Scuola Internazionale Superiore di Studi Avanzati - Trieste

Doctoral Thesis

$T\bar{T}$ Deformations of Quantum Field Theory

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Abstract

Recent work by Zamolodchikov and others has uncovered a kind of "solvable" irrelevant deformation of two dimensional QFT generated by the composite operator $T\bar{T}$. The irrelevant composite operator $T\bar{T}$ is built with the components of the energy-momentum tensor. The $T\bar{T}$ deformation can be regarded as a peculiar kind of integrable perturbation of integrable quantum field theories (IQFT). In the holographic dual, this deformation represents a geometric cutoff of the asymptotic region of AdS. The QFT is placed on a Dirichlet boundary at finite radial distance in the bulk. In this thesis we consider the problem of exact integration of the $T\bar{T}$ deformation of two dimensional quantum field theories, as well as some higher dimensional extensions in the form of $T\bar{T}$ deformations. When the action can be shown to only depend algebraically on the background metric the solution of the deformation equation on the Lagrangian can be given in closed form in terms of solutions of the (extended) Burgers' equation. We present such examples in two and higher dimensions.

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List of publications

This thesis is based on the following publication:

1. G. Bonelli, N. Doroud, and M. Zhu, “$T\bar{T}$-deformations in closed form,” *JHEP* **06** (2018) 149, arXiv:1804.10967 [hep-th]

Other publications during the PhD study are listed below:

2. N. Bai, H.-H. Chen, S. He, J.-B. Wu, W.-L. Yang, and M.-Q. Zhu, “Integrable Open Spin Chains from Flavored ABJM Theory,” *JHEP* **08** (2017) 001, arXiv:1704.05807 [hep-th]

3. B. Chen, J.-B. Wu, and M.-Q. Zhu, “Holographical Description of BPS Wilson Loops in Flavored ABJM Theory,” *JHEP* **12** (2014) 143, arXiv:1410.2311 [hep-th]

Contents

Chapter 1

Introduction

Exact quantification of how Quantum Field Theories react as we vary the coupling constants or dynamical scales is a crucial issue in modern theoretical physics [1, 2]. Cases where such deformations can be integrated exactly and in closed form are extremely rare and often enjoy supersymmetry. In two dimensions there exist examples of non-supersymmetric interacting QFT theories which are integrable and whose renormalisation group flow can be determined explicitly [3].

A generic QFT admits deformations by operators which instigate a flow as we probe the dynamics at different scales. Deformations which drive the flow at lower energies are considered *relevant* whilst deformations that dominate the flow as we probe the dynamics at higher energy scales are considered *irrelevant*. The latter flow is much harder to study as it generally involves reintroducing the high energy degrees of freedom that have been integrated out. Nonetheless there are examples for which the flow can be determined, notably the deformation of any local relativistic QFT in two spacetime dimensions by the irrelevant $T\bar{T}$ operator.

The composite operator $T\bar{T}$ was first studied by Zamolodchikov in [4]. It is constructed from the energy-momentum tensor for a general $D = 2$ QFT. It became a frequent actor in subsequent studies focused on integrable quantum field theory (IQFT) [5–8]. The $T\bar{T}$ deformation is a special case of a more general class of irrelevant integrable deformations of IQFT introducted by Smirnov and Zamolodchikov in [9]. The $T\bar{T}$ deformation is special in that the deformed theories are solvable in a certain sense, even when the original theory is not integrable. Various generalisations of the $T\bar{T}$ deformations of quantum field theories, mostly studied from the viewpoint of the partition functions, were proposed in [10–16]. Applications to holography were studied in [17–20, 22, 23, 26]. The entanglement entropy in $T\bar{T}$ deformed CFT in studied in [25]. Some implications of this irrelevant deformation for the UV theory were considered in [27–29], and in [30] a hydrodynamical approach was considered. A similar Lorentz-breaking deformation of CFT generated by the operator $J\bar{T}$ was proposed in [31] and further studied in [24, 32, 33], where J is a conserved $U(1)$ current.

In this thesis we begin with a review of the background of $T\bar{T}$ deformation and the applications to holography. Then we focus on studying the flow equation for the $T\bar{T}$ deformed QFTs and its extensions to higher dimensions, both for conformal and massive theories.

In Chapter 2, we review the background of $T\bar{T}$ deformation following the work of Smirnov and Zamolodchikov [4, 9] and the paper by Kraus, Liu and Marolf [35]. We first review the construction of the composite $T\bar{T}$ operator in two dimensional spacetime. Let us denote the components of the energy-momentum tensor as $T = -(2\pi)\,T_{zz}$, $\bar{T} = -(2\pi)\,T_{\bar{z}\bar{z}}$ and $\Theta = (2\pi)\,T_{z\bar{z}}$. In the limit $z \to z'$, the operators $T(z)\,\bar{T}(z')$ and $\Theta(z)\,\Theta(z')$ are both divergent but their non-derivative divergent parts cancel each other. Thus the combination $T\bar{T} := T(z)\,\bar{T}(z') - \Theta(z)\,\Theta(z')$ is well defined in the limit $z \to z'$ up to total derivative terms. If the theory is put on a cylinder, the expectation value of the $T\bar{T}$ operator can be computed. The key relation here is the decomposition relation

$$\left\langle n\left|T\bar{T}\right|n\right\rangle = \left\langle n\left|T(z)\right|n\right\rangle \left\langle n\left|\bar{T}(z)\right|n\right\rangle - \left\langle n\left|\Theta(z)\right|n\right\rangle \left\langle n\left|\Theta(z')\right|n\right\rangle. \tag{1.1}$$

The expectation values on the right hand side can all be computed by the definition of the energy-momentum tensor. The $T\bar{T}$ operator turns out to be a special case of a general class of local operators X_s, which are constructed from local conserved currents $(T_{s+1}(z), \Theta_{s-1}(z))$ by $T_{s+1}(z)\,\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\,\bar{\Theta}_{s-1}(z')$. The local currents satisfy the continuity equations $\partial_{\bar{z}}T_{s+1}(z) = \partial_z\Theta_{s-1}(z)$ and local integral of motions (IM) are constructed from the currents

$$P_s = \frac{1}{2\pi}\int_C T_{s+1}(z)\,dz + \Theta_{s-1}(z)\,d\bar{z} \tag{1.2}$$
$$\bar{P}_s = \frac{1}{2\pi}\int_C \bar{T}_{s+1}(z)\,dz + \bar{\Theta}_{s-1}(z)\,d\bar{z}. \tag{1.3}$$

Integrable quantum field theories (IQFT) are characterized by an infinite number of conserved currents and local IM's. Each X_s generates a deformation of the IQFT by

$$S_{QFT} \to S_{QFT} + \sum_s t_s \int d^2x X_s. \tag{1.4}$$

Smirnov and Zamolodchikov[9] proved the above deformation preserves integrability. The idea is that if under the deformation generated by X_s the currents are also deformed as

$$T_{\sigma+1} \to T_{\sigma+1} + \delta g_s \hat{T}_{\sigma+1,s} \tag{1.5}$$
$$\Theta_{\sigma-1} \to \Theta_{\sigma-1} + \delta g_s \hat{\Theta}_{\sigma-1,s}, \tag{1.6}$$

where $\hat{T}_{\sigma+1,s}$ and $\hat{\Theta}_{\sigma-1,s}$ are some local fields with spins $\sigma+1$ and $\sigma-1$, then the local IM P_σ are still conserved and commute with each other. Thus the deformed theory is still integrable. The $X_1 = T\bar{T}$ operator is special in that even if the original theory is not integrable, the deformation generated is in some sense solvable. Since the operator X_s is irrelevant, the deformed theory is not necessarily UV complete. So it should be understood in the effective field theory sense. The UV behaviour of the deformed theories has been studied by Dubovsky et al.[7, 29, 34]. The deformation generated by $T\bar{T}$ can also be defined by the following differential equation

$$\partial_t \mathcal{L}\left(z, \bar{z}, t\right) = \frac{1}{\pi^2} T\bar{T}\left(z, \bar{z}, t\right). \tag{1.7}$$

Put the theory on a cylinder with circumference R. Following the definition of the deformation one finds that the energy levels $E_n = E_n\left(R, t\right)$ of stationary state $|n\rangle$ satisfy the equation

$$\frac{\partial E_n}{\partial t} = \frac{R}{\pi^2}\left\langle n\left|T\bar{T}\right|n\right\rangle. \tag{1.8}$$

Finite-size energy levels are computed based on equation (1.8). Specially if the original theory is a CFT, we have

$$E_t\left(R\right) = F_t R + \frac{R}{2\tilde{t}}\left(1 - \sqrt{1 + \frac{4\tilde{t}C_n}{R^2}}\right), \tag{1.9}$$

where $\tilde{t} = t\left(1 + tF_0\right)$, F_0 is the bulk vacuum energy density. Deformations of IQFT that preserve integrability must also generate integrable deformations of the S-matrix. The space of infinitesimal deformations of the S-matrix contains a finite-dimensional part related to the deformations of solutions of the Yang-Baxter equation and an infinite-dimensional part of deformations of the CDD factor. The two particle S-matrix of the $T\bar{T}$-deformed CFT takes the form

$$S = \exp\left(-i\frac{t}{4}p_1^+ p_2^-\right). \tag{1.10}$$

The $T\bar{T}$-deformed CFT with $t > 0$ has a nice holographic dual description which is AdS_3 with a finite radial cutoff. On the bulk side the boundary lies not at asymptotic infinity but instead at a finite radius. The ability to move the boundary inward is interesting. This could shed some light on the important question of the emergence of bulk locality. The $T\bar{T}$ deformed CFT action is defined by the equation $\frac{dS(t)}{dt} = \int d^2x\sqrt{g}T\bar{T}\left(x\right)$. Assuming that the undeformed theory is a CFT, we can equivalently say that the trace of the deformed stress tensor obeys (up to derivatives of local operators)

$$T_i^i = -4\pi t T\bar{T}. \tag{1.11}$$

For general P_n the finite-size energy is

$$E_n = -\frac{R}{2\pi^2 t}\left(\sqrt{1-\frac{4\pi^2 t}{R}E_{n,0}+\left(\frac{2\pi^2 t}{R}P_n\right)^2}-1\right). \tag{1.12}$$

The energy level can be reproduced in pure AdS$_3$ gravity by considering the quasilocal energy [36, 37] defined on a surface at finite radial location r. The expression appearing under the square root above indeed exhibits similarities to the function appearing in the standard form of the BTZ solution. The quasilocal energy is given as $E = \frac{1}{2\pi}\int d\phi\sqrt{g_{\phi\phi}}u^i u^j T_{ij}$, where u^i is the timelike unit normal to the integration surface, and T_{ij} is the usual boundary stress tensor [36, 38]

$$T_{ij} = \frac{1}{4G}\left(K_{ij} - K g_{ij} + \frac{1}{l}g_{ij}\right) \tag{1.13}$$

where g_{ij} is the boundary metric, K_{ij} is the extrinsic curvature, and l is the AdS scale. Evaluated in BTZ on a surface of fixed r, the quasilocal energy turns out to match the CFT result under the identification $t = \frac{4Gl}{\pi}$, $R = 2\pi r$. As for propagation speeds, if one considers a QFT state in the deformed theory with constant $\langle T_{++}\rangle$ and $\langle T_{--}\rangle$, then small perturbations of the stress tensor can be shown to propagate at speeds

$$v_+ = 1 + 2\pi t\,\langle T_{++}\rangle + \mathcal{O}\left(t^2\right) \tag{1.14}$$

$$v_- = 1 + 2\pi t\,\langle T_{--}\rangle + \mathcal{O}\left(t^2\right). \tag{1.15}$$

The same propagation speeds arise in pure AdS$_3$ gravity by considering perturbations that preserve Dirichlet boundary conditions on the cutoff surface. If we use coordinates such that $ds^2 = d\rho^2 + g_{ij}(x,\rho)\,dx^i dx^j$ with a cutoff surface at fixed ρ, then the $\rho\rho$ component of the Einstein equations is

$$-\frac{1}{2}R^{(2)} + \frac{1}{2}\left[K^2 - K^{ij}K_{ij}\right] - 1 = 0. \tag{1.16}$$

Applying this equation to the stress tensor we get the trace relation (1.11) under the identification $t = \frac{4G}{\pi}$.

