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On numerically flat Higgs bundles

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Introduction

This is a report of research done in the past years, motivated by the following long standing conjecture by Ugo Bruzzo.

The Conjecture Let X be a projective variety with polarization H. Let $\mathcal{E} = (E, \phi)$ be a Higgs bundle. The following facts are equivalent:

1. $\mathcal{E}$ is Higgs-semistable and $\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = 0$,
2. $\forall f : C \to X$, C smooth projective irreducible curve, $f^*\mathcal{E}$ is Higgs-semistable.

The statement above is known in the absence of the Higgs field. In the Higgs case, it is known that that the first condition implies the second. It is also known (*Bogomolov inequality*) that $\Delta \geq 0$ for semistable Higgs bundles, but the converse inequality is yet unproved; it was erroneously assumed in [BH06, lemma 4.2] to give a proof of the above equivalence.

My attempts at a proof of the conjecture have folllowed three differerent lines:

- a reduction of the Higgs case to the classical one;
- a vanishing argument for the Atiyah class;
- some broad algebraic considerations about jump deformations.

In the following there is an account of the work done along these three lines.

Infinitesimal features To solve the conjecture, it can be assumed $c_1 = 0$. Moreover, it is known that $\Delta = c_2 = 0$ implies vanishing of all Chern classes.

The relevance of Higgs bundles with vanishing Chern classes is that they are aliases of representations of the fundamental group, in the form of bundles with flat connection. The Higgs field accounts for the non-unitarity of the representation. The correspondence rests on the existence of a canonical metric, called *harmonic*.

The correspondence is concretely realised through one-parameter families of differential operators. The Higgs structure is the fibre over a point, a rescaled flat connection on the others.

As for the Petersson-Weil metric in Teichmüller theory, the canonical metric on the bundles induce a metric on their moduli space. Another interesting aspect of the moduli space of harmonic bundles is the quaternionic structure that arises from the two interpretations, as Higgs bundles and as flat bundles. Quaternionic structure and metric are compatible, giving a hyperKähler manifold. The one-parameter families above arise as *preferred sections* in the twistor family of the hyperKähler moduli space.

Call *H-nflat* a Higgs bundle with vanishing first Chern class and satisfying condition 2 in the Conjecture. My first hope for solving the conjecture was to be able to identify a preferred deformation of a H-nflat bundle to a flat bundle. This idea prompts the interest in generically trivial deformations (*jump deformations*) and deformations with quaternionic structure. These topics are discussed in chapter 3.

After a longer introduction in chapter 1, chapter 2 addresses the following topics.

Spectral covers A very natural approach would be to reduce the statement about Higgs bundles to the known statement about classical bundles. The first step is well-known and goes through the construction of the *spectral cover*: it is a finite cover of X whose points can be considered as the eigenvalues of the Higgs field. Unfortunately, the spectral cover is often singular, so the known results do not apply. Anyway, there is at least a very simple case in which the conjecture holds.

Fundamental groups The category of nflat Higgs bundles is a neutral Tannakian category, namely, it is a monoidal abelian category equivalent to the category of linear representations of a pro-algebraic group. The group can be reconstructed from the category and a fibre functor, an analogue of the forgetful functor to vector spaces; for bundles, it is given by taking fibres over a fixed point. Numerical flatness is essential to get rigidity (=dualizability).

Langer has studied the classical case (no Higgs field) and called the group the *S-fundamental group*. A verification of the axioms in the Higgs case has been performed in [BBG16]. The proofs in *loc.cit.* work equally well for more generally decorated bundles $E \to E \otimes \mathcal{W}$ and the straightforward definition of nflatness. This means that, for every vector bundles $\mathcal{W}$, we have an exact sequence of affine groups

$$1 \to \pi(\mathcal{W}) \to \pi^{\mathcal{W}}(X, x) \to \pi(X, x) \to 1 \qquad (1)$$

It seems interesting to find an explicit construction of the category of representations of $\pi(\mathcal{W})$, a problem that doesn't seem to have a general solution. What I consider is a way to prove results about the restriction of the bundles to an ample divisor.

Families of harmonic bundles Consider a Higgs bundle (E, ϕ) with vanishing first Chern class satisfying the requirement that its restriction to curves is polystable, not just semistable. Then, taking a family of smooth embedded curves,

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & & \\ \mathbb{P} & & \end{array}$$

the pullback of (E, ϕ) to $\mathcal{C}$ holds a family of harmonic bundles. One way to prove the conjecture would be to use $\mathcal{C} \to X$ to induce a flat connection on E. A side question is whether the relative $\mathcal{C}/\mathbb{P}$ flat connection on $E\big|_{\mathcal{C}}$ changes outside its equivalence class as one moves from one fibre to another.

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Contents

Chapter 1

Higgs bundles

Higgs bundles have been introduced by Hitchin and are largely studied for their connection to representations of the fundamental group of Kähler manifolds.

Narasimhan and Seshadri [NS65] have proved that stable bundles on a Riemann surface are the holomorphic counterpart to unitary representations of the fundamental group, i.e., vector bundles with a flat unitary connection. Hitchin [Hit87] and Donaldson [Don87b] have proved that, through the addition of a Higgs field, the correspondence can be extended to non-unitary representations.

The correspondence is given by a special metric on the bundle: a *harmonic metric*, itself a special case of a *Hermitian-Yang-Mills metric*. On manifolds of arbitrary dimension, the existence of a harmonic metric on flat bundles was proved by Corlette [Cor88]. Hermitian-Yang-Mills metrics on polystable holomorphic bundles were constructed by Uhlenbeck-Yau [UY86], Donaldson [Don87a], and, in the Higgs case, by Simpson [Sim88]. A Hermitian-Yang-Mills metric is harmonic precisely when the Chern classes of the bundle vanish.

The correspondence can be formulated on a categorical level [Sim92]: polystable Higgs bundles form a dg category equivalent to the category of semisimple flat vector bundles. By categorical constructions, this equivalence still holds for all extensions, semistable bundles on one side and flat bundles on the other. Numerical flatness (as in Bruzzo conjecture) in the absence of a Higgs field is known to be equivalent to semistability and vanishing of the Chern classes, so it seems natural to ask if the same is true for Higgs bundles.

With or without the Higgs field, numerically flat (*nflat*) bundles form a Tannakian category, as first considered by Langer [Lan11]. If the vanishing of the Chern classes still holds for the for a nflat Higgs bundle, their category is equivalent to the category of finite dimensional representations of the fundamental group.

This chapter is meant to set the stage, reviewing what is already known and building intuition for the Main Conjecture through different reformulations. Let it end with a definition, that is valid in differential geometry and algebraic geometry too. Let Ω^1 be the sheaf of 1-forms.

Definition 1. A Higgs bundle (E, ϕ) is a vector bundle E with a morphism $\phi : E \to E \otimes \Omega^1$ such that $\phi \wedge \phi : E \to E \otimes \Omega^2_X$ is 0.

$$\phi \wedge \phi : E \xrightarrow{\phi} E \otimes \Omega^1 \xrightarrow{\phi \otimes 1} E \otimes \Omega^1 \otimes \Omega^1 \to E \otimes \Lambda^2 \Omega^1 \tag{1.1}$$

In the following, X is a variety of dimension n and E is a vector bundle of rank r.

1.1 Higgs bundles in differential geometry

On a complex manifold, holomorphic objects are encoded via differential operators.

Let X be a complex manifold. The sheaf of smooth complex differential n-forms $\mathcal{A}^n_X$ has a decomposition $\mathcal{A}^n_X = \bigoplus_{p+q=n} \mathcal{A}^{p,q}_X$ in forms of type (p, q); $\mathcal{A}^{p,q}(E)$ are those forms with values in a vector bundle E.

A holomorphic structure on a bundle E is given by a $\mathbb{C}$-linear morphism $\bar\partial_E : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,1}(E)$ with $\bar\partial_E^2 = 0$ (integrability) and $\bar\partial_E(f\sigma) = f\bar\partial_E\sigma + \sigma\bar\partial f$ ($\bar\partial$-Leibniz rule).

The Higgs field ϕ being holomorphic means $\bar\partial(\phi) := \bar\partial \circ \phi + \phi \circ \bar\partial = 0$. The two operators can be combined in a single $D'' = \bar\partial_E + \phi : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ with $(D'')^2 = 0$.

A Higgs bundle (E, ϕ) fits into a complex

$$E \xrightarrow{\cdot \wedge \phi} E \otimes \Omega^1_X \xrightarrow{\cdot \wedge \phi} E \otimes \Omega^2_X \to \dots \tag{1.2}$$

that is a module over $(\Omega^\cdot_X, 0)$, the dg algebra of holomorphic forms with zero differential; its hypercohomology is called the *Dolbeaut cohomology* of (E, ϕ). The complex $(\mathcal{A}^\cdot_X(E), D'')$ provides a resolution of 1.2. In the classical case of the constant bundle $E = X \times \mathbb{C}$, $\phi = 0$, the complex $\Omega^\cdot_X(E)$ is $\bigoplus \Omega^i_X[-i]$, and one has $H^i_{Dol}(X, \mathbb{C}) = \bigoplus_{p+q=i} H^p(X, \Omega^q_X)$.

Algebroid perspective A Lie algebroid is a vector bundle L with an *anchor* $\cdot^\# : L \to TX$ that is $\mathcal{O}_X$-linear and a Lie bracket on $H^0(X, L)$ linear over the field of constants, satisfying the Leibniz rule

$$[\lambda, f\mu] = f[\lambda, \mu] + \lambda^\#(f)\mu \tag{1.3}$$

There are two prototypes of Lie algebroid. The first is an integrable distribution $L \subset TX$. The second is the Atiyah algebroid $\operatorname{At} P = TP/G$ of a G-principal bundle $P \to X$. If fits in an exact sequence

$$0 \to \operatorname{ad} P \to \operatorname{At} P \to TX \to 0 \tag{1.4}$$

The map $\mathrm{At}\, P \to TX$ is the anchor; algebroids with surjective anchor are called *transitive*. The bundle $\mathrm{ad}\, P = (P \times \mathfrak{g})/G$ is the vertical bundle of the fibration: it has trivial anchor, in fact it is a bundle of Lie algebras with $\mathcal{O}_X$-linear bracket.

A Higgs bundle can also be seen as a representation of the abelian, totally intransitive, holomorphic Lie algebroid $TX^{1,0}$. Analogously to the definition via differential operators, we can see (E, ϕ) as a representation of the associated canonical complex Lie algebroid [LSX08, §4.3] $TX^{1,0} \bowtie TX^{0,1}$

$$\forall X, Y \in TX \otimes \mathbb{C} \qquad [X, Y]_{TX^{1,0} \bowtie TX^{0,1}} := \mathrm{pr}^{0,1}[X, Y]_{TX \otimes \mathbb{C}} \tag{1.5}$$

(the complex vector bundle underlying $TX^{1,0} \bowtie TX^{0,1}$ is just $TX^{1,0} \oplus TX^{0,1}$).

The algebroid $TX^{1,0} \bowtie TX^{0,1}$ is a *contraction* of $TX \otimes \mathbb{C}$: there is a 1-parameter group of automorphisms of the vector bundle $TX \otimes \mathbb{C}$,

$$\lambda \cdot X = \lambda\, \mathrm{pr}^{1,0} X + \mathrm{pr}^{0,1} X \tag{1.6}$$

and we use it to pullback the anchor and the bracket,

$$\begin{aligned} [X, Y]_\lambda &= \lambda\, \mathrm{pr}^{1,0}[X, Y]_{TX \otimes \mathbb{C}} + \mathrm{pr}^{0,1}[X, Y]_{TX \otimes \mathbb{C}} \\ X^\# &= \lambda X^{1,0} + X^{0,1} \end{aligned} \tag{1.7}$$

then we take the limit for $\lambda \to 0$.

The Hitchin-Simpson connection A holomorphic vector bundle E with Hermitian metric K has a *Chern connection* [Kob87, proposition 1.4.9]: it is the unique connection D which preserves the metric

$$d(e, e') = (De, e') + (e, De') \tag{1.8}$$

and whose $(0, 1)$-part is the operator $\bar{\partial}_E$. Its curvature $F = D^2$ is of type $(1, 1)$.

If (E, ϕ) is a Higgs bundle with Hermitian metric K, its *Hitchin-Simpson connection* is the sum of the Chern connection, the Higgs field, and its adjoint,

$$D_K = D + \phi + \bar{\phi} \tag{1.9}$$

Its curvature $F_K = F + [\phi, \bar{\phi}]$ is still of type $(1, 1)$. The metric K is called *harmonic* if $F_K = 0$.

