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RG Flow Irreversibility, C-Theorem and Topological Nature of 4D N=2 SYM

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ABSTRACT

We determine the exact beta function and a RG flow Lyapunov function for $N = 2$ SYM with gauge group $SU(n)$. It turns out that the *classical* discriminants of the Seiberg–Witten curves determine the RG potential. The radial irreversibility of the RG flow in the $SU(2)$ case and the non–perturbative identity relating the u–modulus and the superconformal anomaly, indicate the existence of a four dimensional analogue of the c–theorem for $N = 2$ SYM which we formulate for the full $SU(n)$ theory. Our investigation provides further evidence of the essentially topological nature of the theory.

Recently it has been shown that the exact results about $N = 2$ SUSY Yang–Mills obtained by Seiberg and Witten [1] actually follow from first principles [2]. In particular, in [2] it has been shown that the entire physical content of the $SU(2)$ theory can be extracted from the identity [3]

$$u = \pi i(\mathcal{F} - a\partial_a \mathcal{F}/2). \tag{1}$$

In this context, we observe that uniformization theory is the natural framework for investigating $N = 2$ SYM [2][3]. A basic fact for the derivation in [2] is that the identity (1), first checked up to two–instanton in [4], has been proved to any order in the instanton expansion in [5] and has been obtained as an anomalous superconformal Ward identity in [6] (this also excludes other non–perturbative effects besides instantons). A first consequence of these results is that $u = \langle Tr\phi^2 \rangle$ is actually a good modular invariant global coordinate [3]. In particular, one can consider the complex coupling constant τ as a generally polymorphic function of the independent variable $u \in \mathbf{C} \cup \{\infty\}$. Furthermore, the T^2 symmetry $u(\tau + 2) = u(\tau)$, which rigorously follows from the asymptotic analysis together with the relation (1), and the fact that $\overline{u(\tau)} = u(-\bar{\tau}), u(\tau + 1) = -u(\tau)$, uniquely fix the monodromy group to be $\Gamma(2)$ [2] and therefore the explicit Seiberg–Witten results.

One of the main consequences of the Seiberg–Witten results is that for the first time it has been possible to determine the exact expression of the β–function of a non–trivial four dimensional quantum field theory [7][8]. The exact expression for the β–function in the case of $SU(3)$ was obtained in [9] (see also [10] for related aspects). Very recently the $SU(2)$ β–function has been reconsidered in a series of interesting papers [11][12][13]. We will see that the are non–trivial structures which arise in considering higher rank groups.

The exact solution for the β–function of the theory, provides the possibility of looking for the analogue of the Zamolodchikov c–theorem [14] in the context of four dimensional quantum field theories. The content of the c–theorem is the identification of an RG monotonic quantity, i.e. a Lyapunov function, giving at its fixed points a way to recognize some properties of the conformal limits of the class of 2–D theories. As well known, this quantity is strictly related to the conformal anomaly. In the case of $2 < D < 4$ theories something can be guessed from the speculations in [15] (and reference therein). Our letter tries to give some contributions to the 4–D problem (see for example [16] for related aspects). In the case of the $SU(2)$ Seiberg–Witten theory, the results in [12] can be understood from the c–theorem point of view, since (1) means that u is proportional to the (super)conformal anomaly [6].

In this paper we show that this result fits in a more general framework in which a

Lyapunov function is naturally determined and related to the *classical* discriminant of the Seiberg–Witten curve.

Let us first consider some aspect for the $SU(2)$ case. We refer to [2, 3, 17] for the aspects related to uniformization theory. Since the u quantum moduli space $\mathcal{M}_{SU(2)}$ is the thrice punctured sphere, we have

$$u/\Lambda^2 = J(\tau),$$

where J is the uniformizing map $J : H \to \mathbf{C}\backslash\{\pm 1\}$ and H is the upper half plane.

Since $J(i\infty) = \infty$, $J(\pm 1) = -1$, $J(0) = 1$, the explicit expression of the J map in terms of θ–functions is

$$J(\tau) = 2\frac{\theta_3^4}{\theta_2^4} - 1,$$

connected with the conventions of [8] by $u \to -u$, $\theta_2 \to \alpha\theta_3$, $\theta_3 \to \alpha\theta_2$, $\theta_4 \to \alpha^{-2}\theta_4$, $\alpha^4 = -1$.