In Chapter 3 we propose a simple integration technique for the $T\bar{T}$ flow equation. More specifically, the flow equation induced by the $T\bar{T}$-deformation can be reformulated as a functional equation. Under certain conditions the functional equation reduces to a simple PDE and can be solved exactly. For a QFT with the partition function

$$\mathcal{Z}_\circ[g_{\mu\nu},\lambda] = \int[\mathcal{D}\Phi]\,e^{-S_\circ}, \tag{1.17}$$

where the action is

$$S(t) = \int \mathrm{d}^2x\,\sqrt{g}\,\mathcal{L}(t), \tag{1.18}$$

We assume that the deformed partition function $\mathcal{Z}_t$ stems from a local action $S(t)$. We can define the $T\bar{T}$ flow equation as

$$\left(\frac{\partial}{\partial t} + \Delta\right) \mathcal{Z}_t = 0\,, \tag{1.19}$$

where the functional operator Δ above is defined as

$$\Delta = \lim_{\delta\to 0} \int_M \mathrm{d}^2x \frac{2}{\sqrt{g}} \epsilon^{\mu\nu}\epsilon^{\rho\sigma} \frac{\delta}{\delta g^{\mu\rho}(x+\delta)} \frac{\delta}{\delta g^{\nu\sigma}(x-\delta)}\,. \tag{1.20}$$

The initial condition is $\mathcal{Z}_{t=0} = \mathcal{Z}_\circ$. Since the deformation operator Δ does not generate terms involving derivatives of the metric unless such terms are already present in the undeformed Lagrangian, we can then rewrite the flow equation in local form as

$$\partial_t \mathcal{L} = \mathcal{O}_{T\bar{T}}\,, \tag{1.21}$$

where

$$\mathcal{O}_{T\bar{T}} = \mathcal{L}^2 - 2\mathcal{L} g^{\mu\nu} \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + 2\varepsilon^{\mu\nu}\varepsilon^{\rho\sigma} \frac{\partial\mathcal{L}}{\partial g^{\mu\rho}} \frac{\partial\mathcal{L}}{\partial g^{\nu\sigma}}\,. \tag{1.22}$$

Notice we can always absorb a constant factor to the definition of the parameter t. We will provide explicit solutions in closed form for the flow equation in many examples, notably non-linear σ-models and the massive Thirring model.

Chapter 2

Basics of the $T\bar{T}$ deformation

In this Chapter we review the construction of the composite operator $T\bar{T}$ and the deformation of QFT generated by $T\bar{T}$. We closely follow the analysis in the paper of Smirnov and Zamolodchikov [9] and the paper of Kraus, Liu and Marolf [35].

2.1 The composite operator $T\bar{T}$

In two dimensional quantum field theory, the composite operator $T\bar{T}$ is built from the chiral components $T, \bar{T}$ of the energy-momentum tensor $T_{\mu\nu}$. Point z of the flat $2D$ space with Euclidean signature can be labeled by Cartesian coordinates (x, y). It is more convenient to use complex coordinates $(z, \bar{z}) = (x + iy, x - iy)$. With the CFT convention[39], chiral components of the energy-momentum tensor are defined as follows

$$T = -\left(2\pi\right) T_{zz}, \quad \bar{T} = -\left(2\pi\right) T_{\bar{z}\bar{z}}, \quad \Theta = \left(2\pi\right) T_{z\bar{z}}. \tag{2.1}$$

The expectation value of composite operator $T\bar{T}$ satisfies an important relation

$$\left\langle T\bar{T}\right\rangle = \left\langle T\right\rangle \left\langle \bar{T}\right\rangle - \left\langle \Theta\right\rangle^2. \tag{2.2}$$

This property was first proved by Zamolodchikov [4] and holds for any generic two dimensional quantum field theory with translational symmetry. The proof relies on the conservation equation

$$\partial_{\bar{z}} T\left(z\right) = \partial_z \Theta\left(z\right) \tag{2.3}$$
$$\partial_z \bar{T}\left(z\right) = \partial_{\bar{z}} \Theta\left(z\right). \tag{2.4}$$

Now we consider the difference of two-point functions

$$\Xi\left(z, z'\right) \equiv \left\langle T\left(z\right) \bar{T}\left(z'\right)\right\rangle - \left\langle \Theta\left(z\right) \Theta\left(z'\right)\right\rangle. \tag{2.5}$$

In the limit $z \to z'$ this function $\Xi(z, z')$ formally reduces to the expectation value of the composite operator $T\bar{T}$. One expects that the composite operator $T\bar{T}$ can be obtained from the product $T(z)\bar{T}(z')$ by bringing the points z and z' together. But the operator product expansion of $T(z)\bar{T}(z')$ contains singular terms. As we will see the term $\Theta(z)\Theta(z')$ exactly subtracts the singular terms, such that the limit $z \to z'$ is up to total derivative terms, well defined. Taking $\partial_{\bar{z}}$ derivative of $\Xi(z, z')$ and using the conservation equation we find

$$\left\langle \partial_{\bar{z}} T(z)\bar{T}(z')\right\rangle - \left\langle \partial_{\bar{z}}\Theta(z)\Theta(z')\right\rangle = -\left\langle \Theta(z)\partial_{z'}\bar{T}(z')\right\rangle + \left\langle \Theta(z)\partial_{\bar{z}'}\Theta(z')\right\rangle = 0 \tag{2.6}$$

where we have used the fact that two-point correlation functions depend only on the separations $z - z'$. This shows $\partial_{\bar{z}}\Xi(z, z') = 0$. In the similar way one derives that $\partial_z\Xi(z, z') = 0$. Hence the function $\Xi(z, z')$ is a constant. Now we consider the limit where z and z' are infinitely separated. In this limit $\Xi(z, z')$ becomes $\langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle^2$. This proves equation (2.2). To get more insight we continue to study the combination of the operator products $T(z)\bar{T}(z') - \Theta(z)\Theta(z')$ itself instead of two-point functions. It is assumed that the short-distance behaviour of the field theory is governed by a conformal field theory that certain no-resonance condition is satisfied[4]. We assume the operator product expansions (OPE) as follows

$$\Theta(z)\bar{T}(z') = \sum_i B_i(z - z')\mathcal{O}_i(z') \tag{2.7}$$

$$T(z)\Theta(z') = \sum_i A_i(z - z')\mathcal{O}_i(z') \tag{2.8}$$

and

$$T(z)\bar{T}(z') = \sum_i D_i(z - z')\mathcal{O}_i(z') \tag{2.9}$$

$$\Theta(z)\Theta(z') = \sum_i C_i(z - z')\mathcal{O}_i(z'). \tag{2.10}$$

Using the conservation equation

$$\partial_{\bar{z}}\left(T(z)\bar{T}(z') - \Theta(z)\Theta(z')\right) = (\partial_z + \partial_{z'})\Theta(z)\bar{T}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})\Theta(z)\Theta(z'), \tag{2.11}$$

when $(\partial_z + \partial_{z'})$ acts on a function of $z - z'$ it gives 0. Using the operator product expansions, the above equation reads

$$\sum_i \partial_{\bar{z}} F_i(z - z')\mathcal{O}_i(z') = \sum_i \left(B_i(z - z')\partial_{z'}\mathcal{O}_i(z') - C_i(z - z')\partial_{\bar{z}'}\mathcal{O}_i(z')\right) \tag{2.12}$$

where

$$F_i(z - z') = D_i(z - z') - C_i(z - z'). \tag{2.13}$$

Similarly we obtain

$$\partial_z\left(T(z)\,\bar{T}(z')-\Theta(z)\,\Theta(z')\right)=(\partial_z+\partial_{z'})\,T(z)\,\bar{T}(z')-(\partial_{\bar{z}}+\partial_{\bar{z}'})\,T(z)\,\Theta(z') \tag{2.14}$$

with the operator product expansions, it reads

$$\sum_i \partial_z F_i(z-z')\,\mathcal{O}_i(z')=\sum_i\left(D_i(z-z')\,\partial_{z'}\mathcal{O}_i(z')-A_i(z-z')\,\partial_{\bar{z}'}\mathcal{O}_i(z')\right). \tag{2.15}$$

The right-hand sides of equations (2.12) and (2.15) involve only derivatives of local fields. This implies that in the expansion

$$T(z)\,\bar{T}(z')-\Theta(z)\,\Theta(z')=\sum_i F_i(z-z')\,\mathcal{O}_i(z') \tag{2.16}$$

if $\mathcal{O}_i(z')$ is not a coordinate derivative of another local operator, F_i will be a constant (coordinate-independent). In other words

$$T(z)\,\bar{T}(z')-\Theta(z)\,\Theta(z')=\mathcal{O}_{T\bar{T}}(z')+\text{derivative terms} \tag{2.17}$$

where $\mathcal{O}_{T\bar{T}}(z')$ is some local operator. This can be viewed as a formal definition of the composite operator $T\bar{T}$

$$T\bar{T}(z):=\mathcal{O}_{T\bar{T}}(z). \tag{2.18}$$

One defines $\mathcal{O}_{T\bar{T}}(z')$ modulo derivative terms. Derivative terms do not contribute to the left-hand side of equation (2.2).

