(1 - b^2g^2)/(1 - a^2g^2)}$. The parameters a, b are useful to write the full supergravity solutions. They are determined, in terms of $\mu_{1,2,3}$ and Ξ, by the extra relation

$$\sqrt{(1 - a^2g^2)(1 - b^2g^2)} = \frac{2ab + 2g^{-1}(a + b) + 3g^{-2}}{\mu_1 + \mu_2 + \mu_3 + 3g^{-2}} \ . \tag{3.1.2}$$

[27] For instance, take μ_1 that goes to zero with $\mu_{2,3}$ fixed, then Q_1 becomes negative. One may wonder whether the extra condition that the entropy be real could force the charges to be positive. This is not the case. For instance, setting $\mu_1 = \mu_2^2/3(1 + \mu_2)$ and $\mu_3 = \mu_2$ as well as $\Xi = 1$, one finds (up to constant factors and setting $g = 1$) $Q_1 \sim -\mu_2^2/6 < 0$, $Q_2 = Q_3 \sim \mu_2(\mu_2 + 2)/2 > 0$ and $S_{\text{BH}}^2 \sim \mu_2^4/12 > 0$.

In particular, at most one charge can be zero or negative. Setting $g = 1$ for the sake of clarity, we also have

$$\begin{aligned} Q_I + J_1 &= \frac{\pi}{4G_N}\left[\Big(1+\mu_K\Big)\Big(1+\mu_L\Big)\Big(\mu_I + (1+\mu_I)(\Xi - 1)\Big)\right] > 0 \\ Q_I + J_2 &= \frac{\pi}{4G_N}\left[\Big(1+\mu_K\Big)\Big(1+\mu_L\Big)\Big(\mu_I + (1+\mu_I)\Big(\frac{1}{\Xi} - 1\Big)\Big)\right] > 0 \end{aligned} \tag{3.1.9}$$

for $I \neq K \neq L \neq I$. The two inequalities follow from (3.1.3).

The Bekenstein-Hawking entropy is proportional to the horizon area, and can be written as a function of the black hole charges [169]:

$$S_{\mathrm{BH}} = \frac{\text{Area}}{4G_N} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{\pi}{4G_N g^3}\big(J_1 + J_2\big)}\ . \tag{3.1.10}$$

The constraint (3.1.4) requires the quantity inside the radical to be positive. The BPS solutions have a regular well-defined event horizon only if the angular momenta are non-zero: in other words there is no static limit in gauged supergravity.

In this chapter we will focus on the "self-dual" case $J_1 = J_2 \coloneqq J$ [108]. Since, in general, $\mathcal{J} > 1$ and $\Xi \geq 1$, necessarily $\Xi = 1$. The constraint (3.1.3) simply becomes

$$\mu_I > 0\ . \tag{3.1.11}$$

The charges are

$$Q_I = \frac{\pi}{4G_N}\left[\frac{\mu_I}{g} + \frac{g}{2}\left(\gamma_2 - \frac{2\gamma_3}{\mu_I}\right)\right]\ , \qquad J = \frac{\pi}{4G_N}\left[\frac{g\gamma_2}{2} + g^3\gamma_3\right] > 0\ . \tag{3.1.12}$$

The entropy is

$$S_{\mathrm{BH}} = \frac{\pi^2}{2G_N}\sqrt{(1+g^2\gamma_1)\gamma_3 - \frac{g^2\gamma_2^2}{4}} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{\pi}{2G_N g^3}J}\ . \tag{3.1.13}$$

Once again, the constraint (3.1.4) requires the quantity inside the radical to be positive.[28]

3.2 The dual field theory and its index

A non-perturbative definition of Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is in terms of its boundary dual: 4d $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group [23]. The weak curvature limit in gravity corresponds to the large N and large 't Hooft coupling limit in field theory.

[28] We stress that the entropy is not automatically real. For instance, if we take μ_1 that goes to zero with $\mu_{2,3}$ fixed, then the quantity inside the radical becomes negative.

This follows from the holographic relation

$$N^2 = \frac{\pi\, {\ell_5}^3}{2G_N} = \frac{\pi}{2G_N g^3} \ . \tag{3.2.1}$$

Up to the choice of gauge group, SYM is the unique four-dimensional Lagrangian CFT with maximal supersymmetry. The field content, in $\mathcal{N}=1$ notation, consists of a vector multiplet and three chiral multiplets X, Y, Z, all in the adjoint representation of the gauge group. The quiver diagram is the same as in Figure 1.1 and, furthermore, there is a cubic superpotential $W = \operatorname{Tr} X[Y,Z]$. The R-symmetry is $SO(6)_R$: going to the Cartan $U(1)^3$, we choose a basis of generators $R_{1,2,3}$ each giving R-charge 2 to a single chiral multiplet and zero to the other two, in a symmetric way.

Considering the theory in radial quantization on $\mathbb{R}\times S^3$, we are interested in the states that can be dual to the BPS black holes described in Section 3.1. These are 1/16 BPS states preserving one complex supercharge $\mathcal{Q}$, and characterized by two angular momenta $J_{1,2}$ on S^3 and three R-charges for $U(1)^3 \subset SO(6)_R$. The angular momenta $J_{1,2}$ are semi-integer and each rotates an $\mathbb{R}^2 \subset \mathbb{R}^4$. To comply with the notation of the previous chapter, we set $J_\pm$ as the spins under $SU(2)_+ \times SU(2)_- \cong SO(4)$, so that $J_{1,2} = J_+ \pm J_-$. With respect to the $\mathcal{N}=1$ superconformal subalgebra (SCA) that contains $\mathcal{Q}$, we describe the R-charges in terms of two flavor generators $\mathfrak{q}_{1,2} = \frac{1}{2}(R_{1,2} - R_3)$ commuting with $\mathcal{Q}$, and the R-charge $r = \frac{1}{3}(R_1 + R_2 + R_3)$. All fields in the theory have integer charges under $\mathfrak{q}_{1,2}$. The counting of BPS states is performed by the superconformal index, defined in (2.1.1). In this case it is a function

$$\mathcal{I}(p,q,v_1,v_2) = \operatorname{Tr}\, (-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1+\frac{r}{2}}\, q^{J_2+\frac{r}{2}}\, v_1^{\mathfrak{q}_1}\, v_2^{\mathfrak{q}_2} \tag{3.2.2}$$

of the (complex) angular fugacities p,q and of the (complex) fugacities v_α, with $\alpha = 1,2$, associated with the flavor charges $\mathfrak{q}_\alpha$. The corresponding chemical potentials are τ,σ,ξ_α, defined as in (2.1.9):

$$p = e^{2\pi i\tau} \ , \qquad q = e^{2\pi i\sigma} \ , \qquad v_\alpha = e^{2\pi i\xi_\alpha} \ , \tag{3.2.3}$$

whereas, the fermion number can be expressed as $F = 2(J_+ + J_-) = 2J_1$. Briefly, the index is well-defined for $|p|$, $|q| < 1$, *i.e.* for $\operatorname{Im}\tau$, $\operatorname{Im}\sigma > 0$, and it only counts states annihilated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, thus being independent of β (see Chapter 2 for details).

It will be convenient to redefine the flavor chemical potentials as[29]

$$\Delta_\alpha = \xi_\alpha + \frac{\tau+\sigma}{3} \tag{3.2.4}$$

[29] This choice is similar to the change of variables defined in (2.2.13), although here we do not introduce any redundant chemical potentials.

and use

$$y_\alpha = e^{2\pi i \Delta_\alpha} \ . \tag{3.2.5}$$

The index becomes[30]

$$\mathcal{I}(p,q,y_1,y_2) = \text{Tr}\ (-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1+\frac{1}{2}R_3}\, q^{J_2+\frac{1}{2}R_3}\, y_1^{\mathfrak{q}_1}\, y_2^{\mathfrak{q}_2} \ . \tag{3.2.6}$$

Notice that J_1, J_2, $\frac{1}{2}F$, $\frac{1}{2}R_3$ are all half-integers and correlated according to

$$J_1 = J_2 = \frac{F}{2} = \frac{R_3}{2} \pmod 1 \ . \tag{3.2.7}$$

It is then manifest from (3.2.6) that the index is a single-valued function of the fugacities.

In order to evaluate the large N limit of the index, we find it convenient to use the Bethe Ansatz formula (2.2.10), derived in Chapter 2. Computing the limit with this formula is still challenging, and in this chapter we will restrict ourselves to the case of equal fugacities for the angular momenta:

$$\tau = \sigma \qquad \Longrightarrow \qquad p = q \ , \tag{3.2.8}$$

which implies that $a = b = 1$.

In the case of $\mathcal{N} = 4$ $SU(N)$ SYM and with this restriction, the BA formula of the superconformal index reads:

$$\mathcal{I}(q,y_1,y_2) = \kappa_N \sum_{\hat{u}\,\in\,\mathfrak{M}_{\text{BAEs}}} \mathcal{Z}(\hat{u};\Delta,\tau)\, H(\hat{u};\Delta,\tau)^{-1} \ . \tag{3.2.9}$$

Let us stress again that this is a finite sum over the set of solutions $\{\hat{u}\}$ on the torus of the Bethe Ansatz Equations, given by

$$1 = Q_i(u;\Delta,\tau) = e^{2\pi i\left(\lambda + 3\sum_j u_{ij}\right)} \prod_{j=1}^N \frac{\theta_0(u_{ji}+\Delta_1;\tau)\,\theta_0(u_{ji}+\Delta_2;\tau)\,\theta_0(u_{ji}-\Delta_1-\Delta_2;\tau)}{\theta_0(u_{ij}+\Delta_1;\tau)\,\theta_0(u_{ij}+\Delta_2;\tau)\,\theta_0(u_{ij}-\Delta_1-\Delta_2;\tau)} \tag{3.2.10}$$

for $i = 1,\ldots,N$ and where $u_{ij} = u_i - u_j$. The unknowns are the "complexified $SU(N)$ holonomies" u_i subject to the identifications

$$u_i \ \sim \ u_i + 1 \ \sim \ u_i + \tau \tag{3.2.11}$$

meaning that each one lives on a torus of modular parameter τ, and constrained by

$$\sum_{i=1}^N u_i = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}} \ , \tag{3.2.12}$$

[30] With respect to the notation in [88]: $p = t^3 x\big|_{\text{there}}$, $q = t^3/x\big|_{\text{there}}$, $y_1 = t^2 v\big|_{\text{there}}$, $y_2 = t^2 w/v\big|_{\text{there}}$.

as well as a "Lagrange multiplier" λ. The prefactor in (3.2.9) is

$$\kappa_N = \frac{1}{N!}\left(\frac{(q;q)_\infty^2\, \widetilde{\Gamma}(\Delta_1;\tau,\tau)\, \widetilde{\Gamma}(\Delta_2;\tau,\tau)}{\widetilde{\Gamma}(\Delta_1+\Delta_2;\tau,\tau)}\right)^{N-1} . \tag{3.2.13}$$

The function $\mathcal{Z}$ is the integrand in the integral representation (2.1.3)

$$\mathcal{Z}(u;\Delta,\tau) = \prod_{i\neq j}^{N} \frac{\widetilde{\Gamma}(u_{ij}+\Delta_1;\tau,\tau)\, \widetilde{\Gamma}(u_{ij}+\Delta_2;\tau,\tau)}{\widetilde{\Gamma}(u_{ij}+\Delta_1+\Delta_2;\tau,\tau)\, \widetilde{\Gamma}(u_{ij};\tau,\tau)} . \tag{3.2.14}$$

Finally, the Jacobian H is

$$H\Big|_{\text{BAEs}} = \det\left[\frac{1}{2\pi i}\, \frac{\partial(Q_1,\ldots,Q_N)}{\partial(u_1,\ldots,u_{N-1},\lambda)}\right] \tag{3.2.15}$$

when evaluated on the solutions to the BAEs. Notice once more that both Q_i, κ_N, $\mathcal{Z}$ and H are invariant under integer shifts of τ, Δ_1 and Δ_2, implying that the superconformal index (3.2.9) is a single-valued function of the fugacities. The functions θ_0 and $\widetilde{\Gamma}$ are defined as in the previous chapter (see (2.2.25) and (2.1.4), (2.1.11)) and some of their properties are collected in Appendix A.

Let us add some comments on how (3.2.10) and (3.2.15) are obtained from the general formalism of Chapter 2. The maximal torus of $SU(N)$ is given by the matrices $\text{diag}(z_1,\ldots,z_{N-1},z_N)$ with $\prod_{j=1}^N z_j = 1$ and, setting $z_j = e^{2\pi i u_j}$, is parameterized by $u_1,\ldots,u_{N-1}$. For general gauge group G, the BA operators Q_i in (2.2.4) have an index i that runs over the Cartan subalgebra of G. Let us denote the BA operators of $SU(N)$ as $\widehat{Q}_1,\ldots,\widehat{Q}_{N-1}$, then the BAEs are $\widehat{Q}_j = 1$. The BA operators of $SU(N)$ can be written as $\widehat{Q}_j = Q_j/Q_N$ in terms of the BA operators $Q_1,\ldots,Q_N$ of $U(N)$. Introducing a "Lagrange multiplier" λ, we can set $Q_N = e^{-2\pi i\lambda}$ and write the BAEs as $e^{2\pi i\lambda}Q_j = 1$ for $j = 1,\ldots,N$ (this includes the definition of λ). Absorbing $e^{2\pi i\lambda}$ into Q_i, we end up with (3.2.10).

The Jacobian H for $SU(N)$ is given by

$$H = \det\left[\frac{1}{2\pi i}\, \frac{\partial\widehat{Q}_i}{\partial u_j}\right]_{i,j=1,\ldots,N-1} . \tag{3.2.16}$$

When evaluated on the solutions to the BAEs, we have

$$H\Big|_{\text{BAEs}} = \det\left[\frac{1}{2\pi i}\, \frac{\partial(Q_i - Q_N)}{\partial u_j}\right]_{i,j=1,\ldots,N-1} = (3.2.15) . \tag{3.2.17}$$

To see the last equality, one should notice that $\partial Q_i/\partial\lambda\big|_{\text{BAEs}} = 2\pi i$.

The chemical potentials u_j are defined modulo 1, and the $SU(N)$ condition implies that they should satisfy $\sum_j u_j \in \mathbb{Z}$. However, thanks to (2.2.6), the BAEs (3.2.10) are invariant under shifts of one of the u_j's by the periods of a complex torus of modular parameter τ, namely $u_k \to u_k + n + m\tau$ for a fixed k. Hence, as proven in Chapter 2, the BAEs are well-defined on $N-1$ copies of the torus. Consistently, both H and $\mathcal{Z}$—when evaluated on the solutions to the BAEs—are invariant under shifts of u_j by the periods of the torus. Furthermore, the (3.2.10) are invariant under modular transformations of the torus (see (2.2.7)):

$$T : \begin{cases} \tau \mapsto \tau + 1 \\ u \mapsto u \end{cases} \qquad S : \begin{cases} \tau \mapsto -\dfrac{1}{\tau} \\ u \mapsto \dfrac{u}{\tau} \end{cases} \qquad C : \begin{cases} \tau \mapsto \tau \\ u \mapsto -u \end{cases} . \tag{3.2.18}$$

This has been shown in Section 2.2.2 for the general case and it can be easily checked rewriting the BAEs in terms of the function $\theta(u;\tau)$ (see Appendix A). Doing so, the term $\sum_j u_{ij}$ in the exponential in (3.2.10) disappears and, using the modular properties (A.11), one proves the invariance under the full group $SL(2,\mathbb{Z})$. On the other hand, let us stress once again that the summand $\kappa_N \mathcal{Z} H^{-1}$ in (3.2.9) is not invariant under modular transformations of τ: this is not a symmetry of the superconformal index.

3.2.1 Exact solutions to the BAEs

When evaluating the BA formula (3.2.9), the hardest task is to solve the BAEs (3.2.10). The very same equations appear in the topologically twisted index on $T^2 \times S^2$ [35], and one exact solution was found in [47, 57]:

$$u_{ij} = \frac{\tau}{N}(j-i) \, , \qquad u_j = \frac{\tau(N-j)}{N} + \bar{u} \, , \qquad \lambda = \frac{N-1}{2} \, . \tag{3.2.19}$$

Here $\bar{u}$ is a suitable constant that solves the $SU(N)$ constraint (3.2.12); since all expressions depend solely on u_{ij}, we will not specify that constant. Notice that the solution does not depend on the chemical potentials Δ_α. To prove that it is a solution, we compute

$$\begin{aligned} \prod_{j=1}^N \frac{\theta_0(u_{ji}+\Delta)}{\theta_0(u_{ij}+\Delta)} &= \frac{\prod_{k=0}^{i-1} \theta_0\big(\frac{\tau}{N}k + \Delta\big) \times \prod_{k=i-N}^{-1} \theta_0\big(\frac{\tau}{N}k+\Delta\big)}{\prod_{k=0}^{N-i} \theta_0\big(\frac{\tau}{N}k+\Delta\big) \times \prod_{k=1-i}^{-1} \theta_0\big(\frac{\tau}{N}k+\Delta\big)} = \\ &= \frac{\prod_{k=0}^{N-1} \theta_0\big(\frac{\tau}{N}k+\Delta\big) \times \prod_{k=i-N}^{-1}\big(-q^{k/N}y\big)}{\prod_{k=1}^{N-1} \theta_0\big(\frac{\tau}{N}k+\Delta\big) \times \prod_{k=1-i}^{-1}\big(-q^{k/N}y\big)} = (-1)^{N-1}\, y^{N-2i+1}\, q^{i-\frac{N+1}{2}} \, . \end{aligned} \tag{3.2.20}$$

To go to the second line we used the periodicity relations (A.4). Taking the product over $\Delta = \{\Delta_1, \Delta_2, -\Delta_1 - \Delta_2\}$ we precisely reproduce the inverse of the prefactor of (3.2.10), for every i. Furthermore, notice that the shift $\bar{u} \to \bar{u} + \frac{1}{N}$ generates a new inequivalent solution that solves the $SU(N)$ constraint. Repeating the shift N times, because of the torus periodicities, we go back to the original solution. Therefore, (3.2.19) actually represents N inequivalent solutions.

Because the BAEs are modular invariant, we could transform τ to $\tau' = (a\tau+b)/(c\tau+d)$, then write the solution $u'_{ij} = \tau'(j-i)/N$, and finally go back to $\tau = (d\tau' - b)/(a - c\tau')$. This gives, for any $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$, an $SL(2,\mathbb{Z})$-transformed solution

$$u_{ij} = \frac{a\tau + b}{N}\,(j-i)\;. \tag{3.2.21}$$

However, one should only keep the solutions that are not equivalent—either because of periodicities on the torus or because of Weyl group transformations.

On the other hand, a larger class of inequivalent solutions was found in [57] (we do not know if this is the full set or other solutions exist). For given N, every choice of three non-negative integers $\{m,n,r\}$ that decompose $N = m\cdot n$ and with $0 \leq r < n$ leads to an exact solution

$$u_{\hat{\jmath}\hat{k}} = \frac{\hat{\jmath}}{m} + \frac{\hat{k}}{n}\left(\tau + \frac{r}{m}\right) + \bar{u} \tag{3.2.22}$$

where $\hat{\jmath} = 0,\ldots,m-1$ and $\hat{k} = 0,\ldots,n-1$ are an alternative parameterization of the index $j = 0,\ldots,N-1$. As we show below, the first class is contained into the second class. Once again, (3.2.22) actually represents N inequivalent solutions because of the possibility of shifting $\bar{u}$.

The solutions (3.2.22) organize into orbits of $PSL(2,\mathbb{Z})$ with the following action:

$$T:\{m,n,r\} \mapsto \{m,n,r+m\}\;, \qquad S:\{m,n,r\} \mapsto \left\{\gcd(n,r)\,,\, \frac{m\,n}{\gcd(n,r)}\,,\, \frac{m(n-r)}{\gcd(n,r)}\right\} \tag{3.2.23}$$

where the last entry of $\{m',n',r'\}$ is understood mod n'. One can check that $S^2 = \mathbb{1}$. If $\{m,n,r\}$ have a common divisor, then one can see that also $\{m',n',r'\}$ have that common divisor, and since T, S are invertible, it follows that $d := \gcd(m,n,r)$ is an invariant along $PSL(2,\mathbb{Z})$ orbits.

We can prove that if $\{m,n,r\}$ have $\gcd(m,n,r) = 1$, then they are in the orbit of $\{1,mn,0\}$, *i.e.* there exists a $PSL(2,\mathbb{Z})$ transformation that maps them to $\{1,mn,0\}$. Indeed, let $\tilde{r} = \gcd(m,r)$. We can perform a number of T transformations to reach $\{m,n,\tilde{r}\}$. Necessarily $\gcd(n,\tilde{r}) = 1$, therefore an S transformation gives $\{1,mn,m(n-\tilde{r})\}$. Now a number of T transformations gives $\{1,mn,0\}$. On the other hand, we observe that if $\gcd(m,n,r) = d > 1$, then the orbit under $PSL(2,\mathbb{Z})$ is in one-to-one correspondence with the one of $\{m/d,n/d,r/d\}$, which is generated by $\{1,mn/d^2,0\}$. This shows that the number of orbits is equal to the number of divisors d^2 of N which are also squares. Each orbit is generated by $\{d,N/d,0\}$, and is in one-to-one correspondence with the orbit generated by $\{1,N/d^2,0\}$, which we can regard as the "canonical form".

At this point we recognize that the set of inequivalent solutions in the first class (3.2.21) (neglecting shifts of $\bar{u}$) is precisely the $PSL(2,\mathbb{Z})$ orbit with $\gcd(m,n,r) = 1$ in the second class (3.2.22). Indeed, start with a solution of type (3.2.21) for some N and some coprime integers a, b. Let $m = \gcd(a, N)$ and $n = N/m$. We can write the solution as

$$u_j = -\frac{(a/m)\, j}{n}\tau - \frac{b\, j}{N} + \bar{u} \quad (\text{mod } \mathbb{Z} + \tau\mathbb{Z}) \,. \tag{3.2.24}$$

We can identify $\hat{k} = (a/m)j \mod n$. Since (a/m) and n are coprime, as j runs from 0 to $n-1$, $\hat{k}$ takes all values in the same range once. Moreover there exists $s = (a/m)^{-1} \mod n$, such that $j = s\hat{k} \mod n$. In other words, (a/m) is invertible mod n and its inverse s is coprime with n. We can write

$$j = s\hat{k} + n\hat{\jmath} \tag{3.2.25}$$

and as j runs from 0 to $N-1$, $\hat{\jmath}$ covers a range of length m. Substituting the expression for j we obtain

$$u_j = -\frac{b}{m}\hat{\jmath} - \frac{\hat{k}}{n}\left(\tau + \frac{bs}{m}\right) + \bar{u} \quad (\text{mod } \mathbb{Z} + \tau\mathbb{Z}) \,. \tag{3.2.26}$$

Notice that $\gcd(b, m) = 1$. Indeed, suppose that b and m have a common factor, then this must also be a factor of a, which is a contradiction. Therefore we have the equality of sets $\{b\hat{\jmath} \mod m\} = \{\hat{\jmath} \mod m\}$. Finally, we set $r = bs \mod n$ and we reproduce the expression in (3.2.22). The values $\{m, n, r\}$ obtained this way have $\gcd(m, n, r) = 1$. Indeed, suppose they have a common factor, then this must also be a factor of a but not of (a/m), and thus it must also be a factor of b, which is a contradiction.

