Deformations with a Resonant Irregular Singularity

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Abstract I review topics of my talk in Alcalá, inspired by the paper [5]. An isomonodromic system with irregular singularity at $z=\infty$ (and Fuchsian at $z=0$) is considered, such that $z=\infty$ becomes resonant for some values of the deformation parameters. Namely, the eigenvalues of the leading matrix at $z=\infty$ coalesce along a locus in the space of deformation parameters. I give a complete extension of the isomonodromy deformation theory in this case.

1 Introduction

In these proceedings, I extract some of the main results of [5], which I have presented at the workshop in Alcalá, September 4-8, 2017. In [5] we have studied deformations of a class of linear differential systems when the eigenvalues of the leading matrix at $z=\infty$ coalesce along a locus in the space of deformation parameters. The above class contains, in particular, the $n\times n$ ($n\in\mathbb{N}$) system

$$\frac{dY}{dz}=A(z,t)Y, \qquad A(z,t)=\Lambda(t)+\frac{A_1(t)}{z}. \tag{1}$$

with singularity of Poincaré rank 1 at $z=\infty$. The matrices $\Lambda(t)$ and $A_1(t)$ are holomorphic functions of $t=(t_1,\dots,t_n)$ in a polydisc

$$\mathscr{U}_\varepsilon(0):=\{t\in\mathbb{C}^n \mid |t|\leq\varepsilon\}, \qquad |t|:=\max_{1\leq i\leq m}|t_i|, \tag{2}$$

in $\mathbb{C}^n$, centered at $t=0$. Here, $\Lambda(t)$ is diagonal

$$\Lambda(t):=\mathrm{diag}(u_1(t),\dots,u_n(t)). \tag{3}$$

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In these notes, I will consider only the case when the deformation is isomonodromic, and I refer to [5] for a more general discussion including the non-isomonodromic case.

In some important cases for applications to Frobenius manifolds (like quantum cohomology) and Painlevé equations, it may happen that the eigenvalues coalesce along a certain locus Δ in the t-domain, called the *coalescence locus*, where the matrix $\Lambda(t)$ *remains diagonal*. This means that $u_a(t) = u_b(t)$ for some indices $a \neq b \in \{1,...,n\}$ whenever t belongs to Δ, while $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ are pairwise distinct for $t \notin \Delta$. So, the point $z = \infty$ for $t \in \Delta$ is a *resonant irregular singularity*. I will assume that $A_1(t)$ is holomorphic at Δ, at least up to Theorem 1.

An isomonodromic system as above appears in the analytic approach to semisimple Frobenius manifolds [9] [10] [11], because its monodromy data allow to locally reconstruct the manifold structure (see also [15]). Coalescing eigenvalues arise in Frobenius manifolds *remaining semisimple at the locus of coalescent canonical coordinates*. An important example is the quantum cohomology of Grassmannians (see [3], [6]). For $n = 3$, a special case of system (1) gives an isomonodromic description of the general sixth Painlevé equation, according to [19], and also to [9] [11] for special values of coefficients. Coalescence occurs at critical points of the Painlevé equation, and $A_1(t)$ is holomorphic when the sixth Painlevé transcendents *remain holomorphic at a fixed singularity* of the Painlevé equation (see Section 2.1 below).

Unfortunately, the deformation with coalescence is "non-admissible", because it does not satisfy some of the assumptions of the isomonodromy deformation theory of Jimbo-Miwa-Ueno [18] [13]. Indeed, when t varies in a neighbourhood of the coalescence locus, several problems arise with the behaviour of fundamental matrix solutions and monodromy data. A theory when $\Lambda(t)$ remains diagonal at Δ seems to be missing from the literature (see [5] for a thorough review of the literature, while in these proceedings I have reduced the bibliography to a minimum, for lack of space). Therefore, for the sake of the applications mentioned above, in [5] we have developed a complete deformation theory in this case.