2.2 Expectation value

Consider a QFT on a cylinder with Euclidean coordinates $(x,y)\sim(x+R,y)$. The coordinate y along the cylinder is taken as the Euclidean time. Field theory states are defined on the circle of circumference R and are evolving by the Hamiltonian H. Let P denote the momentum operator and consider a non-degenerate eigenstate $|n\rangle$ of the Hamiltonian and momentum operators,

$$H|n\rangle=E_n|n\rangle\,,\,P|n\rangle=P_n|n\rangle\,. \tag{2.19}$$

We assume the normalization $\langle n|\,n\rangle=1$. It is not difficult to show that the relation (2.2) is still valid if we replace $\langle...\rangle$ by $\langle n|...|\,n\rangle$.

$$\Xi_n(z,z')\equiv\left\langle n\left|T(z)\,\bar{T}(z')\right|n\right\rangle-\langle n|\Theta(z)\,\Theta(z')|\,n\rangle\,. \tag{2.20}$$

Similar to previous discussion one can show

$$\partial_z \Xi_n (z, z') = 0 \tag{2.21}$$
$$\partial_{\bar{z}} \Xi_n (z, z') = 0 \tag{2.22}$$

i.e. $\Xi_n (z, z')$ is independent of z and z' hence it is a constant. But the asymptotic factorization no longer holds. We write down the spectral decompositions of two point functions

$$\left\langle n \left| T(z) \bar{T}(z') \right| n \right\rangle = \sum_{n'} \left\langle n \left| T(z) \right| n' \right\rangle \left\langle n' \left| \bar{T}(z) \right| n \right\rangle \times e^{(E_n - E_{n'})|y-y'| + i(P_n - P_{n'})|x-x'|}. \tag{2.23}$$

Here (x, y) and (x', y') are Cartesian coordinates of the points z and z'. The exponential term appears because we translate the $\bar{T}$ operator from the point z' to z. Here only intermediate states $|n'\rangle$ with $E_{n'} < E_n$ would contribute. Since $\langle n |T(z)| n\rangle$ and $\left\langle n \left|\bar{T}(z)\right| n\right\rangle$ are constants the coordinate dependence is in the exponential term. $\langle n |\Theta(z) \Theta(z')| n\rangle$ would decomposite in the same way

$$\langle n |\Theta(z) \Theta(z')| n\rangle = \sum_{n'} \langle n |\Theta(z)| n'\rangle \langle n' |\Theta(z)| n\rangle \times e^{(E_n - E_{n'})|y-y'| + i(P_n - P_{n'})|x-x'|}. \tag{2.24}$$

In order for $\Xi_n (z, z')$ to be a constant, coordinate dependent terms in the decomposition of $\left\langle n \left|T(z) \bar{T}(z')\right| n\right\rangle$ and $\langle n |\Theta(z) \Theta(z')| n\rangle$ must cancel out. This means that only the term $n' = n$ will remain

$$\Xi_n = \langle n |T(z)| n\rangle \left\langle n \left|\bar{T}(z)\right| n\right\rangle - \langle n |\Theta(z)| n\rangle \langle n |\Theta(z')| n\rangle . \tag{2.25}$$

In the limit $z' \to z$, $T(z) \bar{T}(z') - \Theta(z) \Theta(z')$ becomes $T\bar{T}$ up to derivative terms, so

$$\Xi_n = \left\langle n \left|T\bar{T}\right| n\right\rangle , \tag{2.26}$$

and we arrive at the desired relation

$$\left\langle n \left|T\bar{T}\right| n\right\rangle = \langle n |T(z)| n\rangle \left\langle n \left|\bar{T}(z)\right| n\right\rangle - \langle n |\Theta(z)| n\rangle \langle n |\Theta(z')| n\rangle . \tag{2.27}$$

The energy-momentum tensor components in Euclidean and complex coordinates have the following relations

$$T_{xx} = -\frac{1}{2\pi} \left(\bar{T} + T - 2\Theta\right) \tag{2.28}$$
$$T_{yy} = \frac{1}{2\pi} \left(\bar{T} + T + 2\Theta\right) \tag{2.29}$$
$$T_{xy} = \frac{i}{2\pi} \left(\bar{T} - T\right) . \tag{2.30}$$

We can invert these relations and rewrite equation (2.27) as

$$\left\langle n\left|T\bar{T}\right|n\right\rangle = -\pi^2\left(\left\langle n\left|T_{yy}\right|n\right\rangle\left\langle n\left|T_{xx}\right|n\right\rangle - \left\langle n\left|T_{xy}\right|n\right\rangle\left\langle n\left|T_{xy}\right|n\right\rangle\right). \tag{2.31}$$

Since the y axis is the direction of time, by definition T_{yy} is the energy density and T_{xy} the momentum density

$$\left\langle n\left|T_{yy}\right|n\right\rangle = -\frac{1}{R}E_n(R) \tag{2.32}$$

$$\left\langle n\left|T_{xy}\right|n\right\rangle = -\frac{i}{R}P_n(R) \tag{2.33}$$

where

$$P_n(R) = \frac{2\pi l_n}{R} \tag{2.34}$$

is the corresponding spatial momenta of the states. By momentum quantization condition l_n is an integer. Now let us consider $\left\langle n\left|T_{xx}\right|n\right\rangle$. This can be computed via

$$\left\langle n\left|T_{xx}\right|n\right\rangle = \frac{\delta}{\delta g^{xx}}Z \tag{2.35}$$

with correct boundary conditions. Here Z is the partition function

$$Z = \sum_m e^{-\beta E_m(R)+\mu q(m)} \tag{2.36}$$

where $\mu = \mu(m)$ is the chemical potential. Varying g^{xx} has the same effect as varying R, thus using

$$\frac{d}{dR}Z = -\beta\sum_m \frac{dE_m(R)}{dR}e^{-\beta E_m(R)+\mu q(m)}. \tag{2.37}$$

We have

$$\left\langle n\left|T_{xx}\right|n\right\rangle = -\frac{d}{dR}E_n(R). \tag{2.38}$$

With these relations the expectation value (2.31) can be expressed in terms of $E_n(R), P_n(R)$

$$\left\langle n\left|T\bar{T}\right|n\right\rangle = -\frac{\pi^2}{R}\left(E_n(R)\frac{d}{dR}E_n(R) + \frac{1}{R}P_n^2(R)\right). \tag{2.39}$$

2.3 Local fields X_s and local IM

The previous results can be generalized in Integrable Quantum Field Theories (IQFT). In IQFT there is an infinite set of commutative local Integrals of Motion (IM). Local IM are generated by local currents which are pairs of local fields $(T_{s+1}(z), \Theta_{s-1}(z))$ satisfying the conservation equations

$$\partial_{\bar{z}}T_{s+1}(z) = \partial_z\Theta_{s-1}(z) \tag{2.40}$$

where the subscripts $s+1$ and $s-1$ represent the spins of the corresponding fields. Both $T_{s+1}(z)$ and $\Theta_{s-1}(z)$ have scaling dimension $s+1$. Here s takes values in some set $\{s\} \subset \mathbb{Z}$. These sets $\{s\}$ are different for different theories and can not be empty because at least $s = \pm 1$ exist. They correspond to the components of the energy-momentum tensor $T_{\mu\nu}$. We assume that the theory is P parity invariant. Then the set $\{s\}$ is symmetric under the P-reflection $s \leftrightarrow -s$. We can use separate notation for negative s. For $s > 0$ we denote Θ_{-s-1} as $\bar{T}_{s+1}$ and T_{-s+1} as $\bar{\Theta}_{s-1}$. Now the conservation equations can be written as

$$\partial_{\bar{z}} T_{s+1}(z) = \partial_z \Theta_{s-1}(z) \tag{2.41}$$

$$\partial_z \bar{T}_{s+1}(z) = \partial_{\bar{z}} \bar{\Theta}_{s-1}(z) \tag{2.42}$$

where s is a positive integer. Given the currents $\left(T_{s+1}(z), \bar{T}_{s+1}(z)\right)$ and $\left(\Theta_{s-1}(z), \bar{\Theta}_{s-1}(z)\right)$, we may attempt to construct composite fields by taking the limit $z \to z'$ in the operator products $T_{s+1}(z)\bar{T}_{s+1}(z')$ and $\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')$. Similar to the previous discussion, $T_{s+1}(z)\bar{T}_{s+1}(z')$ and $\Theta_{s-1}(z)\Theta_{s-1}(z')$ will be singular in the limit $z \to z'$. However the non-derivative divergent terms in the OPE of $T(z)\bar{T}(z')$ cancel with those in the OPE of $\Theta(z)\Theta(z')$. Thus for the combination of the operator products

$$T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z') \tag{2.43}$$

the limit $z \to z'$ can be taken. It defines the local fields X_s up to derivative terms

$$\lim_{z\to z'} \left(T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right) = X_s(z') + \text{derivative terms.} \tag{2.44}$$

The derivative terms may involve divergent coefficients but they can nonetheless be ignored. Now we show that the limit $z \to z'$ can indeed be properly taken. The proof is similar to the discussion in the previous section for $s = 1$. Assume the OPEs

$$\Theta_{s-1}(z)\bar{T}_{s+1}(z') = \sum_i B_i(z-z')\,\mathcal{O}_i(z') \tag{2.45}$$

$$\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z') = \sum_i C_i(z-z')\,\mathcal{O}_i(z') \tag{2.46}$$

where the sum is over the complete set of independent fields $\mathcal{O}_i(z')$ of the theory. Now we consider the $\bar{z}$ derivative of the combination (2.43). Using the continuity equations (2.41) we can easily verify

$$\begin{aligned}\partial_{\bar{z}}&\left(T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right) = \\ &(\partial_z + \partial_{z'})\,\Theta_{s-1}(z)\bar{T}_{s+1}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})\,\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\,.\end{aligned} \tag{2.47}$$

Plugging in the OPEs (2.45), we find

$$\begin{aligned}\partial_{\bar{z}}&\left(T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right) = \\ &\sum_i B_i(z-z')\,\partial_{z'}\mathcal{O}_i(z') - \sum_i C_i(z-z')\,\partial_{\bar{z}'}\mathcal{O}_i(z')\,.\end{aligned} \tag{2.48}$$

Since the coefficients $B_i(z-z')$ and $C_i(z-z')$ are annihilated by the derivatives $(\partial_z+\partial_{z'})$ and $(\partial_{\bar{z}}+\partial_{\bar{z}'})$, we conclude that the OPE of the l.h.s in equation (2.48) consists of only derivative terms. Similar calculations show that the ∂_z derivative of (2.43) also consists of only derivative terms. So the OPE of both ∂_z and $\partial_{\bar{z}}$ derivatives of the (2.43) consist only derivative terms. This means that the OPE of equation (2.43) consists mostly of the derivative terms, except for a single term which comes with a constant coefficient. This constant coefficient can be reabsorbed into the definition of the composite operator X_s below. Thus we can write

$$T_{s+1}(z)\bar{T}_{s+1}(z')-\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')=X_s(z')+\text{derivative terms.} \tag{2.49}$$

Thus we see that the limit in equation (2.44) can be taken properly. We can also compute the expectation value $\langle n|X_s|n\rangle$ in a similar way as we discussed for $X_1=T\bar{T}$. Put the theory on an infinite cylinder with circumference R. The energy spectrum is discrete and we assume that the states $|n\rangle$ are generally non-degenerate

$$\langle n|X_s|n\rangle=\left\langle n\left|T_{s+1}(z)\bar{T}_{s+1}(z')\right|n\right\rangle-\left\langle n\left|\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right|n\right\rangle. \tag{2.50}$$

Then write down the spectral decomposition

$$\left\langle n\left|T_{s+1}(z)\bar{T}_{s+1}(z')\right|n\right\rangle=\sum_{n'}\langle n|T_{s+1}(z)|n'\rangle\left\langle n'\left|\bar{T}_{s+1}(z')\right|n\right\rangle\times \\ e^{(E_n-E_{n'})|y-y'|+i(P_n-P_{n'})|x-x'|}. \tag{2.51}$$

As in the $T\bar{T}$ case, the terms with $n'\neq n$ will cancel those in the decomposition of $\left\langle n\left|\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right|n\right\rangle$. Only the term $n'=n$ remain

$$\langle n|X_s|n\rangle=\langle n|T_{s+1}(z)|n\rangle\left\langle n\left|\bar{T}_{s+1}(z)\right|n\right\rangle-\langle n|\Theta_{s-1}(z)|n\rangle\left\langle n\left|\bar{\Theta}_{s-1}(z')\right|n\right\rangle. \tag{2.52}$$

In the limit $R\to\infty$ the expectation value $\langle X_s\rangle$ vanishes for $s>1$ since rotational symmetry forces the r.h.s of (2.52) to vanish.