On the contrary, start with a solution $\{m, n, r\}$ of type (3.2.22) with $\gcd(m, n, r) = 1$. It is easy to see, by repeating the procedure, that it is equivalent to a solution of type (3.2.21) with $a = m$ and $b = r$ (which imply $s = 1$).

3.3 The large N limit

In this section we take the large N limit of the BA formula (3.2.9) for the superconformal index. The first part of the section is technical, and the uninterested reader could directly jump to Section 3.3.3 where the final result is presented.

In the related context of the topologically twisted index on $T^2 \times S^2$ [35, 102], it was shown in [47] that the basic solution (3.2.19) leads to the dominant contribution in the high temperature limit. Assuming that such a solution gives an important contribution in our setup as well, we will start evaluating its large N limit. We will find that it scales as $e^{\mathcal{O}(N^2)}$, therefore in the following we will systematically neglect any factor whose logarithm is subleading with respect to $\mathcal{O}(N^2)$. We will also find that the solution (3.2.19) is not necessarily dominant in our setup, rather other solutions can compete, and we will thus have to include the contributions of some of the solutions (3.2.21).

First of all, consider the prefactor κ_N in (3.2.9) and the multiplicity of the BA solutions, whose contribution does not depend on the particular solution. Each BA solution (3.2.22) has multiplicity $N \cdot N!$, where the first factor comes from shifts of $\bar{u}$ while the second factor from the Weyl group action. Thus, from (3.2.13), we find

$$N \cdot N! \cdot \kappa_N = e^{\mathcal{O}(N)} . \tag{3.3.1}$$

This contribution can be neglected at leading order.

3.3.1 Contribution of the basic solution

Here we consider only the contribution of the basic solution (3.2.19) to the sum in (3.2.9).

The Jacobian. We use the expression in (3.2.15). The derivative of Q_i with respect to u_j can be computed and it gives:

$$\frac{\partial \log Q_i(u;\Delta,\tau)}{\partial u_j} = \sum_{k=1}^{N} \partial_{u_j} u_{ik} \left(6\pi i + \sum_{\Delta\in\{\Delta_1,\Delta_2,-\Delta_1-\Delta_2\}} \frac{\mathcal{G}'(u_{ik};\Delta,\tau)}{\mathcal{G}(u_{ik};\Delta,\tau)} \right) , \tag{3.3.2}$$

with

$$\mathcal{G}(u;\Delta,\tau) = \frac{\theta_0(-u+\Delta;\tau)}{\theta_0(u+\Delta;\tau)} \tag{3.3.3}$$

and $\partial_{u_j} u_{ik} = \delta_{ij} - \delta_{kj} - \delta_{iN} + \delta_{kN}$. This relation holds because we take $u_1, \ldots, u_{N-1}$ as the independent variables, and fix u_N using (3.2.12). Substituting we get

$$\begin{aligned} \frac{\partial \log Q_i(u;\Delta,\tau)}{\partial u_j} = (\delta_{ij} - \delta_{iN}) \left(6\pi i N + \sum_{k=1}^{N} \sum_{\Delta} \frac{\mathcal{G}'(u_{ik};\Delta,\tau)}{\mathcal{G}(u_{ik};\Delta,\tau)} \right) + \\ + \sum_{\Delta} \left(\frac{\mathcal{G}'(u_{iN};\Delta,\tau)}{\mathcal{G}(u_{iN};\Delta,\tau)} - \frac{\mathcal{G}'(u_{ij};\Delta,\tau)}{\mathcal{G}(u_{ij};\Delta,\tau)} \right) \end{aligned} \tag{3.3.4}$$

where Δ is summed over $\{\Delta_1, \Delta_2, -\Delta_1 - \Delta_2\}$.

When we evaluate this expression on $u_{ij} = \tau(j-i)/N$, we notice that—for generic values of Δ_α—the terms in the second line are of order $\mathcal{O}(1)$. Indeed, the distribution of points u_{ij} generically does not hit any zeros or poles of $\mathcal{G}$. Retaining only the terms in the first line, the Jacobian reads

$$H = \det \begin{pmatrix} A_1 & \mathcal{O}(1) & \cdots & \mathcal{O}(1) & 1 \\ \mathcal{O}(1) & A_2 & & \vdots & 1 \\ \vdots & & \ddots & \vdots & \vdots \\ \mathcal{O}(1) & \cdots & \cdots & A_{N-1} & 1 \\ -A_N & -A_N & \cdots & -A_N & 1 \end{pmatrix} \tag{3.3.5}$$

where the diagonal entries are

$$A_i = 3N + \frac{1}{2\pi i} \sum_{k=1}^{N} \sum_{\Delta} \frac{\mathcal{G}'(u_{ik};\Delta,\tau)}{\mathcal{G}(u_{ik};\Delta,\tau)} \,. \tag{3.3.6}$$

Let us estimate the behavior of A_i with N. By the same argument as above, A_i contains the sum of N elements of order $\mathcal{O}(1)$ and thus it scales like $\mathcal{O}(N)$ (or smaller). The determinant can be computed at leading order and it gives

$$H = \sum_{k=1}^{N} \prod_{j\,(\neq k)=1}^{N} A_j + \text{subleading} \,. \tag{3.3.7}$$

This scales as $\mathcal{O}(N^N)$, therefore $\log H = \mathcal{O}(N \log N)$ and can be neglected.

The functions $\widetilde{\Gamma}$. The dominant contribution comes from the function $\mathcal{Z}$ defined in (3.2.14). To evaluate it, let us analyze $\sum_{i\neq j}^{N} \log \widetilde{\Gamma}(u_{ij}+\Delta;\tau,\tau)$ with $\Delta \in \{\Delta_1, \Delta_2, \Delta_1+\Delta_2\}$ separately. Making use of the relation (A.22) proven in [163], we write

$$\widetilde{\Gamma}(u_{ij}+\Delta;\tau,\tau) = \frac{e^{-\pi i \mathcal{Q}(u_{ij}+\Delta;\tau,\tau)}}{\theta_0\Big(\dfrac{u_{ij}+\Delta}{\tau};-\dfrac{1}{\tau}\Big)} \prod_{k=0}^{\infty} \frac{\psi\Big(\dfrac{k+1+u_{ij}+\Delta}{\tau}\Big)}{\psi\Big(\dfrac{k-u_{ij}-\Delta}{\tau}\Big)} \,. \tag{3.3.8}$$

The function $\psi(t)$ is defined in (A.12), while $\mathcal{Q}$ is a cubic polynomial in u:

$$\begin{aligned} \mathcal{Q}(u;\tau,\sigma) = {} & \frac{u^3}{3\tau\sigma} - \frac{\tau+\sigma-1}{2\tau\sigma}u^2 + \\ & + \frac{(\tau+\sigma)^2+\tau\sigma-3(\tau+\sigma)+1}{6\tau\sigma}u + \frac{(\tau+\sigma-1)(\tau+\sigma-\tau\sigma)}{12\tau\sigma} \,. \end{aligned} \tag{3.3.9}$$

To make progress, we perform a series expansion of $\log\theta_0$ and $\log\psi$, evaluate this expansion on the basic solution for u_{ij} in (3.2.19), and perform the sum $\sum_{i\neq j}^{N}$. We define

$$\tilde{z} = e^{2\pi i u/\tau} \,, \qquad \tilde{y} = e^{2\pi i y/\tau} \,, \qquad \tilde{q} = e^{-2\pi i/\tau} \tag{3.3.10}$$

as the modular transformed variables. We have

$$\begin{aligned} \sum_{i\neq j}^{N} \log\,\theta_0\left(\frac{u_{ij}+\Delta}{\tau};-\frac{1}{\tau}\right) &= \sum_{n=0}^{\infty}\sum_{i\neq j}^{N} \log\left[\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\,\tilde{y}\,\tilde{q}^n\right)\left(1-\frac{\tilde{z}_j}{\tilde{z}_i}\,\tilde{y}^{-1}\,\tilde{q}^{n+1}\right)\right] \\ &= -\sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i\neq j}^{N} \frac{1}{\ell}\left[\left(\frac{\tilde{z}_i}{\tilde{z}_j}\,\tilde{y}\,\tilde{q}^n\right)^{\ell} + \left(\frac{\tilde{z}_j}{\tilde{z}_i}\,\tilde{y}^{-1}\,\tilde{q}^{n+1}\right)^{\ell}\right] \\ &= -\sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} \frac{1}{\ell}\Big[A_\ell\,\tilde{y}^\ell\,\tilde{q}^{n\ell} + A_\ell\,\tilde{y}^{-\ell}\,\tilde{q}^{(n+1)\ell}\Big] = -\sum_{\ell=1}^{\infty}\frac{1}{\ell}\,A_\ell\,\frac{\tilde{y}^\ell+\tilde{y}^{-\ell}\tilde{q}^\ell}{1-\tilde{q}^\ell} \,, \end{aligned} \tag{3.3.11}$$

where we introduced A_ℓ which denotes the following sum over i, j:

$$A_\ell \coloneqq \sum_{i\neq j}^{N} \left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^\ell = \sum_{i\neq j}^{N} e^{2\pi i(j-i)\ell/N} = \begin{cases} N^2 - N & \text{for } \ell = 0 \mod N \\ -N & \text{for } \ell \neq 0 \mod N\ . \end{cases} \tag{3.3.12}$$

The series can be resummed to $N \log\left[\theta_0\left(\frac{N\Delta}{\tau}; -\frac{N}{\tau}\right)/\theta_0\left(\frac{\Delta}{\tau}; -\frac{1}{\tau}\right)\right]$, however we do not need that. We collect the terms into two groups:

$$(3.3.11) = N\sum_{\ell=1}^{\infty}\frac{1}{\ell}\,\frac{\tilde{y}^\ell + \tilde{y}^{-\ell}\,\tilde{q}^\ell}{1-\tilde{q}^\ell} - N\sum_{j=1}^{N}\frac{1}{j}\,\frac{\tilde{y}^{Nj} + \tilde{y}^{-Nj}\,\tilde{q}^{Nj}}{1-\tilde{q}^{Nj}} \tag{3.3.13}$$

where the second term comes from $\ell = Nj$. For $|\tilde{q}| < |\tilde{y}| < 1$, namely for

$$0 < \mathbb{Im}\left(\frac{\Delta}{\tau}\right) < \mathbb{Im}\left(-\frac{1}{\tau}\right)\ , \tag{3.3.14}$$

the series converges. The second term is suppressed at large N, whereas the first term is of order $\mathcal{O}(N)$ and can be neglected.

We then perform a similar analysis of $\log\psi$, using the series expansions of the functions log and Li_2. We find

$$\begin{aligned}\sum_{i\neq j}^{N}\sum_{k=0}^{\infty}\log\frac{\psi\left(\frac{k+1+u_{ij}+\Delta}{\tau}\right)}{\psi\left(\frac{k-u_{ij}-\Delta}{\tau}\right)} &= \sum_{i\neq j}^{N}\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\left[-\frac{1}{\ell}\left(\frac{k+1+\Delta}{\tau}\tilde{y}^{-\ell}\,\tilde{q}^\ell - \frac{k-\Delta}{\tau}\tilde{y}^\ell\right) + \right. \\ &\qquad \left. -\frac{1}{\ell}\frac{u_{ij}}{\tau}\left(\tilde{y}^\ell + \tilde{y}^{-\ell}\,\tilde{q}^\ell\right) + \frac{1}{2\pi i}\frac{1}{\ell^2}\left(\tilde{y}^\ell - \tilde{y}^{-\ell}\,\tilde{q}^\ell\right)\right]\left(\frac{\tilde{z}_i}{\tilde{z}_j}\,\tilde{q}^k\right)^\ell \\ &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\left[-\frac{A_\ell}{\ell}\left(\frac{k+1+\Delta}{\tau}\,\tilde{y}^{-\ell}\,\tilde{q}^\ell - \frac{k-\Delta}{\tau}\,\tilde{y}^\ell\right)\tilde{q}^{k\ell} + \frac{1}{2\pi i}\frac{A_\ell}{\ell^2}\left(\tilde{y}^\ell - \tilde{y}^{-\ell}\,\tilde{q}^\ell\right)\tilde{q}^{k\ell}\right]\end{aligned} \tag{3.3.15}$$

where we used that the following sum vanishes:

$$B_\ell \coloneqq \sum_{i\neq j}^{N}\tilde{u}_{ij}\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^\ell = \frac{1}{N}\sum_{i\neq j}^{N}(j-i)\,e^{2\pi i(j-i)\ell/N} = 0\ . \tag{3.3.16}$$

Once again, the expression can be resummed by breaking the sum into two groups (corresponding to generic ℓ and $\ell = Nj$):

$$(3.3.15) = \sum_{k=0}^{\infty}\left[-N\log\frac{\psi\left(\frac{k+1+\Delta}{\tau}\right)}{\psi\left(\frac{k-\Delta}{\tau}\right)} + \log\frac{\psi\left(\frac{N(k+1-\Delta)}{\tau}\right)}{\psi\left(\frac{N(k-\Delta)}{\tau}\right)}\right]\ . \tag{3.3.17}$$

The first term (that comes from setting $A_\ell \to N$) is of order $\mathcal{O}(N)$ and can be neglected.

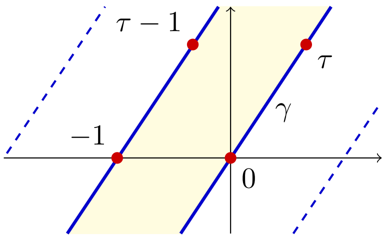

Figure 3.1: In yellow is highlighted the domain (3.3.14) in the complex Δ-plane. The right boundary is the line γ, passing through 0 and τ. The left boundary is the line $\gamma - 1$, passing through -1 and $\tau - 1$. The dashes lines are other elements of $\gamma + \mathbb{Z}$.

The second term is

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left[-\frac{N}{j}\left(\frac{k+1+\Delta}{\tau}(\tilde{q}/\tilde{y})^{Nj}-\frac{k-\Delta}{\tau}\tilde{y}^{Nj}\right)\tilde{q}^{Nkj}+\frac{1}{2\pi i\, j^2}\big(\tilde{y}^{Nj}-(\tilde{q}/\tilde{y})^{Nj}\big)\tilde{q}^{Nkj}\right] . \tag{3.3.18}$$

In the regime of convergence (3.3.14) this series goes to zero as $N \to \infty$. We conclude that the only contribution at leading order in N is from the polynomial $\mathcal{Q}$ in (3.3.9).

The limit we computed is valid as long as Δ satisfies (3.3.14). That inequality has the interpretation that Δ should lie inside an infinite strip, bounded on the left by the line through -1 and $\tau - 1$, and on the right by the line (that we dub γ) through 0 and τ (see Figure 3.1). On the other hand, $\widetilde{\Gamma}(u_{ij} + \Delta; \tau, \tau)$ is a periodic function invariant under shifts $\Delta \to \Delta + 1$. Therefore, unless Δ sits exactly on one image of the line γ under periodic integer shifts, there always exists a shift that brings Δ inside the strip. This means that we can use our computation to extract the limit for all $\Delta \in \mathbb{C} \setminus \{\gamma + \mathbb{Z}\}$.

Let us define the periodic discontinuous function

$$[\Delta]_\tau := \left(\Delta + n \,\middle|\, n \in \mathbb{Z},\ \ 0 < \mathbb{Im}\big(\tfrac{\Delta+n}{\tau}\big) < \mathbb{Im}\big(-\tfrac{1}{\tau}\big)\right) \qquad \text{for } \mathbb{Im}\big(\tfrac{\Delta}{\tau}\big) \notin \mathbb{Z} \times \mathbb{Im}\big(\tfrac{1}{\tau}\big) . \tag{3.3.19}$$

The function is not defined for $\mathbb{Im}(\Delta/\tau) \in \mathbb{Z} \times \mathbb{Im}(1/\tau)$. Essentially, this function is constructed in such a way that $[\Delta]_\tau = \Delta \mod 1$, and $[\Delta]_\tau$ satisfies (3.3.14) when it is defined. It also satisfies

$$[\Delta + 1]_\tau = [\Delta]_\tau , \qquad [\Delta + \tau]_\tau = [\Delta]_\tau + \tau , \qquad [-\Delta]_\tau = -[\Delta]_\tau - 1 . \tag{3.3.20}$$

We use such a function to express the limit as

$$\begin{aligned} \lim_{N\to\infty} \sum_{i\neq j}^{N} \log \widetilde{\Gamma}\big(u_{ij} + \Delta; \tau, \tau\big)\Big|_{(3.2.19)} &= -\pi i \sum_{i\neq j}^{N} \mathcal{Q}\big(u_{ij} + [\Delta]_\tau; \tau, \tau\big) + \mathcal{O}(N) \\ &= -\pi i N^2 \, \frac{\big([\Delta]_\tau - \tau\big)\big([\Delta]_\tau - \tau + \frac{1}{2}\big)\big([\Delta]_\tau - \tau + 1\big)}{3\tau^2} + \mathcal{O}(N) . \end{aligned} \tag{3.3.21}$$

This expression is, by construction, invariant under $\Delta \to \Delta + 1$. The lines

$$\mathbb{Im}(\Delta/\tau) \in \mathbb{Z} \times \mathbb{Im}(1/\tau) \tag{3.3.22}$$

that we have dubbed $\gamma + \mathbb{Z}$, are Stokes lines: they represent transitions between regions in the complex Δ-plane in which different exponential contributions dominate the large N limit, and along which the limit is discontinuous.[31] We do not know what is the limit along the lines, because different contributions compete and a more precise estimate would be necessary to evaluate their sum. We will elaborate on Stokes lines in Section 3.3.3.

The term with $\Delta = 0$ requires a special treatment, because it does not satisfy (3.3.14). We can still use the expansion (3.3.8). The term $\log\theta_0$ is evaluated as

$$\begin{aligned} \sum_{i\neq j}^{N} \log\theta_0\left(\frac{u_{ij}}{\tau}; -\frac{1}{\tau}\right) &= \sum_{i\neq j}^{N}\sum_{k=0}^{\infty} \log\left[\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\,\tilde{q}^k\right)\left(1-\frac{\tilde{z}_j}{\tilde{z}_i}\,\tilde{q}^{k+1}\right)\right] \\ = \sum_{i\neq j}^{N}\log\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\right) &+ 2\sum_{i\neq j}^{N}\sum_{k=1}^{\infty}\log\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\,\tilde{q}^k\right) = N\log N + 2N\log\frac{\left(\tilde{q}^N;\tilde{q}^N\right)_\infty}{(\tilde{q};\tilde{q})_\infty}\ . \end{aligned} \tag{3.3.23}$$

To calculate the first term in the second line, we notice that $x^N - 1 = \prod_{j=1}^{N}\left(x - e^{2\pi i j/N}\right)$. Factoring $(x-1)$ on both sides we get $x^{N-1} + \ldots + x + 1 = \prod_{j=1}^{N-1}\left(x - e^{2\pi i j/N}\right)$, and substituting $x = 1$ we get $N = \prod_{j=1}^{N-1}\left(1 - e^{2\pi i j/N}\right)$. At this point we can shift j by k units and multiply over k:

$$N^N = \prod_{k=1}^{N}\ \prod_{j\,(\neq k)\,=1}^{N}\left(1 - e^{2\pi i (j-k)/N}\right)\ . \tag{3.3.24}$$

To compute the second term we use the series expansion as before. We see that $\log\theta_0$ contributes at order $\mathcal{O}(N\log N)$ and can be neglected. The product of terms $\log\psi$ gives

$$\begin{aligned} \sum_{i\neq j}^{N}\log\prod_{k=0}^{\infty}\frac{\psi\left(\frac{k+1+u_{ij}}{\tau}\right)}{\psi\left(\frac{k-u_{ij}}{\tau}\right)} &= \sum_{i\neq j}^{N}\log\prod_{k=0}^{\infty}\frac{\psi\left(\frac{k+1+u_{ij}}{\tau}\right)}{\psi\left(\frac{k+u_{ij}}{\tau}\right)} = -\sum_{i\neq j}^{N}\log\psi\left(\frac{u_{ij}}{\tau}\right) \\ &= \sum_{i<j}^{N}\pi i\left(\frac{(j-i)^2}{N^2} - \frac{1}{6}\right) = \frac{i\pi}{12}(N-1)\ . \end{aligned} \tag{3.3.25}$$

In the first equality we changed sign to u_{ij} because it is summed over i, j; to go to the second line we used (A.13). This term is of order $\mathcal{O}(N)$ and can be neglected. We conclude that

$$\lim_{N\to\infty}\sum_{i\neq j}^{N}\log\widetilde{\Gamma}(u_{ij};\tau,\tau)\Big|_{(3.2.19)} = \pi i N^2\,\frac{\tau\left(\tau-\frac{1}{2}\right)\left(\tau-1\right)}{3\tau^2} + \mathcal{O}(N\log N)\ . \tag{3.3.26}$$

[31] Stokes lines divide the complex plane into regions in which the limit gives different analytic functions. Because of their origin, Stokes lines have the property that only the imaginary part of the function can jump, while the real part must be continuous. One can indeed check that (3.3.21) satisfies this property.

Total contribution from the basic solution. At this point we can collect the various contributions and obtain the large N limit of $\log \mathcal{Z}$ in (3.2.14) evaluated on the solution (3.2.19). The expression depends on $[\Delta_1]_\tau$, $[\Delta_2]_\tau$ and $[\Delta_1+\Delta_2]_\tau$. We notice the following relation:

$$[\Delta_1+\Delta_2]_\tau = \begin{cases} [\Delta_1]_\tau + [\Delta_2]_\tau & \text{if } 0 < \mathbb{Im}\left(\frac{[\Delta_1]_\tau+[\Delta_2]_\tau}{\tau}\right) < \mathbb{Im}\left(-\frac{1}{\tau}\right) \quad 1^{\text{st}} \text{ case} \\ [\Delta_1]_\tau + [\Delta_2]_\tau + 1 & \text{if } \mathbb{Im}\left(-\frac{1}{\tau}\right) < \mathbb{Im}\left(\frac{[\Delta_1]_\tau+[\Delta_2]_\tau}{\tau}\right) < \mathbb{Im}\left(-\frac{2}{\tau}\right) \quad 2^{\text{nd}} \text{ case.} \end{cases} \tag{3.3.27}$$

The second one can be rewritten as

$$[-\Delta_1-\Delta_2]_\tau = [-\Delta_1]_\tau + [-\Delta_2]_\tau \qquad \text{if } 0 < \mathbb{Im}\left(\frac{[-\Delta_1]_\tau+[-\Delta_2]_\tau}{\tau}\right) < \mathbb{Im}\left(-\frac{1}{\tau}\right) . \tag{3.3.28}$$

The large N limit of the summand is then

$$\lim_{N\to\infty} \log \mathcal{Z}\Big|_{(3.2.19)} = -\pi i N^2\, \Theta(\Delta_1,\Delta_2;\tau) , \tag{3.3.29}$$

where we have introduced the following function for compactness:

$$\Theta(\Delta_1,\Delta_2;\tau) = \begin{cases} \dfrac{[\Delta_1]_\tau[\Delta_2]_\tau\big(2\tau-1-[\Delta_1]_\tau-[\Delta_2]_\tau\big)}{\tau^2} & 1^{\text{st}} \text{ case} \\ \dfrac{\big([\Delta_1]_\tau+1\big)\big([\Delta_2]_\tau+1\big)\big(2\tau-1-[\Delta_1]_\tau-[\Delta_2]_\tau\big)}{\tau^2} - 1 & 2^{\text{nd}} \text{ case.} \end{cases} \tag{3.3.30}$$

The two cases were defined in (3.3.27).

