

Article

Effective Control of Chemical Potentials by Rabi Coupling with RF-Fields in Ultracold Mixtures

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Abstract: We show that a linear term coupling the atoms of an ultracold binary mixture provides a simple method to induce an effective and tunable population imbalance between them. This term is easily realized by Rabi coupling between different hyperfine levels of the same atomic species. The resulting effective imbalance holds for one-particle states dressed by the Rabi coupling and obtained by diagonalizing the mixing matrix of the Rabi term. This way of controlling the chemical potentials applies to both bosonic and fermionic atoms and it also allows for spatially- and temporally-dependent imbalances. As a first application, we show that, in the case of two attractive fermionic hyperfine levels with equal chemical potentials coupled by the Rabi pulse, the same superfluid properties of an imbalanced binary mixture are recovered. We finally discuss the properties of m -species mixtures in the presence of $SU(m)$ -invariant interactions.

Keywords: quantum gases; multicomponent mixtures; rf-fields; Rabi coupling; imbalanced mixtures

1. Introduction

The average density of particles is a central parameter in ultracold experiments [1,2], and in general for condensed matter phenomena. Indeed, most physical systems acquire different features when the interaction energy is varied by changing the number of particles N . This fact arises already at the single-particle level, determining for instance the shape of the Fermi surfaces, in turn related, for example, to conductance or to the critical properties of Bose condensates [3]. The phenomenon is even more pronounced once interactions are included as, for instance, has been shown for superfluidity in fermion mixtures [4]. In the presence of a density imbalance between the pairing species, new types of superfluid phases, different from the standard BCS/BEC ones, can appear [4–12]. Other notable examples are the quantum Hall effect [13], where the filling deeply affects the nature of the ground state, or the transition superfluid–Mott insulator in the Bose–Hubbard model [2].

In recent years, the experimental capability of controlling and measuring local particle densities has greatly improved due to single-site addressing [14,15]. However, typically in ultracold experiments

one controls the density and not the chemical potential μ . In order to go from the canonical description to the grand-canonical one, one needs to exploit the relation between N and μ , generally a non-trivial task. Moreover, standard experiments with imbalanced mixtures are not always straightforward to realize, since often the phases separate or become unstable [2]. Therefore, designing alternative methods to tune imbalances effectively appears a very interesting task. This can be even more relevant if the chemical potential can be tuned in a space- or time-dependent way, which is not easy to realize controlling the atomic populations. For instance, this possibility may be crucial for the synthesis of unconventional superfluid pairings with nonzero momenta or for the creation of space–time defects.

In this paper, we provide a different method to induce and to control effective population imbalances in experiments involving atoms of a binary mixture linearly coupled between them, a major example being different hyperfine levels equally populated coupled by a Rabi rf-field. The obtained imbalances hold for dressed one-particle states, obtained diagonalizing the Rabi term that couples the hyperfine levels pairwise. The method works both for bosonic and fermionic atoms. The Rabi coupling in fermionic and bosonic mixtures is well studied for a variety of applications (see e.g., [16–21]) and widely used in ultracold atoms experiments [2,22,23]. We therefore think that it would be straightforward to perform an experiment along the lines discussed in this paper for realizing an effective imbalance between different hyperfine levels of ultracold mixtures.

In the presence of two-body interactions for the mixture, we argue that, under general conditions, the dynamics can be equally described in terms of the original states or of the dressed states, as the scattering processes do not destroy the coherence of the dressed states. A particularly interesting situation is when the two-body interactions do not depend on the involved hyperfine species in the mixture. This is the case of alkaline earth-like atoms, such as ^{87}Sr [24,25] or ^{173}Yb [26,27]. These atomic species provide instances of $SU(m)$ -invariant mixtures (with $m = 10$ and 6 , respectively). Quite recently, similar mixtures have been found important for the simulation of multispecies (anti-)ferromagnetism [28–31], synthetic dimensions and effective quantum Hall systems [32], and two-flavor symmetry locking [33]. We note also that $SU(m)$ invariance may hold approximately also for atomic species such as ^{87}Rb or ^{40}K [34,35].

The discussion above suggests that our method provides a new method for efficiently probing the physics of interacting imbalanced mixtures. As a first application, we focus on an attractive two-species fermionic mixture, showing that its superfluid properties under Rabi coupling are the same as those of an imbalanced mixture in the absence of a Rabi term. This example is particularly interesting in view of the rich phenomenology of the imbalanced Fermi gases as imbalance and interaction vary, also including FFLO physics (see e.g., [4,5] and references therein). Moreover, we argue that this mapping allows for engineering spatially- and temporally-dependent imbalances. They can be realized, for instance, by driving the intensity of the radio-frequency or of the Raman laser pulse which can induce Rabi oscillation, as recently done in a Floquet spin-orbit coupling experiment [36].

The paper is organized as follows. After a general survey on the proposed method (Section 2) in the presence of two-body interactions, we discuss the application of the Rabi coupling on an atomic mixture of two hyperfine species in a realistic experimental setup, and its generalization to N -species mixtures (Section 3). As a first example, we study a BCS/BEC superfluid in a mixture of two balanced hyperfine species coupled by a Rabi pulse (Section 4), finding agreement, even at the mean-field level, with the known literature on the two-species imbalanced fermionic superfluids in the absence of the Rabi coupling. In Section 5 we reconsider the same problem in the continuous space, discussing the effect of the Rabi coupling on the renormalization of the mean-field self-consistency equations for the nontrivial order parameters. In Section 6 we address further applications, involving the possibility of tuning the effective population imbalances in time and space, showing their experimental feasibility. Finally, in Section 7 we collect and discuss the main results, as well as possible future developments.

2. Two-Species Mixtures Coupled by Rabi Coupling

In this section we describe a general lattice setup where a mixture of two bosonic or fermionic hyperfine species (labeled by $\sigma = \{\uparrow, \downarrow\}$) are coupled by a Rabi term. When not stated otherwise, these two species are supposed to have the same filling: $n_\uparrow = n_\downarrow$, with $n_\sigma \equiv \frac{N_\sigma}{V}$, N_σ being the number of fermions of the species σ on the lattice and V the number of sites of the lattice.

A Rabi term can be induced by microwave or radio-frequency techniques, or by two-photon Raman transitions induced by two opportunely detuned laser beams [22,23]. The latter technique also allows for a spatial dependence of the Rabi frequency, as described in Section 6.