The local Integrals of Motion (IM) are generated by the currents $\left(T_{s+1}(z),\bar{T}_{s+1}(z)\right)$ and $\left(\Theta_{s-1}(z),\bar{\Theta}_{s-1}(z)\right)$

$$P_s=\frac{1}{2\pi}\int_C T_{s+1}(z)\,dz+\Theta_{s-1}(z)\,d\bar{z} \tag{2.53}$$

$$\bar{P}_s=\frac{1}{2\pi}\int_C \bar{T}_{s+1}(z)\,dz+\bar{\Theta}_{s-1}(z)\,d\bar{z}. \tag{2.54}$$

Because of the conservation equation (2.41) the integral P_s and $\bar{P}_s$ do not change under trivial deformations of the integration path C. For integrable theories, $\{P_s\}$ form a commutative set

$$[P_s,P_{s'}]=0 \tag{2.55}$$

for any $s, s' \in \{s\}$. This implies

$$\frac{1}{2\pi}\int_C [P_s, T_{s+1}(z)]\, dz + [P_s, \Theta_{s-1}(z)]\, d\bar{z} = 0. \tag{2.56}$$

The local IMs satisfy

$$[P_s, T_{s+1}(z)] = \partial_z A_{\sigma,s}(z) \tag{2.57}$$
$$[P_s, \Theta_{s-1}(z)] = \partial_{\bar{z}} A_{\sigma,s}(z) \tag{2.58}$$

and

$$\left[P_s, \bar{T}_{s+1}(z)\right] = \partial_{\bar{z}} B_{\sigma,s}(z) \tag{2.59}$$
$$\left[P_s, \bar{\Theta}_{s-1}(z)\right] = \partial_z B_{\sigma,s}(z) \tag{2.60}$$

where $A_{\sigma,s}$ and $B_{\sigma,s}$ are some local fields. There are similar equations for the commutators of $\bar{P}_s$ and local currents. Here the commutators is defined by integrals

$$[P_s, \mathcal{O}(z_0)] = \frac{1}{2\pi}\int_{C_{z_0}} [T_{s+1}(z)\, dz + \Theta_{s-1}(z)\, d\bar{z}]\, \mathcal{O}(z_0). \tag{2.61}$$

These properties will be used in next sections.

2.4 Integrable perturbations of IQFT

Let Σ denote the space of all 2D quantum field theories, then the space of all Integrable Quantum Field Theories (IQFT) is a subspace $\Sigma^{Int} \subset \Sigma$. The infinitesimal deformations of IQFT that preserve integrability form the tangent space $T\Sigma^{Int}|_{IQFT}$, which is a subspace of $T\Sigma|_{IQFT}$. The space $T\Sigma|_{IQFT}$ is given by the span of all local scalar fields (modulo total derivatives) present in the theory. $T\Sigma^{Int}|_{IQFT}$ consists of all fields such that when added as perturbations of IQFT they preserves integrability. Note that the deformation might break the UV completeness of the theory. So such theories should be thought of as effective field theories. The UV behaviour of the deformed theories has been studied by Dubovsky et al. under the name of asymptotic fragility[7, 29, 34]. We will discuss it in more details in Section 2.6. In [9] Smirnov and Zamolodchikov show that every field X_s generates an integrable deformation of IQFT. The set $\{X_s\}$ plus a finite number of additional fields span the whole space of $T\Sigma^{Int}|_{IQFT}$. Now we briefly present the argument for this. Denote $\mathcal{F} = Span\{\mathcal{O}_a(z)\}$ the space of all local fields, $\partial\mathcal{F} = Span\{\partial_z\mathcal{O}_a(z), \partial_{\bar{z}}\mathcal{O}_a(z)\}$ and $\hat{\mathcal{F}} = \mathcal{F}/\partial\mathcal{F}$ the factor space. A generic Lagrangian QFT is described by some action

$$S[\varphi] = \int d^2z \mathcal{L}(\varphi(z), \partial_\mu\varphi(z), \partial_\mu\partial_\nu\varphi(z), ..). \tag{2.62}$$

Denote the set $\{g^i\}$ a full set of coupling constants. Generic variations of the action can be written as

$$\delta S = \int d^2 z \delta\mathcal{L}(z) \tag{2.63}$$

$$\delta\mathcal{L}(z) = \sum_i \delta g^i \mathcal{O}_i(z). \tag{2.64}$$

Under such variations, correlation functions would vary as

$$\begin{aligned}\delta_g \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)...\mathcal{O}_n(z_n)\rangle_g =& -\sum_i \delta g^i \int d^2 z \langle \mathcal{O}_i(z)\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)...\mathcal{O}_n(z_n)\rangle_g \\ &+\sum_k \langle \mathcal{O}_{i_1}(z_1)...\delta_g\mathcal{O}_{i_k}(z_2)...\mathcal{O}_{i_n}(z_n)\rangle_g .\end{aligned} \tag{2.65}$$

To show that X_s generates an integrable deformation, we first need to prove that the commutator of any local IM P_σ with any of the fields X_s is a derivative term

$$[P_\sigma, X_s(z)] \in \partial\mathcal{F}. \tag{2.66}$$

Here the commutator is defined in equation (2.61). $X_s(z)$ is defined by currents

$$\left[P_\sigma, T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right]. \tag{2.67}$$

Consider the terms generated by commuting P_σ with $T_{s+1}(z)$ and $\Theta_{s-1}(z)$. Using the relations (2.57), (2.58) and the conservation equations (2.42) one finds

$$\begin{aligned}&\partial_z A_{\sigma,s}(z)\bar{T}_{s+1}(z') - \partial_{\bar{z}} A_{\sigma,s}(z)\bar{\Theta}_{s-1}(z') \\ &= (\partial_z + \partial_{z'}) A_{\sigma,s}(z)\bar{T}_{s+1}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'}) A_{\sigma,s}(z)\bar{\Theta}_{s-1}(z') \in \partial\mathcal{F}.\end{aligned} \tag{2.68}$$

Similar to previous discussion, in the second line we replace $A_{\sigma,s}(z)\bar{T}_{s+1}(z')$ and $A_{\sigma,s}(z)\bar{\Theta}_{s-1}(z')$ by their OPE. The coefficients are functions of $(z - z')$ hence will be annihilated by the derivatives $(\partial_z + \partial_{z'})$ and $(\partial_{\bar{z}} + \partial_{\bar{z}'})$. So the r.h.s are derivative terms. In the same way we can prove that terms generated by commuting P_σ with $\bar{T}_{s+1}(z)$ and $\bar{\Theta}_{s-1}(z)$ are derivative terms too

$$\begin{aligned}&T_{s+1}(z)\partial_{\bar{z}'}B_{\sigma,s}(z') - \Theta_{s-1}(z)\partial_{z'}B_{\sigma,s}(z') \\ &= (\partial_{\bar{z}} + \partial_{\bar{z}'})T_{s+1}(z)B_{\sigma,s}(z') - (\partial_z + \partial_{z'})\Theta_{s-1}(z)B_{\sigma,s}(z') \in \partial\mathcal{F}.\end{aligned} \tag{2.69}$$

Therefore (2.67) is a combination of derivative terms. Taking the limit $z \to z'$ we get (2.66). Now we need to show that after we deform the theory, there are still local IMs. To this end, there need to be conserved currents. Consider the following correlation function

$$\langle \mathcal{O} \int_C [T_{\sigma+1}(z)\,dz + \Theta_{\sigma-1}(z)\,d\bar{z}]\rangle \tag{2.70}$$

where $\mathcal{O}$ stands for any insertion of operators $\mathcal{O}_{a_1}(z_1)\mathcal{O}_{a_2}(z_2)...\mathcal{O}_{a_n}(z_n)$. The conservation equations for the currents $(T_{\sigma+1},\Theta_{\sigma-1})$ is equivalent to the condition that (2.70) vanish as long as all the insertion points $z_1,z_2,...z_n$ lie outside the contour C. Now we deform the theory by adding the term $\delta g_s\int d^2zX_s(z)$ to the action. (2.70) deforms as

$$\begin{aligned}&-\delta g_s\int d^2w\langle X_s(w)\,\mathcal{O}\int_C\left[T_{\sigma+1}(z)\,dz+\Theta_{\sigma-1}(z)\,d\bar{z}\right]\rangle \\ &+\langle\mathcal{O}\int_C\left[\delta_{g_s}T_{\sigma+1}(z)\,dz+\delta_{g_s}\Theta_{\sigma-1}(z)\,d\bar{z}\right]\rangle.\end{aligned} \tag{2.71}$$

We need to find a way to choose properly $\delta_{g_s}T_{\sigma+1}(z)$ and $\delta_{g_s}\Theta_{\sigma-1}(z)$ so that the variation (2.71) vanishes. If the variation (2.71) vanishes, it means correlation function (2.70) still vanishes after deformation. So for the deformed theory there are still conserved currents and local IM's. Let us look at the first term in (2.71). The insertion points $z_1,z_2,...z_n$ lies outside the contour C. If w also lies outside C, then the expectation value vanishes since all insertion points lie outside the contour. So $\int_{\mathbb{R}^2}d^2w$ can be replaced by $\int_{D(C)}d^2w$ where $D(C)$ is the area lies inside the contour C. By the definition equation (2.61)

$$X_s(w)\int_C\left[T_{\sigma+1}(z)\,dz+\Theta_{\sigma-1}(z)\,d\bar{z}\right]=2\pi\left[P_s,X_s(w)\right]. \tag{2.72}$$