One of the main reasons for this extension of the theory is that we became interested in proving a conjecture formulated by Boris Dubrovin at the ICM 1998 in Berlin (see [10]). The qualitative part of the conjecture says that the quantum cohomology of a smooth projective variety (which is a Frobenius manifold) is semisimple if and only if there exists a full exceptional collection in derived categories of coherent sheaves on the variety. The quantitative part establishes an explicit relation between the monodromy data of the system (1) associated with the quantum cohomology, and certain quantities associated with objects of the exceptional collection. We started our investigation with the quantum cohomology of Grassmannians (for projective spaces, most of the work was done in [14], where there are no coalescences). The problem we had to face is that almost all Grassmannians are coalescent (the meaning of "almost all" is well explained in [3]), and the Frobenius structure, and thus the system (1), are known *only* at coalescence points. So, we can compute monodromy data only at a coalescence point. The question is if these data coincide with the locally constant data (the system must be isomonodromic) in a whole neighbourhood of the coalescence point, so with the data of the Frobenius manifold, as defined in [9][11]. The answer is positive, thanks to the main theorems of [5], which I expose in Section 2 below. As a result, we could prove the conjecture for Grassmannians, in [6] and [7]

The simplest differential system, illustrating our problem with non admissible deformations, is the following *Whittaker Isomonodromic System* (all details of the example are worked out in [8])

$$\frac{dY}{dz} = \left[\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{A_1(u)}{z} \right] Y. \tag{4}$$

Away from $\Delta = \{u_1 = u_2\}$, the system is isomonodromic if and only if

$$A_1(u) = \begin{pmatrix} a & c(u_1-u_2)^{-b} \\ d(u_1-u_2)^b & a-b \end{pmatrix}, \qquad a,b,c,d \in \mathbb{C}. \tag{5}$$

So, we see that for b, c and $d \neq 0$, the points of Δ are *branch points* ($b \notin \mathbb{Z}$) or *poles* ($b \in \mathbb{Z}$). This is what we must expect, following [20].

We leave it as an exercise to solve the system by a standard reduction to the Whittaker equation:

$$\frac{d^2w}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4}-\mu^2}{x^2} \right) w = 0, \quad \mu^2 := \frac{b^2+4cd}{4}, \quad \kappa := -\frac{1+b}{2}.$$

Notice that the eigenvalues $a + 1/2 + \kappa \pm \mu$ of A_1 are independent of u, as it must be in the isomonodromic case. The elements $Y_{11}(z)$ and $Y_{12}(z)$ of the first row of a fundamental matrix solution are obtained by taking two independent solutions $w_1(x)$ and $w_2(x)$ of the Whittaker equation, through the change of variables

$$Y_{1k}(z) = e^{\frac{1}{2}(u_1+u_2)z} z^{a-\frac{b+1}{2}} w_k(x), \quad x = z(u_1-u_2), \qquad k = 1,2.$$

If we use the asymptotic properties of Whittaker functions $W_{\kappa,\mu}(x)$, we can explicitly construct three fundamental solutions $Y_{-1}(z,u)$, $Y_0(z,u)$, $Y_1(z,u)$, which are asymptotic to the following formal solution

$$Y_F(z,u) = \left(I + \frac{F_1}{z} + \frac{F_2}{z^2} + \cdots \right) z^{\operatorname{diag}(A_1)} \begin{pmatrix} e^{u_1 z} & 0 \\ 0 & e^{u_2 z} \end{pmatrix},$$

for $z(u_1-u_2) \to \infty$ in the successive overlapping sectors

$$\mathscr{S}_{-1} := S\Big(-\frac{5\pi}{2}, -\frac{\pi}{2}\Big), \quad \mathscr{S}_0 := S\Big(-\frac{3\pi}{2}, \frac{\pi}{2}\Big), \quad \mathscr{S}_1 := S\Big(-\frac{\pi}{2}, \frac{3\pi}{2}\Big).$$

The matrix coefficients F_k are uniquely determined by the equation and depend holomorphically on $u \notin \Delta$. Moreover, one can do a further exercise and compute the Stokes matrices defined by the connection relations

$$Y_0(z)=Y_{-1}(z)\mathbb{S}_{-1},\quad Y_1(z)=Y_0(z)\mathbb{S}_0,\qquad \mathbb{S}_{-1}=\begin{pmatrix}1 & s_{-1}\\ 0 & 1\end{pmatrix},\quad \mathbb{S}_0=\begin{pmatrix}1 & 0\\ s_0 & 1\end{pmatrix},$$

The result is $s_0 =$

$$\frac{2\pi i}{c\,\Gamma\left(\frac{1}{2}+\kappa+\mu\right)\Gamma\left(\frac{1}{2}+\kappa-\mu\right)}=\frac{2\pi i}{c\,\Gamma\left(\frac{\sqrt{b^2+4cd}}{2}-\frac{b}{2}\right)\,\Gamma\left(-\frac{\sqrt{b^2+4cd}}{2}-\frac{b}{2}\right)},$$

and $s_{-1} =$

$$\frac{2\pi i\, c\, e^{-2\pi i\kappa}}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}=\frac{-2\pi i\, c\, e^{i\pi b}}{\Gamma\left(\frac{\sqrt{b^2+4cd}}{2}+1+\frac{b}{2}\right)\,\Gamma\left(-\frac{\sqrt{b^2+4cd}}{2}+1+\frac{b}{2}\right)}.$$