If (E, D, D'') is a harmonic bundle, for all $\lambda \in \mathbb{C}$, $\bar{\partial}_E + \lambda\bar{\phi} = (\lambda D' + D'')_{(0,1)}$ is a holomorphic structure E_λ, ${}^\lambda\nabla = (\lambda D' + D'')_{(1,0)}$ commutes with $\bar{\partial}_{E_\lambda}$, satisfies a modified Leibniz rule

$${}^\lambda\nabla(fe) = f \cdot {}^\lambda\nabla e + \lambda e \otimes \partial f + e \otimes \bar{\partial} f \tag{1.10}$$

and squares to 0. ${}^1\nabla$ is a flat connection, holomorphic for $\bar{\partial}_E + \bar{\phi}$; for every $\lambda \in \mathbb{C}$, $(E_\lambda, {}^\lambda\nabla)$ is a representation of the Lie algebroid structure on $TX^{1,0} \oplus TX^{0,1}$ given by 1.7. A family $(E_\lambda, {}^\lambda\nabla)$, where $\{E_\lambda\}$ is a holomorphic bundle on $X \times \mathbb{A}^1_{\mathbb{C}}$, and ${}^\lambda\nabla$ satisfies 1.10 and is integrable $(({}^\lambda\nabla)^2 = 0)$ is also called a λ-connection.

Chern classes Let $(E, \bar\partial_E)$ be a holomorphic bundle, and $D = D' + \bar\partial_E$ a connection. The part $F^{1,1} \in \mathcal{A}^{0,1}(\operatorname{End} E \otimes \Omega^1_X)$ of its curvature defines a cohomology class $\operatorname{at}(E) \in H^1(X, \operatorname{End} E \otimes \Omega^1)$ called the *Atiyah class*. It is an obstruction to the splitting of the sequence 1.4.

It is possible to have F itself of type $(1,1)$, for example as the curvature of a Chern connection. The total Chern class $c(E) = c_1(E) + c_2(E) + \dots$ can be represented as

$$c(E) = \det\left(I - \frac{1}{2\pi i}F\right) \tag{1.11}$$

The intersection $c_1(E)\omega^{n-1}$ is the *degree* of E:

$$\deg E = \int_X c_1(E)\omega^{n-1} = \int_X \frac{-\operatorname{tr} F}{2\pi i}\omega^{n-1} = -\frac{1}{2\pi i n}\int_X (\Lambda \operatorname{tr} F)\omega^n \tag{1.12}$$

where we have used the contraction defined by the Kähler form

$$(\Lambda F)^i_j = \frac{1}{\sqrt{-1}}\omega^{\alpha\bar\beta}F^i_{j\alpha\bar\beta} \qquad \alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = (\Lambda\alpha)\frac{\omega^n}{n!} \tag{1.13}$$

Yang-Mills metrics Chern classes provide topological obstructions to the existence of a harmonic metric: by Chern-Weil theory 1.11, if a bundle has a flat connection, its Chern classes vanish in real cohomology.

On Kähler manifolds, given a Higgs bundle (E, ϕ), we can look for a metric K such that

$$\Lambda F_K = -\frac{2\pi i}{(n-1)!}\frac{1}{\int_X \omega^n}\frac{\deg E}{\operatorname{rank} E}\cdot \operatorname{Id}_E \tag{1.14}$$

The constant on the right is the unique possible constant in the equation $\Lambda F_K = \mathit{const} \cdot \operatorname{Id}$, as can be seen taking traces and integrating. These metrics are called *Hermitian-Einstein* in [Kob87] and *Hermitian-Yang-Mills* in [Sim92].

A connection with scalar curvature is called *projectively flat*: it means $F = \alpha \operatorname{Id}$ for some 2-form α. The Chern classes will be

$$c_k = \binom{r}{k}\left(\frac{-\alpha}{2\pi i}\right)^k = \binom{r}{k}\frac{c_1^k}{r^k} \tag{1.15}$$

If E is projectively flat, the bundle $\operatorname{End} E = E \otimes E^\vee$ is a flat bundle. The characteristic class

$$\Delta(E) = c_2(\operatorname{End} E) = 2rc_2(E) - (r-1)c_1(E)^2 \tag{1.16}$$

is an obstruction for E to be projectively flat.

A bundle E with HYM metric satisfies the *Bogomolov inequality* $\Delta(E)\omega^{n-2} \geq 0$, and the equality holds if and only if the Hitchin-Simpson connection is projectively flat. Expressing a tensor T in an orthonormal basis, $\|T\| = \sum |T^{i\dots\alpha\dots}_{j\dots\beta\dots}|^2$. Then [Kob87, §4.4]:

$$\begin{aligned} c_2(E) \wedge \omega^{n-2} &= \frac{\|F\|^2 - \|\Lambda F\|^2 - \|\operatorname{tr} F\|^2 + (\operatorname{tr} \Lambda F)^2}{8\pi^2 n(n-1)} \omega^n \\ c_1(E)^2 \wedge \omega^{n-2} &= \frac{(\operatorname{tr} \Lambda F)^2 - \|\operatorname{tr} F\|^2}{4\pi^2 n(n-1)} \omega^n \end{aligned} \tag{1.17}$$

If the connection is HYM, $\|\Lambda F\|^2 = (\operatorname{tr} \Lambda F)^2/r$. Bogomolov inequality follows from a Cauchy-Schwarz inequality $\|F\|^2 - (\operatorname{tr} \Lambda F)^2/r \geq 0$. Moreover, one gets

$$\frac{r\|F\|^2 \omega^n}{4\pi^2 n(n-1)} = \Delta \wedge \omega^{n-2} + \frac{(\operatorname{tr} \Lambda F)^2 \omega^n}{4\pi^2 n(n-1)} \tag{1.18}$$

and so, if the first two Chern classes vanish, so do F and all other Chern classes.

Semistability For a coherent sheaf E of positive rank on the compact Kähler manifold (X, ω), its *slope* is defined as

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E} = \frac{1}{\operatorname{rank} E} \int_X c_1(E) \omega^{n-1} \tag{1.19}$$

A Higgs bundle (E, ϕ) is *semistable* if, for all ϕ-invariant subsheaves $E' \subset E$ of posite rank, $\mu(E') \geq \mu(E)$; equivalently, if the opposite inequality $\mu(E) \leq \mu(Q)$ holds for all Higgs quotients (i.e., quotients with invariant kernel). If the inequality is always strict, (E, ϕ) is *stable*; if (E, ϕ) is a direct sum of stable bundles of the same slope, (E, ϕ) is *polystable*.

Theorem. *[Sim88] Let X be a compact Kähler manifold. A Higgs bundle (E, ϕ) admits a Hermitian-Yang-Mills metric if and only if it is polystable. If $\operatorname{ch}_1(E) \cdot [\omega]^{n-1} = \operatorname{ch}_2(E) \cdot [\omega]^{n-2} = 0$, D is a flat connection.*

The harmonic metric on a stable Higgs bundle is unique up to a constant, and the Hitchin-Simpson connection is unique.

A bundle with harmonic metric gives a representations of the fundamental group. It is polystable with respect to every polarization, and every restriction to a subvariety is again polystable.

1.2 Numerical classes

On a smooth projective complex curve C, semistability of a bundle E (no Higgs field) is rather geometric. Consider a variant of of the Euler exact sequence for the Grassmannian $\pi : \operatorname{Grass}_s(E) \to C$ of rank s quotients:

$$0 \to \Omega_\pi \to \pi^* E \otimes Q^\vee \to Q \otimes Q^\vee \to 0 \tag{1.20}$$

Then, the relative tautological divisor is $K_\pi = s\pi^* c_1(E) - rc_1(Q)$. Any map $f : C' \to \mathrm{Grass}_s(E)$ corresponds to a quotient $E_{C'} \twoheadrightarrow Q_f$;

$$\mu(Q_f) - \mu(E_{C'}) = [C'] \cdot \left(\frac{f^* c_1(Q)}{s} - \frac{f^* \pi^* c_1(E)}{r} \right) = \frac{1}{rs} f^*(-K_\pi) \cdot [C'] \quad (1.21)$$

Say that a divisor on a projective scheme is *nef* if it has nonnegative intersection with every curve.
For complete irreducible curves C', the composition $C' \to \mathrm{Grass}_s(E) \to C$ is either a finite cover or constant. If E is semistable, any pullback through $C' \to C$ is [Miy87, prop. 3.2], so $-K_\pi$ is nef, and if $-K_\pi$ is nef, then E is semistable. So, the bundle E is semistable if and only if all anti-tautological divisors $\theta_s = -K_\pi$ for $0 < s < r$ are nef. Actually, it is enough to consider any one of them.

Theorem. *[Miy87]) E is semistable if and only if θ_1 is nef.*

For Higgs bundles (E, ϕ) it is necessary to ask nefness of all classes θ_s over the Grassmannian of Higgs quotients (shortly to be defined).

Theorem. *[BH06] On a smooth projective curve, a Higgs bundle (E, ϕ) is semistable if and only if, for all s, $0 < s < r$, the restriction of θ_s to the Higgs Grassmannian $\mathfrak{G}\mathrm{rass}_s$ is nef.*

While cohomology groups are often easy to compute, and we know how to express the cohomology of a projective bundle $\mathbb{P}E$, the cone of effective cycles is a subtler invariant. For the case of projective bundles on a curve, see [Ful11].

Directly from the definition, a bundle E on a variety X is nef if and only if $E\big|_C$ is nef for every curve $C \subset X$. From the above theorem, we see that asking the collection of classes θ_s to be nef implies that the Higgs bundle (E, ϕ) is semistable on every curve. By contrast, the celebrated theorem of Mehta-Ramanathan assures semistability only on a generic curve complete intersection of ample divisors.

For dimension reasons, on a curve all Chern classes higher than c_1 vanish. If it is polystable, it admits a HYM metric, and its Hitchin-Simpson connection is projectively flat. We can formulate one of many equivalent versions of the main conjecture.

Conjecture. *On a smooth projective variety X, if the restriction of a bundle E is projectively flat on every curve, E is projectively flat.*

In the case E admits a projectively flat Hermitian structure, this has been proved by Nakayama [Nak99] and Demailly, Peternell and Schneider [DPS94, thm 1.18]. Passing from E to $\mathrm{End}\, E$, it is enough to consider flat bundles. In conjunction with polystability and the existence of a HYM metric, it is equivalent to asking the vanishing of Chern classes, or just $c_2 \cdot \omega^{n-2} = 0$.

A formulation of the conjecture closer to the original one is as follows.

Definition 2. Let X be a projective scheme, (E, ϕ) a Higgs bundle. (E, ϕ) is H-nflat if, for every curve $f : C \to X$ and every locally free Higgs quotient $f^*E \twoheadrightarrow Q$, $\deg Q = 0$.

The original definition of H-nflat is a formulated on the Higgs Grassmannians, it is iterative, and asks for E and $E^\vee$ to be H-nef [BBG16], as follows.

Let $p : \mathrm{Grass}_s \to X$ be the Grassmannian of quotients of E as a vector bundle. It supports a universal sequence

$$0 \to S_s \to p^*E \to Q_s \to 0 \tag{1.22}$$

Let $\mathcal{G}\mathrm{rass}_s$ the locus of quotients with ϕ-invariant kernel, where the composite morphism vanishes.

$$S_r \to p^*E \to p^*E \otimes p^*\Omega_X \to Q_r \otimes p^*\Omega_X \tag{1.23}$$

Then (E, ϕ) is H-nef if all $Q_s\big|_{\mathcal{G}\mathrm{rass}_s}$ are nef. (E, ϕ) is H-nflat if (E, ϕ) and $(E, \phi)^\vee = (E^\vee, -\phi^\vee)$ are H-nef.

Conjecture. *For any smooth projective complex manifold, if (E, ϕ) is H-nflat, than $c_2(E) = 0$.*

Note that it is enough to prove the statement for projective surfaces, since what is needed is the numerical result $c_2(E)\omega^{n-2} = 0$, and for this we can restrict E to a surface dual to ω^{n-2}.

Nef metrics The notion of nefness arises as a limiting notion of ampleness, after the criterion of Nakai-Mosheizon: a divisor is ample if and only if it has positive intersection with any subvariety, or, equivalently, uniformly positive intersection with any curve [Har70]. On non-projective Kähler manifold there is a shorter supply of subvarieties, so, many notions expressed in terms of intersections should rather be rephrased in terms of metrics.

Let us review some definitions as the occur in [DPS94].

Definition 3. [DPS94, def 1.2] Let (X, ω) be a compact Hermitian manifold. A line bundle L is *nef* if for every $\epsilon > 0$ there exists metric h_ϵ such that $F_{h_\epsilon} \geq -\epsilon\omega$.