We have seen in equation (2.66) that the commutator is a derivative term

$$4\pi i\left[P_s,X_s(w)\right]=\partial_{\bar{w}}\hat{T}_{\sigma+1,s}(w)+\partial_w\hat{\Theta}_{\sigma-1,s}(w) \tag{2.73}$$

where $\hat{T}_{\sigma+1,s}$ and $\hat{\Theta}_{\sigma-1,s}$ are some local fields with spins $\sigma+1$ and $\sigma-1$ respectively. The remaining integral is

$$\begin{aligned}&-\delta g_s\langle\mathcal{O}\int_{D(C)}d^2w\left(\partial_{\bar{w}}\hat{T}_{\sigma+1,s}(w)+\partial_w\hat{\Theta}_{\sigma-1,s}(w)\right)\rangle= \\ &-\delta g_s\langle\mathcal{O}\oint_C\left(\hat{T}_{\sigma+1,s}(w)\,dw+\hat{\Theta}_{\sigma-1,s}(w)\,d\bar{w}\right)\rangle.\end{aligned} \tag{2.74}$$

Thus the variation (2.71) becomes

$$\begin{aligned}&-\delta g_s\langle\mathcal{O}\oint_C\left(\hat{T}_{\sigma+1,s}(z)\,dz+\hat{\Theta}_{\sigma-1,s}(z)\,d\bar{z}\right)\rangle \\ &+\langle\mathcal{O}\int_C\left[\delta_{g_s}T_{\sigma+1}(z)\,dz+\delta_{g_s}\Theta_{\sigma-1}(z)\,d\bar{z}\right]\rangle.\end{aligned} \tag{2.75}$$

It can be set to zero by choosing

$$\delta_{g_s}T_{\sigma+1}=\delta g_s\hat{T}_{\sigma+1,s} \tag{2.76}$$

$$\delta_{g_s}\Theta_{\sigma-1}=\delta g_s\hat{\Theta}_{\sigma-1,s}. \tag{2.77}$$

We find that after the deformation by the operator X_s, the deformed theory still has conserved currents and local IM provided the currents are deformed as

$$T_{\sigma+1} \to T_{\sigma+1} + \delta g_s \hat{T}_{\sigma+1,s} \tag{2.78}$$

$$\Theta_{\sigma-1} \to \Theta_{\sigma-1} + \delta g_s \hat{\Theta}_{\sigma-1,s}. \tag{2.79}$$

In this way we find the entire set of local IM for the deformed theory. We still need to prove that the deformed IMs $\left\{P_s, \bar{P}_s\right\}$ commute with each other. Here there is no rigorous proof yet. One argument is as follows. If two IMs P_s and $P_{s'}$ do not commute with each other, $[P_s, P_{s'}] = Q_{s+s'} \neq 0$, then $Q_{s+s'}$ must be another local IM. This means that in the deformed theory the local IM form a non-abelian algebra. A non-abelian algebra of local higher spin IM means very powerful symmetry structure which is unknown so far outside CFT or free massive QFT. So one can conjecture that in the deformed theory local IM still commute with each other.

2.5 $T\bar{T}$ flow energy spectrum

The operator X_1 is identical to $T\bar{T}$ where $T\bar{T}$ is built from the components of the energy-momentum tensor. So $T\bar{T}$ is present in any QFT. $T\bar{T}$ operator is special because even if the original theory is not integrable, the deformation generated is in some sense solvable. Let us denote by $\mathcal{L}$ the Lagrangian density of the theory. The deformation generated by $T\bar{T}$ can be defined by the following differential equation

$$\partial_t \mathcal{L}\left(z, \bar{z}, t\right) = \frac{1}{\pi^2} T\bar{T}\left(z, \bar{z}, t\right). \tag{2.80}$$

Consider the theory on a cylinder with circumference R. Following the definition of the deformation one finds that the energy levels $E_n = E_n\left(R, t\right)$ of stationary state $|n\rangle$ satisfy the equation

$$\frac{\partial E_n}{\partial t} = \frac{R}{\pi^2}\left\langle n\left|T\bar{T}\right|n\right\rangle. \tag{2.81}$$

Let us show briefly the derivation in the following. Suppose we also compactify the time direction. So our theory is on a torus with characteristic lengths (L, R). For both sides of equation (2.80) we integrate over the torus and then take the vacuum expectation value

$$\left\langle \partial_t S\left[\phi\right]\right\rangle = \frac{1}{\pi^2}\langle \int_0^R dx \int_0^L dy T\bar{T}\rangle \tag{2.82}$$

where $S\left[\phi\right]$ is the action of the theory and ϕ denotes the collection of fields in the theory. The partition function can be written as a path integral

$$\mathcal{Z} = \int D\phi e^{-S[\phi]}. \tag{2.83}$$

In this representation the left hand side of equation (2.82) becomes

$$\frac{1}{\mathcal{Z}}\int D\phi e^{-S[\phi]}\partial_t S\left[\phi\right] = -\partial_t \ln\mathcal{Z}. \tag{2.84}$$

So we get

$$-\partial_t \ln\mathcal{Z} = \frac{1}{\pi^2}\langle\int_0^R dx\int_0^L dy T\bar{T}\rangle. \tag{2.85}$$

The partition can also be expressed as

$$\mathcal{Z} = \sum_n e^{-LE_n}. \tag{2.86}$$

For the left hand side of equation (2.85) we have

$$-\partial_t \ln\mathcal{Z} = \frac{1}{\mathcal{Z}}\sum_n \frac{\partial E_n}{\partial t} L e^{-LE_n}. \tag{2.87}$$

For the right hand side of equation (2.85) we have

$$\frac{1}{\pi^2}\langle\int_0^R dx\int_0^L dy T\bar{T}\rangle = \frac{1}{\pi^2\mathcal{Z}}\sum_n\langle n|\int_0^R dx\int_0^L dy T\bar{T}|n\rangle e^{-LE_n}. \tag{2.88}$$

So we get

$$\frac{1}{\mathcal{Z}}\sum_n \frac{\partial E_n}{\partial t} L e^{-LE_n} = \frac{1}{\pi^2\mathcal{Z}}\sum_n\langle n|\int_0^R dx\int_0^L dy T\bar{T}|n\rangle e^{-LE_n} \tag{2.89}$$

We can see that the equality must hold for every n. This gives equation (2.81).

We have computed $\left\langle n\left|T\bar{T}\right|n\right\rangle$ in equation (2.39). Thus we get a closed differential equation for the energy levels

$$\frac{\partial E_n}{\partial t} + E_n\frac{\partial E_n}{\partial R} + \frac{1}{R}P_n^2 = 0 \tag{2.90}$$

where $P_n\left(R\right) = \frac{2\pi l_n}{R}$ is the corresponding spatial momenta of the states. Now Let us think about the solution of equation (2.90). This equation has the same form for all eigenvalues $E_n\left(R,t\right)$ so we can omit the index n. Now we put the theory on the cylinder again. Let us use $E_t\left(R\right)$ to denote $E_n\left(R,t\right)$. Equation (2.90) is formally identical to the forced inviscid Burgers equation, with $E_t\left(R\right)$ playing the role of velocity field and t as the time. When $P=0$ the solution is well known and is given by

$$E_t\left(R\right) = E_0\left(R - tE_t\left(R\right)\right). \tag{2.91}$$

$E_t\left(R\right)$ is the finite size energy level. We expect it to have the form $E_t\left(R\right)\simeq F_t R$, up to terms bounded at $R\to\infty$. Here F_t is the bulk vacuum energy density. In the limit

$M_t R \gg 1$ the energies are well below inelastic thresholds. We can check that the following solution solves equation (2.90)

$$E_t(R) \simeq F_t R + \sqrt{M_t^2 + P^2(R)} \tag{2.92}$$

where

$$F_t = \frac{F_0}{1 + tF_0} \tag{2.93}$$

and

$$M_t = \frac{M_0}{1 + tF_0}. \tag{2.94}$$

The first term in equation (2.92) is the vacuum energy and the second term is the energy of the particle. We can read off the t-dependence of the bulk vacuum energy and the mass of particles. If the vacuum energy $F_0 = 0$ then we can see the energy levels will not vary with t. In general the solution can be written as

$$E_t(R) = F_t R + \mu_t u(r, \alpha) \tag{2.95}$$

where $r = \mu_t R$, μ_t is a mass scale which satisfies $\mu_t = \mu_0 / (1 + tF_0)$, $\alpha = t\mu_0\mu_t$. For massive theories one can take μ_t to be the mass of the theory. And $u(r, \alpha)$ is a dimensionless function and is bounded at $r \to \infty$. Plugging the ansatz (2.95) into equation (2.90) we can find $u(r, \alpha)$ satisfies the equation

$$\partial_\alpha u + u\partial_r u + \frac{(2\pi l)^2}{r^3} = 0. \tag{2.96}$$

This equation contains only dimensionless quantities.

For a massive theory one can consider a state of two particles with opposite momenta p and $-p$ such that the total momentum is 0. In the limit $M \gg \frac{1}{R}$, $E_t(R)$ has the form

$$E_t(R) = F_t R + 2\sqrt{M_t^2 + p^2} + O\left(e^{-M_t R}\right). \tag{2.97}$$

The momentum p is subject to quantization condition $pR + \Delta_t(p) = 2\pi n$, where $\Delta_t(p)$ is the scattering phase. Now the total momentum is 0. Using equation (2.90) one finds

$$\partial_R p = -\frac{p}{R} \tag{2.98}$$

$$\partial_t p = \frac{2p}{R}\sqrt{M_t^2 + p^2}. \tag{2.99}$$

These indicate that $\Delta_t(p)$ satisfies the condition

$$\Delta_t(p) = \Delta_0(p) - 2tp\sqrt{M_t^2 + p^2}. \tag{2.100}$$

To be more clear we can act ∂_t and ∂_R on both sides of the following equation

$$pR + \Delta_0 (p) - 2tp\sqrt{M_t^2 + p^2} = 2\pi n. \tag{2.101}$$

We obtain

$$R\partial_t p - \partial_p \Delta_0 \partial_t p - 2p\sqrt{M_t^2 + p^2} - 2t\partial_t p\sqrt{M_t^2 + p^2} - \frac{2tp^2\partial_t p}{\sqrt{M_t^2 + p^2}} = 0 \tag{2.102}$$

$$p + R\partial_R p + \partial_R \Delta_0 - 2t\partial_R p\sqrt{M_t^2 + p^2} - \frac{2tp^2\partial_R p}{\sqrt{M_t^2 + p^2}} = 0. \tag{2.103}$$

Combined with equation (2.98) and (2.99) we get constraints for $\partial_p\Delta_0$ and $\partial_R\Delta_0$. It means with proper choice of Δ_0, Δ_t can be expressed in the form of equation (2.100).

The momentum p can be expressed in terms of rapidity difference $\theta = \theta_1 - \theta_2$

$$p = M_t \sinh (\theta/2) , \tag{2.104}$$

and in terms of θ equation (2.100) can be rewriten as

$$\Delta_t = \Delta_0 - tM_t^2 \sinh \theta. \tag{2.105}$$

We see the t-flow adds a CDD factor on the two particle S-matrix

$$S_t (\theta) = S_0 (\theta) \exp \left(-itM_t^2 \sinh \theta\right) . \tag{2.106}$$

We will discuss the CDD factor in the next section.

