Scuola Internazionale Superiore di Studi Avanzati - Trieste

The Abdus Salam International Center for Theoretical Physics
International School for Advanced Studies
Joint Ph.D Programme

Refinement of Kool-Thomas Invariants via Equivariant K-theoretic Invariants

Candidate:
Rizal Afgani

Supervisor:
Lothar Göttsche

Thesis submitted in partial fulfilment of the requirements for the degree of Philosophi Doctor in Geometry and Mathematical Physics

Academic Year 2017/2018

SISSA - Via Bonomea 265 - 34136 TRIESTE - ITALY

Refinement of Kool-Thomas invariants via equivariant K-theoretic invariants

Candidate
Rizal Afgani

Supervisor
Lothar Göttsche

Thesis submitted in partial fulfilment of the requirements for the degree of Philosophiæ Doctor in Geometry and Mathematical Physics

Scuola Internazionale Superiore di Studi Avanzati - Trieste
Academic Year 2017/2018

Abstract

In this thesis we are defining a refinemement of Kool-Thomas invariants of local surfaces via the equivariant K-theoretic invariants proposed by Nekrasov and Okounkov. Kool and Thomas defined the reduced obstruction theory for the moduli of stable pairs $\mathcal{P}_\chi(X, i_*\beta)$ as the degree of the virtual class $[\mathcal{P}_\chi(S,\beta)]^{red}$ afted we apply $\tau([pt])^m \in H^*(\mathcal{P}_\chi(X, i_*\beta), \mathbb{Z})$. $\tau([pt])$ contain the information of the incidence of a point and a curve supporting a $(\mathcal{F}, s)$.

The K-theoretic invariants proposed by Nekrasov and Okounkov is the equivariant holomorphic Euler characteristic of $\mathcal{O}^{vir}_{\mathcal{P}_\chi(X,i_*\beta)} \otimes K^{\frac{1}{2}}_{vir}$. We introduce two classes $\gamma(\mathcal{O}_s)$ and $\bar{\gamma}(\mathcal{O}_s)$ in the Grothendieck group of vector bundles on the moduli space of stable pairs of the local surfaces that contains the information of the incidence of a curve with a point.. Let $\mathcal{P} = \mathcal{P}_\chi(X, i_*\beta)$. By the virtual localization formula the equivariant K theoretic invariant is then

$$P_{X,\beta,\chi}(s_1,\dots,s_m) := R\Gamma\left(\mathcal{P}^G, \frac{\mathcal{O}^{vir}_{\mathcal{P}^G} \otimes K^{\frac{1}{2}}_{vir}\Big|_{\mathcal{P}^G}}{\Lambda^\bullet_{-1}\left(N^{vir}\right)^\vee} \otimes \prod_{i=1}^m \gamma\left(\mathcal{O}_{s_i}\right)\Big|_{\mathcal{P}^G}\right)$$

and

$$\bar{P}_{X,\beta,\chi}(s_1,\dots,s_m) := R\Gamma\left(\mathcal{P}^G, \frac{\mathcal{O}^{vir}_{\mathcal{P}^G} \otimes K^{\frac{1}{2}}_{vir}\Big|_{\mathcal{P}^G}}{\Lambda^\bullet_{-1}\left(N^{vir}\right)^\vee} \otimes \prod_{i=1}^m \bar{\gamma}(\mathcal{O}_{s_i})\Big|_{\mathcal{P}^G}\right).$$

We found that the contribution of $\mathcal{P}_\chi(S,\beta) \subset \mathcal{P}^G$ to $P_{X,\beta,\chi}(s_1,\dots,s_m)$ and to $\bar{P}_{X,\beta,\chi}(s_1,\dots,s_m)$ are the same. Moreover, if we evaluate this contribution at $\mathfrak{t} = 1$ we get the Kool-Thomas invariants.

The generating function of this contribution contain the same information as the generating function of the refined curve counting invariants defined by Göttsche and Shende in [12]. After a change of variable there exist a coefficient $N^\delta_{\delta[S,\mathcal{L}]}(y)$ of the generating function of the refined curve counting that counts the number of δ-nodal curve in $\mathbb{P}^\delta \subset |\mathcal{L}|$. We conjecture that after the same change of variable the corresponding coefficient $M^\delta_{\delta[S,\mathcal{L}]}(y)$ coming from the generating function of the controbution of $\mathcal{P}_\chi(S,\beta)$ to $\bar{P}_{X,\beta,\chi}(s_1,\dots,s_m)$ is identical with $N^\delta_{\delta[S,\mathcal{L}]}(y)$.

Keywords: Kool-Thomas invariants, K-theoretic invariants, Göttsche Shende invariants

Contents

Introduction

Fix a nonsingular projective surface S and a sufficently ample line bundle $\mathcal{L}$ on S. A δ-nodal curve C on S is a 1 dimensional subvariety of S which has nodes at δ points and is regular outside these singular points. For any scheme Y, let $Y^{[n]}$ be the Hilbert scheme of n-points i.e. $Y^{[n]}$ parametrizes subschemes $Z \subset Y$ of length n. Given a family of curves $\mathcal{C} \to B$ over a base B, we denote by $\mathrm{Hilb}^n(\mathcal{C}/B)$ the relative Hilbert scheme of points. Kool, Thomas and Shende showed that some linear combinations $n_{r,C}$ of the Euler characteristic of $C^{[n]}$ counts the number of curves of arithmetic genus r mapping to C. Applying this to the family $\mathcal{C} \to \mathbb{P}^\delta$ where $\mathbb{P}^\delta \subset |\mathcal{L}|$, the number of δ-nodal curves is given by a coefficient of the generating function of the Euler characteristic of $\mathrm{Hilb}(\mathcal{C}/\mathbb{P}^\delta)$ after change of variable[18]. By replacing euler characteristic with Hirzebruch χ_y-genus, Götsche and Shende give a refined counting of δ-nodal curves.

Pandharipande and Thomas showed that a stable pair $(\mathcal{F}, s)$ on a surface S is equivalent to the pair (C, Z) of a curve C on S supporting the sheaf $\mathcal{F}$ with $Z \subset C$ a subscheme of finite length. Thus the moduli space of stable pairs on a surface S is a relative Hilbert scheme of points corresponding to a family of curves on S.

The study of the moduli space of stable pairs on Calabi-Yau threefold Y is an active area of research. This moduli space gives a compactification of the moduli space of nonsingular curves in Y. To get an invariant of the moduli space Behrend and Fantechi introduce the notion of perfect obstruction theory. With this notion we can construct a class in the Chow group of dimension 0 that is invariant under some deformations of Y[1].

The homological invariants of the stable pair moduli space $\mathcal{P}_\chi(X, i_*\beta)$ of the total space X of K_S of some smooth projective surface S contain the information of the number of δ-nodal curves in a hyperplane $\mathbb{P}^\delta \subset |\mathcal{L}|$. Notice that X is Calabi-Yau. There exist a morphism of schemes $\operatorname{div} : \mathcal{P}_\chi(X, i_*\beta) \to |\mathcal{L}|$ that maps a point $(\mathcal{F}, s) \in \mathcal{P}_\chi(X, i_*\beta)$ to a divisor $\operatorname{div}(\pi_*\mathcal{F})$ that support $\pi_*\mathcal{F}$ on S where $\pi : X \to S$ is the structure morphism of X as a vector bundle over S. Using descendents, Kool and Thomas translate the information of the incidence of a curve with a point into cutting down the moduli space by a hypersurface pulledback from $|\mathcal{L}|$ so that after cutting down, we have a moduli space that parameterize Hilbert scheme of curves in $\mathbb{P}^\delta$[19].

The famous conjecture of Maulik, Nekrasov Okounkov and Pandharipande states that the invariants corresponding to the moduli space of stable pairs have the same information as the invariants defined from the moduli space of stable maps and the Hilbert schemes.

The next development in the theory of PT invariants is to give a refinement of the homological invariant. The end product of this homological invariant is a number. A refinement of this invariant would be a Laurent polynomial in a variable t such that when we evaluate t at 1 we get the homological invariant.

There are several methods that have been introduced to give a refinement for DT invariants, for example both motivic and K-theoretic definitions. In this thesis we use the K-theoretic definition which has been proposed by Nekrasov and Okounkov in [23] where we compute the holomorphic Euler characteristic of the twisted virtual structure sheaf of the coresponding moduli space. In the case when $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$ Choi, Katz and Klemm have computed a K-theoretic invariant of the moduli space of stable pairs in the paper [2]. Their computation does not include any information about the incidence of subschemes of S.

In this thesis we will use K-theoretic invariants to define a refinement of the Kool-Thomas invariant in [19]. To do this we introduce the incidence class in $K^G(\mathcal{P}_\chi(X, i_*\beta))$ that will give the information of the incidence of a curve with a point

Here is a summary of this thesis

In Chapter 1 we review some equivariant algebraic geometry that we need in this thesis. In Section 1 we review the definition of equivariant sheaves and principal G-bundles. In Section 2 we review an equivariant version of Chow groups by Graham and Edidin[5]. In Section 3 we review the Grothendieck group of equivariant coherent sheaves and equivariant vector bundles. In Section 4, we are trying to describe a parallelization between the construction of equivariant Chow groups and equivariant K-theory.

In Chapter 2 we review the moduli of stable pair and stable pair invariants defined via virtual fundamental class. We also review the reduced obstruction theory on the moduli of stable pairs. Kool-Thomas invariants are defined using the class constructed using reduced deformation theory.

In chapter 3 we review the definition of K-theoretic invariants proposed by Nekrasov and Okounkov and we also introduce the incidence class. We apply the K-theoretic invariants to the moduli space of stable pairs on K_S.

In chapter 4 we collect the results of our work which are Theorem 4.3.1 and 4.3.2. In Theorem 4.3.2 we compute the contribution of $\mathcal{P}_\chi(S, \beta)$ in the K-theoretic invariants of the moduli space of stable pairs on K_S. In Theorem 4.3.1 we show that this contribution gives a refinement of Kool-Thomas invariants. We also conjecture that our refinement coincide with the refinement defined by Göttsche and Shende.

Acknowledgements

This thesis ends a long period of study I spent at SISSA. It is my pleasure to thank everybody for their support and for what I have learned. Special thanks go to my supervisor Lothar Göttsche for introducing me to this topics and for his support during my PhD. Also special thanks go to Martijn Kool for explaining his paper which I use as the basis of this thesis and also for suggesting problems that I could work on.

Chapter 1

Equivariant algebraic geometry

In this chapter we will review some basic materials concering equivariant K-theory and equivariant intersection theory. For equivariant intersection theory we use [4, 5] as references. And for equivariant K-theory our references are [35, 17, 32] and chapter V of [3].

A group scheme G is a scheme with multiplication map $\mu : G \times G \to G$, inverse $\nu : G \to G$ and identity element $e : Spec\,\mathbb{C} \to G$ satisfying the usual axiom of groups, e.g. associative etc. An example of a group scheme is a torus T_n of dimension n which is defined as the $Spec$ of $R_n := \mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ with multiplication $\mu : T_n \times T_n \to T_n$ defined by $\mu^\natural : R_n \to R_n \otimes_{\mathbb{C}} R_n,\ t_i \mapsto t_i \otimes t_i$, inverse map $\nu : T_n \to T_n$ is defined by $t_i \mapsto t_i^{-1}$ and the identity element $e : Spec\,\mathbb{C} \to T_n$ is defined by $t_i \mapsto 1$. The set of $\mathbb{C}$-valued points of T_n is then $(\mathbb{C}^\times)^n$.

A morphism $\sigma : G \times X \to X$ defines an action of G on X if it satisfies $(\mathrm{id}_G \times \sigma) \circ \sigma = (\mu \times \mathrm{id}_X) \circ \sigma$ and $(e \times \mathrm{id}_X) \circ \sigma = \mathrm{id}_X$. For example, μ defines an action of G on G. If G acts on X we call X a G-scheme. Note that $\sigma_X$and pr_X are flat morphism. Let σ_X and σ_Y define actions of G on X and Y. A morphism $f : X \to Y$ is called a G-equivariant morphism (or G-morphism) if $f \circ \sigma_X = \sigma_Y \circ (\mathrm{id}_G \times f)$. If f is an isomorphism we will say f is a G-isomorphism.

1.1 Equivariant sheaves and principal bundles

In this thesis, any sheaf on a scheme X is an $\mathcal{O}_X$-module.

Definition 1.1.1. [22]Let X be a G-scheme. A G-equivariant structure for an $\mathcal{O}_X$-module $\mathcal{F}$ is an isomorphism α of $\mathcal{O}_{G\times X}$-modules $\alpha : \sigma^*\mathcal{F} \to pr_X^*\mathcal{F}$ satisfying:

1. Its pullbacks by $\mathrm{id}\times\sigma$ and $\mu\times\mathrm{id}$ are related by the equation

$$pr_{23}^*\alpha \circ (\mathrm{id}\times\sigma)^*\,\alpha = (\mu\times\mathrm{id})^*\,\alpha$$

   where $pr_{23} : G\times G\times X \to G\times X$ is the projection to the second and the third factors

2. The restriction of α to $\{e\}\times X \subset G\times X$ is identity .

If $\mathcal{F}$ has a G-equivariant structure, we call the pair $(\mathcal{F},\alpha)$ a G-equivariant $\mathcal{O}_X$-module. Let $(\mathcal{F},\alpha)$ and $(\mathcal{F}',\alpha')$ be two G-equivariant $\mathcal{O}_X$-modules. A G-equivariant morphism $f : (\mathcal{F},\alpha) \to (\mathcal{F}',\alpha)$ of two G-equivariant sheaves is a morphism of $\mathcal{O}_X$-modules $f : \mathcal{F} \to \mathcal{F}'$ satisfying $\alpha'\circ\sigma^* f = pr_X^* f\circ\alpha$. We will drop α from the notation if the equivariant structure is clear.

Let G-act on X. Here is a short list of G-equivariant sheaves and of G-equivariant morphisms:

1. The structure sheaf $\mathcal{O}_X$ of a G scheme has a natural G-equivariant structure induced by the unique isomorphisms $\sigma^*\mathcal{O}_X \simeq \mathcal{O}_{G\times X} \simeq \pi^*\mathcal{O}_X$.

2. For a G-map f the corresponding relative differential ω_f has a natural G-equivariant structure.

3. The usual constructions of sheaves–kernel, cokernel, tensor product, direct sum, internal hom, local $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ and $\mathcal{T}or^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$–have natural G-equivariant structures. In particular, the symmetric algebra $\mathrm{Sym}\mathcal{F} := \bigoplus_{i\geq 0}\mathrm{Sym}^i\mathcal{F}$ has a G-equivariant structure induced from the G-equivariant structure on $\mathcal{F}$. Since *Spec* gives a n equivalence from the opposite category of $\mathcal{O}_X$-algebras to the

category of affine schemes over X, then for a G-equivariant $\mathcal{O}_X$-algebra A, the corresponding affine scheme over X has a natural G-action such that the projection $Spec\, A \to X$ is a G-map by. In particular G-acts on the vector bundle corresponding to a G-equivariant locally free sheaf $\mathcal{F}$.

4. Let $(\mathcal{F}, \alpha)$ be a G-equivariant locally free sheaf and $V = Spec\,(\mathrm{Sym}\mathcal{F}^\vee)$ be the corresponding vector bundle. Let $\mathbb{P}(V) \coloneqq Proj\ (\mathrm{Sym}\mathcal{F}^\vee)$ and let $\pi : \mathbb{P}(V) \to X$ be the structure morphism. Recall that $\mathbb{P}(V)$ represents the functor from the category of schemes over X to the category of sets defined as follows: for each $f : S \to X$ we assign the set of pairs $(\mathcal{L}, \beta)$ where $\mathcal{L}$ is a line bundle on S and $\beta : f^*\mathcal{F}^\vee \to \mathcal{L}$ is a surjection modulo isomorphism i.e we identify $(\mathcal{L}, \beta)$ and $(\mathcal{L}', \beta')$ if there exist an isomorphism $\lambda : \mathcal{L}' \to \mathcal{L}$ such that $\beta = \lambda \circ \beta'$. We will use $\mathbb{P}(V)$ also to denote this functor. Let $\tilde{\beta} : \pi^*\mathcal{F}^\vee \to \mathcal{O}_{\mathbb{P}(V)}(1)$ correspond to the identity morphism $\mathrm{id}_{\mathbb{P}(V)}$. For any morphism $g : X' \to X$ the pullback $f^{-1}\mathbb{P}(V)$ represent the functor from the category of schemes over X' to the categroy of sets defined as follows: for each $f' : S \to X'$ we assign the set of pairs $(\mathcal{L}, \beta)$ where $\mathcal{L}$ is a line bundle on S and $\beta : f^*g^*\mathcal{F}^\vee \to \mathcal{L}$ is a surjection modulo isomorphism. Let $\pi_g : g^{-1}\mathbb{P}(V) \to X'$ be the structure morphism. Any isomorphism $\gamma : \mathcal{F}_1^\vee \to \mathcal{F}_2^\vee$ of locally free sheaves on X corresponds to natural transformation $m_\gamma : \mathbb{P}(V_2) \to \mathbb{P}(V_1)$ by sending the surjection $f^*\mathcal{F}_2^\vee \to \mathcal{L}$ to the surjection $f^*\mathcal{F}_1^\vee \to f^*\mathcal{F}_2^\vee \to \mathcal{L}$. The equivariant structure of $\mathcal{F}$ thus induces an isomorphism $\gamma : \sigma^*\mathcal{F}^\vee \to pr_X^*\mathcal{F}^\vee$ which then induces an isomorphism $m_\gamma : G \times \mathbb{P}(V) = pr_X^{-1}\mathbb{P}(V) \to \sigma^{-1}\mathbb{P}(V)$. One can check that the composition $\sigma_{\mathbb{P}(V)} \coloneqq \pi^{-1}\sigma \circ m_\gamma$ will define an action of G on $\mathbb{P}(V)$ such that the structure morphism π is a G-map. Note that m_γ correspond to the element

$$\left(pr_{\mathbb{P}(V)}^*\mathcal{O}_{\mathbb{P}(V)}(1), \pi_{pr_X}^*\sigma^*\mathcal{F}^\vee \to \pi_{pr_X}^* pr_X^*\mathcal{F}^\vee \to pr_{\mathbb{P}(V)}^*\mathcal{O}_{\mathbb{P}(V)}(1)\right)$$

and also to the element

$$\left(\sigma_{\mathbb{P}(V)}^*\mathcal{O}_{\mathbb{P}(V)}(1) = m_\gamma^*\mathcal{O}_{\sigma^{-1}\mathbb{P}(V)}(1), \pi_{pr_X}^*\sigma^*\mathcal{F}^\vee \simeq m_\gamma^*\pi_\sigma^*\sigma^*\mathcal{F}^\vee \to m_\gamma^*\mathcal{O}_{\sigma^{-1}\mathbb{P}(V)}(1)\right)$$

of $\sigma^{-1}\mathbb{P}(V)(\pi_{pr_X} : G \times \mathbb{P}(V) \to G \times X)$, so that we can conclude the existence of the unique isomorphism

$$\alpha_{\mathcal{O}(1)} : \sigma^*_{\mathbb{P}(V)}\mathcal{O}_{\mathbb{P}(V)}(1) \to pr^*_{\mathbb{P}(V)}\mathcal{O}_{\mathbb{P}(V)}(1)$$

that makes the following diagram commutes.

$$\begin{tikzcd}
\pi^*_{pr_X}\sigma^*\mathcal{F}^\vee \arrow[r] \arrow[d, "\pi^*_{pr_X}\alpha"'] & \sigma^*_{\mathbb{P}(V)}\mathcal{O}_{\mathbb{P}(V)}(1) \arrow[d, "\alpha_{\mathcal{O}(1)}"] \\
\pi^*_{pr_X}pr^*_X\mathcal{F}^\vee \arrow[r] & pr^*_{\mathbb{P}(V)}\mathcal{O}_{\mathbb{P}(V)}(1)
\end{tikzcd} \tag{1.1}$$

One can check that $\alpha_{\mathcal{O}(1)}$ satisfies the cocycle condition so that we can conclude that $\left(\mathcal{O}_{\mathbb{P}(V)}(1), \alpha_{\mathcal{O}(1)}\right)$ is a G-equivariant sheaf. For more details, reader could consult [17]. The above diagram also shows that the canonical morphism $\pi^*\mathcal{F}^\vee \to \mathcal{O}_{\mathbb{P}(V)}(1)$ is an equivariant morphism of sheaves.

5. Given a separated G morphism $f : X \to Y$ of finite type. If $(\mathcal{E}, \alpha)$ (resp. $(\mathcal{F}, \beta)$) is an equivariant sheaf on X (resp. on Y) then $f_*\mathcal{E}$ (resp. $f^*\mathcal{F}$) is an equivariant sheaf on Y (resp. on X) with the folllowing composition

$$\sigma^*_Y f_*\mathcal{E} \simeq (\mathrm{id}_G \times f)_* \sigma^*_X\mathcal{E} \xrightarrow{(\mathrm{id}_G \times f)_*\alpha} (\mathrm{id}_G \times f)_* pr^*_X\mathcal{E} \simeq pr^*_Y f_*\mathcal{E}$$

$$(\text{ resp.} \qquad \sigma^*_X f^*\mathcal{F} \simeq (\mathrm{id}_G \times f)^* \sigma^*_Y\mathcal{F} \xrightarrow{(\mathrm{id}_G \times f)^*\beta} (\mathrm{id}_G \times f)^* pr^*_Y\mathcal{F} \simeq pr^*_X f^*\mathcal{F} \text{)}$$

as the equivariant structure sheaf. Moreover by the naturality of the morphism $f^*f_*\mathcal{F} \to \mathcal{F}$ (resp. $\mathcal{E} \to f_*f^*\mathcal{E}$) we have the following commuttaive diagram

$$\begin{tikzcd}
\sigma^*_X\mathcal{E} \arrow[r] \arrow[d, "\alpha"'] & (\mathrm{id}_G \times f)^* (\mathrm{id}_G \times f)_* \sigma^*_X\mathcal{E} \arrow[d, "(\mathrm{id}_G \times f)^*(\mathrm{id}_G \times f)_*\alpha"] \\
pr^*_X\mathcal{E} \arrow[r] & (\mathrm{id}_G \times f)^* (\mathrm{id}_G \times f)_* pr^*_X\mathcal{E}
\end{tikzcd}$$

$$\left(\text{resp.}\quad \begin{array}{ccc} (\mathrm{id}_G \times f)_* (\mathrm{id}_G \times f)^* \sigma_Y^* \mathcal{F} & \longrightarrow & \sigma_Y^* \mathcal{F} \\ {\scriptstyle (\mathrm{id}_G\times f)_*(\mathrm{id}_G\times f)^*\beta}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ (\mathrm{id}_G \times f)_* (\mathrm{id}_G \times f)^* pr_Y^* \mathcal{F} & \longrightarrow & pr_Y^* \mathcal{F} \end{array}\right) \tag{1.2}$$

Thus we can conclude that $f^* f_* \mathcal{F} \to \mathcal{F}$ (resp. $\mathcal{E} \to f_* f^* \mathcal{E}$) is an equivariant morphism of sheaves. Similarly for higher direct images, $R^i f_* \mathcal{F}$ have a natural equivariant structure.

If $X = Spec\,\mathbb{C}$ and $G = Spec\,R$ for some commutative ring R over $\mathbb{C}$, then an $\mathcal{O}_X$-module $\mathcal{F}$ is a $\mathbb{C}$-vector space V. V is a G-equivariant sheaf if and only if there exist a $\mathbb{C}$-linear map $\gamma_V : V \to R \otimes_{\mathbb{C}} V$ such that $(\mathrm{id}_{R_n} \otimes \gamma) \circ \gamma = (\mu \otimes \mathrm{id}_V) \circ \gamma$ and $(e^\natural \otimes \mathrm{id}_V) \circ \gamma_V = \mathrm{id}_V$. We also call V a G-module and the set of all G-modules over $Spec\,\mathbb{C}$ is a ring denoted by $Rep(G)$. A subvector space $W \subset V$ is called G-invariant if $\gamma_V(W) \subset W \otimes R$. It's easy to see that a G-invariant subvector space is also a G-module.

Let $G = Spec\,R$ .An element $\chi \in R$ is called a character of G if χ is invertible and $\mu^\natural(\chi) = \chi \otimes \chi$. We use $X^*(G)$ to denote the abelian group of characters of G where the group operation is given by the multiplication in G. For example if $G = T_n$, each monomial $\prod_i^n t_i^{a_i}$ is a character of T_n, in fact any character of T_n is a monomial in R_n. Thus $X^*(T_n) \simeq \mathbb{Z}^n$ by identifying the monomials with their degree.

If $\gamma_V(v) = v \otimes \chi$ for a character χ, we call v semi-invariant of weight χ. The set of semi-invariant vectors of weight χ is a G-invariant subspace of V. We call this subspace a weight space and we use V_χ to denote this subspace. It is well known that for any T_n-module V, we can write it as the direct sum of weight spaces i.e $V \simeq \oplus_\chi V_\chi$. Thus a T_n-module is a $\mathbb{Z}^n$-graded vector space. Furthermore, we can conclude that $Rep\,(T_n) \simeq \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$.

For a G-module V of finite $\mathbb{C}$-dimension, the corresponding vector bundle $Spec\,(\mathrm{Sym}V^\vee)$ over $Spec\,\mathbb{C}$ is an affine space with a G-action. We will also use V to denote this affine space and we call V a G-space. For a T_n-module $V = V_\chi$ where $\chi = (\chi_1, \dots, \chi_n)$, the $\mathbb{C}$-valued points of T_n acts on the $\mathbb{C}$-valued points of the T_n-space V by $b.a = b_1^{\chi_1} \dots b_n^{\chi_n} a$ where $b = (b_1, \dots, b_n) \in (\mathbb{C}^\times)^n$.

Definition 1.1.2. For any scheme S, we call $\mu \times \mathrm{id}_S : G \times G \times S \to G \times S$ an action by multiplication. Let G act on X and let $f : X \to Y$ be a morphsim of schemes such that $f \circ \sigma = f \circ pr_X$. Then $f : X \to Y$ is called principal G-bundle if there exist a covering of Y by open subschemes $\{U_i\}$ of Y and G-isomorphisms $\bar{\varphi}_i : G \times U_i \to f^{-1}(U_i)$ for each i such that the following diagram commutes

$$\begin{tikzcd} f^{-1}(U_i) \arrow[dr, "f"'] & & T_n \times U_i \arrow[ll, "\varphi_i"'] \arrow[dl, "pr_{U_i}"] \\ & U_i & \end{tikzcd} \tag{1.3}$$

In this definition $G \times U_i$ is given the action by multiplication and we call the pair $(V_i, \bar{\varphi}_i)_{i\in\Lambda}$ a trivialization of f.

Remark 1.1.3. There is a more general definition of principal bundle for example definition 0.10 of [22] but in the case of $G = T_n$ both definitions are equivalent.

The morphism $\bar{\mu} : G \times G \to G$, $g, h \mapsto hg^{-1}$ also defines a G action on G and also G action on $G \times X$ such that $\bar{\nu}_X : G \times X \to G \times X, (g, x) \to (g^{-1}, x)$ is a G-isomorphism. We call this twisted G-action.

If $f : X \to Y$ is a principal G-bundle and $\mathcal{E}$ a coherent sheaf on Y, the canonical isomorphism $\alpha_{\mathcal{E}} : \sigma^* \circ f^*\mathcal{E} \simeq pr_X^* \circ f^*\mathcal{E}$ induced by the equality $f \circ \sigma = f \circ pr_X$ is a G-equivariant structure for $f^*\mathcal{E}$. If $\xi : \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism of sheaves on Y, by the naturality of $\alpha_{\mathcal{E}}$ we have $\alpha_{\mathcal{E}_2} \circ (f \circ \sigma)^* \xi = (f \circ pr_X)^* \xi \circ \alpha_{\mathcal{E}_1}$,i.e. $f^*\xi$ is an equivariant map of sheaves. Thus there exist a functor $f^* : Coh(Y) \to Coh_G(X)$ and $f^* : Vec(Y) \to Vec_G(X)$ by sending $\mathcal{E}$ to its pullback $f^*\mathcal{E}$. The following proposition is a special case of Theorem 4.46 of **(author?)** [34]. We prove it here using a more elementary technique.

Proposition 1.1.4. *If $f : X \to Y$ is a principal G-bundle then $f^* : Coh(Y) \to Coh_G(X)$ (resp. $f^* : Vec(Y) \to Vec_G(X)$) is an equivalence of categories.*

Proof. From the definition there exist an open cover $\{V_i\}_{i\in\Lambda}$ of Y and G-isomorphism $\bar{\varphi}_i : G \times V_i \to f^{-1}(V_i)$ for each i. Let $\varphi_i := \bar{\varphi}_i \circ \bar{\nu}_{V_i}^{-1}$.

For any (i,j) we will use V_{ij} to denote $V_i \cap V_j$ and for any triple (i,j,k) we will use V_{ijk} to denote $V_i \cap V_j \cap V_k$. Let $(\mathcal{F}, \alpha)$ be a G-equivariant coherent sheaves on X. We will consctruct a coherent sheaves $F(\mathcal{F})$ on Y by gluing

$$\tilde{\mathcal{F}}_i := (e \times \mathrm{id}_{V_i})^* \circ \varphi_i^* \, \mathcal{F}|_{f^{-1}(V_i)} \in Coh(V_i).$$

We will use $\lambda_i : V_i \to f^{-1}(V_i)$ to denote $\varphi_i \circ (e \times \mathrm{id}_{V_i})$ for $i \in \Lambda$. Let $\varphi_{ji} := \varphi_j^{-1} \circ \varphi_i : G \times V_{ij} \to G \times V_{ij}$ and $\psi_{ji} := pr_G \circ \varphi_{ji} \circ (e \times \mathrm{id}_{V_{ij}}) : V_{ij} \to G$. Then since φ_{ji} is a G-isomorphism we can write $\varphi_{ji}(g,v) = (\psi_{ji}(v)g, v)$ and $\varphi_{ji}^{-1}(g,v) = (\psi_{ji}(v)^{-1}g, v)$ for $(g,v) \in G \times V_{ij}$. Furthermore for any triple (i,j,k) we have $\psi_{ki}(v) = \psi_{kj}(v).\psi_{ji}(v)$ where "." is multiplication in G.

Given a pair (i,j). Morphisms φ_i and φ_j are G-morphisms so that $\sigma \circ (\mathrm{id}_G \times \varphi_i) = \varphi_i \circ \bar{\mu}$ and similarly for j. Since $\bar{\mu} \circ \left(\psi_{ji}, e \times \mathrm{id}_{V_{ij}}\right)(v) = (\psi_{ji}^{-1}(v), v)$ we can conclude that $\bar{\mu} \circ \left(\psi_{ji}, e \times \mathrm{id}_{V_{ij}}\right) = \varphi_{ji}^{-1} \circ (e \times \mathrm{id}_{V_{ij}})$ by checking it on each factor of $G \times V_{ij}$. Thus

$$\sigma \circ (\mathrm{id}_G \times \varphi_i) \circ \left(\psi_{ji}, e \times \mathrm{id}_{V_{ij}}\right) = \lambda_j$$

and

$$pr_{V_{ij}} \circ (\mathrm{id}_G \times \varphi_i) \circ \left(\psi_{ji}, e \times \mathrm{id}_{V_{ij}}\right) = \lambda_i$$

so that $\bar{\alpha}_{ji} := \left(\psi_{ji}, e \times \mathrm{id}_{V_{ij}}\right)^* \circ (\mathrm{id}_G \times \varphi_i)^* \, \alpha : \lambda_j^* \mathcal{F} \to \lambda_i^* \mathcal{F}$.

Given any triple (i,j,k) we will show that $\bar{\alpha}_{ji}, \bar{\alpha}_{kj}, \bar{\alpha}_{ki}$ satisfy the gluing condition i.e. $\bar{\alpha}_{kj} \circ \bar{\alpha}_{ji} = \bar{\alpha}_{ki}$. Let $\Psi_{ijk} := (\psi_{kj}, \psi_{ji}, e, \mathrm{id}_{V_{ijk}})$ and $\tilde{\Psi}_{ijk} := (\mathrm{id}_{G \times G} \times \varphi_i) \circ \Psi_{ijk}$. We will show that the pullback of the identity $\left(\mu \times \mathrm{id}^*_{f^{-1}(V_{ijk})}\right)^* \alpha = (\mathrm{id}_G \times \sigma)^* \, \alpha \circ pr_{23}^* \alpha$ by $\tilde{\Psi}_{ijk}$ is $\bar{\alpha}_{kj} \circ \bar{\alpha}_{ji} = \bar{\alpha}_{ki}$. By checking it on each factors of $G \times f^{-1}(V_{ijk})$ and $G \times G \times V_{ijk}$ we can show that $\left(\mu \times \mathrm{id}_{f^{-1}(V_{ijk})}\right) \circ \tilde{\Psi}_{ijk} = (\mathrm{id}_G \times \varphi_i) \circ \left(\mu \times \mathrm{id}_{V_{ijk}}\right) \circ \Psi_{ijk}$ and $\left(\mu \times \mathrm{id}_{V_{ijk}}\right) \circ \Psi_{ijk} = (\psi_{ki}, e, \mathrm{id}_{V_{ijk}})$ so that we can conclude

$$\left(\left(\mu \times \mathrm{id}_{f^{-1}(V_{ijk})}\right) \circ \tilde{\Psi}_{ijk}\right)^* \alpha = (\psi_{ki}, e, \mathrm{id}_{V_{ijk}})^* \circ (\mathrm{id}_G \times \varphi_i)^* \, \alpha = \bar{\alpha}_{ki}.$$

Similarly $pr_{23} \circ \tilde{\Psi}_{ijk} = (\mathrm{id}_G \times \varphi_i) \circ pr_{23} \circ \Psi_{ijk} = (\mathrm{id}_G \times \varphi_i) \circ \left(\psi_{ji}, e, \mathrm{id}_{V_{ijk}}\right)$ so that

$$\left(pr_{23} \circ \tilde{\Psi}_{ijk}\right)^* \alpha = \bar{\alpha}_{ji}.$$

We also can conclude that $(\mathrm{id}_G \times \bar{\mu}) \circ \Psi_{ijk} = (\mathrm{id}_G \times \varphi_{ji}^{-1}) \circ (\psi_{kj}, e, \mathrm{id}_{V_{ijk}})$ by checking it on each factors of $G \times G \times V_{ijk}$. Thus we have

$$\begin{aligned} \left((\mathrm{id}_G \times \sigma) \circ \tilde{\Psi}_{ijk}\right)^* \alpha &= \left((\mathrm{id}_G \times \varphi_i) \circ (\mathrm{id}_G \times \varphi_{ji}^{-1}) \circ \left(\psi_{kj}, e, \mathrm{id}_{V_{ijk}}\right)\right)^* \alpha \\ &= \left(\psi_{kj}, e, \mathrm{id}_{V_{ijk}}\right)^* \circ (\mathrm{id}_G \times \varphi_j)^* \alpha \\ &= \bar{\alpha}_{kj}. \end{aligned}$$

We can conclude that there exist a sheaf $F(\mathcal{F})$ on Y and isomorphism $\gamma_i : F(\mathcal{F})|_{V_i} \to \tilde{\mathcal{F}}_i$ satisfying $\bar{\alpha}_{ji} \circ \gamma_i = \gamma_j$.

For G-maps $\xi : \mathcal{F}_1 \to \mathcal{F}_2$ between equivariant sheaves $(\mathcal{F}_1, \alpha_1)$ and $(\mathcal{F}_2, \alpha_2)$, we want to show that there exist a corresponding morphism of sheaves $F(\xi) : F(\mathcal{F}_1) \to F(\mathcal{F}_2)$ on Y. It is sufficient to show that the pullback of ξ by λ_i and λ_j can be glued for any pair (i, j) i.e. $\bar{\alpha}_{2,ji} \circ \lambda_j^* \xi = \lambda_i^* \xi \circ \bar{\alpha}_{1,ji}$. This is exactly the pullback of the identity $\sigma^* \xi \circ \alpha_1 = \alpha_2 \circ pr_{f^{-1}(V_{ij})}^* \xi$ on $G \times f^{-1}(V_{ij})$ by $(\mathrm{id}_G \times \varphi_i) \circ \left(\psi_{ji}, e, \mathrm{id}_{V_{ij}}\right)$. Finally if $\mathcal{F}$ is an equivariant coherent sheaf (resp. locally free sheaf) on X then $F(\mathcal{F})$ is a coherent sheaf (resp. locally free sheaf) on Y since $F(\mathcal{F})|_{V_i}$ is aisomorphic to a coherent sheaf (resp. locally free sheaf).