We can rewrite the function Θ in a way that will be useful in Section 3.5. Define an auxiliary chemical potential Δ_3, modulo 1, such that[32]

$$\Delta_1+\Delta_2+\Delta_3-2\tau \in \mathbb{Z} . \tag{3.3.31}$$

It follows that $[\Delta_3]_\tau = 2\tau - [\Delta_1+\Delta_2]_\tau - 1$. It is also useful to define the primed bracket

$$[\Delta]'_\tau = [\Delta]_\tau + 1 \qquad \Rightarrow \qquad \mathbb{Im}\left(\frac{1}{\tau}\right) < \mathbb{Im}\left(\frac{[\Delta]'_\tau}{\tau}\right) < 0 . \tag{3.3.32}$$

The primed bracket selects the image of Δ, under integer shifts, that sits inside the strip on the right of the line γ through zero and τ, as opposed to the strip on the left. Hence

$$\Theta(\Delta_1,\Delta_2;\tau) = \begin{cases} \dfrac{[\Delta_1]_\tau[\Delta_2]_\tau[\Delta_3]_\tau}{\tau^2} & \text{if } \; 0 < \mathbb{Im}\left(\frac{[\Delta_1]_\tau+[\Delta_2]_\tau}{\tau}\right) < \mathbb{Im}\left(-\frac{1}{\tau}\right) \\ \dfrac{[\Delta_1]'_\tau[\Delta_2]'_\tau[\Delta_3]'_\tau}{\tau^2} - 1 & \text{if } \; \mathbb{Im}\left(\frac{1}{\tau}\right) < \mathbb{Im}\left(\frac{[\Delta_1]'_\tau+[\Delta_2]'_\tau}{\tau}\right) < 0 . \end{cases} \tag{3.3.33}$$

[32]This definition represents the specialization of the general change of variables (2.2.13) to the case of $\mathcal{N}=4$ SYM.

Irrespective of the integer appearing in (3.3.31), the bracketed potentials satisfy the following constraints:

$$\begin{aligned} &[\Delta_1]_\tau + [\Delta_2]_\tau + [\Delta_3]_\tau - 2\tau + 1 = 0 \qquad && 1^{\text{st}} \text{ case} \\ &[\Delta_1]'_\tau + [\Delta_2]'_\tau + [\Delta_3]'_\tau - 2\tau - 1 = 0 \qquad && 2^{\text{nd}} \text{ case} . \end{aligned} \tag{3.3.34}$$

Such constraints have already appeared in [117, 119, 120].

3.3.2 Contribution of $SL(2,\mathbb{Z})$-transformed solutions

As discussed in Section 3.2.1, (3.2.19) is not the only solution to the BAEs: each inequivalent $SL(2,\mathbb{Z})$ transformation of it, given in (3.2.21), is another solution—and even more generally there are the $\{m,n,r\}$ solutions (3.2.22) found in [57]. Some of those solutions might contribute at the same leading order in N.

A class of inequivalent solutions—particularly simple to study—that contribute at leading order in N is obtained through T-transformations:

$$u_{ij} = \frac{\tau + r}{N}\,(j-i) \qquad \text{for } r = 0, \dots, N-1 . \tag{3.3.35}$$

These are the solutions $\{1, N, r\}$ in the notation of Section 3.2.1. To evaluate their contribution, simply notice that both $\mathcal{Z}$ in (3.2.14) and H are invariant under $\tau \to \tau + r$, thus the contribution of (3.3.35) is the same as in (3.3.29) but with $\tau \to \tau + r$. In the large N limit, r runs over $\mathbb{Z}$.

We have not evaluated the contribution of all other $\{m,n,r\}$ solutions, which is a difficult task. However, in order to have an idea of what their contribution could be, let us estimate the contribution from the S-transformed solution

$$u_{ij} = \frac{j-i}{N} , \tag{3.3.36}$$

which is the $\{N, 1, 0\}$ solution in (3.2.22). The large N limit of κ_N does not depend on the solution, and is subleading. Moreover, the large N limit of $\log H$ is computed in the same way as in Section 3.3.1, and it gives $\mathcal{O}(N \log N)$ or smaller. Let us then analyze $\mathcal{Z}$. In the regime $|q|^2 < |y| < 1$ we can directly expand $\log \widetilde{\Gamma}$ in its plethystic form:

$$\begin{aligned} \sum_{i\neq j}^N \log \widetilde{\Gamma}(u_{ij} + \Delta; \tau, \tau) &= \sum_{i\neq j}^N \sum_{\ell=1}^\infty \sum_{m=0}^\infty \frac{m+1}{\ell} \left(\left(\frac{z_i}{z_j}\right)^\ell y^\ell - \left(\frac{z_j}{z_i}\right)^\ell y^{-\ell} q^{2\ell} \right) q^{m\ell} \\ &= \sum_{\ell=1}^\infty \sum_{m=0}^\infty \frac{m+1}{\ell} A_\ell \left(y^\ell - y^{-\ell} q^{2\ell}\right) q^{m\ell} = N \log \frac{\widetilde{\Gamma}(N\Delta; N\tau, N\tau)}{\widetilde{\Gamma}(\Delta; \tau, \tau)} = \mathcal{O}(N) . \end{aligned} \tag{3.3.37}$$

If $|y|$ is outside the range of convergence of the plethystic expansion, either above or below, the large N estimate of $\widetilde{\Gamma}(N\Delta; N\tau, N\tau)^N$ becomes discontinuous and, in general, it does not produce only exponentially suppressed contributions. To see this we can simply shift $\Delta \to \Delta + k\tau$, for some $k \in \mathbb{Z}$ such that $|y\, q^k|$ is inside the convergence domain. Then, repeatedly using (A.17) and (A.4) gives an additional contribution of the form

$$- \pi i N^2 \Big(k(k-1)\Delta + \frac{2}{3} k(k^2-1)\tau \Big) + Nk \log \theta_0(N(\Delta + k\tau); N\tau) \,. \tag{3.3.38}$$

Here, the last term can be treated similarly to (3.3.37) and it is exponentially suppressed in N. On the other hand, the first terms give a contribution of order $\mathcal{O}(N^2)$. Nevertheless, being these contributions linear in the chemical potentials, they do not play a role in the analysis we perform in Sections 3.4 and 3.5 (they do not modify the extremization problem). Moreover, their appearance may be justified as a consequence of the analytic continuation of the index outside the regime of convergence of its plethystic expansion, where the physical interpretation as a trace is no longer well-defined. For these reasons, we will neglect these problematic terms and work with the estimate (3.3.37) for the regime $|q|^2 < |y| < 1$. The case $\Delta = 0$ requires a special treatment. We have

$$\begin{aligned} \sum_{i \neq j}^{N} \log \widetilde{\Gamma}(u_{ij}; \tau, \tau) &= \sum_{i \neq j}^{N} \left[- \log \left(1 - \frac{z_i}{z_j} \right) + 2 \sum_{\ell, m=1}^{\infty} \frac{1}{\ell} \left(\frac{z_i}{z_j} \right)^{\ell} q^{m\ell} \right] \\ &= -N \log N + 2 \sum_{\ell, m=1}^{\infty} \frac{A_\ell}{\ell}\, q^{m\ell} = -N \log N + 2N \log \frac{(q;q)_\infty}{(q^N; q^N)_\infty} = \mathcal{O}(N \log N) \,. \end{aligned} \tag{3.3.39}$$

Thus, there is no contribution from $\log \mathcal{Z}$ at leading order in N.

In the following we will assume that the only solutions contributing at leading order, namely $\mathcal{O}(N^2)$, are the T-transformed solutions.

3.3.3 Final result and Stokes lines

Since we end up with competing exponentials, the one with the largest real part dominates the large N limit. Assuming that solutions other than the T-transformed of the basic one are of subleading order in N, we find the final formula

$$\lim_{N \to \infty} \log \mathcal{I}(q, y_1, y_2) = \widetilde{\max_{r \in \mathbb{Z}}} \Big(-\pi i N^2\, \Theta(\Delta_1, \Delta_2; \tau + r) \Big) \;\coloneqq\; \log \mathcal{I}_\infty \,. \tag{3.3.40}$$

The function Θ is defined in (3.3.30). The meaning of $\widetilde{\max}$ is that we should choose the value of $r \in \mathbb{Z}$ such that the real part of the argument is maximized. One of the good features of eqn. (3.3.40) is that it is periodic under integer shifts of τ, Δ_1, Δ_2. We already observed that Θ is periodic in $\Delta_{1,2}$ because the functions $[\Delta_{1,2}]_\tau$ are. Taking the $\widetilde{\max}$ over $\tau \to \tau + r$ gives periodicity in τ as well. This implies that the RHS of (3.3.40) is actually a single-valued function of the fugacities q, y_1, y_2. This is a property of the index at finite

N, as manifest in (3.2.6) and (3.2.9), and it is reassuring that the large N expression we found respects the same property.

The function $\mathcal{I}_\infty$ has a complicated structure. The full range of allowed fugacities q, y_1, y_2 gets divided into multiple domains of analyticity, separated by "Stokes lines". In each domain of analyticity, only one exponential contribution (for some value of r) dominates the large N limit: the function $\log \mathcal{I}_\infty$ takes the form of a simple rational function given by $\Theta(\Delta_1, \Delta_2; \tau + r)$. The Stokes lines are real codimension-one surfaces, in the space of fugacities, that separate the different domains. When crossing a Stokes line, a different exponential contribution dominates, and $\log \mathcal{I}_\infty$ takes the form of a different rational function. In particular, on top of a Stokes line there are two (or more) exponential contributions that compete: their exponents have equal real part. This characterizes the locations of Stokes lines. In terms of the function Θ:

$$\mathbb{Im}\, \Theta(\Delta_1, \Delta_2; \tau + r_1) = \mathbb{Im}\, \Theta(\Delta_1, \Delta_2; \tau + r_2) \tag{3.3.41}$$

for some $r_{1,2} \in \mathbb{Z}$.

In fact, also the values of Δ_1, Δ_2 such that $\Theta(\Delta_1, \Delta_2; \tau + r)$ is discontinuous (for the value of r picked up by $\widetilde{\max}$) should be regarded as forming a Stokes line. In this case, the two competing exponents correspond to the values of Θ on the two sides of the discontinuity. There are two possible sources of discontinuity. First, one of the bracket functions, say $[\Delta_1]_\tau$, could be discontinuous. This happens when $\mathbb{Im}(\Delta_1/\tau) \in \mathbb{Z} \times \mathbb{Im}(1/\tau)$, namely when $\alpha := \lim_{\epsilon \to 0^+} [\Delta_1 - \epsilon]_\tau / \tau \in \mathbb{R}$. Taking into account that on the left of the discontinuity we are in the 1$^{\text{st}}$ case, while on the right we are in the 2$^{\text{nd}}$ case—in the terminology of (3.3.27)—and assuming that Δ_2 is generic, we find

$$\lim_{\epsilon \to 0^+} \Big[\Theta(\Delta_1 - \epsilon, \Delta_2; \tau) - \Theta(\Delta_1 + \epsilon, \Delta_2; \tau) \Big] = (\alpha - 1)^2 \ \in \ \mathbb{R} \,, \tag{3.3.42}$$

where the limit is taken with ϵ real positive. Second, we could pass from the 1$^{\text{st}}$ to the 2$^{\text{nd}}$ case of the definition (3.3.30). This happens when $[\Delta_1]_\tau + [\Delta_2]_\tau + 1 = \alpha\, \tau$ for some $\alpha \in \mathbb{R}$. Assuming that $\Delta_{1,2}$ are otherwise generic, we find

$$\Delta\Theta = (\alpha - 1)^2 \ \in \ \mathbb{R} \,. \tag{3.3.43}$$

In both cases we confirm that the codimension-one surface of discontinuity is a Stokes line, because $\mathbb{Im}\, \Theta$ is equal on the two sides.

When we sit exactly on a Stokes line, two (or more) exponential contributions compete, and in order to compute the large N limit we should sum them. However we do not know the relative phases, because they are affected by all subleading terms and a more accurate analysis would be required. Therefore, we cannot determine the large N limit of the index along Stokes lines.

It turns out that a value of r that maximizes the real part of the argument of $\widetilde{\max}$ may or may not exist. We can estimate the behavior of the real part at large r by noticing that

$$\lim_{r\to\pm\infty} \frac{[\Delta]_{\tau+r}}{\tau+r} = \frac{\mathbb{Im}\,\Delta}{\mathbb{Im}\,\tau} \,. \tag{3.3.44}$$

This implies that

$$\lim_{r\to\pm\infty} \mathbb{Im}\,\Theta(\Delta_1,\Delta_2;\tau+r) = \frac{\mathbb{Im}\,\Delta_1\ \mathbb{Im}\,\Delta_2\ \mathbb{Im}(2\tau-\Delta_1-\Delta_2)}{(\mathbb{Im}\,\tau)^2} \,. \tag{3.3.45}$$

Thus, the real part of the argument of $\widetilde{\max}$ approaches a constant value. If there is no maximum but rather the constant value is a supremum, then our computation is not finished: All contributions from the T-transformed solutions should be summed, however for large $|r|$ they form an infinite number of competing exponentials, whose sum crucially depends on how they interfere. In order to determine such a sum we would need more accurate information.

We conclude by stressing that—even though only the dominant exponential determines the large N limit of the index—we expect that all exponential contributions, including the subdominant ones, have some physical meaning. Each of them plays the role of a "saddle point", although our treatment is not the standard saddle-point approximation. We will make this comment more concrete in Section 3.5, when comparing the large N limit of the index with BPS black hole solutions in supergravity.

3.3.4 Comparison with previous literature

The large N limit of the superconformal index of $\mathcal{N}=4$ SYM was already computed in [88]. There, it was found that the large N limit does not depend on N, and therefore it does not show a rapid enough growth of the number of states to reproduce the black hole entropy. In this section we would like to explain how the results here and there can be compatible.

The authors of [88] took the large N limit of the index, for *real* fugacities. Their result, in our notation and restricted to the case $p=q$, is

$$\lim_{N\to\infty} \mathcal{I}(q,y_1,y_2) = \prod_{n=1}^{\infty} \frac{1}{1-f\big(q^n,y_1^n,y_2^n\big)} \tag{3.3.46}$$

with

$$1-f(q,y_1,y_2) = \frac{(1-y_1)(1-y_2)(1-q^2/y_1y_2)}{(1-q)^2} \,. \tag{3.3.47}$$

In particular, $\log\mathcal{I}$ is of order $\mathcal{O}(1)$. On the contrary, we computed the large N limit for generic *complex* fugacities, and found that $\log\mathcal{I}$ is of order $\mathcal{O}(N^2)$. It was already discussed in [120], in a double-scaling Cardy-like limit, that the large N limit of the index is completely different for real and complex fugacities, and it was observed in [121] that there exists a deconfinement transition once complex fugacities are taken into account.

The resolution we propose relies on the fact that, for complex fugacities, the limit shows Stokes lines. As we described, along those codimension-one surfaces multiple exponentials compete. In order to know what the limit is there, we would need to sum those competing exponentials, but this requires a more accurate knowledge of the subleading terms.

What we notice, though, is that the codimension-three subspace of real fugacities is precisely within a Stokes line. Therefore, although we cannot prove it, it is conceivable that the competing terms cancel exactly, leaving the $\mathcal{O}(1)$ result (3.3.46). Indeed, in Appendix E we prove the following result, which is stronger than the statement that we sit on a Stokes line. Take the angular fugacity q to be real positive, namely $0 < q < 1$ and set $\tau \in i\mathbb{R}_{\geq 0}$ for concreteness, and take the flavor fugacities $y_{1,2}$ to be real. Then $\Theta(\Delta_1, \Delta_2; \tau)$ is along a Stokes line and is not defined, while

$$\Theta(\Delta_1, \Delta_2; \tau - r) = -\,\overline{\Theta(\Delta_1, \Delta_2; \tau + r)} - 1 \tag{3.3.48}$$

for $r > 0$. On the other hand, take the angular fugacity real negative, namely $-1 < q < 0$ and set $\tau \in -\frac{1}{2} + i\mathbb{R}_{\geq 0}$, and take again the flavor fugacities to be real. Then

$$\Theta(\Delta_1, \Delta_2; \tau - r) = -\,\overline{\Theta(\Delta_1, \Delta_2; \tau + r + 1)} - 1 \tag{3.3.49}$$

for $r \geq 0$. Therefore, among the various contributions from T-transformed solutions parameterized by $r \in \mathbb{Z}$, there is an exact pairing of all well-defined terms where, in each pair, two terms have the same real part and can conceivably cancel. In other words, not only the term with maximal real part can cancel, but also all other terms we computed at order $\mathcal{O}(N^2)$. This scenario is a strong check of our result, that makes it compatible with [88].

3.4 Statistical interpretation and $\mathcal{I}$-extremization

We wish to extract the number of BPS states, for given electric charges and angular momenta, from the large N limit of the exact expression (3.2.9) of the superconformal index. Since the latter counts states weighted by the fermion number $(-1)^F$, one may worry that strong cancelations take place and that the total number of states is not accessible. However, following the argument of [29,30]—reviewed in Section 1.2—one can assert that the index (3.2.2) or (3.2.6) is equal to

$$\mathcal{I}(p, q, y_1, y_2) = \operatorname{Tr}\, e^{i\pi R_{\text{trial}}(\tau,\sigma,\Delta_1,\Delta_2)}\; e^{-2\pi\, \mathbb{Im}\left[\tau C_1 + \sigma C_2 + \Delta_1 C_3 + \Delta_2 C_4\right]}\; e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\,, \tag{3.4.1}$$

where the trace is taken in the IR $\mathcal{N}=2$ super quantum mechanics (QM) obtained by reducing the 4d theory on S^3. Here R_{trial} is a trial R-symmetry, and $C_{1,2,3,4}$ are the charges appearing in (3.2.6):

$$C_1 = J_1 + \frac{R_3}{2}\,, \qquad C_2 = J_2 + \frac{R_3}{2}\,, \qquad C_3 = \mathfrak{q}_1\,, \qquad C_4 = \mathfrak{q}_2\,. \tag{3.4.2}$$

Indeed, because of the relations (3.2.7), we can represent the fermion number as $(-1)^F = e^{i\pi R_3}$. Substituting in (3.2.6) and separating the chemical potentials into real and imaginary part, we obtain the expression (3.4.1) with

$$R_{\text{trial}}(\tau,\sigma,\Delta_1,\Delta_2) = R_3 + 2\,\mathbb{R}\mathrm{e}\,\tau\, C_1 + 2\,\mathbb{R}\mathrm{e}\,\sigma\, C_2 + 2\,\mathbb{R}\mathrm{e}\,\Delta_1\, C_3 + 2\,\mathbb{R}\mathrm{e}\,\Delta_2\, C_4\,. \tag{3.4.3}$$

From the point of view of the super QM, R_3 is an R-symmetry while the other four operators are flavor charges, hence R_{trial} is an R-symmetry. We see in (3.4.1) that only the first exponential can produce possibly-dangerous phases, while the other two are real positive.

Now, for a single-center black hole in the microcanonical ensemble, the near-horizon AdS_2 region is dual to an $\mathcal{N}=2$ superconformal QM. The black hole states are vacua of the $\mathfrak{su}(1,1|1)$ 1d SCA. Since we are in the microcanonical ensemble, each of those states is invariant under the global conformal algebra $\mathfrak{su}(1,1)\cong\mathfrak{so}(2,1)$ (because AdS_2 is) as well as under the fermionic generators (because the black hole is supersymmetric). This necessarily implies that those states are invariant under the superconformal R-symmetry $\mathfrak{u}(1)_{\text{sc}}\subset\mathfrak{su}(1,1|1)$, *i.e.* that they have vanishing IR superconformal R-charge R_{sc}. Thus, when R_{trial} is tuned to R_{sc}, the index counts the black hole states with no extra signs or phases (this is similar to [41, 42]). Of course, in a given charge sector there will be more BPS states than just the single-center black hole, but assuming that the single-center black hole dominates, the index captures its entropy. It remains to understand how to identify R_{sc}. At large N, the entropy is extracted from the index with a Legendre transform, and this operation can be argued to effectively select R_{sc} among the R_{trial}'s. This large N principle is the $\mathcal{I}$-extremization principle of [29, 30], originally proposed for dyonic black holes in AdS_4 and now extended to magnetically neutral black holes in AdS_5.

Let us elaborate on this point. The index is the grand canonical partition function of BPS states. Introducing an auxiliary variable Δ_3 and the corresponding fugacity $y_3 = e^{2\pi i\Delta_3}$ such that $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma+1\in 2\mathbb{Z}$, we can rewrite (3.4.1) as

$$\mathcal{I}(p,q,y_1,y_2) = \mathrm{Tr}_{\,\text{BPS}}\, p^{J_1}\, q^{J_2}\, y_1^{Q_1}\, y_2^{Q_2}\, y_3^{Q_3}\,. \tag{3.4.4}$$

Here the trace is over states with $\{\mathcal{Q},\mathcal{Q}^\dagger\}=0$, and we have identified $Q_I = R_I/2$ (for $I=1,2,3$) with the electric charges in supergravity. We recognise that the black hole angular momenta $J_{1,2}$ are associated with the chemical potentials τ,σ and the charges $Q_{1,2,3}$ with

$\Delta_{1,2,3}$. The microcanonical degeneracies at fixed quantum numbers are extracted by computing the Fourier transform of (3.4.4). However, since Δ_3 is not an independent variable, what we obtain are the degeneracies for fixed values of the four charge operators appearing in (3.2.6), summed over Q_3. Using the supergravity notation, those four fixed charge operators are

$$C_1 = J_1 + Q_3 \,, \qquad C_2 = J_2 + Q_3 \,, \qquad C_3 = Q_1 - Q_3 \,, \qquad C_4 = Q_2 - Q_3 \,. \tag{3.4.5}$$

Thus, what we can compute is

$$\sum_{Q_3} d(J,Q)\Big|_{C_{1,2,3,4}} = \int d\tau\, d\sigma\, d\Delta_1\, d\Delta_2\;\; \mathcal{I}(p,q,y_1,y_2)\, p^{-J_1} q^{-J_2} \prod_{I=1}^{3} y_I^{-Q_I} \tag{3.4.6}$$

where $d(J,Q)$ are the (weighted) degeneracies with all charges $J_{1,2}$ and $Q_{1,2,3}$ fixed.

Nevertheless, we can take advantage of the fact, reviewed in Section 3.1, that the charges of BPS back holes are constrained, and for fixed $C_{1,2,3,4}$ there is at most one black hole—for a certain value of the fifth charge Q_3. We can then use (3.4.6) to extract its degeneracy $d(J,Q) = \exp S_{\text{BH}}(J,Q)$ at leading order because the latter will dominate the sum over Q_3.

In the large N limit, the integral (3.4.6) reduces by saddle point approximation to a Legendre transform with respect to the independent variables $\{\tau, \sigma, \Delta_1, \Delta_2\}$:

$$\begin{aligned} S_{\text{BH}}(J,Q) &= \log \mathcal{I}\big(\widehat{\tau}, \widehat{\sigma}, \widehat{\Delta}_1, \widehat{\Delta}_2\big) - 2\pi i\Big(\widehat{\tau} J_1 + \widehat{\sigma} J_2 + \sum_{I=1}^{3} \widehat{\Delta}_I Q_I\Big) \\ &= \log \mathcal{I}\big(\widehat{\tau}, \widehat{\sigma}, \widehat{\Delta}_1, \widehat{\Delta}_2\big) - 2\pi i\Big(\widehat{\tau} C_1 + \widehat{\sigma} C_2 + \widehat{\Delta}_1 C_3 + \widehat{\Delta}_2 C_4\Big) + 2\pi i Q_3 \,, \end{aligned} \tag{3.4.7}$$

where hatted variables denote the critical point. In this approach, Q_3 can be determined as the unique value that makes the entropy $S_{\text{BH}}(J,Q)$ real [29, 30].