When the undeformed theory is a CFT, the energy levels have standard form

$$E_0 (R) = F_0 R - \frac{C_n}{R} \tag{2.107}$$

with

$$C_n = \frac{\pi}{16} \left(c - 12 \left(\Delta_n + \bar{\Delta}_n\right)\right) \tag{2.108}$$

where c is the central charge and Δ_n, $\bar{\Delta}_n$ are the eigenvalues of the operators L_0, $\bar{L}_0$. For the case $P_n = 0$ (i.e. $\Delta_n = \bar{\Delta}_n$), one can use equation (2.91) to find a solution for $E_t (R)$

$$E_t (R) = F_t R + \frac{R}{2\tilde{t}} \left(1 - \sqrt{1 + \frac{4\tilde{t}C_n}{R^2}}\right) \tag{2.109}$$

where $\tilde{t} = t (1 + tF_0)$. When $\tilde{t}C_n$ is negative and $R < 2\sqrt{-\tilde{t}C_n}$, $E_t (R)$ gets a non-zero imaginary part. These algebraic square-root singularities of $E_t (R)$ are called "shocks". Shocks are not specific to the cases when the undeformed theory is a CFT. Such shock singularities might have some connection with the UV completeness of the $T\bar{T}$ deformed theories.

2.6 Two particle S-matrix

Massive 2D QFTs are uniquely characterized by the spectrum of stable particles and their S-matrix. For IQFT, the presence of an infinite set of higher spin local IM implies that the S-matrix is purely elastic which means the number of particles N and the set of their individual momenta are preserved after scattering. Thus $N \to N$ S-matrix can be expressed as the product of $2 \to 2$ S-matrices. In other words, the S-matrices are factorizable. We need only to focus on the two particle S-matrix $\hat{S}(\theta)$. The two particle S-matrix is only a function of the particles' rapidity difference. Deformations of IQFT that preserve integrability must also generate deformation of the S-matrix preserving elasticity. The two particle S-matrix $\hat{S}(\theta)$ satisfies the Yang-Baxter equation and constrains of analyticity, crossing symmetry and unitarity. Thus it is fixed up to CDD factors

$$\hat{S}(\theta) \to \hat{S}(\theta)\,\Phi(\theta)\,. \tag{2.110}$$

$\Phi(\theta)$ satisfies the equations

$$\Phi(\theta)\,\Phi(-\theta) = 1 \tag{2.111}$$

$$\Phi(i\pi+\theta)\,\Phi(i\pi-\theta) = 1. \tag{2.112}$$

The generally allowed phase takes the form

$$\Phi(\theta) = \exp\left\{ i \sum_{s\in\{s\}} t_s \sinh(s\theta) \right\} \tag{2.113}$$

where the set $\{s\}$ coincides with the set of spins local IM of the IQFT. Therefore the space of infinitesimal deformations of the S-matrix contains a finite-dimensional part related to the deformations of solutions of the Yang-Baxter equation and an infinite-dimensional part corresponding to deformations by CDD factors

$$\delta\hat{S}(\theta) = \left(i \sum_{s\in\{s\}} \delta t_s \sinh(s\theta) \right) \hat{S}(\theta)\,. \tag{2.114}$$

For the $T\bar{T}$ deformation only the lowest spin deformation is turned on and the CDD factor simplifies to

$$\Phi(\theta) = \exp\left(-i\frac{t}{4} m_1 m_2 \sinh(\theta) \right) \tag{2.115}$$

with m_1 and m_2 denoting the mass of each particle. If the original theory is a CFT, it contains only massless left- and right-moving excitations. Thus one needs to boost particle

1 and 2 in opposite directions to the speed of light and sending the masses to zero. In this limit the CDD factor reduces to

$$\Phi(\theta) = \exp\left(-i\frac{t}{8}m_1 m_2 e^{\theta_1-\theta_2}\right) = \exp\left(-i\frac{t}{4}p_1^+ p_2^-\right) \tag{2.116}$$

where p_1^+, p_2^- are the light-cone momenta of the two particles. Thus the two particle S-matrix of the $T\bar{T}$-deformed CFT takes the form

$$S = \exp\left(-i\frac{t}{4}p_1^+ p_2^-\right). \tag{2.117}$$

The S-matrix definition of the $T\bar{T}$ deformation is closely related to gravitational dressing procedure [29]. For a two dimensional quantum field theory defined by its S-matrix elements $S(\{p_i\})$, the gravitationally dressed S-matrix is defined as

$$\tilde{S}(\{p_i\}) = S(\{p_i\})e^{\frac{il^2}{4}\sum_{i<j}\epsilon_{\alpha\beta}p_i^\alpha p_j^\beta} \tag{2.118}$$

where l^2 is a parameter characterizing the dressing. The momenta are ordered according to the corresponding rapidities. If i and j are both incoming particles and $\beta_i > \beta_j$, or if they are both outgoing and $\beta_i < \beta_j$ or if i is incoming and j is outgoing we have $i < j$. The dressed amplitudes satisfy all the requirements of S-matrix. But their high energy behaviour is incompatible with the existence of a UV fixed point. This novel type of high energy behaviour is dubbed asymptotic fragility. A dressed theory exhibits many features expected from a gravitational theory. There are some evidence strongly indicating that the $T\bar{T}$ deformation is equivalent to the gravitational dressing[29]. With the definition of rapidity one finds

$$\frac{il^2}{4}\epsilon_{\alpha\beta}p_i^\alpha p_j^\beta = -\frac{il^2}{4}m_1 m_2 \sinh(\theta) \tag{2.119}$$

By comparing equation (2.115) with equation (2.118) we find $t = l^2$ so t needs to be positive.

2.7 Holographic description of the $T\bar{T}$ deformation

The $T\bar{T}$-deformed CFT has a nice holographic dual description which is AdS_3 with a finite radial cutoff [17, 35]. This becomes a case of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. On the bulk side the boundary lies not at asymptotic infinity but instead at a finite radius. The idea of some deformation of CFT dual to gravity with a cutoff boundary surface is studied in [40–42]. [43, 44] proposed the duality of $T\bar{T}$-deformed CFT and AdS_3 with finite radial cutoff. The motivation is the agreement between several quantities computed on the two sides including the deformed energy spectrum and the propagation speeds of small perturbations. Stress tensor correlators can also be matched for the deformed CFT on the boundary and classical pure gravity in the bulk.

Now let us discuss the matching of the deformed energy spectrum. Starting from the original theory with action S_0. The $T\bar{T}$ deformed CFT action is defined by the equation

$$\frac{dS\left(t\right)}{dt}=\int d^2x\sqrt{g}T\bar{T}\left(x\right) \tag{2.120}$$

with boundary condition

$$S\left(t\right)=S_0. \tag{2.121}$$

For a theory with a single mass scale μ dimensional analysis yields

$$\mu\frac{dS}{d\mu}=\frac{1}{2\pi}\int d^2x\sqrt{g}T^i_i. \tag{2.122}$$

A CFT deformed by $T\bar{T}$ has single scale for finite t

$$t=\frac{1}{\mu^2}. \tag{2.123}$$

Comparing equation (2.120) and equation (2.122) we get the trace relation

$$T^i_i=-4\pi tT\bar{T}. \tag{2.124}$$

We have discussed that the finite size energy levels for the $T\bar{T}$ deformed theory on a cylinder of circumference satisfy the equation

$$\frac{\partial E_n}{\partial t}+E_n\frac{\partial E_n}{\partial R}+\frac{1}{R}P_n^2=0. \tag{2.125}$$

The solution for general P_n is

$$E_n=-\frac{R}{2\pi^2t}\left(\sqrt{1-\frac{4\pi^2t}{R}E_{n,0}+\left(\frac{2\pi^2t}{R}P_n\right)^2}-1\right). \tag{2.126}$$

The dual gravity of the $T\bar{T}$-deformed CFT is a BTZ blackhole in a region of AdS with finite radial cutoff $r=r_c$. The gravitational action for pure gravity in AdS_3 is

$$S=-\frac{1}{16\pi G}\int_M d^3x\sqrt{g}\left(R+2l^{-2}\right)-\frac{1}{8\pi G}\int_{\partial M}d^2x\sqrt{h}\left(K-l^{-1}\right). \tag{2.127}$$

Here Euclidean signature is chosen and AdS radius is set to be 1 ($l=1$). The curvature conventions are that $R\left(\mathrm{AdS}_3\right)=-6$. h_{ij} is the metric on the boundary. In a coordinate system with the metric

$$ds^2=d\rho^2+g_{ij}\left(x,\rho\right)dx^idx^j \tag{2.128}$$

the extrinsic curvature is

$$K_{ij}=\frac{1}{2}\partial_\rho g_{ij}. \tag{2.129}$$

The Einstein's equations are

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - g_{\mu\nu} = 0 \tag{2.130}$$

takes the form

$$\begin{aligned} E^i_j &= -\partial_\rho\left(K^i_j - \delta^i_j K\right) - KK^i_j + \frac{1}{2}\delta^i_j\left(K^{mn}K_{mn} + K^2\right) - \delta^i_j = 0 &(2.131)\\ E^\rho_j &= \nabla^i\left(K_{ij} - Kg_{ij}\right) = 0 &(2.132)\\ E^\rho_\rho &= -\frac{1}{2}R^{(2)} + \frac{1}{2}\left(K^2 - K^{ij}K_{ij}\right) - 1 = 0. &(2.133)\end{aligned}$$

The boundary stress tensor can be obtained by varying the action

$$\delta S = \frac{1}{4\pi}\int d^2x\sqrt{h}T^{ij}\delta h_{ij}. \tag{2.134}$$

The stress tensor is

$$T_{ij} = \frac{1}{4G}\left(K_{ij} - Kg_{ij} + g_{ij}\right). \tag{2.135}$$

It obeys $\nabla^i T_{ij} = 0$ by the equation $E^\rho_j = 0$. So we can compute the trace of the stress tensor

$$T^i_i = \frac{1}{4G}\left(2 - K\right) \tag{2.136}$$

and

$$T\bar{T} = \frac{1}{8}\left(T^{ij}T_{ij} - \left(T^i_i\right)^2\right) = -\frac{1}{64G^2}\left(2-K\right) - \frac{R^{(2)}}{128G^2}. \tag{2.137}$$

On a flat boundary metric we have

$$T^i_i = -16GT\bar{T}. \tag{2.138}$$

By comparing with the trace relation of the deformed CFT (2.124), we read the dictionary

$$t = \frac{4G}{\pi}. \tag{2.139}$$

By solving the bulk equation of motion with boundary condition, the action (2.127) becomes a functional of the boundary metric. The gravitational energy of the blackhole can be computed from the boundary stress tensor

$$\mathcal{E} = EL = \frac{L}{2\pi}\int d\phi\sqrt{g_{\phi\phi}}u^i u^j T_{ij} \tag{2.140}$$

where ϕ is the coordinate of the spatial circle on the boundary and L is the size of the spatial circle. The result is

$$\begin{aligned}\mathcal{E} &= \frac{L^2}{2\pi^2 t}\left(1 - r_c^{-1}f\left(r\right)\right) &(2.141)\\ &= \frac{L^2}{2\pi^2 t}\left(1 - \sqrt{1 - \frac{4\pi^2 t}{L}M + \left(\frac{2\pi^2 t}{L}J\right)^2}\right).\end{aligned}$$

This agrees with the energy spectrum of $T\bar{T}$-deformed CFT (2.126) under the identification $M = E_0,\ J = p$.