For simplicity, we will consider a three-dimensional cubic lattice, but, as discussed in the following, no significant changes arise in the continuum space or in lower dimensions. The atoms can hop between nearest neighbor sites with hopping energy t ($t > 0$) and interact via a contact attractive potential with magnitude U . The Rabi coupling flips the spin- σ orientations of the atoms, with frequency $\Omega(t, \mathbf{r})$ which can be a complex value and may depend on time and position. We consider a real valued $\Omega(t, \mathbf{r})$ of a factorized form in time and space, $\Omega(t, \mathbf{r}) = \Omega f(t)g(\mathbf{r})$. While by microwave or radio-frequency pulses a non-constant $g(\mathbf{r})$ is hard to obtain, by back-reflecting on a mirror the lasers inducing the Raman transitions, one can achieve a sinusoidal profile $g(\mathbf{r}) \propto \cos(\mathbf{r} \cdot \mathbf{q})$, where \mathbf{q} is the difference between the wave-vectors of the two lasers.

In the discussion below we omit the space and time dependences, for which consequences will be discussed at the end of the paper, in Section 6.

2.1. Fermions

We focus first on fermions. Thus, the Hamiltonian describing the system that we are interested in is

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} + \Omega \sum_i \left(e^{i\phi} c_{i\uparrow}^\dagger c_{i\downarrow} + e^{-i\phi} c_{i\downarrow}^\dagger c_{i\uparrow} \right) - U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (1)$$

where $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$ are the number operators, and ϕ a generic phase.

Here we describe the system in the canonical ensemble, as for a single experimental realization, where the number of the atoms is fixed. The Hamiltonian (1) can be mapped by the unitary transformation

$$a_{i\pm} = \frac{e^{-i\frac{\phi}{2}} c_{i\uparrow} \pm e^{i\frac{\phi}{2}} c_{i\downarrow}}{\sqrt{2}}. \quad (2)$$

to the following Hamiltonian

$$H_{\text{ROT}} = -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} + \Omega \sum_i (n_{i+} - n_{i-}) - U \sum_i n_{i+} n_{i-}, \quad (3)$$

where $\alpha = \pm$ and $n_{i\pm} = a_{i\pm}^\dagger a_{i\pm}$. We find that the original Rabi coupling $\propto \Omega$ is mapped to an energy imbalance $h = 2\Omega$, as for a Zeeman term. Even more importantly, the interaction $\propto U$ transforms covariantly under the same transformation [37], allowing us to probe the Hubbard physics on imbalanced Hamiltonians also (an example will be given subsequently). We finally mention that, due to the Rabi coupling, an imbalance at $\Omega = 0$ does not produce any qualitative difference, as compared to the balanced case, once Ω is switched on (see e.g., [38]).

The Zeeman term can be also interpreted as a chemical potential term. Indeed, as we will describe in Section 4, although the total number of particles $N_+ + N_- = N_\uparrow + N_\downarrow \equiv N$ remains constant in the present canonical scheme, the number N_\pm is not fixed but can fluctuate, and therefore N_\pm can be considered as quantum averages. In particular, we will discuss the behaviour of the average difference ($N_+ - N_-$) as a function of U and Ω , non vanishing in a normal state.

The calculations above are also valid in the grand canonical ensemble, introducing a chemical potential μ and then letting N_\uparrow ($n_{i,\uparrow}$) and N_\downarrow ($n_{i,\downarrow}$) fluctuate, which is then fixed only in the average.

In real experiments the μ -term in (4) can be ascribed to the average occupation numbers for different experimental realizations. The resulting Hamiltonian reads:

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} + \Omega \sum_i \left(c_{i\uparrow}^\dagger c_{i\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow} \right) - U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i,\sigma} c_{i\sigma}^\dagger c_{i\sigma}, \quad (4)$$

and the transformation (2) yields:

$$\begin{aligned} H_{\text{ROT}} &= -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} - (\mu - \Omega) \sum_i n_{i+} - (\mu + \Omega) \sum_i n_{i-} - U \sum_i n_{i+} n_{i-} \\ &= -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} - \mu \sum_i (n_{i+} + n_{i-}) + \Omega \sum_i (n_{i+} - n_{i-}) - U \sum_i n_{i+} n_{i-}. \end{aligned} \quad (5)$$

We find that the Rabi term in the old basis $\{\uparrow, \downarrow\}$ becomes, after the transformation in (2), an imbalance term $\delta\mu = 2\Omega$ for the chemical potentials of the new species a_{\pm} .

2.2. Bosons

For a bosonic system, three Hubbard interactions are possible: the intra-species interactions $U_{\uparrow\uparrow}$ and $U_{\downarrow\downarrow}$, and the inter-species interaction $U_{\uparrow\downarrow}$, so that we get

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} + \Omega \sum_i \left(e^{i\varphi} c_{i\uparrow}^\dagger c_{i\downarrow} + e^{-i\varphi} c_{i\downarrow}^\dagger c_{i\uparrow} \right) - \sum_i \left(U_{\uparrow\uparrow} n_{i\uparrow}^2 + U_{\downarrow\downarrow} n_{i\downarrow}^2 + U_{\uparrow\downarrow} n_{i\uparrow} n_{i\downarrow} \right). \quad (6)$$

The interacting term of this Hamiltonian transforms covariantly under the rotation in Equation (2) iff $U_{\uparrow\uparrow} = U_{\downarrow\downarrow} = U_{\uparrow\downarrow}/2$, which is the case for alkaline earth atoms [28,29]. In this case, with these relations for the interaction strengths, applying Equation (2), Equation (6) maps to

$$H_{\text{ROT}} = -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} + \Omega \sum_i (n_{i+} - n_{i-}) - \frac{U_{\uparrow\downarrow}}{2} \sum_i (n_{i+} + n_{i-})^2. \quad (7)$$

2.3. Trap Effects

In current ultracold atoms experiments, both for bosons and fermions, a typical ingredient is the external trapping potential of the form

$$\sum_i V(\mathbf{r}_i) (n_{i\uparrow} + n_{i\downarrow}), \quad (8)$$

where $V(\mathbf{r}_i) = \frac{m}{2}\omega^2|\mathbf{r}_i|^2$ and \mathbf{r}_i is the vector distance of the lattice site at position i from the center of the trap. We assume the trapping frequency ω and the mass m equal for both the species $\{\uparrow, \downarrow\}$, as in most of the experiments involving different hyperfine levels of the same atom. It is straightforward to show that also the potential in (8) transforms covariantly under the rotation in Equation (2):

$$\sum_i V(\mathbf{r}_i) (n_{i+} + n_{i-}), \quad (9)$$

and then all the previous discussions are not spoiled by the presence of this term.