Definition 4. [DPS94, def 1.9, thm 1.12] A vector bundle E is nef over X if the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ is nef over $\mathbb{P}E$.
Equivalently, E is nef if it is possible to fix metrics h_m on every S^mE such that, for all $\epsilon > 0$ and $m \gg 0$,

$$F_{(S^mE, h_m)} \geq -m\epsilon\omega \otimes \mathrm{id}_{S^mE} \tag{1.24}$$

A simpler and stronger definition was posed in [Cat98].

Definition 5. [Cat98, def 3.1.3] The bundle E is 1-nef if, for every $\epsilon > 0$, there exists h_ϵ with $F_{(E,h_\epsilon)} \geq -\epsilon\omega \otimes \mathrm{id}_E$.

E 1-nef implies E nef [Cat98, prop 3.2.4], and they are equivalent on smooth complete curves [*loc.cit*, thm 3.3.1]. It is unknown whether they are equivalent in general.

The same definitions can be stated for Higgs bundles using the curvature of the Hitchin-Simpson connection [BG07]. E (1-)(Higgs-)nflat is defined as E and $E^\vee$ both (1-)(Higgs-)nef.

Assuming the stronger notion of 1-H-nflat, it is possible to prove that Chern classes vanish [*loc.cit*, lemma 3.15]. Since a harmonic metric gives immediately the required inequality, the other conjectures are equivalent to

Conjecture. *On a projective manifold, a stable H-nflat bundle is 1-H-nflat.*

1.3 Further thoughts

The Kobayashi-Hitchin correspondence about the existence of HYM metric on polystable stables has been extended much beyond Higgs bundles.

The functional $F \mapsto \Lambda F - c\,\mathrm{Id}$ can be seen as a moment map; on the other hand, there is the holomorphic notion of stability. So, these results fit into the Kempf-Ness framework.

After a case by case extension to web of bundles and morphisms between them, a comprehensive generalization appears in [Mun00; LT06].

Let X, F be Kähler manifolds, with X compact. Consider an exact sequence of complex reductive groups

$$1 \to G \to \hat{G} \to G_0 \to 1 \tag{1.25}$$

and a holomorphic action $\hat{\alpha} : \hat{G} \times F \to F$. The goal is to classify pairs $(\hat{\mathcal{Q}}, \phi)$

- $\hat{\mathcal{Q}}$ is a holomorphic principal $\hat{G}$-bundle such that $\hat{\mathcal{Q}}/G \cong \mathcal{Q}_0$ is been fixed;
- $\phi \in H^0(X, \hat{\mathcal{Q}} \times^{\hat{G}} F)$

modulo the gauge group $\mathcal{G} = \mathrm{Aut}_{\mathcal{Q}_0}(\hat{\mathcal{Q}})$. The stability condition consists of an inequality on the degree for every meromorphic reduction (meromorphic reduction are the analogous of torsion-free subsheaves of vector bundles).

In the Higgs bundles case, $\dim X = n$, $\mathrm{rank}\, E = r$, we would look at the exact sequence $1 \to \mathrm{GL}_r \to \mathrm{GL}_r \times \mathrm{GL}_n \to \mathrm{GL}_n \to 1$ and the space $F = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^r \oplus \mathbb{C}^n)$. The principal $\mathrm{GL}_r \times \mathrm{GL}_n$-bundle with fixed GL_r reduction is $E \oplus \Omega^1_X \to \Omega^1_x$; a section of $\hat{\mathcal{Q}} \times^{\hat{G}} F$ would be the Higgs field.

The holomorphic picture has a symplectic counterpart. First, we have to assume the action $\hat{\alpha}$ is hamiltonian: it must be fixed a maximal compact subgroup $\hat{K} \subset G$, a $\hat{K}$-invariant Kähler metric on F, and a moment map μ for the action of $\hat{K}$. Then, we make use of the Chern correspondence between reduction to compact subgroup of holomorphic bundles (e.g., Hermitian metrics) and connection on principal bundles for compact groups (e.g., unitary connections) to produce a pair $(\hat{A}, \phi)$:

- $\hat{A}$ is a $\hat{K}$-connection on a $\hat{K}$-reduction $\hat{P} \subset \hat{Q}$, $\hat{K} \subset \hat{G}$ maximal compact, inducing the fixed connection A_0 on $\hat{P}/K$ $(K = \hat{K} \cap G)$;
- $\phi \in H^0(X, \hat{P} \times^{\hat{K}} F)$.

The *Hermitian-Einstein equation* for the setting above is

$$\mathrm{pr}_{i\,\mathrm{Lie}(K)}[i\Lambda_\omega F_{\hat{A}}] + i\mu(\phi) = 0 \tag{1.26}$$

and, together with the holomorphicity condition $\bar{\partial}_{\hat{A}}\phi = 0$, describes the minimums of the generalized Yang-Mills-Higgs functional

$$\begin{aligned} YMH(\hat{A}, \phi) &= \|F_{\hat{A}}\|^2 + \|d_{\hat{A}}\phi\|^2 + \|\mu(\phi)\|^2 \\ &= \|\Lambda F_{\hat{A}} + \mu(\phi)\|^2 + 2\|\bar{\partial}_{\hat{A}}\phi\|^2 + \\ &\quad + 2\int_X \phi^*\omega_{\hat{P}(F)} \wedge \omega^{n-1} - \int_X B_2(F_{\hat{A}}, F_{\hat{A}}) \wedge \omega^{n-2} \end{aligned} \tag{1.27}$$

where the last two summands are topological quantities.

The universal Kobayashi-Hitchin correspondence states that a HE reduction exists if and only if the holomorphic pair is polystable. Moreover, the following generalization of the Bogomolov inequality is satisfied [Mun00, corollary 7.10]:

$$\int_X \phi^*\omega_{\hat{P}(F)} \wedge \omega^{n-1} - \frac{1}{2}\int_X B_2(F_{\hat{A}}, F_{\hat{A}}) \wedge \omega^{n-2} \geq 0 \tag{1.28}$$

The correspondence produces a real-analytic isomorphism

$$\mathcal{M}^{\mathrm{stable}} = \frac{\{(\hat{\mathcal{Q}}, \phi)\}^{\mathrm{stable}}}{\mathcal{G}} \cong \frac{\{(\hat{A}, \phi)\}^{\mathrm{HE,irreducible}}}{\mathcal{K}} = \mathcal{M}^{\mathrm{HE,irr}} \tag{1.29}$$

and the space carries a canonical metric.

In our context, it would be very interisting to characterize those pairs for which the left-hand side in 1.28 vanish: they would play the role of projectively flat bundles.

There is a further aspect too, related to the understanding of schematic homotopy types. Flat bundles depend only on the 1-homotopy type of a topological space. For projective varieties X, their moduli space of harmonic bundles $\mathcal{M}(X)$ form hyperKähler manifolds that embed, according to the Lefschetz-Bott theorem, as hyperKähler submanifolds in $\mathcal{M}(C)$ for any curve $C \subset X$ that is complete intersection of ample divisors. The moduli space (or moduli stack) $\mathcal{M}(X)$, with its hyperKähler structure and the action of the discrete group $\mathbb{C}^\times$, is what is called the *non-abelian Hodge structure* on the fundamental group of a projective variety [Sim97]. Since the moduli space of holomorphic pairs are objects of algebraic geometry, it seems important to develop suitable restriction theorems and study their contribution to the 1-homotopy type of X.

Formality A harmonic bundle (E, D, D'') is a smooth vector bundle E with a flat connection D and an operator $D'' = \bar{\partial} + \phi$ defining a Higgs bundle, D and D'' related by a harmonic metric (that is not part of the datum). The two operators define two differentials on the same complex of fine sheaves,

$$\mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \dots \tag{1.30}$$

The complex $(\mathcal{A}^\cdot(E), D)$ resolves the locally constant sheaf of flat sections; its cohomology is called *de Rham cohomology*. The complex $(\mathcal{A}^\cdot(E), D'')$ resolves the complex $(E \otimes \Omega^\cdot_X, \cdot \wedge \phi)$; its cohomology is called *Dolbeaut cohomology*. For E the constant bundle $X \times \mathbb{C}$, one recovers the classical definitions $H^i_{\mathrm{dR}}(X, \mathbb{C})$, $\bigoplus H^p(X, \Omega^q_X)$, and the Hodge decomposition $H^i(X, \mathbb{C}) = \bigoplus H^{p,q}$.

The *principle of two types* holds for a harmonic bundle and it helds formality results on the complex above [Sim92, §2]

$$\begin{array}{ccccc} (\mathcal{A}^\cdot(E), D) & \xleftarrow{\sim} & (\ker(D'), D'') & \xrightarrow{\sim} & (\mathcal{A}^\cdot(E), D'') \\ & & \| & & \\ H^\cdot_{\mathrm{dR}}(E) & \xleftarrow{\sim} & (\ker(D'), D'') & \xrightarrow{\sim} & H^\cdot_{\mathrm{Dol}}(E) \end{array} \tag{1.31}$$

In particular, we have formality of the dg Lie algebra $(\mathcal{A}^\cdot(\operatorname{End} E), \bar{\partial} + \mathrm{ad}_\phi)$ that governs the deformations of (E, ϕ) as Higgs bundle.

HyperKähler structure The smooth locus M of the moduli space of harmonic bundles has a hyperKähler structure [Sim97, thm 3.1]. Consider the twistor space $TW(M) \sim M \times \mathbb{P}^1$, where at each point $\lambda \in \mathbb{P}^1$ is associated a complex structure I_λ on M.
The holomorphic structure $J = I_1$ is the one on moduli space of flat bundles. By the Riemann-Hilbert correspondence, it is the same complex space as $\mathrm{Hom}(\pi_1(X, *), GL_r)/GL_r$, the character variety of the fundamental group modulo the conjugation action.
The structure $I = I_0$ is the one on moduli space of Higgs bundles.
Any other structure is given by

$$I_{u+iv} = (1 - uIJ + vJ)^{-1} I (1 - uIJ + vJ) \tag{1.32}$$

For $\lambda \neq 0, \infty$, $I_\lambda \cong J$.

Chapter 2

Attempts at a proof

This chapter consists of three sections. We introduce the spectral cover of a Higgs bundle and study how to relate Higgs quotients to classical quotients on a different scheme. Then we show how to prove restriction theorems for Tannakian fundamental groups.

In the last section we explore the idea of constructing a flat connection on a H-nflat bundle. Moreover, we try to compute when the representation of the topological fundamental group stays constant in a family of harmonic bundles.

2.1 Spectral covers

Let $T^*X = \operatorname{Spec}_X \operatorname{Sym} T_X \xrightarrow{\pi} X$ be the cotangent bundle of X. A Higgs bundle (E, ϕ) comes equipped with a morphism $\operatorname{Sym} T_X \to \mathcal{E}\mathrm{nd}\, E$, thus corresponds to a sheaf $\mathcal{E}$ on T^*X with $E = \pi_*\mathcal{E}$. There is a tautological section $\lambda \in H^0(T^*X, \pi^*\Omega^1_X)$ corresponding to the map $T_X \otimes \operatorname{Sym} T_X \to \operatorname{Sym} T_X$. The characteristic polynomial of ϕ is

$$\det(\lambda \operatorname{Id} - \pi^*\phi) \in H^0(T^*X, \pi^* \operatorname{Sym}^r \Omega^1_X) \tag{2.1}$$

The *spectral cover* of (E, ϕ) is the vanishing locus $Z \subset T^*X$ of $\det(\lambda \operatorname{Id} - \pi^*\phi)$. It is defined by $\dim \operatorname{Sym}^r \Omega^1_X = \binom{n+r-1}{r}$ equations. At any point $z \in Z$ in the cover, it is known that three properties are equivalent: $\mathcal{O}_{Z,z}$ is Cohen-Macauley, $\mathcal{O}_{Z,z}$ is flat over $\mathcal{O}_{X,\pi(z)}$, $\mathcal{O}_{Z,z}$ is a free $\mathcal{O}_{X,\pi(z)}$-module.

Remark. The scheme Z is generally larger than the support of $\mathcal{E}$ on T^*X. Consider, for example, $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$, $T^*X = \operatorname{Spec} \mathbb{C}[x,t]$. The section $\lambda : \mathbb{C}[x,t] \to \mathbb{C}[x,t] \otimes_{\mathbb{C}[x]} \mathbb{C}[x]dx$ is $\lambda(\sum a_i(x)t^i) = \sum a_i(x)t^{i+1}dx$. If E is free of rank r and the Higgs field is 0, the annihilator of $\mathcal{E}$ is $t\mathbb{C}[x,t]$, and its support is the zero section $X \subset T^*X$. On the other hand, the scheme $Z = Z(\det(\lambda \operatorname{Id}))$ is the non-reduced zero-section $X \otimes \mathbb{C}[t]/(t^r)$.