The exact β–function is [7, 8]

$$\beta(\tau) = \Lambda\frac{\partial\tau}{\partial\Lambda}|_u = -2\frac{J(\tau)}{J'(\tau)}, \tag{2}$$

that in terms of θ–functions has the form

$$\beta(\tau) = -\frac{i}{\pi}\left(\frac{1}{\theta_3^4} + \frac{1}{\theta_4^4}\right). \tag{3}$$

This expression has been recently rederived in [11] and further investigated in [12][13] where it has been observed that by (2)

$$\frac{d\tau}{\beta(\tau)} = \partial\Psi_2(\tau), \tag{4}$$

where

$$\Psi_2(\tau) = -\frac{1}{2}\ln|J|^2, \tag{5}$$

with $\partial = d\tau\partial_\tau$.

The radial irreversibility of the RG flow is proved just by noticing that

$$\Lambda\partial_\Lambda|J|^2 = -4|J|^2, \tag{6}$$

which means that $|u/\Lambda^2|^2$ is a non–increasing function along the RG flow. In other words

$$L_2 = |J|^2 = e^{-2\Psi_2}, \tag{7}$$

is a Lyapunov function for the RG flow. Note that the only stable fixed point is $u = 0$ which is $\mathbf{Z}_2$ invariant. It corresponds to the zero locus $\tau_0 = \{\tau \in H | J(\tau) = 0\}$, that is $\tau_0 = \left\{\gamma \cdot (\frac{i\pm 1}{2}) | \gamma \in \Gamma(2)\right\}$, where $\Gamma(2)$ acts linearly fractionally on τ.

It is clear that in view of the c–theorem, a basic step should be to prove the existence of the potential for the β–function. While in the $SU(2)$ case the derivation of this potential reduces to a simple integration, this is a non–trivial task for higher rank groups. We will see that the β–function potential exists for $SU(n)$ for any $n \geq 2$. Actually, it turns out that the structure introduced in [9] is the natural one to explicitly solve this problem. As we will see, somewhat surprisingly, the potential is determined by the *classical* discriminant of the Seiberg–Witten curves.

In [9] it has been shown that the basic structures of the $SU(2)$ case are naturally extended to $SU(3)$ if one introduces the modular invariant quantities

$$I_\beta{}^\gamma = (\partial_k z)(\partial_\beta \tau)^{-1\, kl} \partial_l u^\gamma, \tag{8}$$

where $\beta, \gamma = 2, 3$, z is the modular invariant $z = a^k \partial_k \mathcal{F} - 2\mathcal{F}$, $\partial_k = \partial_{a^k}$, $\partial_\alpha = \partial_{u^\alpha}$ and $u^2 \equiv u$, $u^3 \equiv v$. These expressions, which have been given in [9] for $SU(3)$, trivially extend to $SU(n)$ for any $n \geq 2$. For the modular invariant z we have $z = \frac{3i}{\pi} u$, that is [9] $u = \frac{2\pi i}{3} \left(\mathcal{F} - \frac{a^k}{2} \partial_k \mathcal{F} \right)$, which is the generalization of (1) and has been derived by other means and also for higher rank groups in [18].

The above framework is the natural one to properly investigating the extension to $N = 2$ SYM with higher rank gauge groups [19][20]. For example, the modular invariant quantities $I_\beta{}^\gamma$ allow us to find the analogue of the identity (1) in the case of v.

Let us consider the beta function (matrix)

$$\beta_{ij} = \Lambda \frac{\partial \tau_{ij}}{\partial \Lambda}|_{u^2, u^3, \ldots}. \tag{9}$$

Since under modular transformations

$$(a^D, a) \to (a^{D'}, a') = (A a^D + B a, C a^D + D a),$$

$\begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp(2n-2, \mathbf{Z})$, we have

$$\beta \to (\tau C^t + D^t)^{-1} \beta (\tau C + D)^{-1},$$

and

$$d\tau \to (\tau C^t + D^t)^{-1} d\tau (\tau C + D)^{-1}.$$

It follows that

$$b = \beta^{ij} d\tau_{ij} = \beta^{\alpha\delta} J_{\alpha\gamma\delta} du^\gamma, \tag{10}$$

is a modular invariant one–form. Here β^{ij} and $\beta^{\alpha\gamma}$ denote the inverse of the matrices β_{ij} and $\beta_{\alpha\gamma}$ respectively, and

$$J_{\alpha\beta\gamma}=\partial_\alpha a^i\partial_\beta\tau_{ij}\partial_\gamma a^j, \tag{11}$$

are the modular invariant quantities we introduced in [9]. For $SU(3)$ these are related to the I_β^{γ}'s by

$$I_\beta^{\gamma}J_{\gamma\beta 2}=\frac{\pi}{3i}, \qquad I_\beta^{\gamma}J_{\gamma\beta 3}=0, \tag{12}$$