In the light of the above consideration, we expect that the properties of balanced mixtures (both in the canonical and in the grandcanonical ensemble) under the Rabi coupling and possibly of two-body interactions (provided that the interactions do not to spoil the rotated states (2)) are equal to the physics of imbalanced interacting mixtures. This will be further studied in Section 4, considering the superfluid properties of the Hamiltonian (Equation (3)).

2.4. Comments on the Experimental Implementation

In this subsection, we comment on the limits of tunability of the Rabi coupling, also in relation to the other parameters, such as the typical hopping amplitude $\{t\}$ and the strengths of the interaction $\{U_i\}$. Indeed our method, to be effective, requires the Rabi coupling to be tunable over a range of energies at least comparable with the smallest energy scales between the hopping amplitude and the interactions. This condition is easily fulfilled for constant (or smoothly varying) Ω . Indeed, it is possible to achieve $\Omega \sim t$ within the validity of the tight-binding approximation. This favourable situation can change if the Rabi frequency depends on the position or on the time. Roughly speaking, the allowed dependence should fulfill the adiabatic theorem. In Section 6.2, we quantitatively discuss the limits imposed by the presence of a sinusoidal behavior for Ω induced by Raman lasers. In particular we will show that the related momentum transfer does not considerably limit further the range of allowed values for Ω . Thus, it turns out that there are neither conceptual nor technical limitations in the use of the Rabi coupling technique. Nowadays intensities $\Omega \sim \text{kHz}$ are realistic, which are of the same order of magnitude of the Fermi energies in typical experiments, both in the continuous space and on the lattice. We conclude that unbalancing an attractive two-species hyperfine mixture by Rabi coupling allows us to investigate (at least) both the BCS and BEC limits of the superfluid regime for ultracold fermionic mixtures.

2.5. Further Generalizations

In principle, spin-dependent hopping t_σ can be added to (4), for instance exploiting superlattice configurations (see, e.g., [2] and references therein). Interestingly, if we apply the transformation in (2) to the resulting Hamiltonian, we obtain:

$$H_{t, \text{ROT}} = -\frac{t_\uparrow + t_\downarrow}{2} \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} - \frac{t_\uparrow - t_\downarrow}{2} \sum_{\langle i,j \rangle} (a_{i+}^\dagger a_{j-} + a_{i-}^\dagger a_{j+}) - (\mu - \Omega) \sum_i n_{i+} - (\mu + \Omega) \sum_i n_{i-} - U \sum_i n_{i+} n_{i-}, \quad (10)$$

We find that, in the rotated frame, in addition to the spin-dependent chemical potential, a spin-orbit-like coupling term appears. Therefore, this scheme can be used to simulate a spin-orbit coupling, at least in one-dimensional lattices. If we proceed further with the diagonalization, we end up with the Hamiltonian

$$H_{t, \text{ROT}}^{(\text{diag})} = \sum_{\mathbf{k}} \left[\lambda_+(\mathbf{k}) a_{\mathbf{k}+}^\dagger a_{\mathbf{k}+} + \lambda_-(\mathbf{k}) a_{\mathbf{k}-}^\dagger a_{\mathbf{k}-} - U n_{\mathbf{k}+} n_{\mathbf{k}-} \right], \quad (11)$$

with

$$\lambda_\pm(k) = \varepsilon_{\mathbf{k}} - \mu \pm \sqrt{\delta\varepsilon_{\mathbf{k}}^2 + \Omega^2}, \quad (12)$$

where $\varepsilon_{\mathbf{k}} \equiv \frac{\varepsilon_{\mathbf{k}\uparrow} + \varepsilon_{\mathbf{k}\downarrow}}{2}$ and $\delta\varepsilon_{\mathbf{k}} \equiv \frac{\varepsilon_{\mathbf{k}\uparrow} - \varepsilon_{\mathbf{k}\downarrow}}{2}$, posing $\varepsilon_{\mathbf{k}\uparrow} = -t_\uparrow \sum_{s=x,y,z} \cos k_s$, $\varepsilon_{\mathbf{k}\downarrow} = -t_\downarrow \sum_{s=x,y,z} \cos k_s$.

3. Generalization to N Species

The model described in the previous section can be easily extended to a N -mixture ($N = 2M$) of bosonic or fermionic atoms in several ways, for instance by coupling the various hyperfine species pairwise. The non-interacting grand canonical Hamiltonian with possible imbalances in the densities (with different chemical potentials) and in the hopping amplitudes reads in this case:

$$H_{\{\mu_l\}, \{t_l\}} = - \sum_{\langle i,j \rangle, l} t_{2l} c_{i,2l}^\dagger c_{j,2l} - \sum_{\langle i,j \rangle, l} t_{2l+1} c_{i,2l+1}^\dagger c_{j,2l+1} - \sum_{i,l} (\mu_{2l} c_{i,2l}^\dagger c_{i,2l} + \mu_{2l+1} c_{i,2l+1}^\dagger c_{i,2l+1}) + \sum_{i,l} \Omega_l (c_{i,2l}^\dagger c_{i,2l+1} + c_{i,2l+1}^\dagger c_{i,2l}), \quad (13)$$

where $l = 1, \dots, M$ labels the pairs of hyperfine species coupled by the Rabi coupling.

This Hamiltonian, after the unitary transformation

$$a_{i\pm}^{(l)} = \frac{c_{i,2l} \pm c_{i,2l+1}}{\sqrt{2}}, \quad (14)$$

becomes

$$H_{\{\mu_l\}, \{t_l\}} \text{ROT} = - \sum_{\langle i,j \rangle, \alpha, l} \frac{t_{2l} + t_{2l+1}}{2} a_{i\alpha}^{(l)\dagger} a_{j\alpha}^{(l)} - \sum_{\langle i,j \rangle, l} \frac{t_{2l} - t_{2l+1}}{2} \left(a_{i+}^{(l)\dagger} a_{j-}^{(l)} + a_{i-}^{(l)\dagger} a_{j+}^{(l)} \right) \quad (15)$$

$$- \sum_{i,l} \left[\left(\frac{\mu_{2l} + \mu_{2l+1}}{2} - \Omega_l \right) n_{i+}^{(l)} + \left(\frac{\mu_{2l} + \mu_{2l+1}}{2} + \Omega_l \right) n_{i-}^{(l)} \right] - \sum_{i,l} \frac{\mu_{2l} - \mu_{2l+1}}{2} \left(a_{i+}^{(l)\dagger} a_{i-}^{(l)} + a_{i-}^{(l)\dagger} a_{i+}^{(l)} \right).$$