The propagation speed of stress energy perturbations is derived using the conservation equation and trace relation. On a flat metric $ds^2 = dzd\bar{z}$, these equations can be written as

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{z\bar{z}} = 0 \tag{2.142}$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0 \tag{2.143}$$

$$T_{z\bar{z}} + \pi t \left(T_{zz} T_{\bar{z}\bar{z}} - \left(T_{z\bar{z}}\right)^2 \right) = 0. \tag{2.144}$$

By converting these equation to Lorentzian signature and linearizing round constant background $\langle T_{ij} \rangle$ we get the perturbation propagation speed as

$$v_+ = 1 + 2\pi t \left\langle T_{++} \right\rangle + \mathcal{O}\left(t^2\right) \tag{2.145}$$

$$v_- = 1 + 2\pi t \left\langle T_{--} \right\rangle + \mathcal{O}\left(t^2\right). \tag{2.146}$$

This agrees with the CFT results stated in [17]. The superluminal nature of these speeds for $\lambda > 0$ has been discussed in [42, 45]. We see the results of the deformed energy spectrum and propagation speeds match on both side. In addition, it has been shown that the results of the stress tensor correlators also agree with each other[35]. We refrain from going to the details here.

Chapter 3

$T\bar{T}$-deformation in closed form

3.1 The $T\bar{T}$ flow equation

We have discussed the $T\bar{T}$ flow equation in equation (2.80). Now we discuss it from another point of view. Let M denote a two dimensional manifold equipped with a (Euclidean) metric tensor $g_{\mu\nu}$ with $\mu, \nu = 1, 2$ and consider a QFT on M whose dynamics is governed by the local action

$$S_\circ = \int_M \mathrm{d}^2x \sqrt{g}\, \mathcal{L}_\circ(\Phi, g_{\mu\nu}, \lambda)\,. \tag{3.1}$$

Here $\mathcal{L}_\circ$ denotes the Lagrangian for the local fields which we have collectively denoted by Φ. The coupling constants, denoted by λ, control the strength of interactions among the fields as well as with local sources. The partition function of this theory,

$$\mathcal{Z}_\circ[g_{\mu\nu}, \lambda] = \int [\mathcal{D}\Phi]\, e^{-S_\circ}\,, \tag{3.2}$$

thus depends on the constants λ as well as the background metric $g_{\mu\nu}$. The $T\bar{T}$-flow equation is the first order differential equation in a real deformation parameter t,

$$\left(\frac{\partial}{\partial t} + \Delta\right) \mathcal{Z}_t = 0\,, \tag{3.3}$$

where the functional operator Δ above is defined as

$$\Delta = \lim_{\delta \to 0} \int_M \mathrm{d}^2x \frac{2}{\sqrt{g}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \frac{\delta}{\delta g^{\mu\rho}(x+\delta)} \frac{\delta}{\delta g^{\nu\sigma}(x-\delta)}\,. \tag{3.4}$$

The initial condition for (3.3) is provided by the undeformed theory $\mathcal{Z}_{t=0} = \mathcal{Z}_\circ$. Once the initial condition is given, then the solution is uniquely determined.

For Lagrangian theories, equation (3.3) becomes the equation for the action functional

$$\frac{\partial S}{\partial t} = (S, S) \tag{3.5}$$

where the pairing $(\,\cdot\,,\,\cdot\,)$ is defined via

$$(X, Y) = \lim_{\delta\to 0} \int_M \mathrm{d}^2x \frac{2}{\sqrt{g}} \epsilon^{\mu\nu}\epsilon^{\rho\sigma} \frac{\delta X}{\delta g^{\mu\rho}(x+\delta)} \frac{\delta Y}{\delta g^{\nu\sigma}(x-\delta)}\,, \tag{3.6}$$

for *local* functionals X and Y. Equation (3.5) is derived in [9], where the absence of contact terms in the $T\bar{T}$ composite operator is proven to follow from general assumptions[1]. This implies that the point splitting regulator δ in the definition of the $T\bar{T}$ composite operator can be removed after the regularisation of the QFT and does not compete with its UV regulator.

Our approach to integrating $T\bar{T}$-variations is concretely obtained by giving a class of solutions of (3.5). The computation of the path integral for the deformed theory is a different issue which we do not address here and we restrict our analysis to the deformation problem of the classical action. We propose a simple integration technique for equation (3.5) which follows from the locality of the action, the absence of space-time derivatives in Δ and covariance under diffeomorphisms Diff(M).

Let us summarise our logic. The first observation is that equation (3.5), being first order in t, has a *unique* solution for any initial (undeformed) local action. Locality of the operator Δ therefore suggests that we should look for a solution which can be expressed as a local functional $S(t) = \int_M \mathrm{d}^2x \sqrt{g}\,\mathcal{L}(t)$ at finite t. We next observe that the deformation operator Δ does not generate terms involving derivatives of the metric unless such terms are already present in the undeformed Lagrangian[2]. Consider a theory $\mathcal{T}_\circ$ on a two dimensional manifold M endowed with the Euclidean metric tensor $g_{\mu\nu}$ whose dynamics is captured by the local action $S_\circ = \int \mathrm{d}^2x\,\sqrt{g}\,\mathcal{L}_\circ$. We are interested in finding a solution to the flow equation (3.5) in terms of a local functional

$$S(t) = \int \mathrm{d}^2x\,\sqrt{g}\,\mathcal{L}(t)\,, \tag{3.7}$$

[1]This is analogous to the absence of contact terms for the YM curvature which facilitates the derivation of Migdal's loop equations in Yang-Mills theories. Furthermore, the analogue of Polyakov's loop Laplacian for YM is Δ above (See §7.2 in [46]).

[2]In the absence of such terms therefore the deformed Lagrangian can be viewed as a function only of certain combinations of the dynamical fields and couplings. The form of these "*invariants*" is dictated by the flow equation (3.12) as well as the explicit dependence of the undeformed Lagrangian on the metric. This, as it will be shown later, induces very strict dependences on the metric and allows the complete integration of the $T\bar{T}$-flow equation.

with the initial condition $\mathcal{L}(t=0) = \mathcal{L}_\circ$. Plugging (3.7) into the *rhs* of (3.5) we find

$$(S,S) = \int \mathrm{d}^2x\, \mathcal{O}_{T\bar{T}}\,, \tag{3.8}$$

where the (local) $T\bar{T}$-operator is given by

$$\mathcal{O}_{T\bar{T}} = \frac{1}{2}\varepsilon^{\mu\nu}\varepsilon^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}\,. \tag{3.9}$$

Since the theory described by $S(t)$ is coupled to the background metric tensor $g_{\mu\nu}$, the associated energy momentum tensor can be extracted by looking at small variations of the metric,

$$T_{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\delta S(t)}{\delta\, g^{\mu\nu}} = g_{\mu\nu}\mathcal{L}(t) - 2\frac{\partial\mathcal{L}(t)}{\partial g^{\mu\nu}}\,, \tag{3.10}$$

where the second equality holds under the condition that the undeformed Lagrangian depends algebraically on the metric tensor. The unique solution of the $T\bar{T}$-flow equation will, as already mentioned, enjoy the same property. Using this expression for the energy momentum tensor enables us to recast the $T\bar{T}$-operator as

$$\mathcal{O}_{T\bar{T}} = \mathcal{L}^2 - 2\mathcal{L}g^{\mu\nu}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + 2\varepsilon^{\mu\nu}\varepsilon^{\rho\sigma}\frac{\partial\mathcal{L}}{\partial g^{\mu\rho}}\frac{\partial\mathcal{L}}{\partial g^{\nu\sigma}}\,. \tag{3.11}$$

Therefore eq.(3.5) reads

$$\partial_t\mathcal{L} = \mathcal{O}_{T\bar{T}}\,, \tag{3.12}$$

Here the notation is a bit different from equation (2.80). This is not a problem since we can always absorb a constant into the definition of the parameter t. We will discuss the implementation of this method in the specific cases in the next sections. The upshot is that equation (3.12) reduces to a partial differential equation in the deformation variable t and invariants formed from the metric and the dynamical fields. We will show that in many examples the flow equation can be recast as the (extended) Burgers' equation. Since the Burgers' equation can be reduced to quadratures via the method of characteristics, we can present the explicit solution depending on the form of the initial condition.

We remark that the link between the Burgers' equation and the $T\bar{T}$-deformed action was already observed in [47], where the appearance of its characteristic curve was rebuilt from the assumption of validity of the non-linear integral equation for the theory. Our approach leads directly to the Burgers' equation and in a more general setting.

In the following we will first analyze the example of a single massless scalar field to familiarise the reader with our approach and to set up the notation. We then solve the case of an interacting scalar field in closed form and for an arbitrary potential before considering a general σ-model with an arbitrary target metric and B-field and the WZW model. We also discuss the result of a power expansion of the solution of the $T\bar{T}$-deformation equation in the case in which a curvature coupling is turned on and show the proliferation of higher order derivatives at higher orders in the deformation parameter. In section 3.7 we explicitly solve the $T\bar{T}$-deformation of a massive Dirac fermion with quartic interaction, i.e. the massive Thirring model, and show that the solution is in this case given by a finite power series in t. We dedicate section 3.8 to possible higher dimensional generalisations.