We can understand the Higgs quotients of E through the quotients of $\mathcal{E}$. For any morphism of schemes $t : Y \to X$, there is a fibre product $\widetilde{Y}$

$$\begin{array}{ccc} \widetilde{Y} = Y \times_X T^*X & \xrightarrow{\tilde{t}} & T^*X \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ Y & \xrightarrow{t} & X \end{array} \tag{2.2}$$

Since $\pi : T^*X \to X$ is affine, for any quasicoherent $\mathcal{F}$ on T^*X, $t^*\pi_*\mathcal{F} \cong \widetilde{\pi}_*\tilde{t}^*\mathcal{F}$ [GD71, cor I.9.3.3], and $\tilde{Y} \cong \operatorname{Spec}_Y t^*\pi_*\mathcal{O}_{T^*X}$.
Any $t^*\pi_*\mathcal{O}_Z$-invariant exact sequence $0 \to S_M \to t^*E \to M \to 0$ is an exact sequence of $\widetilde{\pi}_*\tilde{t}^*\mathcal{O}_{T^*X}$-modules, and it induces an analogous sequence on $\widetilde{Y}$, where the quotient $\mathcal{M}$ is locally free (=flat) over Y.

All in all, the Higgs Grassmannian is the Quot scheme $\operatorname{Quot}_{\mathcal{E}/Z/X} \subset \operatorname{Grass}(E)$. For any $Y \to X$, $\mathcal{E}' = \mathcal{E}|_{Z\times_X Y}$, we have $\operatorname{Quot}_{\mathcal{E}'/Z\times_X Y/Y} \cong \operatorname{Quot}_{\mathcal{E}/Z/X} \times_X Y$.

If $Z \times_X Y \to Y$ is flat, a quotient of $\mathcal{E}'$ is flat on Y if and only if it is flat on $Z \times_X Y$. Assuming $\mathcal{E}'$ is a line bundle, we have a correspondence between locally free Higgs quotients of (E, ϕ) and irreducible components of $Z \times_X Y$.

Theorem 1. *Let X be a smooth projective variety, (E, ϕ) a H-nflat Higgs bundle. Assume the spectral cover Z of (E, ϕ) is étale over X, and (E, ϕ) is induced by a line bundle $\mathcal{E}$ on Z. Then, the Chern classes of E vanish.*

Proof. The Chern character of E can be calculated via the Grothendieck-Riemann-Roch formula $\operatorname{ch}(\pi_*\mathcal{E}) = \pi_*(\operatorname{ch}(\mathcal{E}) \operatorname{td}(T_{Z/X}))$. Since $Z \to X$ is étale, $\operatorname{td}(T_{Z/X}) = 1$. The Higgs bundle (E, ϕ) is H-nflat if and only if $\deg \mathcal{E} = 0$. So, $\operatorname{ch}_i(E) = 0$ for $i > 0$. □

2.2 Fundamental groups

The many known properties of Higgs-nflat bundles [BBG16] generalize immediately to bundles with a Higgs field $\phi : E \to E \otimes \mathcal{W}$, $\phi \wedge \phi = 0$. We indicate by $\operatorname{NF}(X, \mathcal{W})$ the category of $\mathcal{W}$-nflat bundles; with $\operatorname{NF}(X)$ the category of nflat bundles with no Higgs field.

Theorem. *[BBG16] The category* $\operatorname{NF}(X, \mathcal{W})$ *is a neutral Tannakian category.*

Properties of affine groups and morphisms between them can deduced from their categories of representations and functors. Let us recall the characterization of quotients and closed subgroups [DM82, prop 2.21].

- A morphism of affine groups $f : G \to H$ is faithfully flat if and only if the functor $f^* : \operatorname{Rep} H \to \operatorname{Rep} G$ is fully faithful and, for every $W \subset f^*(V)$, there exists $U \subset V$ with $W \cong f^*(U)$;

- $f : K \to G$ is a closed embedding if and only if every object of $\operatorname{Rep} K$ is isomorphic to a subquotient of some $f^*(V)$, V in $\operatorname{Rep} G$.

For any $\mathcal{W}$, there is an obvious inclusion $\mathrm{NF}(X) \to \mathrm{NF}(X, \mathcal{W})$. It is immediate that it satisfies the properties to define a faithfully flat morphism: if the Higgs-like field vanishes on E, it vanishes on every subobject, and the morphisms are just the same. So, there is a surjection $\pi(X, \mathcal{W}) \to \pi(X)$; the last group is the *S-fundamental group scheme* of Langer [Lan11].

On the other hand, it is easily seen that $\mathrm{NF}(X) \subset \mathrm{NF}(X, \mathcal{W})$ needs not be closed under extensions: if

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.3}$$

is an extensions of nflat bundles, any morphism $\phi : E_2 \to E_1 \otimes \mathcal{W}$ gives an element in $\mathrm{NF}(X, \mathcal{W})$, the easiest example being $E_1 = E_2 = \mathcal{W} = \mathcal{O}_X$, $\phi = const$.

The categories $\mathrm{NF}(X) \subset \mathrm{NF}(X, \mathcal{W})$ coincide in case $\mathcal{W}$ is semistable of negative degree, since any Higgs-like field must vanish.

It would be nice to see what the category of representations of the kernel of $\pi(X, \mathcal{W}) \to \pi(X)$ could look like. Since the subcategory $\operatorname{Rep} \pi(X)$ is not closed under extensions, it is not a Serre subcategory, and the usual construction of quotients of abelian categories does not apply. After the example in [EHS08] and the theory of [Mil07], no general construction is known. Moreover, it would be nice to understand the behaviour under deformations of $\mathcal{W}$.

Lefschetz theorems are vanishing theorems Here we make use of the same machinery of [Lan11] to understand the behaviour of $\pi(X, \mathcal{W})$ under restriction to an ample divisor $D \subset X$.

The fundamental sequence of a divisor is

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \tag{2.4}$$

It can be tensored with any vector bundle E to give the restriction of E. Since $\mathrm{NF}(X, \mathcal{W})$ is closed monoidal, morphism $(F_1, \psi_1) \to (F_2, \psi_2)$ are sections of a $\mathcal{W}$-nflat bundle $(E, \phi) = (F_1, \psi_1)^\vee \otimes (F_2, \psi_2)$, and we have a sequence

$$H^0_{\mathcal{W}}\big(X, E(-D)\big) \to H^0_{\mathcal{W}}\big(X, E\big) \to H^0_{\mathcal{W}|_D}\big(D, E|_D\big) \to H^1_{\mathcal{W}}\big(X, E(-D)\big) \tag{2.5}$$

where, $H^i_{\mathcal{W}}$ is the $\mathcal{W}$-Dolbeault cohomology, i.e., the hypercohomology of the complex

$$0 \to E \to E \otimes \mathcal{W} \to E \otimes \Lambda^2 \mathcal{W} \to \dots \tag{2.6}$$

So, if we wish to prove that $\pi(X, \mathcal{W}) \to \pi\left(D, \mathcal{W}|_D\right)$ is fully faithful, we can prove a vanishing result for $H^i_{\mathcal{W}}(X, E(-D))$, $i = 0, 1$.

The space $H^0_{\mathcal{W}}(X, E(-D))$ is simply $H^0(X, (\ker\phi)(-D))$. The sheaf $\ker\phi$ is a $\mathcal{W}$-invariant subsheaf of E with $\phi = 0$, so it is nflat, and $(\ker\phi)(-D)$ is semistable of negative degree, so it has no sections.

Next, we can use both the clever and the naive filtration on the $\mathcal{W}$-Dolbeaut complex to get

$$H^1((\ker\phi)(-D)) \to H^1_{\mathcal{W}}(E(-D)) \to H^1(\tau_{\geq 1}(E \otimes \Lambda^{\cdot}\mathcal{W})(-D)) \tag{2.7}$$

$$H^0\big(\ker(E \otimes \mathcal{W}(-D) \xrightarrow{\wedge\phi} E \otimes \Lambda^2\mathcal{W}(-D))\big) \to \\ H^1\big(\tau_{\geq 1}(E \otimes \Lambda^{\cdot}\mathcal{W})(-D)\big) \to H^1\big((\operatorname{im}\phi)(-D)\big) \tag{2.8}$$

If $-D$ is sufficiently negative, the sheaf $E \otimes \mathcal{W}(-D)$ has no sections, and $H^1((\ker\phi)(-D))$, $H^1((\operatorname{im}\phi)(-D))$ vanish by the lemma of Enriques-Severi-Zariski [Har77, p. III.7.8].

If the Chern classes of all bundles in $\mathrm{NF}(X, \mathcal{W})$ vanish, the degree of D can be choosed uniformly, and we obtain the Lefschetz theorem: $\pi(D, \mathcal{W}) \to \pi(X, \mathcal{W})$ is surjective.

2.3 Families of harmonic bundles

Let X be a smooth projective complex surface and (E, ϕ) a tentative harmonic bundle on it. By Lefschetz-Bott theorem, for $C \subset X$ a smooth ample divisor, $\pi_1(C) \to \pi_1(X)$ is surjective. The Higgs bundle (E, ϕ) restricted to C gives, by the assumption on semistability and degree, a linear representation of $\pi_1(C)$, which we wish to descend to $\pi_1(X)$. In the same mindset, (E, ϕ) restricted to C allows for a flat connection, and we wish to say that E on X supports a flat connection.

By the geometric characterization of [Har77, p. II.7.8.2], a linear system giving an embedding into projective space must separate points and tangent vectors; in other words, through every point of X must run a divisor, and at no point x can all divisors be tangent to each other, so their common tangent spaces span the tangent space of X at x. The tautological divisor $\mathcal{C} \subset X \times \mathbb{P}H^0(X, \mathcal{L})$ is the locus $\{(x, s) : s(x) = 0\}$: it is a $\mathbb{P}^{N-2}$-bundle on X, $N = \dim H^0(X, \mathcal{L})$, so it is smooth. The projection $p : \mathcal{C} \to \mathbb{P}V$, $V = H^0(X, \mathcal{L})$, is flat, and the fibres are smooth curves over an open dense of $\mathbb{P}V$, by Bertini's theorem. Their arithmetic genus can be calculated by the adjunction formula: if $C \subset X$ is a divisor, $p_a(C) = \frac{1}{2}C(C + K_X) + 1$.

A strategy to show flatness of the Hitchin-Simpson connection can be summarized as follows:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ \mathbb{P}H^0(X, \mathcal{L}) & & \end{array}$$

Take the complete linear system of a very ample line bundle. Assume for the moment that all divisors are smooth.

0. Simpson correspondence gives a flat relative $\mathcal{C}/\mathbb{P}V$-connection on the pullback of E to $\mathcal{C}$,
1. Extend the relative connection to a $\mathcal{C}$-connection;
2. show that the extended connection descends to X.

The existence of a holomorphic connection implies the vanishing of the Atiyah class, and of the Chern classes too. Actually, the Hitchin-Simpson $\mathcal{C}/\mathbb{P}V$-connection is not holomorphic on $E\big|_{\mathcal{C}}$ (unless the Higgs field vanishes) but on a deformation of it; since the numerical Chern classes are deformation invariant, this is a minor problem.