We notice that the sectors $\mathscr{S}_r$, $r=-1,0,1$ in the x-plane determine sectors in the z-plane which depend on $\arg(u_1-u_2)$. For example, $\mathscr{S}_1=S(-\frac{\pi}{2},\frac{3\pi}{2})$ gives

$$-\frac{\pi}{2}-\arg(u_1-u_2)<\arg z<-\arg(u_1-u_2)+\frac{3\pi}{2}$$

The boundaries of these z-sectors rotate with varying u. Also, the Stokes rays $\Re(z(u_1-u_2))=0$ rotate. Therefore, for u in some small open domain $\mathscr{V}$ of the (u_1,u_2)-plane, we can *fix* a z-sector of central opening angle greater than π, which is *independent of* $u\in\mathscr{V}$, and where the asymptotic behaviour holds. But if u varies too much outside $\mathscr{V}$, then the asymptotc behaviour will no longer hold in the previously fixed sector of the z-plane.

We do not have to worry about this problem only if the Stokes matrices are trivial. Triviality, namely $s_0=s_{-1}=0$, occurs if and only if one of the following conditions is satisfied

1) $c=d=0$ and $b\in\mathbb{C}$,
2) $cd=mn,\quad b=n-m$,
3) either $d=0$ and $b=-m$, or $c=0$ and $b=n$,

for $n\geq 1$ and $m\geq 1$ be integers. Now, we look back at the expression (5) for A_1 and immediately conclude that the first part of the following proposition holds.

Proposition 1. *If $s_{-1}=s_0=0$, then the points of Δ are not branch points, namely:*

1) $A_1(u)$ is single-valued for a loop $(u_1-u_2)\mapsto(u_1-u_2)e^{2\pi i}$ around the coalescence locus $u_1=u_2$;
2) the fundamental matrix solutions $Y_r(z,u)$, $r=-1,0,1$ are also single-valued.

The second part of the proposition requires some additional work with the monodromy properties of Whittaker functions at $x=0$, and we refer to [8]. Moreover, the following stronger converse statement holds (see [8] for the proof):

Proposition 2. *If $A_1(u)$ is holomorphic at Δ and both $(A_1)_{12}$ and $(A_1)_{21}$ vanish as $u_1-u_2\to 0$, then the $Y_r(z,u)$'s are single-valued in u_1-u_2 and holomorphic at Δ. Moreover, the Stokes matrices have entries $s_{-1}=s_0=0$.*

In conclusion, the example teaches us three things

- the locus Δ is in general of *branch points* for both A_1 and the fundamental matrix solutions. Also the coefficients F_k of the formal solution may have poles or branch points at Δ. Moreover, for a fundamental matrix solution, the canonical asymptotic behaviour for $z\to\infty$ holds in a *fixed* (big) sector of central opening angle greater than π in the z-plane provided that u varies in a sufficiently small domain $\mathscr{V}$ of the u-space. The asymptotics in the fixed sector is lost otherwise (precisely, when u goes around Δ).
- if the entries of the Stokes matrices, with indices corresponding to those of the coalescing eigenvalues, vanish a Δ, then the points of Δ are not branch points. This exemplifies one main result of [5], which is Theorem 2 below.
- if $A_1(t)$ is holomorphic in a domain containing Δ, and if its entries, with indices corresponding to those of the coalescing eigenvalues, vanish at Δ, then the fundamental matrix solutions are holomorphic also at Δ and the entries (as above) of the Stokes matrices vanish. This exemplifies another main result of [5], namely Theorem 1 below.

2 Main Results

No loss of generality occurs if we assume that $t=0$ is a coalescence point in the polydisc (2). In the isomonodromic case, it is known [18] that we can take the eigenvalues of $\Lambda(t)$ to be the deformation parameters, as I did in the previous example. Hence, we can assume that the eigenvalues $u_1(t),...,u_n(t)$ are linear in t:

$$u_a(t)=u_a(0)+t_a,\quad 1\le a\le n. \tag{6}$$

Therefore,

$$\Lambda(t)=\Lambda(0)+\mathrm{diag}(t_1,...,t_n),$$

where $\Lambda(0)$ has $s<n$ distinct eigenvalues of multiplicities p_1, ..., p_s respectively, so that $p_1+\cdots+p_s=n$. In this case, Δ is a union of hyperplanes.