Now we have constructed a functor $F : Coh_G(X) \to Coh(\mathrm{Y})$, $\mathcal{F} \mapsto F(\mathcal{F})$. Since $f \circ \varphi_i \circ (e, \mathrm{id}_{V_i}) = \mathrm{id}_{V_i}$, then locally there is a canonical isomorphism $\eta_{\mathcal{E}} : F(f^* \mathcal{E})|_{V_i} \simeq \mathcal{E}|_{V_i}$ for any coherent sheaf $\mathcal{E}$ on Y. Since the isomorphism is canonical it can be glued to isomorphism on Y. We leave it to the reader to show that $\eta : F f^* \to \mathrm{id}_{Coh(Y)}$(resp. $F f^* \to \mathrm{id}_{Vec(Y)}$) is a natural transformation.

It remains to show that there exist a natural transformation $\epsilon : \mathrm{id}_{Coh_G(X)} \to f^* F$ (resp. $\epsilon : \mathrm{id}_{Vec_G(X)} \to f^* F$. Let $\beta_i : f^{-1}(V_I) \to G$ defined as $pr_G \circ \varphi_i^{-1}$ so that $\varphi^{-1}(x) = (\beta_i(x), f(x)) \in G \times V_i$. It's easy to show that $\beta_j(x) = \psi_{ji}(f(x))\beta_i(x)$ and $x = \beta_i(x)^{-1} \varphi_i(e, f(x))$ for all $x \in X$. Define a morphism $\delta_i : f^{-1}(V_i) \to G \times f^{-1}(V_i)$

as $x \mapsto (\beta_i(x)^{-1}, \varphi_i(e, f(x)))$. Thus $\sigma \circ \delta_i(x) = \beta_i(x)^{-1}(\varphi_i(e, f(x)) = x$ and $pr_{f^{-1}(V_i)} \circ \delta_i(x) = \varphi_i(e, f(x))$ so that $\delta_i^* \sigma^* \mathcal{F}|_{f^{-1}(V_i)} = \mathcal{F}|_{V_i}$ and $\delta_i^* pr_{f^{-1}(V_i)}^* \mathcal{F}|_{f^{-1}(V_i)} = f^* \tilde{\mathcal{F}}_i$. We will show that

$$\delta_i^* \alpha : \mathcal{F}|_{f^{-1}(V_i)} \to f^* \tilde{\mathcal{F}}_i$$

can be glued to a G-morphism $\epsilon_{\mathcal{F}} : \mathcal{F} \to f^* F(\mathcal{F})$. Define a morphism $\Delta_{ji} : f^{-1}(V_i) \to G \times G \times f^{-1}(V)$, $x \mapsto (\beta_j(x)^{-1}, \beta_j(x).\beta_i(x)^{-1}, \varphi_i(e, (x)))$. It's easy to show that $\left(\mu \times \mathrm{id}_{f^{-1}(V_{ij})}\right) \circ \Delta_{ji} = \delta_i$, $(\mathrm{id}_G \times \sigma) \circ \Delta_{ji} = \delta_j$ and $pr_{23} \circ \Delta_{ij} = (\mathrm{id}_G \times \varphi_i) \circ \left(\psi_{Ji}, e \times \mathrm{id}_{V_{ij}}\right) \circ f$. The pullback of the cocycle condition by Δ_{ji} gives us the gluing condition for $\delta_I^* \alpha$, i.e.

$$\delta_i^* \alpha = f^* \bar{\alpha}_{ji} \circ \delta_J^* \alpha.$$

We leave it to the reader to show that $\epsilon_{\mathcal{F}}$ will give a natural transformation. To show that $\epsilon_{\mathcal{F}}$ is a G-morphism, it's enough to show that for each i, we have $\tilde{\alpha}_i \circ \sigma^* \delta_I^* \alpha = pr_{f^{-1}(V_i)}^* \delta_i^* \alpha \circ \alpha$ where $\tilde{\alpha}_i$ is the canonical isomorphism induced by the equality $f \circ \sigma = f \circ pr_{f^{-1}(V_i)}$. It's easy to show that $\left(\mu \times \mathrm{id}_{f^{-1}(V_i)}\right) \circ (\mathrm{id}_G \times \delta_I) = \delta_i \circ \sigma$, $(\mathrm{id}_G \times \sigma) \circ (\mathrm{id}_G \times \delta_i) = \mathrm{id}_{G \times f^{-1}(V_i)}$, and $pr_{23} \circ (\mathrm{id}_G \times \delta_i) = \delta_i \circ pr_{f^{-1}(V_i)}$. One can show that the pullback of the cocylce condition of α by $(\mathrm{id}_G \times \delta_i)$ gives the desired identity. □

We will use the following Lemmas in the next section in the construction of equivariant Chow groups.

Lemma 1.1.5. *Let $f : X \to Y$ be a G-morphism. Assume that $\pi_X : X \to X_G$ and $\pi_Y : Y \to Y_G$ are principal bundles. Then there exist a unique map $f_G : X_G \to Y_G$ such that*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X_G \arrow[r, "f_G"'] & Y_G \end{tikzcd} \tag{1.4}$$

is a cartesian diagram.

Proof. From the definition of principal bundle we have a covering by open subschemes $\{W_i\}_{i \in \Lambda}$ of Y_G such that $\pi_Y|_{\pi_Y^{-1}(W_i)}$ is trivial bundle. Thus we have a G-isomorphism $\varphi_i : G \times W_i \to \pi_Y^{-1}(W_i)$ such that $\pi_Y \circ \varphi_i = pr_{W_i}$. Let $\delta_i := pr_G \circ \varphi_i^{-1} \circ f : (\pi_Y \circ f)^{-1} (W_i) \to$

G. Let $V_i := \pi_X((\pi_Y \circ f)^{-1}(W_i))$. One can show that $\psi_i := (\delta_i, \pi_X) : (\pi_Y \circ f)^{-1}(W_i) \to G \times V_i$ is an isomorphism. Thus we have a morphism $g_i := \pi_Y \circ f \circ \psi_i^{-1} \circ (e \times \mathrm{id}_{V_i}) : V_i \to W_i$. One can show that g_i can be glued to a morphism $g : X_G \to Y_G$. Let $f_i : G \times V_i \to G \times W_i$ defined as $\varphi_i \circ f \circ \psi_i^{-1}$. Any morphism $g' : X_G \to Y_G$ that makes equation (1.4) commute must satisfy $\pi_Y \circ f_i = g'|_{V_i} \circ \pi_X$. It's easy to show that $g'\big|_{V_i} = g_i$ and we can conclude that g is unique.

To show that diagram (1.4) is cartesian, it is sufficient to show it for any of the open subschemes W_i of Y_G. By checking it on each factor of $G \times W_i$ we have $f_i = \mathrm{id}_G \times g_i$. Locally diagram (1.4) is isomorphic to

$$\begin{tikzcd} G \times V_i \arrow[r, "f_i"] \arrow[d, "pr_{V_i}"'] & G \times W_i \arrow[d, "pr_{V_i}"] \\ V_i \arrow[r, "g_i"'] & W_i \end{tikzcd}$$

which is clearly cartesian. Thus we can conclude that diagram (1.4) is cartesian. □

Remark 1.1.6. From Lemma 1.1.5 if $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are principal G-bundles then there exists a unique isomorphism $g : Y_1 \to Y_2$ such that $g \circ f_1 = f_2$. We can conclude that if $f : X \to Y$ is a principal G-bundles then f is the initial object in the category of morphsims $g : X \to Y$ satisfying $g \circ \sigma = g \circ pr_X$. We call Y the quotient of X by G we will use X/G or X_G to denote Y.

Let σ_X and σ_Y defines G-action on X and Y. For any two schemes S_1, S_2 let $\tau_{S_1,S_2} : S_1 \times S_S \to S_2 \times S_1$, $(s_1, s_2) \mapsto (s_2, s_1)$ and let $\Delta_{S_1} : S_1 \to S_1 \times S_1, s_1 \mapsto (s_1, s_1)$. And we define $\sigma_{X\times Y}$ to be the morphism $(\Delta_G \times \mathrm{id}_{X\times Y}) \circ (\mathrm{id}_G \times \tau_{X\times G} \times \mathrm{id}_Y) \circ (\sigma_X \times \sigma_Y)$. One can show that $\sigma_{X\times Y}$ defines an action of G on $X \times Y$ and we say that G acts diagonally on $X \times Y$.

Lemma 1.1.7. *If G acts on X and $\pi : U \to U/G$ is a principal G-bundle. There exist a principal G-bundle $\pi_X : X \times U \to (X \times U)/G$ where G acts on $X \times U$ diagonally. By Lemma 1.1.5 there exist a morphism $g : X \times_G U \to U/G$ induced from the projection $pr_U : X \times U \to U$. Moreover, the fiber of g is X i.e. $g^{-1}(u) = X$ for any closed point $u \in U_G$.*

Proof. Let $(\{V_i\}_{i\in\Lambda}, \bar{\varphi}_i)$ be a trivialization of $\pi : U \to U/G$ and let $\varphi_i := \bar{\varphi}_i \circ \bar{\nu}^{-1}$ so that $\mathrm{id}_X \times \varphi_i : X \times G \times V_i \to X \times \pi^{-1}(V_i)$ is a G-isomorphisms where both the domain and the target of $\mathrm{id}_G \times \varphi$ has diagonal G-actions. Let $\bar{\sigma} := \sigma \circ \tau_{X\times G} : X \times G \to X$, then we have a G-isomorphism $\gamma : G \times X \to X \times G$, $(g, x) \mapsto (gx, g^{-1})$ where $G \times X$ has a trivial G action and $X \times G$ has a diagonal action. By simple calculation we have $\bar{\sigma} \circ \gamma = pr_X$. Given a pair (i, j), let $\varphi_{ji} = \varphi_j^{-1} \circ \varphi_i$, $\psi_{ji} := (e \times \mathrm{id}_{V_{ij}}) \circ \varphi_{ji} \circ pr_G$ and $\gamma_{ji} := \bar{\sigma} \circ \big(\mathrm{id}_X \times (\psi_{ji}, \mathrm{id}_{V_{ij}})\big)$: $X \times V_{ij} \to X \times V_{ij}$. Recall that for any triple (i, j, k) we have $\psi_{ki} = \psi_{kj}.\psi_{ji}$ so that $\gamma_{ki} = \gamma_{kj} \circ \gamma_{ji}$ so that there exist a scheme Y and open immersions $\gamma_i : X \times V_j \to Y$ such that $\gamma_i = \gamma_j \circ \gamma_{ji}$and for any point $y \in Y$ there exist i and $(x, v) \in X \times V_i$ satisfying $\gamma_i(x, v) = y$. Let Y_i be the image of γ_i and let γ_i^{-1} be the inverse of $\gamma_i : X \times V_i \to Y_i$.

Let $\pi_i : X \times \pi^{-1}(V_i) \to Y_i$ be defined by $\gamma_i \circ \bar{\sigma} \circ (\mathrm{id}_G \times \varphi_i)^{-1}$. From the definition of φ_{ji} we have $\pi_j|_{X\times\pi^{-1}(V_{ij})} = \pi_i|_{X\times\pi^{-1}(V_{ij})}$ so that π_i can be glued to $\pi_X : X \times U \to Y$. One can show that $\pi_X \circ (\mathrm{id}_X \times \varphi_i) \circ (\gamma \times \mathrm{id}_{V_i}) \circ (\mathrm{id}_G \times \gamma_i^{-1}) = pr_{Y_i}$ and we can conclude that $(Y_i, (\mathrm{id}_X \times \varphi_i) \circ (\gamma \times \mathrm{id}_{V_i}) \circ (\mathrm{id}_G \times \gamma_i^{-1}))_{i\in\Lambda}$ is a trivialization of g. It's clear that the restriction of $g : X \times_G U \to U_G$ to V_I is isomorphic to the projection $pr_{V_i} : X \times V_i \to V_i$ so that the fiber of g is X. □

1.2 Equivariant chow group and Its completion

In this section we review the definition of equivariant Chow groups given in [4, 5]. We will use g to denote the dimension of our group G as a scheme over $\mathbb{C}$.

Given $i \in \mathbb{Z}$. Let X be a G-scheme with $\dim X = d$. Let V be G-vector space of dimension l. Assume that there exists an open subscheme $U \subset V$ and a principal G-bundle $\pi : U \to U_G$.By giving $X \times V$ a diagonal action of G, assume furthermore that there exist a principal G-bundle $\pi_X : X \times U \to (X \times U)/G$. We will use $X \times_G U$ to denote $(X \times U)/G$. Assume also that $V \smallsetminus U$ has codimension greater than $d - i$, then the equivariant Chow group is defined as

$$A_i^G(X) := A_{i+l-g}(X \times_G U).$$

The definition is independent up to isomorphism of the choice of a representation as long as $V \setminus U$ is of codimension greater than $d - i$.

For a G-equivariant map $f : X \to Y$ with property P where P is either proper, flat, smooth, or regular embedding the G-equivariant map $f \times 1 : X \times U \to Y \times U$ has the property P since all of these properties are preserved by a flat base change. Moreover, the corresponding morphism $f_G : X \times_G U \to Y \times_G U$ also has property P. In fact, these properties are local on the target in the Zariski topology and for any trivialization $(V_i, \bar{\varphi}_i)_{i\in\Lambda}$ of $\pi : U \to U_G$ the restriction of f_G on $\pi_X(X \times \pi^{-1}(V_i))$ is isomorphic to $f \times \mathrm{id}_{V_i}$. So from the definition, for a flat G -map $f : X \to Y$ of codimension l we can define pullback map $f^* : A_i^G(Y) \to A_{i+l}^G(X)$ for equivariant Chow groups. Similarly, for regular embedding $f : X \to Y$ of codimension d we have a Gysin homomorphism $f^* : A_i^G(Y) \to A_{i-d}^G(X)$ and for proper G-map $f : X \to Y$ we can define pushforward $f_* : A_i^G(X) \to A_i^G(Y)$ for equivariant Chow groups.

For $G = T_1$ and an $l + 1$-dimensional weight space V_χ we have a principal G-bundle $\pi_U := V_\chi \setminus \{0\} \to \mathbb{P}(V_\chi)$. By Lemma 1.1.7, there exist a principal G-bundle $\pi_X : X \times U \to X \times_G U$. And since $\operatorname{codim} V_\chi \setminus U$ is $l + 1$, for each $i \in \mathbb{Z}$ we can take $A_{i+l}(X \times_G U))$ to represent $A_i^G(X)$ if $l + i \geq d$. We can also fix χ to be -1 to cover all i.

Thus we fix the following notation. For each positive integer l let V_l be a T_1-space of weight -1 with coordinate $x_0, \dots, x_l$. Thus V_{l-1} is the zero locus of the last coordinate of V_l. We use U_l to denote $V_l \setminus \{0\}$ and X_l to denote $X \times_G U_l$ and $\pi_{X,l} : X \times U_l \to X_l$ the corresponding principal bundle. Thus we have the following direct system

$$\dots \longrightarrow X_{l-1} \xrightarrow{j_{X,l-1}} X_l \xrightarrow{j_{X,l}} X_{l+1} \xrightarrow{j_{X,l+1}} \dots \tag{1.5}$$

There is a projection from $\xi : V_{l+1} \to V_l$ by forgetting the last coordinate such that $j_l : V_l \to V_{l+1}$ is the zero section of ξ. By removing the fiber of $p := (0 : 0 : \dots : 0 : 1) \in \mathbb{P}(V_{l+1})$, the corresponding projection $\xi : X_{l+1} \setminus \pi_X^{-1}(p) \to X_l$ is a line bundle over X_l such that $j_{X,l} : X_l \to X_{l+1} \setminus \pi_{X,l+1}^{-1}(p)$ is the zero section. Note that $\dim \pi_{X,l+1}^{-1}(x)$=$\dim X = d$. Thus for $i \geq d - l$ the restriction map

$A_{i+l+1}(X_{l+1}) \to A_{i+l+1}(X_{l+1} \smallsetminus \pi^{-1}_{X,l+1}(p))$ is an isomorphism. In general this restriction is a surjection. Since $\hat{j}_{X,l} : X_l \to X_{l+1} \smallsetminus \pi^{-1}_{X,l+1}(p)$ is the zero section of ξ , the Gysin homomorphism $\hat{j}^!_{X,n} : A_{k+1}(X_{l+1} \smallsetminus \pi^{-1}_{X,l+1}(p)) \to A_k(X_l)$ is an isomorphism. Since j is a regular embedding we have a Gysin homomorphism $j^! : A_{k+1}\,(X_{l+1}) \to A_k(X_l)$ which is the composition of the above homomorphisms.

Lemma 1.2.1. *The Gysin homomorphism $j^!_{X,l} : A_{k+1}(X_{l+1}) \to A_k(X_l)$ is a surjection. Furthermore, $j^!_{X,l}$ is an isomorphism for $k \geq d - l$.*

The direct system 1.5 induces an inverse system

$$\dots \longleftarrow A_*(X_{l-1}) \xleftarrow{j^!_{X,l-1}} A_*(X_l) \xleftarrow{j^!_{X,l}} A_*(X_{l+1}) \qquad \dots$$

of abelian groups. Let $(\varprojlim A(X_l), \lambda_l)$ be the inverse limit of the above inverse system. From the definition of equivariant Chow groups, $A^G_i(X) = A_{i+n}(X_n)$ for $i \geq d - n$ so that we can identify $\prod_{i=d-n}^{d} A^G_i(X)$ with the group $\prod_{i=d}^{d+n} A_i(X_n)$. Recall that $\left(\prod\limits_{i=-\infty}^{d} A^G_i(X), \nu_i\right)$ where $\nu_n : \prod\limits_{i=-\infty}^{d} A^G_i(X) \to \prod_{i=d-n}^{d} A^G(X)$ is defined by $(a_d, a_{d-1}\dots) \mapsto (a_d, \dots, a_{d-n})$ is the inverse limit of the inverse system defined by the projection $p_{X,n} : \prod_{i=d-n-1}^{d} A^G(X) \to \prod_{i=d-n}^{d} A^G(X)$, $(a_d, \dots, a_{d-n}, a_{d-n-1}) \mapsto (a_d, \dots, a_{d-n})$. By Lemma 1.2.1, after indentifying $\prod_{i=d-n}^{d} A^G_i(X)$ with $\prod_{i=d}^{d+n} A_i(X_n)$, $p_{X,n}$ and $j^!_{X,n}$ are the same homomorphism. The compostion of the projections $\hat{\xi}_n : A_*(X_n) \to \prod_{i=d}^{d+n} A(X_n)$ with $\lambda_n : \varprojlim A_*(X_l) \to A_*(X_n)$ are homorphisms $\xi_i : \varprojlim A_*(X) \to \prod_{i=d-n}^{d} A^G_i(X)$ satisfying $p_{X,n+1} \circ \xi_n = p_{X,n}$ so that by the universal property of inverse limit we have a group homomorphism $\xi : \varprojlim A_*(X_n) \to \prod\limits_{i=-\infty}^{d} A^G_i(X)$ satisfying $p_{X,n} \circ \xi = \xi_n$.

Proposition 1.2.2. $\xi : \varprojlim A_*(X_l) \to \prod\limits_{i=-\infty}^{d} A^G_i(X)$ *is an isomorphism .*

Proof. We will show that for each $a = (a_d, a_{d-1}, \dots) \in \prod_{i=-\infty}^{d} A^G_i(X)$ there exist a unique $b \in \varprojlim A_*(X_l)$ such that $\xi(b) = a$. $b \in \varprojlim A_*(X_l)$ can be written as $(b_1, b_2, \dots,)$ such that $j^! b_{l+1} = b_l$. For each l, let $\hat{b}_l = \sum_{i=0}^{l+d} a_{d-i} \in A_*(X_{l+d})$ where we identify $A^G_k(X)$ with $A_{k+l+d}(X_{l+d})$ for $-l \leq k \leq d$. Set b_l as the restriction of $\hat{b}_l$ to $A_*(X_l)$ by succesively applying $j^!_l$, d times. Since $\delta := j^!_l \hat{b}_{l+1} - \hat{b}_l \in A_{d-1}(X_{l+d})$ its restriction to $A_{-1}(X_l) = 0$

must be zero so that $j_l^! b_{l+1} = b_l$. For $-l + d \leq k \leq d$ we can still identify $A_k^G(X)$ with $A_{k+l}(X_l)$ even after applying $j_l^!$, d times. Thus the projection $A_*(X_l) \to \prod_{i=d}^{l+d} A_i(X_l)$ send b_l to $\sum_{i=0}^{l} a_{d-i}$. We can conclude that $\xi(b) = a$.

To prove injectivity we will show that if $\xi(b) = 0$ then $b = 0$. For any l, $(b_l)_i \in A_i(X_l)$ is the restriction of $(b_{l+d})_{i+d} \in A_{i+d}(X_{l+d})$ which we can identify as an element of $A_{i+d-l}^G(X)$. Since $\xi(b) = 0$ $(b_{l+d})_{i+d}$ is also zero which implies that $(b_l)_i = 0$. □

1.3 Equivariant K-theory

1.3.1 $K^G(X)$ and $G^G(X)$

Let A be an abelian category. A full subcategory B of A is called closed under extension if for any short exact sequence

$$0 \to a \to b \to c \to 0 \tag{1.6}$$

(1.6) $a, c \in B$ implies that b is also an object of B. On the other hand, a full subcategory B is called closed under kernels of surjections if for any short exact sequence (1.6) $b, c \in B$ implies $a \in B$. If a, b and c of (1.6) are in B we call (1.6) an exact sequence in B. We call a full subcategory B of an abelian category A an exact category if B is closed under extension. In particular, the abelian category A is an exact category. The Grothendieck group $K_0(B)$ of an exact category B is defined as the free abelian group $\mathbb{Z}[B]$ generated by the objects of B modulo the relation $a + c = b$ for every short exact sequence (1.6) in B. We will use $[a]_B$ to denote a class in $K_0(B)$ represented by the object a of B. We will drop the subscript if the corresponding exact category is clear from the context.

A functor $F : A \to B$ between exact categories is called exact if F maps exact sequences into exact sequences. From the definition, an exact functor induces a group homomorphism between Grothendieck groups of exact categories. For example the inclusion $B \subset A$ defines the group homomorphism $i : K_0(B) \to K_0(A)$ by mapping the class $[a]_B \in K_0(B)$ to its class $[a]_A \in K_0(A)$ as an object of A.

Another condition that lets us have a group homomorphism from $K_0(A)$ to $K_0(B)$ is if there exist a group homomorphism $\bar{f}$ from $\mathbb{Z}[A]$ to $K_0(B)$ such that $\bar{f}(b) = \bar{f}(a) + \bar{f}(c)$ for any short exact sequence (1.6) in A. Thus the kernel of $\bar{f}$ contains the subgroup of $\mathbb{Z}[A]$ generated by the element $a + c - b$ for every exact sequence (1.6) so that $\bar{f}$ factors through a unique group homomorphism $f : K_0(A) \to K_0(B)$.

The category $Coh(X)$ of coherent sheaves on X is an abelian category. The full subcategory $Vec(X)$ of locally free shevaes is an exact category since $Vec(X)$ is closed under extension. Moreover, $Vec(X)$ is also closed under kernels of surjection. $G(X)$(resp. $K(X)$) is defined as the Grotendieck group of $Coh(X)$(resp. of $Vec(X)$).

Similarly, the category $Coh_G(X)$ of G-equivariant coherent sheaves with G-equivariant morphism is an abelian category and the full subcategory $Vec_G(X)$ of locally free sheaves is an exact category. Moreover $Vec_G(X)$ is also closed under kernel of surjection. We will use $G^G(X)$ (resp. $K^G(X)$) to denote $K_0(Coh_G(X))$ (resp. $K_0(Vec_G(X))$.

The inclusion $Vec_G(X) \subset Coh_G(X)$ induce a group homomorphism $i : K^G(X) \to G^G(X)$ by sending the class of a locally free sheaf to its class as a coherent sheaf. This map in general is not injective nor surjective. For any G-equivariant morphism of schemes $f : X \to Y$, the pullback f^* induces a morphism $f^* : K^0_G(Y) \to K^0_G(X)$ since f^* map exact sequence of locally free sheaves into exact sequence of locally free sheaves. For any flat morphism $f : X \to Y$ of a G-equivariant schemes, the pullback functor induces a group homomorphism $f^* : G^G(Y) \to G^G(X)$. For any finite morphism f, the pushforward $f_* : Coh_G(X) \to Coh_G(X)$ is an exact functor, thus it induces the pushforward map $f_* : G^G(X) \to G^G(Y)$. If f is projective i.e. f is the composition of a closed embedding $i : X \to \mathbb{P}_Y(\mathcal{E})$ and the projection $\varphi : \mathbb{P}_Y(\mathcal{E}) \to Y$, then $f_* : G^G(X) \to G^G(Y), [\mathcal{F}] \mapsto \sum(-1)^{-i}[R^i f_* \mathcal{F}]$ is a group homomorphism.

1.3.1.1 Pushforward for $K^G(X)$

We will skecth the construction of pushforward map $f_* : K^G(X) \to K^G(Y)$ in some special cases. For more details, readers should consult chapter 2 of [35] or section 7 and 8 of [28].

First we need the following Lemma.

Lemma 1.3.1. *Let $\mathcal{N}_X$ be a full subcategory of $Coh_G(X)$ staisfying the following conditions:*

1. $\mathcal{N}_X$ contains $Vec_G(X)$

2. $\mathcal{N}_X$ is closed under extension

3. Each objects of $\mathcal{N}_X$ has a resolution by a bounded complex of elements in $Vec_G(X)$

4. $\mathcal{N}_X$ is closed under kernels of surjections.

Then

1. $\mathcal{N}_X$ is exact and the inclusion $Vec_G(X) \subset \mathcal{N}_X$ induces the group homomorphism $i : K^G(X) \to K_0(\mathcal{N}_X)$ by mapping the class $[\mathcal{P}]_{Vec_G(X)}$ of any locally free sheaf $\mathcal{P}$ to its class $[\mathcal{P}]_{\mathcal{N}_X}$ in $K_0(\mathcal{N}_x)$

2. all resolutions of $\mathcal{F}$ by equivariant locally free sheaves

$$0 \longrightarrow \mathcal{P}_n \longrightarrow \mathcal{P}_{n-1} \longrightarrow \dots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

define the same element $\chi(\mathcal{F}) := \sum_{i=0}^{n} (-1)^{-i} [\mathcal{P}_i]$ in $K^G(X)$. Furthermore, χ define a group homomorphism $\chi : K_0(\mathcal{N}_X) \to K^G(X)$ which is the inverse of $i : K^G(X) \to K_0(\mathcal{N}_X)$.

Proof. 1. It's imeediate from the definition.

2. The first statement can be conclude from Lemma 7.6.1 and corollary 7.5.1 of chapter II of [35] so that for any object $\mathcal{F}$ of $\mathcal{N}_X$ the class $\chi(\mathcal{F}) := \sum_{i=0}^{n} (-1)^{-i} [\mathcal{P}_i] \in K^G(X)$ is well defined. For any short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

we have $\chi(\mathcal{E}) + \chi(\mathcal{G}) = \chi(\mathcal{F})$. Thus there exist a homomorphism of abelian groups $\chi : K_0(\mathcal{N}_X) \to K^G(X), [\mathcal{F}] \mapsto \chi(\mathcal{F})$. Since for any locally free sheaf $\mathcal{P}$ the identity morphism $\mathrm{id}_{\mathcal{P}}$ is a resolution for $\mathcal{P}$ then $\chi(\mathcal{P}) = [\mathcal{P}]$ and $\chi \circ i = \mathrm{id}$. Since $[\mathcal{F}] = \sum_{i=0}^{n} (-1)^{-i} [\mathcal{P}_i] \in K_0(\mathcal{N}_X)$ we can conclude that $i \circ \chi = \mathrm{id}$. □

Corollary 1.3.2. *Let $f : X \to Y$ be a finite G-morphism such that $f_* : Vec_G(X) \to Coh_G(X)$ factors through a subcatcategory $\mathcal{N}_Y \subset Coh_G(Y)$ satisfying all 4 conditions of Lemma 1.3.1 above. Then there exist a group homomorphism $f_* : K^G(X) \to K^G(Y)$ such that $f_*[\mathcal{E}] = \chi(f_*\mathcal{E})$ for any locally free sheaf $\mathcal{E}$ on X.*

Proof. Since $f_* : Vec_G(X) \to \mathcal{N}_Y$ is exact we can define the pushforward map $f_* : K^G(X) \to K^G(Y)$ as the composition $K^G(X) \to K_0(\mathcal{N}_Y) \simeq K^G(Y)$ where the last isomorphism is $\chi : K_0(\mathcal{N}_Y) \to K^G(Y)$. □

Now let $X = \mathbb{P}_Y(\mathcal{E})$ and $f : X \to Y$ be the projection where $\mathcal{E}$ is an equivariant locally free sheaf on Y of rank $r+1$. Let $\mathcal{O}_X(1)$ be the dual of the tautological line bundle on X with its natural G-equivariant structure. Let $\mathcal{M}_X \subset Vec_G(X)$ be the full subacategory of locally free sheaves $\mathcal{F}$ such that $R^q f_*\mathcal{F}(-q) = 0$ for all $q > 0$ i.e $\mathcal{F}$ is Mumford regular. Here, we suse $\mathcal{F}(n)$ to denote $\mathcal{F} \otimes \mathcal{O}_X(n)$. In the following Lemma we collect some properies of Mumford-regular vector bundles.

Lemma 1.3.3. *Let $\mathcal{F}$ be a vector bundle on X.*

1. There exist a large enough integer n depending on $\mathcal{F}$ such that $\mathcal{F}(n)$ is Mumford-regular.

2. If $\mathcal{F}$ is Mumford-regular then $\mathcal{F}(n)$ is also Mumford regular for all $n > 0$.

3. If $\mathcal{F}$ is Mumford-regular then $R^i f_*\mathcal{F} = 0$ for all $i > 0$ and $f_*\mathcal{F}$ is a vector bundle on Y.

Proof. The first and the third staments are consequences of Lemma 1.12 of sSection 8 of [28]. The second statement is Lemma 1.3 of Section 8 of [28] □

By Lemma 8.7.4 of [35] $\mathcal{M}_X$ is an exact subacategory of $Vec_G(X)$. By Lemma 1.3.3 there exist a functor $f_* : \mathcal{M}_X \to Vec_G(X)$, $\mathcal{F} \mapsto f_*\mathcal{F}$ which is exact so that there is a homomorphism $\bar{f}_* : K_0(\mathcal{M}_X) \to K^G(Y)$. In the next several paragraphs, we will show that the group homomorphism $i : K_0(\mathcal{M}_X) \to K^G(X)$ induced by the inclusion $\mathcal{M}_X \subset Vec_G(X)$ is an isomorphim. The pushforward map $f_* : K^G(X) \to K^G(Y)$ is then defined as $i^{-1} \circ \bar{f}_*$.

Let $\mathcal{M}_X(l)$ be the full subcategory of $Vec_G(X)$ of objects $\mathcal{F}$ such that $\mathcal{F}(l)$ is Mumford-regular. Since tensoring by line bundle is exact, $\mathcal{M}_X(l)$ are exact for all l. By Lemma 1.3.3 the following nested inclusion of exact categories

$$\mathcal{M}_X \subset \ldots \mathcal{M}_X(l) \subset \mathcal{M}_X(l+1) \subset \ldots Vec_G(X)$$

satisfies $Vec_G(X) = \bigcup_l \mathcal{M}(l)$. This implies that $K^G(X) = \lim_{l\to\infty} K_0(\mathcal{M}(l))$. By the following Lemma the inclusion $\mathcal{M}_X(l) \subset \mathcal{M}_X(l+1)$ induces isomorphisms $i_l : K_0(\mathcal{M}_X(l)) \to K_0(\mathcal{M}_X(l+1))$ so that we can conclude that $i : K_0(\mathcal{M}_X) \to K^G(X)$ is an isomorphism.

Lemma 1.3.4. $i_l : K_0(\mathcal{M}_X(l)) \to K_0(\mathcal{M}_X(l+1)$ *is an isomorphism*

Proof. By Lemma 1.3.5, we can follow the proof of Proposition 8.7.10 of [35]. □

Let $\mathcal{A} = \oplus_{i\in\mathbb{Z}} \mathcal{A}_i$ be a graded $\mathcal{O}_Y$-module. The graded $\mathcal{O}_Y$-module $\mathcal{A}(n)$ is defined as follows :$\mathcal{A}(n) \coloneqq \oplus_{i\geq 0}\mathcal{A}(n)_i$ where $\mathcal{A}(n)_i = \mathcal{A}_{i+n}$. Recall the definition of graded $\mathcal{O}_Y$-algebra $\Gamma_*(\mathcal{O}_X) \coloneqq \oplus_{i\in\mathbb{Z}} f_*\mathcal{O}_X(i)$ then $\Gamma_*(\mathcal{O}_X) = \oplus_{i=0}^{\infty}\mathrm{Sym}^i\mathcal{E}^\vee$. Consider a morphism of graded $\Gamma_*(\mathcal{O}_X)$-modules $d_0 : \mathcal{E}^\vee \otimes \Gamma_*(\mathcal{O}_X)(-1) \to \Gamma_*(\mathcal{O}_X)$, $\xi \otimes 1 \mapsto \xi$ where we have identified $\mathrm{Sym}^1\mathcal{E}^\vee$ with $\mathcal{E}^\vee$. If we fortget the shift, this morphism of $\mathcal{O}_X$-modlues define the zero section of V. This morphism then induces a Koszul resolution

$$0 \to \wedge^{r+1}\mathcal{E}^\vee \otimes \Gamma_*(\mathcal{O}_X)(-r-1) \overset{d_r}{\to} \ldots \to \mathcal{E}^\vee \otimes \Gamma_*(\mathcal{O}_X)(-1) \overset{d_0}{\to} \Gamma_*(\mathcal{O}_X) \to 0 \qquad (1.7)$$

where $d_n : \bigwedge^{n+1} \mathcal{E}^\vee \otimes \Gamma_*(\mathcal{O}_X)(-n-1) \to \bigwedge^{n} \mathcal{E}^\vee \otimes \Gamma_*(\mathcal{O}_X)(-n)$ is given by

$$d_n((\xi_1 \wedge \ldots \wedge \xi_{n+1}) \otimes 1) = \sum_{i=1}^{n+1}(-1)^i(\xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_{n+1}) \otimes \xi_i$$

where $\xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_{n+1}$ means that we ommit the factor ξ_i from $\xi_1 \wedge \ldots \wedge \xi_{n+1}$. By taking the $Proj$ of (1.7) we get a resolution of $\mathcal{O}_X$ by equivariant locally free sheaves.