If I is the ideal of the tautological divisor $\mathcal{C}$ in $X \times \mathbb{P}V$, I/I^2 is the pullback of the dual of the ample line bundle $\mathcal{L}$ to $\mathcal{C}$. Since the fibres of $\mathcal{C} \to \mathbb{P}V$ are embedded in X, there is a short exact sequence

$$0 \to I/I^2 \to \Omega_X\big|_{\mathcal{C}} \to \Omega_{\mathcal{C}/\mathbb{P}V} \to 0 \tag{2.9}$$

fitting in the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_{\mathbb{P}V}\big|_{\mathcal{C}} & = & \Omega_{\mathbb{P}V}\big|_{\mathcal{C}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & I/I^2 & \to & (\Omega_X \boxplus \Omega_{\mathbb{P}V})\big|_{\mathcal{C}} & \to & \Omega_{\mathcal{C}} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & I/I^2 & \longrightarrow & \Omega_X\big|_{\mathcal{C}} & \longrightarrow & \Omega_{\mathcal{C}/\mathbb{P}V} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The Atiyah class [Ill71] for a morphism $Y \to Z$ and a Y-module M, is $\mathrm{at}_{Y/Z}(M) : M \to M \otimes^{\mathbb{L}} L_{Y/Z}[1]$, where $L_{Y/Z}$ is the cotangent complex. The Atiyah class is the obstruction to the existence of a relative (Y/Z)-connection on M. For $\mathcal{C} \subset X \times \mathbb{P}V$ the closed embedding of a smooth subvariety, $L_{\mathcal{C}/X\times\mathbb{P}V} = I/I^2[1]$. Look at the diagram below

$$\begin{array}{ccc}
 & & E\big|_{\mathcal{C}} \otimes L_{\mathcal{C}/X\times\mathbb{P}V} \\
 & \overset{a}{\nearrow} & \downarrow \\
E\big|_{\mathcal{C}} & \xrightarrow{\ \mathrm{at}_X(E)\big|_{\mathcal{C}}\ } & \big(E \otimes L_X\big)\big|_{\mathcal{C}}[1] \\
 & \underset{\mathrm{at}_{\mathcal{C}/\mathbb{P}V}\left(E\big|_{\mathcal{C}}\right)}{\searrow} & \downarrow \\
 & & E\big|_{\mathcal{C}} \otimes L_{\mathcal{C}/\mathbb{P}V}[1]
\end{array} \tag{2.10}$$

Since $\mathrm{at}_{\mathcal{C}/\mathbb{P}V}(E\big|_{\mathcal{C}}) = 0$ ($E\big|_{\mathcal{C}}$ has a relative connection), $\mathrm{at}_X(E)\big|_{\mathcal{C}}$ must factor through $E\big|_{\mathcal{C}} \otimes L_{\mathcal{C}/X\times\mathbb{P}V} \to E\big|_{\mathcal{C}} \otimes L_X\big|_{\mathcal{C}}[1]$;

call $a : E\big|_{\mathcal{C}} \to E\big|_{\mathcal{C}} \otimes L_{\mathcal{C}/X\times\mathbb{P}V}$ the morphism so induced. Then we compute:

$$\begin{aligned}\operatorname{Hom}\left(E\big|_{\mathcal{C}}, E\big|_{\mathcal{C}} \otimes L_{\mathcal{C}/X\times\mathbb{P}V}\right) &= H^1\left(\mathcal{C}, I/I^2 \otimes \operatorname{End} E\big|_{\mathcal{C}}\right)\\ &= H^1\left(\mathcal{C}, \mathcal{L}^\vee \otimes \operatorname{End} E\big|_{\mathcal{C}}\right)\end{aligned} \tag{2.11}$$

We have the projection formula $\mathbb{R}^i q_*\left(\mathcal{M} \otimes q^*\mathcal{N}\right) \cong \left(\mathbb{R}^i q_*\mathcal{M}\right) \otimes \mathcal{N}$, for $\mathcal{N}$ on X locally free of finite rank. By the Leray spectral sequence,

$$\begin{aligned}0 \to H^1(X, \mathcal{L}^\vee \otimes \operatorname{End} E) \to H^1\left(\mathcal{C}, \mathcal{L}^\vee \otimes \operatorname{End} E\big|_{\mathcal{C}}\right) \to\\ \to H^0\left(X, \mathbb{R}^1 q_*\mathcal{O}_{\mathcal{C}} \otimes \mathcal{L}^\vee \otimes \operatorname{End} E\right)\end{aligned} \tag{2.12}$$

The first group is trivial by the lemma of Enriques-Severi-Zariski [Har77, p. III.7.8] as soon as $\mathcal{L}$ is sufficiently positive, while the last one is trivial because $\mathbb{R}^1 q_*\mathcal{O}_{\mathcal{C}} = 0$ ($\mathcal{C} \to X$ is a projective bundle). So, $\operatorname{at}_X(E)\big|_{\mathcal{C}} = 0$, and, by the projection formula, $\operatorname{at}_X(E) = 0$.

Remark We can not expect all divisors in a linear system to be smooth. On the other hand, instead of looking at the complete linear system, we can look at a Lefschetz pencil, i.e., a fibration $\tilde{X} \to \mathbb{P}^1$ where the only singularities are double points, they are in a finite number, and no two of them occur on the same fibre. The total space $\tilde{X}$ is the blowup of X at finitely many points.

The Gauss-Manin derivative Analogously to the desire of having a flat connection descend from $\mathcal{C}$ to X, we may wish that a linear representation descends along $\pi_1(C) \twoheadrightarrow \pi_1(X)$. Here we make some considerations about the change of representation of the fundamental group, tantamount to the change of flat connection outside its gauge equivalence class, in a family of polystable Higgs bundles. This is related to the Weil-Petersson metric studied in [BS99; BS06].

De Rham side Fix a differentiable manifold C and a vector bundle E, and let D be a flat connection on E. We can have a one-parameter (or multiple-parameters) deformation of D as a power series

$$D_u = D + u\alpha_1 + u^2\alpha_2 + \dots \tag{2.13}$$

where $\alpha_i \in A^1(C, \operatorname{End} E)$ (smooth forms in $\operatorname{End} E$). If D_u is flat for every u, then $D(\alpha_1) = 0$.

A deformation as above is the same as a connection on $\operatorname{pr}^* E$ over $C \times U$ relative to U,

$$D : \mathcal{E} \to \mathcal{E} \otimes A^1(C \times U/U) \tag{2.14}$$

(D satisfies the Leibniz rule relative to $d_{C\times U/U}$). The form α_1 is the first Taylor coefficient in the direction u,

$$\alpha_1 = \frac{\partial}{\partial u}\Big|_{u=0} D \tag{2.15}$$

In a twisted situation $\mathcal{C} \to U$, the partial derivative $\partial/\partial u$ must be replaced by a covariant derivative $\nabla_u^{\mathcal{C}}$ for some choice of a connection on $\mathcal{C}$ over U (for example, $\mathcal{C} \to U$ could be taken to be a Riemannian submersion).

Definition/Lemma Let $(\mathcal{E}, D)$ be relatively flat connection on $\mathcal{C} \to U$ ($\mathcal{E}$ is a vector bundle on $\mathcal{C}$, D takes values in the relative 1-forms tensor the endomorphisms, and $D^2 = 0$). Choose a connection $\nabla : T\mathcal{C} \to T\mathcal{C} \otimes A^1(U)$.

Define the first variation of D at $u \in U$,

$$\begin{aligned} T_uU &\to H^1_{dR}(\mathcal{C}_u, \operatorname{End}\mathcal{E}\big|_{\mathcal{C}_u}) \\ v &\mapsto \big[\nabla_v\ D\big] \end{aligned} \tag{2.16}$$

- $D_u\big(\nabla_v\ D\big) = 0$ (a cohomology class is defined)
- if ∇' is a different connection on $\mathcal{C}/U$, then $\nabla' - \nabla = a \in T\mathcal{C}^{vert}$ and $\nabla' D - \nabla D = D(a \otimes \mathrm{id}_E)$, so the variation of D is intrinsically defined.

Gauss-Manin derivative We see the assignment above as the Gauss-Manin connection on the first de Rham nonabelian cohomology space.

Given a submersion (smooth map) $\mathcal{C} \to U$, there is a fibration $\pi : M_{dR}(\mathcal{C}/U) \to U$, where the fibre over $u \in U$ is the moduli space of vector bundles with flat connections on $\mathcal{C}_u$. The tangent space at a point $[(E, D)] \in M_{dR}(\mathcal{C}_u) = M_{dR}(\mathcal{C}/U)_u$ is canonically identified with $H^1_{dR}(\mathcal{C}_u, \operatorname{End} E)$.

If $\sigma : U \to M_{dR}(\mathcal{C}/U)$ is a section of π, it classifies the datum $(\mathcal{E}, D)$ of a vector bundle $\mathcal{E}$ on $\mathcal{C}$ with a flat relative connection w.r.t. $\mathcal{C} \to U$. The *Gauss-Manin connection* computes the covariant derivative of such a (local) section σ:

$$\begin{aligned} \nabla^{GM}\sigma : T_uU &\to \sigma^* TM_{dR}(\mathcal{C}/U)^{vert} \\ v &\mapsto \big[\nabla_v^{\mathcal{C}} D_{\sigma(u)}\big] \in H^1_{dR}(\mathcal{C}_u, \sigma(u)) \end{aligned} \tag{2.17}$$

A section σ is flat if and only if, locally on $\mathcal{C}$, there is some gauge for $(\mathcal{E}, D)$ such that D is constant in the U-direction (the choice of gauge for $\mathcal{E}$ depends on the choice of a connection $\nabla^{\mathcal{C}}$).

The Dolbeaut side On the Dolbeaut side there is the annoyance that the holomorphic structure on $\mathcal{C}_u$ can vary from fibre to fibre.

For a start, let us recall how the diffeomorphism $M_{Dol}(C) \cong M_{dR}(C)$ works at the level of tangent spaces. On the Dolbeaut side, a tangent vector at a point represented by the Higgs bundle (E, ϕ) is (the class of) a 1-form

$\alpha \in A^1(X, \operatorname{End} E)$ such that $D''\alpha = 0$ ($D'' = \bar\partial + \mathrm{ad}_\phi$), or, decomposing in types $\alpha = \alpha' + \alpha''$,

$$\bar\partial\alpha'' = 0, \quad [\phi, \alpha'] = 0, \quad \bar\partial\alpha' + [\phi, \alpha''] = 0 \tag{2.18}$$

Suppose (E, ϕ) has been given a harmonic metric, so that $D' = \partial + \bar\phi$ is obtained by the requirements that $\partial + \bar\partial$ is unitary and $\bar\phi$ is adjoint to ϕ,

$$\begin{aligned} d\langle s, t\rangle &= \langle(\partial + \bar\partial)s, t\rangle + \langle s, (\partial + \bar\partial)t\rangle \\ \langle\phi(s), t\rangle &= \langle s, \bar\phi(t)\rangle \end{aligned} \tag{2.19}$$

and $D = D' + D'' = \bar\partial + \bar\phi + \partial + \phi$ is a flat connection. We can assume the Yang-Mills metric is fixed through the deformation, because all possible metrics are related by a complex gauge transformation, and α would still represent the same class thereafter. Then, the deformation α of D'' is complemented by a deformation $\beta = \beta' + \beta''$ of D' such that

$$\begin{aligned} d\langle s, t\rangle &= \langle(\partial + \bar\partial + \beta' + \alpha'')s, t\rangle + \langle s, (\partial + \bar\partial + \beta' + \alpha'')t\rangle \\ \langle(\phi + \alpha')(s), t\rangle &= \langle s, (\bar\phi + \beta'')(t)\rangle \end{aligned} \tag{2.20}$$

from which we get, as in [Sim97, pag. 235] ($\alpha^\dagger = -\bar\alpha^*$),

$$\begin{aligned} \langle\beta'(s), t\rangle + \langle s, \alpha''(t)\rangle &= 0 & \beta' &= -(\alpha'')^\dagger \\ \langle\alpha'(s), t\rangle &= \langle s, \beta''(t)\rangle & \beta'' &= (\alpha')^\dagger \end{aligned} \tag{2.21}$$

Families $\mathcal{C} \to U$ The deformations of a pair $(complex\ manifold, Higgs\ bundle)$ are governed by the Dolbeaut cohomology of the Atiyah bundle [Mar12]:

$$0 \to \operatorname{At} E \to \operatorname{End} E \otimes \Omega^1 \to \operatorname{End} E \otimes \Omega^2 \to \dots \tag{2.22}$$

This means that first order deformations are given by classes $\Psi + \alpha'$ where $\alpha' \in A^{1,0}(\operatorname{End} E)$ as before, $\Psi \in A^{0,1}(\operatorname{At} E)$ is a form with values in first-order differential operators with scalar symbols $\sigma(\Psi)$, and

$$\bar\partial\Psi = 0, \quad [\phi, \alpha'] = 0, \quad \bar\partial\alpha' + [\phi, \Psi] = 0 \tag{2.23}$$

Since all possible Hermitian metrics are equivalent under the complex gauge group, if needed, we can change the deformation at hand to an equivalent one such that the Hermitian-Yang-Mills metric is the same. Then, the equation for the adjoint does not change,

$$\langle\alpha' s, t\rangle = \langle s, \overline{\alpha'} t\rangle \tag{2.24}$$

but we need to understand explicitly how the change in the holomorphic structure depends on Ψ.

Complex structures Let V be a real vector space. A *(linear) complex structure* on V is an endomorphism J such that $J^2 = -1$. The pair (V, J) gives a complex vector space V_J via $z \cdot v = (\mathrm{Re}\ z)v + (\mathrm{Im}\ z)Jv$.
The group of automorphisms of V as a real vector space acts transitively on the set of possible complex structures by conjugation, and the stabilizer of J coincides with the automorphisms of V_J as a complex vector space.

$$\{\textit{cpx str.s on } V\} \cong \mathrm{Aut}(V)/\,\mathrm{Stab}(J) = \mathrm{GL}_{\mathbb{R}} V/\,\mathrm{GL}_{\mathbb{C}} V_J$$

The endomorphism J is clearly invertible on V; its adjoint action on $\mathrm{End}\, V$, $X \mapsto JXJ^{-1}$, has eigenvalues ± 1. The $(+1)$-eigenspace, $\{X : JX = XJ\}$, is the Lie algebra of $\mathrm{GL}_{\mathbb{C}}(V, J)$. The (-1)-eigenspace, $\{X : JX = -XJ\}$, is a set of representatives for the tangent space at J to the space of complex structures; a complex structure J' close to J can be represented as $J' = e^X J e^{-X} = e^{2X} J = J e^{-2X}$.
If J is a complex structure, $-J$ is the *conjugate (linear) complex structure*; we will write $\overline{V_J} := V_{-J}$.