Possible density–density interactions have the same form also in the dressed basis (14), provided that the interactions involve only the hyperfine species pairwise $2l$ – $(2l + 1)$:

$$\sum_{l=1}^M \sum_{i,j} V_l(i-j) \left(n_{i,2l} + n_{i,2l+1} \right) \left(n_{j,2l} + n_{j,2l+1} \right) \quad (16)$$

$$(n_{i,\alpha} = c_{i,\alpha}^\dagger c_{i,\alpha}).$$

In this case, the transformation (14) yields

$$\sum_{l=1}^M \sum_{i,j} V_l(i-j) \left(n_{i+}^{(l)} + n_{i-}^{(l)} \right) \left(n_{j+}^{(l)} + n_{j-}^{(l)} \right), \quad (17)$$

where $n_{i\pm}^{(l)} = a_{i\pm}^{(l)\dagger} a_{i\pm}^{(l)}$.

Another case, even more interesting from the experimental point of view, occurs when the interactions pair all the hyperfine species in the mixtures, with the same strength and the same space dependence (as for alkaline earth-like atoms [28–30,32], see also the Introduction):

$$\sum_{i,j} V(i-j) \sum_{s,s'=1}^{2M} n_{i,s} n_{j,s'} = \sum_{i,j} V(i-j) \left(\sum_{s=1}^{2M} n_{i,s} \right) \left(\sum_{s'=1}^{2M} n_{j,s'} \right). \quad (18)$$

This interaction still transforms covariantly under (14):

$$\sum_{i,j} V(i-j) \sum_{l,l'=1}^M \left(n_{i+}^{(l)} + n_{i-}^{(l)} \right) \left(n_{j+}^{(l')} + n_{j-}^{(l')} \right). \quad (19)$$

In these cases, the dynamics can be described easily in terms of dressed states as well, since the scattering processes related to the interactions do not spoil them.

A clarification of the latter point can be obtained considering the example of two species, \uparrow and \downarrow (labeled by the momentum), scattering in the s -wave channel. Due to the Pauli principle, the low-energy scattering, supposedly elastic, can occur only in the channel

$$\frac{1}{\sqrt{2}} \left[\frac{\uparrow + \downarrow}{\sqrt{2}} \otimes \frac{\uparrow - \downarrow}{\sqrt{2}} - \frac{\uparrow - \downarrow}{\sqrt{2}} \otimes \frac{\uparrow + \downarrow}{\sqrt{2}} \right] (\mathbf{k}) = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} (\mathbf{k}) \rightarrow e^{i\phi_{\mathbf{k}}} \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} (\mathbf{k}), \quad (20)$$

with $\phi_{\mathbf{k}}$ being the momentum depending scattering phase. We find that $e^{i\phi_{\mathbf{k}}}$ globally multiplies the final scattering state, without spoiling the relative coherence of the states $\uparrow\downarrow$ and $\downarrow\uparrow$. This is an effect of the linearity of the scattering matrix [39] and it immediately holds if $N = 2$. The phase $e^{i\phi_{\mathbf{k}}}$ is nothing but the phase resulting from the scattering at $\Omega = 0$ of \uparrow and \downarrow particles, with momentum \mathbf{k} in the spin singlet state.

In the general case, with $N \geq 2$, the conditions that we imposed above on the interactions assure that every scattering process changes the initial state $\frac{|l\rangle \pm |l+1\rangle}{\sqrt{2}}$ just by multiplying it by a pure phase $e^{i\phi_{lk}}$, as for the case reported in Equation (20).

4. Two-Species Superfluidity in the Presence of Rabi Coupling

In this section, we exemplify the effect of an imbalance induced by Rabi coupling, focusing on a two-species fermionic mixture and studying the properties of its superfluid phase. We consider the Hamiltonian in (4):

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} - \mu \sum_{i,\sigma} c_{i\sigma}^\dagger c_{i\sigma} + \Omega \sum_i \left(c_{i\uparrow}^\dagger c_{i\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow} \right) - U \sum_i n_{i\uparrow} n_{i\downarrow}. \quad (21)$$

As discussed in Section 2, we explicitly include the chemical potential term. In the following, this Hamiltonian and similar ones with an Hubbard interaction will be denoted as “full” Hamiltonians, in contrast to the mean field quadratic Hamiltonians.

In the presence of a nonzero superfluid gap Δ (to be verified a posteriori), the corresponding (mean field) BCS-projected Hamiltonian is:

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} - \left(\mu + \frac{Un}{2} \right) \sum_{i,\sigma} c_{i\sigma}^\dagger c_{i\sigma} + (\Omega + \gamma) \sum_i \left(c_{i\uparrow}^\dagger c_{i\downarrow} + c_{i\downarrow}^\dagger c_{i\uparrow} \right) - \Delta \sum_i \left(c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + c_{i\downarrow} c_{i\uparrow} \right). \quad (22)$$

In Equation (22) we assumed the presence of a further bilinear order parameter, also coming from the Wick decomposition of the Hubbard interaction term in Equation (21):

$$\gamma = -U \langle c_{i\uparrow}^\dagger c_{i\downarrow} \rangle = -U \langle c_{i\downarrow}^\dagger c_{i\uparrow} \rangle, \quad (23)$$

due to the presence of the Rabi coupling. Moreover, we fix $(\Omega + \gamma) > 0$, up to a phase redefinition of the $c_{i\sigma}$ operators. Going to momentum space, the Hamiltonian (22) can be easily diagonalized, finding the eigenvalues:

$$\lambda_{\mathbf{k}}^{(\pm)} = E_{\mathbf{k}} \pm (\Omega + \gamma), \quad (24)$$

where $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$ and

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \tilde{\mu}, \quad \varepsilon_{\mathbf{k}} = -2t \sum_{l=x,y,z} \cos k_l, \quad \tilde{\mu} = \mu + \frac{UN}{2V}. \quad (25)$$