Assume that $A_1(t)$ is holomorphic in $\mathscr{U}_\varepsilon(0)$, so that for $t\notin\Delta$ there is a unique formal solution

$$Y_F(z,t):=\Big(I+\sum_{k=1}^{\infty}F_k(t)z^{-k}\Big)z^{\mathrm{diag}(A_1(t))}e^{\Lambda(t)z}, \tag{7}$$

where the matrices $F_k(t)$ are uniquely determined by the equation and are holomorphic on $\mathscr{U}_\varepsilon(0)\backslash\Delta$. The well-known result of [16] states that, if t varies in a sufficiently small domain of $\mathscr{U}_\varepsilon(0)\backslash\Delta$ (actually, very small in [16]), there exists a

t-independent sector and a fundamental matrix solution whose asymptotic representation is Y_F. But if t varies too much in $\mathscr{U}_\varepsilon(0)\backslash\Delta$, namely goes around Δ, this is no longer true, and the reason is that the Stokes rays

$$\{z\in\mathscr{R}\mid \Re e[(u_a(t)-u_b(t))z]=0\},\qquad \mathscr{R}:=\text{ universal covering of }\mathbb{C}\backslash\{0\},$$

associated with $\Lambda(t)$ rotate with t varying.

To be more precise, suppose we have fixed $t\notin\Delta$, so Stokes rays are frozen. We can consider a half plane $\Pi_1:=\{z\in\mathscr{R}\mid \tau-\pi<\arg z<\tau\}$, having chosen τ so that no Stokes rays associated with $\Lambda(t)$ have the direction τ (and so $\tau+k\pi$, $k\in\mathbb{Z}$). Then, we consider a big sector $\mathscr{S}_1(t):=$ the open sector containing Π_1 and extending up to the closest Stokes rays of $\Lambda(t)$ outside Π_1. Then, there is a unique fundamental solution $Y_1(z,t)\sim Y_F(z,t)$ for $z\to\infty$ in $\mathscr{S}_1(t)$ (this follows from [1]). Just to fix τ once and for all, we choose it so that, in particular, no Stokes rays associated with $\Lambda(0)$ lie on the ray $\arg z=\tau$.

Now, let t vary, so that Stokes rays start to rotate. First, we let t vary only inside a domain $\mathscr{V}$, or better in its closure $\overline{\mathscr{V}}$, sufficiently small that no Stokes rays cross the direction τ. This is what an *admissible deformation* is. We can take $\mathscr{S}_1(\mathscr{V}):=\bigcap_{t\in\overline{\mathscr{V}}}\mathscr{S}_1(t)$. This sector has central opening angle greater than π, by construction. We conclude (and we prove in [5]) that the unique fundamental solution $Y_1(z,t)$ has asymptotic expansion $Y_F(z,t)$ for $z\to\infty$ in $\mathscr{S}_1(\mathscr{V})$ and $t\in\overline{\mathscr{V}}$.

We can repeat the construction of a family of actual solutions $Y_r(z,t)$, $r\in\mathbb{Z}$, having the asymptotic representation $Y_F(z,t)$ in big sectors $\mathscr{S}_r(\mathscr{V})$ constructed as above, starting from the half planes $\Pi_r:=\{z\in\mathscr{R}\mid \tau+(r-2)\pi<\arg z<\tau+(r-1)\pi\}$. Notice that $\mathscr{S}_r(\mathscr{V})\cap\mathscr{S}_{r+1}(\mathscr{V})\neq\emptyset$. This allows to define for $t\in\mathscr{V}$ the Stokes matrices $\mathbb{S}_r(t)$ by the following relations

$$Y_{r+1}(z,t)=Y_r(z,t)\,\mathbb{S}_r(t). \tag{8}$$

On the other hand, if t varies too much, leaving $\mathscr{V}$, Stokes rays may cross $\arg z=\tau$. Since the dominance relations, which determine the change of asymptotics, depend on the behaviour of the exponents $\exp\{z(u_a-u_b)\}$, we see that when a ray $\Re e[(u_a(t)-u_b(t))z]=0$ has crossed $\arg z=\tau$, the asymptotic relation $Y_r(z,t)\sim Y_F(z,t)$ for $z\to\infty$ in $\mathscr{S}_r(\mathscr{V})$ generally must fail. Maybe, it may not fail if $(\mathbb{S}_r)_{ab}=(\mathbb{S}_r)_{ba}=0$. This is what actually happens.