If we extend J to the complexification $V \otimes \mathbb{C}$ of V, we get an eigenspace decomposition $V \otimes \mathbb{C} = V^+ \oplus V^-$, with eigenvalues $\pm i$. There arise two complex linear embeddings onto the two eigenspaces,

$$\begin{aligned} V_J \ni v &\mapsto \frac{1}{2}v - \frac{1}{2}iJv \in V^+ \\ \overline{V_J} \ni v &\mapsto \frac{1}{2}v + \frac{1}{2}iJv \in V^- \end{aligned} \tag{2.25}$$

With respect to the eigenspace decomposition, the extensions of J and X have the form

$$J \otimes \mathbb{C} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad X \otimes \mathbb{C} = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \tag{2.26}$$

The two orthogonal projectors $\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\bar{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ onto the eigenspaces for J get transformed by

$$\begin{aligned} \exp\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} e^X + e^{-X} & e^X - e^{-X} \\ e^X - e^{-X} & e^X + e^{-X} \end{pmatrix} \\ \pi' = \exp\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & -X \\ -X & 0 \end{pmatrix} &= \\ = \frac{1}{4}\begin{pmatrix} 2 + e^{2X} + e^{-2X} & -e^{2X} + e^{-2X} \\ e^{2X} - e^{-2X} & 2 - e^{2X} - e^{-2X} \end{pmatrix} \\ \bar{\pi}' = \exp\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & -X \\ -X & 0 \end{pmatrix} &= \\ = \frac{1}{4}\begin{pmatrix} 2 - e^{2X} - e^{-2X} & e^{2X} - e^{-2X} \\ -e^{2X} + e^{-2X} & 2 + e^{2X} + e^{-2X} \end{pmatrix} \end{aligned} \tag{2.27}$$

The deformation from J to J' can as well be described by a map $\psi : \overline{V_J} \to V_J$ (i.e., an antilinear map $V_J \to V_J$). For small X, the projection

$$\bar{\pi} : (V^-)' = e^{X\otimes\mathbb{C}}\, V^- \subset V \otimes \mathbb{C} \to V^- \tag{2.28}$$

is an isomorphism; $(V^-)'$ is then the graph of a map $\psi : \bar{V} \cong V^- \to V^+ \cong V$. Since, by the previous calculations, $(V^-)' = \frac{1}{2}\begin{pmatrix} e^X - e^{-X} \\ e^X + e^{-X} \end{pmatrix} V$, we see that

$$\psi = \frac{e^X - e^{-X}}{e^X + e^{-X}} = \tanh(X) \qquad X = \tanh^{-1}(\psi) = \frac{1}{2}\log\frac{1+\psi}{1-\psi} \tag{2.29}$$

Notice that, looking for the "conjugate" map $\bar{\psi} : V \to \bar{V}$, we find $\bar{\psi} = \psi$. We have thus found that

$$\begin{aligned} \bar{\pi}' - \bar{\pi} &= \frac{1}{4}\begin{pmatrix} 2 - e^{2X} - e^{-2X} & e^{2X} - e^{-2X} \\ -e^{2X} + e^{-2X} & -2 + e^{2X} + e^{-2X} \end{pmatrix} = \\ &= \frac{1}{1-\psi^2}\begin{pmatrix} -\psi^2 & \psi \\ -\psi & \psi^2 \end{pmatrix} = -\pi' + \pi \end{aligned} \tag{2.30}$$

Notice that occurences of ψ^2 can be read either as $\psi\bar{\psi}$ or $\bar{\psi}\psi$, and they are complex linear endomorphisms of $V, \bar{V}$.

If we apply the computations above to the space of differential forms on a complex manifold ($\partial = \pi d$, $\bar{\partial} = \bar{\pi} d$), we find

$$\begin{cases} \partial' = \partial + \frac{\psi}{1-\psi}(\partial - \bar{\partial}) \\ \bar{\partial}' = \bar{\partial}' - \frac{\psi}{1-\psi}(\partial - \bar{\partial}) \end{cases} \tag{2.31}$$

2.4 Addendum

Cech cohomology Cohomology computations can be performed using local trivialisations and transition functions.

Fix a Stein/affine open cover $\mathcal{U} = (U_\alpha)$. Let two vector bundles E, E' be given through collections of transition functions $(g_{\alpha\beta})$, $(g'_{\alpha\beta})$. Then

$$\mathrm{Hom}(E, E') = \left\{ (f_\alpha) \in \prod \mathrm{Hom}(\mathbb{C}^{\mathrm{rk}\,E}, \mathbb{C}^{\mathrm{rk}\,E'}) \colon f_\alpha g_{\alpha\beta} = g'_{\alpha\beta} f_\beta \right\}$$

Tentative theorem Ext-groups can be computed via a pseudo-Cech complex (we suppose the set of indices $\{\alpha\}$ is ordered)

$$0 \to \prod_\alpha \mathrm{Hom}(\mathbb{C}^{\mathrm{rk}\,E}, \mathbb{C}^{\mathrm{rk}\,E'}) \to \prod_{\alpha<\beta} \mathrm{Hom}(\mathbb{C}^{\mathrm{rk}\,E}, \mathbb{C}^{\mathrm{rk}\,E'}) \to \dots$$

where the differential δ is given as

$$
\begin{aligned}
(\delta f)_{\alpha\beta} &= f_\alpha g_{\alpha\beta} - g'_{\alpha\beta} f_\beta \\
(\delta f)_{\alpha\beta\gamma} &= f_{\alpha\beta} g_{\beta\gamma} - f_{\alpha\gamma} + g'_{\alpha\beta} f_{\beta\gamma} \\
(\delta f)_{\alpha\beta\gamma\delta} &= f_{\alpha\beta\gamma} g_{\gamma\delta} - f_{\alpha\beta\delta} + f_{\alpha\gamma\delta} - g'_{\alpha\beta} f_{\beta\gamma\delta} \\
&\dots
\end{aligned}
$$

Aside from the Ext^0 case mentioned above, we recover the Ext^1:

Given an extension

$$0 \to E' \to F \to E \to 0$$

the transition functions of F can be put in the form

$$\widetilde{g}_{\alpha\beta} = \begin{pmatrix} g'_{\alpha\beta} & f_{\alpha\beta} \\ 0 & g_{\alpha\beta} \end{pmatrix}$$

The cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ translates to $\delta f = 0$. Changing the transition functions as

$$S_\alpha \widetilde{g}_{\alpha\beta} S_\beta^{-1} = \begin{pmatrix} 1 & s_\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g'_{\alpha\beta} & f_{\alpha\beta} \\ 0 & g_{\alpha\beta} \end{pmatrix} \begin{pmatrix} 1 & -s_\beta \\ 0 & 1 \end{pmatrix}$$

changes $f \mapsto f + \delta s$.

Longer extension sequences (higher extension classes) are obtained composing short exact sequences. The Ext^2 case in instructive: as is well known, a 4-terms exact sequence is quasi-isomorphic to a split sequence

$$0 \to E' \xrightarrow{=} E' \xrightarrow{0} E \xrightarrow{=} E \to 0$$

if and only if it arises from a filtered object $E' \subset E'' \subset E'''$ as

$$0 \to E' \to E'' \to E'''/E' \to E'''/E'' \to 0$$

In this case, writing the transition functions of E''',

$$G_{\alpha\beta} = \begin{pmatrix} g'_{\alpha\beta} & f'_{\alpha\beta} & h_{\alpha\beta} \\ 0 & \hat{g}_{\alpha\beta} & f_{\alpha\beta} \\ 0 & 0 & g_{\alpha\beta} \end{pmatrix}$$

we find $f'_{\alpha\beta} f_{\beta\gamma} = -(\delta h)_{\alpha\beta\gamma}$, conferming that the cohomology class is 0.

Chapter 3

Questions in deformation theory

The deformation theory and the moduli space of harmonic bundles possess some interisting features. If Bruzzo's conjecture holds true, an irreducible nflat Higgs bundle should present the same features: its endomorphism algebra should be formal and its deformation space should have a quaternionic structure.

Moreover, we have presented Higgs bundles through the contraction of a Lie algebroid (the complexified tangent bundle). Going in the opposite direction (from Higgs bundles to flat connections) the algebroid together with its representations undertake a generically trivial deformation. We get in the situation where the dg category of representations of an algebroid embeds fully faithfully in the representations of a contracted one.

In this chapter we offer just a minor consideration about jump deformations, linking the definition of jump cycles of Griffiths [Gri65] to the the notion of jump deformation of algebraic structures by Gerstenhaber [Ger74]. Let us mention that the central equation in deformation theory, the Maurer-Cartan equation 3.25, is clearly non homogeneous, but it is homogeneous when the dg Lie algebra is formal ($d = 0$). Jump deformations have a sort of homogeneity.

3.1 Contractions

Recall the notion of *contraction of algebraic structure*, already employed for Lie algebroids in chapter 1. Here we repeat it in a colloquial style.

It is given a set V with some operations $m : V \times \cdots \times V \to V$ satisfying certain axioms. It is given a 1-parameter group of self-bijections of V, $\lambda(t) : V \to V$, $t \neq 0$. It is possible to conjugate the operations of V as $m_t = \lambda(t)^{-1} m(\lambda(t)x_1, \dots, \lambda(t)x_n)$. Of course, for any t, $\lambda(t)$ intertwines the structure of $(V, \{m_t\})$ and $(V, \{m\})$, so it realizes an isomorphism in the

appropriate category.
We suppose to be able to take the limit for $t \to 0$. The resulting $(V, \{m_0\})$ is a *contraction* of $(V, \{m\})$ (it is not assumed to be able to give a sense to $\lambda(0)$, just to the various operations m_0).

Deformation to the associated graded A contraction as defined above is an example of degeneration to the orbit closure: the group of self bijections of V acts on the variety of tuples of maps $V \times \cdots \times V \to V$ satisfying the axioms of the theory; then we take a structure $\mathcal{V}_0$ in the closure of the orbit of the original structure $\mathcal{V}$. A requirement for a contraction is that the degeneration from $\mathcal{V}$ to $\mathcal{V}_0$ is realized by a 1-parameter subgroup. In the linear algebraic case there is an easy result:

$\mathcal{V}_0$ is a degeneration of $\mathcal{V}$ via a 1-parameter group if and only if $\mathcal{V}$ admits a filtration such that $\mathcal{V}_0$ is the associated graded.
This theorem already appear in the literature in the case of modules over an associative algebra [Kra82, §II.4.3] and for Lie algebras [GO88, theorem 1.2]. The proof is simple: the filtration is by eigenspaces, ordered by their weight.

First example Let R be a ring and consider an extension of R-modules

$$\eta: \ 0 \to E'' \to E \to E' \to 0, \qquad [\eta] \in \operatorname{Ext}^1_R(E', E'') \tag{3.1}$$

We can see E as a deformation of $E_0 = E'' \oplus E'$ in the following way. Take $\mathbb{E} = E'' \oplus E \oplus E \oplus E \oplus \dots$ as a $R[x]$-module, where the action of x is $E'' \hookrightarrow E = E = E = \dots$. Reducing modulo x^2 we find $E'' \oplus E \oplus E'$; modulo x it is $E'' \oplus E'$. So we have found

$$\widetilde{\eta}: \quad 0 \to E_0 \to E'' \oplus E \oplus E' \to E_0 \to 0 \tag{3.2}$$

correponding to the composition $E_0 \twoheadrightarrow E' \xrightarrow{\eta} E''[1] \hookrightarrow E_0[1]$.

Second example Let consider a 3-dimensional Lie algebra over $k[x]$, with structure constants $f: (\bigwedge^2 k^3) \otimes k[x] \to k^3 \otimes k[x]$

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & x^2 \end{pmatrix}: \quad \begin{matrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{matrix} \quad \mapsto \quad \begin{matrix} 2e_2 \\ 2e_1 \\ x^2 e_3 \end{matrix} \tag{3.3}$$

At $x = 1$, this is the algebra $\mathrm{sl}_2 k$; at $x = 0$, it is a solvable Lie algebra $\mathfrak{r}$. The contraction is via $\lambda(t) = \operatorname{diag}(t, t, 0)$; the associated weight filtration $V_i = \operatorname{span}\{v\colon \operatorname{wt} v \le i\}$ is $V_0 = \langle e_3 \rangle \subset V_1 = \mathrm{sl}_2 k$.