Lemma 1.3.5. *For any equivariant locally free sheaf $\mathcal{F}$ on X we have the following exact complex of equivariant locally free sheaves induced from the Koszul resolution*

(1.7)

$$0 \to \mathcal{F} \to \mathcal{F}(1) \otimes \mathcal{E} \to \ldots \to \mathcal{F}(r+1) \otimes \bigwedge^{r+1} \mathcal{E} \to 0 \tag{1.8}$$

Proof. It is sufficient to prove it for the case $\mathcal{F} = \mathcal{O}_X$. Diagram 1.1 shows that the canonical morphism of $\mathcal{O}_X$-modules $\lambda : f^*\mathcal{E}^\vee \to \mathcal{O}_X(1)$ is equivariant so that its dual is also equivariant. One can show that the contraction morphism of $\mathcal{O}_Y$-modules $\delta_n : \bigwedge^{n+1} \mathcal{E}^\vee \otimes \mathcal{E} \to \bigwedge^n \mathcal{E}^\vee$, $(\xi_1 \wedge \ldots \wedge \xi_{n+1}) \otimes v \mapsto \sum_{i=1}^n (-1)^i \xi_i(v) \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_{n+1}$ is equivariant. By checking it locally one can show that d_n is the composition $\delta_n \circ \left(\mathrm{id}_{\wedge^{n+1} f^*\mathcal{E}^\vee} \otimes \lambda(n)\right)$ where $\lambda(n) := \lambda \otimes \mathrm{id}_{\mathcal{O}(n)}$. Thus we can conclude that d_n is equivariant for all n. □

We summarise the above discussion in the following corollary

Corollary 1.3.6. *Let G act on Y and $\mathcal{E}$ is an equivariant locally free sheaf. Let $f : Proj\ (Sym\mathcal{E}^\vee) \to Y$ be the structure morphism. Then there exist a group homomorphism $f_* : K^G(Proj\ (Sym\mathcal{E}^\vee)) \to K^G(Y)$ such that $f_*[\mathcal{E}] = [f_*\mathcal{E}]$ for Mumford regular vector bundle $\mathcal{E}$.*

In the case when f is the composition $p \circ i$ where i is a finite morphism satisfying the conditions of corollary 1.3.2 and p is the structure morphism $Proj\ (\mathrm{Sym}\mathcal{E}^\vee) \to Y$, we do not know if $p_* \circ i_* : K^G(X) \to K^G(Y)$ is independent of the factorization $p \circ i$. However, in the case when i is a regular embedding, by Lemma 2.7 of [16] we have an affirmative answer so that we can define f_* as the composition $p_* \circ i_*$.

Beside addition, $K^G(X)$ has multiplication structure given by tensor product with $[\mathcal{O}_X]$ as the identity element. For any morphism of scheme $f : X \to Y$, the pullback $f^* : K^G(Y) \to K^G(X)$ is a ring homomorphism. In particular, $K^G(X)$ has a $K^G(Y)$-module structure via f^*. Moreover, given a morphism satisfying the condition of corollary 1.3.2 or being the projection $\varphi : \mathbb{P}_Y(V) \to Y$, by the following proposition, f_* is a morphism of $K(Y)$-modules.

Proposition 1.3.7 (Projection Formula). *Let $f : X \to Y$ be a morphism satisfying the condition in corollary 1.3.2 or the projection $\varphi : \mathbb{P}_Y(V) \to Y$ where V is a G-*

equivariant vector bundle. Then for any $x \in K^G(X)$ and $y \in K^G(Y)$ we have

$$f_*\left(x.f^*y\right) = \left(f_*x\right).y \in K^G(Y).$$

Proof. Since all operations involved is $\mathbb{Z}$-linear, we can assume that x and y are represented by G-equivariant locally free sheaves $\mathcal{E}$ and $\mathcal{F}$. For a G morphism $f : X \to Y$ and G-equivariant locally free sheaves $\mathcal{E}$ on X and $\mathcal{F}$ on Y the canonical morphism

$$f_*\mathcal{E} \otimes \mathcal{F} \to f_*(\mathcal{E} \otimes f^*\mathcal{F}) \tag{1.9}$$

is G-equivariant and is an isomorphism. Since $f^*\mathcal{F}$ is a vector bundle and $\mathcal{N}_Y$ is closed under extension, $f_*\mathcal{E} \otimes \mathcal{F}$ and $f_*\left(\mathcal{E} \otimes f^*\mathcal{F}\right)$ are objects of $\mathcal{N}_Y$. This conclude the first case. If f is the structure morphism $\varphi : \mathbb{P}_Y(V) \to Y$, since $K_0(\mathcal{M}_X) \simeq K(X)$ we can assume that $\mathcal{E} \in \mathcal{M}_X$. Since the canonical morphism $R^i f_*(\mathcal{E} \otimes f^*\mathcal{F}) \to R^i f_*\mathcal{E} \otimes \mathcal{F}$ is an isomorphism, if $\mathcal{E}$ is Mumford-regular, then $\mathcal{E} \otimes f^*\mathcal{F}$ is also Mumford-regular so that $f_*[\mathcal{E} \otimes f^*\mathcal{F}] = [f_*\left(\mathcal{E} \otimes f^*\mathcal{F}\right)]$ and we can conclude that $f_*([\mathcal{E}].f^*[\mathcal{F}]) = (f_*[\mathcal{E}].[\mathcal{F}])$. □

Proposition 1.3.8 (Base change formula)**.**

1. *Consider the following cartesian diagram*

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & X \arrow[d, "f"] \\ \bar{Y} \arrow[r, "g"'] & X \end{tikzcd}$$

   such that f and f' are G-regular embeddings of codimension r. Then $g^* \circ f_* = \bar{f}_* \circ \bar{g}^* : K^G(X) \to K^G(\bar{Y})$

2. *Let A be a smooth projective variety and let $p : A \times Y \to Y$ be the projection to the second factor. Let $g : \bar{Y} \to Y$ be any morphism and consider the following*

cartesian diagram

$$\begin{tikzcd} A\times\bar{Y} \arrow[r, "\bar{g}"] \arrow[d, "\bar{p}"'] & A\times Y \arrow[d, "p"] \\ \bar{Y} \arrow[r, "g"'] & Y \end{tikzcd}.$$

Then the pushforward maps $p_* : K^G(A\times Y) \to K^G(Y)$ *and* $\bar{p}_* : K^G(A\times\bar{Y}) \to K^G(\bar{Y})$ *are well defined and* $\bar{p}_*\circ\bar{g}^* = g^*\circ p_* : K^G(A\times Y) \to K^G(\bar{Y})$*. Let* $d : D \to A\times Y$ *be a* G*-closed embedding such that* D *is flat over* Y *and let* $d' : D' \to A\times Y'$ *be the corresponding pullback so that we have the following cartesian diagram*

$$\begin{tikzcd} \bar{D} \arrow[r, "\hat{g}"] \arrow[d, "\bar{d}"'] & D \arrow[d, "d"] \\ A\times\bar{Y} \arrow[r, "\bar{g}"'] & A\times Y. \end{tikzcd}$$

Then $\bar{g}^*[\mathcal{O}_D] = [\mathcal{O}_{\bar{D}}] \in K^G(A\times\bar{Y})$*.*

Proof. 1. Since $f, \bar{f}$ are closed embeddings both of them are affine morphisms so that $\bar{f}_*\bar{g}^*\mathcal{F} = g^*f_*\mathcal{F}$. Given a finite resolution $\mathcal{E}^\bullet \to f_*\mathcal{F}$ of $f_*\mathcal{F}$, we need to show that $g^*\mathcal{E}^\bullet \to g^*f_*\mathcal{F} \simeq \bar{f}_*\bar{g}^*\mathcal{F}$ is a resolution of $\bar{f}_*\bar{g}^*\mathcal{F}$. Let $\mathcal{F}$ be an equivariant locally free sheaf and given a finite resolution $\mathcal{E}^\bullet \to f_*\mathcal{F}$ of $f_*\mathcal{F}$ the proof of Proposition 4.5 of [9] shows that $g^*\mathcal{E}^\bullet \to g^*f_*\mathcal{F}$ is a resolution of $g^*f_*\mathcal{F}$.

2. For the first assertion, since A is smooth and projective, we can factorize p into a regular embedding $i : A\times Y \to \mathbb{P}^N_Y$ and a projection $\pi : \mathbb{P}^N_Y \to Y$. In the case of the projection π, It's sufficient to check it for a Mumford-regular vector bunlde $\mathcal{F}$ on $\mathbb{P}^N_Y$. Since $R^i\pi_*\mathcal{F} = 0$ for all $i > 0$ we have $g^*\pi_*\mathcal{F} = \bar{\pi}_*\hat{g}^*\mathcal{F}$ on $\bar{Y}$ where $\bar{\pi}$ is the projection $\mathbb{P}^N_{\bar{Y}} \to \bar{Y}$ and $\hat{g}$ is the canonical morphism $\mathbb{P}^N_{\bar{Y}} \to \mathbb{P}^N_Y$. For i we can use the assertion in point 1. of this Lemma.

For the second assertion it is sufficient to show that for a resolution $F^\bullet \to \mathcal{O}_D$ of $\mathcal{O}_D$ by a bounded complex of equivariant locally free sheaves, $\bar{g}^*F^\bullet \to \bar{g}^*\mathcal{O}_D = \mathcal{O}_{D'}$ is exact. Since the question is local, we can assume that all schemes are affine, let $A\times Y = SpecR$, $Y = SpecS$ and $\bar{Y} = Spec\bar{S}$. Let $F^\bullet \to \mathcal{O}_D$ be given by $\tilde{M}^\bullet \to \tilde{M}$ for some S-modules M, M^i. Note that M and M^i are flat as S-modules. By the natural isomorphism $\big(\bar{S} \otimes_S R\big) \otimes_R N \simeq \bar{S} \otimes_S N$ for all R -modules N so that $\bar{g}^* \tilde{N} \simeq \widetilde{\big(\bar{S} \otimes_S R\big) \otimes_R N} \simeq \widetilde{\bar{S} \otimes_S N}$. So we can conclude that $\bar{g}^* \tilde{M}^\bullet \to \bar{g}^* \tilde{M}$ is exact. $\square$

Tensor product defines on $G^G(X)$ a $K^G(X)$-module structure. If $f : X \to Y$ is a flat morphism, the pullback $f^* : G^G(Y) \to G^G(X)$ is a morphism of $K^G(X)$-modules. If $f : X \to Y$ is a proper morphism, by replacing $x \in K^G(X)$ with $\hat{x} \in G^G(X)$ in Proposition 1.3.7, we can conclude that $f_* : G^G(X) \to G^G(Y)$ is a morphism of $K^G(Y)$-modules.

1.3.2 $G^G(X)$ with support

Let $i : X \to Y$ be a G-equivariant closed embdedding and let $U = Y \setminus X$ with open embedding $j : U \to Y$. Then there exist group homomorphism $i_* : G^G(X) \to G^G(Y)$ and $j^* : G^G(Y) \to G^G(U)$. These two homomorphism is related as follows

Lemma 1.3.9. *The following complex of abelian groups is exact*

$$G^G(X) \xrightarrow{i_*} G^G(Y) \xrightarrow{j^*} G^G(U) \longrightarrow 0\,.$$

Proof. This is Theorem 2.7 of [32]. $\square$

We call a class $\beta \in G^G(Y)$ is supported on X if β is in the image of i_*. Equivalently β is supported on X if $j^*\beta = 0$.

Let $Coh_G^X(Y)$ be the abelian group of coherent sheaves supported on X. Note that $\mathcal{F} \in Coh_G^X(Y)$ is not necessarily an $\mathcal{O}_X$-module. Let $G_X^G(Y)$ be the corresponding Grothendieck group. The pushforward functor $i_* : Coh_G(X) \to Coh_G(Y)$ factors through $Coh_G^X(Y)$ so that there exist a group homomorphism $\bar{i} : G^G(X) \to G_X^G(Y)$, $[\mathcal{F}] \mapsto [i_*\mathcal{F}]$. There exist an inverse of $\bar{i}$ described as follows.

Let $\mathcal{F} \in Coh_G^X(Y)$ and let $\mathcal{I}$ be the ideal of X. Then there exist positive integer n such that $\mathcal{I}^n\mathcal{F} = 0$ so that we have a filtration

$$\mathcal{F} \supseteq \mathcal{I}\mathcal{F} \supseteq \mathcal{I}^2\mathcal{F} \supseteq \ldots \supseteq \mathcal{I}^{n-1}\mathcal{F} \supseteq \mathcal{I}^n\mathcal{F} = 0.$$

Note that each $\mathcal{I}^r\mathcal{F}/\mathcal{I}^{r+1}\mathcal{F}$ is an $\mathcal{O}_X$-module. One can show that $[\mathcal{F}] \mapsto \sum_{r=0}^{n-1}[\mathcal{I}^r\mathcal{F}/\mathcal{I}^{r+1}\mathcal{F}]$ defines a group homomorphism $\bar{i}^{-1} : G^G_X(Y) \to G^G(X)$. For a coherent sheaf $\mathcal{F}$ supported on X we will use $[\mathcal{F}]_Y$ to denote its class in $G^G(Y)$ and we will use $[\mathcal{F}]_X$ to denote $\sum_{r=0}^{n-1}[\mathcal{I}^r\mathcal{F}/\mathcal{I}^{r+1}\mathcal{F}]$. Observe that if $W \xrightarrow{i} X \xrightarrow{j} Y$ with i and j are closed embedding and a coherent sheaf $\mathcal{F}$ supported on W then $i_*[\mathcal{F}]_W = [\mathcal{F}]_X$ and $j_*i_*[\mathcal{F}]_W = j_*[\mathcal{F}]_X = [\mathcal{F}]_Y$.

Lemma 1.3.10. *$\bar{i} : G^G(X) \to G^G_X(Y)$ is an isomorphism.*

Given a cartesian diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ {\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with i, f are closed embeddings and a coherent sheaf $\mathcal{E}$ on X such that $f_*\mathcal{E}$ has a finite resolution by a complex of locally free sheaves. Then we can define a group homomorphism $f^{[\mathcal{E}]} : G^G(\bar{Y}) \to G^G(\bar{X})$, described as follows. Let $\mathcal{F}$ be a coherent sheaf on Y supported on $\bar{Y}$. For each $y \in Y$, the stalk of $\mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F})$ on y is $\mathcal{T}or^i_{\mathcal{O}_{Y,y}}\left((f_*\mathcal{E})_y, \mathcal{F}_y\right)$ so that $\mathcal{T}or^i_y(f_*\mathcal{E},\mathcal{F})$ is supported on $\bar{X}$. For any exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}" \to 0$$

of coherent sheaves on $\bar{Y}$ we have a long exact sequence

$$\mathcal{T}or^{i+1}_Y(f_*\mathcal{E},\mathcal{F}") \to \mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F}') \to \mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F}) \to \mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F}') \to$$

so that

$$\sum_{i\geq 0}(-1)^i[\mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F})] = \sum_{i\geq 0}(-1)^i[\mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F}')+\sum_{i\geq 0}(-1)^i[\mathcal{T}or^i_Y(f_*\mathcal{E},\mathcal{F}")] \in G^G_{\bar{X}}(Y).$$

Thus there exist a group homomorphism $\bar{f}^{[\mathcal{E}]} : G^G(\bar{Y}) \to G^G_{\bar{X}}(Y)$. By Lemma 1.3.10, we can define $f^{[\mathcal{E}]}$ as the composition $\bar{i}^{-1} \circ \bar{f}^{[\mathcal{E}]}$.

Lemma 1.3.11. *Let $f : X \to Y$ be a closed embedding and a coherent sheaf $\mathcal{E}$ on X such that $f_*\mathcal{E}$ has a finite resolution by locally free sheaves. For any closed embedding $i : \bar{Y} \to Y$, there exist a group homomorphism $f^{[\mathcal{E}]} : G^G(\bar{Y}) \to G^G(\bar{Y} \cap X)$ that maps $[\mathcal{F}]$ to $\sum_{i=0}(-1)^{-1}[\mathcal{T}or^i_Y(f_*\mathcal{E}, \mathcal{F})]_{\bar{Y}\cap X}$. Furthermore, $k_* f^{[\mathcal{E}]}([\mathcal{F}]) = \sum_{i=0}(-1)^{-1}[\mathcal{T}or^i_Y(f_*\mathcal{E}, \mathcal{F})]_Y$.*

1.4 $\underleftarrow{\lim} K(X_l)$

Let G be the torus T_1 and let X be a G-scheme. Recall that by Proposition 1.2.2there exist an isomorphism $\xi : \underleftarrow{\lim} A_*(X_n) \to \prod_{i=-\infty}^{d} A_i^G(X)$. In this section we want to recall some results of the corresponding $\underleftarrow{\lim} K(X_n)$.

From the direct system 1.5, we have the inverse system

$$\dots \longleftrightarrow K(X_{l-1}) \xleftarrow{j^*_{X,l-1}} K(X_l) \xleftarrow{j^*_{X,l}} K(X_{l+1}) \qquad \dots$$

We denote the inverse limit of the above inverse system as $\underleftarrow{\lim} K(X_l)$ and use $\rho_{X,l}$ to denote the canonical morphism $\underleftarrow{\lim} K(X_l) \to K(X_l)$. The pullback functor induced from the projection map $pr_X : X \times U_l \to X$ and the equivalence between $Vec_G(X \times U_l)$ and $Vec(X_l)$ induces group homomorphims $\kappa_{X,l} : K^G(X) \to K(X_l)$. It's easy to show that $\kappa_{X,l} = j^*_{X,l} \circ \kappa_{X,l+1}$ so that we have a uniqe group homomorphism $\kappa_X : K^G(X) \to \underleftarrow{\lim} K(X_l)$ such that $\kappa_{X,l} = \rho_{X,l} \circ \kappa_X$. In this section, to distinguish bertween the ordinary and the equivariant version of pullback and pushforward map, we will use superscript ^{G} to denote the equivariant version, for example we will use $f^{G,*}$ to denote the pullback in the equivariant setting.

1.4.1 Derived category and K-theory

The ordinary K theory of a scheme X is defined in the same way as in subsection 1.3.1. For any morphism $f : X \to Y$ there exist a group homomorphism $f^* : K(Y) \to K(X)$, $[\mathcal{F}] \mapsto [f^*\mathcal{F}]$. Furthermore, for ordinary morphism $f : X \to Y$ satisfying the condition of corollary 1.3.2 and for g the structure morphism $Y := \mathbb{P}_Z(V) \to Z$ there are group homomorphisms $f_* : K(X) \to K(Y)$ and $g_* : K(Y) \to K(Z)$. Certainly when h is the compoisition $g \circ f$ we can define $h_* := g_* \circ f_*$. In this section we want to show that this definition is independent of the factorization of h. In order to do this we will use the derived category of coherent sheaves and derived functor to define the group homomorphism between the correspnding K-groups.

Right derived functors Rf_* between derived categories of bounded complex of coherent sheaves maps exact sequence of coherent sheaves to an exact triangle. This properties allow us to define morphism between the corresponding Grothendieck groups. For more general morphism we will use derived functor to define the group homomorphis between K-groups.

Let $(\mathcal{T}, T)$ be a triangulated category with shift functor $T : \mathcal{T} \to \mathcal{T}$. The Grothendieck group of a triangulated category $\mathcal{T}$is the quotient of a free abelian group generated by the objects of $\mathcal{T}$ modulo $[A] + [C] - [B]$ for any exact triangle $A \to B \to C \to TA$.

One can show that the inclusion $Coh(X) \to D^b(X)$ defined by identifying a coherent sheaf as a complex concentrated in 0^{th}-order, gives an isomorphism of abelian group $G(X) \to K_0(D^b(X))$ with inverse $[A^\bullet] \mapsto \sum_{i \in Z}[h^i A^\bullet]$ where $h^i A^\bullet$ is the i^{th}-homology of the complex $A^\bullet$.

We recall the definition and some results about perfect complexes from section 2 of [33]. Let X be a noetherian, quasi compact and quasiseparated scheme. The complex $C^\bullet \in D^b(X)$ is called perfect if for each $x \in X$ there exists an open neighborhood U of x such that $C^\bullet$ is quasi isomorphic to a bounded complex of free sheaves $E^\bullet \in D^b(U)$. If we also assume that X is quasiprojective then $C^\bullet$ is perfect if and only if $C^\bullet$ is quasiisomorphic to a bounded complex of locally free sheaves. The fullsubcategory $X_{perf} \subset D^b(Coh(X))$of perfect complexes is a tringulated subcategory. By identifiying a locally free sheaves as a complex concentrated in the 0^{th}-order, we have a ring homomorphism $\iota_X : K(X) \to K_0(X_{perf})$. For a perfect complex $C^\bullet$, there exist a quasiisomorphism $\alpha : \bar{C}^\bullet \to C^\bullet$ from a bounded complex of locally free sheaves $\bar{C}^\bullet$. Moreover, if $\alpha' : \tilde{C}^\bullet \to C^\bullet$ is another such quasiisomorphism then one can show that $\sum_i (-1)^i \left[\tilde{C}^i\right] = \sum (-1)^i \left[\bar{C}^i\right] \in K(X)$. Thus there exist a group homomorphism

$\bar{\chi}:\mathbb{Z}[X_{perf}]\to K(X)$,$C^\bullet\mapsto\sum_i(-1)^i[C^i]$. One can show that for any exact triangle $C_1^\bullet\to C_2^\bullet\to C_3^\bullet\to TC_1^\bullet$, $\bar{\chi}(C_1^\bullet)-\bar{\chi}(C_2^\bullet)+\bar{\chi}(C_3^\bullet)=0$ so that we have a group homomorphism $\chi:K_0(X_{perf})\to K(X)$. Since $[C^\bullet]=\sum(-1)^i[C^i]\in K_0(X_{perf})$ it's easy to show that χ is the inverse of ι.

For any morphism $f:X\to Y$, the derived pullback $L^\bullet f$ maps bounded complex of locally free sheaves to bounded complex of locally free sheaves, indeed $L^\bullet f^*(C^\bullet)=f^*C^\bullet$ for $C^\bullet$ any bounded complex of locally free sheaves. Since the properties of being perfect is local we can check it on open subscheme on which $C^\bullet$is quasi isomorphic to a bounded complex of locally free sheaves. Thus there exist a group homomorphsim $\mathbf{f}^*:K_0(Y_{perf})\to K_0(X_{perf}),[C^\bullet]\mapsto[L^\bullet f^*C^\bullet]$. If X and Y are quasi isomorphism we can define a group homomorphism $\hat{f}^*:K(Y)\to K(X)$,$[\mathcal{E}]\mapsto\chi[L^\bullet f^*\mathcal{E}]$ which coincide with the one we have defined before.

Let $f:X\to Y$ be a proper morphism between quasiprojective scheme with the property that there exist an open cover $\{U_i\}$ of Y such that the restriction f_i of f to $W_i:=f^{-1}(U_i)$ maps perfect complex $C^\bullet\in D^b(W_i)$ to perfect complex $R^\bullet f_{i,*}C^\bullet\in D^b(U_i)$. Since being perfect is local, we can conclude that $R^\bullet f_*C^\bullet\in D^b(Y)$ is perfect if $C^\bullet\in D^b(X)$ is perfect. Furthermore, $Rf_*C^\bullet$ maps exact triangle to exact triangle so that there is a group homomorphsim $\mathbf{f}_*:K_0(X_{perf})\to K_0(Y_{perf})$,$[C^\bullet]\mapsto[R^\bullet f_*C^\bullet]$. Then we can define a pushforward map $\hat{f}_*:K(X)\to K(Y)$ as $\hat{f}_*=\chi\circ\mathbf{f}_*\circ\iota_X$. The following gives an example when $R^\bullet f_*$ maps perfect complex to perfect complex.

Proposition 1.4.1. *Let $f:X\to Y$ be a morphism between quasi projective scheme over $\mathbb{C}$. If $f:X\to Y$ is a finite morphism satisfying condition in corollary 1.3.2 or f is the projection $\varphi:\mathbb{P}_Y(V)\to Y$ where V is a vector bundle of rank $r+1$, then $R^\bullet f_*C^\bullet$ is a perfect complex for any perfect complex $C^\bullet$.*

Proof. Let f be a finite morphism satisfying condition in corrolary 1.3.2 and since X is quasiprojective, we can assume that $C^\bullet$ is a bounded complex of locally free sheaves. Since f is finite, $f_*$is exact and presereves quasiisomorphism. By Lemma 7.6.1 of [35], there exist a double complex $P^{\bullet,\bullet}$ with horizontal morphism $d^{i,j}:P^{i,j}\to P^{i+1,j}$ and vertical differential $\delta^{i,j}:P^{i,j+1}\to P^{i,j}$ and a morphism of complex $\beta^\bullet:P^{\bullet,0}\to f_*C^\bullet$

such that for each i,

$$\ldots \longrightarrow P^{i,n+1} \xrightarrow{\delta^n} P^{i,n} \xrightarrow{\delta^{n-1}} \ldots \xrightarrow{\delta^0} P^{i,0} \xrightarrow{\beta^i} \bar{C}^i \longrightarrow 0 \tag{1.10}$$

is exact. Note that $P^{i,j} = 0$ for almost all $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ except for a finitely many (i,j). Let $\tilde{C}^\bullet$ be the total complex of $P^{i,j}$ and let $\tilde{\beta} : \tilde{C}^\bullet \to f_*C^\bullet$ be a morphism of complex defined on the m$^{\text{th}}$-order by the composition $\tilde{C}^m = \oplus_{i-j=m} P^{i,j} \to P^{m,0} \to f_*C^m$ where the first arrow is the projection to the factor $P^{m,0}$. By Lemma 12 of section III.7 of [10] we can conclude that $\tilde{\beta}$ is a quasi isomorphism. Thus we can conclude that $R^\bullet f_* C^\bullet = f_* C^\bullet$ is perfect.

Let $\mathcal{P} \subset Vec(X)$ be the subcategory of locally free sheaves $\mathcal{F}$ satisfying $R^i f_* \mathcal{F} = 0$ for $i \neq r$ and $R^r f_* \mathcal{F}$ is a vector bundle. It's easy to see that $\mathcal{P}$ is closed under extension. Proposition 2.1.10 of [14] implies that we can apply Lemma 7.6.1 of [35] and conclude that for a bounded complex of locally free sheaves there exist a double complex $P^{\bullet,\bullet}$ with horizontal morphism $d^{i,j} : P^{i,j} \to P^{i+1,j}$ and vertical differential $\delta^{i,j} : P^{i,j+1} \to P^{i,j}$ and a morphism of complex $\beta^\bullet : P^{\bullet,0} \to C^\bullet$ such that $P^{i,j} \in \mathcal{P}$ for all (i,j) and for each i, the complex (1.10) is exact. Let $\tilde{C}^\bullet$ be the total complex of the double complex $P^{\bullet,\bullet}$ and let $\tilde{\beta} : \tilde{C}^\bullet \to C^\bullet$ be a morphism of complex defined on the m$^{\text{th}}$-order by the composition $\tilde{C}^m = \oplus_{i-j=m} P^{i,j} \to P^{m,0} \to C^m$ where the first arrow is the projection to the factor $P^{m,0}$. By Lemma 12 of section III.7 of [10] we can conclude that $\tilde{\beta}$ is a quasi isomorphism so that $R^\bullet f_* \tilde{C}^\bullet \simeq R^\bullet f_* C^\bullet$. Again by Lemma 12 of section III.7 of [10] we can conclude that that $R^\bullet f_* \tilde{C}^\bullet$ is quasi isomorphic to $R^r f_* \tilde{C}^\bullet$. This conclude the prove of the second case. □

For the case when f is a finite morphism satisfying the condition in corrolary 1.3.2, since on the objects of $Vec(X)$, $f_*, \hat{f}_* : K(X) \to K(Y)$ are the same we can conclude that $f_* = \hat{f}_*$ since $Vec(X)$ generates $K(X)$. Similarly for the case when f is the structure morphism $\mathbb{P}_Y(V) \to Y$. So we will use f_* to denote $\hat{f}_*$ even when f_* is not defined.

Corollary 1.4.2. *Let $f : X \to Y$ be a morphism between quasiprojective scheme such that f can be factorized into $i : X \to Z$ and $p : Z \to Y$ where i is a finite* *morphism satisfying the condition in corollary 1.3.2 and* p *is the structure morphism* $\varphi : \mathbb{P}_Y(V) \to Y$ *of a projectivied vector bundle. Then for any perfect complex* $C^\bullet \in D^b(Coh(X))$ *,* $R^\bullet f_* C^\bullet \in D^b(Y)$ *is also perfect.*

Proof. For any complex $C^\bullet \in D^b(Coh(X)$, there exist a canonical quasi isomorphism $R^\bullet(p \circ i)_* C^\bullet \to R^\bullet p_* \circ R^\bullet i_* C^\bullet$. From the above proposition we can conclude that $R^\bullet f_* C^\bullet$ is perfect if $C^\bullet$ is perfect. □

Given a factorization $f = p \circ i$, by corrolary 1.4.2 we can conclude that for any vector bundle $\mathcal{E}$ in X we have

$$\begin{aligned} p_* \circ i_* [\mathcal{E}] &= \chi \mathbf{p}_* \iota_Y \chi \mathbf{i}_* \iota_X [\mathcal{E}] \\ &= \chi \mathbf{p}_* \mathbf{i}_* \iota_X [\mathcal{E}] \\ &= \chi \mathbf{p}_* [i_* \mathcal{E}] \\ &= \chi [R^\bullet p_* (i_* \mathcal{E})] \\ &= \chi [R^\bullet p_* \circ R^\bullet i_* \mathcal{E}] \\ &= \chi [R^\bullet (f)_* \mathcal{E}] \end{aligned}$$

so that if we define $f_* := p_* \circ i_* : K(X) \to K(Y)$, it is independent of the factorization.

1.4.2 Pullback for $\varprojlim K(X_l)$

Let $f : X \to Y$ be a G-map of G-schemes. Recall that for each G-equivariant map $f : X \to Y$ the induced map $f_n : X_n \to Y_n$ is flat (resp. smooth, proper, regular embedding) if f is flat (resp. smooth, proper, regular embedding). By the functoriality of the pullback we have a commutative diagram

$$\begin{tikzcd} K^G(Y) \arrow[r, "f^*"] \arrow[d, "\kappa_n"'] & K^G(X) \arrow[d, "\kappa_n"] \\ K(Y_n) \arrow[r, "f_n^*"'] & K(X_n) \end{tikzcd}$$

Again by the functoriality of the pullback and the universal property of inverse limit we have ring homomorphisms $\overleftarrow{f^*} : \varprojlim K(Y) \to \varprojlim K(X)$, $\kappa_X : K^G(X) \to \varprojlim K(X_n)$ and $\kappa_Y : K^G(X) \to \varprojlim K(Y_n)$. Futhermore these maps satisfy $\overleftarrow{f^*} \circ \kappa_Y = \kappa_X \circ f^{G,*}$.

1.4.3 Pushforward for $\varprojlim K(X_n)$

Let G be the torus T_1 and let $f : X \to Y$ be a G-morphism between quasiprojective schemes. Recall that $U_l = \mathbb{C}^{l+1} \setminus \{0\}$ where $\mathbb{C}^{l+1}$ is a G-space of weight 1.

First assume that f is a finite morphism satisfying the condition in the corrolary 1.3.2. For any G-morphism $g : Z \to Y$, the pullback $f' : Z \times_Y X \to Z$ of f by g is also finite. Assume also that f' satisfies the condition in corollary 1.3.2 when $g = pr_X : X \times U_l \to X$. In particular, $\mathrm{id}_{U_l} \times f : X \times U_l \to Y \times U_l$ induce a group homomorphism $(\mathrm{id}_{U_l} \times f)_* : K^G(U_l \times X) \to K^G(U_l \times Y)$. Since the pullback functor $\pi^*_{X,l}: Vec(X_l) \to Vec_G(X \times U_l)$ and $\pi_{Y,l} : Vec(Y_l) \to Vec_G(U_l \times Y))$ are equivalence of abelian categories, then there exist a group homomorphism $f_{l,*} : K(X_l) \to K(Y_l)$ which one can show that it maps $[\mathcal{E}] \in K(X_l)$ to the class $[f_{l,*}\mathcal{E}] \in K(Y_l)$ for any locally feee sheaf $\mathcal{E}$ on X.

Next we will show that $f_{l,*}$ ascend to a homomorphism $\overleftarrow{f_*} : \varprojlim K(X_l) \to \varprojlim K(Y_l)$ which satifies $\kappa_Y \circ f^G_* = \overleftarrow{f_*} \circ \kappa_X$. First, by working locally on Y_{l+1}, one can show that $f_{l,*}$ satisfy the identity $f_{l,*} \circ j^*_{X,l} = j^*_{Y,l} \circ f_{l+1,*}$ so that $j^*_{Y,l} \circ (f_{l+1,*} \circ \rho_{X,l+1}) = (f_{l,*} \circ \rho_{X,l})$ so that there exist a group homomorphism $\overleftarrow{f_*} : \varprojlim K(X_l) \to \varprojlim K(Y_l)$ such that $\rho_{Y,l} \circ \overleftarrow{f_*} = f_{l,*} \circ \rho_{X,l}$. The canonical morphism $pr_Y^{G,*} \circ f^G_*\mathcal{E} \to (f \times \mathrm{id}_{U_l})^G_* \circ pr_X^{G,*}\mathcal{E}$ induced from the following cartesian diagram

$$\begin{tikzcd} X \times U_l \arrow[r, "pr_X"] \arrow[d, "(f\times\mathrm{id})"'] & X \arrow[d, "f"] \\ Y \times U_l \arrow[r, "pr_Y"'] & D \end{tikzcd}$$

is an isomorphism so that $f_{l,*} \circ \kappa_{X,l} = \kappa_{Y,l} \circ f^G_*$. Since for any l ,

$$\rho_{Y,l} \circ \kappa_Y \circ f^G_* = \kappa_{Y,l} \circ f^G_*$$

$$= f_{l,*} \circ \kappa_{X,l}$$

$$= f_{l,*} \circ \rho_{X,l} \circ \kappa_X$$

$$= \rho_{Y,l} \circ \overleftarrow{f}_* \circ \kappa_X$$

we can conclue that $\kappa_Y \circ f_*^G = \overleftarrow{f_*} \circ \kappa_X$. Similarly for the case when f is the projection $\mathbb{P}_Y(V) \to Y$. In this case we use the fact that the canonical morphism $L^\bullet j_{Y,l}^* \circ R^\bullet f_{l+1,*} \to R^\bullet f_{l,*} \circ L^\bullet j_{X,l}^*$ is a quasiisomorphism.

We summarise the above discussion in the following Lemma:

Lemma 1.4.3. *Let $f : X \to X$ be a G morphism.*

1. If $f : X \to Y$ is a finite G-morphism satisfying the condition in corrolary 1.3.2. Assume also that for all l , $(f \times id_{U_l})$ also satisfies the condition in corrolary 1.3.2 . Then there exist a group homomorphism $\overleftarrow{f_*} : \varprojlim K(X_l) \to \varprojlim K(Y_l)$ satisfying the identity $\kappa_Y \circ f_*^G = \overleftarrow{f_*} \circ \kappa_X$.

2. If $f : X \to Y$ is the structure morphism $\mathbb{P}_Y(V) \to Y$ where V is a G-equivariant vector bundle. Then there exist a group homomorphism $\overleftarrow{f}_* : \varprojlim K(X_l) \to \varprojlim K(Y_l)$ satisfying the identity $\kappa_Y \circ f_*^G = \overleftarrow{f_*} \circ \kappa_X$.

3. If $f : X \to Y$ is a G-morphism that can be factorized into $p \circ i$ where $i : X \to Z$ is a finite morphism satisfying the condition 1. and p satisfies condition 2. then the group homomorphsim $\overleftarrow{f_*} := \overleftarrow{p_*} \circ \overleftarrow{i_*}$:$\varprojlim K(X) \to \varprojlim K(Y)$ is independent of the factorization.

1.5 Equivariant operational chow ring, Chern class and Chern character

An element of the operational Chow group $A_G^i(X)$ is defined as a class of maps $c_G^l(f : Y \to X) : A_i^G(Y) \to A_{i-l}^G(Y)$ for each G-map $f : X \to Y$ satisfying 3 conditions in chapter 18 of [8]:

1. It commutes with proper pushforward,

2. It commutes with flat pullback
3. It commutes with the refined Gysin map induced by a regular embedding.

Similar to the non-equivariant case , we can also define product, pushforward by proper map, and pullback on the equivariant operational Chow groups. The direct sum $A^*_G(X) := \oplus_{i=0}^{\infty} A^i_G(X)$ and its completion $\prod_{i=0}^{\infty} A^i_G(X)$ are rings with the product operation as the multiplication.