The Bogoliubov coefficients turn out to be the same as in the purely ($\Omega = 0$) BCS case. However, we find that, introducing the Rabi term, the quasiparticle excitation spectrum is split into two bands, shifted by $\pm(\Omega + \gamma)$ with respect to the usual BCS case. Correspondingly, the minimum of the excitation spectrum is

$$\Delta_G = \Delta - |\Omega + \gamma|. \quad (26)$$

The ground-state energy reads:

$$E_{\text{SUP}} = \frac{V \Delta^2}{U} + \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{D}} \left[\left(\xi_{\mathbf{k}} - \lambda^{(+)}(\mathbf{k}) \right) + \left(\xi_{\mathbf{k}} - \lambda^{(-)}(\mathbf{k}) \right) \right] + \sum_{\mathbf{k} \in \mathcal{D}} \lambda^{(-)}(\mathbf{k}), \quad (27)$$

where \mathcal{D} is a domain in the first Brillouin zone defined as

$$\mathcal{D} = \{ \mathbf{k} \in 1^{\text{st}}\text{BZ} : E_{\mathbf{k}} > \Omega + \gamma \} \quad (28)$$

and its complementary domain as

$$\bar{\mathcal{D}} = \{\mathbf{k} \in 1^{\text{st}}\text{BZ} : E_{\mathbf{k}} < \Omega + \gamma\} . \quad (29)$$

The explicit calculation of the self-consistent equations for the order parameters Δ, γ , and μ is performed by setting to 0 the derivatives of $E_{\text{SUP}} + \mu N + \gamma \delta N$, $\delta N \equiv (N_- - N_+)$, with regards to Δ, γ and μ , respectively.

The introduction of the quantity $\gamma \delta N$ stems from the Wick decomposition of the Hubbard interaction in (21), similar to the first term in E_{SUP} [40]. The final result is:

$$1 = \frac{U}{2V} \sum_{\mathbf{k} \in \mathcal{D}} \frac{1}{E_{\mathbf{k}}} , \quad (30)$$

$$\delta N = - \sum_{\mathbf{k} \in \bar{\mathcal{D}}} 1 , \quad (31)$$

$$n = \frac{1}{V} \sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) + \frac{1}{V} \sum_{\mathbf{k} \in \bar{\mathcal{D}}} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} = 1 - \frac{1}{V} \sum_{\mathbf{k} \in \mathcal{D}} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} . \quad (32)$$

From (31), we conclude that $\gamma \leq 0$, with $\gamma = 0$ (as well as $\bar{\mathcal{D}} = 0$) iff $\Omega = 0$; moreover in (32) an additional term is present, compared to the $\Omega = 0$ case.

We observe that the domain \mathcal{D} , as well as its complementary $\bar{\mathcal{D}}$, differs from the definition given implicitly in [5], where the contribution of the imbalance δN has been neglected in the Hartree terms $-UN_- \sum_i a_{i+}^\dagger a_{i+} - UN_+ \sum_i a_{i-}^\dagger a_{i-}$. In this case, it holds that $\mathcal{D} = \{\mathbf{k} \in 1^{\text{st}}\text{BZ} : E_{\mathbf{k}} > \Omega + \gamma\}$ and Equation (31) is simplified. This aspect will be discussed in Section 5, where we will infer that, if $U = 0$, the parameter δN equals the imbalance of two fermionic species with chemical potentials $\mu_{\pm} \equiv \mu \pm \Omega$. The same identification holds for the ground-state energy (27).

4.1. Comparison at Mean-Field Level

The mean field ground-state energy (27), resulting from the full Hamiltonian (21), equals the same quantity for a imbalanced mixture under an onsite attraction [5], as described by the Hamiltonian (5). Indeed, we found that the Hamiltonian (21) maps exactly to (5) under the transformation in (2); for this reason the spectra and the phenomenologies at these two full Hamiltonians must be the same. However, one can ask whether the equivalence exactly established is valid also at mean-field level.

In the following we show, as may be expected, that the equivalence is valid as well for the corresponding Hamiltonians obtained in mean-field approximations, i.e., that the previous result holds at the mean-field level (where the self-consistency conditions have to be enforced).

The mean-field Hamiltonian from (5) reads:

$$H_{\text{ROT}} = -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} - (\mu - \Omega) \sum_i a_{i+}^\dagger a_{i+} - (\mu + \Omega) \sum_i a_{i-}^\dagger a_{i-} - UN_- \sum_i a_{i+}^\dagger a_{i+} - UN_+ \sum_i a_{i-}^\dagger a_{i-} + \Delta \sum_i (a_{i+} a_{i-} + \text{h.c.}) + \mu N + \Theta \delta N , \quad (33)$$

Similar to the previous subsection, the self-consistent equations for $\Delta, N, \delta N$ can be found deriving $E_{\text{SUP}} + \mu N + \Omega \delta N$ with respect to Δ, Θ , and μ [5]. The results are the same as in Equations (30)–(32).

We can check at this point the mean-field Hamiltonian obtained from (22) by the transformation in (2)

$$H_{\text{ROT}} = -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^\dagger a_{j\alpha} - (\mu - \Omega - \gamma) \sum_i a_{i+}^\dagger a_{i+} - (\mu + \Omega + \gamma) \sum_i a_{i-}^\dagger a_{i-} - \frac{U}{2} N \sum_i (a_{i+}^\dagger a_{i+} + a_{i-}^\dagger a_{i-}) + \Delta \sum_i (a_{i+} a_{i-} + \text{h.c.}) + \mu N + (\Omega + \gamma) \delta N , \quad (34)$$

coincides with the mean-field Hamiltonian (33). Interestingly, this comparison sheds light on the physical meaning of the parameter γ , yielding indeed:

$$\gamma = \frac{U}{2} \delta N. \quad (35)$$

Summing up, we found that the relation in (35) for the mean-field parameter γ is imposed by the request of equality between the mean-field Hamiltonians in (33) and (34), a fact physically motivated, but not trivially implied, by the equivalence of the full Hamiltonians in (21) and (5).

4.2. Discussions

Since the Hamiltonians in Equations (5) and (33) have been studied extensively in various papers devoted to imbalanced fermionic mixtures [4,41], both at zero and at finite temperatures, we do not further study their properties, and refer to the pertinent literature. For our purposes, indeed, the main point we want to stress here is the equivalence of these Hamiltonians in the presence of Rabi couplings with those describing imbalanced Fermi mixtures. In the experiments, one performs measurements on the fermionic species c 's; from the obtained findings for the quantities $\langle c^\dagger c \rangle$ and $\langle c^\dagger c^\dagger \rangle$, and via the relation between the c 's and the a 's, one can then reconstruct the phase diagram of the (a -)imbalanced mixture. The resulting main feature, based on the available results, is the appearance, both on the BCS and BEC sides, of a coexistence of normal and superfluid phases for suitable effective imbalances $\delta\mu = 2\Omega$. Moreover, unconventional forms of superconductivity, such as FFLO, are conjectured (for an extended review, see [6]). By the definition of γ , we expect that $\gamma = 0$ in a superfluid phase, where $N_+ = N_-$.