The Stokes rays cross $\arg z=\tau$ for t along a certain locus in the polydisc, that we call $X(\tau)$. From the above discussion, it is clear that everything is nice in $\mathscr{U}_\varepsilon(0)\backslash(\Delta\cup X(\tau))$. In [5]) we have proved that $\Delta\cup X(\tau)$ is a union of real hyperplanes, which disconnect $\mathscr{U}_\varepsilon(0)$. Every connected component is simply connected and homeomorphic to a ball in $\mathbb{R}^{2n}$. Thus, it is a cell in the topological sense, so we call it *a τ-cell*. Summarising, we have the following general facts:

i) The deformation is called *admissible in $\mathscr{V}$* if t varies in a domain $\mathscr{V}\subset\mathscr{U}_\varepsilon(0)$, such that its closure $\overline{\mathscr{V}}$ is properly contained in a τ-cell.
ii) If $t\in\mathscr{V}$, there is a family of actual fundamental matrix solutions $Y_r(z,t)$, $r\in\mathbb{Z}$, uniquely determined by the canonical asymptotic representation $Y_r(z,t)\sim$

$Y_F(z,t)$, for $z \to \infty$ in sectors $\mathscr{S}_r(\mathscr{V})$. Each $Y_r(z,t)$ is holomorphic within $\mathscr{R}$ for large $|z|$, and in $t \in \mathscr{V}$. The asymptotic series $I + \sum_{k=1}^{\infty} F_k(t) z^{-k}$ is uniform in $\overline{\mathscr{V}}$.

Moreover, we have the following problems, to be solved in Theorem 1 below:

iii) When t crosses $X(\tau)$ and leaves the cell of $\mathscr{V}$, which means that some Stokes ray $\Re e\,[(u_a(t) - u_b(t))z] = 0$, associated with $\Lambda(t)$, cross the admissible direction τ, then the asymptotic representation $Y_r(z,t) \sim Y_F(z,t)$ for $z \to \infty$ in $\mathscr{S}_r(\mathscr{V})$ does no longer hold.
iv) The locus Δ is expected to be a locus of poles or branch points for the coefficients $F_k(t)$ and for the $Y_r(z,t)$'s.
v) The *Stokes matrices* $\mathbb{S}_r(t)$ in (8) are expected to diverge as t approaches Δ.

Notice that in order to completely describe the Stokes phenomenon, it suffices to consider only three fundamental matrix solutions, for example $Y_r(z,t)$ for $r = 1,2,3$, and $\mathbb{S}_1(t)$, $\mathbb{S}_2(t)$ (this has been done in example, with $r = -1,0,1$ and $\mathbb{S}_{-1}$ and $\mathbb{S}_0$).

To complete the general picture, we will define the monodromy data. For given t, a matrix $G(t)$ puts $A_1(t)$ in Jordan form $J(t) := G^{-1}(t)\, A_1(t)\, G(t)$. For a given t, the system (1) has a fundamental solution represented in Levelt form

$$Y^{(0)}(z,t) = G(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big) z^{D(t)} z^{S(t)+R(t)}. \tag{9}$$

in a neighbourhood of $z = 0$. The matrix coefficients $\Psi_l(t)$ of the convergent expansion are constructed by a recursive procedure. $J(t) = D(t) + S(t)$, where $D(t) = \mathrm{diag}(d_1(t), \ldots, d_n(t))$ is a matrix of integers, piecewise constant in t, $S(t)$ is a Jordan matrix whose eigenvalues have real part in $[0,1[$. The nilpotent matrix $R(t)$ has non-vanishing entries only if some eigenvalues of $A_1(t)$ differ by non-zero integers. It is proved in Theorem 1 that the solution (9) turns out to be holomorphic in $t \in \mathscr{U}_\varepsilon(0)$. Chosen $Y^{(0)}(z,t)$, a *central connection matrix* $C^{(0)}$ is defined by the relation

$$Y_1(z,t) = Y^{(0)}(z,t)\, C^{(0)}(t), \qquad z \in \mathscr{S}_1(\mathscr{V}). \tag{10}$$

The *essential* monodromy data (the name is inspired by a similar definition in [18]) are then

$$\mathbb{S}_1(t), \quad \mathbb{S}_2(t), \quad \mathrm{diag}(A_1(t)), \quad C^{(0)}(t), \quad J(t), \quad R(t). \tag{11}$$

Now, when t tends to a point $t_\Delta \in \Delta$, the limits of the above data may not exist. If the limits exist, they *do not in general give the monodromy data of the system with matrix* $A(z,t_\Delta)$ (see [2], [5]).