Let $(\varprojlim A^*(X_n), \beta_n)$ be the inverse limit of the following inverse system

$$\ldots \longleftarrow A^*(X_{n-1}) \xleftarrow{j^*_{X,n-1}} A^*(X_n) \xleftarrow{j^*_{X,n}} A^*(X_{n+1}) \longleftarrow \ldots.$$

The pullback by the composition $X \times U_n \subset X \times \mathbb{C}^{n+1} \to X$ gives a ring homomorphism $\gamma : \prod_{i=0}^{\infty} A^i_G(X) \to \prod_{i=0}^{\infty} A^i_G(X \times U_n)$ and for any principal G-bundle $Y \to Y_G$, $A^G_i(Y) \simeq A_i(Y_G)$. Then by the definition of operational Chow groups, we have a ring homomorphism $\bar{\alpha}_n : \prod\limits_{i=0}^{\infty} A^i_G(X \times U_n) \to \prod\limits_{i=0}^{\infty} A^i(X_n)$ and the composition $\alpha_n = \bar{\alpha}_n \circ \gamma$ is a ring homomorphism $\prod\limits_{i=0}^{\infty} A^i_G(X) \to A^*(X_n)$. One can show that the ring homomorphisms α_n satisfy $\alpha_n = j^*_{X,n} \circ \alpha_{n+1}$. By the universal property of inverse limit, we have a map $\alpha : \prod_{i=0}^{\infty} A^i_G(X) \to \varprojlim A^*(X_n)$ such that $\beta_n \circ \alpha = \alpha_n$.

Let $\rho_n : A^*(X_n) \times A_*(X_n) \to A_*(X_n)$ be the action of $A^*(X_n)$ on $A_*(X_n)$ defined by $\rho_n(c, a) = c(a)$ for $(c, a) \in A^*(X_n) \times A_*(X_n)$. Since the elements of operational Chow groups commute with the Gysin map induced by regular embedding $j_n : X_n \to X_{n+1}$ we have $j^!_{X,n} c(\alpha) = c(j^!_{X,n}\alpha)$ where both $j^!_{X,n}$ are the refined Gysin homomorphism. By the definition of the pullback $j^*_{X,n} : A^*(X_{n+1}) \to A^*(X_n)$ we have $c(j^!_{X,n}\alpha) = j^*_{X,n} c(j^!_{X,n}\alpha)$ and we have the following commutative diagram

$$\begin{array}{ccc} A^*(X_{n+1}) \times A_*(X_{n+1}) & \xrightarrow{\rho_{n+1}} & A_*(X_{n+1}) \\ \Big\downarrow{\scriptstyle j^*_n \times j^!_n} & & \Big\downarrow{\scriptstyle j^!_n} \\ A^*(X_n) \times A_*(X_n) & \xrightarrow[\rho_n]{} & A_*(X_n). \end{array} \tag{1.11}$$

By 1.11 we have the action map

$$\varprojlim A^*(X_n) \times \varprojlim A_*(X_n) \simeq \varprojlim \left(A^*(X_n) \times A_*(X_n)\right) \to \varprojlim A_*(X_n)$$

as the unique map induced by the universal property of inverse limit. Note that j_n^* is a graded morphism of order 0. Thus $\varprojlim A^*(X_n)$ is also graded.

For each equivariant vector bundle $\mathcal{E}$ on X, its pullback $\tilde{\mathcal{E}}$ to $X \times U_n$ correspond to a vector bundle $\mathcal{E}_n$ on X_n such that $\pi^*\mathcal{E}_n = \tilde{\mathcal{E}}$. By the identification $A_j^G(X) = A_{j+n}(X_n)$, $c_G^i(\mathcal{E}) : A_j^G(X) \to A_{j-i}^G(X)$ is given by $c^i(\mathcal{E}_n) : A_{j+n}(X_n) \to A_{j-i+n}(X_n)$. Since Chern class commutes with pullback this definition is well defined. Furthermore, $c_G^j(\mathcal{E})$ is an element of $A_G^i(X)$.

In the non equivariant case, each vector bundle $\mathcal{E}$ of rank r has Chern roots $x_1, \dots, x_r$ such that $c^i(\mathcal{E}) = e_i(x_1, \dots, x_r)$ where e_i is the i^{th} symmetric polynomial. Furthermore, its Chern character is defined as $ch(\mathcal{E}) = \sum_{i=1}^r e^{x_i}$. From this definition, we have the following formula of Chern chararacter in terms of Chern classes

$$\begin{aligned} ch(\mathcal{E}) &= r + c^1(\mathcal{E}) + \frac{1}{2}\left(c^1(\mathcal{E})^2 - 2c^2(\mathcal{E})\right) + .. \\ &= \sum_{i=0}^{\infty} P_j(c^1(\mathcal{E}), \dots, c^i(\mathcal{E})) \end{aligned}$$

where $P_j\left(c^1(\mathcal{E}), \dots, c^j(\mathcal{E})\right)$ is a polynomial of order j with $c^i(\mathcal{E})$ has weight i.

In [5], Edidin and Graham define an equivariant Chern character map $ch^G : K^G(X) \to \prod_{i=0}^{\infty} A_G^i(X)$ by the following formula

$$ch^G(\mathcal{E}) = \sum_{i=0}^{\infty} P_i(c_G^1(\mathcal{E}), \dots, c_G^i(\mathcal{E})).$$

One can show that ch^G is a ring homomorphism. Let $\overleftarrow{ch} : K^G(X) \to \varprojlim A^*(X_n)$ denote the composition $\alpha \circ ch^G$.

For each n there is a Chern character map $ch_n : K(X_n) \to A^*(X_n)$ which commutes with refined Gysin homomorphisms. By the universal property of inverse limits we have a ring homomorphism $\widehat{ch} : \varprojlim K(X_n) \to \varprojlim A^*(X_n)$. Since each ch_n is a ring homomorphis, $\widehat{ch}$ is also a ring homomorphism. Furthermore the following diagram commutes

$$\begin{tikzcd} K^G(X) \arrow[r, "ch^G"] \arrow[d, "\kappa"'] & \prod_{i=0}^{\infty} A^i_G(X) \arrow[d, "\alpha"'] \\ \varprojlim K(X_n) \arrow[r, "\widehat{ch}"'] & \varprojlim A^*(X_n) \end{tikzcd} \tag{1.12}$$

Recall the group homomorphism ξ from subsection 1.2.

Lemma 1.5.1. *For all $x \in \varprojlim A_*(X_n)$ and for any $\beta \in K^G(X)$ we have $\xi\left(\overleftarrow{ch}(\beta)(x)\right) = ch^G(\beta)(\xi x)$.*

Proof. An element $x \in \varprojlim A_*(X_n)$ can be written as infinite tuples $(x_0, x_1, \ldots)$ where $x_i \in A_*(X_i)$ satisfying $j^!_{X,i}(x_{i+1}) = x_i$. An element $y \in \prod_{i=-\infty}^{d} A^G_i(X)$ can be written as infinite tuple $(y_d, y_{d-1}, y_{d-2}, \ldots)$ where $y_i \in A^G_i(X)$.

It's sufficient to prove it for an equivariant vector bundle $\mathcal{E}$ on X. Let $x = (x_0, x_1, \ldots) \in \varprojlim A_*(X_n)$, then $\overleftarrow{ch}(\mathcal{E})(x) = (ch(\mathcal{E}_0)(x_0), ch(\mathcal{E}_1)(x_1), \ldots)$. For each k there exist n big enough such that $\nu_k \xi\left(\overleftarrow{ch}(\mathcal{E})(x)\right) = ((ch(\mathcal{E}_n)x_n)_d, (ch(\mathcal{E}_n)x_n)_{d-1}, \ldots, (ch(\mathcal{E}_n)x_n)_{d-k})$ where ν_k is the projection $\nu_k : \prod_{i=-\infty}^{d} A^G_i(X) \to \prod_{i=d-k}^{d} A^G_i(X)$ and $(ch(\mathcal{E}_n)x_n)_i$ is the homogeneous component of $ch(\mathcal{E}_n)x_n$ in degree i.

On the other hand, for each x_i there exist large enough n_i such that $\xi(x) = ((x_{n_0})_d, (x_{n_1})_{d-1}, \ldots)$ where $(x_{n_i})_{d-i}$ is the homogeneous component of x_{n_i} of degree $d-i$. We can also choose n_i large enough so that if $ch^G(\mathcal{E})(\xi x) = (y_d, y_{d-1}, \ldots)$ then

$$y_{d-i} = \sum_{0 \le l \le i} P_l\left(c^1(\mathcal{E}_{n_i}), c^2(\mathcal{E}_{n_i}), \ldots, c^l(\mathcal{E}_{n_i})\right)(x_{n_i})_{d-i+l}.$$

Since Chern class commutes with Gysin homomorhism for $i \le i'$ we have

$$\begin{aligned} y_{d-i} &= \sum_{0 \le l \le i} P_l\left(c^1(\mathcal{E}_{n_{i'}}), c^2(\mathcal{E}_{n_{i'}}), \ldots, c^l(\mathcal{E}_{n_{i'}})\right)(x_{n_{i'}})_{d-i+l} \\ &= \left(ch(\mathcal{E}_{n_{i'}})x_{n_{i'}}\right)_{d-i} \end{aligned}$$

and we can conclude that $\nu_k \xi \left(\overleftarrow{ch}(\mathcal{E})(x) \right) = \nu_k ch^G(\mathcal{E})(\xi x)$. Thus by the universal property of inverse limit $\xi \left(\overleftarrow{ch}(\beta)(x) \right) = ch^G(\beta)(\xi x)$. $\square$

From previous Lemma we can write $ch^G(\alpha)(x) = \overleftarrow{ch}(\alpha)(x)$ after indentifying elements of $\varprojlim A_*(X_n)$ with $\prod_{i=0}^{\infty} A_i^G(X)$ by ξ .

Chapter 2

Kool-Thomas Invariants

The moduli space of stable pairs attempts to compactify the space of embedded curves in a nonsingular projective variety X. It was shown that the moduli of stable pairs have a perfect obstruction theory and thus a virtual fundamental class. Pandharipande-Thomas invariants are defined as the degree of the virtual fundamental class. Historically, there were moduli of stable maps and Hilbert scheme which leads to Gromov-Witten invariants and Donaldson-Thomas Invariants. It was conjectured that if X is a threefold all of these invariants contain the same informations.

In this chapter we will review the definition of stable pair invariants defined in [24] and the reduced obstruction theory of [19] its relation to δ-nodal curve counting [19, 18]. Our reference is [24, 21, 19, 18]

Before we continue we want to fix some notations that we will use later. For a flat morphism $f : X \to Y$ of schemes and for any closed subscheme Z of Y with the closed embedding $g : Z \to Y$, we will use X_Z to denote the fiber product $X \times_Y Z$ and $f^Z : X_Z \to Z$ to denote the corresponding morphism so that we have the following cartesian diagram

$$\begin{tikzcd} X_Z \arrow[r, "\bar{g}"] \arrow[d, "f^Z"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

For a sheaf $\mathcal{F}$ on X, we will use $\mathcal{F}_Z$ to denote the sheaf $\bar{g}^*\mathcal{F}$ on X_Z. For a closed subscheme $Z \subset X$ of X, we will use $\bar{g}^{-1}(Z) \subset X_Z$ to denotes its pullback by $\bar{g}$.

2.1 Pandharipande-Thomas Invariants

2.1.1 Stable Pairs

Let X be a smooth projective variety of dimension 3 with an ample line bundle $\mathcal{L}$. The dimension of a coherent sheaf $\mathcal{F}$ on X is the dimension of its support. A coherent sheaf $\mathcal{F}$ on X is called pure of dimension d if for any subsheaf $\mathcal{E} \subset \mathcal{F}$ of $\mathcal{F}$, $\mathcal{E}$ is of dimension d. In particular, the supporting subscheme has no embedded components.

Definition 2.1.1. Let X be a projective smooth varietiy of dimension 3. A pair $(\mathcal{F}, s)$ where $\mathcal{F}$ is a coherent sheaf of dimension 1 and s is a section of $\mathcal{F}$ is called stable if the following two conditions holds:

1. $\mathcal{F}$ is pure
2. The cokernel Q of s is of dimension 0.

Remark 2.1.2. In [26], Le Potier described the stability condition for the GIT problem of pairs $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ using a polynomial $q \in \mathbb{Q}[k]$ as a parameter. For sufficicently large q the semistable condition is equivalent to the above 2 conditions. Furthermore for sufficiently large q semistable pairs are stable.

For every stable pair $(\mathcal{F}, s)$ we then have 2 exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{F} \longrightarrow Q \longrightarrow 0\,.$$

Lemma 1.6 of [24] tells us that $\mathcal{I}$ is the ideal describing the scheme theoretic support of $\mathcal{F}$. By the purity of $\mathcal{F}$, the scheme theoretic support $C_{\mathcal{F}}$ of $\mathcal{F}$ is a Cohen Macaulay curve i.e. $C_{\mathcal{F}}$ has no embedded points.

Here are some examples of stable pairs on X:

1. Every structure sheaf of a Cohen-Macaulay curve is a stable pair. A divisor D on a Cohen Macaulay curve C in X correspond to a section $s : \mathcal{O}_C \to \mathcal{O}_C(\mathcal{D})$ with cokernel $\mathcal{O}_{\mathcal{D}}$. Thus $\mathcal{O}_X \to \mathcal{O}_C \xrightarrow{s} \mathcal{O}_C(D)$ is a stable pair. This is the prototype for stable pairs

2. This is example from Martijn Kool. Let $C = \{xy = 0\} \subset \mathbb{C}^2$ be the node and let $C_1 = \{y = 0\}$ and $C_2 = \{x = 0\}$. Let $p = (0,0)$. Then p is a divisor for C_1 and C_2. $\mathcal{O}_{C_1}(p)$ can be identified with $\mathbb{C}[x]$ as $\mathcal{O}_{C_1}$ module with section $\mathbb{C}[x] \to \mathbb{C}[x]$, $1 \mapsto x$. Similarly for $\mathcal{O}_{C_2}(p)$. Let $i_1 : C_1 \to C$ and $i_2 : C_2 \to C$ be the closed embedding. Consider the morphism $\mathcal{O}_X \to \mathcal{O}_C \to i_{1*}\mathcal{O}_{C_1}(p) \oplus i_{2*}\mathcal{O}_{C_2}(p)$ which after the identification

$$\mathbb{C}[x,y] \longrightarrow \frac{\mathbb{C}[x,y]}{(xy)} \xrightarrow{(x,y)} \frac{\mathbb{C}[x,y]}{(y)} \oplus \frac{\mathbb{C}[x,y]}{(x)}.$$

The cokernel of the above morphism is supported on p and is generated by $(1,0)$ and $(0,1)$ and $(x,0)$. There is no surjective map from $\mathbb{C}[x,y]$ to the cokernel. If we map $1 \in \mathbb{C}[x,y]$ to $(1,0)$, there is no element of $\mathbb{C}[x,y]$ that we can map to $(0,1)$. This gives an example that the cokernel of the stable pairs might not be a structure sheaf of a subscheme. In particular, it cannot be a section of a divisor on the curve.

2.1.2 Moduli of Stable Pairs

Definition 2.1.3. A family of stable pairs on X over a base scheme B is a the pair $(\mathcal{F}, s)$ where $\mathcal{F}$ is a coherent sheaf on $B \times X$ flat over B and s is a section of $\mathcal{F}$ such that for each closed point b of B, $(\mathcal{F}_b, s_b)$ is a stable pair on X where $\mathcal{F}_b$ and s_b are the restriction of $\mathcal{F}$ and s to b . Two families $(\mathcal{F}_1, s_1)$ and $(\mathcal{F}_2, s_2)$ are isomorphic if there exists an isomorphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ such that $s_2 = \varphi \circ s_1$.

Let X be a smooth projective 3-fold and let χ be an interger and β be a class in $H_2(X, \mathbb{Z})$. Let $\mathfrak{P}_\chi(X, \beta)$ be the functor from the category of scheme to the category of sets that assign to a scheme S the set of families of stable pairs $(\mathcal{F}, s)$ over S modulo isomorphism such that for each closed point $s \in S$ we have $\chi(\mathcal{F}_s) = \chi$ and the scheme theoretic support $C_{\mathcal{F}_s}$ of $\mathcal{F}_s$ is of class β. Then there exists a projective scheme $\mathcal{P}_\chi(X, \beta)$ representing the functor $\mathfrak{P}_\chi(X, \beta)$[26]. Furthermore on the product $\mathcal{P}_\chi(X, \beta) \times X$ there exists a universal sheaf $\mathbb{F}$ and a universal section $\mathbb{S}$ of $\mathbb{F}$. We denote by p and q the projection from $\mathcal{P}_\chi(X, \beta) \times X$ to the factor $\mathcal{P}_\chi(X, \beta)$ and X

respectively.

Let $\mathcal{P}$ be the moduli space $\mathcal{P}_\chi(X, i_*\beta)$. If G acts on X there is a natural G action on $\mathcal{P}$ described as follows: Let $f : G \times \mathcal{P} \times X \to \mathcal{P} \times X$, $(g, p, x) \to (p, g^{-1}x)$. Then $(f^*\mathbb{F}, f^*\mathbb{S})$ is a family of stable pairs over $G \times \mathcal{P}$. So there exists a morphism $\sigma_P : G \times \mathcal{P} \to \mathcal{P}$ such that $\left((\sigma_{\mathcal{P}} \times \mathrm{id}_X)^* \mathbb{F}, (\sigma_{\mathcal{P}} \times \mathrm{id}_X)^* \mathbb{S}\right)$ is isomorphic to $(f^*\mathbb{F}, f^*\mathbb{S})$. Moreover, if G acts diagonally on $\mathcal{P} \times X$ i.e. $\sigma_{\mathcal{P}\times X} : G \times \mathcal{P} \times X \to \mathcal{P} \times X$, $(g, p, x) \mapsto (g.p, g.x)$, then the universal sheaf $\mathbb{F}$ is an equivariant sheaf and $\mathbb{S} : \mathcal{O}_{\mathcal{P}\times X} \to \mathbb{F}$ is an equivariant morphism of sheaves. Let $\hat{\sigma}_X : G \times \mathcal{P} \times X \to G \times \mathcal{P} \times X$,$(g, p, x) \mapsto (g, p, gx)$ so that $\sigma_{\mathcal{P}\times X} = (\sigma_{\mathcal{P}} \times \mathrm{id}_X) \circ \hat{\sigma}_X$. Since $f^*\mathbb{F} \simeq (\sigma_{\mathcal{P}} \times \mathrm{id}_X)^* \mathbb{F}$ and $f \circ \hat{\sigma}_X = pr_{\mathcal{P}\times X}$ where $pr_{\mathcal{P}\times X} : G \times \mathcal{P} \times X \to \mathcal{P} \times X$ is the projection, there exists a canonical isomorphism $\sigma^*_{\mathcal{P}\times X}\mathbb{F} \simeq \hat{\sigma}^*_X (\sigma_{\mathcal{P}} \times \mathrm{id}_X)^* \mathbb{F} \simeq \hat{\sigma}_X f^*\mathbb{F} \simeq pr^*_{\mathcal{P}\times X}\mathbb{F}$. Since the isomorphism is the canonical isomorphism induced from the functoriality of the pullback functor, it automatically satisfies the cocyle condition. This isomorphism is the natural equivariant structure of $\mathbb{F}$.

2.1.3 Perfect obstruction theory and virtual fundamental class

First we recall the notions of perfect obstruction theory of [1] and the construction of virtual fundamental class.

Let Y be a scheme and assume that there exists a closed embedding $\iota : Y \to M$ to a smooth scheme. Let $\mathcal{J}$ be the ideal sheaf describing the closed embedding ι. Let $\{\mathcal{J}/\mathcal{J}^2 \to \iota^*\Omega_M\} \in D^b(X)$ be a complex concentrated in degree -1 and 0 where Ω_M is the cotangent bundle of M . Given another such embedding $\hat{\iota} : X \to \hat{M}$ with ideal $\hat{\mathcal{J}}$, the complex $\{\mathcal{J}/\mathcal{J}^2 \to \iota^*\Omega_M\}$ and $\{\hat{\mathcal{J}}/\hat{\mathcal{J}}^2 \to \hat{\iota}^*\Omega_{\hat{M}}\}$ are quasiisomorphic. We will use $\mathbb{L}_X$ to denote the complex $\{\mathcal{J}/\mathcal{J}^2 \to \iota^*\Omega_M\}$ and we call it the truncated cotangent complex of X. Note that $H^0(\mathbb{L}_X)$ is the sheaf of Kähler differentials of X.

Definition 2.1.4 (Behrend-Fantechi)**.** Let $E^\bullet \in D^b(Y)$ be a two term complex of vector bundles concentrated in degree -1 and 0. A morphism $\phi : E^\bullet \to \mathbb{L}_Y$ in $D^b(X)$ is called a perfect obstruction theory if the induced morphism on homology $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

There exists a two term complex of vector bundles $\hat{E}^\bullet$ quasi isomorphic to $E^\bullet$ and a morphism of complexes $\hat{\phi} : \hat{E}^\bullet \to \mathbb{L}_Y$ representing ϕ. So we can assume that ϕ is a morphism of complexes and write ϕ as the following commutative diagram

$$\begin{tikzcd} E^{-1} \arrow[r, "\partial"] \arrow[d, "\phi^{-1}"'] & E^0 \arrow[d, "\phi^0"] \\ \mathcal{J}/\mathcal{J}^2 \arrow[r, "d"'] & \Omega_M|_Y \end{tikzcd} \tag{2.1}$$

Given a perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}_Y$, Behrend and Fantechi construct a class $[Y]^{vir} \in A_{\mathrm{rk}E^0 - \mathrm{rk}E^{-1}}(Y)$ called virtual fundamental class[1]. We call $vd := \mathrm{rk}E^0 - \mathrm{rk}E^{-1}$ the virtual dimension of Y. The virtual fundamental class is the image of a cone in a vector bundle E_0 over Y by the refined Gysin homomorphism corresponding to the embedding of Y to E_0 as the zero section. In [1], the above cone is constructed using the notion of stacks. Here we will review the construction of the virtual fundamental class in [30], which only uses schemes.

A cone over a scheme Y is a scheme over Y of the form $Spec \oplus_{i\geq 0} \mathcal{S}$ where $\oplus_{i\geq 0}\mathcal{S}_i$ is a graded $\mathcal{O}_Y$-algebra such that $\mathcal{S}_0 = \mathcal{O}_Y$ and $\oplus_{i\geq 0}\mathcal{S}_i$ is generated by the coherent sheaf $\mathcal{S}_1$. For any coherent sheaf $\mathcal{F}$ on Y the scheme $Spec(\mathrm{Sym}\mathcal{F})$ over Y is a cone and we denote it by $C(\mathcal{F})$. If $\iota : Y \to \bar{Y}$ is a closed embedding, then $N_{Y|\bar{Y}} := C(I/I^2)$ is called the normal space of Y in $\bar{Y}$. And we call $C_{Y|\bar{Y}} := Spec\ (\oplus_{i\geq 0} I^i/I^{i+1})$ the normal cone to Y in $\bar{Y}$.

The morphism of sheaves $\varphi : \mathcal{E} \to \mathcal{F}$ induces a morphism of schemes $C(\varphi) : C(\mathcal{F}) \to C(\mathcal{E})$. Let $\mathcal{F}$ be a locally free sheaf. The morphism $C(\varphi)$ gives an action of $C(\mathcal{F})$ on $C(\mathcal{E})$ defined by $f \bullet e = e + C(\varphi)(f)$ for every $e \in C(\mathcal{E})_x$ and $f \in C(\mathcal{F})_Y$. If a cone C is embedded in $C(\mathcal{E})$ such that C is invariant under the action of $C(\mathcal{F})$ we call C a $C(\mathcal{F})$ cone. For example, $C_{Y|M}$ is a $T_M|_Y$ cone where action of $T_M|_Y$ is defined through the morphism $d : \mathcal{J}/\mathcal{J}^2 \to \Omega_M|_Y$.

Let the morphism of complexes $\phi : \mathbb{E}^\bullet \to \mathbb{L}_Y$ be a perfect obstruction theory. Then the following sequence is exact

$$E^{-1} \xrightarrow{(\partial,\phi^{-1})^T} E^0 \oplus \mathcal{J}/\mathcal{J}^2 \xrightarrow{(\phi^0,-d)} \Omega_M|_Y \longrightarrow 0. \tag{2.2}$$

We will use E_i to denote $C(E^{-i})$ for $i = 0, 1$. Let Q be the kernel of $(\phi^0, -d)$. Since $E_0 \times_Y C_{Y|M}$ is a $T_M|_Y$ cone, by Proposition 2.7 of [30] there exists a unique cone D embedded in $C(Q)$ such that locally there exists an isomorphism $E_0 \times_Y C_{Y|M} \to T_M \times_Y D$. Moreover the following diagram is cartesian

$$\begin{tikzcd}
TM|_Y \arrow[r] \arrow[d] & C(E^0) \times_Y C_{Y|M} \arrow[d] \\
Y \arrow[r, "0_{E_1}"] & E_1
\end{tikzcd} \tag{2.3}$$

Since $C_{Y|M}$ is equidimensional of dimension $\dim M$, D is equidimensional of dimension $\mathrm{rk}E^0$. Since $C(Q)$ is embedded in E_1, we can send the class in $A_{\mathrm{rk}E^0}(E_1)$ represented by the cycle of D to a class $A_{\mathrm{rk}E^0-\mathrm{rk}E^{-1}}(Y)$ using the refined Gysin homomorphism corresponding to the zero section $0_{C(E^{-1})} : Y \to E_1$. The resulting class $[Y]^{vir} := 0^!_{E_1}[D]$ is shown in [30] to be independent of the embedding $\iota : Y \to M$ and also independent of the representation of $E^\bullet$. Moreover, Theorem 4.6 of [30] tells us that $[Y]^{vir}$ only depend on the K-theory class $[E_0] - [E_1]$ if Y is projective.

If ϕ is an equivariant perfect obstruction theory i.e. ϕ_i for $i = 0, -1$ and $d : E^{-1} \to E^0$ are equivariant map and the closed embedding ι is also equivariant then the same construction can be carried out equivariantly and we have $[Y]^{\mathrm{vir}} \in A^G_{vd}(Y)$.

In the remaining we will review the perfect obstruction theory of the moduli of stable pairs defined in [24]. Let p, q be the projections $\mathcal{P}_\chi(X,\beta) \times X \to \mathcal{P}_\chi(X,\beta)$ and $\mathcal{P}_\chi(X,\beta) \times X \to X$.

Pandharipande and Thomas showed that $\mathcal{P}_\chi(X,\beta)$ parameterizes objects in the derived category $D^b(X)$ with fixed determinant. Each stable pair $(\mathcal{F}, s)$ corresponds to a complex $I^\bullet := \{\mathcal{O}_X \xrightarrow{s} \mathcal{F}\} \in D^b(X)$. On $\mathcal{P}_\chi(X, i_*\beta) \times X$ the universal pair $(\mathbb{F}, \mathbb{S})$ defines a complex $\mathbb{I}^\bullet := \{\mathcal{O}_{P\times X} \xrightarrow{\mathbb{S}} \mathbb{F}\}$. Let ω_p be the dualizing sheaf of p, which is the pullback $q^*\omega_X$ of the canonical bundle of X.

As a moduli space of objects in the derived category with fixed determinant the deformation-obstruction theory is described in [15] as follows. For any scheme Y and any complex of locally freee sheaf $E^\bullet$ there are morphisms $\iota : \mathcal{O} \to R\mathcal{H}om\,(E^\bullet, E^\bullet)$, $1 \to \mathrm{id}_{E^\bullet}$ and $\mathrm{tr} : R\mathcal{H}om\,(E^\bullet, E^\bullet) \to \mathcal{O}_Y$ such that $\mathrm{tr} \circ \iota = \mathrm{rk}(E^\bullet)\mathrm{id}_{E^\bullet}$. The traceless part $R\mathcal{H}om(E^\bullet, E^\bullet)_0[1]$ of $R\mathcal{H}om(E^\bullet, E^\bullet)$ is the cone of the tr morphism. If $\mathrm{rk}E^\bullet > 0$ then $R\mathcal{H}om(E^\bullet, E^\bullet) \simeq R\mathcal{H}om(E^\bullet, E^\bullet)_0 \oplus \mathcal{O}_Y$.

Consider the following diagram

$$\begin{tikzcd} & \mathcal{P}_\chi(X,\beta) \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{P}_\chi(X,\beta) & & X \end{tikzcd} \tag{2.4}$$

To save space we use $\mathcal{P}$ to denote $\mathcal{P}_\chi(X,\beta)$. Let $A(\mathbb{I}^\bullet) \in \mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes \mathbb{L}_{\mathcal{P}\times X}) = \mathrm{Hom}\,(R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet), \mathbb{L}_{\mathcal{P}\times X}))$ be the truncated Atiyah class of $\mathbb{I}^\bullet$ defined in [15]. The composition of $A(\mathbb{I}^\bullet)$ with the canonical morphisms $R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \to R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$ and the canonical morphism $\mathbb{L}_{\mathcal{P}\times X} \to \mathbb{L}_{\mathcal{P}\times X/X} \simeq p^*\mathbb{L}_{\mathcal{P}}$ is an element in $\mathrm{Ext}^1(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0, p^*\mathbb{L}_{\mathcal{P}})$. Here $\mathbb{L}_{\mathcal{P}\times X/X}$ is the relative cotangent complex corresponding to the morphism q. Since X is projective we can apply Verdier duality so that the above element corresponds to an element in $\mathrm{Ext}^{-2}(Rp_*R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_X, \mathbb{L}_{\mathcal{P}})$. By the identification $\mathrm{Ext}^2(Rp_*R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_X, \mathbb{L}_{\mathcal{P}}) = \mathrm{Hom}\,(Rp_*\,(R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_X)\,[2], \mathbb{L}_{\mathcal{P}})$ we have a morphism $\phi : Rp_*\,(R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_X)\,[2] \to \mathbb{L}_{\mathcal{P}}$.

Pandharipande and Thomas have shown that $Rp_*\,(R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_X)\,[2]$ is a two term complex of locally free sheaves. We will use $\mathbb{E}^\bullet$ to denote the complex $Rp_*\,(R\mathcal{H}om\,(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_p)\,[2]$. The virtual dimension of $\mathcal{P}_\chi(X,\beta)$ is then $-\chi(R\mathcal{H}om\,(I^\bullet, I^\bullet)_0) = \int_\beta c_1(X)$. If X is Calabi-Yau $\omega_X \simeq \mathcal{O}_X$ so that by Serre duality $vd = 0$. If $vd = 0$ then $P_{X,\beta,\chi} := \int_{[\mathcal{P}]^{vir}} 1 \in \mathbb{Z}$ is invariant along a deformation of X. $P_{X,\beta,\chi}$ is called Pandharipande-Thomas invariant or PT-invariant.

One technique to compute PT-invariants is using the virtual localization formula by Graber and Pandharipande. If $G = \mathbb{C}^\times$ acts on $\mathcal{P}_\chi(X,\beta)$ then $\mathbb{L}_{\mathcal{P}_\chi(X,\beta)}$ has a natural equivariant structure. If all morphisms in (2.1) are equivariant, we call ϕ an equivariant perfect obstruction theory. Let $\mathcal{P}^G$ be the fixed locus of $\mathcal{P}$, then $\mathbb{E}^\bullet$ has a sub-bundle $(\mathbb{E}^\bullet|_{\mathcal{P}^G})^{fix}$ which has weight 0 and a sub-bundle $(\mathbb{E}^\bullet|_{\mathcal{P}^G})^{mov}$ with non zero weight such that $\mathbb{E}^\bullet|_{\mathcal{P}^G} = (\mathbb{E}^\bullet|_{\mathcal{P}^G})^{fix} \oplus (\mathbb{E}^\bullet|_{\mathcal{P}^G})^{mov}$. Graber and Pandharipande showed that there exists a canonical morphism $\hat{\phi} : (\mathbb{E}^\bullet|_{\mathcal{P}^G})^{fix} \to \mathbb{L}_{\mathcal{P}^G}$ that induces a perfect obstruction theory for $\mathcal{P}^G$. So that we have the virtual fundamental class $\left[\mathcal{P}^G\right]^{vir}$ of $\mathcal{P}^G$. Graber and Pandaripandhe gives a formula that relates $[\mathcal{P}^G]^{vir}$ with $[\mathcal{P}]^{vir}$ as follows :

$$[\mathcal{P}]^{vir} = i_*\left(\frac{[\mathcal{P}^G]^{vir}}{e(N^{vir})}\right) \in A_*^G \otimes_{\mathbb{Z}} \mathbb{Q}[t, t^{-1}]$$

where $e\left(N^{vir}\right)$ is the top Chern class of the vector bundle $N^{vir} = ((\mathbb{E}^\bullet|_{\mathcal{P}^G})^{mov})^\vee$ and t is the first Chern class of the equivariant line bundle with weight 1.

2.2 Kool-Thomas Invariants

2.2.1 Stable Pairs on Local Surfaces

Let S be a nonsingular projective surface with canonical bundle ω_S and let X be the total space of ω_S i.e. $X = Spec\left(\mathrm{Sym}(\omega_S^\vee)\right)$. Then there is a closed embedding i of S into X as the zero section. Let $\pi : X \to S$ be the structure morphism. Since $\omega_X \simeq \pi^*\omega_S \otimes \pi^*\omega_S^\vee \simeq \mathcal{O}_X$, X is Calabi-Yau. Let $\bar{X} = \mathbb{P}(X \oplus \mathbb{A}^1_S)$, then X is an open subscheme of $\bar{X}$ and let $j : X \to \bar{X}$ be the inclusion and $\bar{\pi} : \bar{X} \to S$ be the structure morphism of $\bar{X}$ as a projective bundle over S. Since S is projective, $\bar{i} := j \circ i : S \to \bar{X}$ is a closed embedding.

Let $\beta \in H_2(S, \mathbb{Z})$ be an effective class and $\chi \in \mathbb{Z}$. By [24] there is a projective scheme $\mathcal{P}_\chi\left(\bar{X}, \bar{i}_*\beta\right)$ parametrizing stable pairs $(\mathcal{F}, s)$ with $\chi(\mathcal{F}) = \chi$ and the cycle $[C_{\mathcal{F}}]$ of the supporting curve is in class β. By removing the pairs $(\mathcal{F}, s)$ with supporting curve $C_{\mathcal{F}}$ which intersect the closed subschem $\bar{X} \smallsetminus X$, we have an open subscheme $\mathcal{P}_\chi(X, i_*\beta)$ that parametrize stable pairs $(\mathcal{F}, s)$ with $\mathcal{F}$ supported on X and let $\hat{j} : \mathcal{P}_\chi(X, i_*\beta) \to \mathcal{P}_\chi\left(\bar{X}, \bar{i}_*\beta\right)$ be the inclusion. Let $\bar{\mathbb{F}}$ be the universal sheaf on $\mathcal{P}_\chi\left(\bar{X}, \bar{i}_*\beta\right) \times \bar{X}$ and $\bar{\mathbb{S}} : \mathcal{O}_{\mathcal{P}_\chi(\bar{X}, \bar{i}_*\beta)\times\bar{X}} \to \bar{\mathbb{F}}$ be the universal section, then their restriction $\mathbb{F}$, $\mathbb{S}$ to $\mathcal{P}_\chi(X, i_*\beta) \times X$ is the universal sheaf and the universal section corresponding to the moduli space $\mathcal{P}_\chi(X, i_*\beta)$. Notice that $\left(\mathrm{id}_{\mathcal{P}_\chi(X,i_*\beta)} \times j\right)_* \mathbb{F} = \left(\hat{j} \times \mathrm{id}_{\bar{X}}\right)^* \bar{\mathbb{F}}$ on $\mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$. We also use $\mathbb{F}$ to denote $\left(\mathrm{id}_{\mathcal{P}_\chi(X,i_*\beta)} \times j\right)_* \mathbb{F}$ on $\mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$.

There exists an action of $G = \mathbb{C}^\times$ on $\bar{X}$ by scaling the fiber such that X is an invariant open subscheme. In Section 2.1.2 we described the canonical action of G on $\mathcal{P}_\chi\left(\bar{X}, \bar{i}_*\beta\right)$. Since X is an invariant open subscheme, $\mathcal{P}_\chi(X, i_*\beta)$ is also invariant in $\mathcal{P}_\chi\left(\bar{X}, \bar{i}_*\beta\right)$. Thus $\bar{\mathbb{F}}$ and $\mathbb{F}$ are equivariant sheaves and $\bar{\mathbb{S}}$ and $\mathbb{S}$ are equivariant morphism of sheaves.