5. The Continuous Case

In this section we deal with the analogous equation (1) in the continuum space. In this situation, most of the features are qualitatively equal to the lattice case, and then we detail only the formulation and discussion of the equations for γ , μ , and Δ . In particular, we discuss the removal of the infinities encountered during their solution. We again assume work in three dimensions. The Hamiltonian is:

$$\begin{aligned} H^{(\text{cont})} = & \frac{1}{V} \sum_{\sigma} \int d\mathbf{r} c_{\sigma}^{\dagger}(\mathbf{r}) \left(-\hbar^2 \frac{\nabla^2}{2m} - \mu \right) c_{\sigma}(\mathbf{r}) \\ & + \frac{\Omega}{V} \int d\mathbf{r} \left(c_{\uparrow}^{\dagger}(\mathbf{r}) c_{\downarrow}(\mathbf{r}) + c_{\downarrow}^{\dagger}(\mathbf{r}) c_{\uparrow}(\mathbf{r}) \right) - \frac{U}{V} \int d\mathbf{r} n_{\uparrow}(\mathbf{r}) n_{\downarrow}(\mathbf{r}), \end{aligned} \quad (36)$$

with V denoting in this section the volume. The eigenstates of (36) in the absence of the interaction term $H_i = -\frac{U}{V} \int d\mathbf{r} n_{\uparrow}(\mathbf{r}) n_{\downarrow}(\mathbf{r})$ are superpositions of plane waves with momentum \mathbf{k} , of the form $a_{\pm}(\mathbf{k}) = \frac{c_{\uparrow} \pm c_{\downarrow}}{\sqrt{2}}(\mathbf{k})$, as in (2). These states are assumed to be asymptotical in the scattering evolution [34], and the interaction to arise between them without coherence-spoiling effects. Since here $N = 2$ in the notation of Section 3, this requirement is automatically fulfilled (see the discussion therein). The Hamiltonian transformed by (2) reads:

$$\begin{aligned} H_{\text{ROT}}^{(\text{cont})} = & -\frac{\hbar^2}{V} \sum_{\alpha=\pm} \int d\mathbf{r} a_{\alpha}^{\dagger}(\mathbf{r}) \frac{\nabla^2}{2m} a_{\alpha}(\mathbf{r}) - \frac{(\mu - \Omega)}{V} \int d\mathbf{r} a_{+}^{\dagger}(\mathbf{r}) a_{+}(\mathbf{r}) \\ & - \frac{(\mu + \Omega)}{V} \int d\mathbf{r} a_{-}^{\dagger}(\mathbf{r}) a_{-}(\mathbf{r}) - \frac{U}{V} \int d\mathbf{r} n_{+}(\mathbf{r}) n_{-}(\mathbf{r}). \end{aligned} \quad (37)$$

Gap Equations

We consider the zero-temperature self-consistency equations for γ , μ , and Δ . These equations have the same functional form as in the lattice case:

$$1 = \frac{U}{2} \frac{V}{(2\pi\hbar)^3} \int_{\mathcal{D}} d\mathbf{k} \frac{1}{E_{\mathbf{k}}} \tag{38}$$

$$\gamma = -\frac{U}{2} \frac{V}{(2\pi\hbar)^3} \int_{\bar{\mathcal{D}}} d\mathbf{k} 1 \tag{39}$$

$$n = 1 - \frac{1}{(2\pi\hbar)^3} \int_{\mathcal{D}} d\mathbf{k} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}, \tag{40}$$

similarly, the domains \mathcal{D} and $\bar{\mathcal{D}}$ are the same as in Section 4. Indeed, no assumption has been made about the precise form of $\varepsilon_{\mathbf{k}}$ in the derivation of the equations for γ , μ , and Δ . Clearly, \mathcal{D} and $\bar{\mathcal{D}}$ now take values into the infinite set of all the possible three-dimensional momenta, and not any longer in the first Brillouin zone. In order to derive (38)–(40) we also used the fact that $\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi\hbar)^3} \int d\mathbf{k}$, passing to the continuum limit.

In the absence of Rabi coupling, Equation (38) is known to be divergent and need regularization by the introduction of the scattering lengths a [4]. Let us study what happens to (38)–(40) in the presence of a Rabi term. To do this, it is useful to reconsider their derivation in the presence of a finite range potential $U(\mathbf{r})$, generalizing the equation in Section 4 (where a $\delta(\mathbf{r})$ interaction has been assumed instead). The result is:

$$\Delta_{\mathbf{k}'} = \frac{1}{2} \frac{1}{(2\pi\hbar)^3} \int_{\mathcal{D}} d\mathbf{k} \frac{\tilde{U}(\mathbf{k}, \mathbf{k}')}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}} \Delta_{\mathbf{k}}, \tag{41}$$

$$\gamma_{\mathbf{k}'} = -\frac{1}{2} \frac{1}{(2\pi\hbar)^3} \int_{\bar{\mathcal{D}}} d\mathbf{k} \tilde{U}(\mathbf{k}, \mathbf{k}'), \tag{42}$$

$$n = 1 - \frac{1}{(2\pi\hbar)^3} \int_{\mathcal{D}} d\mathbf{k} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}, \tag{43}$$

where $\tilde{U}(\mathbf{k}, \mathbf{k}') = \tilde{U}(\mathbf{k} - \mathbf{k}')$ denotes the Fourier transform of $U(\mathbf{r})$, $\tilde{U}(\mathbf{q}) = \int d\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} U(\mathbf{r})$. We insert into these equations the scattering amplitude (for the moment not restricted to the $l = 0$ contribution) $f(\mathbf{k}, \mathbf{k}')$, via the formula [39,42]:

$$f(\mathbf{k}, \mathbf{k}') = \frac{m}{4\pi\hbar^2} \tilde{U}(\mathbf{k}, \mathbf{k}') + \frac{1}{(2\pi\hbar)^3} \int d\mathbf{q} \frac{\tilde{U}(\mathbf{k}', \mathbf{q}) f(\mathbf{k}, \mathbf{q})}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{q}} - i0^+)}. \tag{44}$$