In the particular case of 4d $\mathcal{N} = 4$ SYM, the large N limit of the index is a function with multiple domains of analyticity, separated by Stokes lines. This makes things more interesting. In each domain we should perform the Legendre transform, and whenever the critical point falls inside the domain itself, we obtain a self-consistent contribution to the total entropy. Even more generally, we have written the index as a sum of competing exponentials (one for each Bethe Ansatz solution) and we can compute the Legendre transform of each of those exponentials—irrespective of which one dominates. We expect each contribution to represent the entropy of some classical solution—very similarly to a standard saddle point—even when the entropy is smaller than that of the dominant solution.

3.5 Black hole entropy from the index

In this section we show that the contribution of the basic solution (3.2.19) to the superconformal index at large N, in the domain of analyticity that we called "1^{st} case" in (3.3.27), given by

$$-\pi i N^2\,\Theta(\Delta_1,\Delta_2;\tau)\Big|_{1^{\text{st}}\text{ case}} = -\pi i N^2\,\frac{[\Delta_1]_\tau[\Delta_2]_\tau\big(2\tau-1-[\Delta_1]_\tau-[\Delta_2]_\tau\big)}{\tau^2}\;, \tag{3.5.1}$$

precisely reproduces the Bekenstein-Hawking entropy (3.1.13) of single-center black holes in AdS_5 (this is in line with the result of [120] in a double-scaling Cardy-like limit). It amounts to show that the Legendre transform of (3.5.1) is the black hole entropy (this will be reviewed below), and that the critical point involved in the Legendre transform consistently lies within the domain of analyticity in which (3.5.1) holds.

Recall that the contribution of the basic solution corresponds to the $r = 0$ sector in (3.3.40). For black holes with large charges, *i.e.* for black holes that are large compared with the AdS_5 scale, that is indeed the dominant contribution to the index. However, intriguingly enough, as we reduce the charges the contribution of the single-center black hole may cease to dominate. We will highlight this phenomenon in Section 3.5.1 in the very special case of black holes with equal charges. This seems to suggest that, below a certain threshold, the BPS black holes may develop instabilities, possibly towards hairy or multi-center black holes. Indications that this is the case have also been given in [120, 121]. It would be nice if there was a connection between this observation and recently constructed hairy black holes in AdS_5 [122–126].

The entropy function. The Legendre transform of the black hole entropy (3.1.10) in the general case, also called entropy function, was obtained in [117]. Let us review it, following the detailed discussion in Appendix B of [119]. The entropy function is

$$\mathcal{S} = -2\pi i\nu\,\frac{X_1\,X_2\,X_3}{\omega_1\,\omega_2} \qquad \text{with} \qquad \nu = \frac{N^2}{2} = \frac{\pi}{4G_N g^3} \tag{3.5.2}$$

and with the constraint

$$\sum_{a=1,2,3} X_a - \sum_{i=1,2}\omega_i + 1 = 0\;. \tag{3.5.3}$$

Because of the constraint, $\mathcal{S}$ is really a function of four variables. The entropy S_{BH} is the Legendre transform of $\mathcal{S}$ with its constraint. We can compute it as the critical point of

$$\widehat{\mathcal{S}} = \mathcal{S} - 2\pi i\Big(\sum\nolimits_a Q_a\,X_a + \sum\nolimits_i J_i\,\omega_i\Big) - 2\pi i\Lambda\Big(\sum\nolimits_a X_a - \sum\nolimits_i \omega_i + 1\Big) \tag{3.5.4}$$

in which the constraint is imposed with a Lagrange multiplier Λ. The equations for the critical point are

$$Q_a + \Lambda = \frac{1}{2\pi i}\,\frac{\partial \mathcal{S}}{\partial X_a}\;, \qquad\qquad J_i - \Lambda = \frac{1}{2\pi i}\,\frac{\partial \mathcal{S}}{\partial \omega_i}\;, \tag{3.5.5}$$

and the constraint (3.5.3). In details,

$$\begin{aligned} Q_1 + \Lambda &= -\nu\,\frac{X_2\,X_3}{\omega_1\,\omega_2}\;, \qquad & Q_2 + \Lambda &= -\nu\,\frac{X_1\,X_3}{\omega_1\,\omega_2}\;, \qquad & Q_3 + \Lambda &= -\nu\,\frac{X_1\,X_2}{\omega_1\,\omega_2} \\ J_1 - \Lambda &= \nu\,\frac{X_1\,X_2\,X_3}{\omega_1^2\,\omega_2}\;, \qquad & J_2 - \Lambda &= \nu\,\frac{X_1\,X_2\,X_3}{\omega_1\,\omega_2^2}\;. \end{aligned} \tag{3.5.6}$$

It follows that

$$0 = (Q_1+\Lambda)(Q_2+\Lambda)(Q_3+\Lambda) + \nu(J_1-\Lambda)(J_2-\Lambda) = \Lambda^3 + p_2\Lambda^2 + p_1\Lambda + p_0 \tag{3.5.7}$$

with

$$\begin{aligned} p_2 &= Q_1 + Q_2 + Q_3 + \nu \\ p_1 &= Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \nu(J_1+J_2) \\ p_0 &= Q_1Q_2Q_3 + \nu J_1 J_2\;. \end{aligned} \tag{3.5.8}$$

It turns out that we can find the value of $\widehat{\mathcal{S}}$ at the critical point without knowing the exact solution for the critical point. We use the fact that $\mathcal{S}$ is homogeneous of degree 1 (it is a monomial), and thus

$$\sum\nolimits_a X_a\,\frac{\partial \mathcal{S}}{\partial X_a} + \sum\nolimits_i \omega_i\,\frac{\partial \mathcal{S}}{\partial \omega_i} = \mathcal{S}\;. \tag{3.5.9}$$

Substituting into (3.5.4) we find

$$S_{\text{BH}} = \widehat{\mathcal{S}}\Big|_{\text{crit}} = -2\pi i\Lambda\;. \tag{3.5.10}$$

Since Λ is the solution to the cubic equation (3.5.7), it looks like there are three possible values for the entropy. However, since for real charges the cubic equation has real coefficients, we either find 3 real roots or 1 real and 2 complex conjugate roots for Λ. Imposing the entropy to be *real positive*, we require that there is 1 real and 2 imaginary conjugate roots, then only one of them—the one along the positive imaginary axis—leads to an acceptable value for the entropy. Since $(\Lambda-\beta)(\Lambda-i\alpha)(\Lambda+i\alpha) = \Lambda^3 - \beta\Lambda^2 + \alpha^2\Lambda - \beta\alpha^2$, we obtain the following constraint on the charges:

$$p_0 = p_1 p_2 \qquad \text{and} \qquad p_1 > 0\;. \tag{3.5.11}$$

One can check that the parameterization (3.1.6) automatically solves the first equation. Then the roots of (3.5.7) are $\Lambda \in \{-p_2, \pm i\sqrt{p_1}\}$. The physical solution is

$$\Lambda = i\sqrt{p_1} \qquad \Rightarrow \qquad S_{\text{BH}} = 2\pi\sqrt{p_1} \;, \tag{3.5.12}$$

which is precisely eqn. (3.1.10). We stress that the conditions (3.5.11) are necessary, but not sufficient, to guarantee that the supergravity solution is well-defined.[33]

It is not difficult to write the values of the chemical potentials at the critical point. To simplify the notation, let us define

$$P_{1,2,3} = Q_{1,2,3} + \Lambda \;, \qquad P_{4,5} = J_{1,2} - \Lambda \;, \qquad \Phi_{1,2,3} = X_{1,2,3} \;, \qquad \Phi_{4,5} = -\omega_{1,2} \tag{3.5.13}$$

and use an index $A = 1, \ldots, 5$. The equations (3.5.6) imply that

$$\Phi_A P_A \quad \text{are all equal for } A = 1, \ldots, 5 \;. \tag{3.5.14}$$

Implementing the constraint (3.5.3), the solution is

$$\Phi_A = -\frac{1}{P_A}\left(\sum_{B=1}^{5} \frac{1}{P_B}\right)^{-1} \;. \tag{3.5.15}$$

Since, even for real charges, the P_A's are complex, the solutions Φ_A are in general complex.

Equal angular momenta. Let us specialize the formulas to the case $J_1 = J_2 \coloneqq J$, and determine useful inequalities satisfied by the chemical potentials at the critical point. First of all, from the constraint (3.5.3) it immediately follows

$$-\frac{1}{\omega} = \frac{X_1}{\omega} + \frac{X_2}{\omega} + \frac{X_3}{\omega} - 2 \;. \tag{3.5.16}$$

At the critical point (3.5.15) one finds

$$\frac{X_a}{\omega} = -\frac{J - \Lambda}{Q_a + \Lambda} \;, \qquad \mathbb{Im}\left(\frac{X_a}{\omega}\right) = \sqrt{p_1}\,\frac{Q_a + J}{Q_a^2 + p_1} > 0 \;. \tag{3.5.17}$$

[33] As an example, take $Q_1 = Q_2 = Q_3 \coloneqq Q$ and $J_1 = J_2 \coloneqq J$. The first equation in (3.5.11) is solved by $J = -3Q - 1 \pm (2Q+1)^{3/2}$, and both branches are covered by the parameterization (3.1.6) as $Q = \mu + \mu^2/2$, $J = 3\mu^2/2 + \mu^3$. Then $p_1 = 3Q^2 + 6Q + 2 \mp 2(2Q+1)^{3/2}$ and one can check that, for $Q > 0$, both branches have $p_1 > 0$. However, only the branch with upper sign satisfies also (3.1.3)—here $\mu > 0$—and corresponds to well-defined supergravity solutions, while the branch with lower sign does not.

To obtain the last inequality we used that $Q_a + J > 0$ for the BPS black holes, as we showed in (3.1.9). This implies that

$$0 < \mathbb{I}\mathrm{m}\left(\frac{X_a}{\omega}\right) < \mathbb{I}\mathrm{m}\left(-\frac{1}{\omega}\right) \qquad \text{for } a = 1,2,3 \; . \tag{3.5.18}$$

Using the explicit parameterization (3.1.12) presented in Section 3.1 (and setting $g = 1$ for the sake of clarity), one can also show that

$$\mathbb{R}\mathrm{e}(\omega) = \frac{1}{2\,(1+\gamma_1)} \;, \qquad \mathbb{I}\mathrm{m}(\omega) = \frac{\nu\,\gamma_2}{4\,(1+\gamma_1)\,\sqrt{p_1}} \;, \tag{3.5.19}$$

where $p_1 = \nu^2\left((1+\gamma_1)\gamma_3 - \frac{1}{4}\gamma_2^2\right)$. In particular, the first equation shows that

$$0 < \mathbb{R}\mathrm{e}(\omega) < \frac{1}{2} \; . \tag{3.5.20}$$

Entropy from the index. Finally, we compare the contribution to the index from the basic solution in the 1st case, given in (3.5.1), with the entropy function $\mathcal{S}$ in (3.5.2). The latter, after eliminating X_3 with the constraint (3.5.3) and restricting to equal angular fugacities, reads

$$\mathcal{S} = -\pi i N^2\,\frac{X_1\,X_2\,\big(2\omega - 1 - X_1 - X_2\big)}{\omega^2} \; . \tag{3.5.21}$$

We see that it is exactly equal to (3.5.1), as long as we can identify

$$\tau = \omega \;, \qquad [\Delta_a]_\tau = X_a \qquad \text{for} \quad a = 1,2,3 \; . \tag{3.5.22}$$

This is not obvious, but we can check that it is indeed possible. First of all, X_1 and X_2 should satisfy the strip inequalities that $[\,\cdot\,]_\tau$ does, at least in a neighbourhood of the critical point. This is precisely what we proved in (3.5.18). Second, the fugacities at the critical point should also satisfy the inequalities (3.3.27) that define the 1st case. Because of the constraint, this is the same as requiring that also X_3 satisfies (3.5.18), which is true. Thus, this concludes our proof. Let us stress that, in our approach, the constraint (3.5.3) with the correct constant term simply comes out of the large N limit.

One could wonder what is the physics described by the domain of analyticity named 2nd case in (3.3.30). It appears that it reproduces the very same black hole entropy as the 1st case. Indeed, as apparent from (3.3.33), in the two cases Θ takes almost the same form, the only difference being that $[\,\cdot\,]_\tau$ and $[\,\cdot\,]'_\tau$ satisfy opposite strip inequalities and a constraint with opposite constant term. It was already observed in [119] that the entropy function $\mathcal{S}$ reproduces the black hole entropy with either one of the two constraints imposed. We leave for future work to understand what is the role of such a twin contribution.

3.5.1 Example: equal charges and angular momenta

In order to make some of the previous statements more concrete, we now study in detail a very special case in which the index counts states with equal charges $Q \coloneqq Q_1 = Q_2 = Q_3$ and angular momenta $J \coloneqq J_1 = J_2$. This will be instructive to elucidate the structure of Stokes lines.

Let us first quickly summarize the properties of black holes and their entropy in this case [107]. We set $\nu = 1$ (all charges are in "units" of ν) so that

$$p_0 = Q^3 + J^2 \,, \qquad p_1 = 3Q^2 - 2J \,, \qquad p_2 = 3Q + 1 \,, \tag{3.5.23}$$

and the charge constraint is

$$p_1 p_2 - p_0 = 8Q^3 + 3Q^2 - 2(3Q+1)J - J^2 = 0 \,. \tag{3.5.24}$$

This is quadratic in J and potentially leads to two branches of solutions. However only one of them satisfies (3.1.3) when parameterized in terms of μ (we also set $g = 1$):

$$\begin{aligned} Q &= \mu + \frac{1}{2}\mu^2 & \Lambda &= i\sqrt{p_1} \,, \qquad S = 2\pi\sqrt{p_1} \\ J &= (2Q+1)^{3/2} - 3Q - 1 = \frac{3}{2}\mu^2 + \mu^3 & & \\ p_1 &= 3Q^2 + 6Q + 2 - 2(2Q+1)^{3/2} = \mu^3 + \frac{3}{4}\mu^4 \,. & & \end{aligned} \tag{3.5.25}$$

The entropy is positive for $Q > 0$, and in this range $J > 0$.

The extremization problem (3.5.4) simplifies because we only have two chemical potentials, $X \coloneqq X_1 = X_2 = X_3$ and ω with the constraint (3.5.3). The critical point is

$$\omega = \frac{Q + \Lambda}{2Q + 3J - \Lambda} \,, \qquad X = -\frac{J - \Lambda}{2Q + 3J + \Lambda} \,. \tag{3.5.26}$$

Let us mention that in the alternative extremization problem in which the constraint (3.5.3) is modified by changing $+1$ into -1, the critical values of X and ω are given by the same expressions, however the critical value of the Lagrange multiplier becomes $\Lambda = -i\sqrt{p_1}$.

We now turn to the index. Given the identifications $X_a = [\Delta_a]_\tau$ and $\omega = \tau$, we can restrict to chemical potentials such that $[\Delta]_\tau \coloneqq [\Delta_1]_\tau = [\Delta_2]_\tau = [\Delta_3]_\tau$, where Δ_3 is defined through the general constraint (3.3.31). Up to integer shifts, this amounts to

$$\Delta \coloneqq \Delta_1 = \Delta_2 = \Delta_3 = \frac{2\tau - 1}{3} \,. \tag{3.5.27}$$

The critical points (3.5.26) indeed satisfy this relation. We have thus reduced to a single independent chemical potential τ. Notice that the function $\mathcal{I}\big(\Delta(\tau);\tau\big) = \mathcal{I}\big(\frac{2\tau-1}{3};\tau\big)$ is periodic under $\tau \to \tau + 3$, therefore we will restrict to $0 \leq \mathbb{R}\mathrm{e}\,\tau < 3$.

We study the large N formula (3.3.40) for the index, in particular we want to determine the structure of the leading contributions as τ is varied, and where the Stokes lines are. To do so, we need the values of the bracketed potentials $[\Delta]_{\tau+r}$ for $r \in \mathbb{Z}$. We find

$$[\Delta]_{\tau+r} = \left[\frac{2\tau - 1}{3}\right]_{\tau+r} = \begin{cases} \Delta + \dfrac{2r}{3} & \text{if } r = 0 \mod 3 \\ \text{undefined} & \text{if } r = 1 \mod 3 \\ \Delta + \dfrac{2r-1}{3} & \text{if } r = 2 \mod 3\,. \end{cases} \tag{3.5.28}$$

In the second case the bracket is not defined because $\mathbb{I}\mathrm{m}\big(\Delta/(\tau+r)\big) \in \mathbb{Z} \times \mathbb{I}\mathrm{m}\big(1/(\tau+r)\big)$, *i.e.* because Δ sits exactly on the boundary of a strip defined by $\tau + r$. We can however consider $[\Delta]_{\tau+r}$ for values of Δ that are a bit off the boundary of the strip in the real direction. We consider the values $\Delta_{(\pm)} = \Delta \pm \epsilon$ with infinitesimal $\epsilon > 0$ and find

$$\big[\Delta_{(+)}\big]_{\tau+r} \xrightarrow[\epsilon\to 0]{} \Delta + \frac{2r-2}{3}\,, \qquad \big[\Delta_{(-)}\big]_{\tau+r} \xrightarrow[\epsilon\to 0]{} \Delta + \frac{2r+1}{3} \qquad \text{if } r = 1 \mod 3\,. \tag{3.5.29}$$

Using these formulas, the values of $\Theta(\Delta, \tau + r)$ are easily computed.[34] In particular, the imaginary parts of Θ computed on $\Delta_{(\pm)}$ are the same.

The dominant contribution to the index is determined by comparing the absolute values of $\exp\big(-\pi i N^2 \Theta(\Delta; \tau + r)\big)$—or equivalently the imaginary parts of Θ—as we vary r. When there is a particular value $\widehat{r}$ for which $\mathbb{I}\mathrm{m}\,\Theta(\Delta; \tau + \widehat{r})$ is maximum, there is one dominant contribution which leads to a concrete estimate of the leading behavior of the index. When, instead, there is no maximum, we are left with an infinite number of competing contributions and more detailed information would be needed to resum them. We obtain the following values for the imaginary part of Θ:

$$\mathbb{I}\mathrm{m}\,\Theta(\Delta; \tau + r) = \begin{cases} \dfrac{2\,\mathbb{I}\mathrm{m}\,\tau}{27}\left(4 + \dfrac{\mathbb{R}\mathrm{e}\,\tau + r}{|\tau + r|^4} - \dfrac{3}{|\tau + r|^2}\right) & \text{if } r = 0 \mod 3 \\ \dfrac{8\,\mathbb{I}\mathrm{m}\,\tau}{27} & \text{if } r = 1 \mod 3 \\ \dfrac{2\,\mathbb{I}\mathrm{m}\,\tau}{27}\left(4 - \dfrac{\mathbb{R}\mathrm{e}\,\tau + r}{|\tau + r|^4} - \dfrac{3}{|\tau + r|^2}\right) & \text{if } r = 2 \mod 3\,. \end{cases} \tag{3.5.30}$$

Notice that the limiting value for large $|r|$ (equal to the value for $r = 1 \mod 3$) is as in (3.3.45). If there is a value of r that maximizes $\mathbb{I}\mathrm{m}\,\Theta$, it must come from the first or third case. In particular, there exists $\widehat{r}$ with $\widehat{r} = 0 \mod 3$ if τ satisfies the following relation:

$$\mathbb{R}\mathrm{e}\,\tau + \widehat{r} > 3\,|\tau + \widehat{r}|^2 \qquad \text{with} \qquad \widehat{r} = 0 \mod 3\,. \tag{3.5.31}$$

[34] For $r = 0 \mod 3$ one has to use the 1st case of Θ, while for $r = 2 \mod 3$ the 2nd case.

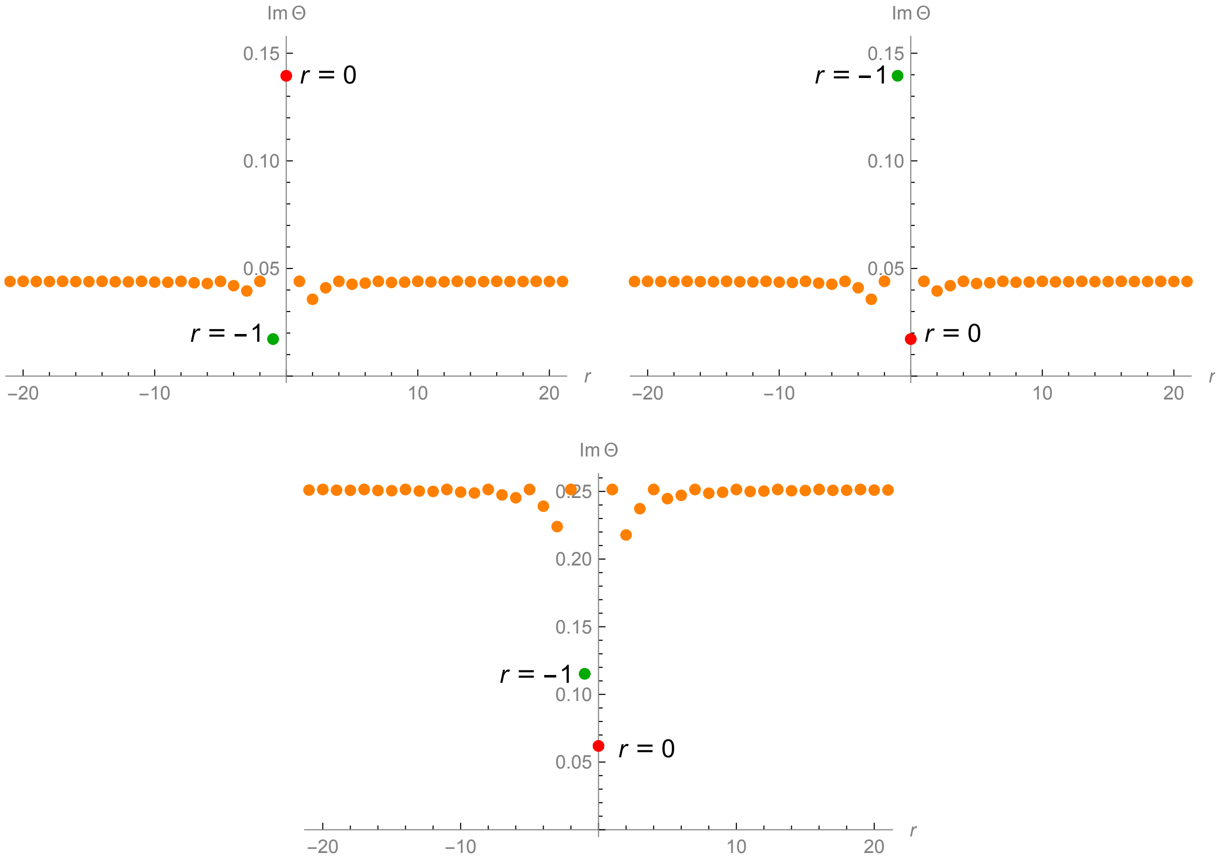

Figure 3.2: The upper left plot shows the values of $\mathbb{I}\mathrm{m}\,\Theta(\Delta;\tau+r)$ as a function of r, for τ inside the semi-circle (3.5.31). The red dot corresponds to $r=0$, and is the dominant contribution in this case. The upper right plot shows $\mathbb{I}\mathrm{m}\,\Theta(\Delta;\tau+r)$ for τ inside the semi-circle (3.5.32). The green dot corresponds to $r=-1$, and is the dominant contribution in this case. The lower plot shows the values of $\mathbb{I}\mathrm{m}\,\Theta(\Delta;\tau+r)$ for τ outside the two semicircles, where there is no dominant contribution.