Definition 1. An admissible deformation in a small domain $\mathscr{V}$ is **isomonodromic in** $\mathscr{V}$ if the essential monodromy data (11) do not depend on $t \in \mathscr{V}$.

For admissible isomonodromic deformations as defined in Definition 1, the classical theory of Jimbo-Miwa-Ueno [18] applies in $\mathscr{V}$. In Theorem 1 and Corollary 1 below we have extended the theory to the whole $\mathscr{U}_\varepsilon(0)$, including the coalescence locus Δ. In the statement of the theorem, we will not explain how small ε is, since this is a little bit technical point (see [5] and [8]). We also skip the construction of new sectors $\widehat{\mathscr{S}_r}(t)$ and $\widehat{\mathscr{S}_r} = \bigcap_{t\in\mathscr{U}_\varepsilon(0)} \widehat{\mathscr{S}_r}(t)$, which appear in the theorem. They are bigger than $\mathscr{S}_r(\mathscr{V})$, namely $\mathscr{S}_r(\mathscr{V}) \subset \widehat{\mathscr{S}_r} \subset \widehat{\mathscr{S}_r}(t)$, with $t \in \mathscr{U}_\varepsilon(0)$.

Theorem 1. *Consider the system (1), with eigenvalues of $\Lambda(t)$ linear in t as in (6), and with $A_1(t)$ holomorphic on a closed polydisc $\mathscr{U}_\varepsilon(0)$ centred at $t = 0$, with sufficiently small radius ε (as specified in [5]). Let Δ be the coalescence locus in $\mathscr{U}_\varepsilon(0)$, passing through $t = 0$. Let the deformation be isomonodromic in $\mathscr{V} \subset \mathscr{U}_\varepsilon(0)$ as in Definition 1. If the matrix entries of $A_1(t)$ satisfy in $\mathscr{U}_\varepsilon(0)$ the vanishing conditions*

$$\Big(A_1(t)\Big)_{ab} = O(u_a(t) - u_b(t)), \qquad 1 \le a \ne b \le n, \tag{12}$$

whenever $u_a(t)$ and $u_b(t)$ coalesce as t tends to a point of Δ, then the following results hold:

• *The coefficients $F_k(t)$ of the formal solution $Y_F(z,t)$ in (7) are holomorphic on the whole $\mathscr{U}_\varepsilon(0)$.*

• *The three fundamental matrix solutions $Y_r(z,t)$, $r = 1,2,3$, initially defined on $\mathscr{V}$, with asymptotic representation $Y_F(z,t)$ for $z \to \infty$ in the sectors $\mathscr{S}_r(\mathscr{V})$ introduced above, can be t-analytically continued as single-valued holomorphic functions on $\mathscr{U}_\varepsilon(0)$, with asymptotic representation*

$$Y_r(z,t) \sim Y_F(z,t) \quad \text{for } z \to \infty \text{ in wider sectors } \widehat{\mathscr{S}_r},$$

for any $t \in \mathscr{U}_{\varepsilon_1}(0)$ and any $0 < \varepsilon_1 < \varepsilon$. In particular, they are defined at any $t_\Delta \in \Delta$ with asymptotic representation $Y_F(z,t_\Delta)$. The fundamental matrix solution $Y^{(0)}(z,t)$ is also holomorphic on $\mathscr{U}_\varepsilon(0)$

• *The constant matrices $\mathbb{S}_1$, $\mathbb{S}_2$, and $C^{(0)}$, initially defined for $t \in \mathscr{V}$, are actually* globally defined on $\mathscr{U}_\varepsilon(0)$. *They coincide with the Stokes and connection matrices of the fundamental solutions $Y_r(z,0)$ and $Y^{(0)}(z,0)$ of the system*

$$\frac{dY}{dz} = A(z,0)Y, \qquad A(z,0) = \Lambda(0) + \frac{A_1(0)}{z}. \tag{13}$$

Also the remaining t-independent monodromy data in (11) coincide with those of (13).