The solution of this equation can be obtained by iteration on $f(\mathbf{p}, \mathbf{p}')$ [39] in the second term to the right side:

$$f(\mathbf{k}, \mathbf{k}') = \frac{m}{4\pi\hbar^2} \left[\tilde{U}(\mathbf{k}, \mathbf{k}') + \frac{1}{(2\pi\hbar)^3} \int d\mathbf{q} \frac{\tilde{U}(\mathbf{k}', \mathbf{q}) \tilde{U}(\mathbf{k}, \mathbf{q})}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{q}} - i0^+)} + \dots \right], \tag{45}$$

the approximation keeping only the first term of (45) quoted as the (first) Born approximation [39]. Multiplying now both of the terms in (44) for $d\mathbf{k}'$, performing this integration, and exploiting (44), we obtain:

$$\Delta_{\mathbf{k}'} = \frac{1}{2} \frac{4\pi\hbar^2}{m} \frac{1}{(2\pi\hbar)^3} \left(\int_{\mathcal{D}} d\mathbf{k} \frac{f(\mathbf{k}, \mathbf{k}')}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}} - \int d\mathbf{k} \frac{f(\mathbf{k}, \mathbf{k}')}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}} \right) \Delta_{\mathbf{k}}. \tag{46}$$

Let us restrict now the scattering in the low-energy limit, where $\mathbf{k}, \mathbf{k}' \rightarrow 0$: in this regime $\Delta_{\mathbf{k},\mathbf{k}'} \rightarrow \Delta$ and $f(\mathbf{k}, \mathbf{k}') \rightarrow -a$ (a being the scattering length in s -wave, supposed negative) [39], so that we obtain:

$$1 = -\frac{1}{2} \frac{4\pi\hbar^2 a}{m} \frac{1}{(2\pi\hbar)^3} \left(\int_{\mathcal{D}} d\mathbf{k} \frac{1}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}} - \int d\mathbf{k} \frac{1}{\varepsilon_{\mathbf{k}}} \right). \quad (47)$$

This equation is the renormalized version of (38). Note that the integration on the second term in the right part of (47) is onto all the possible momenta, independent on the range of integration \mathcal{D} of the first term, as in the absence of the Rabi term. The result is expected in the light of the mapping of (4) on (5): the effective imbalance in the last Hamiltonian does not affect the interaction.

We deal now with (42): this is finite since $\bar{\mathcal{D}}$ is finite. Using the Born approximation $f(\mathbf{k}, \mathbf{k}') \approx \frac{m}{4\pi\hbar^2} \tilde{U}(\mathbf{k}, \mathbf{k}')$, one gets:

$$\gamma = -\frac{2\pi\hbar^2 a}{m} \frac{1}{(2\pi\hbar)^3} \int_{\mathcal{D}} d\mathbf{k}. \quad (48)$$

We stress that this approximation, well motivated here, is instead a source of divergencies if used in (38) and (41). Different approximations in (38) and (39) for the scattering amplitudes are possible since the two equations are decoupled: the latter one can be solved once the former and (40) have been solved at the same time (see the end of this subsection). Finally we observe that (40) is finite and independent on $\tilde{U}(\mathbf{k}, \mathbf{k}')$, and thus it does not require any further handling prior to being solved.

We consider now the explicit solution of (47), (48) and (40): we already showed that Δ and γ cannot be non-vanishing at the same time, for this reason we consider separately the super-conductive ($\gamma = 0, \Delta \neq 0$) and normal state regimes ($\gamma \neq 0, \Delta = 0$). Notice that in the case of coexistence of superfluid and normal state, as for suitable imbalanced fermionic mixtures (see [6] and references therein), the two cases can be discussed separately.

In the superconductive regime, \mathcal{D} involves all the possible momenta and $\bar{\mathcal{D}} = 0$, and one has:

$$\frac{m}{4\pi\hbar^2 a} = -\frac{1}{2} \frac{1}{(2\pi\hbar)^3} \int d\mathbf{k} \left(\frac{1}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} - \frac{1}{\varepsilon_{\mathbf{k}}} \right) \quad (49)$$

$$n = 1 - \frac{1}{(2\pi\hbar)^3} \int_{\mathcal{D}} d\mathbf{k} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}, \quad (50)$$

i.e., the usual equations describing the BEC/BCS crossover (see [43,44] and references therein, as well as, for example, [45] for the two-dimensional case).

As occurs for (41) and (43), the parameter γ does not appear in (49) and (50). For this reason, in the normal state, (48) can be solved independently, putting inside it the value $\tilde{\mu}$ obtained for each pair of external parameters (a, Ω), from the solution of (50) with $\Delta = 0$. Here $\bar{\mathcal{D}} = \{\mathbf{k} : |\xi_{\mathbf{k}}| < \Omega + \gamma\} = \{\mathbf{k} : |\frac{\mathbf{k}^2}{2m} - \tilde{\mu}| < \Omega + \gamma\}$. The set $\bar{\mathcal{D}}$ can be rewritten as $\bar{\mathcal{D}} = \{\{\tilde{\mu} < \frac{\mathbf{k}^2}{2m} < \tilde{\mu} + \Omega + \gamma\} \vee \{\tilde{\mu} - (\Omega + \gamma) < \frac{\mathbf{k}^2}{2m} < \tilde{\mu}\}\}$, so that a straightforward integration of (48) leads to:

$$\gamma = -\frac{4\sqrt{2}\pi}{3h^3} UVm^{\frac{3}{2}} \left[(\tilde{\mu} + \Omega + \gamma)^{\frac{3}{2}} - (\tilde{\mu} - \Omega - \gamma)^{\frac{3}{2}} \right]. \quad (51)$$

This equation can be solved numerically or analytically after some algebra, however the explicit solution is not very enlightening. A considerable simplification, together with a deeper insight, can be achieved neglecting the Hartree terms from the interaction $\propto a$. In this way, exploiting (35), we also obtain:

$$\delta N = -\frac{8\sqrt{2}\pi}{3h^3} Vm^{\frac{3}{2}} \left[(\mu + \Omega)^{\frac{3}{2}} - (\mu - \Omega)^{\frac{3}{2}} \right]. \quad (52)$$

By direct inspection, it is easy to check that (52) is equal to the imbalance between two free Fermi gases with chemical potential $\mu_{\pm} \equiv \mu \pm \Omega$. This result, valid also in the presence of a lattice, clarifies even further the meaning of δN and γ .

6. Further Applications

Various applications and extensions of the Hamiltonian (1) are interesting and experimentally feasible. In the past sections, we described two examples, based on imbalances in the initial densities of two hyperfine species in the considered mixture or in their hopping terms. In this section, we describe two possible further applications of the Rabi coupling, also feasible in present experiments.