This corresponds to the interior of a semi-circle in the upper half τ-plane, centered at the boundary point $\tau = 1/6 - \widehat{r}$ and with radius $1/6$. Similarly, there exists $\widehat{r}$ with $\widehat{r} = 2 \mod 3$ if τ satisfies

$$-\,\mathbb{R}\mathrm{e}\,\tau - \widehat{r} > 3\,|\tau + \widehat{r}\,|^2 \qquad \text{with} \qquad \widehat{r} = 2 \mod 3\,. \tag{3.5.32}$$

This corresponds to the interior of another semi-circle of radius $1/6$, centered at $\tau = -1/6 - \widehat{r}$. The two inequalities (3.5.31) and (3.5.32) define two semi-circles in the fundamental range $0 \le \mathbb{R}\mathrm{e}\,\tau < 3$, for $\widehat{r} = 0$ and $\widehat{r} = -1$ respectively, as well as all their images under the periodicity $\tau \to \tau + 3$. On the other hand, outside the two regions there is no dominant contribution because, for all values of r, $\mathbb{I}\mathrm{m}\,\Theta$ is smaller than the limiting value. In Figure 3.2 we provide plots of $\mathbb{I}\mathrm{m}\,\Theta(\Delta;\tau+r)$ as r is varied, both for τ insides the semi-circle (3.5.31), inside the semi-circle (3.5.32), and outside those two.

In Figure 3.3 we represent the fundamental range $0 \le \mathbb{R}\mathrm{e}\,\tau < 3$ of the upper half

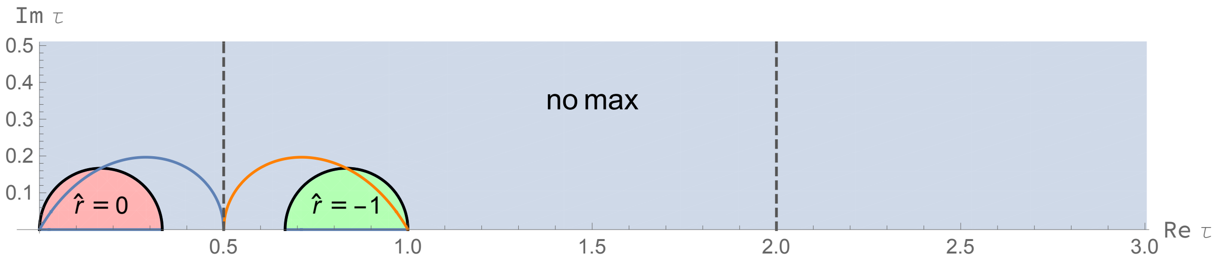

Figure 3.3: Stokes lines in the τ-plane. The red semicircle corresponds to the domain of analyticity where the $r = 0$ contribution dominates the index. The green semicircle instead corresponds to the domain where the $r = -1$ contribution dominates. In the remaining region we do not have a dominant contribution. The blue and orange lines are the critical points of the entropy function for BPS black holes, in the two possible formulations. The dashed lines indicate the subspace where both fugacities q and y are real and the computation of [88] applies.

τ-plane, dividing it into regions according to the dominant contribution. In Figure 3.4 we represent the same information in the q-plane, using $q^{1/3}$ as the variable. The red semi-circle (3.5.31) corresponds to the values of τ in which $\widehat{r} = 0$, while the green semi-circle (3.5.32) corresponds to $\widehat{r} = -1$. These are two different domains of analyticity. The remaining "no max" region, in blue, corresponds to values of τ for which there is no dominant contribution. The three regions are separated by Stokes lines (in black).

Inside the red semi-circle (3.5.31) the large N limit of the superconformal index is

$$\log \mathcal{I}_\infty(\Delta;\tau) = -\pi i N^2\, \Theta(\Delta;\tau) = -\pi i N^2 \frac{[\Delta]_\tau^3}{\tau^2} = -\pi i N^2 \frac{(2\tau-1)^3}{27\tau^2} \,. \tag{3.5.33}$$

This expression exactly matches the entropy function (3.5.2) of black holes with equal charges Q and angular momenta J, and its Legendre transform selects the critical points (3.5.26). We represent the line of critical points, as $\mu > 0$ is varied, by a blue solid line in Figures 3.3 and 3.4. As we see from there, for $\mu > \mu_*$ the blue line lies inside the red semi-circle, meaning that the entropy of the single-center black hole is the dominant contribution to the index. This seems to confirm that "large" BPS black holes, with $Q > Q_*$ or equivalently $J > J_*$, are stable. On the contrary, for $0 < \mu < \mu_*$ the blue line plunges into the "no max" region. We can still identify the black hole entropy with the contribution of the basic solution (3.2.19) to the index, however such a contribution is no longer dominant. This suggests that "small" BPS black holes with $Q < Q_*$ might be unstable towards other supergravity configurations. We find the following values at the transition point:

$$\mu_* = \frac{2}{3} \,, \qquad \tau_* = \frac{1+i}{6} \,, \qquad Q_* = \frac{8}{9} \,, \qquad J_* = \frac{26}{27} \,, \qquad S_* = \frac{4\pi}{3} \,, \tag{3.5.34}$$

where Q, J, S are in units of ν. It would be nice to derive these values from supergravity.

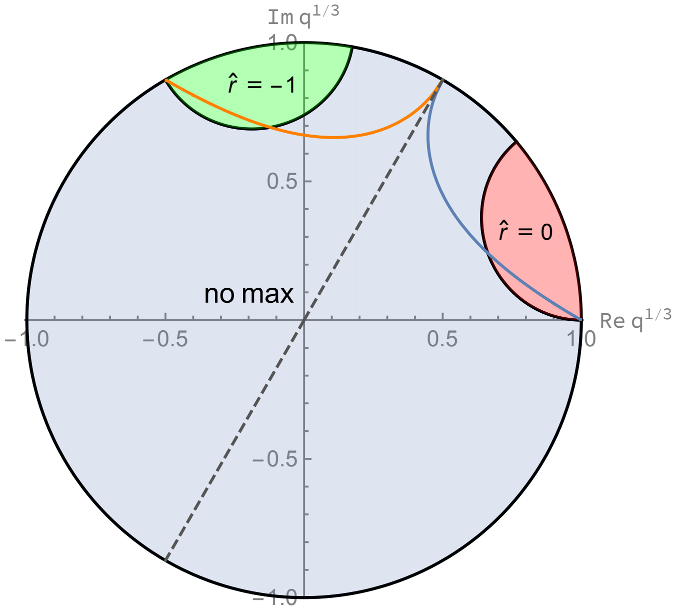

Figure 3.4: Stokes lines in the q-plane, where the variable is $q^{1/3}$. The notation is the same as in Figure 3.3. The dashed line indicates the subspace where both fugacities q and y are real and the computation of [88] applies.

The green semicircle in Figures 3.3 and 3.4 corresponds to values of τ for which the $r = -1$ contribution dominates. In this domain we find

$$\begin{aligned}\log \mathcal{I}_\infty(\Delta;\tau) = -\pi i N^2\, \Theta(\Delta;\tau-1) &= -\pi i N^2 \left(\frac{\left([\Delta]_{\tau-1}+1\right)^3}{(\tau-1)^2} - 1 \right) \\ &= -\pi i N^2 \left(\frac{(2\tau-1)^3}{27(\tau-1)^2} - 1 \right) \,. \end{aligned} \tag{3.5.35}$$

This also reproduces the entropy of single-center black holes: this expression matches the entropy function (3.5.2) with the alternative constraint among the chemical potentials, given by (3.5.3) with $+1$ substituted with -1. In the figures we have indicated with a solid orange line the critical points obtained with the alternative extremization principle.

It is interesting to draw the subspace where both fugacities q and y are real and the computation of [88] applies. We include this subspace, in terms of $q^{1/3}$, in Figure 3.4. We see that the real subspace does not intercept the black hole lines: it only asymptotically reaches them, at the tail that describes black holes much smaller than the AdS radius.

3.6 Conclusions

In this chapter we have successfully studied the microscopic origin of the Bekenstein-Hawking entropy of electrically charged rotating black holes in Type IIB on $AdS_5 \times S^5$. This has been done in the holographic dual $\mathcal{N} = 4$ SYM theory by computing the large

N limit of the superconformal index. Due to technical difficulties, we have restricted ourselves to the case of equal angular momenta.

To perform the large N limit we have used the Bethe Ansatz formula derived in Chapter 2. As we have shown in Section 3.3, the asymptotic expansion enjoys Stokes phenomena in the space of complex fugacities. This space is then divided into various chambers—separated by Stokes lines—in each of which the leading behavior of the index is captured by a different analytic function. The latter have the form of many exponentially large contributions and their interplay seems to justify the discrepancy with the computation of [88], that was performed with real fugacities. This is because the subspace of real fugacities lies on a particular Stokes line in which all the exponentially large contributions organize into competing pairs, making possible their cancelation to an order $\mathcal{O}(1)$ contribution.

We have furthermore shown that there exists one particular contribution from which the Bekenstein-Hawking entropy of the dual BPS black holes can be extracted via a refined version of the $\mathcal{I}$-extremization principle. On the other hand, the presence of other exponentially large contributions points towards the interpretation as saddle points in the dual gravitational path integral.

We have been able to shed more light in this direction in the simple example of BPS black holes with equal charges and angular momenta. Here, the entropy dominates the superconformal index only for large black holes. One possible explanation invokes the presence of instabilities for small black holes, possibly towards hairy or multi-center black holes. We leave the task of making these observations more precise for future works.

Appendices

A Special functions

We review the properties and main identities of special functions that have been used throughout this thesis.

q-Pochhammer symbol. The function is defined as

$$(z;q)_\infty \coloneqq \prod_{k=0}^{\infty}\left(1 - zq^k\right) . \tag{A.1}$$

Here and in the following we set $z = e^{2\pi i u}$, $q = e^{2\pi i \tau}$ and take $|q| < 1$.

A plethystic representation is

$$(z;q)_\infty = \exp\left[-\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k}{1-q^k}\right] , \tag{A.2}$$

which converges for $|z|, |q| < 1$.

Function θ_0. This function, also called q-theta function, is defined as [163]

$$\theta_0(u;\tau) \coloneqq (z;q)_\infty \,(q/z;q)_\infty \, . \tag{A.3}$$

It has simple zeros at $z = q^m$,for $m \in \mathbb{Z}$ and no singularities except for $z = 0$ or ∞. It satisfies the following relations:

$$\begin{aligned} \theta_0(u + n + m\tau;\tau) &= (-z)^{-m}\, q^{-\frac{m(m-1)}{2}}\, \theta_0(u;\tau) \\ \theta_0(u;\tau) &= \theta_0(\tau - u;\tau) = -z\,\theta_0(-u;\tau) \, , \end{aligned} \tag{A.4}$$

valid for any $m, n \in \mathbb{Z}$. The modular transformations are

$$\theta_0(u;\tau+1) = \theta_0(u;\tau) \, , \qquad \theta_0\left(\frac{u}{\tau};-\frac{1}{\tau}\right) = e^{\pi i\left(\frac{u^2}{\tau} - u + \frac{u}{\tau} + \frac{\tau}{6} + \frac{1}{6\tau} - \frac{1}{2}\right)}\,\theta_0(u;\tau) \, . \tag{A.5}$$

To derive them, one can relate θ_0 to the Dedekind η and Jacobi θ_3 functions:

$$\theta_0(u;\tau) = q^{\frac{1}{24}}\,\frac{\theta_3\big(u - \frac{\tau}{2} - \frac{1}{2};\tau\big)}{\eta(\tau)} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{+\infty}(-z)^n\,q^{n(n-1)/2}\,, \tag{A.6}$$

where

$$\theta_3(u;\tau) = \sum_{n=-\infty}^{+\infty} z^n\,q^{\frac{n^2}{2}} = (q;q)_\infty\,(-zq^{1/2};q)_\infty\,(-z^{-1}q^{1/2};q)_\infty\,. \tag{A.7}$$

The function θ_3 is also called ϑ_{00} in the literature.

Thanks to (A.2), there is a plethystic representation

$$\theta_0(u;\tau) = \exp\left[-\sum_{k=1}^{\infty}\frac{1}{k}\frac{z^k + z^{-k}q^k}{1-q^k}\right]\,, \tag{A.8}$$

which converges for $|q| < |z| < 1$.

Function $\boldsymbol{\theta}$. This is a modification of the function θ_0, defined as

$$\theta(u;\tau) \coloneqq e^{-\pi i u + \pi i \tau/6}\,\theta_0(u;\tau)\,. \tag{A.9}$$

Its periodicity relations are

$$\begin{aligned} \theta(u+n+m\tau;\tau) &= (-1)^{n+m}\,z^{-m}\,q^{-m^2/2}\,\theta(u;\tau) \\ \theta(-u;\tau) &= -\theta(u;\tau)\,, \end{aligned} \tag{A.10}$$

again valid for $m,n\in\mathbb{Z}$. The modular transformations are

$$\theta(u;\tau+1) = e^{\pi i/6}\,\theta(u;\tau)\,, \qquad \theta\left(\frac{u}{\tau};-\frac{1}{\tau}\right) = -i\,e^{\pi i u^2/\tau}\,\theta(u;\tau)\,. \tag{A.11}$$

Function $\boldsymbol{\psi}$. Following [163], we define the function

$$\psi(t) \coloneqq \exp\left[t\log\big(1-e^{-2\pi i t}\big) - \frac{1}{2\pi i}\,\mathrm{Li}_2\big(e^{-2\pi i t}\big)\right]\,. \tag{A.12}$$

Within $\mathbb{I}\mathrm{m}\,t < 0$, the definition is analytic and single-valued. The branch of the logarithm is determined by the series expansion $\log(1-z) = -\sum_{k=0}^{\infty} z^k/k$, whereas $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$. One can show that the branch cut ambiguities of the logarithm and the dilogarithm, that appear for $\mathbb{I}\mathrm{m}\,t \geq 0$, cancel in the definition of $\psi(t)$. This means that the latter function can be analytically continued to the whole complex plane yielding a meromorphic function.

Two useful properties of $\psi(t)$ are:

$$\psi(t)\,\psi(-t) = e^{-\pi i(t^2-1/6)}\,, \qquad \psi(t+n) = \big(1-e^{-2\pi i t}\big)^n \psi(t) \qquad \forall\, n\in\mathbb{Z}\,, \tag{A.13}$$

valid for any $t \in \mathbb{C}$.

Function $\widetilde{\Gamma}$. Setting now $p = e^{2\pi i\tau}$, $q = e^{2\pi i\sigma}$, the elliptic gamma function [163] is

$$\widetilde{\Gamma}(u;\tau,\sigma) \coloneqq \Gamma\big(z = e^{2\pi i u}\,;\, p = e^{2\pi i\tau}\,,\, q = e^{2\pi i\sigma}\big) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}z^{-1}}{1-p^m q^n z}\,, \tag{A.14}$$

defined for $|p|, |q| < 1$. The function $\Gamma(z;p,q)$ is meromorphic in z, with simple zeros at $z = p^{m+1}q^{n+1}$ and simple poles at $z = p^{-m}q^{-n}$, for $m,n \in \mathbb{Z}_{\geq 0}$. A plethystic representation is

$$\Gamma(z;p,q) = \exp\left[\sum_{k=1}^{\infty} \frac{1}{k}\, \frac{z^k - p^k q^k z^{-k}}{(1-p^k)(1-q^k)}\right]\,, \tag{A.15}$$

which is convergent for $|pq| < |z| < 1$.

The basic periodicity relations are

$$\widetilde{\Gamma}(u+1;\tau,\sigma) = \widetilde{\Gamma}(u;\tau,\sigma) = \widetilde{\Gamma}(u;\sigma,\tau) \tag{A.16}$$

and

$$\widetilde{\Gamma}(u+\tau;\tau,\sigma) = \theta_0(u;\sigma)\,\widetilde{\Gamma}(u;\tau,\sigma) \qquad \widetilde{\Gamma}(u+\sigma;\tau,\sigma) = \theta_0(u;\tau)\,\widetilde{\Gamma}(u;\tau,\sigma)\,. \tag{A.17}$$

There is also an inversion formula:

$$\widetilde{\Gamma}(u;\tau,\sigma) = \frac{1}{\widetilde{\Gamma}(\tau+\sigma-u;\tau,\sigma)}\,. \tag{A.18}$$

For $\tau, \sigma, \tau/\sigma, \tau+\sigma \in \mathbb{C}\setminus\mathbb{R}$ there exist modular transformations (Theorem 4.1 of [163]):

$$\widetilde{\Gamma}(u;\tau,\sigma) = e^{-\pi i \mathcal{Q}(u;\tau,\sigma)}\, \frac{\widetilde{\Gamma}\big(\frac{u}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma}\big)}{\widetilde{\Gamma}\big(\frac{u-\sigma}{\tau};-\frac{1}{\tau},-\frac{\sigma}{\tau}\big)} = e^{-\pi i \mathcal{Q}(u;\tau,\sigma)}\, \frac{\widetilde{\Gamma}\big(\frac{u}{\tau};-\frac{1}{\tau},\frac{\sigma}{\tau}\big)}{\widetilde{\Gamma}\big(\frac{u-\tau}{\sigma};-\frac{\tau}{\sigma},-\frac{1}{\sigma}\big)} \tag{A.19}$$

where

$$\begin{aligned}\mathcal{Q}(u;\tau,\sigma) = {}& \frac{u^3}{3\tau\sigma} - \frac{\tau+\sigma-1}{2\tau\sigma}u^2 + \\ & + \frac{(\tau+\sigma)^2+\tau\sigma-3(\tau+\sigma)+1}{6\tau\sigma}u + \frac{(\tau+\sigma-1)(\tau+\sigma-\tau\sigma)}{12\tau\sigma}\end{aligned} \tag{A.20}$$

is a cubic polynomial in u, as defined in (3.3.9).

In the degenerate case $\tau = \sigma$ the definition (A.14) simplifies to

$$\Gamma(z; q, q) = \prod_{m=0}^{\infty} \left(\frac{1 - q^{m+2} z^{-1}}{1 - q^m z} \right)^{m+1} , \qquad |q| < 1 \ . \tag{A.21}$$

Furthermore, the identities (A.19) do not apply. However, there exists an alternative version (Theorem 5.2 of [163]):

$$\widetilde{\Gamma}(u; \tau, \tau) = \frac{e^{-\pi i \mathcal{Q}(u;\tau,\tau)}}{\theta_0\left(\frac{u}{\tau}; -\frac{1}{\tau}\right)} \prod_{k=0}^{\infty} \frac{\psi\left(\frac{k+1+u}{\tau}\right)}{\psi\left(\frac{k-u}{\tau}\right)} , \tag{A.22}$$

valid for $u \in \mathbb{C} \setminus \{\mathbb{Z} + \tau\mathbb{Z}\}$. The function $\psi(t)$ is defined in (A.12).

In the case where the periods satisfy $\tau/\sigma \in \mathbb{Q}$, a useful relation holds (Theorem 5.4 of [163]):

$$\widetilde{\Gamma}(u; a\tau, b\tau) = \prod_{r=0}^{a-1} \prod_{s=0}^{b-1} \widetilde{\Gamma}(u + (as + br)\tau; ab\tau, ab\tau) \ , \tag{A.23}$$

valid for $a, b \in \mathbb{Z}$.

Additionally selecting $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$, the following identity applies (Theorem 5.5 of [163]):

$$\widetilde{\Gamma}(u; a\tau, b\tau)^{ab} = \widetilde{\Gamma}(u; \tau, \tau) \prod_{k=0}^{ab-1} \theta_0(u + k\tau; ab\tau)^{\alpha_k} \ , \tag{A.24}$$

where $\alpha_k = -ab + k + 1$ if $k = ar + bs$ for some $r, s \in \mathbb{Z}_{\geq 0}$ and $\alpha_k = k + 1$ otherwise. In the language of numerical semigroups (see Appendix B) the definition of α_k can be given as:

$$\alpha_k = \begin{cases} -ab + k + 1 & \text{if } k \in \mathcal{R}(a, b) \\ k + 1 & \text{if } k \in \overline{\mathcal{R}}(a, b) \ . \end{cases} \tag{A.25}$$

B Numerical semigroups and the Frobenius problem

Given a set of non-negative integer numbers $\{a_1, \dots, a_r\}$, the Frobenius problem consists in classifying which integers can (or cannot) be written as non-negative integer linear combinations of those. This problem has deep roots in the theory of numerical semigroups.

A "semigroup" is an algebraic structure $\mathcal{R}$ endowed with an associative binary operation. Analogously to groups, we denote it as $(\mathcal{R}, *)$. On the other hand, differently from the case of a group, no requirement on the presence of identity and inverse elements is made. A "numerical semigroup" is an additive semigroup $(\mathcal{R}, +)$, where $\mathcal{R}$ consists of all non-negative integers $\mathbb{Z}_{\geq 0}$ except for a finite number of positive elements (thus $0 \in \mathcal{R}$). The set $\{n_1, \dots, n_t\}$ is called a "generating set" for $(\mathcal{R}, +)$ if all elements of $\mathcal{R}$ can be written as non-negative integers linear combinations of $n_1, \dots, n_t$. We then denote the semigroup with the presentation

$$\mathcal{R} = \langle n_1, \dots, n_t \rangle \, . \tag{B.1}$$

Among all possible presentations of $\mathcal{R}$, there exists a unique minimal presentation, which contains the minimal number of generators. Such a number is called the "embedding dimension" $e(\mathcal{R})$ of the semigroup. We now define other important quantities associated with numerical semigroups:

- The "multiplicity" $m(\mathcal{R})$ is the smallest non-zero element of $\mathcal{R}$.
- The "set of gaps" $\overline{\mathcal{R}} = \mathbb{N} \setminus \mathcal{R}$ is the set of positive integers which are not contained in $\mathcal{R}$. Equivalently, the gaps are defined as all natural numbers which cannot be written as non-negative integer linear combination of the generators $n_1, \dots, n_t$ of $\mathcal{R}$.
- The set of gaps $\overline{\mathcal{R}}$ is always a finite set. Its largest element is the "Frobenius number" $F(\mathcal{R})$. Alternatively, given a presentation $\langle n_1, \dots, n_t \rangle$, the Frobenius number is defined as the largest integer which cannot be written as a non-negative integer linear combination of the generators.
- The "genus" $\chi(\mathcal{R})$ is the number of gaps, *i.e.* the order of the set of gaps: $\chi(\mathcal{R}) = \left|\overline{\mathcal{R}}\right|$.

- The "weight" $w(\mathcal{R})$ is the sum of all gaps: $w(\mathcal{R}) = \sum_{k\in\overline{\mathcal{R}}} k$.
- The following inequalities hold:

$$e(\mathcal{R}) \leq m(\mathcal{R}) \qquad\qquad F(\mathcal{R}) \leq 2\chi(\mathcal{R}) - 1. \tag{B.2}$$

  In particular, if $x \in \mathcal{R}$, then $F(\mathcal{R}) - x \notin \mathcal{R}$.