The Chevalley-Eilenberg complex of $\mathfrak{r}$ is $CE^{\cdot}(\mathfrak{r}; \mathfrak{r}) = \operatorname{Hom}_k(\bigwedge^{\cdot} \mathfrak{r}, \mathfrak{r})[1]$. The bracket and differential of 1-cochains (bilinear maps, since we have shifted the complex) is

$$\begin{aligned}\llbracket\phi,\psi\rrbracket(v_1,v_2,v_3) &= \sum_{\text{cycl}}\phi(\psi(u_i,u_j),u_k)+\sum_{\text{cycl}}\psi(\phi(u_i,u_j),u_k)\\ d\phi(v_1,v_2,v_3) &= \sum_{\text{cycl}}[\phi(u_i,u_j),u_k]+\sum_{\text{cycl}}\phi([u_i,u_j],u_k)\end{aligned} \tag{3.4}$$

The infinitesimal of the deformation is

$$\phi = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} : \begin{matrix} e_2\wedge e_3 \\ e_3\wedge e_1 \\ e_1\wedge e_2\end{matrix} \quad \mapsto \quad \begin{matrix}0\\0\\e_3\end{matrix} \tag{3.5}$$

It is a cocycle ($d\phi = 0$) and it satisfies also $\llbracket\phi,\phi\rrbracket = 0$, so it solves the Maurer-Cartan equation

$$d\phi + \frac{1}{2}\llbracket\phi,\phi\rrbracket = 0 \tag{3.6}$$

formally at any order.

3.2 The Rees construction

Let $V^{\cdot}$ be a filtered k-module

$$\dots \hookrightarrow V^{n-1} \hookrightarrow V^n \hookrightarrow V^{n+1} \hookrightarrow \dots \tag{3.7}$$

We can see it as a functor $(\mathbb{Z},\leq)\to \mathrm{Mod}_k$. Let's call all maps x; then the direct sum $\mathbb{V}=\bigoplus_{n\in\mathbb{Z}} V^n$ is a module over the polynomial algebra $k[x]$. $\mathbb{V}$ is graded, so it carries an action of the algebraic group $\mathbb{G}_{m,k} = \operatorname{Spec} k[t,t^{-1}]$. The action of $x\in k[x]$ on $\mathbb{V}$ increases the weight by 1. Summing up, $\mathbb{V}$ gives a $\mathbb{G}_m$ equivariant sheaf on $\mathbb{A}^1_k$, or, which is the same, a sheaf on the stack quotient $\mathbb{A}^1/\mathbb{G}_m$, for $\mathbb{G}_m$ acting on $\mathbb{A}^1$ with weight 1.

En passant, we note that, for a generic functor $V : (\mathbb{Z},\leq)\to\mathrm{Mod}_k$, a *generalized filtered module*, $\ker x\subset\mathbb{V}$ gives the torsion subsheaf, which is supported at the origin. Torsion-free sheaves on the line are actually the flat sheaves.

In the flat case, we have $\mathbb{V}_0 = \mathbb{V}/x\mathbb{V} = \bigoplus_{n\in\mathbb{Z}} V^n/xV^{n-1} = \operatorname{gr} V^{\cdot}$, the associated graded of the filtered module $V^{\cdot}$, and,

$$\begin{aligned}\mathrm{Hom}_k(\mathbb{V}/(x-1)\mathbb{V},M) &= \{f:\bigoplus V^n\to M,\ \forall v\ f(xv-v)=0\}\\ &= \{(f_n:V^n\to M)_n,\ \forall n\forall v\in V^n\ f_{n+1}(xv)=f_n(v)\}\\ &= \mathrm{Hom}_k(\varinjlim V, M)\end{aligned} \tag{3.8}$$

so $\mathbb{V}_1 = \varinjlim V = \bigcup V^n$.

There is a k-linear morphism

$$\begin{aligned} \mathbb{V}_1 = \varinjlim V &\to \mathbb{V} \otimes_{k[x]} k[x, x^{-1}] \\ V^i \ni v &\mapsto v \otimes x^{-i} \end{aligned} \tag{3.9}$$

and it induces an isomorphism $\mathbb{V}_1 \otimes_k k[x, x^{-1}] \to \mathbb{V} \otimes_{k[x]} k[x, x^{-1}]$. On the free $k[x, x^{-1}]$-module on the left hand side, there is the standard flat connection $d(v \otimes 1) = 0$. This connection gets transported to

$$\begin{aligned} \nabla : \mathbb{V} \otimes_{k[x]} k[x, x^{-1}] &\to \mathbb{V} \otimes_{k[x]} k[x, x^{-1}]dx \\ V^i \otimes 1 \;\ni\; v \otimes 1 &\mapsto iv \otimes \frac{dx}{x} \end{aligned} \tag{3.10}$$

We conclude that a $\mathbb{G}_m$-equivariant sheaf on $\mathbb{A}^1$ (for an action on the line of weight 1) has a natural logarithmic connection with pole at the origin. The monodromy must be trivial, as can be checked. The original module $\mathbb{V}$ with its filtration can be read out of the space of flat sections and their order of vanishing in the origin.

Perverse sheaves If we ignore any issue about dimension, we could present equivariant sheaves on the affine line through the formalisms of D-modules and perverse sheaves. The interest is limited by the fact that the equivariance condition means the monodromy is trivial.

The algebra of differential operators on the affine line is $k\langle x, \partial\rangle$, $[\partial, x] = 1$.

- The module $k[x]$, $\partial x = 1$, is irreducible; it corresponds to the trivial local system $\underline{k}$.
- The module $k[x, x^{-1}]$, $\partial 1 = nx^{-1}$, for $n \in \mathbb{N}$, is not irreducible; $k[x]x^{-n}$ is a submodule isomorphic to the previous one. Since n is an integer, the monodromy is trivial. If $j : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ is the inclusion, we find an exact sequence of D-modules and correponding local systems

$$\begin{gathered} 0 \to k[x]x^{-n} \to k[x, x^{-1}] \to \frac{k[x, x^{-1}]}{k[x]x^{-n}} \to 0 \\ 0 \to \underline{k}_{\mathbb{A}^1} \to j_* \left(\underline{k}_{\mathbb{A}^1 \setminus \{0\}}\right) \to \underline{k}_{\{0\}} \to 0 \end{gathered} \tag{3.11}$$

A perverse sheaf on the line with respect to the fixed stratification $(\{0\}, \mathbb{A}^1 \setminus \{0\})$ is encoded by a diagram of vector spaces

$$\Phi \underset{u}{\overset{v}{\rightleftarrows}} \Psi \qquad\qquad T = 1 + uv \text{ is invertible} \tag{3.12}$$

The data corresponding to the exact sequence 3.11 is

$$\begin{array}{cccccc} \Phi : & 0 & \longrightarrow & k & \longrightarrow & k \\ & \uparrow\downarrow & & {\scriptstyle 0}\uparrow\downarrow{\scriptstyle 1} & & \uparrow\downarrow \\ \Psi : & k & \longrightarrow & k & \longrightarrow & 0 \end{array} \tag{3.13}$$

The (shifted) tangent complex to $\mathbb{A}^1/\mathbb{G}_m$ The action of $\mathbb{G}_m$ on $\mathbb{A}^1$ is described by $k[x] \to k[x, t^{\pm 1}]$, $x \mapsto tx$. In the description 3.7, it corresponds to the diagram $k \xrightarrow{\sim} k \xrightarrow{\sim} \dots$ starting at weight 0.

The action has an associated Lie algebroid $\mathcal{O}_{\mathbb{A}^1} \otimes \mathfrak{gl}_1 \to T_{\mathbb{A}^1}$. Let $\lambda = \partial/\partial t$ be a generator of $\mathfrak{gl}_1$, $\xi = \partial/\partial x$ for $T_{\mathbb{A}^1}$. The anchor correponds to the fundamental vector field $\lambda \mapsto x\xi$. The bracket is trivial on constant sections, since $\mathfrak{gl}_1$ is abelian, but is extended according to the Leibniz rule, $[f\lambda, g\lambda] = x(fg' - f'g)\lambda$.

The complex $\big[k[x]\lambda \to k[x]\xi\big]$ in degrees $[-1, 0]$, once considered with its natural $\mathbb{G}_m$-equivariant structure, is the tangent complex to $\mathbb{A}^1/\mathbb{G}_m$. This complex is indeed quasi-isomorphic to 0 on the open subset $\mathbb{A}^1 \setminus \{0\}$, while the closed point 0 has codimension 1 and one-dimensional stabilizer. The grading is: $\deg \lambda = 0$ because the adjoint action is trivial; $\deg \xi = -1$, so $\lambda \to x\xi$ is homogeneous.

The Chevalley-Eilenberg complex of the Lie algebroid is the analogous to the de Rham complex of a manifold; in this case it is

$$k[x] \to k[x]d\lambda \qquad f \mapsto xf'd\lambda \tag{3.14}$$

After the work of Kapranov, Markarian, and Hennion, we know that, for an algebraic derived stack X, locally of finite presentation over a field k of characteristic zero, the Atiyah class makes the shifted tangent complex $T_X[-1]$ a sheaf of homotopy Lie algebras, with a natural action on any quasichoherent sheaf of modules. Moreover, the fibres of $T_X[-1]$ govern the deformation theory of the points of X (some notions of derived deformation theory will be recalled in the next section).

The power series expansion of the anchor and the bracket of an action Lie algebroid

$$\begin{aligned} a &: \operatorname{Sym} T_{X,x} \otimes \mathfrak{g} \to T_{X,x} \\ b &: \operatorname{Sym} T_{X,x} \otimes \bigwedge^2 \mathfrak{g} \to \mathfrak{g} \end{aligned} \tag{3.15}$$

are exactly the maps needed to turn $(O_X \otimes \mathfrak{g} \to T_X)[-1] \otimes k(x)$ into a L_∞-algebra. The one we are interested in is the one associated to $\Theta = (\mathbb{A}^1/\mathbb{G}_m)^\wedge_0$, describing the formal moduli problem associated to the closed point of $\mathbb{A}^1/\mathbb{G}_m$. Call this algebra θ. In order to compute $\operatorname{Map}_{\mathrm{Ho(Lie)}}(\theta, L)$ for any dg Lie algebra L, we need a cofibrant model of θ, to be computed as in [Hin01, §2.2].

The composite morphism $\operatorname{Spec} k[[x]] \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is faithfully flat, because both maps are flat and the two points of $[\mathbb{A}^1/\mathbb{G}_m]$ are in the image [Hei18, lemma 1.5].

For any locally Noetherian quasi-geometric k-stack $\mathcal{M}$ in the sense of Lurie, $\mathcal{M}\big([\mathbb{A}^1/\mathbb{G}_m]\big) \cong \lim \mathcal{M}\big([\operatorname{Spec}(k[x]/x^n)/\mathbb{G}_m]\big)$ [BH17, remark 8.3]. Once we fix the image of $0 \in [\mathbb{A}^1/\mathbb{G}_m]$, the problem can be approached via deformation theory.

3.3 Derived deformation theory

We quickly review some results by Lurie [Lur]. Over a field k of characteristic 0, hereby fixed, and working consistently with ∞-categories and their dg models, his treatment of formal moduli problems of commutative algebras is based on the adjunction

$$\begin{gathered} C^* : \mathrm{Lie} \rightleftarrows (\mathrm{CAlg}^{\mathrm{aug}})^{\mathrm{op}} : \mathfrak{D} \\ \mathrm{Hom}_{\mathrm{CAlg}^{\mathrm{aug}}}(A, C^*(\mathfrak{g})) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{D}(A)) \end{gathered} \tag{3.16}$$

The functor C^* is the *cohomological Chevalley-Eilenberg complex* $\mathrm{Ext}^*_{\mathfrak{g}}(k,k)$. A representative can be constructed taking $\mathrm{Cone}(\mathfrak{g}) = \mathrm{Cone}(\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}) \simeq 0$, so $U(\mathrm{Cone}(\mathfrak{g})) \simeq k$; the map $\mathfrak{g} \to \mathrm{Cone}(\mathfrak{g})$ makes $U(\mathrm{Cone}(\mathfrak{g}))$ a $U(\mathfrak{g})$-resolution of k. Then

$$C^*(\mathfrak{g}) = \mathrm{Hom}_{U(\mathfrak{g})}\big(U(\mathrm{Cone}(\mathfrak{g})), k\big) \cong \widehat{\mathrm{Sym}}\big(\mathfrak{g}[1]\big)^\vee = \prod_{n \geq 0} \big(\bigwedge^n \mathfrak{g}\big)^\vee[-n] \tag{3.17}$$

with a differential that is a sum of the differential on $\mathfrak{g}$ and the dual of the Lie bracket.