Consider the following diagrams

$$\begin{tikzcd} & \mathcal{P}_\chi(X, i_*\beta) \times \bar{X} \arrow[dl, "\bar{p}"'] \arrow[dr, "\bar{q}"] & \\ \mathcal{P}_\chi(X, i_*\beta) & & \bar{X} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{P}_\chi(X, i_*\beta) \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{P}_\chi(X, i_*\beta) & & X. \end{tikzcd} \tag{2.5}$$

Let $\bar{\mathbb{I}}^\bullet$ be the complex $[\,\mathcal{O}_{\mathcal{P}_\chi(X,i_*\beta)\times\bar{X}} \xrightarrow{\bar{\mathbb{S}}} \mathbb{F}]$ in $D(\mathcal{P}_\chi(X, i_*\beta) \times \bar{X})$ and let $\mathbb{I}^\bullet$ be the complex $[\,\mathcal{O}_{\mathcal{P}_\chi(X,i_*\beta)\times X} \xrightarrow{\mathbb{S}} \mathbb{F}]$ in $D(\mathcal{P}_\chi(X, i_*\beta) \times X)$. Since $\mathbb{F}$ is supported on $\mathcal{P}_\chi(X, i_*\beta) \times X$ one can show that $Rp_*\left(R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \simeq R\bar{p}_*\left(R\mathcal{H}om\left(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet\right)_0\right)\right.$ and $Rp_*\left(R\mathcal{H}om\left(\mathbb{I}^\bullet, \mathbb{I}^\bullet\right)_0 \otimes \omega_X\right) \simeq R\bar{p}_*\left(R\mathcal{H}om\left(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet\right)_0 \otimes \omega_{\bar{X}}\right)$. Thus, the dual of the morphism $\mathbb{L}^\vee_{\mathcal{P}_\chi(X,i_*\beta)} \to Rp_*R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1]$ induced by the Atiyah class $A(\mathbb{F})$ is a perfect obstruction theory on $\mathcal{P}_\chi(X, i_*\beta)$. Let $\mathbb{E}^\bullet$ be the complex $Rp_*\left(R\mathcal{H}om\left(\mathbb{I}^\bullet, \mathbb{I}^\bullet\right)_0 \otimes \omega_X\right)[2]$. Notice that $\omega_X \simeq \mathcal{O}_X \otimes \mathfrak{t}^*$. By Serre duality we have an isomorphism $\left(\mathbb{E}^\bullet\right)^\vee \to \mathbb{E}^\bullet[-1] \otimes \mathfrak{t}$ and $\mathbb{E}$ is a symmetric equivariant obstruction theory.

Let $\mathcal{P}_\chi(S, \beta)$ be the scheme parameterizing stable pairs $(\mathcal{F}, s)$ on S such that the support $C_\mathcal{F}$ of $\mathcal{F}$ is in class β and $\mathcal{F}$ has Euler characteristic $\chi(\mathcal{F}) = \chi$. On $\mathcal{P}_\chi(S, \beta) \times S$ there exists a universal sheaves $\mathbb{F}$ and universal section $\mathbb{S}$. With the closed embedding $\hat{i} := \mathrm{id}_{\mathcal{P}_\chi(S,\beta)} \times i \colon \mathcal{P}_\chi(S, \beta) \times S \to \mathcal{P}_\chi(S, \beta) \times X$, $\mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times X} \longrightarrow \hat{i}_*\mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times S} \xrightarrow{\hat{i}_*\mathbb{S}} \hat{i}_*\mathbb{F}$ is a family of pairs over $\mathcal{P}_\chi(S, \beta)$. This family induces a closed embedding $\mathcal{P}_\chi(S, \beta) \to \mathcal{P}_\chi(X, i_*\beta)$. Indeed, $\mathcal{P}_\chi(S, \beta)$ is a connected component of $\mathcal{P}_\chi(X, i_*\beta)^G$.

Let $\mathbb{I}^\bullet_S$ denote the complex $[\mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times S} \to \mathbb{F}]$ and $\mathbb{I}^\bullet$ denotes the complex

$[\mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times X} \to \hat{i}_*\mathbb{F}]$. Proposition 3.4 of [19] gives us the decomposition of $\mathbb{E}|_{\mathcal{P}_\chi(S,\beta)}$ into its fixed and moving part as follows:

$$\left(\mathbb{E}^\bullet|_{\mathcal{P}_\chi(S,\beta)}\right)^{fix} \simeq R\hat{p}_* R\mathcal{H}om\left(\mathbb{I}_S^\bullet, \mathbb{F}\right)^\vee \qquad \left(\mathbb{E}^\bullet|_{\mathcal{P}_\chi(S,\beta)}\right)^{mov} \simeq R\hat{p}_* R\mathcal{H}om\left(\mathbb{I}_S^\bullet, \mathbb{F}\right)[1] \otimes \mathfrak{t}^* \tag{2.6}$$

We will use $\mathcal{E}^\bullet$ to denote $\left(\mathbb{E}^\bullet|_{\mathcal{P}_\chi(S,\beta)}\right)^{fix}$.

2.2.1.1 Reduced obstruction theory

If there is a deformation of S such that the class β is no longer algebraic, then the virtual fundamental class will be zero because the the virtual class is deformation invariant. If we restrict the deformation inside the locus when β is always algebraic we get the reduced obstruction theory.

Recall that $\mathcal{E}xt^2_{\bar{p}}\left(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet\right)_0$ is the obstruction sheaf of the Pandaripandhe-Thomas obstruction theory. We also use β to enote the Poincaré dual of $\beta \in H_2(S,\mathbb{Z})$. Assume that the map $\cup\beta : H^1(T_S) \to H^2(\mathcal{O}_S)$ induced by the pairing $\Omega_S \otimes T_S \to \mathcal{O}_S$ is surjective. Then Theorem 2.7 of [19] tells us that the following composition is surjective

$$\begin{aligned} \mathcal{E}xt^2_{\bar{p}}\left(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet\right)_0 \longrightarrow \mathcal{E}xt^2_{\bar{p}}(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet) \xrightarrow{\cup A(\bar{\mathbb{I}}^\bullet)} \mathcal{E}xt^3_{\bar{p}}\left(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet \otimes \mathbb{L}_{\mathcal{P}_\chi(X,i_*\beta)\times\bar{X}}\right) \longrightarrow \\ \mathcal{E}xt^3_{\bar{p}}(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet \otimes \bar{q}^*\Omega_{\bar{X}}) \xrightarrow{\mathrm{tr}} R^3\bar{p}_*\bar{q}^*\Omega_{\bar{X}} \simeq H^{1,3}(\bar{X}) \otimes \mathcal{P}_\chi(X, i_*\beta) \end{aligned} \tag{2.7}$$

Theorem 2.7 of [19] also tells us that there exists a perfect obstruction theory $\hat{\phi}$: $\mathbb{E}^\bullet_{red} \to \mathbb{L}_{\mathcal{P}_\chi(X,i_*\beta)}$ where $\mathbb{E}^\bullet_{red}$ is the cone of the morphism $H^3(\Omega_{\bar{X}}) \otimes \mathcal{O}_{\mathcal{P}_\chi(X,i_*\beta)}[1] \to \mathbb{E}^\bullet$ constructed as the composition of the dual of (4.2) shifted by 1 and the canonical morphism $\mathcal{E}xt^2_{\bar{p}}(\bar{\mathbb{I}}^\bullet, \bar{\mathbb{I}}^\bullet)_0^\vee[1] = H^1((\mathbb{E}^\bullet)^\vee)^\vee[1] \to \mathbb{E}^\bullet$. $\hat{\phi}$ is called reduced obstruction theory for $\mathcal{P}_\chi(X, i_*\beta)$.

Proposition 3.4 of [19] gives us the decomposition of $\mathbb{E}^\bullet_{red}|_{\mathcal{P}_\chi(S,\beta)}$ into fixed part and moving part as follows:

$$\left(\left(\mathbb{E}^\bullet_{red}|_{\mathcal{P}_\chi(S,\beta)}\right)^{fix}\right)^\vee = \mathrm{Cone}\left(R\hat{p}_* R\mathcal{H}om\,(\mathbb{I}^\bullet_S, \mathbb{F}) \xrightarrow{\psi} H^2(\mathcal{O}_S)\otimes \mathcal{O}_{\mathcal{P}_\chi(S,\beta)}[-1] \right)$$

$$\left(\mathbb{E}^\bullet_{red}|_{\mathcal{P}_\chi(S,\beta)}\right)^{mov} = R\hat{p}_* R\mathcal{H}om(\mathbb{I}^\bullet_S, \mathbb{F})[1]\otimes \mathfrak{t}$$

where ψ is the composition

$$R\hat{p}_* R\mathcal{H}om\,(\mathbb{I}^\bullet_S, \mathbb{F}) \longrightarrow R\hat{p}_* R\mathcal{H}om(\mathbb{F},\mathbb{F})[1] \xrightarrow{\mathrm{tr}} R\hat{p}_*\mathcal{O}[1] \longrightarrow R^2\hat{p}_*\mathcal{O}[-1]$$

We will use $\mathcal{E}^\bullet_{red}$ to denote $\left(\mathbb{E}^\bullet_{red}|_{\mathcal{P}_\chi(S,\beta)}\right)^{fix}$.

2.2.1.2 div map and point insertions

We will give a proof of the existence of the map div : $\mathcal{P}_\chi(X, i_*\beta) \to \mathrm{Hilb}_\beta(S)$ that maps $(\mathcal{F}, s) \in \mathcal{P}_\chi(X, i_*\beta)$ to a divisor $D \in \mathrm{Hilb}_\beta(S)$ such that $\pi_*\mathcal{F}$ is supported on D. The morphism has been used by Kool and Thomas in [19]. We prove it here because we could not find the proof in the literature.

First we review the construction of a divisor $\operatorname{div}\mathcal{F}$ from a coherent sheaf $\mathcal{F}$ on Y or more generally from a bounded complex of locally free sheaves $\mathcal{F}^\bullet$ defined in [22] and [7]. Recall the notion of depth of a Noetherian local ring R with maximal ideal m. A sequence $(a_1, \dots, a_n)$ of elements of m is called R-regular if for all $0 \le i \le n$, a_i is not a zero divisor for the R-module $\frac{R}{\langle a_1,\dots,a_{i-1}\rangle R}$ and n is called the length of the sequence. The length of the longest R-regular sequence is called the depth of R. Equivalently the depth of R is the smallest p such that $\mathrm{Ext}^p\,(R/m, R) \neq 0$. The depth of a point $p \in Y$ is the depth of the local ring $\mathcal{O}_p$. If Y is nonsingular then the generic point of Y is the only point of depth 0 and the points of depth 1 are exactly those that correspond to the generic point of codimension 1 irreducible subscheme.

Let $\mathcal{F}^\bullet$ be bounded complex of free sheaves on a scheme U such that $\mathcal{F}^\bullet$ is torsion i.e. the support of $\mathcal{F}^\bullet$ does not contain any point of depth 0. Then $\det\mathcal{F}^i \simeq \mathcal{O}_U$ so that there is an isomorphism $\kappa : \bigotimes_{i\in\mathbb{Z}} (\det\mathcal{F}^i)^{(-1)^i} \simeq \mathcal{O}_U$. Outside the support V of $\mathcal{F}^\bullet$, $\mathcal{F}^\bullet$ is exact so that we have a canonical isomorphism $\lambda : \bigotimes_{i\in\mathbb{Z}} (\det\mathcal{F}^i)^{(-1)^i} \to \mathcal{O}_{U\setminus V}$. Thus

$\lambda \circ \kappa^{-1} : \mathcal{O}_{U \smallsetminus V} \to \mathcal{O}_{U \smallsetminus V}$ is an isomorphism so it correspond to unit $f \in \Gamma(U \smallsetminus V, \mathcal{O}_U)$. Since $U \smallsetminus V$ contains all points of depth 0 of U, by Lemma 1 of [7] f defines a Cartier divisor div $(\mathcal{F}^\bullet)$ on U. div $(\mathcal{F}^\bullet)$ has the following properties

Proposition 2.2.1 (Proposition 1 of [7]). *Let $\mathcal{F}^\bullet$ be a torsion bounded complex of free sheaves on a scheme U. Then div$(\mathcal{F}^\bullet)$ satisfies the following properties:*

1. *If $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ are quasi isomorphic then div$^\bullet \mathcal{F}_1$ and div$^\bullet \mathcal{F}_2$ are equal.*
2. *If $g : U' \to U$ is a morphism of schemes then if $g^*\mathcal{F}^\bullet$ is torsion then $g^{-1}(div \mathcal{F}^\bullet)$ is a Cartier divisor and $div(g^*\mathcal{F}^\bullet) = g^{-1} div \mathcal{F}^\bullet$*
3. *If $H^0(\mathcal{F}^\bullet) = \mathcal{F}$ and $H^i(\mathcal{F}^\bullet) = 0$ for $i \neq 0$ then div$(\mathcal{F}^\bullet)$ is an effective Cartier divisor. Moreover if $H^0(\mathcal{F}^\bullet) = \mathcal{O}_D$ of an effective Cartier divisor D then div$(\mathcal{F}^\bullet) = D$.*
4. *Given a morphism $\phi : \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet$ of complexes and let Cone(ϕ) be the mapping cone of ϕ then div$(Cone(\phi)) = div(\mathcal{F}_2^\bullet) - div(\mathcal{F}_1^\bullet)$.*

Let $\mathcal{F}^\bullet$ be a torsion bounded complex of locally free sheaves on a scheme Y. Then locally $\mathcal{F}^\bullet$ is a bounded complex of free sheaves so that div $(\mathcal{F}^\bullet)$ can be defined. By point 1. and 2. of the above proposition we can define div $(\mathcal{F}^\bullet)$ globally by gluing the locally constructed divisors. If $\mathcal{F}$ is a torsion coherent sheaf with a resolution $\mathcal{F}^\bullet$, we can define div $(\mathcal{F}) :=$ div$(\mathcal{F}^\bullet)$. In the above proposition we can replace free sheaves by locally free sheaves.

Let $f : Y' \to Y$ be a projective morphism of Noetherian schemes such that (i) $R^i f_* \mathcal{O}_{Y'} = 0$ for $i > 0$, (ii) $f_*\mathcal{O}_{Y'}$ has a resolution by a bounded complex of locally free sheaves and (iii) if $y \in Y$ has depth 0 (resp. depth 1) then $f^{-1}(y)$ is empty (resp. finite). Then div(f) is defined as div$(f_*\mathcal{O}_{Y'})$. If Y' is a closed subscheme of a scheme $\bar{Y}$ with a projective morphism $\bar{f} : \bar{Y} \to Y$ such that $\bar{f}|_{Y'} = f$ then $\bar{f}_* \mathrm{cycle}_{\bar{Y}}(Y') = \mathrm{cycle}_Y(\mathrm{div}(f))$ where $\mathrm{cycle}_{\bar{Y}}(Y') \in Z_*(\bar{Y})$ is the corresponding cycle of Y' as a subscheme of $\bar{Y}$.

Proposition 2.2.2. *There exists a G-equivariant morphism of schemes div : $\mathcal{P}_\chi(X, i_*\beta) \to |\mathcal{L}|$ that maps the closed point $(\mathcal{F}, s)$ to div$(\bar{\pi}^{\mathcal{P}}_* \mathcal{F})$ where π is the projection $\pi : X \to S$.*

Proof. Let $\mathcal{F}$ be the universal sheaf. Let $\bar{\pi}^{\mathcal{P}} := \mathrm{id}_P \times \pi : \mathcal{P}_\chi(X, i_*\beta) \times \bar{X} \to \mathcal{P}_\chi(X, i_*\beta) \times S$. We will show that $\operatorname{div} \bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ is a flat family of effective Cartier divisors of S such that for every $p \in \mathcal{P}_\chi(X, i_*\beta)$, the class of $\mathrm{cycle}_S \left(\operatorname{div} \bar{\pi}_* \mathbb{F}\right)_p$ in $H_2(S, \mathbb{Z})$ is β. The support $\mathcal{C}$ of $\mathbb{F}$ is proper relative to $\mathcal{P}_\chi(X, i_*\beta)$ so that $\bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ is coherent. $\bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ is also flat over $\mathcal{P}_\chi(X, i_*\beta)$ so that $\bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ has a resolution by a complex of locally free sheaves of finite length. Moreover for each closed point $p : Spec\, \mathbb{C} \to \mathcal{P}_\chi(X, i_*\beta)$, the restriction of $\bar{\pi}^{p}_* \mathbb{F}$ to $\{x\} \times X$ do not contain an points of depth 0 so that by L emma 5 of [7], $\bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ do not contain any points of depth 0 and we can construct $\operatorname{div} \bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ as an effective Cartier divisor of $\mathcal{P}_\chi(X, i_*\beta) \times S$. By the functoriality of the div construction for each point p of $\mathcal{P}_\chi(X, i_*\beta)$, $\left(\operatorname{div} \bar{\pi}^{\mathcal{P}}_* \mathbb{F}\right)_p = \operatorname{div} \left(\bar{\pi}^{\mathcal{P}}_* \mathbb{F}\right)_p = \operatorname{div} \left(\bar{\pi}_* \left(\mathbb{F}_p\right)\right)$ is an effective Cartier divisor of S so that $\operatorname{div} \bar{\pi}^{\mathcal{P}}_* \mathbb{F}$ is flat by Lemma 2.2.3.

It remains to show that the the corresponding cycle of $\operatorname{div} \bar{\pi}_* \mathbb{F}_p$ is in class β. Since $\mathbb{F}_p$ is supported on X the composition $\mathcal{C}_{\mathbb{F}_p} \to \bar{X} \to S$ is an affine morphism so that we have an exact sequence

$$0 \longrightarrow \bar{\pi}_* \mathcal{O}_{\mathcal{C}_p} \longrightarrow \bar{\pi}_* \mathbb{F}_p \longrightarrow \bar{\pi}_* Q_p \longrightarrow 0$$

where $\bar{\pi}_* Q_p$ is supported on subscheme of codimension 2. Then we have $\operatorname{div} \pi_* \mathbb{F}_p = \operatorname{div} \pi_* \mathcal{O}_{\mathcal{C}_p}$. By the proof of Lemma 5.9 of [[22]] we have

$$\mathrm{cycle}_S \left(\operatorname{div} \bar{\pi}_* \mathbb{F}_p\right) = \mathrm{cycle}_S \left(\operatorname{div} \bar{\pi}_* \mathcal{O}_{\mathcal{C}_p}\right) = \bar{\pi}_* \mathrm{cycle}_{\bar{X}} \left(\mathcal{C}_p\right).$$

Notice that $\mathrm{cycle}_{\bar{X}} \left(\mathcal{C}_p\right)$ is in class $\bar{i}_*\beta \in H_2(\bar{X}, \mathbb{Z})$. Since $\bar{\pi}_* \circ \bar{i}_*$ is identity we can conclude that $\mathrm{cycle} \left(\operatorname{div} \bar{\pi}_* \mathbb{F}_p\right)$ is in class β.

It remains to show that $\operatorname{div} : \mathcal{P}_\chi(X, i_*\beta) \to |\mathcal{L}|$ is an equivariant morphism where the action of G on $|\mathcal{L}|$ is described in Lemma 2.2.4. Consider the following cartesian diagram

$$\begin{tikzcd}
G \times \mathcal{P} \times X \arrow[r, "f"] \arrow[d, "\pi^{G\times\mathcal{P}}"'] & \mathcal{P} \times X \arrow[d, "\pi^{\mathcal{P}}"] \\
G \times \mathcal{P} \times S \arrow[r, "\dot{f}"] \arrow[d, "\mathrm{id}_G\times\mathrm{div}"'] & \mathcal{P} \times S \arrow[d, "\mathrm{div}"] \\
G \times \mathrm{Hilb}_\beta(S) \times S \arrow[r, "\hat{f}"] & \mathrm{Hilb}_\beta(S) \times S
\end{tikzcd}$$

where $\dot{f} : G \times \mathcal{P} \times S \to \mathcal{P} \times S$,$(g, p, s) \mapsto (p, g^{-1}s)$. Since

$$\begin{aligned}
(\sigma_{\mathcal{P}} \times \mathrm{id}_S)^{-1} \mathrm{div}^{-1}\mathcal{D} &= (\sigma_{\mathcal{P}} \times \mathrm{id}_S)^{-1} \mathrm{div}\left(\pi_*^{\mathcal{P}}\mathbb{F}\right) \\
&= \mathrm{div}\,(\sigma_{\mathcal{P}} \times \mathrm{id}_S)^* \,\pi_*^{\mathcal{P}}\mathbb{F} \\
&= \mathrm{div}\left(\pi_*^{G\times\mathcal{P}}\,(\sigma_{\mathcal{P}} \times \mathrm{id}_X)^*\,\mathbb{F}\right) \\
&= \mathrm{div}\left(\pi_*^{G\times\mathcal{P}} f^*\mathbb{F}\right) \\
&= \mathrm{div}\left(\dot{f}^*\pi_*^{\mathcal{P}}\mathbb{F}\right) \\
&= \dot{f}^{-1}\mathrm{div}\left(\pi_*^{\mathcal{P}}\mathbb{F}\right) \\
&= \dot{f}^{-1}\mathrm{div}^{-1}\mathcal{D} \\
&= (\mathrm{id}_G \times \mathrm{div})^{-1}\,\hat{f}^{-1}\mathcal{D} \\
&= \mathrm{div}^{-1}(\sigma_{\mathrm{Hilb}_\beta(S)} \times \mathrm{id}_S)^{-1}\mathcal{D}
\end{aligned}$$

we can conclude that $\mathrm{div} \circ \sigma_{\mathcal{P}} = \sigma_{\mathrm{Hilb}_\beta(S)} \circ (\mathrm{id}_G \times \mathrm{div})$. □

Lemma 2.2.3. *If $\mathcal{D} \subset B \times S$ be an effective Cartier divisor, then $\mathcal{D}$ is flat over B if and only if $\mathcal{D}_b$ is an effective Cartier divisor for all closed point $b \in B$.*

Proof. Since $\mathcal{D}$ is a Cartier divisor, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}(-\mathcal{D}) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\mathcal{D}} \longrightarrow 0$$

If $\mathcal{O}_D$ is flat over B then for each point $b \in B$, the restriction of the above exact sequence to b is still exact the ideal sheaf of $\mathcal{D}_b$ is the line bundle $\mathcal{O}(\mathcal{D})_b$. For the converse, since $\mathcal{D}_b$ is a Cartier divisor, the restriction to b of the above exact sequence is exact, in particular $\mathcal{O}(-\mathcal{D})_b \to (\mathcal{O}_{B\times s})_b$ is injective. Since $\mathcal{O}(-\mathcal{D})$ is a line bundle, it is flat over B and by Lemma 2.14 of [14] we can conclude that $\mathcal{O}_{\mathcal{D}}$ is flat over B. $\square$

Lemma 2.2.4. *Let G act on a surface S and $\beta \in H_2(S,\mathbb{Z})$. Let $\mathcal{D} \subset Hilb_\beta(S) \times S$ be the universal divisor. Let $\hat{f} : G \times Hilb_\beta(S) \times S \to Hilb_\beta \times S$, $(g,h,s) \mapsto (h, g^{-1}s)$. Since $\hat{f}$ is flat $\hat{f}^{-1}\mathcal{D} \subset G \times Hilb_\beta(S) \times S$ is an effective divisor and induces a morphism $\sigma_{Hilb_\beta(S)} : G \times Hilb_\beta(S) \to Hilb_\beta(S)$ since $Hilb_\beta(S)$ is a fine moduli space. Then $\sigma_{Hilb_\beta(S)}$ defines an action of G on $Hilb_\beta(S)$.*

For a cohomology classes $\sigma_i \in H^*(X,\mathbb{Z})$, $i = 1,\dots,m$ Kool and Thomas assign a class $\tau(\sigma_i) := p_*\left(c^2(\mathbb{F})\, q^*\sigma\right) \in H^*\left(\mathcal{P}_\chi(X, i_*\beta)\right)$ where $c^2(\mathbb{F})$ is the second Chern class of $\mathbb{F}$ and define the reduced invariants as

$$\mathcal{P}^{red}_{\beta,\chi}(X,\sigma_1,\dots,\sigma_m) := \int_{[\mathcal{P}_\chi(X,i_*\beta)^G]^{vir}} \frac{1}{e\left(N^{vir}\right)} \prod_{i=1}^{m} \tau(\sigma_i).$$

Assume that $b_1(S) = 0$ so that $\mathrm{Hilb}_\beta = |\mathcal{L}|$. It was shown that if for all i , σ_i is the pullback of the Poincaré dual of the $[pt] \in H^4(S,\mathbb{Z})$ represented by a closed point then

$$\mathcal{P}^{red}_{\beta,\chi}(X,[pt]^m) = \int_{j^![\mathcal{P}_\chi(X,i_*\beta)^G]^{vir}} \frac{1}{e\left(N^{vir}\right)}$$

where $j^!$ is the refined Gysin homomorphim corresponding to the following cartesian diagram

$$\begin{tikzcd} \mathbb{P}^\epsilon \times_{|\mathcal{L}|} \mathcal{P}_\chi(X,i_*\beta) \arrow[r] \arrow[d] & \mathcal{P}_\chi(X,i_*\beta) \arrow[d, "\mathrm{div}"] \\ \mathbb{P}^\epsilon \arrow[r, "j"] & |\mathcal{L}| \end{tikzcd}$$

where j is a regular embedding $\mathbb{P}^\epsilon \subset |\mathcal{L}|$ of a sublinear system and $\epsilon = \dim|\mathcal{L}| - m$.

2.2.2 δ-nodal Curve Counting via Kool-Thomas invariants

Recall that a line bundle $\mathcal{L}$ on a surface S is n-very ample if for any subscheme Z with length $\le n+1$ the natural morphsim $H^0(X,\mathcal{L}) \to H^0(Z, \mathcal{L}|_Z)$ is surjective.

We assume that $b_1(S) = 0$ and let $\mathcal{L}$ be $(2\delta+1)$-very ample line bundle on S with $H^1(\mathcal{L}) = 0$. We also assume that the first Chern class $c_1(\mathcal{L}) = \beta \in H^2(S,\mathbb{Z})$ of $\mathcal{L}$

satisfies the condition that the the morphism $\cup\beta : H^1(T_S) \to H^2(\mathcal{O}_S)$ is surjective; in particular then $H^2(\mathcal{L}) = 0$ also. Given a curve C not necessarily reduced and connected, we let $g(C)$ to denote its arithmetic genus, defined by $1 - g(C) := \chi(\mathcal{O}_C)$. If C is reduced its geometric genus $\bar{g}(C)$ is defined to be the $g(\bar{C})$ the genus of its normalisation. And let h denote the arithmetic genus of curves in $|\mathcal{L}|$, so that $2h - 2 = \beta^2 - c_1(S)\beta$.

Proposition 2.1 of [18] and Proposition 5.1 of [19] tells us that the general δ-dimensional linear system $\mathbb{P}^\delta \subset |\mathcal{L}|$ only contains reduced and irreducible curves. Moreover $\mathbb{P}^\delta$ contains finitely many δ-nodal curves with geometric genus $h - \delta$ and other curves has geometric genus $> h - \delta$.

Kool and Thomas also define

$$\mathcal{P}^{red}_{\chi,\beta}(S, [pt]^m) := \int_{[\mathcal{P}_\chi(S,\beta)]^{red}} \frac{1}{e\left(N^{vir}\right)} \tau\left([pt]\right)^m.$$

They compute $P^{red}_{\chi,\beta}(S, [pt]^m)$ in [20] and $P^{red}_{\chi,\beta}(S, [pt]^m)$ is given by the following expression

$$t^{n+\chi(\mathcal{L})-\chi(\mathcal{O}_S)} \left(-\frac{1}{t}\right)^{n+\chi(\mathcal{L})-1-m} \int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1-m}} c_n(\mathcal{L}^{[n]}(1)) \frac{c_\bullet(T_{S^{[n]}}) c_\bullet\left(\mathcal{O}(1)^{\oplus\chi(\mathcal{L})}\right)}{c_\bullet\left(\mathcal{L}^{[n]}(1)\right),} \tag{2.8}$$

where $\mathcal{L}^{[n]}$ is the vector bundle of rank n on $S^{[n]}$ with fiber $H^0(\mathcal{L}|_Z)$ for a point $Z \in S^{[n]}$ and $\mathcal{L}^{[n]}(1) = \mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)$.

Under the above assumption, only the contribution from $\mathcal{P}_\chi(S, \beta)$ counts for $\mathcal{P}^{red}_{\beta,\chi}(X, [pt]^m)$ so $\mathcal{P}^{red}_{\beta,\chi}(X, [pt]^m) = \mathcal{P}^{red}_{\chi,\beta}(S, [pt]^m)$. Define the generating function for $\mathcal{P}^{red}_{\beta,\chi}(X, [pt]^m)$ as

$$\sum_{\chi\in\mathbb{Z}} \mathcal{P}^{red}_{\beta,\chi}(X, [pt]^m) q^\chi$$

then define $\bar{q} = q^{1-i}(1+q)^{2i-2}$ then the coefficient of $\bar{q}^{h-\delta}$ is $n_\delta(\mathcal{L}) t^{h-\delta-1+\int_\beta c_1(S)}$ where $n_\delta(\mathcal{L})$ is the number of δ-nodal curves in $\mathbb{P}^\delta$.

$n_\delta(\mathcal{L})$ has been studied for example in [11] and [18]. In [18], it is shown that after the same change of variable $n_\delta(\mathcal{L})$ can be computed as the coefficient of $\bar{q}^{h-\delta}$ of the generating function

$$\sum_{i=0}^{\infty} e(\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta))q^{i+1-h}$$

where $e(\mathrm{Hilb}^i(\mathcal{C}/\mathbb{P}^\delta)$ is the Euler characteristic of the relative Hilbert scheme of points. Moreover $e(\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta))$ can be computed as

$$\int_{S^{[n]}\times\mathbb{P}^\delta} c_i(\mathcal{L}^{[n]}(1))\frac{c_\bullet(T_{S^{[n]}})\, c_\bullet(\mathcal{O}(1)^{\oplus\delta+1})}{c_\bullet(\mathcal{L}^{[n]}(1))}.$$

In [18], we have to assume that $\mathcal{L}$ is sufficiently ample and $H^i(\mathcal{L}) = 0$ for $i > 0$ so that $\mathrm{Hilb}^n(\mathcal{C}/\mathbb{P}^\delta)$ are smooth. While in [19], $\mathcal{P}^{red}_{\chi,\beta}(S,[pt]^m)$ can be defined under the assumption that $H^2(\mathcal{L}) = 0$ for all $\mathcal{L}$ with $c_1(\mathcal{L}) = 0$. We can think $n_\delta(\mathcal{L})$ as a generalization of the one studied in [18]. In particular, we can think $n_\delta(\mathcal{L})$ as a virtual count of δ-nodal curves for not necessarily ample line bundle $\mathcal{L}$.

Chapter 3

Equivariant K-theoretic PT invariants of local surfaces

In this chapter we will recall the K-theoretic invariants proposed by Nekrasov and Okounkov in [23] and introduce a class that will account for the incidence of the supporting curve of a stable pairs and a point. The definition of this class is motivated by the definition of points insertions in [19].

3.1 $K_{vir}^{1/2}$ and twisted virtual structure sheaf

Let $\phi : E^\bullet \to \mathbb{L}_Y$ be a perfect obstruction theory. Let $\phi : E_1 \to Y$ be the structure morphism of E_1 and let $0_{E_1} : Y \to E_1$ be the zero section. In Section 2.1.3 we describe the construction of the virtual fundamental class $[Y]^{\mathrm{vir}} \in A_{vd}(Y)$ where $vd := \mathrm{rk}E^\bullet = \mathrm{rk}E^0 - \mathrm{rk}E^{-1}$ as the image of the class in $A_{\mathrm{rk}E^0}(E_1)$ represented by the cycle of a cone $D \subset E_1$ by the Gysin homomorphism $0^!_{E_1} : A_{\mathrm{rk}E^0}(E_1) \to A_{\mathrm{rk}E^0 - \mathrm{rk}E^{-1}}(Y)$. As the zero section of E_1, the Koszul sequence gives a resolution for $0_{E_1*}\mathcal{O}_X$ so that we can map the class of $\mathcal{O}_D$ in $G(E_1)$ to a class $\mathcal{O}_X^{vir}$ in $G(Y)$ defined in [6] as

$$\mathcal{O}_X^{vir} := \sum_i^\infty (-1)^i [\mathcal{T}or^i_{\mathcal{O}_{E_1}}(\mathcal{O}_X, \mathcal{O}_D)]_Y \in G(Y).$$

We call $\mathcal{O}_Y^{vir}$ the virtual structure sheaf of Y. Note that $\mathcal{O}_Y^{vir}$ is not a sheaf but a class in the Grothendieck group of coherent sheaves on Y. If ϕ is an equivariant perfect deformation theory, D is an invariant subscheme of E_1 so that we can construct $\mathcal{O}_Y^{vir} \in G^G(Y)$. If Y is proper over $\mathbb{C}$, the virtual fundamental class and virtual structure sheaf are related by the following virtual Riemann-Roch formula by Fantechi and Göttsche in [6]

$$\chi(\mathcal{O}_Y^{vir}) = \int_{[Y]^{vir}} \mathrm{td}(T^{vir}) \tag{3.1}$$

where $T_Y^{vir} := [E_0] - [E_1] \in K(Y)$. We call T_Y^{vir} the virtual tangent bundle and the dual of it's determinant $K_{Y,vir} := (\det E_0)^{-1} \otimes \det E_1 = \det E^0 \otimes (\det E^{-1})^{-1} \in \mathrm{Pic}(Y)$ the virtual canonical bundle.

If $vd = 0$, by equation (3.1) we have

$$\chi(\mathcal{O}_Y^{vir}) = \int_{[Y]^{vir}} 1 \in \mathbb{Z} \tag{3.2}$$

so that we can use either virtual structure sheaf or virtual fundamental class to define a numerical invariant. If there exist an isomorphism $\theta : E^\bullet \to (E^\bullet)^\vee [1]$ then $\mathrm{rk}E^\bullet = \mathrm{rk}\left((E^\bullet)^\vee [1]\right) = -\mathrm{rk}E^\bullet$ so that $vd = 0$

The next development in enumerative geometry is to give refinements of these numerical invariants. In [23], Nekrasov and Okounkov propose that we should choose a square root of K^{vir} and work with the twisted virtual structure sheaf [28]

$$\hat{\mathcal{O}}_Y^{vir} := K_{Y,vir}^{\frac{1}{2}} \otimes \mathcal{O}_Y^{vir}.$$

To get a refinement of (3.2), we have to consider the action of the symmetry group of Y so that $\chi\left(\hat{\mathcal{O}}_Y^{vir}\right)$ is a function with the equivariant parameter as variables. For example let Y be the moduli space of stable pairs on a toric 3-folds X and $(\mathbb{C}^\times)^3$acts on Y. Choi, Katz and Klemm have calculated $\chi(\hat{\mathcal{O}}_Y^{vir})$ where X is the total space of the canonical bundle K_S for $S = \mathbb{P}^2$ and $S = \mathbb{P}^1 \times \mathbb{P}^1$ in [2]. They have shown that the generating function with coefficients $\chi(\hat{\mathcal{O}}_Y^{vir})$ calculates a refinement of BPS invariants.