We now study the case where the embedding dimension is $e(\mathcal{R}) = 2$, *i.e.* the minimal presentation is defined by two positive integers a, b with $\gcd(a, b) = 1$. The associated numerical semigroup is denoted by $\mathcal{R}(a, b) = \langle a, b\rangle$ and the set of gaps is $\overline{\mathcal{R}}(a, b) = \mathbb{N}\backslash\mathcal{R}(a, b)$. The multiplicity is simply $m(a, b) = \min\{a, b\}$, whereas the Frobenius number is given by

$$F(a, b) = ab - a - b\ . \tag{B.3}$$

The genus and the weight are

$$\chi(a, b) = \frac{(a-1)(b-1)}{2} \qquad\qquad w(a, b) = \frac{(a-1)(b-1)(2ab-a-b-1)}{12}\ . \tag{B.4}$$

Thanks to the properties of $\mathcal{R}(a, b)$, one can prove the following identities:

$$\begin{aligned}
&\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(zh^{as+br}; h^{ab})_\infty = \frac{(z; h)_\infty}{\prod_{k\in\overline{\mathcal{R}}(a,b)}(1-zh^k)}\\
&\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(z^{-1}h^{ab-as-br}; h^{ab})_\infty = (z^{-1}h; h)_\infty \prod_{k\in\overline{\mathcal{R}}(a,b)} (1 - z^{-1}h^{-k})\ ,
\end{aligned} \tag{B.5}$$

which are useful in the analysis of Chapter 2.

Proof. We begin with the first identity. Using the definition (A.1) of the q-Pochhammer symbol we can write:

$$\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(zh^{as+br}; h^{ab})_\infty = \prod_{n=0}^{\infty}\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(1 - zh^{abn+as+br})\ . \tag{B.6}$$

Using that a, b are coprime, the set of integers $\{as+br \,|\, r = 0, \ldots, a-1,\ s = 0, \ldots, b-1\}$ covers once and only once every class modulo ab. It follows that the set of exponents $\{abn + as + br\}$ is precisely $\mathcal{R}(a, b)$. Then

$$\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(zh^{as+br}; h^{ab})_\infty = \prod_{k\in\mathcal{R}(a,b)} (1 - zh^k) = \frac{\prod_{k=0}^{\infty}(1-zh^k)}{\prod_{k\in\overline{\mathcal{R}}(a,b)}(1-zh^k)} = \frac{(z; h)_\infty}{\prod_{k\in\overline{\mathcal{R}}(a,b)}(1-zh^k)}\ , \tag{B.7}$$

which proves the first equality in (2.2.36).

The proof of the second identity is a bit trickier. The key point is to notice that the set $\{as + br\}$ does not contain any element of $\overline{\mathcal{R}}(a,b)$ and thus

$$\{as+br\} = \left\{k + \beta_k ab \,\middle|\, k = 0,\dots,ab-1\right\} \qquad \text{with} \qquad \beta_k := \begin{cases} 0 & \text{if } k \in \mathcal{R}(a,b) \\ 1 & \text{if } k \in \overline{\mathcal{R}}(a,b)\;. \end{cases} \tag{B.8}$$

This implies that $\{ab - as - br\} = \left\{-k + (1-\beta_k)ab \,\middle|\, k = 0,\dots,ab-1\right\}$. Finally, including the freedom of choosing $n \geq 0$, we find that the set of exponents is

$$\{abn + ab - as - ar\} = (-\overline{\mathcal{R}}) \cup \mathbb{Z}_{>0}\;. \tag{B.9}$$

Then

$$\begin{aligned} \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}(z^{-1}h^{ab-as-br}; h^{ab})_\infty &= \prod_{n=0}^{\infty}\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\left(1 - z^{-1}h^{ab(n+1)-as-br}\right) \\ &= \prod_{k\in\overline{\mathcal{R}}(a,b)}(1 - z^{-1}h^{-k}) \times \prod_{k=1}^{\infty}(1 - z^{-1}h^{k}) = (z^{-1}h;h)_\infty \prod_{k\in\overline{\mathcal{R}}(a,b)}(1 - z^{-1}h^{-k})\;. \end{aligned} \tag{B.10}$$

This completes the proof of (2.2.36). □

Thanks to the definition of $\theta_0(u;\omega)$, we can apply (2.2.36) and we obtain that

$$\prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\theta_0\big(u + (as+br)\omega;\omega\big) = \prod_{k\in\overline{\mathcal{R}}(a,b)}\frac{(1 - z^{-1}h^{-k})}{(1 - zh^{k})}\,\theta_0(u;\omega) = \frac{1}{(-z)^{\chi(a,b)}\,h^{w(a,b)}}\,\theta_0(u;\omega)\;. \tag{B.11}$$

C | A dense set

Here we show that the set of points (p, q) such that

$$q^a = p^b \qquad \text{for coprime } a, b \in \mathbb{N} \tag{C.1}$$

is dense in $\{|p| < 1,\ |q| < 1\}$. We write the fugacities in terms of chemical potentials, $p = e^{2\pi i\sigma}$ and $q = e^{2\pi i\tau}$ with $\operatorname{Im}\sigma, \operatorname{Im}\tau > 0$, and for the sake of this argument we choose the determination on the "strip" $0 \leq \operatorname{Re}\sigma, \operatorname{Re}\tau < 1$. Then the condition (C.1) is equivalent to

$$a(\tau + n) = b(\sigma + m) \tag{C.2}$$

for some $m, n \in \mathbb{Z}$ and $a, b \in \mathbb{N}$ coprime.

We choose an arbitrary point (τ_0, σ_0) in the strip and ask if we can find another point (τ, σ), arbitrarily close, that satisfies (C.2). Consider a straight line in the complex plane that starts from 0 and goes through $\tau_0 + n$ for some integer n. When winding once around the strip, this line has an imaginary excursion

$$\Delta y = \frac{\operatorname{Im}\tau_0}{\operatorname{Re}\tau_0 + n} \ . \tag{C.3}$$

We can make this quantity arbitrarily small by choosing n sufficiently large. We define σ' as the closest point to σ_0 that lies on the image of the line on the strip modulo 1, and has $\operatorname{Re}\sigma' = \operatorname{Re}\sigma_0$. It is clear that

$$|\sigma' - \sigma_0| = \left|\operatorname{Im}\sigma' - \operatorname{Im}\sigma_0\right| \leq \Delta y/2 \ , \tag{C.4}$$

and, by construction, $(\sigma' + m) = t(\tau_0 + n)$ for some $m, n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$. We see that $|\sigma' - \sigma_0|$ can be made arbitrarily small by increasing n. Next, we approximate t by a fraction $a/b \in \mathbb{Q}_+$. This, for a/b sufficiently close to t, defines a point σ in the strip by

$$(\sigma + m) = \frac{a}{b}\,(\tau_0 + n) \ . \tag{C.5}$$

It is clear that σ can be made arbitrarily close to σ' by approximating t sufficiently well with a/b. We have thus found a pair $(\sigma, \tau = \tau_0)$, arbitrarily close to (σ_0, τ_0), that satisfies the constraint (C.2).

D | Weyl group fixed points

In this appendix we prove that $\mathcal{Z}_{\text{tot}}(u;\xi,\nu_R,a\omega,b\omega)$ vanishes when evaluated at a point $\hat{u}$ which is fixed, on a torus of modular parameter ω, by a non-trivial element w of the Weyl group $\mathcal{W}_G$:

$$w\cdot[\hat{u}]=[\hat{u}]\;. \tag{D.1}$$

This implies that the solutions to the BAEs (2.2.3) which are fixed points on the torus of an element of the Weyl group, can be excluded from the set $\mathfrak{M}_{\text{BAE}}$—as is done in (2.2.9)—because they do not contribute to the BA formula (2.2.10) for the superconformal index.

D.1 The rank-one case

Let us first consider the case that the gauge group G has rank one, *i.e.*, that $\mathfrak{g}=\mathfrak{su}(2)$. Then there are only two roots, α and $-\alpha$, and the Weyl group is $\mathcal{W}_G=\{1,s_\alpha\}\cong\mathbb{Z}_2$ where s_α is the unique non-trivial Weyl reflection along the root α:

$$s_\alpha(u)=-u \qquad\qquad \forall\, u\in\mathfrak{h}\;. \tag{D.1}$$

We choose a basis element $\{H\}$ for the Cartan subalgebra $\mathfrak{h}$ such that $\rho(H)\coloneqq\rho\in\mathbb{Z}$ for any weight $\rho\in\Lambda_{\text{weight}}$. In this canonical basis $\alpha=2$ (while the fundamental weight is $\lambda=1$). The solutions to $s_\alpha\cdot[\hat{u}]=[\hat{u}]$ are given by[35]

$$\hat{u}=\frac{p+q\omega}{2} \qquad\qquad \text{with}\quad p,q\in\mathbb{Z}\;. \tag{D.2}$$

Choosing a representative for $[\hat{u}]$ in the fundamental domain of the torus, the inequivalent solutions are with $p=0,1$ and $q=0,1$.

The representations of $\mathfrak{su}(2)$ are labelled by a half-integer spin $j\in\mathbb{N}/2$ and their weights are $\rho\in\big\{\ell\alpha\,\big|\,\ell=-j,-j+1,\ldots,j-1,j\big\}$. Therefore, exploiting the expression

[35] The integers p,q appearing in this appendix should not be confused with the complex angular fugacities appearing in the rest of the thesis.

in (2.2.58), the function $\mathcal{Z}$ reduces to

$$\begin{aligned}&\mathcal{Z}(u;\xi,\nu_R,a\omega,b\omega)=\\&\qquad=\theta_0\big(\alpha(u);a\omega\big)\,\theta_0\big(-\alpha(u);b\omega\big)\prod_a\prod_{\ell_a=-j_a}^{j_a}\widetilde{\Gamma}\big(\ell_a\alpha(u)+\omega_a(\xi)+r_a\nu_R;a\omega,b\omega\big)\;.\end{aligned}\tag{D.3}$$

Moreover, the function $\mathcal{Z}_{\text{tot}}$ defined in (2.2.11) is a single sum over $m=1,\dots,ab$.

We want to prove that $\mathcal{Z}_{\text{tot}}(\hat{u};\xi,\nu_R,a\omega,b\omega)=0$. To do that, we construct an involutive map $\gamma : m\mapsto m'$ acting on the set of integers $\{1,\dots,ab\}$ according to

$$m'=m\mod b\,,\qquad\qquad m'=q-m\mod a\,,\tag{D.4}$$

which define m' uniquely. It will be convenient to introduce the numbers $r,s\in\mathbb{Z}$ such that $m'=m+sb=q-m+ra$. The map γ has the property that

$$m'-q/2=\begin{cases}m-q/2 & \mod b\,,\\-(m-q/2) & \mod a\,,\end{cases}\quad=m-q/2+sb=-(m-q/2)+ra\;.\tag{D.5}$$

We will prove that

$$\mathcal{Z}\big(\hat{u}-m'\omega;\xi,\nu_R,a\omega,b\omega\big)=-\mathcal{Z}\big(\hat{u}-m\omega;\xi,\nu_R,a\omega,b\omega\big)\;.\tag{D.6}$$

In particular, the sum over m inside $\mathcal{Z}_{\text{tot}}$ splits into a sum over the fixed points of γ and a sum over the pairs of values related by γ. The property (D.6) guarantees that each term in those sums vanishes, implying that $\mathcal{Z}_{\text{tot}}$ vanishes.

Let us adopt the notation

$$\mathcal{Z}_m\;\coloneqq\;\mathcal{Z}(\hat{u}-m\omega;\xi,\nu_R,a\omega,b\omega)=\mathcal{Z}\big(p/2-(m-q/2)\omega;\xi,\nu_R,a\omega,b\omega\big)\;.\tag{D.7}$$

We define the vector multiplet and the chiral multiplet contribution, respectively, as

$$\begin{aligned}\mathcal{A}_m\;&\coloneqq\;\theta_0\big(\alpha(p/2)-\alpha(m-q/2)\omega;a\omega\big)\,\theta_0\big(-\alpha(p/2)+\alpha(m-q/2)\omega;b\omega\big)\\\mathcal{B}_m\;&\coloneqq\;\prod_a\prod_{\ell_a=-j_a}^{j_a}\widetilde{\Gamma}\Big(\ell_a\alpha(p/2)-\ell_a\alpha(m-q/2)\omega+\omega_a(\xi)+r_a\nu_R;a\omega,b\omega\Big)\;,\end{aligned}\tag{D.8}$$

such that $\mathcal{Z}_m=\mathcal{A}_m\,\mathcal{B}_m$. Then $\mathcal{Z}_{\text{tot}}$ evaluated on $\hat{u}$ can be expressed as

$$\mathcal{Z}_{\text{tot}}\Big(\frac{p+q\omega}{2};\xi,\nu_R,a\omega,b\omega\Big)=\sum_{m=1\,:\,m'=m}^{ab}\mathcal{Z}_m+\sum_{(m,m')\,:\,m'\neq m}\big(\mathcal{Z}_m+\mathcal{Z}_{m'}\big)\;.\tag{D.9}$$

Our goal is to show that $\mathcal{Z}_{m'}=-\mathcal{Z}_m$.

We begin by considering the contribution of $\mathcal{A}_m$. Using (D.5) we can write

$$\begin{aligned}\mathcal{A}_{m'} &= \theta_0\big(p + (2m-q)\omega - 2ra\omega; a\omega\big)\, \theta_0\big(-p + (2m-q)\omega + 2sb\omega; b\omega\big) \\ &= \theta_0\big(-p - (2m-q)\omega + (2r+1)a\omega; a\omega\big)\, \theta_0\big(-p + (2m-q)\omega + 2sb\omega; b\omega\big)\ .\end{aligned} \tag{D.10}$$

In the second equality we used the second relation in (A.4). Using the first relation in (A.4), the identity $2m - q - ra + sb = 0$ and reinstating α, with some algebra we obtain

$$\mathcal{A}_{m'} = -\, e^{-2\pi i\, \alpha(r)\, \alpha(s)\, \nu_R}\, \mathcal{A}_m\ . \tag{D.11}$$

Then we turn to $\mathcal{B}_m$ and, using (D.5), write

$$\begin{aligned}\mathcal{B}_{m'} &= \prod_a \prod_{\ell_a=-j_a}^{j_a} \widetilde{\Gamma}\Big(\ell_a p + \ell_a(2m-q)\omega + \omega_a(\xi) + r_a\nu_R - 2\ell_a ra\omega; a\omega, b\omega\Big) \\ &= \prod_a \prod_{\ell_a=-j_a}^{j_a} \widetilde{\Gamma}\Big(\ell_a p - \ell_a(2m-q)\omega + \omega_a(\xi) + r_a\nu_R + 2\ell_a ra\omega; a\omega, b\omega\Big)\ .\end{aligned} \tag{D.12}$$

We recall that j_a can be integer or half-integer. In the second equality we simply redefined $\ell_a \to -\ell_a$ and shifted the argument by the integer $2\ell_a p$. Using the identity (A.17) repeatedly and distinguishing the cases $\ell_a \lesseqgtr 0$, we obtain

$$\mathcal{B}_{m'} = \eta \times \mathcal{B}_m \tag{D.13}$$

where the factor η equals

$$\eta \coloneqq \prod_a \prod_{\ell_a>0}^{j_a} \prod_{k=0}^{2\ell_a r-1} \frac{\theta_0\big(\ell_a p - \ell_a(2m-q)\omega + \omega_a(\xi) + r_a\nu_R + ka\omega; b\omega\big)}{\theta_0\big(-\ell_a p + \ell_a(2m-q)\omega + \omega_a(\xi) + r_a\nu_R + (k-2\ell_a r)a\omega; b\omega\big)}\ . \tag{D.14}$$

The second product starts from 1 or $\frac{1}{2}$ depending on j_a being integer or half-integer. Using $2m - q - ra + sb = 0$ at denominator and shifting the arguments by integers, we rewrite

$$\eta = \prod_a \prod_{\ell_a>0}^{j_a} \prod_{k=0}^{2\ell_a r-1} \frac{\theta_0\big(\ell_a p - \ell_a(2m-q)\omega + \omega_a(\xi) + r_a\nu_R + ka\omega; b\omega\big)}{\theta_0\big(\ell_a p - \ell_a(2m-q)\omega + \omega_a(\xi) + r_a\nu_R + ka\omega - 2\ell_a sb\omega; b\omega\big)}\ . \tag{D.15}$$

Finally we use the first relation in (A.4) at denominator, to obtain

$$\eta = \prod_a \prod_{\ell_a>0}^{j_a} (-1)^{4\ell_a^2 rs + 8\ell_a^3 rsp}\, e^{-8\pi i \ell_a^2 rs\, \omega_a(\xi)}\, e^{-8\pi i \ell_a^2 rs(r_a-1)\nu_R}\ . \tag{D.16}$$

Reinstating the root α, this factor can be written as

$$\eta = \prod_a \prod_{\ell_a>0}^{j_a} (-1)^{\ell_a^3 \alpha(r)\alpha(s)\alpha(p)} \times \prod_{a,\, \rho_a\in\mathfrak{R}_a} e^{\pi i \rho_a(r)\rho_a(s)\left(\frac{1}{2} - \omega_a(\xi) - (r_a-1)\nu_R\right)}\ . \tag{D.17}$$

Combining with (D.11), the factor picked up by $\mathcal{Z}$ can be expressed in terms of the anomaly coefficients (2.2.18) and (2.2.19):

$$\mathcal{Z}_{m'} = -\, e^{2\pi i \phi}\, e^{\pi i r s\left(\frac{1}{2}\mathcal{A}^{ii} - \mathcal{A}^{ii\alpha}\xi_\alpha - \mathcal{A}^{iiR}\nu_R\right)}\, \mathcal{Z}_m \,. \tag{D.18}$$

Here i is the gauge index taking a single value. We recall the anomaly cancelation conditions $\mathcal{A}^{ii\alpha} = \mathcal{A}^{iiR} = 0$ and $\mathcal{A}^{ii} \in 4\mathbb{Z}$, implying that the second exponential equals 1. In the first exponential we defined

$$\phi \,:=\, \frac{1}{2}\,\alpha(r)\,\alpha(s)\,\alpha(p) \sum_a \sum_{\ell_a>0}^{j_a} \ell_a^3 = 4rsp \sum_a \sum_{\ell_a>0}^{j_a} \ell_a^3 \,. \tag{D.19}$$

It remains to show that $\phi \in \mathbb{Z}$, so that also the first exponential equals 1.

For each chiral multiplet in the theory, indicized by a, in order to evaluate the second sum in (D.19) we should distinguish different cases:

$$\psi_j \,:=\, 4\sum_{\ell>0}^{j} \ell^3 = \begin{cases} j^2(j+1)^2 & \in 4\mathbb{Z} \qquad \text{if } j\in\mathbb{Z} \\ 2(k+1)^2(8k^2+16k+7) & \in 2\mathbb{Z} \qquad \text{if } j = 2k+\frac{3}{2} \in 2\mathbb{Z}+\frac{3}{2} \\ \frac{1}{2}\,(2k+1)^2(8k^2+8k+1) & \in 4\mathbb{Z}+\frac{1}{2} \quad \text{if } j = 2k+\frac{1}{2} \in 2\mathbb{Z}+\frac{1}{2}\,. \end{cases} \tag{D.20}$$

Therefore, chiral multiplets whose gauge representation has spin $j \in \mathbb{Z}$ or $j \in 2\mathbb{Z} + \frac{3}{2}$ give integer contribution to ϕ. On the other hand, chiral multiplets with $j \in 2\mathbb{Z} + \frac{1}{2}$ can give half-integer contribution. However, because of the Witten anomaly [164], the total number of such multiplets must be even. This is reproduced by the condition (2.2.21) on the pseudo-anomaly coefficient $\mathcal{A}^{ii}$. Indeed, the contribution of a chiral multiplet to the pseudo-anomaly is

$$\mathcal{A}^{ii}_{(j)} = \sum_{\ell=-j}^{j} (2\ell)^2 = \frac{4}{3}\, j(j+1)(2j+1) \,\in\, \begin{cases} 4\mathbb{Z} & \text{if } j\in\mathbb{Z} \text{ or } j \in 2\mathbb{Z}+\frac{3}{2} \\ 4\mathbb{Z}+2 & \text{if } j \in 2\mathbb{Z}+\frac{1}{2}\,, \end{cases} \tag{D.21}$$

and the condition $\mathcal{A}^{ii} \in 4\mathbb{Z}$ requires that the total number of chiral multiplets with $j \in 2\mathbb{Z} + \frac{1}{2}$ be even. This implies that $\phi \in \mathbb{Z}$, and thus that $\mathcal{Z}_{m'} = -\mathcal{Z}_m$. In turn, using (D.9), this implies that

$$\mathcal{Z}_{\text{tot}}(\hat{u}; \xi, \nu_R, a\omega, b\omega) = 0 \tag{D.22}$$

whenever $\hat{u}$ is fixed on the torus by the non-trivial element s_α of the Weyl group of $\mathfrak{su}(2)$.

D.2 The higher-rank case

Let us now move to the case of a generic semi-simple gauge algebra $\mathfrak{g}$ of rank $\text{rk}(G)$. The Weyl group $\mathcal{W}_G$ is a finite group generated by the Weyl reflections

$$s_\alpha(u) = u - 2\frac{\alpha(u)}{(\alpha,\alpha)}\,\widetilde{\alpha} \qquad\qquad \forall\, u \in \mathfrak{h}\ , \tag{D.1}$$

where $\widetilde{\alpha}$ is the image of the root α under the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ induced by the non-degenerate scalar product $(\cdot,\cdot)$ on $\mathfrak{h}^*$. Suppose that there exists a non-trivial element w of $\mathcal{W}_G$ such that $w \cdot \hat{u} = \hat{u}$. It is a standard theorem that the Weyl group acts freely and transitively on the set of Weyl chambers. Therefore, $\hat{u}$ cannot belong to a Weyl chamber but must instead lie on a boundary between two or more chambers. Such boundaries are the hyperplanes fixed by the Weyl reflections, $\{u | s_\alpha(u) = u\}$, and their intersections. We conclude that there must exist at least one root $\hat{\alpha}$ such that $s_{\hat{\alpha}}(\hat{u}) = \hat{u}$.