6.1. Time Modulation

An interesting set of possible applications of the Rabi coupling applied to atomic mixtures opens when a time dependence on the time τ is considered in Equation (1). The resulting Hamiltonian can be mapped by the transformation in (2) to:

$$H_{\text{ROT}}(\tau) = -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^{\dagger} a_{j\alpha} - \sum_i [(\mu - \Omega(\tau))n_{i+} + (\mu + \Omega(\tau))n_{i-}] - U \sum_i n_{i+} n_{i-}. \quad (53)$$

This Hamiltonian can be realized in the present experiments by varying in time the intensity of the lasers inducing the Raman transitions at the basis of the Rabi coupling. The allowed profile of variation is controllable in experiments, as well as the rate of variation. It can be changed from scales much larger to those much smaller than any intrinsic timescale of the experiment [46]. In this way, different situations, including quenches or adiabatic evolutions, can be probed.

Note at the end that such a type of imbalance cannot be achieved easily without Rabi coupling acting on the numbers of atoms for each hyperfine species of the mixture, since these numbers are not easily controllable in time.

6.2. Spatial Modulations

Another interesting extension is the study of the space-dependence of the Rabi coupling $\Omega(i)$ in Equation (1). Again, in our knowledge such a type of imbalance cannot be achieved following other techniques. Experimentally feasible spatial dependences $\Omega(i)$ are (1) Gaussian: $\Omega(\mathbf{r}, \mathbf{r}_0) = \Omega e^{-\frac{|\mathbf{r}-\mathbf{r}_0|^2}{\sigma^2}}$, with $\sigma \approx 10\text{--}100 \mu\text{m}$ and then of the order of 10–100 lattice sites for a typical lattice; and (2) sinusoidal along a direction: $\Omega(\mathbf{r}) = \Omega \sin \mathbf{k}_R \cdot \mathbf{r}$, where \mathbf{k}_R can be tuned varying the angle between the lasers inducing the Rabi coupling, with its maximum magnitude being $|\mathbf{k}_R|^{(\text{max})} = \frac{2\pi}{\lambda_R}$ (λ_R is the wavelength of the lasers). In this interesting case, we obtain, via the transformation in (2), a spatially-modulated chemical potential:

$$H_{\text{ROT}}(\tau) = -t \sum_{\langle i,j \rangle, \alpha} a_{i\alpha}^{\dagger} a_{j\alpha} - \sum_i [(\mu - \Omega \cos \vec{\phi} \cdot \mathbf{r}_i)n_{i+} + (\mu + \Omega \cos \phi \cdot \mathbf{r}_i)n_{i-}] - U \sum_i n_{i+} n_{i-}. \quad (54)$$

It would particularly be interesting to study the effect of a generic modulation $\vec{\phi}$ on a superfluid phase.

As addressed at the end of Section 2, since the Rabi coupling with spatial dependence also transfers momentum $\delta \mathbf{p}$ to the atoms, in the presence of an optical lattice, its intensity is limited by the requirement that excited Wannier functions are not populated significantly [46]. The Rabi transition width involving different Wannier states reads $\Omega_{i,j} \propto \langle w_i(\mathbf{r}) | e^{i(\delta \mathbf{p}) \cdot \mathbf{r}} | w_j(\mathbf{r}) \rangle$, being $w_m(\mathbf{r})$ the m -th Wannier function at each site [1]. Considering just the first two Wannier functions, $w_0(\mathbf{r})$ and $w_1(\mathbf{r})$, the probability $P_{0,1}$ of transition between them by a two-photon Raman transition giving rise to the Rabi coupling is $P_{0,1} = \left(\frac{\Omega}{\sqrt{\Omega^2 + E_g^2}} \right)^2$, E_g being the energy difference between the two Wannier states. Imposing this quantity as much less than 1, we obtain $\Omega \ll E_g$. For a deep optical lattice potential, where the confinement on the lattice is approximately harmonic, $E_g \sim 2E_R \left(\frac{V_0}{E_R} \right)^{\frac{1}{2}}$ [1], so that

$\frac{\Omega}{E_R} \lesssim 2 \left(\frac{V_0}{E_R}\right)^{\frac{1}{2}}$. V_0 is the maximum intensity of the confining lattice potential and $E_R = \frac{\hbar^2 k^2}{2m}$ the recoil energy, $k = \frac{2\pi}{\lambda}$ being in turn the wave vector of the laser light creating the lattice and m the mass of the atoms. The lower bound for the ratio $\frac{V_0}{E_R}$, such that the tight-binding approximation holds, is $\frac{V_0}{E_R} \sim 5$. The scattering amplitude t for a n -dimensional hyper cubic lattice in the regime $\left(\frac{V_0}{E_R}\right) \gg 1$ can be estimated as [1]

$$t \approx \frac{4}{\sqrt{\pi}} E_R \left(\frac{V_0}{E_R}\right)^{\frac{3}{4}} \exp\left[-2\left(\frac{V_0}{E_R}\right)^{\frac{3}{4}}\right]. \quad (55)$$

From this formula, with $\frac{V_0}{E_R} \gtrsim 5$, we also obtain $\frac{t}{E_R} \lesssim 0.08$ and $\frac{\Omega}{t} \lesssim 23.2 \left(\frac{V_0}{E_R}\right)^{\frac{1}{2}}$. We find that basically the ratio $\frac{\Omega}{t}$ is only limited by the achievable intensity for V_0 before reaching appreciable heating regimes. As written at the end of Section 2, the reachable intensities by present laser techniques can be of the same order of the Fermi energies in typical experiments, both in continuous space and on a lattice. Thus, we conclude that our method is effective in a very wide range of intensities for Rabi coupling.

7. Conclusions

We have shown that Rabi coupling applied to an atomic mixture of different hyperfine levels gives a simple method to design and control effective population imbalances. These effective imbalances hold for dressed one-particle states obtained diagonalizing the mixing matrix related to the Rabi coupling. The method works equally for bosonic and fermionic atoms.

The presented way of controlling the chemical potential opens up the possibility of very effectively probing the physics of interacting imbalanced mixtures. Indeed, as shown explicitly for a balanced and interacting two-species fermionic mixture, the superfluid properties in the presence of Rabi coupling are the same as those for an imbalanced mixture in the absence of the Rabi term. It would be very interesting to have spin-dependent tunneling, in order to create spin-orbit-like couplings, a case we plan to study in future work.

Notably, the proposed method can be also exploited to engineer spatially and/or temporally-dependent effective population imbalances, generally not achievable in the present ultracold atoms experiments. We hope that the discussion presented in this paper will stimulate the design of new experiments in the near future.

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