• *The entries (a,b) of the Stokes matrices are characterised by the following vanishing property whenever $u_a(0) = u_b(0)$, $1 \le a \ne b \le n$:*

$$(\mathbb{S}_1)_{ab} = (\mathbb{S}_1)_{ba} = (\mathbb{S}_2)_{ab} = (\mathbb{S}_2)_{ba} = 0. \tag{14}$$

Thus, under the only condition (12), we have no more problems with rotating Stokes rays, with loss of asymptotic representation in fixed big sectors, and with the appearance of branch points at Δ. I think that this is a remarkable fact.

It is now time to explain when and how system (13) suffices to compute the monodromy data of (1). Let the assumptions of Theorem 1 hold. Then, (13) has a formal solution

$$\mathring{Y}_F(z) = \Big(I + \sum_{k=1}^{\infty} \mathring{F}_k z^{-k}\Big) z^{\mathrm{diag}(A_1(0))} e^{\Lambda(0) z}. \tag{15}$$

Actually, there is a family of formal solutions (15): the coefficients $\mathring{F}_k$ can be recursively constructed from the differential system, but *there is not a unique choice for them*. To each element of the family there correspond unique actual solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$ such that $\mathring{Y}_r(z) \sim \mathring{Y}_F(z)$ for $z \to \infty$ in a sector $\mathscr{S}_r \supset \mathscr{S}_r(\mathscr{V})$, $r = 1,2,3$, with Stokes matrices defined by

$$\mathring{Y}_{r+1}(z) = \mathring{Y}(z)\, \mathring{\mathbb{S}}_r, \qquad r = 1,2.$$

Notice that only one element of the family of formal solutions (15) satisfies the condition $\mathring{F}_k = F_k(0)$ for any $k \geq 1$, so that $\mathbb{S}_r = \mathring{\mathbb{S}}_r$.

To complete the picture, let us also choose a fundamental matrix solution $\mathring{Y}^{(0)}(z)$ of (13), in Levelt form in a neighbourhood of $z = 0$, and define the corresponding central connection matrix $\mathring{C}^{(0)}$ such that $\mathring{Y}_1(z) = \mathring{Y}^{(0)}(z)\, \mathring{C}^{(0)}$. The following holds

Corollary 1. *Let the assumptions of Theorem 1 hold. If the diagonal entries of $A_1(0)$ do not differ by non-zero integers, then there is a unique formal solution (15) of the system (13), whose coefficients necessarily satisfy the condition*

$$\mathring{F}_k \equiv F_k(0).$$

Hence, the corresponding fundamental matrix solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$ of are such that

$$Y_1(z,0) = \mathring{Y}_1(z), \quad Y_2(z,0) = \mathring{Y}_2(z), \quad Y_3(z,0) = \mathring{Y}_3(z).$$

Moreover, for any $\mathring{Y}^{(0)}(z)$ there exists $Y^{(0)}(z,t)$ such that $Y^{(0)}(z,0) = \mathring{Y}^{(0)}(z)$. The following equalities hold:

$$\mathbb{S}_1 = \mathring{\mathbb{S}}_1, \quad \mathbb{S}_2 = \mathring{\mathbb{S}}_2, \quad C^{(0)} = \mathring{C}^{(0)}.$$

Corollary 1 has a practical computational importance: the *constant monodromy data* (11) of the system (1) on the whole $\mathscr{U}_\varepsilon(0)$ are computable just by considering the system (13) at the coalescence point $t = 0$. This is useful for applications. For example, it allows to compute the monodromy data of a semisimple Frobenius manifold, such as the quantum cohomology of Grassmannians [3], [6] mentioned in the Introduction, just by considering the Frobenius structure at a coalescence point.

In [5], we also prove the (weaker) converse of Theorem 1. Assume that the deformation is admissible and isomonodromic on a simply connected domain $\mathscr{V} \subset \mathscr{U}_\varepsilon(0)$, and that $A_1(t)$ is holomorphic (only) in $\mathscr{V}$. As a result of [20], the fundamental matrix solutions $Y_r(z,t)$, $r = 1,2,3$, and $A_1(t)$ can be analytically continued as *multi-valued functions* on $\mathscr{U}_\varepsilon(0) \backslash \Delta$, with movable poles. Nevertheless, if the vanishing condition (14) holds, then Δ *does not contain branch points* and the *asymptotic behaviour is preserved on big sectors*, according to the following