On the other hand, we are interested in points $\hat{u}$ such that their equivalence class on the torus is fixed by a non-trivial element of the Weyl group, $w \cdot [\hat{u}] = [\hat{u}]$. In this case, for each w we can always identify (at least) one root $\hat{\alpha}$ such that $s_{\hat{\alpha}}[\hat{u}] = [\hat{u}]$, and moreover we can choose a set of simple roots that contains $\hat{\alpha}$. Let us fix a basis of simple roots $\{\alpha_l\}_{l=1,\ldots,\text{rk}(G)}$ for $\mathfrak{g}$ that contains $\hat{\alpha}$. The fundamental weights λ_l are defined by

$$2\,\frac{(\lambda_k,\alpha_l)}{(\alpha_l,\alpha_l)} = \delta_{kl}\ . \tag{D.2}$$

We choose a basis $\{H^i\}$ for the Cartan subalgebra $\mathfrak{h}$ such that the fundamental weights have components $\lambda_l{}^i = \lambda_l(H^i) = \delta_l^i$. In this basis $\rho(H^i) := \rho^i \in \mathbb{Z}$ for any weight $\rho \in \Lambda_{\text{weight}}$. Moreover, the double periodicity of the gauge variables $u = u_i H^i$ is $u_i \sim u_i + 1 \sim u_i + \omega$. From (D.1), the fixed points should satisfy

$$2\frac{\hat{\alpha}(\hat{u})}{(\hat{\alpha},\hat{\alpha})}\,\widetilde{\alpha} = p + q\omega \qquad \text{for} \qquad p = p_i H^i\ , \quad q = q_i H^i \quad \text{and} \quad p_i,\, q_i \in \mathbb{Z}\ . \tag{D.3}$$

Here $\widetilde{\alpha}$ is dual to $\hat{\alpha}$. It is clear that p, q should be aligned with $\widetilde{\alpha}$, therefore we set

$$p = \frac{2\hat{p}}{(\hat{\alpha},\hat{\alpha})}\widetilde{\alpha}\ , \qquad\qquad q = \frac{2\hat{q}}{(\hat{\alpha},\hat{\alpha})}\widetilde{\alpha}\ , \qquad\qquad \text{with } \hat{p},\, \hat{q} \in \mathbb{Z}\ . \tag{D.4}$$

In the basis $\{H^i\}$ we have choosen, the components of $\widetilde{\alpha}$ are $(\lambda_i,\hat{\alpha}) = \delta_{il}\,(\hat{\alpha},\hat{\alpha})/2$, where l is such that $\hat{\alpha} = \alpha_l$ and we have used (D.2). Only one component of $\widetilde{\alpha}$ is non-zero, which implies that the integer components of p, q are $p_i = \hat{p}\,\delta_{il}$ and $q_i = \hat{q}\,\delta_{il}$. This proves integrality of $\hat{p}$, $\hat{q}$. The general solution to (D.3) can then be written as

$$\hat{u} = \hat{u}_0 + \frac{p + q\omega}{2}\ , \tag{D.5}$$

where $\hat{u}_0$ is such that $\hat{\alpha}(\hat{u}_0) = 0$.

Now, consider the explicit expression (2.2.11) for $\mathcal{Z}_{\text{tot}}$, in terms of $\mathcal{Z}$ given in (2.1.12). Given any representation $\mathfrak{R}$ of $\mathfrak{g}$, we can always decompose it into irreducible representations of the $\mathfrak{su}(2)_{\hat{\alpha}}$ subalgebra associated with $\hat{\alpha}$. The set of weights (with multiplicities) $\Lambda_{\mathfrak{R}}$ corresponding to $\mathfrak{R}$ can be organized as a union $\Lambda_{\mathfrak{R}} = \cup_I \Lambda_{\mathfrak{R},I}$ of subsets $\Lambda_{\mathfrak{R},I}$, each corresponding to a representation of $\mathfrak{su}(2)_{\hat{\alpha}}$. Concretely, each $\Lambda_{\mathfrak{R},I}$ is associated to a representation of $\mathfrak{su}(2)_{\hat{\alpha}}$ of spin j_I, so that its elements can be expressed as an $\hat{\alpha}$-chain:

$$\Lambda_{\mathfrak{R},I} = \left\{ \hat{\rho}_I + \ell_I \hat{\alpha} \;\middle|\; \ell_I = -j_I, -j_I + 1, \ldots, j_I - 1, j_I \right\} . \tag{D.6}$$

Here $\hat{\rho}_I$ is the central point, which is orthogonal to $\hat{\alpha}$, *i.e.* such that $(\hat{\rho}_I, \hat{\alpha}) = 0$. Notice that, in general, $\hat{\rho}_I$ is not a weight.[36] The product over all weights ρ of the representation $\mathfrak{R}$ can then be expressed as a product over the representations of $\mathfrak{su}(2)_{\hat{\alpha}}$ contained in $\mathfrak{R}$. In particular we can write

$$\begin{aligned}\prod_a \prod_{\rho_a \in \mathfrak{R}_a} \widetilde{\Gamma}\big(\rho_a(u) + \omega_a(\xi) + r_a \nu_R; a\omega, b\omega\big) = \\ = \prod_{a,I} \prod_{\ell_{aI}=-j_{aI}}^{j_{aI}} \widetilde{\Gamma}\big(\hat{\rho}_{aI}(u) + \ell_{aI}\hat{\alpha}(u) + \omega_a(\xi) + r_a\nu_R; a\omega, b\omega\big) \, .\end{aligned} \tag{D.7}$$

When specifying $\mathfrak{R}$ to the adjoint representation, we obtain a similar decomposition for the roots of $\mathfrak{g}$. Besides the roots $\hat{\alpha}$ and $-\hat{\alpha}$ of $\mathfrak{su}(2)_{\hat{\alpha}}$, the other roots organize into $\hat{\alpha}$-chains that we indicate as

$$\Lambda_{\text{roots},J} = \left\{ \hat{\beta}_J + \ell_J \hat{\alpha} \;\middle|\; \ell_J = -j_J, -j_J + 1, \ldots, j_J - 1, j_J \right\} , \tag{D.8}$$

where $\hat{\beta}_J$ is the non-vanishing central point orthogonal to $\hat{\alpha}$ (once again, $\hat{\beta}_J$ is in general not a weight). Notice that, for each subset $\Lambda_{\text{roots},J}$ of the set of roots, there is a disjoint conjugate subset $\overline{\Lambda}_{\text{roots},J}$ with the same spin j_J but opposite central point $-\hat{\beta}_J$.[37] For this reason, we have that

$$\begin{aligned}\mathcal{Z}(u; \xi, \nu_R, a\omega, b\omega) = \theta_0\big(\hat{\alpha}(u); a\omega\big)\, \theta_0\big(-\hat{\alpha}(u); b\omega\big) \times \\ \times \frac{\prod_{a,I} \prod_{\ell_{aI}=-j_{aI}}^{j_{aI}} \widetilde{\Gamma}\big(\hat{\rho}_{aI}(u) + \ell_{aI}\hat{\alpha}(u) + \omega_a(\xi) + r_a\nu_R; a\omega, b\omega\big)}{\prod_J \prod_{\ell_J=-j_J}^{j_J} \widetilde{\Gamma}\big(\hat{\beta}_J(u) + \ell_J\hat{\alpha}; a\omega, b\omega\big)\, \widetilde{\Gamma}\big(-\hat{\beta}_J(u) + \ell_J\hat{\alpha}; a\omega, b\omega\big)} \, .\end{aligned} \tag{D.9}$$

Similarly to the rank one case, we want to prove that $\mathcal{Z}_{\text{tot}}(\hat{u}; \xi, \nu_R, a\omega, b\omega) = 0$ for $\hat{u}$ in

[36] Indeed, $\hat{\rho}_I$ is guaranteed to be a weight (and in particular a root) only if the spin j_I is integer.

[37] It is easy to prove that $\Lambda_{\text{roots},J}$ and $\overline{\Lambda}_{\text{roots},J}$ are disjoint. Suppose, on the contrary, that there exists some common element $\hat{\beta}_J + \ell_J\hat{\alpha} = -\hat{\beta}_J + k_J\hat{\alpha}$ for some ℓ_J, k_J. This would imply that $\hat{\beta}_J = (k_J - l_J)\hat{\alpha}/2$, but since $(\hat{\beta}_J, \hat{\alpha}) = 0$, then $\hat{\beta}_J = 0$. Since the only roots proportional to $\hat{\alpha}$ are $-\hat{\alpha}$ and $\hat{\alpha}$ itself, we have reached a contradiction.

(D.5). Thus, we construct an involutive map $\gamma : m \mapsto m'$, acting on the set $\mathcal{M}$ of vectors $m = m_i H^i$ with integer components $1 \leq m_i \leq ab$. The map is constructed in such a way that it leaves m invariant along the directions orthogonal to $\widetilde{\alpha}$, whereas it shifts the component parallel to $\widetilde{\alpha}$ by an integer amount. To be precise, take two vectors r, $s \in \mathfrak{h}$ such that

$$r = \frac{2\hat{r}}{(\hat{\alpha}, \hat{\alpha})}\, \widetilde{\alpha} \,, \qquad s = \frac{2\hat{s}}{(\hat{\alpha}, \hat{\alpha})}\, \widetilde{\alpha} \,, \qquad \text{with } \hat{r},\, \hat{s} \in \mathbb{Z} \,, \tag{D.10}$$

meaning that r, s are parallel to $\widetilde{\alpha}$ and have integer components $r_i = \hat{r}\, \delta_{il}$, $s_i = \hat{s}\, \delta_{il}$. Then, we construct m' as

$$m' = m + s\, b \,, \tag{D.11}$$

which implies that m' differs from m only by integer shifts along the direction of $\widetilde{\alpha}$. For $\hat{s}$ we take the unique integer such that $m' \in \mathcal{M}$ and

$$\hat{\alpha}(m') = \hat{\alpha}(m) + \hat{\alpha}(s)\, b = \hat{\alpha}(q - m) + \hat{\alpha}(r)\, a \,. \tag{D.12}$$

Indeed, consider the following equation in r and s: $2\hat{\alpha}(m) - \hat{\alpha}(q) = \hat{\alpha}(r)\, a - \hat{\alpha}(s)\, b$. Using (D.4) and(D.10), it reduces to $\hat{\alpha}(m) - \hat{q} = \hat{r}a - \hat{s}b$. Since a, b are coprime, this equation always admits an infinite number of solutions in the pair $(\hat{r}, \hat{s})$, which can be parametrized as $(\hat{r}_0 + kb, \hat{s}_0 + ka)$ with $k \in \mathbb{Z}$. There is however one and only one solution such that m' has components $1 \leq m'_i \leq ab$. We define $\gamma(m) = m'$ in such a way. One can easily check that it is an involution.

As in the rank-one case, we adopt the notation

$$\mathcal{Z}_m \coloneqq \mathcal{Z}\big(\hat{u} - m\omega; \xi, \nu_R, a\omega, b\omega\big) = \mathcal{Z}\big(\hat{u}_0 + p/2 - (m - q/2)\omega; \xi, \nu_R, a\omega, b\omega\big) \,, \tag{D.13}$$

and, for later convenience, split $\mathcal{Z}$ into the vector multiplet and chiral multiplet contributions:

$$\begin{aligned}
\mathcal{A}_m &\coloneqq \theta_0\big(\hat{\alpha}(p/2) - \hat{\alpha}(m - q/2)\omega; a\omega\big)\, \theta_0\big(-\hat{\alpha}(p/2) + \hat{\alpha}(m - q/2)\omega; b\omega\big) \\
\mathcal{C}_m^{\pm} &\coloneqq \prod_J \prod_{\ell_J = -j_J}^{j_J} \widetilde{\Gamma}\big(\pm\hat{\beta}_J(\hat{u}_0 - m\omega) + \ell_J \hat{\alpha}(p/2) - \ell_J \hat{\alpha}(m - q/2)\omega; a\omega, b\omega\big) \\
\mathcal{B}_m &\coloneqq \prod_{a,I} \prod_{\ell_{aI} = -j_{aI}}^{j_{aI}} \widetilde{\Gamma}\Big(\hat{\rho}_{aI}(\hat{u}_0 - m\omega) + \ell_{aI}\hat{\alpha}(p/2) - \ell_{aI}\hat{\alpha}(m - q/2)\omega + \omega_a(\xi) + r_a \nu_R; a\omega, b\omega\Big)
\end{aligned} \tag{D.14}$$

such that $\mathcal{Z}_m = \mathcal{A}_m \, \mathcal{B}_m / \mathcal{C}_m^+ \, \mathcal{C}_m^-$. We will prove that $\mathcal{Z}_{m'} = -\mathcal{Z}_m$, which implies that $\mathcal{Z}_{\text{tot}}(\hat{u})$ vanishes because γ is an involution.

We begin by considering the contribution of $\mathcal{A}_{m'}$. Following the same steps as in (D.10) and using (D.12) and (A.4), we can show

$$\mathcal{A}_{m'} = -e^{-2\pi i\, \hat{\alpha}(r)\, \hat{\alpha}(s)\, \nu_R}\, \mathcal{A}_m \,. \tag{D.15}$$

We also used that $\hat{\alpha}(r)$, $\hat{\alpha}(s)$, $\hat{\alpha}(q) \in 2\mathbb{Z}$, which is guaranteed by (D.4) and (D.10). We now turn to $\mathcal{B}_{m'}$. Eqn. (D.11) implies that $\hat{\rho}_{aI}(m') = \hat{\rho}_{aI}(m)$ for any $\hat{\rho}_{aI}$ orthogonal to $\hat{\alpha}$. Using the identity (A.17) repeatedly and distinguishing the cases $\ell_{aI} \lessgtr 0$, we obtain

$$\mathcal{B}_{m'} = \prod_{a,I} \prod_{\ell_{aI}>0}^{j_{aI}} (-1)^{\ell_{aI}^3 \hat{\alpha}(r)\hat{\alpha}(s)\hat{\alpha}(p)} \times \\ \times \prod_{a,\,\rho_a} (-1)^{\frac{1}{2}\rho_a(r)\rho_a(s)} \, e^{-\pi i \rho_a(r)\rho_a(s)\left(\rho_a(\hat{u}_0 - m\omega) + \omega_a(\xi) + (r_a - 1)\nu_R\right)} \, \mathcal{B}_m \,. \quad \text{(D.16)}$$

The analysis of $\mathcal{C}_m^{\pm}$ is analogous to the one for $\mathcal{B}_m$ and it gives the following:

$$\mathcal{C}_{m'}^{\pm} = \prod_J \prod_{\ell_J>0}^{j_J} (-1)^{\ell_J^3 \hat{\alpha}(r)\hat{\alpha}(s)\hat{\alpha}(p)} \prod_{\alpha\neq\pm\hat{\alpha}} (-1)^{\frac{1}{2}\alpha(r)\alpha(s)} \, e^{\pi i \alpha(r)\alpha(s)\nu_R} \times \mathcal{C}_m^{\pm} \,. \quad \text{(D.17)}$$

Combining (D.15) with the latter, we obtain that the vector-multiplet contribution is

$$\mathcal{A}_{m'}/\mathcal{C}_{m'}^{+}\mathcal{C}_{m'}^{-} = -e^{-2\pi i \sum_{\alpha\in\Delta} \alpha(r)\alpha(s)\nu_R} \, \mathcal{A}_m/\mathcal{C}_m^{+}\mathcal{C}_m^{-} \,. \quad \text{(D.18)}$$

We used $\hat{\alpha}(r)\hat{\alpha}(s) \in 4\mathbb{Z}$, as well as $\sum_{\alpha\in\Delta} \alpha(r)\alpha(s) \in 4\mathbb{Z}$ for any semi-simple Lie algebra $\mathfrak{g}$, and that $2\ell_J^3 \hat{\alpha}(r)\hat{\alpha}(s)\hat{\alpha}(p) \in 2\mathbb{Z}$ for any integer or half-integer spin. Including now also the contribution from $\mathcal{B}_m$, the factor picked up by $\mathcal{Z}$ can be expressed in terms of the anomaly coefficients (2.2.18) and (2.2.19):

$$\mathcal{Z}_{m'} = -\, e^{2\pi i \phi} \, e^{\pi i \, r_i s_j \left(\frac{1}{2}\mathcal{A}^{ij} - \mathcal{A}^{ijk}(\hat{u}_0 - m\omega)_k - \mathcal{A}^{ij\alpha}\xi_\alpha - \mathcal{A}^{ijR}\nu_R\right)} \, \mathcal{Z}_m \,. \quad \text{(D.19)}$$

The anomaly cancelation conditions $\mathcal{A}^{ijk} = \mathcal{A}^{ij\alpha} = \mathcal{A}^{ijR} = 0$ and $\mathcal{A}^{ij} \in 4\mathbb{Z}$ imply that the second exponential equals 1. In the first exponential we defined

$$\phi \coloneqq \frac{1}{2}\,\hat{\alpha}(r)\,\hat{\alpha}(s)\,\hat{\alpha}(p) \sum_{a,I} \sum_{\ell_{aI}>0}^{j_{aI}} \ell_{aI}^3 = 4\hat{r}\hat{s}\hat{p} \sum_{a,I} \sum_{\ell_{aI}>0}^{j_{aI}} \ell_{aI}^3 \,. \quad \text{(D.20)}$$

Once again, in an anomaly-free theory $\phi \in \mathbb{Z}$. Indeed, labelling the chiral multiplets by a, their $\mathfrak{su}(2)_{\hat{\alpha}}$ representations by I and dubbing their spin j_{aI}, the only non-integer contributions to ϕ come from representations with $j_{aI} \in 2\mathbb{Z} + \frac{1}{2}$. On the other hand, the contribution of an $\mathfrak{su}(2)_{\hat{\alpha}}$ representation to the pseudo-anomaly coefficient is

$$\mathcal{A}_{aI}^{ij} = \sum_{\ell=-j_{aI}}^{j_{aI}} (\hat{\rho}_{aI} + \ell\hat{\alpha})^i (\hat{\rho}_{aI} + \ell\hat{\alpha})^j \,. \quad \text{(D.21)}$$

Since generic vectors r, s (D.10) have integer components, the condition $\mathcal{A}^{ij} \in 4\mathbb{Z}$ implies that also $\mathcal{A}^{ij} r_i s_j \in 4\mathbb{Z}$ for any choice of r, s. Contracting with the vectors, we obtain

$$\mathcal{A}^{ij}_{aI} r_i s_j = \frac{4}{3} \hat{r}\hat{s}\, j_{aI}(j_{aI}+1)(2j_{aI}+1) \in \begin{cases} 4\mathbb{Z} & \text{if } j_{aI} \in \mathbb{Z} \text{ or } j_{aI} \in 2\mathbb{Z}+\frac{3}{2} \\ 4\mathbb{Z}+2 & \text{if } j_{aI} \in 2\mathbb{Z}+\frac{1}{2}\ . \end{cases} \tag{D.22}$$

Therefore, the condition $\mathcal{A}^{ij} \in 4\mathbb{Z}$ requires that the number of $\mathfrak{su}(2)_{\hat{\alpha}}$ representations with $j_{aI} \in 2\mathbb{Z}+\frac{1}{2}$ be even, and this guarantees that $\phi = 0$.

E | Real fugacities

In Section 3.3 we evaluated, in the large N limit, the contribution of some of the solutions to the BAEs to the sum in (3.2.9). In particular we found that all T-transformed solutions (3.3.35) of the basic solution, parameterized by the integer r, contribute at the same order in N, and their contributions are the arguments of $\widetilde{\max}$ in the final formula (3.3.40):

$$-\pi i N^2\, \Theta(\Delta_1, \Delta_2; \tau + r)$$

in terms of Θ defined in (3.3.30) and with $r \in \mathbb{Z}$.

Here we show that when we take the fugacities q, y_1, y_2 to be all real, we end up precisely on a Stokes line. More precisely, we show that all contributions for $r \in \mathbb{Z}$ organize into pairs, except for those elements that already sit on a Stokes line determined by the discontinuity of one of the functions $[\,\cdot\,]_\tau$. In each pair, the two contributions have equal real part and compete. We cannot compute the sum of the two terms, as this would require more accurate information about the subleading corrections. Yet, this makes our result compatible with the result of [88]. There it was found that, for real fugacities, the index scales as $\mathcal{O}(1)$ at large N, implying that all $\mathcal{O}(N^2)$ contributions cancel out. This point was also stressed in [120, 121].

Real fugacities corresponds to chemical potentials whose real part is either zero or $-1/2$ modulo 1. We distinguish the various possibilities into two major cases: the case that $0 < q < 1$, corresponding to $\tau \in i\mathbb{R}_{\geq 0}$, and the case that $-1 < q < 0$, corresponding to $\tau \in -\frac{1}{2} + i\mathbb{R}_{\geq 0}$. Each case is further divided into subcases, according to the number of positive flavor fugacities $y_{1,2}$.

E.1 The case $0 < q < 1$

We start with the case of positive angular fugacity, $0 < q < 1$. We take $\tau \in i\mathbb{R}_{\geq 0}$ and write

$$\tau = it\,, \qquad \text{with } t > 0\,. \tag{E.1}$$

We distinguish three different subcases, corresponding to y_1, y_2 being both positive, one positive and one negative, or both negative.

If one of the flavor fugacities—that we call y—is real positive, we set the corresponding chemical potential

$$\Delta = i\delta \;, \qquad \text{with } \delta \in \mathbb{R} \;. \tag{E.2}$$

We immediately see that $[\Delta]_\tau$ is not defined, because the argument sits precisely along one of the lines of discontinuity. On the other hand, for $r > 0$ and generic δ, the functions $[\Delta]_{\tau\pm r}$ are well-defined and we would like to evaluate them. We can precisely determine their values by splitting the imaginary axis in the Δ-plane into a series of intervals

$$I_k = (k, k+1) \times \frac{t}{r} \qquad \text{with } k \in \mathbb{Z} \;. \tag{E.3}$$

Assuming that $\delta \in I_k$, we see that Δ can be brought inside the strip corresponding to $\tau + r$ by shifting it by k, and inside the strip corresponding to $\tau - r$ by shifting it by $-k-1$. In formulas:

$$[\Delta]_{\tau+r} = i\delta + k \;, \qquad [\Delta]_{\tau-r} = i\delta - k - 1 \qquad \text{for } \delta \in I_k \;. \tag{E.4}$$

If δ is equal to an extremum of I_k, *i.e.* if $\delta = k\,t/r$ for some $k \in \mathbb{Z}$, then $[\Delta]_{\tau\pm r}$ are not defined.

On the other hand, if one of the flavor fugacities—that we keep calling y—is real negative, we set its chemical potential

$$\Delta = -\frac{1}{2} + i\delta \;, \qquad \text{with } \delta \in \mathbb{R} \;. \tag{E.5}$$

For $r > 0$ and generic δ, both functions $[\Delta]_{\tau\pm r}$ are well-defined. As before, we can determine their values by splitting the imaginary axis in intervals. This time the intervals are

$$\widetilde{I}_k = (2k-1, 2k+1) \times \frac{t}{2r} \qquad \text{with } k \in \mathbb{Z} \;. \tag{E.6}$$

We then find

$$[\Delta]_{\tau+r} = i\delta + k - \frac{1}{2} \;, \qquad [\Delta]_{\tau-r} = i\delta - k - \frac{1}{2} \qquad \text{for } \delta \in \widetilde{I}_k \;. \tag{E.7}$$

If $\delta = (2k-1)t/2r$ for some $k \in \mathbb{Z}$, then $[\Delta]_{\tau\pm r}$ are not defined.

We now proceed to applying these formulas to the three subcases.

The subcase $y_1, y_2 > 0$ with $0 < q < 1$

We take both flavor fugacities $y_{1,2}$ to be positive. Correspondingly, we set purely imaginary chemical potentials:

$$\Delta_a = i\delta_a \;, \qquad \text{with } \delta_a \in \mathbb{R} \text{ and } a = 1, 2 \;. \tag{E.8}$$

We immediately see that neither $[\Delta_1]_\tau$, $[\Delta_2]_\tau$ nor $[\Delta_1+\Delta_2]_\tau$ are defined because their arguments sit precisely along one of the lines of discontinuity. This means that the contribution $r=0$ is already along a Stokes line.