More generally, for every $\mathfrak{g}$-represention V, the complex

$$C^*(\mathfrak{g}; V) = \mathrm{Hom}_{U(\mathfrak{g})}\big(U(\mathrm{Cone}(\mathfrak{g})), V\big) \tag{3.18}$$

is quasi-isomorphic to $\mathrm{Ext}^*_{U(\mathfrak{g})}(k, V)$ and there is a diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Spf}\, A) & \hookrightarrow & \mathrm{Rep}(\mathfrak{D}(A)) \\ & \searrow\!\!\nwarrow \qquad \swarrow_{C^*(\mathfrak{D}(A),\ \cdot\ )} & \\ & \mathrm{Mod}(C^*\mathfrak{D}(A)) & \end{array} \tag{3.19}$$

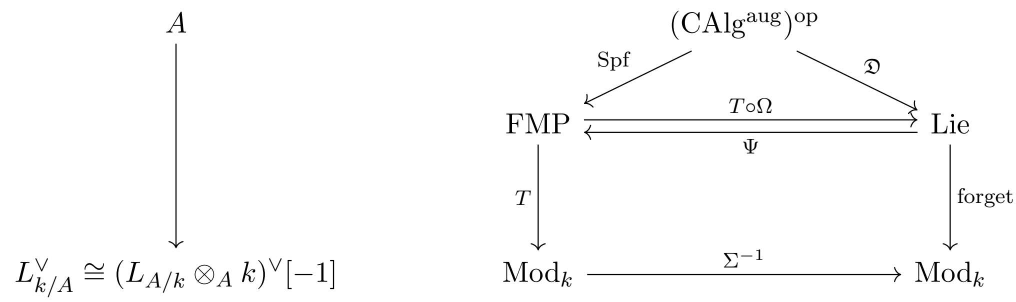
The functor $\mathfrak{D}$ of *Koszul duality* is defined by the adjunction. When A is local artinian, the unit $A \to C^*(\mathfrak{D}(A))$ is a quasi-isomorphism, and the functor $C^*(\mathfrak{D}(A);\ \cdot\ )$ gives an equivalence $\mathrm{Perf}(\mathrm{Rep}(\mathfrak{D}A)) \to \mathrm{Mod}(A)$.

On the other hand, for any dg Lie algebra $\mathfrak{g}$, when A is local artinian,

$$\Psi_{\mathfrak{g}} : A \mapsto \mathrm{Map}_{\mathrm{Lie}}(\mathfrak{D}(A), \mathfrak{g}) \tag{3.20}$$

is analogous to the space of Maurer-Cartan elements in $\mathfrak{m}_A \otimes \mathfrak{g}$, $\Psi_{\mathfrak{g}}$ is a (derived) formal moduli problem (*FMP*), and every (derived) formal moduli problem arises uniquely in this way. The converse equivalence is

$X \mapsto T_{\Omega X} \simeq \Sigma^{-1} T_X$.



A dg commutative artinian local k-algebra A can be presented as

$$A = A_n \to A_{n-1} \to \cdots \to A_0 = k \tag{3.21}$$

where each A_i is a square-zero extension of A_{i-1} by $k[n_i]$, $n \leq 0$, and

$$0 = \mathfrak{D}(A_0) \to \mathfrak{D}(A_1) \to \cdots \to \mathfrak{D}(A) \tag{3.22}$$

is obtained by routinely adjoining a cell in dimension $n_i + 1$.

For the basic example $A = k[t]/(t^{r+1})$ we get a semi-free Lie algebra (it is free as a graded Lie algebra)

$$\begin{gathered} \mathfrak{D}(A) = \mathrm{Free\,Lie}(k[x_1, \ldots, x_r]), \quad \deg x_i = 1 \\ dx_h + \frac{1}{2} \sum_{i+j=h} [x_i, x_j] = 0 \end{gathered} \tag{3.23}$$

In the limit, we get $\mathrm{Free\,Lie}(k[x_1, x_2, \ldots]) \xrightarrow{\sim} k[-1]$ sending all generators but x_1 to 0. $k[-1]$ is the abelian Lie algebra concentrated in degree 1 and it is Koszul dual to the formal line $k[[t]]$.

$$\begin{gathered} \pi_0\big(\Psi_{\mathfrak{g}}(k[t]/(t^2))\big) = \pi_0\big(\mathrm{Map}_{\mathrm{Lie}}(\mathrm{Free\,Lie}(k[-1]), \mathfrak{g})\big) = H^1(\mathfrak{g}) \\ \pi_0\big(\Psi_{\mathfrak{g}}(k[[t]])\big) = \pi_0\big(\mathrm{Map}_{\mathrm{Lie}}(k[-1], \mathfrak{g})\big) = \mathrm{MC}(t\mathfrak{g}[[t]])/\mathrm{gauge} \end{gathered} \tag{3.24}$$

3.4 Jump deformations

Consider a complete dg Lie algebra L over the complex numbers (we use cohomological conventions). A Maurer-Cartan element is a solution $x \in L^1$ to the equation

$$dx + \frac{1}{2}[x, x] = 0 \tag{3.25}$$

so that $d_x = d + \mathrm{ad}_x$ is a new differential on L.

The sub Lie algebra L^0 acts on L^1 via the vector fields ($a \in L^0$, $x \in L^1$)

$$\hat{a}_x = [a, x] - da \tag{3.26}$$

In unit time, x flows along a to

$$a\mathcal{G}x := e^{\mathrm{ad}_a}x - \frac{e^{\mathrm{ad}_a}-1}{\mathrm{ad}_a}da \tag{3.27}$$

Writing $y = a\mathcal{G}x$, the equation above is equivalent to (B_n are the Bernoulli numbers)

$$\begin{aligned} da &= \frac{-a}{e^{-a}-1}x - \frac{a}{e^{a}-1}y \\ &= \frac{1}{2}[a, x+y] + \sum_{n\geq 0}\frac{B_{2n}}{(2n)!}\,\mathrm{ad}_a^{2n}(x-y) \end{aligned} \tag{3.28}$$

The gauge action preserves Maurer-Cartan elements; if $y = a\mathcal{G}x$, then [Bui+18, proposition 1.5]

$$e^a : (L, d_x) \to (L, d_y) \text{ is an isomorphism.}$$

Jump deformations Let $L = \ell \otimes \mathfrak{m}_{\mathbb{C}[[t]]}$ a Lie algebra of formal power series, and let $x(t) = tx_1 + t^2x_2 + \ldots$ be a Maurer-Cartan element. Its first derivative is a cycle in (L, d_x):

$$dx' + [x, x'] = 0 \tag{3.29}$$

If x_r is the first non-zero coefficient of $x(t)$ (the *infinitesimal* of the deformation $x(t)$), x_r is a 1-cycle in ℓ and x'/r is an extension/lift of x_r to a cycle of L.

The cohomology of L is a $k[[t]]$-module, and we can ask whether the class of x' is t-torsion, i.e., whether there exists $a \in L^0$ such that, for some $m \geq 0$

$$t^m x' = -d_x a \tag{3.30}$$

According to Griffiths [Gri65], we would say that the class of x_r above is a *jump class*; it is easy to prove, and it is shown in both *loc. cit.* and [Ger74] that jump classes are the obstruction classes.

The reader will note that $-d_x a = -da - [x, a]$ is the right-hand side of eq. 3.26. Interpreting both sides of the equation above as vector fields for the evolution of x in time-variable u, and exponentiating, we find

$$e^{ut^m \frac{\partial}{\partial t}}x = (ua)\mathcal{G}x \tag{3.31}$$

For low m we can make more explicit the expression on the left. For $m = 0$,

$$\sum_{k\geq 0}\frac{u^k}{k!}x^{(k)}(t) = x(t+u) \tag{3.32}$$

and we find that, for every t, $x(t)$ is isomorphic to $x(0)$, so the deformation is trivial, as it was expected, since its infinitesimal is a coboundary.

For $m = 1$, we find

$$\sum_{k\geq 0} \frac{u^k}{k!} \left(t\frac{\partial}{\partial t}\right)^k x(t) = \sum_{n,k\geq 0} \frac{u^k}{k!} n^k t^n x_n = \sum_{n\geq 0} e^{un} t^n x_n = x(e^u t) \qquad (3.33)$$

In this case, we find that $x(t_1) \cong x(t_2)$ for $t_1 t_2 \neq 0$, as in the definition of *jump deformation* according to Gerstenhaber [Ger74].

Remark While I used the field of complex numbers to make sense of the expression $x(e^u t)$, Gerstenhaber considers as a base ring any $\mathbb{Q}$ or $\mathbb{F}_p$-commutative algebra, and looks at deformations of the form $x(t + ut)$. But, $t \mapsto t(1 + u)$ is not a one-parameter group action, so the necessary gauge transformation cannot be the exponential of a single $\hat{a}$.
The condition *"x' is a torsion class"* is equivalent to *"$x(t)$ is generically trivial"* (i.e., after passage to the field of quotients $\mathbb{C}((t))$), but it is not clear whether they are equivalent to *"$x(t) \cong x(f(u, t))$ for some f non constant in u"*.

It is clear that is not possible to reconstruct a deformation x from an automorphism a that becomes obstructed. Think of ℓ as the Chevalley-Eilenberg deformation complex of a Lie algebra $\mathfrak{g}$. The scalar action of $\mathbb{C}^\times$ on a subspace $V \subset \mathfrak{g}$ is a one-parameter group of Lie algebra automorphisms if and only if the Lie bracket vanishes on $V \wedge V$. If $\mathfrak{g} = A \oplus V$ as a vector space, with A a subalgebra, then there is a degeneration/contraction of $\mathfrak{g}$ to $\mathfrak{g}_0 = A \oplus V$ where V is an abelian subalgebra, independently of its original bracket.

Rational homotopy theory Maurer-Cartan elements in a complete dg Lie algebra L are points for some geometric realization $|L|$ with homotopy groups $\pi_m(|L|, x) = H^{1-m}(L, d_x)$.

We can introduce $L' = L \oplus \mathbb{C} \cdot D$ where $D = t^m \frac{\partial}{\partial t}$ sits in degree 0, and $[d, D] = 0$, $[D, z] = Dz$.

$$H^0(L', d_x) = H^0(L, d_x) \oplus \mathbb{C} \cdot \{a \in L^0 \colon t^m x' = [a, x] - da\} \qquad (3.34)$$

A computation Let us look at the L_∞-algebra

$$\mathfrak{g} = \left[k\lambda \to k\xi\right] \qquad \text{degrees} = \{0, 1\} \qquad (3.35)$$

The differential must be 0, otherwise the complex would be quasi isomorphic to 0. Assume $\mathfrak{g}$ is actually a dg Lie algebra: the only ambiguity is $[\lambda, \xi] = c\xi$; rescaling λ, it is a matter of $c = 0$ or $c \neq 0$.

Let $S = \mathrm{Sym}\left(\mathfrak{g}[1]\right) \cong \left[k[x]y \oplus k[x]\right]$ be a free graded cocommutative coalgebras in degrees $\{-1, 0\}$. Its differential is $x^n y \mapsto cnx^n$ (up to a sign). If we were rather computing the Chevalley-Eilenberg algebra

$$\widehat{C}(\mathfrak{g}) = S^\vee = \widehat{\mathrm{Sym}}\ \mathfrak{g}^\vee[-1] \cong \left[k[[t]] \to k[[t]]u\right] \tag{3.36}$$

the differential would be $t^n \mapsto cnt^n u$ (up to a sign).

Then, for a dg Lie algebra L,

$$\mathrm{Map}_{\mathrm{Ho}(Lie)}\left(\mathfrak{g}, L\right) \cong \mathrm{MC}\left(\mathrm{Hom}\left(S^+, L\right)\right)/gauge \cong MC\left(L \otimes \mathfrak{m}_{\widehat{C}}\right)/gauge \tag{3.37}$$

(in the middle, there is the convolution Lie algebra $[f, g] := [\ ,\ ] \circ f \otimes g \circ \Delta$). At the end of the day, solutions are given by power series

$$\begin{aligned} &B = \sum_{i>0} b_i t^i, \quad A = \sum_{i\geq 0} a_i t^i, \qquad b_i \in L^1,\ a_i \in L^0 \\ &\begin{cases} dB + \frac{1}{2}[B, B] = 0 \\ dA + [B, A] = ctB' \end{cases} \end{aligned} \tag{3.38}$$

In the case $c = 0$, $(d + \mathrm{ad}_B)A = 0$ means A is a formal automorphism.
In the case $c = -1$, $tB' = -(d + \mathrm{ad}_B)A$ is eq. 3.30 for $m = 1$.

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