One advantage of working equivariantly is that to compute $\chi\left(\hat{\mathcal{O}}_Y^{vir}\right)$, we can use the virtual localization formula for the Grothendieck group of coherent sheaves from [27] by Qu . Let $G = \mathbb{C}^\times$ act on Y and $\phi : \mathbb{E}^\bullet \to \mathbb{L}_Y$ be an equivariant perfect obstruction theory. Similar to the virtual localization formula by Graber and Pandaripandhe, it states that, the virtual structure sheaf equals a class coming from the fixed locus. On Y^G we can decompose $\mathbb{E}^\bullet$ into $(\mathbb{E}^\bullet)^{fix} \oplus (\mathbb{E}^\bullet)^{mov}$ where $(\mathbb{E}^\bullet)^{fix}$ is a two term complex with zero weight and $(\mathbb{E}^\bullet)^{mov}$ is a two term complex with non zero weight. Let $i : Y^G \to Y$ be the closed embedding and let $N^{vir} = ((\mathbb{E}^\bullet)^{mov})^\vee$. Then the virtual localization formula can be stated as

$$i_*\left(\frac{\mathcal{O}_{Y^G}^{vir}}{\bigwedge^\bullet (N^{vir})^\vee}\right) = \mathcal{O}_Y^{vir} \qquad \in G^G(Y) \otimes_{\mathbb{Z}[\mathfrak{t},\mathfrak{t}^{-1}]} \mathbb{Q}(\mathfrak{t}) \tag{3.3}$$

where for a two term complex $F^\bullet = [F^{-1} \to F^0]$, $\bigwedge^\bullet F^\bullet = \frac{\sum_{i=0}^{r_0}(-1)^i \bigwedge^i F^0}{\sum_{j=0}^{r_1}(-1)^j \bigwedge^j F^{-1}}$ with $r_i = \mathrm{rk} F^{-i}$. On the fixed locus, the Grothendieck group of coherent sheaves is isomorphic to the tensor product $G(Y^G) \otimes_{\mathbb{Z}} K^G(pt)$ which is easier to work with.

To incorporate $K_{Y,vir}^{\frac{1}{2}}$ in our computation we will consider a double cover G' of G so that $\mathfrak{t}^{\frac{1}{2}}$ is a representation of G'. Explicitly let $\zeta : G' := \mathbb{C}^\times \to \mathbb{C}^\times = G$, $z \mapsto z^2$ be the double cover. Then G' acts on Y via ζ by defining $\sigma'_Y : G' \times Y \to Y, (g', y) \mapsto \sigma_Y(\zeta(g'), y)$ where $\sigma : G \times Y \to Y$ is the morphism defining the action of G on Y. Also via ζ any G-equivariant sheaf $\mathcal{F}$ on Y is a G'-equivariant sheaf by pulling back the equivariant structure via ζ. This gives an exact functor $Coh^G(Y) \to Coh^{G'}(Y)$ and a group homomorphism $\hat{\zeta} : G^G(Y) \to G^{G'}(Y)$. Moreover $\hat{\zeta}$ is a morphism of $K^G(\mathrm{pt})$-modules. For example, the primitive representation $\mathfrak{t}$ of G has weight 2 at G' module. We can take the primitive representation of G' as the canonical square root of $\mathfrak{t}$ and denote it by $\mathfrak{t}^{\frac{1}{2}}$.

Next we have to compute the restriction of $K_{Y,vir}^{\frac{1}{2}}$ on the fixed locus. Notice that $Y^{G'} = Y^G$. Assume that there exist an isomorphism $\theta : E^\bullet \to (E^\bullet)^\vee [1] \otimes \mathfrak{t}$.

The follwing argument by Richard Thomas in [31] shows that on Y^G, $K_{Y,vir}^{\frac{1}{2}}$ has a canonical equivariant structure.

We decompose $E^\bullet|_{Y^G}$ into its weight spaces so that

$$E^\bullet|_{Y^{\mathbb{C}^\times}} = \bigoplus_{i\in\mathbb{Z}} F^i \mathfrak{t}^i$$

where F^i are two-term complex of non-equivariant vector bundle which only finitely many of them are nonzero and $\mathfrak{t}$ is a representation of G of weight 1. $\det E^\bullet$ can be computed as the determinant of its class in $K^G(Y)$.The isomorphism θ implies that $[(F^i)^\vee] = [F^{-i-1}[-1]]$ in $K^G(Y)$. Thus $K_{Y,vir}$ is a squre twisted by a power of $\mathfrak{t}$, explicitly

$$K_{Y,vir} = \left(\bigotimes_{i\geq 0} \det\left(F^i\mathfrak{t}^i\right)\right)^{\otimes 2} \mathfrak{t}^{r_0+r_1+\ldots}$$

where $r_i = \mathrm{rk} F^i$. Thus the canonical choice for $\left.K_{Y,vir}^{\frac{1}{2}}\right|_{Y^G}$ is

$$\bigotimes_{i\geq 0} \det\left(F^i\mathfrak{t}^i\right) \otimes \mathfrak{t}^{\frac{1}{2}(r_0+r_1+\ldots)} \in K^G(Y^G) \otimes_{\mathbb{Z}[\mathfrak{t},\mathfrak{t}^{-1}]} \mathbb{Z}[\mathfrak{t}^{\frac{1}{2}}, \mathfrak{t}^{-\frac{1}{2}}].$$

Recall that N^{vir} is the moving part of the dual of $E^\bullet|_{Y^G}$ so that in our case $(N^{vir})^\vee = \bigoplus_{i\neq 0} F^i\mathfrak{t}^i$.

After choosing a square root of $K_{Y,vir}$, and assuming that the square root has an equivariant structure, by equation (3.3) we then have

$$i_*\left(\frac{\mathcal{O}_{Y^G}^{vir} \otimes \left.K_{Y,vir}^{\frac{1}{2}}\right|_{Y^G}}{\Lambda^\bullet\left(N^{vir}\right)^\vee}\right) = \hat{\mathcal{O}}_Y^{vir} \qquad \in K^G(Y) \otimes_{\mathbb{Z}[\mathfrak{t},\mathfrak{t}^{-1}]} \mathbb{Q}(\mathfrak{t}^{\frac{1}{2}})$$

If Y is compact we can apply $R\Gamma$ to both sides of the above equation and we have

$$R\Gamma\left(Y^G, \frac{\mathcal{O}_{Y^G}^{vir} \otimes \left.K_{Y,vir}^{\frac{1}{2}}\right|_{Y^G}}{\Lambda^\bullet\left(N^{vir}\right)^\vee}\right) = R\Gamma\left(Y, \hat{\mathcal{O}}_Y^{vir}\right) \in \mathbb{Q}(\mathfrak{t}^{\frac{1}{2}}). \tag{3.4}$$

Thomas has proved the above identity in [31] without using equation (3.3). Furthermore Thomas has shown that

$$R\Gamma\left(Y^G, \frac{\mathcal{O}^{vir}_{Y^G}\otimes K^{\frac{1}{2}}_{Y,vir}\Big|_{Y^G}}{\bigwedge^\bullet (N^{vir})^\vee}\right)\Bigg|_{\mathsf{t}=1} = \int_{[M]^{vir}}\frac{1}{e(N^{vir})}\in\mathbb{Q}$$

In the case that we are interested on, the moduli space Y is not compact. Thus we will use the left hand side of equation (3.4) to define our invariants.

3.2 Equivariant K-theoretic PT invariants of local surfaces

3.2.1 Equivariant K-theoretic invariants

Let Y be the moduli space of stable pairs on the canonical bundle $X := Spec\,(\mathrm{Sym}\,\omega_S^\vee)$ of a smooth projective surface i.e. $Y = \mathcal{P}_\chi(X, i_*\beta)$ for some $\chi\in\mathbb{Z}$ and $\beta\in H_2(S,\mathbb{Z})$ where $i : S\to X$ is the zero section. We will use π to denote the structure map $X\to S$ of X as a vector bundle over S. Note that $\mathcal{P}_\chi(X, i_*\beta)$ is a quasiprojective scheme over $\mathbb{C}$. In particular, $\mathcal{P}_\chi(X, i_*\beta)$ is separated and of finite type.

Let $G = \mathbb{C}^\times$ act on X by scaling the fiber of π. Consider the following diagram:

$$\begin{tikzcd} & \mathcal{P}_\chi(X, i_*\beta)\times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{P}_\chi(X, i_*\beta) & & X \end{tikzcd} \tag{3.5}$$

Recall form Chapter 2 that $\mathcal{P}_\chi(X, i_*\beta)$ has an equivariant perfect obstructrion theory $\phi : \mathbb{E}^\bullet\to\mathbb{L}_{\mathcal{P}_\chi(X,i_*\beta)}$ where $\mathbb{E}^\bullet$ is the complex $Rp_*\left(R\mathcal{H}om\,(\mathbb{I}^\bullet,\mathbb{I}^\bullet)_0\otimes\omega_p\right)[2]$ with $\omega_P = q^*\omega_X$. Since X is Calabi-Yau $\omega_X\simeq\mathcal{O}\otimes\mathsf{t}^*$ Serre duality gives us the isomorphism

$$(\mathbb{E}^\bullet)^\vee\simeq\mathbb{E}^\bullet[-1]\otimes\mathsf{t}. \tag{3.6}$$

So that by Proposition 2.6 of [31] we have an equivariant line bundle

$K^{\frac{1}{2}}_{\mathcal{P}_\chi(X,i_*\beta),vir}\Big|_{\mathcal{P}_\chi(X,i_*\beta)^G}$ on $\mathcal{P}_\chi(X,i_*\beta)^G$.

We want to study how to define a class that contains the information about the incidence between a K-theory class in $K^T(X)$ and the class of the universal sheaf $\mathbb{F}$. From another direction we also want to give a refinement for the Kool-Thomas invariants. In [19], Kool and Thomas take the cup product of the second Chern class of the universal sheaf $\mathbb{F}$ with the cohomology class coming from X. Informally we could think that as taking the intersection between the universal supporting curve and the Poincaré dual of the supporting curve.

In this thesis we are exploring two approaches. In the first approach we are trying to immitate the definition of descendent used in the article [19]. In [19] the authors are cupping the cohomology class coming from X with the second Chern class of $\mathbb{F}$. Since we are unfamiliar on how to define Chern classes as a K-theory class, we are considering to take the class of the structure sheaf of the supporting scheme $\mathcal{O}_{\mathcal{C}_\mathbb{F}}$ and take the tensor product of $\mathcal{O}_{\mathcal{C}_\mathbb{F}}$ with the the class coming from X through the projection $q : \mathcal{P}_\chi(X,i_*\beta) \times X \to X$. In the second approach we use the K-theory class on $\mathcal{P}_\chi(X,i_*\beta) \times S$ of the structure sheaf of the divisor $\operatorname{div} \pi_*\mathbb{F}$ and take the tensor product of $\mathcal{O}_{\operatorname{div}\pi_*\mathbb{F}}$ with the class coming from S through the projection $q_S : \mathcal{P}_\chi(X,i_*\beta) \times S \to S$.

The following proposition is an equivariant version of Proposition 2.1.0 in [14] which we will use to define the K-theory class.

Proposition 3.2.1. *Let $f : Y \to T$ be a smooth projective G-map of relative dimension n with G-equivariant f-very ample line bundle $\mathcal{O}_Y(1)$. Let $\mathcal{F}$ be a G-equivariant sheaf flat over T. Then there is a resolution of $\mathcal{F}$ by a bounded complex of G-equivariant locally free sheaves :*

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow \dots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}$$

where all morphisms are G-equivariant such that $R^n f_* F_\nu$ is locally free for $\nu = 0, \dots, n$ and $R^i f_* F_\nu = 0$ for $i \neq n$ and $\nu = 0, \dots, n$.

Proof. The equivariant structure of all sheaves constructed in the proof of Proposition

2.1.10 in [14] can be defined canonically. $\square$

If $\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ is flat over $\mathcal{P}_\chi(X, i_*\beta)$ then $\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ define a K-theory class in $\mathcal{P}_\chi(X, i_*\beta) \times X$. To push the tensor product down to a K-theory class in $\mathcal{P}_\chi(X, i_*\beta)$, we push forward $\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ to $\mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$ where $\bar{X}$ is $\mathbb{P}(K_S \oplus \mathcal{O}_S)$ the projective completion of X. Since $\mathcal{C}_{\mathbb{F}}$ is proper relative to $\mathcal{P}_\chi(S, \beta)$ the push forward $i_*\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ by the open embedding $i : \mathcal{P}_\chi(X, i_*\beta) \times X \to \mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$ is a coherent sheaf on $\mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$. Then Proposition 3.2.1 implies that $\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ has a resolution by a finite complex of locally free sheaf $F^\bullet$ on $\mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$ so that we can take $[\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}] \coloneqq \sum_i (-1)^i [F^i]$. The class $[\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}]$ is independent of the resolution.

In Chapter 1 we have described the ring homomorphism $f^* : K^G(\bar{Y}) \to K^G(Y)$ for any morphism of sheaves $f : Y \to \bar{Y}$. We also described the group homomorphism $f_* : K^G(Y) \to K^G(\bar{Y})$ when f is the structure morphism of a projective bundle or when f is finite and $f_*\mathcal{F}$ has a resolution by locally free sheaves.

Consider the following diagram

$$\begin{tikzcd} & \mathcal{P}_\chi(X, i_*\beta) \times \bar{X} \arrow[dl, "\bar{p}"'] \arrow[dr, "\bar{q}"] & \\ \mathcal{P}_\chi(X, i_*\beta) & & \bar{X} \end{tikzcd} \tag{3.7}$$

Let $\bar{\pi} : \bar{X} \to S$ be the structure morphism of $\bar{X}$ as a projective bundle over S. We assign for each class $\alpha \in K^T(X)$ a class $\gamma(\alpha)$ in $K^T(\mathcal{P}_\chi(X, i_*\beta))$ as follows. The pullback map $\pi^* : K^T(S) \to K^T(X)$ is an isomorphism. Thus there exist a unique class $\hat{\alpha} \in K^T(S)$ such that $\pi^*\hat{\alpha} = \alpha$. We define $\gamma(\alpha) \coloneqq \bar{p}_*\left(\left[\mathcal{O}_{\mathcal{C}_{\bar{\mathbb{F}}}}\right] . [\bar{q}^* \circ \bar{\pi}^* \hat{\alpha}]\right)$. By Proposition 3.2.1, $\left[\mathcal{O}_{\mathcal{C}_{\bar{\mathbb{F}}}}\right] \in K^T(\mathcal{P}_\chi(X, i_*\beta) \times \bar{X})$ and since $\bar{X}$ is smooth and projective over $\mathbb{C}$, $\bar{p}_*$ can be defined as the composition of i_* and r_* where i is a regular embedding and r is the structure morphism $\mathbb{P}^N_{\mathcal{P}_\chi(X, i_*\beta)} \to \mathcal{P}_\chi(X, i_*\beta)$. Thus the class $\gamma(\alpha)$ is well defined. In particular for every subscheme $Z \subset X$, $\gamma(\mathcal{O}_Z)$ is an element in $K^T(\mathcal{P}_\chi(X, i_*\beta))$.

For the second approach, $\operatorname{div} \pi_*\mathbb{F}$ is a Cartier divisor on $\mathcal{P}_\chi(X, i_*\beta) \times S$ so that we have a line bundle $\mathcal{O}(\operatorname{div}\pi_*\mathbb{F})$ and exact sequence

$$0 \longrightarrow \mathcal{O}(-\operatorname{div}\pi_*\mathbb{F}) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\operatorname{div}\pi_*\mathbb{F}} \longrightarrow 0\,.$$

Thus the K-theory class of $\mathcal{O}_{\operatorname{div}\pi_*\mathbb{F}}$ is $1-[\mathcal{O}(-\operatorname{div}\pi_*\mathbb{F})]$.

Consider the following diagram

$$\begin{tikzcd} & \mathcal{P}_\chi(X, i_*\beta)\times S \arrow[dl, "\hat{p}"'] \arrow[dr, "q_S"] & \\ \mathcal{P}_\chi(X, i_*\beta) & & S. \end{tikzcd} \tag{3.8}$$

Similar to the first approach we assign for each $\alpha \in K^T(X)$ the class $\bar{\gamma}(\alpha) := \hat{p}_*\left([\mathcal{O}_{\operatorname{div}\pi_*\mathbb{F}}].q_S^*\hat{\alpha}\right)$.

In this thesis we only working for the case when α is represented by the class of the pullback of a closed point $s \in S$. Instead of $\gamma\left(\pi^*[\mathcal{O}_s]\right)$ we will use $\gamma\left([\mathcal{O}_s]\right)$ to denote this class. We also assume that $b_1(S)=0$ so that Hilb_β is simply $|\mathcal{L}|$ for a line bundle $\mathcal{L}$ on S with $c_1(\mathcal{L})=\beta$. In this thesis, we want to study the following invariants

$$R\Gamma\left(\mathcal{P}^G, \frac{\mathcal{O}^{vir}_{\mathcal{P}^G}}{\bigwedge^\bullet (N^{vir})^\vee} \otimes K^{\frac{1}{2}}_{\mathcal{P},vir}|_{\mathcal{P}^G} \otimes \prod_{i=1}^m \beta_i\bigg|_{\mathcal{P}^G}\right) \in \mathbb{Q}\left(\mathfrak{t}^{\frac{1}{2}}\right) \tag{3.9}$$

where β_i is either $\gamma\left(\mathcal{O}_{s_i}\right)$ or $\bar{\gamma}(\mathcal{O}_{s_i})$ with $\mathcal{O}_{s_i}$ are the classes of the structure sheaves of closed points $s_i \in S$. In a special case that we have worked out in this thesis, in order to make the invariant coincide with Kool-Thomas invariant when we evaluate it at $\mathfrak{t}=1$ we have to replace $\gamma(\mathcal{O}_{s_i})$ by $\frac{\gamma(\mathcal{O}_{s_i})}{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}}$ and $\bar{\gamma}(\mathcal{O}_{s_i})$ with $\frac{\bar{\gamma}(\mathcal{O}_{s_i})}{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}}$. Thus we define the following invariants

$$P_{X,\beta,\chi}(s_1,\dots,s_m) := R\Gamma\left(\mathcal{P}^G, \frac{\mathcal{O}^{vir}_{\mathcal{P}^G}}{\bigwedge^\bullet (N^{vir})^\vee} \otimes K^{\frac{1}{2}}_{\mathcal{P},vir}|_{\mathcal{P}^G} \otimes \prod_{i=1}^m \frac{\gamma(\mathcal{O}_{s_i})}{\mathfrak{t}^{-\frac{1}{2}}-\mathfrak{t}^{\frac{1}{2}}}\bigg|_{\mathcal{P}^G}\right)$$

when $\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ is flat and

$$\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m) := R\Gamma\left(\mathcal{P}^G, \frac{\mathcal{O}^{vir}_{\mathcal{P}^G}}{\bigwedge^\bullet (N^{vir})^\vee} \otimes K^{\frac{1}{2}}_{\mathcal{P},vir}|_{\mathcal{P}^G} \otimes \prod_{i=1}^m \frac{\bar{\gamma}(\mathcal{O}_{s_i})}{\mathsf{t}^{-\frac{1}{2}} - \mathsf{t}^{\frac{1}{2}}}\bigg|_{\mathcal{P}^G}\right)$$

3.2.2 Vanishing of contribution of pairs supported on a thickening of S in X

In this subsection we will prove that under the assumption that all curve that pass through all the m points are reduced and irreducible the contribution the invariants $P_{X,\beta,\chi}(s_1,\ldots,s_m)$ and $\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m)$ of the curves not supported on S is zero.

Proposition 2.1 of [18] tells us that if $\mathcal{L}$ is a $2\delta+1$-very ample line bundle on S then the δ-dimensional general sublinear system $\mathbb{P}^\delta \subset |\mathcal{L}|$ only contain reduced curves. Proposition 5.1 of [19] also implies that these curves are also irreducible. Thus our assumption that all curves passing through all m points are reduced and irreducible is more likely to happen. If for all s_i, $\mathcal{O}_{s_i}$ are in the same class, our assumption automatically holds since we can replace $\{s_i\}$ by $\{s_i'\}$ that satisfies our assumption.

First we work for $P_{X,\beta,\chi}(s_1,\ldots,s_m)$.

Let $\bar{\pi}^{\mathcal{P}} : \mathcal{P}_\chi(X, i_*\beta) \times \bar{X} \to \mathcal{P}_\chi(X, i_*\beta) \times S$ be the pullback of $\bar{\pi}$ and let $i : \mathcal{C} \to \mathcal{P}_\chi(X, i_*\beta) \times \bar{X}$ be the closed embedding of the universal curve. As the composition of projective morphisms is projective then the composition $\bar{\pi}^{\mathcal{P}} \circ i$ is also projective. Notice the above composition equals to the composition $\mathcal{C} \to \mathcal{P}_\chi(X, i_*\beta) \times X \to \mathcal{P}_\chi(X, i_*\beta) \times S$ which is affine. Thus we can conclude that $\bar{\pi}^{\mathcal{P}} \circ i$ is a finite morphism. We denote this morphism by ρ.

Recall the morphism $\mathrm{div} : \mathcal{P}_\chi(X, i_*\beta) \to |\mathcal{L}|$ from Chapter 2 that maps the stable pairs $(\mathcal{F}, s)$ to the supporting curve $C_{\mathcal{F}} \in |\mathcal{L}|$ of $\mathcal{F}$. Let $\mathcal{D} \subset |\mathcal{L}| \times S$ be the universal divisor and let $\mathcal{D}_{\mathcal{P}} \subset \mathcal{P} \times S$ be the family of divisors that correspond to the morphism $\mathrm{div} : \mathcal{P}_\chi(X, i_*\beta) \to |\mathcal{L}|$ and let $j : \mathcal{D}_{\mathcal{P}} \to \mathcal{P} \times S$ be the closed embedding. Equivalently $\mathcal{D}_{\mathcal{P}} = \mathrm{div}^{-1}\mathcal{D}$.

Lemma 3.2.2. *ρ factors through j.*

Proof. The ideal I in $\mathcal{O}_{\mathcal{P}_\chi(X,i_*\beta)\times S}$ corresponding to the divisor $\mathcal{D}_{\mathcal{P}}$ is flat over $\mathcal{P}_\chi(X,i_*\beta)$ and ρ factorize through j if the composition $I \to \mathcal{O}_{\mathcal{P}_\chi(X,i_*\beta)\times S} \to \rho_*\mathcal{O}_{\mathcal{C}}$ is zero. By Nakayama's Lemma it is sufficient to check whether the composition is zero for each $p \in \mathcal{P}_\chi(X,i_*\beta)$. Or equivalently, we can check whether ρ factorize through j at each point $p \in \mathcal{P}_\chi(X,i_*\beta)$.

Let $\rho^p : \mathcal{C}_p \to \{p\}\times S = S$ be the restriction of ρ to the point $p \in \mathcal{P}_\chi(X,i_*\beta)$ and let $W \subset S$ be the scheme theoretic support of $\rho^p_*\mathcal{O}_{\mathcal{C}_p}$. Notice that $|W| = \mathrm{Supp}(\rho_*\mathcal{O}_{\mathcal{C}_p})$ is a curve. We claim that W is a Cartier Divisor. We will show that W is a subscheme of $\operatorname{div}\mathcal{F} = \operatorname{div}\rho_*\mathcal{O}_{\mathcal{C}_p}$ so that ρ^p factorize through j^p. Let $\sigma : \mathcal{O}_S \to \rho^p_*\mathcal{O}_{\mathcal{C}_p}$ be the morphism of sheaves corresponding to the morphism $\rho^p : \mathcal{C}_p \to S$. Then $\mathcal{O}_W$ is the image of σ so that we have an injection $\mathcal{O}_W \to \rho^p_*\mathcal{O}_{\mathcal{C}_p} \to \rho^p_*\mathcal{F}_p$. By Proposition 2.2.1 we have $\operatorname{div}\rho^p_*\mathcal{F}_p = \operatorname{div}\mathcal{O}_W + D$ where D is some effective divisor. Since W is a Cartier divisor then $\operatorname{div}\mathcal{O}_W = W$. So that we can conclude that W is a subscheme of $\operatorname{div}\mathcal{F}$.

It remains to show that W is a Cartier divisor. Let $I \subset \mathcal{O}_S$ be the ideal sheaf of W. It is sufficient to show that I_x is a free $\mathcal{O}_{S,x}$-module of rank 1 for every $x \in X$. For $U = S \setminus W$, the inclusion $I \subset \mathcal{O}_S$ is an isomorphism so that if $x \notin W$, I_x is isomorphic to $\mathcal{O}_{S,x}$. Since S is nonsingular $\mathcal{O}_{S,x}$ is a domain so that it is sufficient to show that I_x is generated by one element $f \in \mathcal{O}_{S,x}$.

Note that the morphism $\rho : \mathcal{C}_p \to S$ is a finite morphism so that $\left(\rho^p_*\mathcal{O}_{\mathcal{C}_p}\right)_x$ is a finitely generated $\mathcal{O}_{S,x}$-module. In particular, $\left(\rho^p_*\mathcal{O}_{\mathcal{C}_p}\right)_x$ is a Cohen-Macaulay $\mathcal{O}_{S,x}$-module. By Proposition IV.13 of [29], any prime $\mathfrak{p} \subset \mathcal{O}_{S,x}$ such that $\mathcal{O}_{S,x}/\mathfrak{p}$ is isomorphic to a submodule of $\left(\rho^p_*\mathcal{O}_{\mathcal{C}_p}\right)_x$ must be generated by a single irreducible element $g \in \mathcal{O}_{S,x}$. There are finitely many of such $\mathfrak{p}$ and we denote them by $\mathfrak{p}_1,\dots,\mathfrak{p}_k$. Let g_i generate $\mathfrak{p}_i$. By Proposition IV.11 of [29], I_x is the intersection $\bigcap_{i=1}^k \mathfrak{q}_i$ where $\mathfrak{q}_i$ is an ideal of $\mathcal{O}_{S,x}$ such that $\mathfrak{p}_i^{n_i} \subset \mathfrak{q}_i \subset \mathfrak{p}_i$ for some positive integer n_i. Since $\mathcal{O}_{S,x}$ is a domain, $\mathfrak{q}_i$ must be generated by a single element $g_i^{m_i}$ for some positive integer m_i. Thus we conclude that I_x is generated by a single element $\prod_{i=1}^k g_i^{m_i}$. □

Let $R \subset \mathcal{P}_\chi(X,i_*\beta)^G$ be a connected component different from $\mathcal{P}_\chi(S,\beta)$. We denote the inclusion $R \subset \mathcal{P}_\chi(X,i_*\beta)$ by ι. For every $(F,s) \in R$ the supporting curve $C \subset X$ is not supported by S but F is supported on an infinitesimal thickening of S

in X. So we have the following diagram where all square are Cartesian

$$\begin{tikzcd}
\mathcal{C}_R \arrow[r] \arrow[d, "i^R"'] & \mathcal{C} \arrow[d, "i"'] & \\
R \times \bar{X} \arrow[r, "\iota_{\bar{X}}"] \arrow[d, "\bar{\pi}^R"'] & \mathcal{P}_\chi(X, i_*\beta) \times \bar{X} \arrow[r, "\bar{q}"] \arrow[d, "\bar{\pi}^{\mathcal{P}}"'] & \bar{X} \arrow[d, "\bar{\pi}"] \\
R \times S \arrow[r, "\iota_S"] \arrow[d, "\hat{p}^R"'] & \mathcal{P}_\chi(X, i_*\beta) \times S \arrow[r, "q_S"] \arrow[d, "\hat{p}"'] & S \\
R \arrow[r, "\iota"] & \mathcal{P}_\chi(X, i_*\beta) & .
\end{tikzcd} \tag{3.10}$$

By base change formula 1.3.8 and projection formula 1.3.7 we have

$$\begin{aligned}
\iota^*\gamma\left(\mathcal{O}_s\right) &= \iota^*\left(\hat{p}\circ\bar{\pi}^{\mathcal{P}}\right)_*\left(\left[\mathcal{O}_{\mathcal{C}}\right].\bar{q}^*\bar{\pi}^*\left[\mathcal{O}_s\right]\right) \\
&= \left(\hat{p}^R\circ\bar{\pi}^R\right)_*\iota_{\bar{X}}^*\left(\left[\mathcal{O}_{\mathcal{C}}\right].\bar{q}^*\bar{\pi}^*\left[\mathcal{O}_s\right]\right) \\
&= \left(\hat{p}^R\circ\bar{\pi}^R\right)_*\left(\iota_{\bar{X}}^*[\mathcal{O}_{\mathcal{C}}].\iota_{\bar{X}}^*\bar{q}^*\bar{\pi}^*\left[\mathcal{O}_s\right]\right) \\
&= \hat{p}_*^R\bar{\pi}_*^R\left([\mathcal{O}_{\mathcal{C}_R}].\left(\pi^R\right)^*\iota_S^*q_S^*\left[\mathcal{O}_s\right]\right) \\
&= \hat{p}_{x*}\left(\bar{\pi}_*^R\left[\mathcal{O}_{\mathcal{C}_R}\right].\iota_S^*q_S^*\left[\mathcal{O}_s\right]\right).
\end{aligned} \tag{3.11}$$

Now we restrict ρ from 3.2.2 to $R \subset \mathcal{P}_\chi(X, i_*\beta)$. By the Lemma 3.2.2 we can write ρ^R as the composition $j^R \circ \lambda^R$. So now we have the following diagram

$$\begin{tikzcd}
\mathcal{C}_R \arrow[r, "\lambda^R"] & \mathcal{D}_R \arrow[r, "j^R"] & R \times S \arrow[r, "q_S \circ \iota_S"] \arrow[d, "\hat{p}^R"'] & S \\
& & R &
\end{tikzcd}$$

By Proposition 3.2.1 the subcategory of flat coherent sheaves on $\mathcal{D}_R$ satisfies all conditions in Lemma 1.3.1 so that by Corollary 1.3.2 we have a group homomorphism $\lambda_*^R : K^G(\mathcal{C}_R) \to K^G(\mathcal{D}_R)$ that maps $[\mathcal{F}]$ to $\chi(\lambda_*^R\mathcal{F})$. By the same argument we can conclude the existence of the group homomorphism $j_*^R : K^G(\mathcal{D}_R) \to K^G(R \times S)$.

Recall the definition of the ring homomorphism $\kappa : K^G(Y) \to \lim K(Y_l)$ from Section 1.4. Although we have not proved that $\pi_*^R \circ i_*^R[\mathcal{O}_{\mathcal{C}}] = j_*^R \circ \lambda_*^R[\mathcal{O}_{\mathcal{C}}]$, by Lemma 1.4.3 we still have $\kappa_{R\times S} \circ \pi_*^R \circ i_*^R = \kappa_{R\times S} \circ j_*^R \circ \lambda_*^R$.

Lemma 3.2.3.

$$\begin{aligned}\kappa_R\left(\gamma\left(\mathcal{O}_s\right)|_R\right) &:= \kappa_R\left(\hat{p}^R_*\left(\bar{\pi}^R_* \circ i^R_*[\mathcal{O}_{\mathcal{C}_R}] \otimes \iota^*_S q^*_S\left[\mathcal{O}_s\right]\right)\right)\\ &= \kappa_R\left(\hat{p}^R_*\left(\left(j^R_* \circ \lambda^R_*[\mathcal{O}_{\mathcal{C}}]\right) \otimes \iota^*_S q^*_S\left[\mathcal{O}_s\right]\right)\right)\end{aligned}$$

We will use $\hat{\gamma}\left(\mathcal{O}_s\right)|_R$ to denote $\hat{p}^R_*\left(\left(j^R_* \circ \lambda^R_*[\mathcal{O}_{\mathcal{C}_R}]\right) \otimes \iota^*_S q^*_S\left[\mathcal{O}_s\right]\right)$ and $[\mathcal{O}_{\mathcal{C}_R}]$ to denote $\lambda_*[\mathcal{O}_{\mathcal{C}_R}]$.

Lemma 3.2.4.

$$R\Gamma\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^{\frac{1}{2}}_{vir}|_R}{\bigwedge^\bullet\left(N^\bullet_{vir}\right)^\vee} \prod_{i=1}^m \gamma\left(\mathcal{O}_{s_i}\right)\Big|_R\right) = R\Gamma\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^{\frac{1}{2}}_{vir}\Big|_R}{\bigwedge^\bullet\left(N^\bullet_{vir}\right)^\vee} \prod_{i=1}^m \hat{\gamma}\left(\mathcal{O}_{s_i}\right)\Big|_R\right)$$

Proof. The Chern character map $ch^G : \mathbb{Q}(\mathfrak{t}^{\frac{1}{2}}) \to \mathbb{Q}((t))$, $\mathfrak{t}^{\frac{1}{2}} \mapsto e^{\frac{1}{2}t}$ where t is the equivariant first Chern class of $\mathfrak{t}$ is an injection since $e^{\frac{1}{2}t}$ is invertible in $\mathbb{Q}((t))$. By virtual Riemann-Roch theorem of [6], Lemma 1.5.1 and Lemma 3.2.3 we have

$$\begin{aligned}ch^G R\Gamma\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^{\frac{1}{2}}_{vir}|_R}{\wedge\left(N^\bullet_{vir}\right)^\vee} \prod_{i=1}^m \gamma\left(\mathcal{O}_{s_i}\right)\Big|_R\right) &= \int_{[R]^{vir}} ch^G\left(\frac{K^{\frac{1}{2}}_{vir}\Big|_R}{\bigwedge^\bullet(N^\bullet_{vir})^\vee} \prod_{i=1}^m \gamma\left(\mathcal{O}_{s_i}\right)\Big|_R\right) \mathrm{td}^G\left(T^{vir}_R\right)\\ &= \int_{[R]^{vir}} \overleftarrow{ch} \circ \kappa\left(\frac{K^{\frac{1}{2}}_{vir}\Big|_R}{\bigwedge^\bullet(N^\bullet_{vir})^\vee} \prod_{i=1}^m \gamma\left(\mathcal{O}_{s_i}\right)\Big|_R\right) \mathrm{td}^G\left(T^{vir}_R\right)\\ &= \int_{[R]^{vir}} \overleftarrow{ch} \circ \kappa\left(\frac{K^{\frac{1}{2}}_{vir}\Big|_R}{\bigwedge^\bullet(N^\bullet_{vir})^\vee} \prod_{i=1}^m \hat{\gamma}\left(\mathcal{O}_{s_i}\right)\Big|_R\right) \mathrm{td}^G\left(T^{vir}_R\right)\\ &= \int_{[R]^{vir}} ch^G\left(\frac{K^{\frac{1}{2}}_{vir}\Big|_R}{\bigwedge^\bullet(N^\bullet_{vir})^\vee} \prod_{i=1}^m \hat{\gamma}\left(\mathcal{O}_{s_i}\right)\Big|_R\right) \mathrm{td}^G\left(T^{vir}_R\right)\\ &= ch^G R\Gamma\left(R, \frac{\mathcal{O}^{vir}_R \otimes K^{\frac{1}{2}}_{vir}\Big|_R}{\bigwedge^\bullet\left(N^\bullet_{vir}\right)^\vee} \prod_{i=1}^m \hat{\gamma}\left(\mathcal{O}_{s_i}\right)\Big|_R\right).\end{aligned}$$

The injectivity of $ch^G : \mathbb{Q}(\mathfrak{t}^{\frac{1}{2}}) \to \mathbb{Q}((t))$ implies the lemma. $\square$

The above lemma also holds if we replace $K_{vir}^{\frac{1}{2}}\Big|_R$ by any class $\alpha \in K^G(R)$.

By the above lemma we can replace $\gamma(\mathcal{O}_s)$ with $\hat{\gamma}(\mathcal{O}_s) = \hat{p}_*(\rho_*[\mathcal{O}_{\mathcal{C}}].q_S^*[\mathcal{O}_s])$. The advantage of using $\hat{\gamma}(\mathcal{O}_s)$ will become clear later.

Lemma 3.2.5. *Let $\mathcal{L}$ be a globally generated line bundle on S. Let* $\dim|\mathcal{L}| = n$ *and $\mathcal{D} \subset |\mathcal{L}| \times S$ be the universal divisor. Then for any point $s \in S$ the fiber product $\mathcal{D} \times_{|\mathcal{L}|\times S} (|\mathcal{L}| \times \{s\})$ is a hyperplane $\mathbb{P}^{n-1} \subset |\mathcal{L}| \times \{s\}$.*

Proof. Let $\mathcal{L}$ be globally generated line bundle on S and let $f : S \to Spec\,\mathbb{C}$ be the structure morphism. Then $S \times |\mathcal{L}| = Proj\left(\mathrm{Sym} f^*(f_*\mathcal{L})^\vee\right)$ and the canonical morphism $\xi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective. Let $\xi^\vee : \mathcal{L}^\vee \to f^*(f_*\mathcal{L})^\vee$ be the dual of ξ. Let e_i be the basis of $f_*\mathcal{L}$ and let $e_i^\vee \in (f_*\mathcal{L})^\vee$ defined as $e_i^\vee(e_j) = 1$ if $i = j$ and 0 if $i \neq j$. Then $\xi^\vee$ sends a local section ψ of $\mathcal{L}^\vee$ to $\xi^\vee(\psi) : \sum_i a_i e_i \mapsto a_i\psi(e_i)\, e_i^\vee$.