Let us now consider $r>0$. For generic δ_a, the functions $[\Delta_a]_{\tau\pm r}$ are well-defined. Precisely, for $\delta_a \in I_{k_a}$ the functions $[\Delta_a]_{\tau\pm r}$ are given by (E.4). Turning to $\Delta_1+\Delta_2$, we have two possibilities:

$$\delta_1+\delta_2 \in I_{k_1+k_2} \quad \Rightarrow \quad \begin{aligned} [\Delta_1+\Delta_2]_{\tau+r} &= [\Delta_1]_{\tau+r}+[\Delta_2]_{\tau+r} &&= i(\delta_1+\delta_2)+k_1+k_2 \\ [\Delta_1+\Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r}+[\Delta_2]_{\tau-r}+1 &&= i(\delta_1+\delta_2)-k_1-k_2-1 \end{aligned} \tag{E.9}$$

or

$$\delta_1+\delta_2 \in I_{k_1+k_2+1} \quad \Rightarrow \quad \begin{aligned} [\Delta_1+\Delta_2]_{\tau+r} &= [\Delta_1]_{\tau+r}+[\Delta_2]_{\tau+r}+1 &&= i(\delta_1+\delta_2)+k_1+k_2+1 \\ [\Delta_1+\Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r}+[\Delta_2]_{\tau-r} &&= i(\delta_1+\delta_2)-k_1-k_2-2\,, \end{aligned} \tag{E.10}$$

whereas $[\Delta_1+\Delta_2]_{\tau\pm r}$ are not defined if $\delta_1+\delta_2 = n\,t/r$ with $n\in\mathbb{Z}$.

In the first case, given by (E.9), we compute

$$\begin{aligned} \Theta(\Delta_1,\Delta_2;\tau+r) &= \frac{[\Delta_1]_{\tau+r}[\Delta_2]_{\tau+r}\big(2(\tau+r)-1-[\Delta_1]_{\tau+r}-[\Delta_2]_{\tau+r}\big)}{(\tau+r)^2} \\ &= \frac{(i\delta_1+k_1)(i\delta_2+k_2)(2r-1-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r+it)^2} \\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{\big([\Delta_1]_{\tau-r}+1\big)\big([\Delta_2]_{\tau-r}+1\big)\big(2(\tau-r)-1-[\Delta_1]_{\tau-r}-[\Delta_2]_{\tau-r}\big)}{(\tau-r)^2}-1 \\ &= \frac{(i\delta_1-k_1)(i\delta_2-k_2)(-2r+1+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r+it)^2}-1\,. \end{aligned} \tag{E.11}$$

In the second case, given by (E.10), we compute

$$\begin{aligned} \Theta(\Delta_1,\Delta_2;\tau+r) &= \frac{\big([\Delta_1]_{\tau+r}+1\big)\big([\Delta_2]_{\tau+r}+1\big)\big(2(\tau+r)-1-[\Delta_1]_{\tau+r}-[\Delta_2]_{\tau+r}\big)}{(\tau+r)^2}-1 \\ &= \frac{(i\delta_1+k_1+1)(i\delta_2+k_2+1)(2r-1-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r+it)^2}-1 \\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{[\Delta_1]_{\tau-r}[\Delta_2]_{\tau-r}\big(2(\tau-r)-1-[\Delta_1]_{\tau-r}-[\Delta_2]_{\tau-r}\big)}{(\tau-r)^2} \\ &= \frac{(i\delta_1-k_1-1)(i\delta_2-k_2-1)(-2r+1+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r+it)^2}\,. \end{aligned} \tag{E.12}$$

From these expression we see that, in both cases,

$$\Theta(\Delta_1,\Delta_2;\tau-r) = -\,\overline{\Theta(\Delta_1,\Delta_2;\tau+r)}\,-1\,. \tag{E.13}$$

This implies that

$$\mathbb{Im}\,\Theta(\Delta_1, \Delta_2; \tau + r) = \mathbb{Im}\,\Theta(\Delta_1, \Delta_2; \tau - r) \tag{E.14}$$

and thus

$$\left| e^{-\pi i N^2 \Theta(\Delta_1,\Delta_2;\tau+r)} \right| = \left| e^{-\pi i N^2 \Theta(\Delta_1,\Delta_2;\tau-r)} \right| \,, \tag{E.15}$$

yielding to a competition between the two terms for each $r > 0$.

The subcase $y_1 < 0$, $y_2 > 0$ with $0 < q < 1$

We take one flavor fugacity to be positive and one negative, say $y_1 < 0$ and $y_2 > 0$ (recall that the index is symmetric in the two flavor fugacities). Correspondingly, we set

$$\Delta_1 = -\frac{1}{2} + i\delta_1 \,, \qquad \Delta_2 = i\delta_2 \,, \qquad \text{with } \delta_{1,2} \in \mathbb{R} \,. \tag{E.16}$$

Similarly to the previous case, $[\Delta_2]_\tau$ is not defined and the contribution $r = 0$ is already along a Stokes line.

For $r > 0$ and generic δ_a, instead, both functions $[\Delta_{1,2}]_{\tau\pm r}$ are well-defined. Assuming $\delta_1 \in \widetilde{I}_{k_1}$ the functions $[\Delta_1]_{\tau\pm r}$ are given by (E.7), and assuming $\delta_2 \in I_{k_2}$ the functions $[\Delta_2]_{\tau\pm r}$ are given by (E.4). Turning to $\Delta_1 + \Delta_2$ we have two possibilities:

$$\delta_1 + \delta_2 \in \widetilde{I}_{k_1+k_2} \quad \Rightarrow \quad \begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r} &= [\Delta_1]_{\tau+r} + [\Delta_2]_{\tau+r} &&= i(\delta_1+\delta_2) + k_1 + k_2 - \frac{1}{2} \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} + 1 &&= i(\delta_1+\delta_2) - k_1 - k_2 - \frac{1}{2} \end{aligned} \tag{E.17}$$

or

$$\delta_1 + \delta_2 \in \widetilde{I}_{k_1+k_2+1} \quad \Rightarrow \quad \begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r} &= [\Delta_1]_{\tau+r} + [\Delta_2]_{\tau+r} + 1 &&= i(\delta_1+\delta_2) + k_1 + k_2 + \frac{1}{2} \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} &&= i(\delta_1+\delta_2) - k_1 - k_2 - \frac{3}{2} \,, \end{aligned} \tag{E.18}$$

whereas $[\Delta_1 + \Delta_2]_{\tau\pm r}$ are not defined if $\delta_1 + \delta_2 = (2n-1)\,t/2r$ with $n \in \mathbb{Z}$.

In the first case, given by (E.17), we compute

$$\begin{aligned} \Theta(\Delta_1, \Delta_2; \tau + r) &= \frac{(i\delta_1 + k_1 - \frac{1}{2})(i\delta_2 + k_2)(2r - \frac{1}{2} - k_1 - k_2 + i(2t - \delta_1 - \delta_2))}{(r+it)^2} \\ \Theta(\Delta_1, \Delta_2; \tau - r) &= \frac{(i\delta_1 - k_1 + \frac{1}{2})(i\delta_2 - k_2)(-2r + \frac{1}{2} + k_1 + k_2 + i(2t - \delta_1 - \delta_2))}{(-r+it)^2} - 1 \,. \end{aligned} \tag{E.19}$$

In the second case, given by (E.18), we compute

$$\begin{aligned}\Theta(\Delta_1,\Delta_2;\tau+r) &= \frac{(i\delta_1+k_1+\frac{1}{2})(i\delta_2+k_2+1)(2r-\frac{1}{2}-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r+it)^2}-1\\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{(i\delta_1-k_1-\frac{1}{2})(i\delta_2-k_2-1)(-2r+\frac{1}{2}+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r+it)^2}\;.\end{aligned} \tag{E.20}$$

Hence, in both cases we find $\Theta(\Delta_1,\Delta_2;\tau-r) = -\overline{\Theta(\Delta_1,\Delta_2;\tau+r)}-1$, meaning that the two terms compete.

The subcase $y_1, y_2 < 0$ with $0 < q < 1$

We take both flavor fugacities $y_{1,2}$ to be negative. Correspondingly we set

$$\Delta_a = -\frac{1}{2}+i\delta_a\;, \qquad \text{with } \delta_a\in\mathbb{R} \text{ and } a=1,2\;. \tag{E.21}$$

In this case $[\Delta_a]_\tau$ are defined but $[\Delta_1+\Delta_2]_\tau$ is not. Therefore the contribution $r=0$ is along a Stokes line.

For $r>0$ and generic δ_a, the functions $[\Delta_a]_{\tau\pm r}$ are well-defined. Assuming $\delta_a\in\widetilde{I}_{k_a}$ then $[\Delta_a]_{\tau\pm r}$ are given by (E.7). We have two possibilities for $\Delta_1+\Delta_2$:

$$\delta_1+\delta_2\in I_{k_1+k_2-1} \quad\Rightarrow\quad \begin{aligned}[\Delta_1+\Delta_2]_{\tau+r} &= [\Delta_1]_{\tau+r}+[\Delta_2]_{\tau+r} &&= i(\delta_1+\delta_2)+k_1+k_2-1\\ [\Delta_1+\Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r}+[\Delta_2]_{\tau-r}+1 &&= i(\delta_1+\delta_2)-k_1-k_2\end{aligned} \tag{E.22}$$

or

$$\delta_1+\delta_2\in I_{k_1+k_2} \quad\Rightarrow\quad \begin{aligned}[\Delta_1+\Delta_2]_{\tau+r} &= [\Delta_1]_{\tau+r}+[\Delta_2]_{\tau+r}+1 &&= i(\delta_1+\delta_2)+k_1+k_2\\ [\Delta_1+\Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r}+[\Delta_2]_{\tau-r} &&= i(\delta_1+\delta_2)-k_1-k_2-1,\end{aligned} \tag{E.23}$$

whereas $[\Delta_1+\Delta_2]_{\tau\pm r}$ are not defined if $\delta_1+\delta_2 = n\,t/r$ with $n\in\mathbb{Z}$. In the first case (E.22)

$$\begin{aligned}\Theta(\Delta_1,\Delta_2;\tau+r) &= \frac{(i\delta_1+k_1-\frac{1}{2})(i\delta_2+k_2-\frac{1}{2})(2r-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r+it)^2}\\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{(i\delta_1-k_1+\frac{1}{2})(i\delta_2-k_2+\frac{1}{2})(-2r+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r+it)^2}\;,\end{aligned} \tag{E.24}$$

while in the second case, given by (E.23), we compute

$$\begin{aligned}\Theta(\Delta_1,\Delta_2;\tau+r) &= \frac{(i\delta_1+k_1+\frac{1}{2})(i\delta_2+k_2+\frac{1}{2})(2r-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r+it)^2}-1\\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{(i\delta_1-k_1-\frac{1}{2})(i\delta_2-k_2-\frac{1}{2})(-2r+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r+it)^2}\;.\end{aligned} \tag{E.25}$$

Hence in both cases $\Theta(\Delta_1, \Delta_2; \tau - r) = -\overline{\Theta(\Delta_1, \Delta_2; \tau + r)} - 1$, meaning that the two terms compete.

E.2 The case $-1 < q < 0$

Now we move to the case of negative angular fugacity, $-1 < q < 0$, and set

$$\tau = -\frac{1}{2} + it\ , \qquad \text{with } t > 0\ . \tag{E.1}$$

Once again, we distinguish three different subcases corresponding to y_1, y_2 being both positive, one positive and one negative, or both negative. First, let us discuss the new intervals we need.

If a flavor fugacity y is real positive, as before we set $\Delta = i\delta$ with $\delta \in \mathbb{R}$. Taking $r \geq 0$ and generic δ, the functions $[\Delta]_{\tau+r+1}$ and $[\Delta]_{\tau-r}$ are well-defined. To evaluate them, we split the imaginary axis into intervals

$$I_k = (k, k+1) \times \frac{2t}{2r+1} \qquad \text{with } k \in \mathbb{Z}\ . \tag{E.2}$$

We find

$$[\Delta]_{\tau+r+1} = i\delta + k\ , \qquad [\Delta]_{\tau-r} = i\delta - k - 1 \qquad \text{for } \delta \in I_k\ . \tag{E.3}$$

If $\delta = n\, 2t/(2r+1)$ for some $n \in \mathbb{Z}$, then $[\Delta]_{\tau+r+1}$ and $[\Delta]_{\tau-r}$ are not defined.

On the other hand, if a flavor fugacity y is real negative, we set $\Delta = -\frac{1}{2} + i\delta$ with $\delta \in \mathbb{R}$. For $r \geq 0$ and generic δ, the functions $[\Delta]_{\tau+r+1}$ and $[\Delta]_{\tau-r}$ are once again well-defined. We split the imaginary axis into intervals

$$\widehat{I}_k = (2k-1, 2k+1) \times \frac{t}{2r+1} \qquad \text{with } k \in \mathbb{Z}\ . \tag{E.4}$$

This time we find

$$[\Delta]_{\tau+r+1} = i\delta + k - \frac{1}{2}\ , \qquad [\Delta]_{\tau-r} = i\delta - k - \frac{1}{2} \qquad \text{for } \delta \in \widehat{I}_k\ . \tag{E.5}$$

If $\delta = (2n_1 - 1)\, t/(2r+1)$ for some $n \in \mathbb{Z}$, then $[\Delta]_{\tau+r+1}$ and $[\Delta]_{\tau-r}$ are not defined.

The subcase $y_1, y_2 > 0$ with $-1 < q < 0$

We take both flavor fugacities $y_{1,2}$ to be positive and set

$$\Delta_a = i\delta_a\ , \qquad \text{with } \delta_a \in \mathbb{R} \text{ and } a = 1, 2\ . \tag{E.6}$$

For $r \geq 0$ and generic $\delta_a \in I_k$, the functions $[\Delta_a]_{\tau+r+1}$ and $[\Delta_a]_{\tau-r}$ are well-defined and given by (E.3). There are then two possibilities. If $\delta_1 + \delta_2 \in I_{k_1+k_2}$ then

$$\begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r+1} &= [\Delta_1]_{\tau+r+1} + [\Delta_2]_{\tau+r+1} = i(\delta_1+\delta_2) + k_1 + k_2 \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} + 1 = i(\delta_1+\delta_2) - k_1 - k_2 - 1 \end{aligned} \tag{E.7}$$

and

$$\begin{aligned} \Theta(\Delta_1, \Delta_2; \tau + r + 1) &= \frac{(i\delta_1 + k_1)(i\delta_2 + k_2)(2r - k_1 - k_2 + i(2t - \delta_1 - \delta_2))}{(r + \frac{1}{2} + it)^2} \\ \Theta(\Delta_1, \Delta_2; \tau - r) &= \frac{(i\delta_1 - k_1)(i\delta_2 - k_2)(-2r + k_1 + k_2 + i(2t - \delta_1 - \delta_2))}{(-r - \frac{1}{2} + it)^2} \ . \end{aligned} \tag{E.8}$$

If $\delta_1 + \delta_2 \in I_{k_1+k_2+1}$ then

$$\begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r+1} &= [\Delta_1]_{\tau+r+1} + [\Delta_2]_{\tau+r+1} + 1 = i(\delta_1+\delta_2) + k_1 + k_2 + 1 \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} \qquad\qquad = i(\delta_1+\delta_2) - k_1 - k_2 - 2 \end{aligned} \tag{E.9}$$

and

$$\begin{aligned} \Theta(\Delta_1, \Delta_2; \tau + r + 1) &= \frac{(i\delta_1 + k_1 + 1)(i\delta_2 + k_2 + 1)(2r - k_1 - k_2 + i(2t - \delta_1 - \delta_2))}{(r - \frac{1}{2} + it)^2} - 1 \\ \Theta(\Delta_1, \Delta_2; \tau - r) &= \frac{(i\delta_1 - k_1 - 1)(i\delta_2 - k_2 - 1)(-2r + k_1 + k_2 + i(2t - \delta_1 - \delta_2))}{(-r - \frac{1}{2} + it)^2} \ . \end{aligned} \tag{E.10}$$

If $\delta_1 + \delta_2 = n\, 2t/(2r+1)$ with $n \in \mathbb{Z}$, then $[\Delta_1 + \Delta_2]_{\tau+r+1}$ and $[\Delta_1 + \Delta_2]_{\tau-r}$ are not defined.

In both well-defined cases, we find

$$\Theta(\Delta_1, \Delta_2; \tau - r) = -\overline{\Theta(\Delta_1, \Delta_2; \tau + r + 1)} - 1 \tag{E.11}$$

This implies that

$$\left| e^{-\pi i N^2 \Theta(\Delta_1, \Delta_2; \tau+r+1)} \right| = \left| e^{-\pi i \Theta(\Delta_1, \Delta_2; \tau - r)} \right| \ , \tag{E.12}$$

yielding to a competition between the two terms for each $r \geq 0$.

The subcase $y_1 < 0$, $y_2 > 0$ with $-1 < q < 0$

We take one flavor fugacities to be positive and the other one to be negative. Hence we set

$$\Delta_1 = -\frac{1}{2} + i\delta_1 \ , \qquad \Delta_2 = i\delta_2 \ , \qquad \text{with } \delta_{1,2} \in \mathbb{R} \ . \tag{E.13}$$

For $r \geq 0$ and generic δ_a, the functions $[\Delta_a]_{\tau+r+1}$ and $[\Delta_a]_{\tau-r}$ are well-defined. Assuming $\delta_1 \in \widehat{I}_{k_1}$ and $\delta_2 \in I_{k_2}$, those functions are given by (E.5) and (E.3), respectively. If

$\delta_1 + \delta_2 \in \widehat{I}_{k_1+k_2}$ then

$$\begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r+1} &= [\Delta_1]_{\tau+r+1} + [\Delta_2]_{\tau+r+1} = i(\delta_1+\delta_2) + k_1 + k_2 - \frac{1}{2} \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} + 1 \ = i(\delta_1+\delta_2) - k_1 - k_2 - \frac{1}{2} \end{aligned} \tag{E.14}$$

and

$$\begin{aligned} \Theta(\Delta_1, \Delta_2; \tau + r + 1) &= \frac{(i\delta_1 + k_1 - \frac{1}{2})(i\delta_2 + k_2)(2r + \frac{1}{2} - k_1 - k_2 + i(2t - \delta_1 - \delta_2))}{(r + \frac{1}{2} + it)^2} \\ \Theta(\Delta_1, \Delta_2; \tau - r) &= \frac{(i\delta_1 - k_1 + \frac{1}{2})(i\delta_2 - k_2)(-2r - \frac{1}{2} + k_1 + k_2 + i(2t - \delta_1 - \delta_2))}{(-r - \frac{1}{2} + it)^2} \ . \end{aligned} \tag{E.15}$$

If $\delta_1 + \delta_2 \in \widehat{I}_{k_1+k_2+1}$ then

$$\begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r+1} &= [\Delta_1]_{\tau+r+1} + [\Delta_2]_{\tau+r+1} + 1 = i(\delta_1+\delta_2) + k_1 + k_2 + \frac{1}{2} \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} \qquad\qquad = i(\delta_1+\delta_2) - k_1 - k_2 - \frac{3}{2} \end{aligned} \tag{E.16}$$

and

$$\begin{aligned} \Theta(\Delta_1, \Delta_2; \tau + r + 1) &= \frac{(i\delta_1 + k_1 + \frac{1}{2})(i\delta_2 + k_2 + 1)(2r + \frac{1}{2} - k_1 - k_2 + i(2t - \delta_1 - \delta_2))}{(r - \frac{1}{2} + it)^2} - 1 \\ \Theta(\Delta_1, \Delta_2; \tau - r) &= \frac{(i\delta_1 - k_1 - \frac{1}{2})(i\delta_2 - k_2 - 1)(-2r - \frac{1}{2} + k_1 + k_2 + i(2t - \delta_1 - \delta_2))}{(-r - \frac{1}{2} + it)^2} \ . \end{aligned} \tag{E.17}$$

If $\delta_1 + \delta_2 = (2n-1)t/(2r+1)$ with $n \in \mathbb{Z}$, then $[\Delta_1 + \Delta_2]_{\tau+r+1}$ and $[\Delta_1 + \Delta_2]_{\tau-r}$ are not defined. In both well-defined cases we find $\Theta(\Delta_1, \Delta_2; \tau - r) = -\overline{\Theta(\Delta_1, \Delta_2; \tau + r + 1)} - 1$, meaning that the two terms compete.

The subcase $y_1, y_2 < 0$ with $-1 < q < 0$

Finally, we consider both flavor fugacities to be negative and set

$$\Delta_a = -\frac{1}{2} + i\delta_a \ , \qquad \text{with } \delta_a \in \mathbb{R} \text{ and } a = 1, 2 \ . \tag{E.18}$$

For $r \geq 0$ and generic $\delta_a \in \widehat{I}_{k_a}$, the functions $[\Delta_a]_{\tau+r+1}$ and $[\Delta_a]_{\tau-r}$ are well-defined and given by (E.5). If $\delta_1 + \delta_2 \in I_{k_1+k_2-1}$ then

$$\begin{aligned} [\Delta_1 + \Delta_2]_{\tau+r+1} &= [\Delta_1]_{\tau+r+1} + [\Delta_2]_{\tau+r+1} = i(\delta_1+\delta_2) + k_1 + k_2 - 1 \\ [\Delta_1 + \Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r} + [\Delta_2]_{\tau-r} + 1 \ = i(\delta_1+\delta_2) - k_1 - k_2 \end{aligned} \tag{E.19}$$

and

$$\begin{aligned}\Theta(\Delta_1,\Delta_2;\tau+r+1) &= \frac{(i\delta_1+k_1-\frac{1}{2})(i\delta_2+k_2-\frac{1}{2})(2r+1-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r+\frac{1}{2}+it)^2}\\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{(i\delta_1-k_1+\frac{1}{2})(i\delta_2-k_2+\frac{1}{2})(-2r-1+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r-\frac{1}{2}+it)^2}\ .\end{aligned} \tag{E.20}$$

If $\delta_1+\delta_2 \in I_{k_1+k_2+1}$ then

$$\begin{aligned}[][\Delta_1+\Delta_2]_{\tau+r+1} &= [\Delta_1]_{\tau+r+1}+[\Delta_2]_{\tau+r+1}+1 = i(\delta_1+\delta_2)+k_1+k_2\\ [\Delta_1+\Delta_2]_{\tau-r} &= [\Delta_1]_{\tau-r}+[\Delta_2]_{\tau-r} \qquad\qquad = i(\delta_1+\delta_2)-k_1-k_2-1\end{aligned} \tag{E.21}$$

and

$$\begin{aligned}\Theta(\Delta_1,\Delta_2;\tau+r+1) &= \frac{(i\delta_1+k_1+\frac{1}{2})(i\delta_2+k_2+\frac{1}{2})(2r+1-k_1-k_2+i(2t-\delta_1-\delta_2))}{(r-\frac{1}{2}+it)^2}-1\\ \Theta(\Delta_1,\Delta_2;\tau-r) &= \frac{(i\delta_1-k_1-\frac{1}{2})(i\delta_2-k_2-\frac{1}{2})(-2r-1+k_1+k_2+i(2t-\delta_1-\delta_2))}{(-r-\frac{1}{2}+it)^2}\ .\end{aligned} \tag{E.22}$$

If $\delta_1+\delta_2 = n\,2t/(2r+1)$ with $n\in\mathbb{Z}$, then $[\Delta_1+\Delta_2]_{\tau+r+1}$ and $[\Delta_1+\Delta_2]_{\tau-r}$ are not defined. In both well-defined cases: $\Theta(\Delta_1,\Delta_2;\tau-r) = -\overline{\Theta(\Delta_1,\Delta_2;\tau+r+1)}-1$, meaning that the two terms compete.

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