whose solution is

$$J_{222}=-\frac{1}{3}AP, \qquad J_{223}=12uvA, \qquad J_{233}=-\frac{1}{u}AP, \qquad J_{333}=36vA, \tag{13}$$

where $A=\frac{3i}{\pi}[(12uv)^2-P^2/3u]^{-1}$ and $P=27(v^2-\Lambda^6)+4u^3$. Since the explicit expression of $\beta_{\alpha\gamma}$ is [9]

$$\beta_{22}=\frac{2Au}{3}[P-54v^2], \qquad \beta_{23}=\beta_{32}=\frac{3Av}{u}[P-8u^3], \qquad \beta_{33}=2A[P-54v^2], \tag{14}$$

it follows by (10) and (13) that ($\partial=du\partial_u+dv\partial_v$)

$$b=\partial\Psi_3, \tag{15}$$

where

$$\Psi_3=-\frac{1}{3}\ln\left|\frac{27v^2-4u^3}{\Lambda^6}\right|^2=-\frac{1}{3}\ln\left|27\mathtt{v}^2(\tau)-4\mathtt{u}^3(\tau)\right|^2, \tag{16}$$

with $\mathtt{u}(\tau)=u/\Lambda^2$, $\mathtt{v}(\tau)=v/\Lambda^3$. Eq.(15) shows that the RG flow is gradient. Furthermore, as

$$\Lambda\partial_\Lambda e^{-3\Psi_3}=-12e^{-3\Psi_3}, \tag{17}$$

it follows that

$$L_3=e^{-3\Psi_3}, \tag{18}$$

is a Lyapunov function for the RG flow.

In [9] it has been observed that also in the $SU(3)$ case there is the uniformization mechanism which generalizes the structure underlying the $SU(2)$ case [3]. In particular, the structure of the covering of the quantum moduli space $\mathcal{M}_{SU(3)}$ is encoded in the properties of the Appell's functions. The fact that τ_{ij} is dimensionless implies that

$$(\Lambda\partial_\Lambda+\Delta_{u,v})\tau_{ij}=0, \tag{19}$$

where $\Delta_{u,v}=2u\partial_u+3v\partial_v$ is the scaling invariant vector field. Eq.(19) implies that $\tau=\tau(\mathtt{u},\mathtt{v})$. Therefore, the τ–space is a subvariety $\mathcal{S}$ of the genus 2 Siegel upper–half space of complex codimension one covering $\mathcal{M}_{SU(3)}$. In particular, the Picard–Fuchs equations (not to be confused with the reduced ones) are the uniformizing equations for $\mathcal{M}_{SU(3)} \cong \mathcal{S}/M_{SU(3)}$ where $M_{SU(3)} \subset Sp(4, \mathbf{Z})$ is the monodromy group of the polymorphic matrix function τ seen as the inverse of the uniformizing map $\tau_{jk} = \tau_{jk}(\mathtt{u}, \mathtt{v})$.

For the higher rank case it is immediate to read out the general structure we are looking for. In fact our RG potentials Ψ_2 and Ψ_3 are simply related to the *classical* discriminants of the Seiberg–Witten curves [19]. These are defined as

$$\Delta_{cl.}^{SU(n)}(u^\gamma) = \prod_{i<j}^{n} (e_i - e_j)^2,$$

where $\{e_i\}$ are the zeros in x of the polynomial $\mathcal{W}_{A_{n-1}}(x; u^2, .., u^n) = x^n - \sum_{\gamma=2}^{n} u^\gamma x^{n-\gamma}$. Explicitly, for $n = 2, 3$

$$\Delta_{cl.}^{SU(2)}(u) = u, \qquad \Delta_{cl.}^{SU(3)}(u, v) = 4u^3 - 27v^2.$$

Therefore, there is strong evidence that for the $SU(n)$ case ($\partial = \sum_{\gamma=2}^{n} du^\gamma \partial_\gamma$)

$$b = \partial \Psi_n, \tag{20}$$

where

$$\Psi_n = -\frac{1}{n} \ln \left| \frac{\Delta_{cl.}^{SU(n)}(u^\gamma)}{\Lambda^{n(n-1)}} \right|^2 = -\frac{1}{n} \ln \left| \hat{\Delta}_{cl.}^{SU(n)}(\tau) \right|^2, \tag{21}$$