Sections of $f^*(f_*\mathcal{L})^\vee$ are linear combinations v of $\{e_i^\vee\}$ with coefficient in $\mathcal{O}_S$ and sections of $\mathrm{Sym} f^*(f_*\mathcal{L})^\vee$ are polynomials P in $\{e_i^\vee\}$ with coefficient in $\mathcal{O}_S$. There is a canonical graded morphism $\phi : f^*(f_*\mathcal{L})^\vee \otimes \mathrm{Sym} f^*(f_*\mathcal{L})^\vee(-1) \to \mathrm{Sym} f^*(f_*\mathcal{L})^\vee$, that sends $v \otimes P$ to the products of the polynomials $v.P$. The composition of $\xi^\vee \otimes \mathrm{id}_{\mathrm{Sym} f^*(f_*\mathcal{L})^\vee(-1)}$ with ϕ sends $\psi \otimes P$ to $\xi^\vee(\psi).P$. Let θ be this composition. This composition is injective since $\xi^\vee$is injective. This composition correspond to the morphism $\sigma : \mathcal{L}^\vee \boxtimes \mathcal{O}(-1) \to \mathcal{O}$ on $S \times |\mathcal{L}|$ which is injective because θ is injective and $Proj$ construction preserve injective morphism. The cokernel σ is the structure sheaf of the universal divisor $\mathcal{D} \subset S \times |\mathcal{L}|$.

For any closed point $s \in S$, we want to show that the restriction of σ to $|\mathcal{L}|$ is still injective. In this case $\mathcal{D} \times_{|\mathcal{L}|\times S} (|\mathcal{L}| \times \{s\})$ is an effective divisor with ideal $\mathcal{O}(-1)$ so that $\mathcal{D} \times_{|\mathcal{L}|\times S} (|\mathcal{L}| \times \{s\})$ is a hyperplane $\mathbb{P}^{n-1}$. Since ξ is surjective, its restriction to s is also surjective. Any element $\alpha \in \mathcal{L}^\vee|_s$ is the restriction of a local section $\psi \in \mathcal{L}^\vee$. Thus if α is not zero there exist $\psi \in \mathcal{L}^\vee$ such that its restriction to s is $\alpha$and e_i such that the $\psi(e_i)|_s = \psi|_s(e_i|_s)$ is not zero. We can conclude that $\xi^\vee|_s$ is injective. Because $\sigma|_s : \psi|_s \otimes P|_s \mapsto \xi^\vee(\psi)|_s\, P|_s$ we can conclude that $\sigma|_s$ is injective. $\square$

We will use $\mathbb{P}^{n-1}_{s_i}$ to denote $\mathcal{D} \times_{|\mathcal{L}|\times S} (|\mathcal{L}| \times \{s\})$.

Lemma 3.2.6. *Let $c_1(\mathcal{L}) = \beta$ and let $\mathcal{P} = \mathcal{P}_\chi(X, i_*\beta)$. Then all squares in the following diagram are Cartesian.*

$$
\begin{tikzcd}
& \mathcal{D}_{\mathcal{P}} \times_{\mathcal{D}} \mathbb{P}^{n-1} \arrow[rr] \arrow[dl, "\bar{j}"] \arrow[dd, "\bar{h}"] & & \mathbb{P}^{n-1} \arrow[dl] \arrow[dd] \\
\mathcal{P} \times \{s\} \arrow[rr, crossing over] \arrow[dd, "h"] & & |\mathcal{L}| \times \{s\} & \\
& \mathcal{D}_{\mathcal{P}} \arrow[rr] \arrow[dl, "j"] & & \mathcal{D} \arrow[dl] \\
\mathcal{P} \times S \arrow[rr] & & |\mathcal{L}| \times S \arrow[from=uu, crossing over] &
\end{tikzcd}
\tag{3.12}
$$

Lemma 3.2.7. *If $\beta \in G^T(\mathcal{P})$ is supported on $V \subset \mathcal{P}$ then $\beta.\hat{\gamma}(\mathcal{O}_s)$ is supported on $V \times_{\mathcal{P}} W_s$ where $W_s := \mathcal{D}_{\mathcal{P}} \times_{\mathcal{P}\times S} (\mathcal{P} \times \{s\})$.*

Proof. Recall the morphism $\hat{p}$ from diagram (3.10) and $h, \bar{h}$ from (3.12). Since $\hat{p} \circ h = \mathrm{id}_{\mathcal{P}}$ we can conclude that $\hat{\gamma}(\mathcal{O}_s) = h^* j_* [\mathcal{O}_{\mathcal{C}}] = h^* [j_* \mathcal{O}_{\mathcal{C}}]$. Let $E^\bullet$ be a finite resolution of $j_*\mathcal{O}_{\mathcal{C}}$ by locally free sheaves. It's sufficient prove the statement for the case when β is the class of a coherent sheaf $\mathcal{F}$ on V. By Lemma 1.3.11, we have

$$
\begin{aligned}
[\mathcal{F}].\hat{\gamma}(\mathcal{O}_s) &= \sum_i (-1)^i \left[\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{P}\times\{s\}}} \left(\mathcal{O}_{\mathcal{P}\times\{s\}} \otimes E^i\right)\right]_{\mathcal{P}\times\{s\}} \\
&= \sum_i (-1)^i \left[\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{P}\times S}} E^i\right]_{\mathcal{P}\times\{s\}} \\
&= \sum_i (-1) \left[\mathcal{T}or^i_{\mathcal{P}\times S}(\mathcal{F}, j_*\mathcal{O}_{\mathcal{C}})\right]_{\mathcal{P}\times\{s\}} \\
&= \bar{j}_* k_* j^{[\mathcal{O}_{\mathcal{C}}]}(\mathcal{F}).
\end{aligned}
$$

where $j^{[\mathcal{O}_{\mathcal{C}}]}$ is the refined Gysin homomorphism from Chapter 1 and k is the closed embedding $V \times_{\mathcal{P}\times\{s\}} W_s \to W_s$ where $W_s = \mathcal{D}_{\mathcal{P}} \times_{\mathcal{D}} \mathbb{P}^{n-1}_s$.

□

Lemma 3.2.8. *Given m points $s_1, \dots, s_m \in S$ in general position such that all curves in $|\mathcal{L}|$ that passes through all m points are reduced and irreducible, then for any component $R \subset \mathcal{P}^G$ different from $\mathcal{P}_\chi(S,\beta)$ we have $\iota_* \mathcal{O}_R^{vir} . \prod_{i=1}^m \hat{\gamma}(\mathcal{O}_{s_i}) = 0$.*

Proof. Let $\beta_l = \iota_* \mathcal{O}_R^{vir} . \prod_{i=1}^l \hat{\gamma}(\mathcal{O}_{s_i})$. By Lemma 3.2.7, β_1 is supported on $R \times_{\mathcal{P}} W_s =$

$R \times_{|\mathcal{L}|} \mathbb{P}^{n-1}_{s_1}$. Our assumptions implies that for any $1 \le l \le m$, $\bigcap_{i=1}^{l-1} \mathbb{P}^{n-1}_{s_1}$ is not contained in $\mathbb{P}^{n-1}_{s_l}$. In particular, $\bigcap_{i=l}^{l} \mathbb{P}^{n-1}_{s_l} = \mathbb{P}^{n-m}$ and by induction we can conclude that β_m is supported on $R \times_{|\mathcal{L}|} \mathbb{P}^{n-m}$. Note that all curves in $\mathbb{P}^{n-m}$ is reduced and irreducible.

We will show that for any $(\mathcal{F}, s) \in R$, div $(\mathcal{F}, s)$ is not in $\mathbb{P}^{n-m}$. Let $C_{\mathcal{F}}$ be the curve on X supporting an element $(\mathcal{F}, s) \in R$. Note that the reduced subscheme $C_{\mathcal{F}}^{red}$ of $C_{\mathcal{F}}$ is a curve on S so that if $C_{\mathcal{F}}$ is reduced and irreducible then $C_{\mathcal{F}} = C_{\mathcal{F}}^{red}$ is a curve on S and $(\mathcal{F}, s)$ can't be in R. If $C_{\mathcal{F}}$ is not irreducible, then the support of $\pi_* \mathcal{O}_{C_{\mathcal{F}}}$ is not irreduble so that $\operatorname{div}(\mathcal{F}, s)$ is not in $\mathbb{P}^{n-m}$. So we are left with the case when $C_{\mathcal{F}}$ is irreducible. Let C be the reduced subscheme of $C_{\mathcal{F}}$. Let $Spec\, A \subset S$ be an open subset such that K_S is a free line bundle over $Spec\, A$. We can write $C = Spec\, A/(f)$ for an irreducible element $f \in A$ and $X|_{Spec\, A} = Spec\, A[x]$. Then $\mathcal{O}_{C_{\mathcal{F}}}$ can be written as $M := \oplus_{i=0}^{r} A/(f^{n_i}) x^i$ for some positive integers r, n_i and $\operatorname{div} M$ is described by the ideal $(f^{\sum_i n_i})$. Since $C_{\mathcal{F}}$ is not supported on S, then $\sum_i n_i \ge 2$ and $\operatorname{div} M$ is not reduced. Thus in this case $\operatorname{div}(\mathcal{F}, s)$ is not in $\mathbb{P}^{n-m}$.

Since $\operatorname{div}(R)$ is disjoint from $\mathbb{P}^{n-m}$, we can conclude that $R \times_{|\mathcal{L}|} \mathbb{P}^{n-m}$ is empty. By lemma 1.3.9, β_m is zero. □

Following the proof of Lemma 3.2.7 and Lemma 3.2.8and by replacing $[\mathcal{O}_{\mathcal{C}}]$ with $[\mathcal{O}_{\mathcal{D}}]$ we can prove that the contribution to $\bar{P}_{X,\beta,\chi}(s_1, \ldots, s_m)$ of the component $R \subset \mathcal{P}^G$ where $R \neq \mathcal{P}_\chi(S, \beta)$ is zero when $s_1, \ldots, s_m$ is in general position and all curves on S that passthrough all m points are reduced and irreducible.

Actually we have a stronger result for $\bar{P}_{X,\beta,\chi}(s_1, \ldots, s_m)$. By Proposition 4.2.2 for any point $s \in S$, $\bar{\gamma}(\mathcal{O}_s)$ is $1 - [\operatorname{div}^* \mathcal{O}(-1)]$. In particular it's independent from the choosen point.

Proposition 3.2.9. *Given a positive integer δ, let S be a smooth projective surface with $b_1(S) = 0$. Let $\mathcal{L}$ be a $2\delta + 1$-very ample line bundle on S with $c_1(\mathcal{L}) = \beta$ and $H^i(\mathcal{L}) = 0$ for $i > 0$. Let $X = K_S$ be the canonical line bundle over S. Then for any connected component R of $\mathcal{P}_\chi(X, i_*\beta)^{\mathbb{C}^\times}$ different from $\mathcal{P}_\chi(S, \beta)$ and for $m \ge$*

$H^0(\mathcal{L}) - 1 - \delta$, *we have*

$$R\Gamma\left(R, \frac{\mathcal{O}_R^{vir}}{\wedge^\bullet (N_{vir}^\bullet)^\vee} K_{vir}^{\frac{1}{2}}|_R \otimes \prod_{i=1}^m \frac{\bar{\gamma}(\mathcal{O}_{s_i})}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right) = 0$$

where $s_1, \dots s_m$ are closed points of S which can be identical. We then can conclude that

$$\bar{P}_{X,\beta,\chi}(s_1, \dots, s_m) = R\Gamma\left(\mathcal{P}_\chi(S,\beta), \frac{\mathcal{O}_{\mathcal{P}_\chi(S,\beta)}^{vir}}{\wedge^\bullet (N_{vir}^\bullet)^\vee} K_{vir}^{\frac{1}{2}}|_{\mathcal{P}_\chi(S,\beta)} \otimes \prod_{i=1}^m \frac{\bar{\gamma}(\mathcal{O}_{s_i})}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right).$$

The same result also holds for $P_{X,\beta,\chi}(s_1, \dots, s_m)$ under additional assumption that the structure sheaf $\mathcal{O}_{\mathcal{C}_\mathbb{F}}$ of the universal supporting curve $\mathcal{C}_\mathbb{F}$ is flat over $\mathcal{P}_\chi(X, i_*\beta)$ and $s_1, \dots, s_m$ are closed points in S in general position such that all curves in $|\mathcal{L}|$ passing through all the given m points are irreducible.

3.2.3 The contribution of $\mathcal{P}_\chi(S, \beta)$

The component $\mathcal{P}_\chi(S,\beta)$ of $\mathcal{P}_\chi(X, i_*\beta)^G$ parametrize stable pairs (F, s) supported on $S \subset X$ where S is the zero section. The restriction of $\mathbb{I}^\bullet$ to $\mathcal{P}_\chi(S,\beta) \times X$ is $\mathbb{I}_X^\bullet := \{\mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times X} \to \mathcal{F}\}$, where $\mathcal{F}$ is the universal sheaf restricted to $\mathcal{P}_\chi(S,\beta) \times X$, so that the restriction of $\mathbb{E}^\bullet$ to $\mathcal{P}_\chi(S,\beta)$ is $Rp_* R\mathcal{H}om\left(\mathbb{I}_X^\bullet, \mathbb{I}_X^\bullet \otimes \mathfrak{t}^*\right)_0 [2]$. Thomas and Kool showed that on $\mathcal{P}_\chi(S,\beta)$, the decomposition of $\mathbb{E}^\bullet|_{\mathcal{P}_\chi(S,\beta)}$ into fixed and moving part is

$$(\mathbb{E}^\bullet)^{mov} \simeq Rp_* R\mathcal{H}om\left(\mathbb{I}_S^\bullet, \mathcal{F}\right)[1] \otimes \mathfrak{t}^* \qquad (\mathbb{E}^\bullet)^{fix} \simeq \left(Rp_* R\mathcal{H}om\left(\mathbb{I}_S^\bullet, \mathcal{F}\right)\right)^\vee \tag{3.13}$$

where $\mathbb{I}_S^\bullet = \{\mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times S} \to \mathcal{F}\}$. $(\mathbb{E}^\bullet)^{fix}$ gives $\mathcal{P}_\chi(S,\beta)$ a perfect obstruction theory. We will use $\mathcal{E}^\bullet$ to denote $(\mathbb{E}^\bullet)^{fix}$. From equation (3.13) and (3.6) we have $(\mathbb{E}^\bullet)^{mov} \simeq (\mathcal{E}^\bullet)^\vee [1] \otimes \mathfrak{t}^*$.

Proposition 3.2.10. *On $\mathcal{P}_\chi(S,\beta)$ we have*

$$\frac{K_{vir}^{\frac{1}{2}}|_{\mathcal{P}_\chi(S,\beta)}}{\wedge^\bullet (N_{vir}^\bullet)^\vee} = \left(-\mathfrak{t}^{-\frac{1}{2}}\right)^v \wedge_{-\mathfrak{t}} \mathcal{E}^\bullet$$

where $vd = rk\mathcal{E}^\bullet$ *and* $\wedge_{-\mathfrak{t}}\mathcal{E}^\bullet = \frac{\sum_{i=0}^{rkE^0}(-\mathfrak{t})^i\wedge^i\mathcal{E}^0}{\sum_{j=0}^{rkE^{-1}}(-\mathfrak{t})^j\wedge^j\mathcal{E}^{-1}}$ *for* $\mathcal{E}^\bullet = [\mathcal{E}^{-1}\to\mathcal{E}^0]$.

Proof. By equation (3.13) and (3.6) we have

$$K_{vir}|_{\mathcal{P}_\chi(S,\beta)} = \det\mathcal{E}^\bullet\det\left((\mathcal{E}^\bullet)^\vee\otimes\mathfrak{t}^*\right)^\vee = \det\mathcal{E}^\bullet\det\mathcal{E}^\bullet\mathfrak{t}^v$$

where $v = rk\mathcal{E}^\bullet$. Thus we can take $K_{vir}^{\frac{1}{2}}|_{\mathcal{P}_\chi(S,\beta)} = \det\mathcal{E}^\bullet\mathfrak{t}^{\frac{1}{2}v}$. Let $\mathcal{E}^\bullet = [\mathcal{E}^{-1}\to\mathcal{E}^0]$ so that $(\mathcal{E}^\bullet)^\vee[1]\otimes\mathfrak{t}^* = [(\mathcal{E}^0)^\vee\otimes\mathfrak{t}^*\to(\mathcal{E}^{-1})^\vee\otimes\mathfrak{t}^*]$ in the place of -1 and 0. Let $r_i = rk\mathcal{E}^i$ for $i=-1$ and $i=0$. Thus in $K^G(\mathcal{P}_\chi(S,\beta))$ we have

$$\begin{aligned}
\frac{K_{vir}^{\frac{1}{2}}|_{\mathcal{P}_\chi(S,\beta)}}{\wedge^\bullet(N_{vir}^\bullet)^\vee} &= \frac{\det\mathcal{E}^0\wedge^\bullet\left((\mathcal{E}^0)^\vee\otimes\mathfrak{t}^*\right)}{\det\mathcal{E}^{-1}\wedge^\bullet\left((\mathcal{E}^{-1})^\vee\otimes\mathfrak{t}^*\right)}\mathfrak{t}^{\frac{1}{2}vd}\\
&= \frac{\sum_{i=0}^{r_0}(-1)^i\wedge^{r_0-i}\mathcal{E}^0\otimes\mathfrak{t}^{-i}}{\sum_{j=0}^{r_1}(-1)^j\wedge^{r_1-j}\mathcal{E}^{-1}\otimes\mathfrak{t}^{-j}}\mathfrak{t}^{\frac{1}{2}vd}\\
&= \frac{\sum_{i=0}^{r_0}(-1)^{r_0-i}\wedge^{r_0-i}\mathcal{E}^0\otimes\mathfrak{t}^{r_0-i}}{\sum_{j=0}^{r_1}(-1)^{r_1-j}\wedge^{r_1-j}\mathcal{E}^{-1}\otimes\mathfrak{t}^{r_1-j}}\left(-\mathfrak{t}^{-\frac{1}{2}}\right)^{vd}\\
&= \left(-\mathfrak{t}^{-\frac{1}{2}}\right)^{vd}\wedge_{-\mathfrak{t}}\mathcal{E}^\bullet
\end{aligned}$$

□

The calculation of the contribution from this component is given in the next Chapter. We recall Corollary 4.3.3 here.

Under the assumption of Proposition 3.2.9 we have the following formula for $\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m)$

$$\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m) = (-1)^{vd}\int_{[\mathcal{P}_\chi(S,\beta)]^{red}}\frac{X_{-\mathfrak{t}}(TS^{[n]})\,X_{-\mathfrak{t}}(\mathcal{O}(1))^{\delta+1}}{X_{-\mathfrak{t}}(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1))}\left(\frac{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}e^{-H(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2})}}{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}}\right)^m H^m$$

where vd is the virtual dimension of $\mathcal{P}_\chi(S,\beta)$ and $\mathcal{O}(1)$ is the dual of the pullback by the morphism $\mathrm{div}:\mathcal{P}_\chi(X,i_*\beta)\to|\mathcal{L}|$ of the tautological line bundle and $H = c_1(\mathcal{O}(1))$

and for any vector bundle E of rank r with Chern roots $x_1, \dots, x_r$,

$$X_{-\mathsf{t}}(E) = \prod_{i=1}^{r} \frac{x_i \left(\mathsf{t}^{-1/2} - \mathsf{t}^{1/2} e^{-x_i\left(\mathsf{t}^{-1/2} - \mathsf{t}^{1/2}\right)}\right)}{1 - e^{-x_i\left(\mathsf{t}^{-1/2} - \mathsf{t}^{1/2}\right)}}.$$

We have the same formula for $P_{X,\beta,\chi}(s_1, \dots, s_m)$ whenever $P_{X,\beta,\chi}(s_1, \dots, s_m)$ can be defined. This is because the restriction of $\gamma(\mathcal{O}_{s_i})$ and $\bar{\gamma}(\mathcal{O}_{s_i})$ to $\mathcal{P}_\chi(S, \beta)$ are identical.

We can observe from the above formula that $P_{X,\beta,\chi}(s_1, \dots, s_m)$ is independent from the choosen points. It's natural to ask if without assuming that $s_1, \dots, s_m$ are in general positions such that all curves passing through all these points are reduced and irreducible the above proposition still holds.

Chapter 4

Refinement of Kool-Thomas Invariant

Let S be a smooth projective surface and let $\mathcal{L}$ be a a line bundle on S. Then $|\mathcal{L}| = \mathbb{P}(H^0(\mathcal{L}))$ parameterizes curves C with $\mathcal{O}(C) \cong \mathcal{L}$. For a sufficiently ample line bundle $\mathcal{L}$, Kool, Shende and Thomas showed that for the general δ-dimensional linear system $\mathbb{P}^\delta \subset |\mathcal{L}|$, there are finitely many δ-nodal curves in $\mathbb{P}^\delta$. They also compute this number as BPS numbers of the generating function of the Euler characteristic of smooth relative Hilbert scheme of points. In [19], Kool and Thomas compute this number as the reduced stable pair invariants using reduced obstruction theory which is invariant under the deformation of S such that β is always algebraic. Here we will give a refinement of these numbers as a K-theoretic invariants and compare it to the refinement given by Göttsche and Shende in [12]. We only consider the case when $h^2(\mathcal{O}_S) = 0$. In this case, the full obstruction theory coincide with the reduced obstruction theory.

4.1 Reduced obstruction theory of moduli space of stable pairs on surface

In Chapter 2 we have reviewed the construction of reduced obstruction theory by Kool and Thomas in [19]. In this section we will review the description of it's restriction to $\mathcal{P}_\chi(S,\beta)$ as a two term complex of locally free sheaves following Appendix A of [19]. The Appendix is written by Martijn Kool, Richard P. Thomas and Dmitri Panov.

Pandharipande and Thomas showed that $\mathcal{P}_\chi(S,\beta)$ is isomorphic to the relative Hilbert scheme of points $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$ where $\mathcal{C} \to \mathrm{Hilb}_\beta(S)$ is the universal family of curves C in S in class $\beta \in H_2(S,\mathbb{Z})$ and $\chi = n+1-h$ where h is the arithmetic genus of C. Notice that for $n=1$, $\mathcal{P}_\chi(S,\beta) = \mathrm{Hilb}^1(\mathcal{C}/\mathrm{Hilb}_\beta(S)) = \mathrm{Hilb}_\beta(S)$.

We will review first the description of $\mathcal{P}_\chi(S,\beta)$ as the zero locus of a vector bundle on a smooth scheme. We assume that $b_1(S)=0$ for simplicity and also because we are only working for this case in this thesis. The following construction does not need this assumption.

For $n=0$, pick a sufficiently ample line divisor A on S such that $\mathcal{L}(A) = \mathcal{L}\otimes\mathcal{O}(A)$ satisfies $H^i(\mathcal{L}(A)) = 0$ for $i>0$. Let $\gamma = \beta + [A]$. Then $\mathrm{Hilb}_\gamma(S) = |\mathcal{L}(A)| = \mathbb{P}^{\chi(\mathcal{L}(A)-1}$ has the right dimension. The map that send $C \in |\mathcal{L}|$ to $C+A \in |\mathcal{L}(A)|$ defines a closed embedding $\mathrm{Hilb}_\beta(S) \to \mathrm{Hilb}_\gamma(S)$.

Let $\mathcal{D} \subset H_\gamma(S) \times S$ be the universal divisor and let $\hat{p}$ and q_S be the projections $H_\gamma(S)\times S \to H_\gamma(S)$ and $H_\gamma(S)\times S \to S$ respectively. Let $s_\mathcal{D} \in H^0(\mathcal{O}(\mathcal{D}))$ be the section defining $\mathcal{D}$ and restrict it to $H_\gamma(S)\times A$ and consider the section

$$\zeta := s_\mathcal{D}|_{\pi_S^{-1}A} \in H^0(H_\gamma(S)\times A, \mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A}) = H^0(H_\gamma(S), \pi_{H*}(\mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A}))$$

where for a point $D \in H_\gamma(S)$ we have $\zeta|_D = s_D|_A \in H^0(A,\mathcal{L}(A))$ where s_D is the section of $\mathcal{L}(A)$ defining D. $s_D|_A = 0$ if and only if $A \subset D$ i.e $D = A + C$ for some effective divisor C with $\mathcal{O}(C)\otimes\mathcal{O}(A) = \mathcal{L}(A)$. Thus the zero locus of ζ is the image of the closed embedding $\mathrm{Hilb}_\beta(S) \to \mathrm{Hilb}_\gamma(S)$. If $H^2(\mathcal{L}) = 0$ then $F = \pi_{H*}(\mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A})$ is a vector bundle of rank $\chi(\mathcal{L}(A)) - \chi(\mathcal{L}) = h^0(\mathcal{L}(A)) - h^0(\mathcal{L}) + h^1(\mathcal{L})$ on $\mathrm{Hilb}_\gamma(S)$

since $R^i\pi_{H_*}(\mathcal{O}(\mathcal{D})|_{\pi_S^{-1}A}) = 0$ for $i > 0$. Consider the following diagram

$$\begin{array}{rccc} F^\bullet_{red} = & \{F^* & \xrightarrow{d\circ\zeta^*} & \Omega_{H_\gamma(S)}|_{H_\beta(S)}\} \\ & \downarrow{\scriptstyle \zeta^*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{L}_{H_\beta(S)} = & \{I/I^2|_{H_\beta(S)} & \xrightarrow{d} & \Omega_{H_\gamma(S)}|_{H_\beta(S)}\}. \end{array}$$

The above morphism is a perfect obstruction theory for $\mathrm{Hilb}_\beta(S)$.

Next, we embed $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$ into $S^{[n]}\times\mathrm{Hilb}_\beta(S)$. Let $\mathcal{Z} \subset S^{[n]}\times\mathrm{Hilb}_\beta(S)\times S$ be the pullback of the universal length n subscheme of $S^{[n]} \times S$. Let $\mathcal{C} \subset S^{[n]} \times \mathrm{Hilb}_\beta(S) \times S$ be the pullback of the universal divisor of $\mathrm{Hilb}_\beta \times S$ and let $\pi : S^{[n]} \times \mathrm{Hilb}_\beta(S) \times S \to S^{[n]} \times \mathrm{Hilb}_\beta(S)$ be the projection. Then $\mathcal{C}$ correspond to a section $s_\mathcal{C}$ of the line bundle $\mathcal{O}(\mathcal{C})$ on $S^{[n]} \times \mathrm{Hilb}_\beta(S) \times S$. A point $(Z, C) \in S^{[n]} \times \mathrm{Hilb}_\beta(S)$ is in the image of $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$ if $Z \subset C$. We denote by $\mathcal{O}(\mathcal{C})^{[n]}$ the vector bundle $\pi_*(\mathcal{O}(\mathcal{C})|_\mathcal{Z})$ of rank n. Let $\sigma_\mathcal{C}$ be the pushforward of $s_\mathcal{C}$ so that $\sigma_\mathcal{C}|_{(Z,C)} = s_C|_Z \in H^0(\mathcal{L}|_Z)$. Thus a point $(Z, C) \in S^{[n]} \times \mathrm{Hilb}_\beta(S)$ is in the image of $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$ if and only if $\sigma_\mathcal{C}|_{(Z,C)} = s_C|_Z = 0$. Thus we get a perfect relative obstruction theory :

$$\begin{array}{rccc} E^\bullet = & \{(\mathcal{O}(\mathcal{C})^{[n]})^* & \xrightarrow{d\circ s^*} & \Omega_{S^{[n]}}\} \\ & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{L}_{\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))/\mathrm{Hilb}_\beta(S)} = & \{J/J^2 & \xrightarrow{d} & \Omega_{S^{[n]}}\} \end{array}$$

where J is the ideal describing $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$ as a subscheme of $S^{[n]} \times \mathrm{Hilb}_\beta(S)$. Notice that in general $|\mathcal{L}|$ is not of the right dimension.

Appendix A of [19] shows how to combine the above obstruction theories to define an absolute perfect obstruction theory for $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$. To do it we have to consider the embedding of $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$ into $S^{[n]} \times \mathrm{Hilb}_\gamma(S)$. $E^\bullet$ is the restriction of $[(\mathcal{O}(\mathcal{D} - A)^{[n]})^* \to \Omega_{S^{[n]}}]$ to $\mathrm{Hilb}^n(\mathcal{C}/\mathrm{Hilb}_\beta(S))$. It was shown that the complex $E^\bullet_{red}$ that correspond to the combined obstruction theory sits in the following exact triangle

$$F^\bullet_{red} \longrightarrow E^\bullet_{red} \longrightarrow E^\bullet \,.$$

Also in Appendix A of [19], it was shown that the combination of the above obstruction theory have the same K-theory class with the reduced obstruction theory $\mathcal{E}^\bullet_{red}$. Thus we can conclude that the K-theory class of $\mathcal{E}^\bullet_{red}$ is

$$[\Omega_{S^{[n]}\times \mathrm{Hilb}_\gamma(S)}] - [(\mathcal{O}(\mathcal{D}-A)^{[n]})^*] - [F^*] \tag{4.1}$$

Moreover, Theorem A.7 of [19] gives the virtual class corresponding to the reduced obstruction theory $[\mathcal{P}_\chi(S,\beta)]^{red}$ as the class $c_n\left(\mathcal{O}(\mathcal{D}-A)^{[n]}.c_{top}(F)\cap[S^{[n]}\times \mathrm{Hilb}_\gamma(S)]\right.$.

4.2 Point insertion and linear subsystem

In this section we assume that $h^{0,1}(S)=0$ i.e. $\mathrm{Pic}_\beta=\{\mathcal{L}\}$ and $\mathrm{Hilb}_\beta(S)=|\mathcal{L}|$.

Let $\mathcal{D}\subset S\times|\mathcal{L}|$ be the universal curve. Pandharipande and Thomas showed in [25] that $\mathcal{P}_\chi(S,\beta)$ is isomorphic to the relative Hilbert scheme of points $\mathrm{Hilb}^n(\mathcal{D}\to|\mathcal{L}|)$. There is an embedding of $\mathrm{Hilb}^n(\mathcal{D}\to|\mathcal{L}|)$ into $S^{[n]}\times|\mathcal{L}|$ and the projection $\mathrm{Hilb}^n(\mathcal{D}\to |\mathcal{L}|)\to|\mathcal{L}|$ gives a morphism $\mathrm{div}:\mathcal{P}_\chi(S,\beta)\to|\mathcal{L}|$ that maps $(\mathcal{F},s)\in\mathcal{P}_\chi(S,\beta)$ to the supporting curve $C_\mathcal{F}\in|\mathcal{L}|$ of $\mathcal{F}$.

Fix $\chi\in\mathbb{Z}$ and let $\mathcal{C}$ be the universal curve supporting the universal sheaf $\mathcal{F}$ on $S\times\mathcal{P}_\chi(S,\beta)$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{P}_\chi(S,\beta)\times S \arrow[r, "q_S"] \arrow[d, "\hat{p}"'] & S \\
\mathcal{P}_\chi(S,\beta)
\end{tikzcd}$$

Of course when $n=1$, $\mathcal{P}_\chi(S,\beta)$ is $|\mathcal{L}|$ and $\mathcal{C}=\mathcal{D}$.

Here we will compute explicitly the class $\gamma(\mathcal{O}_s)$ restricted to $\mathcal{P}_\chi(S,\beta)\to\mathcal{P}_\chi(X,i_*\beta)^G$. Note that G acts trivially on S and on $\mathcal{P}_\chi(S,\beta)$. Let $\mathcal{C}\subset\mathcal{P}_\chi(S,\beta)\times\bar{X}$ be the support of the universal sheaf. Note that $\mathcal{C}$ is supported on $\mathcal{P}_\chi(S,\beta)\times S$ where S is the zero section of the bundle $X\to S$. Thus $\pi\circ i:\mathcal{C}\to\bar{\mathcal{P}}_\chi(S,\beta)\times S$ is a closed embedding. By equation (3.11), $\gamma(\mathcal{O}_s)=\hat{p}_*\left([\mathcal{O}_\mathcal{C}]\otimes q_S^*[\mathcal{O}_s]\right)$. Notice that G acts on $\mathcal{O}_s$ and $\mathcal{O}_\mathcal{C}$ trivially.

Proposition 4.2.1. *Let $s\in S$ be a point with structure sheaf $\mathcal{O}_s$. Let $[\mathcal{O}_s]$ be its class* *in* $K(S)$*. Then*

$$\hat{p}_*\left([\mathcal{O}_{\mathcal{C}}].q_S^*[\mathcal{O}_s]\right) = 1 - [div^*\mathcal{O}(-1)].$$

where $\mathcal{O}(-1)$ *is the tautological line bundle on* $|\mathcal{L}|$*.*

Proof. First consider the following diagram

$$\begin{tikzcd} |\mathcal{L}| \times S \arrow[r, "q_S"] \arrow[d, "\hat{p}^{|\mathcal{L}|}"] & S . \\ |\mathcal{L}| & \end{tikzcd}$$

We will show that $\hat{p}_*\left(q_S^*[\mathcal{O}_z].[\mathcal{O}_{\mathcal{D}}]\right) = 1 - [\mathcal{O}(-1)]$. Since q_S is a flat morphism $q_S^*[\mathcal{O}_z] = [q_S^*\mathcal{O}_z] = k_*\left[\mathcal{O}_{|\mathcal{L}|\times\{z\}}\right]$ where k is the inclusion $k : |\mathcal{L}| \times \{z\} \to |\mathcal{L}| \times S$. $\mathcal{C}$ is the universal divisor with $\mathcal{L}^* \boxtimes \mathcal{O}(-1)$ as the defining ideal. By the projection formula $q_S^*[\mathcal{O}_s].[\mathcal{O}_{\mathcal{D}}]$ is equal to

$$k_*\left[\mathcal{O}_{|\mathcal{L}|\times\{s\}}\right].\left(1 - [\mathcal{L}^* \boxtimes \mathcal{O}(-1)]\right) = k_*\left([k^*\mathcal{O}_{|\mathcal{L}|\times S}] - [k^*q_S^*\mathcal{L}^* \otimes k^*\hat{p}^*\mathcal{O}(-1)]\right).$$

$k^*q_S^*\mathcal{L}^* = q_s^*\mathcal{L}^*|_s = \mathcal{O}_{|\mathcal{L}|\times\{s\}}$ where $q_s = q_S|_{|\mathcal{L}|\times\{s\}}$ and $k^*\hat{p}^*\mathcal{O}(-1) = \mathcal{O}(-1)$ since $\hat{p} \circ k$ is the identity morphism. Thus we conclude that

$$\hat{p}_*\left(q_S^*[\mathcal{O}_s].[\mathcal{O}_{\mathcal{D}}]\right) = \hat{p}_*k_*\left(\left[\mathcal{O}_{|\mathcal{L}|\times\{s\}}\right] - [\mathcal{O}(-1)]\right) = 1 - [\mathcal{O}(-1)]$$

Now we are working on $\mathcal{P}_\chi(S,\beta)$. Consider the following Cartesian diagram

$$\begin{tikzcd} \mathrm{div}^{-1}\mathcal{D} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{P}_\chi(S,\beta) \times S \arrow[r, "\mathrm{div}"] \arrow[d, "\hat{p}^{\mathcal{P}_\chi(S,\beta)}"'] & |\mathcal{L}| \times S \arrow[d, "\hat{p}^{|\mathcal{L}|}"'] \\ \mathcal{P}_\chi(S,\beta) \arrow[r, "\mathrm{div}"] & |\mathcal{L}|. \end{tikzcd}$$

$\mathrm{div}^{-1}\mathcal{D}$ is the family of effective Cartier divisor corresponding to the morphism $\mathrm{div} : \mathcal{P}_\chi(S,\beta) \to |\mathcal{L}|$, For each point $p \in \mathcal{P}_\chi(S,\beta)$, $\mathrm{div}^{-1}\mathcal{D}|_p$ is the corresponding curve $\mathcal{C}_{\mathcal{F}_p}$

supporting the sheaf $\mathcal{F}_p$. We conclude that $\mathcal{C}$ and $\operatorname{div}^{-1}\mathcal{C}$ are the same families of divisors on S so that we have a short exact sequence

$$0 \longrightarrow \operatorname{div}^*(\mathcal{L}^* \boxtimes \mathcal{O}(-1)) \longrightarrow \mathcal{O}_{\mathcal{P}_\chi(S,\beta)\times S} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow 0$$

and $[\mathcal{O}_{\mathcal{C}}] = \operatorname{div}^*[\mathcal{O}_{\mathcal{D}}]$. Thus we have

$$\begin{aligned}\hat{p}_*^{\mathcal{P}_\chi(S,\beta)}([\mathcal{O}_{\mathcal{C}}]\,q_S^*[\mathcal{O}_s]) &= \hat{p}_*^{\mathcal{P}_\chi(S,\beta)}(\operatorname{div}^*[\mathcal{O}_{\mathcal{D}}].\operatorname{div}^* q_S^*[\mathcal{O}_s]) \\ &= \operatorname{div}^*\hat{p}_*^{|\mathcal{L}|}([\mathcal{O}_{\mathcal{C}}].q_S^*[\mathcal{O}_s]) \\ &= \operatorname{div}^*(1-[\mathcal{O}(-1)])\end{aligned}$$

□

We also have similar result for $\mathcal{P}_\chi(X, i_*\beta)$ if we replace $\mathcal{O}_{\mathcal{C}}$ with $\mathcal{O}_{\operatorname{div}\pi_*\mathcal{F}}$.