Theorem 2. *Let $A_1(t)$ be holomorphic on an open simply connected domain $\mathscr{V} \subset \mathscr{U}_\varepsilon(0)$, where the deformation is admissible and isomonodromic as in Definition 1. Let ε be sufficiently small (as specified in [5]). If*

$$(\mathbb{S}_1)_{ab} = (\mathbb{S}_1)_{ba} = (\mathbb{S}_2)_{ab} = (\mathbb{S}_2)_{ba} = 0 \quad \textit{whenever } u_a(0) = u_b(0), \quad 1 \le a \ne b \le n,$$

then, the fundamental matrix solutions $Y_r(z,t)$ and $A_1(t)$ admit single-valued analytic continuation on $\mathscr{U}_\varepsilon(0)\backslash\Delta$ *as meromorphic functions of t. Moreover, for any $t \in \mathscr{U}_\varepsilon(0)\backslash\Delta$ which is not a pole of $Y_r(z,t)$ we have*

$$Y_r(z,t) \sim Y_F(z,t) \ \textit{for } z \to \infty \textit{ in } \widehat{\mathscr{S}_r}(t), \quad r = 1,2,3,$$

and $Y_{r+1}(z,t) = Y_r(z,t)\,\mathbb{S}_r$, $r = 1,2$. The sectors $\widehat{\mathscr{S}_r}(t)$ are described in [5].

2.1 Applications

We have no space to explain the applications of Theorem 1 and Corollary 1 to Frobenius manifolds (see [6]). As for Painlevé equations, they provide an alternative to Jimbo's approach for the computation of the monodromy data associated with Painlevé VI transcendents holomorphic at a critical point. As an example, we consider the A_3-algebraic solution of Dubrovin-Mazzocco [12]

$$y(s) = \frac{(1-s)^2\,(1+3s)\,(9s^2-5)^2}{(1+s)\,(243s^6+1539s^4-207s^2+25)}, \qquad t(s) = \frac{(1-s)^3\,(1+3s)}{(1+s)^3\,(1-3s)}, \quad (16)$$

with $s \in \mathbb{C}$, which solves the Painlevé VI equation

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right]\left(\frac{dy}{dt}\right)^2 - \left[\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right]\frac{dy}{dt} + \\ + \frac{1}{2}\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\frac{9}{4}+\frac{t(t-1)}{(y-t)^2}\right].$$

The above equation is the isomonodromicity condition for a 3×3 system

$$\frac{dY}{dz} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A_1(t)}{z}\right] Y, \qquad A_1(t) =: \begin{pmatrix} 0 & \Omega_2 & -\Omega_3 \\ -\Omega_2 & 0 & \Omega_1 \\ \Omega_3 & -\Omega_1 & 0 \end{pmatrix}; \qquad (17)$$

$$\Omega_1 = i\frac{\sqrt{y-1}\sqrt{y-t}}{\sqrt{t}}\left[\frac{A}{(y-1)(y-t)}+\mu\right], \;\; \Omega_2 = i\frac{\sqrt{y}\sqrt{y-t}}{\sqrt{1-t}}\left[\frac{A}{y(y-t)}+\mu\right],$$

$$\Omega_3 = -\frac{\sqrt{y}\sqrt{y-1}}{\sqrt{t}\sqrt{1-t}}\left[\frac{A}{y(y-1)}+\mu\right],\ A := \frac{1}{2}\left[\frac{dy}{dt}t(t-1)-y(y-1)\right].$$

The above formulae are in [15]. A holomorphic branch is obtained by letting $s \to -\frac{1}{3}$ in (16), which gives convergent Taylor expansions

$$\Omega_1(t) = \frac{i\sqrt{2}}{8} - \frac{i\sqrt{2}t}{256} + O(t^2), \quad \Omega_3(t) = \frac{i\sqrt{2}}{8} + \frac{i\sqrt{2}t}{256} + O(t^2),$$
$$\Omega_2(t) = -\frac{t}{32} + O(t^2).$$

Since $\lim_{t\to 0}\Omega_2(t) = 0$, Theorem 1 holds. Since $\mathrm{diag}(A_1) = (0,0,0)$, also Corollary 1 holds. Accordingly, the Stokes matrices can be computed using (17) at fixed $t = 0$, which is integrable by reduction to a Bessel equation. Thus, its Stokes matrices can be computed:

$$\mathbb{S}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbb{S}_2 = \mathbb{S}_1^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

This result is in accordance with [12]. Notice that $(\mathbb{S}_r)_{12} = (\mathbb{S}_r)_{21} = 0$, as Theorem 1 predicts.

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