with b defined as in (10) and

$$\hat{\Delta}_{cl.}^{SU(n)}(\tau) = \Delta_{cl.}^{SU(n)}(\mathtt{u}^\gamma),$$

where

$$\mathtt{u}^\gamma = u^\gamma / \Lambda^\gamma,$$

$\gamma = 2, \ldots, n$. Furthermore, we have the equation

$$\Lambda \partial_\Lambda L_n = -2n(n-1) L_n, \tag{22}$$

where

$$L_n = e^{-n\Psi_n} = |\hat{\Delta}_{cl.}^{SU(n)}(\tau)|^2. \tag{23}$$

Observe that by (10) it follows that Eq.(20) is equivalent to

$$\partial_\gamma \Delta_{cl.}^{SU(n)}(u^\sigma) = \beta^{\alpha\delta} J_{\alpha\gamma\delta} \Delta_{cl.}^{SU(n)}(u^\sigma), \tag{24}$$

so that the integrability condition

$$\partial_\gamma\partial_\sigma\Delta_{cl.}^{SU(n)}(u^\sigma)=\partial_\sigma\partial_\gamma\Delta_{cl.}^{SU(n)}(u^\sigma), \tag{25}$$

yields

$$\partial_\sigma(\beta^{\alpha\delta}J_{\alpha\gamma\delta})=\partial_\gamma(\beta^{\alpha\delta}J_{\alpha\sigma\delta}), \tag{26}$$

$\gamma,\sigma=1,\ldots,n-1$.

It is interesting to note that the previous analysis implies the following scaling law for the classical discriminant

$$\hat\Delta_{cl.}^{SU(n)}(\tau)=\hat\Delta_{cl.}^{SU(n)}(\tau_0)e^{-n\int_{\tau_0}^{\tau}b}, \tag{27}$$

which is the higher rank version of the scaling law for the u–modulus derived in [8]. In this context we observe that due to the presence of other moduli besides u, while in the $SU(2)$ case the scaling law (27) implied, in view of (1), the RG equation for $\mathcal{F}$, this is no the case for $SU(n)$, $n\geq 3$.

As L_n reaches its minimum, we have $\hat\Delta_{cl.}^{SU(n)}(\tau)=0$, meaning that the system naturally tends to flow through the classical locus of gauge symmetry restoring: this restoring in fact does not really happen since the quantum moduli space is dramatically different from the semiclassical one. In any case, all this means that the classical symmetry restoring locus continues playing a non trivial attracting röle in the full theory.

Notice that the above result may cause some doubts: it is stated that the exact quantum RG flow of a given theory follows some *classically* determined character. However observe that the classical character concerns the dependence of the potential on the quantum moduli rather than the moduli themselves. It seems that a full explanation of the above phenomenon should be found in a deeper understanding of non–renormalization theorems intertwined with the essentially topological nature of the theory [21][9][22][23][24].

The fact that the theory has an essentially topological structure has been suggested [9] where the WDVV equations [25] for the SU(3) case have been derived in the framework of the Picard–Fuchs equations. Using a different method, the WDVV equations for higher rank groups have been obtained in [26].[1]

To show a possible connection with the WDVV equations, we note that by (10) it follows that

$$b=\beta^{ij}d\tau_{ij}=\beta^{ij}\mathcal{F}_{ijk}da^k, \tag{28}$$

[1]The problem of deriving the WDVV equations for higher rank groups from the Picard–Fuchs equations has been recently solved in the interesting paper by Ito and Yang [27].

where

$$\mathcal{F}_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}.$$

Using $\sum_\alpha du^\alpha \partial_\alpha = \sum_k da^k \partial_k$, one has that the integrability condition (25) implies

$$(\mathcal{F}_{ijk}\partial_l - \mathcal{F}_{ijl}\partial_k)\beta^{ij} = 0, \tag{29}$$

$k, l = 1, \ldots, n-1$. The point is that in [28] the WDVV equations have been obtained as consistency condition for a system of differential equations whose structure is reminiscent of Eq.(29). The appearance of the β–function suggests that Eq.(29) corresponds to the version of WDVV equations derived in [24] (see also [28, 29] for related aspects). The fact that the WDVV equations can be extended by considering the RG scale Λ as modulus [30], provides further evidence for the topological nature of $N = 2$ SYM (see also [27] for related results).

A crucial point about our Lyapunov functions is whether they encode in some way any physical information about the structure of the massless sector of the theory at the critical points. We refer the reader to [13] and references therein for a more general discussion about this aspect which is general enough to extend also to the higher rank case.

Finally, we observe that the approach in [17][31][10] should be useful to extend our results to the case with matter. Another interesting aspect is that, as observed in [13], the above structures are related to the quantum Hall system [32] and non–linear sigma models [33].

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