Proposition 4.2.2. *Let $\mathcal{O}_s$ be the structure sheaf of the points $s \in S$. Then $\hat{p}([\mathcal{O}_{div\pi_*\mathcal{F}}].q_S^*[\mathcal{O}_s]) = 1 - div^*(\mathcal{O}(-1))$ where $\mathcal{O}(-1)$ is the tautological bundle of $|\mathcal{L}|$ and $\hat{p}$, q_S are morphism from diagram 3.8.*

Proof. From the definition of the morphism div, $\operatorname{div}\pi_*\mathcal{F}$ is exactly $\operatorname{div}^{-1}\mathcal{D}$. Thus we can use exactly the same proof as the previous Proposition. □

Later we will drop div*from $\operatorname{div}^*\mathcal{O}(-1)$ for simplicity.

4.3 Refinement of Kool-Thomas invariants

Assume that $b_1(S) = 0$. From Proposition 4.2.1 and Proposition 4.2.2 , the contribution of $\mathcal{P}_\chi(S,\beta)$ to $P_{X,\beta,\chi}(s_1,\ldots,s_m)$ and to $\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m)$ are equal. Consider the contribution of $\mathcal{P}_\chi(S,\beta)$ to $\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m)$ invariants, i.e.

$$\Xi = R\Gamma\left(\mathcal{P}_\chi(S,\beta), \frac{\mathcal{O}^{vir}_{\mathcal{P}_\chi(S,\beta)} \otimes K_{vir}^{\frac{1}{2}}}{\bigwedge^\bullet(N^{vir})^\vee}\prod_{i=1}^m \frac{\bar{\gamma}(\mathcal{O}_{s_i})}{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}}\right).$$

On $\mathrm{Hilb}_\beta(S) \times S$ we have the following exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{s_{\mathcal{C}}} \mathcal{O}(\mathcal{C}) \longrightarrow \mathcal{O}_{\mathcal{C}}(\mathcal{C}) \longrightarrow 0 \tag{4.2}$$

which induces the exact sequence

$$H^1(\mathcal{O}_{\mathcal{C}}(\mathcal{C})) \xrightarrow{\hat{\phi}} H^2(\mathcal{O}_S) \longrightarrow H^2(\mathcal{L}) \ .$$

If $H^2(\mathcal{L}) = 0$ then $\hat{\phi}$ is surjective. Observe that $R\pi_{H*}\mathcal{O}_{\mathcal{C}}(\mathcal{C})$ is the complex $\mathcal{E}^\bullet$ from Subsection 2.2.1 when $\chi = 2 - h$ or equivalently when $n = 1$. For $n > 1$, it was shown in Appendix A of [19] that $\mathcal{E}^\bullet$ sits in the exact triangle

$$R\pi_{H*}\mathcal{O}_{\mathcal{C}}(\mathcal{C}) \longrightarrow \mathcal{E}^\bullet \longrightarrow E^\bullet \ .$$

Thus if $h^2(\mathcal{O}_S) > 0$ then $\mathcal{E}^\bullet$ contain a trivial bundle so that $[\mathcal{P}_\chi(S,\beta)]^{vir}$ vanish. In particular, by virtual Riemann-Roch the contribution of $\mathcal{P}_\chi(S,\beta)$ is zero.

If $H^2(\mathcal{O}_S) = 0$, $\mathcal{E}^\bullet_{red}$ and $\mathcal{E}^\bullet$ are quasi isomorphic. Let P be the moduli space $\mathcal{P}_\chi(S,\beta)$. By the virtual Riemann-Roch theorem and by Lemma 4.2.2 we then have

$$ch^G(\Xi) = \left(-\mathfrak{t}^{-\frac{1}{2}}\right)^{vd} \int_{[P]^{red}} \mathrm{ch}\left(\Lambda_{-\mathfrak{t}}\mathcal{E}^\bullet_{red}\left(\frac{\Lambda_{-1}\mathcal{O}(-1)}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m\right).\mathrm{td}\left(T_P^{red}\right)$$

where T_P^{red} is the derived dual of $\mathcal{E}^\bullet_{red}$ and $\left(-\mathfrak{t}^{-\frac{1}{2}}\right)^{vd}$ should be understood as $\left(-e^{-\frac{1}{2}t}\right)^{vd}$ where t is the equivariant first Chern class of $\mathfrak{t}$. Observe that $ch^G(\Xi)$ can be computed whenever $H^2(\mathcal{L}) = 0$ without assuming $h^2(\mathcal{O}_S) = 0$. Thus for S with $b_1(S) = 0$ and a line bundle $\mathcal{L}$ with $H^2(\mathcal{L}) = 0$, we define $P_{S,\mathcal{L},m,\chi} = ch^G(\Xi)$.

The K-theory class of $\mathcal{E}^\bullet_{red}$ is given by equation (4.1). Since $\mathcal{O}(\mathcal{C}) = \mathcal{L} \boxtimes \mathcal{O}(1)$, by the projection formula we have $F = H^0(\mathcal{L}(A)|_A) \otimes \mathcal{O}(1)$. From the exact sequence

$$0 \longrightarrow \mathcal{O}(\mathcal{C}) \longrightarrow \mathcal{O}(\mathcal{C} + A) \longrightarrow \mathcal{O}_{\pi_S^{-1}A}(\mathcal{C} + \pi_S^{-1}A) \longrightarrow 0$$

on P, and since $H^{i>0}(\mathcal{L}(A)|_A) = 0$, we conclude that

$$F = \mathcal{O}(1)^{\oplus\chi(\mathcal{L}(A))-\chi(\mathcal{L})} \tag{4.3}$$

And again by projection formula we have $\mathcal{O}(\mathcal{C})^{[n]} = \mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)$. By Theorem A.7 of [19] we then can compute $P_{S,\mathcal{L},m,\chi}$ as

$$\left(-\mathsf{t}^{-\frac{1}{2}}\right)^v \int_{S^{[n]}\times|\mathcal{L}(A)|} H^{\chi(\mathcal{L}(A))-\chi(\mathcal{L})} c_n(\mathcal{O}(\mathcal{D}-A)^{[n]}) \mathrm{ch}\left(\frac{\wedge_{-\mathsf{t}}\mathcal{E}^{\bullet}_{red}\left(\wedge_{-1}\mathcal{O}(-1)\right)^m}{\left(\mathsf{t}^{-1/2}-\mathsf{t}^{1/2}\right)^m}\right) \mathrm{td}\left(T^{red}_P\right) \tag{4.4}$$

where $H = c_1(\mathcal{O}(1)$ and $n = \chi + h - 1$.

Theorem 4.3.1. $P_{S,\mathcal{L},m,\chi}|_{\mathsf{t}=1} = (-1)^{vd} \int_{S^{[n]}\times\mathbb{P}^{\varepsilon}} c_n(\mathcal{L}^{[n]} \otimes \mathcal{O}(1)) \frac{c_\bullet(TS^{[n]})c_\bullet(\mathcal{O}(1))^{\chi(\mathcal{L})}}{c_\bullet(\mathcal{L}^{\{n\}}\boxtimes\mathcal{O}(1))}$ *where* $\varepsilon = \chi(\mathcal{L}) - 1 - m$. *Thus we can relate Kool-Thomas invariants with our invariants as follows:*

$$\mathcal{P}^{red}_{\chi,\beta}\left(S,[pt]^m\right) \quad = \quad (-1)^m\, t^{m+1-\chi(\mathcal{O}_S)}\, P_{S,\mathcal{L},m,\chi}|_{\mathsf{t}=1}\,.$$

Proof. Let $\mathcal{X}_{-\mathsf{t}}(T^{red}_P) := \mathrm{ch}\left(\wedge_{-\mathsf{t}}\mathcal{E}^{\bullet}_{red}\right) \mathrm{td}\left(T^{red}_P\right)$ and let $d := \mathrm{rk}\mathcal{E}^{\bullet}_{red} = n + \chi(\mathcal{L}) - 1$ be the virtual dimension of P so that we can rewrite (4.4) as

$$(-1)^m \left(-\mathsf{t}^{-\frac{1}{2}}\right)^{d-m} \int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n\left(\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)\right) \mathcal{X}_{-\mathsf{t}}\left(T^{red}_P\right) \mathrm{ch}\left(\frac{\wedge_{-1}\left(\mathcal{O}(-1)\right)}{1-\mathsf{t}}\right)^m \tag{4.5}$$

By Proposition 5.3 of [6] we can write

$$\mathcal{X}_{-\mathsf{t}}(T^{red}_P) = \sum_{l=0}^{d} (1-\mathsf{t})^{d-l}\, \mathcal{X}^l$$

where $\mathcal{X}^l = c_l(T^{red}_P) + b_l$where $b_l \in A^{>l}(P)$. Then we can write $P_{S,\mathcal{L},m,\chi}$ as

$$(-1)^m \left(-\mathsf{t}^{-\frac{1}{2}}\right)^{d-m} \int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n(\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)) \sum_{l=0}^{d} (1-\mathsf{t})^{d-m-l} \mathcal{X}^l \mathrm{ch}\left(\wedge_{-1}\left(\mathcal{O}(-1)\right)\right)^m.$$

Note that $\mathrm{ch}\left(\wedge_{-1}\left(\mathcal{O}(-1)\right)\right)^m = H^m + O\left(H^{m+1}\right)$ so that

$$\int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n\left(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1)\right)\mathcal{X}^l\mathrm{ch}\left(\wedge_{-1}\left(\mathcal{O}(-1)\right)\right)^m = 0$$

for $l > d-m$. Thus the summation ranges from $l=0$ to $l=d-m$. In this range the power of $(1-\mathfrak{t})$ is positive except when $l = d-m$ in which the power of $(1-\mathfrak{t})$ is zero. Thus we can conclude that $P_{S,\mathcal{L},m,\chi}|_{\mathfrak{t}=1}$equals to

$$(-1)^m\left(-\mathfrak{t}^{-\frac{1}{2}}\right)^{d-m}\int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n\left(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1)\right)\mathcal{X}^{d-m}\mathrm{ch}\left(\wedge_{-1}\left(\mathcal{O}(-1)\right)\right)^m.$$

Since $b_{d-m}\in A^{>d-m}(P)$ and $c_{d-m}(T_P^{red})\in A^{d-m}(P)$ we have

$$\int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n\left(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1)\right)b_{d-m}\mathrm{ch}\left(\wedge_{-1}\left(\mathcal{O}(-1)\right)\right)^m = 0$$

and

$$\int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n\left(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1)\right)c_{d-m}(T_P^{red})H^k = 0$$

for $k > m$ and we can conclude that

$$P_{S,\mathcal{L},m,\chi}|_{\mathfrak{t}=1} = (-1)^{\frac{1}{2}d}\int_{S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}} c_n\left(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1)\right).H^m.c_{d-m}(T_P^{red})$$

From (4.1) and (4.3) we have

$$T_P^{red} = T\left(S^{[n]}\right) + \mathcal{O}(1)^{\chi(\mathcal{L}(A))} - \mathcal{O} - \mathcal{L}^{[n]}\boxtimes\mathcal{O}(1) - \mathcal{O}(1)^{\chi(\mathcal{L}(A))-\chi(\mathcal{L})}$$

and

$$c_{d-m}(T_P^{red}) = \mathrm{Coeff}_{t^{d-m}}\left[\frac{c_t\left(TS^{[n]}\right)c_t\left(\mathcal{O}(1)\right)^{\chi(\mathcal{L})}}{c_t\left(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1)\right)}\right].$$

Finally we conclude that

$$P_{S,\mathcal{L},m,\chi}|_{\mathfrak{t}=1} = (-1)^{-\frac{1}{2}d}\int_{S^{[n]}\times\mathbb{P}^{\delta}} c_n(\mathcal{L}^{[n]}\otimes\mathcal{O}(1))\frac{c_\bullet\left(TS^{[n]}\right)c_\bullet(\mathcal{O}(1))^{\chi(\mathcal{L})}}{c_\bullet\left(\mathcal{L}^{\{n\}}\boxtimes\mathcal{O}(1)\right)}$$

□

Let $X_{-y}(x) \in \mathbb{Q}[[x,y]]$ defined by

$$X_{-y}(x) := \frac{x\left(y^{-\frac{1}{2}} - y^{\frac{1}{2}}e^{-x(y^{-\frac{1}{2}}-y^{\frac{1}{2}})}\right)}{1-e^{-x\left(y^{-\frac{1}{2}}-y^{\frac{1}{2}}\right)}}.$$

For a vector bundle E on a scheme $Y$of rank r with Chern roots $x_1,\ldots,x_r$ we will use $X_{-y}(E)$ to denote

$$\prod_{i=1}^{r} \frac{x_i\left(y^{-\frac{1}{2}} - y^{\frac{1}{2}}e^{-x_i(y^{-\frac{1}{2}}-y^{\frac{1}{2}})}\right)}{1-e^{-x_i\left(y^{-\frac{1}{2}}-y^{\frac{1}{2}}\right)}}.$$

Observe that X_{-y} is additive on an exact sequence of vector bundle. Thus we can extend X_y to $K(Y)$. For a class $\beta \in K(Y)$ we can write $\beta = \sum_i [E_i^+] - \sum_j [E_j^-]$ for vector bundles E_i^+, E_j^- and we can define $X_{-y}(\beta) = \frac{\prod_i X_{-y}(E_i^+)}{\prod_j X_{-y}(E_j^-)}$. For a proper nonsingular scheme Y with tangent bundle T_Y

$$\int_Y X_{-y}\left(T_Y\right) = \left(\frac{1}{y}\right)^{\frac{1}{2}d} \sum_i (-1)^{p+q} y^q h^{p,q}(Y)$$

where $h^{p,q}(Y)$ are the Hodge number of Y i.e. $\int_Y X_{-y}\left(T_Y\right)$ is the normalized χ_{-y} genus.

Theorem 4.3.2.

$$P_{S,\mathcal{L},m,\chi} =$$
$$(-1)^{vd} \int\limits_{[P]^{red}} \frac{X_{-\mathfrak{t}}\left(TS^{[n]}\right)}{X_{-\mathfrak{t}}\left(\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)\right)} X_{-\mathfrak{t}}\left(\mathcal{O}(1)\right)^{\delta+1} \left(\frac{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}e^{-H\left(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}\right)}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m H^m \quad (4.6)$$

where $[P]^{red}$ *is* $c_n\left(\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)\right) \cap \left[S^{[n]} \times \mathbb{P}^{\chi(\mathcal{L})-1}\right]$.

Proof. $P_{S,\mathcal{L},m,\chi}$ equals to (4.5), and we can rewrite it as

$$P_{S,\mathcal{L},m,\chi} = (-1)^{vd} \int_{[P]^{red}} \frac{\prod_{i=1}^{2n+\chi(\mathcal{L})-1} \frac{\phi_{-\mathfrak{t}}(\alpha_i)}{\mathfrak{t}^{1/2}}}{\prod_{i=1}^{n} \frac{\phi_{-\mathfrak{t}}(\beta_i)}{\mathfrak{t}^{1/2}}} \left(\frac{1-e^{-H}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m$$

where $\phi_{-\mathfrak{t}}(x) = \frac{x(1-\mathfrak{t}e^{-x})}{1-e^{-x}}$ and α_i are the Chern roots of $T(S^{[n]} \times \mathbb{P}^{\chi(\mathcal{L})-1})$ and β_i are the Chern roots of $\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)$. Let's define

$$\bar{\phi}_{-\mathfrak{t}}(x) := \frac{\phi_{-\mathfrak{t}}(x)}{\mathfrak{t}^{1/2}} = \frac{x\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}e^{-x}\right)}{1 - e^{-x}} = \sum_{i\geq 0} \bar{\phi}_i x^i.$$

Note that this power series starts with $\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}$. By substituting x with $x\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}\right)$ and dividing it by $\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}\right)$ we have the power series

$$X_{-\mathfrak{t}}(x) = \frac{x\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}e^{-x\left(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}\right)}\right)}{1 - e^{-x\left(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}\right)}} = \sum_{i\geq 0} \xi_i x^i$$

such that $\xi_0 = 1$ and $\xi_i = \bar{\phi}_i\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}\right)^{i-1}$. Thus by substituting x in

$$\frac{\prod_{i=1}^{2n+\chi(\mathcal{L})-1} \frac{\phi_{-y}(\alpha_i)}{\mathfrak{t}^{1/2}}}{\prod_{i=1}^{n} \frac{\phi_{-y}(\beta_i)}{\mathfrak{t}^{1/2}}} \left(\frac{1 - e^{-H}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m$$

with $x\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}\right)$ whenever $x = \alpha_i, \beta_i, H$ and dividing it by

$$\left(\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}\right)^{n+\chi(\mathcal{L})-1}$$

so that the coefficients of $q^{n+\chi(\mathcal{L})-1}$ in

$$\frac{\prod_{i=1}^{2n+\chi(\mathcal{L})-1} X_{-\mathfrak{t}}(\alpha_i q)}{\prod_{i=1}^{n} X_{-\mathfrak{t}}(\beta_i q)} \left(\frac{1 - e^{-Hq\left(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}\right)}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m$$

and

$$\frac{\prod_{i=1}^{2n+\chi(\mathcal{L})-1} \bar{\phi}_{-\mathfrak{t}}(\alpha_i q)}{\prod_{i=1}^{n} \bar{\phi}_{-\mathfrak{t}}(\beta_i q)} \left(\frac{1 - e^{-Hq}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m$$

are the same. Since $\left[T\mathbb{P}^{\chi(\mathcal{L})-1}\right] = \left[\oplus_{i=1}^{\chi(\mathcal{L})}\mathcal{O}(1)\right] - \left[\mathcal{O}_{\mathbb{P}^{\chi(\mathcal{L})-1}}\right]$, $P_{S,\mathcal{L},m,\chi}$ equals

$$(-1)^{vd}\int_{[P]^{red}} \frac{X_{-\mathfrak{t}}\left(TS^{[n]}\right) X_{-\mathfrak{t}}\left(\mathcal{O}(1)\right)^{\chi(\mathcal{L})}}{X_{-\mathfrak{t}}\left(\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)\right)} \left(\frac{1 - e^{-H\left(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}\right)}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m =$$

$$(-1)^{vd}\int_{[P]^{red}} \frac{X_{-\mathfrak{t}}\left(TS^{[n]}\right) X_{-\mathfrak{t}}\left(\mathcal{O}(1)\right)^{\delta+1}}{X_{-\mathfrak{t}}\left(\mathcal{L}^{[n]} \boxtimes \mathcal{O}(1)\right)} \left(\frac{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}e^{-H\left(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}\right)}}{\mathfrak{t}^{-1/2} - \mathfrak{t}^{1/2}}\right)^m H^m$$

□

In the following Corollary we want to complete the computation of $P_{X,\beta,\chi}(s_1,\ldots,s_m)$.

Corollary 4.3.3. *Given a positive integer δ, let S be a smooth projective surface with $b_1(S)=0$. Let $\mathcal{L}$ be a $2\delta+1$-very ample line bundle on S with $c_1(\mathcal{L})=\beta$ and $H^i(\mathcal{L})=0$ for $i>0$. Let $X=K_S$ be the canonical line bundle over S. Then for $m=\chi(\mathcal{L})-1-\delta$ points $s_1,\ldots,s_m$ which is not necessarily different*

$$\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m)=(-1)^{vd}\int_{[P]^{red}}\frac{X_{-\mathfrak{t}}(TS^{[n]})\,X_{-\mathfrak{t}}(\mathcal{O}(1))^{\delta+1}}{X_{-\mathfrak{t}}(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1))}\left(\frac{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}e^{-H(\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2})}}{\mathfrak{t}^{-1/2}-\mathfrak{t}^{1/2}}\right)^m H^m$$

where $[P]^{red}=c_n(\mathcal{L}^{[n]}\boxtimes\mathcal{O}(1))\cap[S^{[n]}\times\mathbb{P}^{\chi(\mathcal{L})-1}]$ for $m\geq H^0(\mathcal{L})-1-\delta$.

If additionally $\mathcal{O}_{\mathcal{C}_{\mathbb{F}}}$ is flat over $\mathcal{P}_\chi(X,i_*\beta)$ and $s_1,\ldots,s_m$ are closed points of S in general position such that all curves on S that pass through all m points are reduced and irredcible then $P_{X,\beta,\chi}(s_1,\ldots,s_m)$ is given by the same formula.

Proof. By Proposition 3.2.9 $\bar{P}_{X,\beta,\chi}(s_1,\ldots,s_m) = P_{S,\mathcal{L},m,\chi}$. Similarly for $P_{X,\beta,\chi}(s_1,\ldots,s_m)$. □

In [12, 13], for every smooth projective surface S and line bundle $\mathcal{L}$ on S, Göttsche and Shende defined the following power series

$$D^{S,\mathcal{L}}(x,y,w):=\sum_{n\geq 0}w^n\int_{S^{[n]}}X_{-y}\left(TS^{[n]}\right)\frac{c_n\left(\mathcal{L}^{[n]}\otimes e^x\right)}{X_{-y}\left(\mathcal{L}^{[n]}\otimes e^x\right)}\in\mathbb{Q}[\![x,y,w]\!]$$

where e^x denotes a trivial line bundle with nontrivial $\mathbb{C}^\times$ action with equivariant first Chern class x. Motivated by this power series we define a generating function

$$P_{S,\mathcal{L},m}:=\sum_{n\geq 0}(-w)^n\,P_{S,\mathcal{L},m,n+1-h}. \tag{4.7}$$

where h is the arithmetic genus of the curve C in S with $\mathcal{O}(C)\simeq\mathcal{L}$ so that for the pair $(\mathcal{F},s)\in\mathcal{P}_\chi(S,\beta)$, $n=\chi-1+h$.

By Theorem 4.3.2, after substituting $\mathfrak{t}$ by y we can rewrite $P_{S,\mathcal{L},m}$ as

$$\mathrm{Coeff}_{x^\delta}\left[D^{S,\mathcal{L}}(x,y,w)X_{-y}(x)^{\delta+1}\left(\frac{y^{-1/2}-y^{1/2}e^{-x\left(y^{-1/2}-y^{1/2}\right)}}{y^{-1/2}-y^{1/2}}\right)^m\right]$$

Note that

$$Q_{S,\mathcal{L},m} := \mathrm{Coeff}_{x^\delta}\left[D^{S,\mathcal{L}}(x,y,w)X_{-y}(x)^{\delta+1}\right]$$

is equation (2.1) of [13] and

$$\left(\frac{y^{-1/2}-y^{1/2}e^{-x\left(y^{-1/2}-y^{1/2}\right)}}{y^{-1/2}-y^{1/2}}\right)^m$$

is a power series starting with 1.

In [13], Gottsche and Shende defined the power series $N^i_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y)$ by the following equation:

$$\sum_{i\in\mathbb{Z}} N^i_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y)\left(\frac{w}{(1-y^{-1/2}w)(1-y^{1/2}w)}\right)^{i+1-g} = Q_{S,\mathcal{L},m} \tag{4.8}$$

Motivated by this we also define $M^i_{\chi(\mathcal{L})-1-m,[S,\mathcal{L}]}(y)$ as

$$\sum_{i\in\mathbb{Z}} M^i_{\chi(\mathcal{L})-1-m,[S,\mathcal{L}]}(y)\left(\frac{w}{(1-y^{-1/2}w)(1-y^{1/2}w)}\right)^{i+1-g} = P_{S,\mathcal{L},m} \tag{4.9}$$

Let's define $\frac{1}{Q} = \frac{(1-y^{-1/2}w)(1-y^{1/2}w)}{w} = w + w^{-1} - y^{-1/2} - y^{1/2}$ and recall a conjecture from [12].

Conjecture 4.3.4 (Conjecture 55 of [12])**.**

$$\left(\frac{w(Q)}{Q}\right)^{1-g(\mathcal{L})} D^{S,\mathcal{L}}(x,y,w(Q)) \in \mathbb{Q}[y^{-1/2},y^{1/2}]\llbracket x, xQ\rrbracket$$

Motivated by the conjecture above we define another power series

$$\tilde{D}^{S,\mathcal{L}}(x,y,Q) := \left(\frac{w(Q)}{Q}\right)^{1-g(\mathcal{L})} D^{S,\mathcal{L}}(x,y,w(Q)).$$

Proposition 4.3.5. *Assume* Conjecture 4.3.4. For $\chi(\mathcal{L}) - 1 \geq k \geq 0$ we have

1. $M^i_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y) = 0$ and $N^i_{\chi(\mathcal{L})-1-k}(y) = 0$ for $i > \chi(\mathcal{L}) - 1 - k$ and for $i \leq 0$.
2. $M^i_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y)$ and $N^i_{\chi(\mathcal{L})-1-k}(y)$ are Laurent polynomials in $y^{1/2}$.
3. Furthermore $M^{\chi(\mathcal{L})-1-k}_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y) = N^{\chi(\mathcal{L})-1-k}_{\chi(\mathcal{L})-1-k,[S,\mathcal{L}]}(y)$. Moreover

$$\sum_{i\geq 0} M^{\delta}_{\delta,[S,\mathcal{L}]}(y)\,(s)^{\delta} = \tilde{D}^{S,\mathcal{L}}(x, y, \frac{s}{x})|_{x=0} = \sum_{\delta\geq 0} N^{\delta}_{\delta,[S,\mathcal{L}]}(y) s^{\delta}$$

Proof. After substituting w by $w(Q)$ we rewrite equation (4.8) and (4.9)

$$\sum_{i\in\mathbb{Z}} N^i_{\delta,[S,\mathcal{L}]}(y) x^{\delta-i}\,(xQ)^i = \left[\tilde{D}^{S,\mathcal{L}}(x,y,Q) X_{-y}(x)^{\delta+1}\right]_{x^\delta}$$

$$\sum_{i\in\mathbb{Z}} M^i_{\delta,[S,\mathcal{L}]}(y) x^{\delta-i}\,(xQ)^i =$$
$$\left[\tilde{D}^{S,\mathcal{L}}(x,y,Q) X_{-y}(x)^{\delta+1} \left(\frac{y^{1-/2} - y^{1/2} e^{-x\left(y^{-1/2}-y^{1/2}\right)}}{y^{-1/2} - y^{1/2}}\right)^m\right]_{x^\delta}.$$

By Conjecture 4.3.4

$$\sum_{i\in\mathbb{Z}} N^i_{\delta,[S,\mathcal{L}]}(y) x^{\delta-i}\,(xQ)^i, \sum_{i\in\mathbb{Z}} M^i_{\delta,[S,\mathcal{L}]}(y) x^{\delta-i}\,(xQ)^i \in \mathbb{Q}[y^{-1/2}, y^{1/2}][\![x, xQ]\!]$$

so that the only possible power of Q that could appear is $i = 0, \ldots, \delta$. We can directly conclude that $N^i_{\delta,[S,\mathcal{L}]}, M^i_{\delta,[S,\mathcal{L}]}$ are Laurent polynomial in $y^{1/2}$. Set $s = xQ$, so that by Conjecture 4.3.4 we can write $\tilde{D}^{S,\mathcal{L}}(x,y,Q)$ as power series of x and s i.e $\tilde{D}^{S,\mathcal{L}}(x,y,\frac{s}{x}) \in \mathbb{Q}[y^{-1/2}, y^{1/2}][\![x, s]\!]$. And since

$$X_{-y}(x=0) = 1$$
$$\left(\frac{y^{1-/2} - y^{1/2} e^{-x\left(y^{-1/2}-y^{1/2}\right)}|_{x=0}}{y^{-1/2} - y^{1/2}}\right)^m = 1$$

we can conclude that

$$\sum_{i\geq 0} M^{\delta}_{\delta,[S,\mathcal{L}]}(y)\,(s)^{\delta} = \tilde{D}^{S,\mathcal{L}}(x, y, \frac{s}{x})|_{x=0}$$

$$= \sum_{\delta \geq 0} N^{\delta}_{\delta,[S,\mathcal{L}]}(y) s^{\delta}$$

$\square$

If $H^i(\mathcal{L}) = 0$ for $i > 0$ and $\mathcal{L}$ is δ-very ample, then $N^{\delta}_{\delta,[S,\mathcal{L}]}(y)$ is the refinement defined by Goettsche and Shende in [12] of $n_{\delta}(\mathcal{L})$ that computes the number of δ-nodal curves in $|\mathcal{L}|$. Theorem 4.3.2 and Theorem 4.3.1 gives geometric argument for the equality $M^{\delta}_{\delta,[S,\mathcal{L}]}(y)|_{y=1} = N^{\delta}_{\delta,[S,\mathcal{L}]}(y)|_{y=1}$. Without assuming the conjecture above we would like to know if Proposition 4.3.5 still true.

Bibliography

[1] Kai Behrend and Barbara Fantechi. The intrinsic normal cone. *Inventiones Mathematicae*, **128**:45–88, 1997.

[2] Jinwon Choi, Sheldon Katz, and Albrecht Klemm. The refined BPS index from stable pair invariants. *Communications in Mathematical Physics*, **328**(3):903–954, 2014.

[3] Neil Chriss and Victor Ginzburg. *Representation Theory and Complex Geometry*. Birkhäuser, 1997.

[4] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. math.*, **131**:595–634, 1998.

[5] Dan Edidin and William Graham. Riemann-roch for equivariant chow groups. *Duke Mathematical Journal*, **102**(3):567–594, 2000.

[6] Barbara Fantechi and Lothar Göttsche. Riemann-roch theorems and elliptic genus for virtually smooth schemes. *Geometry & Topology*, **14**(1):83–115, 2010.

[7] Barbara Fantechi and Rahul Pandaripandhe. Stable maps and branch divisor. *Compositio Mathematica*, **130**(3):345–364, 2000.

[8] William Fulton. *Intersection Theory*. Springer-Verlag, Boston, MA, USA, 2nd edition, 1984.

[9] William Fulton and Serge Lang. *Riemann-Roch Algebra*. Grunlehren de mathematischen Wissenschaften, 277. Springer Verlag, New York, 1985.

[10] Sergei I. Gelfand and Yuri I. Manin. *Methods of Homological Algebra*. Springer Verlag, Berlin, Heidelberg, 1996.

[11] Lothar Göttsche. A conjectural generating function for numbers of curves on surfaces. *Commun. Math. Phys.*, **196**(1):523–533, 1998.

[12] Lothar Göttsche and Vivek Shende. Refined curve counting on complex surfaces. *Geometry & Topology*, **18**(4):2245–2307, 10 2014.

[13] Lothar Göttsche and Vivek Shende. The χ_y genera of relative hilbert schemes for linear systems on abelian and $K3$ surfaces. *Algebraic Geometry*, **2**(4):405–421, 2015.

[14] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*. Cambridge University Press, New York, 2nd edition, 2010.

[15] Daniel Huybrechts and Richard P. Thomas. Deformation-obstruction theory for complexes via atiyah and kodaira-spencer classes. *Mathematische Annalen*, pages 346–545, 2010.

[16] Bernhard Köck. Das adams-riemann-roch-theorem in der höheren äquivarianten K-theorie. *J. Reine Angew. Math.*, **421**:189–217, 1991.

[17] Bernhard Köck. The grothendieck-riemann-roch theorem for group scheme actions. *Ann. scient. Éc. Norm. Sup.*, **31**(3):415–458, 1998.

[18] Martijn Kool, Vivek Shende, and Richard P Thomas. A short proof of the Göttsche conjecture. *Geometry & Topology*, **15**(1):397–406, 2011.

[19] Martijn Kool and Richard Thomas. Reduced classes and curve counting on surfaces I: theory. *Algebraic Geometry*, **1**(3):334–383, 2014.

[20] Martijn Kool and Richard Thomas. Reduced classes and curve counting on surfaces II: calculations. *Algebraic Geometry*, **1**(3):384–399, 2014.

[21] Davesh Maulik, Rahul Pandaripandhe, and Richard P. Thomas. Curves on $K3$ surfaces and modular forms. *Ann. scient. Éc. Norm. Sup.*, **31**(3):415–458, 2010.

[22] David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory.* Erg. Math. 34. Springer Verlag, Berlin, Heidelberg, 3rd edition, 1984.

[23] Nikita Nekrasov and Andrei Okounkov. Membranes and sheaves. *Algebraic Geometry*, **3**(3):320–369, 2016.

[24] Rahul Pandharipande and Richard Thomas. The 3-fold vertex via stable pairs. *Geometry & Topology*, **13**(4):83–115, 2009.

[25] Rahul Pandharipande and Richard P. Thomas. Stable pairs and bps invariants. *Journal of American Mathematical Society*, **23**(1), January 2009.

[26] Joseph Le Potier. Systèmes cohérents et structures de niveau. *Astérique*, **214**:143, 1993.

[27] Feng Qu. Virtual pullbacks in K-theory. *Annales de l'Institut Fourier*, **68**(4):1609–1641, 2018.

[28] Daniel Gray Quillen. Higher algebraic K-theory: I. In H. Bass, editor, *Algebraic K-theory I*, Lecture Notes in Math. 341, pages 85–147. Springer, new York, 1973.

[29] Jean-Pierre Serre. *Local Algebra.* Springer-Verlag, 2000.

[30] Bernd Siebert. Virtual fundamental classes, global normal cones and fulton's canon- ical classes. In K. Hertling and M. Marcolli, editors, *Frobenius Manifolds*, Aspects Math. 36, pages 341–358. Vieweg, Wiesbaden, Germany, 2004.

[31] Richard Thomas. Equivariant K-theory and refined vafa-witten invariants. 2018. preprint, https://arxiv.org/abs/1810.00078.

[32] R. W. Thomason. Algebraic k-theory of group scheme actions. In William Browder, editor, *Algebraic Topology and Algebraic K-theory*, Annals of Mathematics Studies 113, chapter 20, pages 539–563. Princeton University Press, Princeton, New Jersey, 1987.

[33] R. W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived category. In *The Grothendieck Festschift III*, Progress in Math. 88, pages 247–435. Birkhäuser, 1990.

[34] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theories. In *Fundamental Algebraic Geometry: Grothendieck's FGA's Explained*, Mathematical Surves and Monographs no. 123, chapter 1-4, pages 1–103. American Mathematical Society, Rhode Island, 2005.

[35] Charles A. Weibel. *The K-book : an introduction to algebraic K-theory.* GSM 145. American Mathematical Society, Providence, Rhode Island, 2013.