3.2 Free massless scalar field

As our first example we would like to find the unique solution to equation (3.12) with the initial condition provided by the action for a free real scalar field

$$S_\circ = \frac{1}{2}\int \mathrm{d}^2x\,\sqrt{g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,. \tag{3.13}$$

In the following we find it convenient to define the symmetric – and metric independent – tensor

$$X_{\mu\nu} := \partial_\mu\phi\partial_\nu\phi \tag{3.14}$$

whose trace we denote by $X = g^{\mu\nu}X_{\mu\nu}$. The initial Lagrangian is

$$\mathcal{L}_\circ = \frac{1}{2}X\,, \tag{3.15}$$

If we expand the deformed Lagrangian in terms of t

$$\mathcal{L} = \sum_{j=0}^{\infty} t^j\mathcal{L}^{(j)}. \tag{3.16}$$

Then $\mathcal{L}^{(j)}$ can be computed order by order.

$$\partial_t\mathcal{L} = \sum_{j=1}^{\infty} jt^{j-1}\mathcal{L}^{(j)}. \tag{3.17}$$

The flow equation can be expanded as

$$\sum_{j=0}(j+1)\,t^j\mathcal{L}^{(j+1)} = \sum_{k=0}^{j}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{1}{\sqrt{g}}\frac{\delta S^{(k)}}{\delta g^{\mu\nu}}\right)\left(\frac{1}{\sqrt{g}}\frac{\delta S^{(j-k)}}{\delta g^{\rho\sigma}}\right). \tag{3.18}$$

The first few orders can be computed as follows

$$
\begin{aligned}
\mathcal{L}^{(1)} &= \frac{1}{2}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(0)}}{\delta g^{\mu\nu}}\right)\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(0)}}{\delta g^{\rho\sigma}}\right) \\
&= \frac{1}{2}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}^{(0)} + \partial_\mu\phi\partial_\upsilon\phi\right)\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}^{(0)} + \partial_\mu\phi\partial_\upsilon\phi\right) \\
&= -\frac{1}{4}X^2.
\end{aligned}
\tag{3.19}
$$

Based on $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ we can compute $\mathcal{L}^{(2)}$

$$
\begin{aligned}
2\mathcal{L}^{(2)} &= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(0)}}{\delta g^{\mu\nu}}\right)\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(1)}}{\delta g^{\rho\sigma}}\right) \\
&= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(-\frac{1}{2}g_{\mu\nu}X + \partial_\mu\phi\partial_\upsilon\phi\right)\left(\frac{1}{4}g_{\rho\sigma}X^2 - X\partial_\rho\phi\partial_\sigma\phi\right) \\
&= \frac{1}{2}X^3
\end{aligned}
\tag{3.20}
$$

$$
\mathcal{L}^{(2)} = \frac{1}{4}X^3. \tag{3.21}
$$

For $\mathcal{L}^{(3)}$ there are two terms

$$
\begin{aligned}
3\mathcal{L}^{(3)} &= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(0)}}{\delta g^{\mu\nu}}\right)\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(2)}}{\delta g^{\rho\sigma}}\right) + \frac{1}{2}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(1)}}{\delta g^{\mu\nu}}\right)\left(\frac{2}{\sqrt{g}}\frac{\delta S^{(1)}}{\delta g^{\rho\sigma}}\right) \\
&= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(-\frac{1}{2}g_{\mu\nu}X + \partial_\mu\phi\partial_\upsilon\phi\right)\left(-\frac{1}{4}g_{\rho\sigma}X^3 + \frac{3}{2}X^2\partial_\rho\phi\partial_\sigma\phi\right) \\
&\quad + \frac{1}{2}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{1}{4}g_{\rho\sigma}X^2 - X\partial_\rho\phi\partial_\sigma\phi\right)\left(\frac{1}{4}g_{\rho\sigma}X^2 - X\partial_\rho\phi\partial_\sigma\phi\right) \\
&= -\frac{15}{16}X^4
\end{aligned}
\tag{3.22}
$$

$$
\mathcal{L}^{(3)} = -\frac{5}{16}X^4. \tag{3.23}
$$

So the first few terms in t are

$$
\mathcal{L} = \frac{1}{2}X - \frac{1}{4}tX^2 + \frac{1}{4}t^2X^3 - \frac{5}{16}t^3X^4 + ... \tag{3.24}
$$

Now we show how to directly solve the flow equation and derive the closed form for the deformed Lagrangian. we expect the deformed Lagrangian to depend on the fields only through $X_{\mu\nu}$. Moreover, since any diffeomorphism invariant function of $X_{\mu\nu}$ and the metric is only a function of the scalar X we conclude that the deformed Lagrangian is only a function of two scalar variables t and X, *i.e.* $\mathcal{L} = \mathcal{L}(t, X)$. Consequently the deformation operator (3.11) takes the simple form

$$\mathcal{O}_{T\bar{T}} = \mathcal{L}^2 - 2\mathcal{L}X\partial_X\mathcal{L}\,, \tag{3.25}$$

yielding the flow equation

$$\begin{aligned}\partial_t\mathcal{L} &= (1 - X\partial_X)\,\mathcal{L}^2 \\ &= -X^2\partial_X\left(\frac{1}{X}\mathcal{L}^2\right).\end{aligned} \tag{3.26}$$

Set

$$\mathcal{L} = -\sqrt{X}f \tag{3.27}$$

$$Y = \frac{1}{\sqrt{X}} \tag{3.28}$$

so that

$$\frac{\partial Y}{\partial X} = -X^{-3/2} \tag{3.29}$$

$$-2X^{3/2}\partial_X = \partial_Y. \tag{3.30}$$

The flow equation becomes

$$2\partial_t f = -\partial_Y f^2 \tag{3.31}$$

$$f\,(0, Y) = F_0\,(Y) \tag{3.32}$$

As discussed in the appendix this is Burger's equation and is solved by

$$f\,(t, Y) = f_0\,(Y - f\,(Y, t)\,t)\,. \tag{3.33}$$

So we have

$$\begin{aligned}\mathcal{L}_0 &= -\sqrt{X}f_0 \\ &= \frac{1}{2}X \\ &= \sqrt{X}\frac{1}{2Y}\end{aligned} \tag{3.34}$$

which means

$$f_0\,(Y) = -\frac{1}{2Y}. \tag{3.35}$$

Thus we get equation for $f\,(t, Y)$

$$f = \frac{-1}{2(Y - Ft)}. \tag{3.36}$$

We can solve for $F\left(t, Y\right)$

$$f = \frac{-Y \pm \sqrt{Y^2 + 2t}}{-2t}. \tag{3.37}$$

So we get the deformed Lagrangian

$$\begin{aligned} \mathcal{L} &= -\sqrt{X} f \\ &= \frac{-1 + \sqrt{1 + 2tX}}{2t}. \end{aligned} \tag{3.38}$$

Note that the solution (3.38) is smooth for $t \geq 0$ but can become imaginary for $t < 0$. This is closely related to the fact that the spectrum of the deformed theory on a circle exhibits Hagedorn behavior for $t < 0$. The analysis above extends to more general boundary conditions which we discuss below.

3.3 Interacting scalar field

An immediate generalisation of the above result follows from the altered boundary condition

$$\mathcal{L}(0, X) = \frac{1}{2} X + V \tag{3.39}$$

where $V = V(\phi)$ is an arbitrary potential so long as it is independent of the background metric. Now we expect the deformed Lagrangian to have to have the form $\mathcal{L} = \mathcal{L}(t, X, V)$. The flow equation is the same as the free massless scalar case

$$2\partial_t f = -\partial_Y f^2 \tag{3.40}$$

$$f\left(0, Y\right) = f_0\left(Y\right). \tag{3.41}$$

The initial condition now is

$$\begin{aligned} \mathcal{L}_0 &= -\sqrt{X} f_0 \\ &= \frac{1}{2} X + V \\ &= \sqrt{X} \left(\frac{1}{2}\sqrt{X} + \frac{V}{\sqrt{X}} \right) \\ &= \sqrt{X} \left(\frac{1}{2Y} + YV \right) \end{aligned} \tag{3.42}$$

which gives

$$f_0\left(Y\right) = -\frac{1}{2Y} - YV. \tag{3.43}$$

So we get $f\left(t, Y\right)$

$$f = \frac{-1}{2(Y - ft)} - V\left(Y - ft\right). \tag{3.44}$$

This is a simple quadratic equation for f. We can solve for f

$$f = \frac{1}{Yt}\frac{1-2tV}{1-tV} - \frac{1}{Yt}\sqrt{\left(\frac{1-2tV}{1-tV}\right)^2 + 2t\frac{2+VY^2}{Y^2-4t}}. \tag{3.45}$$

Thus we get the deformed Lagrangian

$$\mathcal{L} = -\frac{1}{2t}\frac{1-2tV}{1-tV} + \frac{1}{2t}\sqrt{\left(\frac{1-2tV}{1-tV}\right)^2 + 2t\frac{X+2V}{1-tX}}. \tag{3.46}$$

This agrees with (and significantly simplifies) the expression obtained in [47] whose first few terms were first presented in [34].

3.4 Curvature couplings

Another generalisation of (3.38) is obtained by imposing as the boundary condition a Lagrangian with curvature couplings. As an example, consider the undeformed Lagrangian

$$\mathcal{L}(t=0) = \frac{1}{2}X + \alpha_\circ \phi R \tag{3.47}$$

where R denotes the Ricci scalar associated with the background metric $g_{\mu\nu}$. This Lagrangian describes a theory with central charge $c = 1 + 6Q^2$, where $\alpha_\circ = \sqrt{2\pi}Q$. We may think of (3.47) as a deformation of the free theory and thus expand the solution to the flow equation in powers of $\alpha_\circ$,

$$\mathcal{L} = \sum_{n=0}^{N} \alpha_\circ^n \mathcal{L}^{(n)} \qquad \text{and} \qquad T_{\mu\nu} = \sum_{n=0}^{N} \alpha_\circ^n T_{\mu\nu}^{(n)} \,,$$

where

$$T_{\mu\nu}^{(n)} = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^2x \sqrt{g}\mathcal{L}^{(n)} \,.$$

Using this expansion we can solve the flow equation,

$$\partial_t \mathcal{L} = \frac{1}{2}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{\mu\nu}T_{\rho\sigma} \,,$$

order by order in $\alpha_\circ$. At order $\alpha_\circ^0$ we recover (3.38), while the flow equation at order $\alpha_\circ$ reads

$$\partial_t \mathcal{L}^{(1)} = \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{\mu\nu}^{(0)}T_{\rho\sigma}^{(1)} \,.$$

This equation can in turn be solved order by order in t with the first few terms given by

$$\mathcal{L}^{(1)} = \phi R - 2tX\Box\phi + 2t^2X^2\Box\phi - \frac{8}{3}t^3X^3\Box\phi + \mathcal{O}\left(t^3\right) \,.$$

This leads us to consider the following ansatz

$$\mathcal{L}^{(1)} = \phi R + f\,(tX)\,\Box\phi\,.$$

Plugging this ansatz in the flow equation yields

$$2\sqrt{1+2y} + f'\,(y)\,y\sqrt{1+2y} + 2q\,(y) - 2 = 0\,,$$

where $y = tX$, and

$$q\,(y) = 1 + \int \mathrm{d}y\,\frac{y f'(y)}{2\sqrt{1+2y}}\,.$$

Solving for $f\,(y)$ we obtain the deformed Lagrangian

$$\mathcal{L}(t) = -\frac{1}{2t} + \frac{1}{2t}\sqrt{1+2tX} + \alpha_\circ\left[\phi R - \log\left(1+2tX\right)\Box\phi\right] + \mathcal{O}\left(\alpha_\circ^2\right). \tag{3.48}$$

As one might have expected, upon deformation, the Ricci scalar term induces higher derivative corrections with the second order derivative term $\Box\phi = \nabla^\mu\partial_\mu\phi$ appearing at order $\alpha_\circ$. The $\alpha_\circ$-expansion of the deformed Lagrangian therefore takes the form of an expansion in higher derivative terms which have proved too cumbersome to determine.

3.5 Non-linear σ-model

Now that the logic is clear lets see if we can generalise the above analysis to multiple scalar fields described by the σ-model action

$$S_\circ = \frac{1}{2}\int \mathrm{d}^2x\,\sqrt{g}\,\left[g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j G_{ij}(\phi) + \varepsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j B_{ij}(\phi)\right]\,. \tag{3.49}$$

As before we define a set of metric independent tensors

$$X^{ij}_{\mu\nu} = \partial_\mu\phi^i\partial_\nu\phi^j\,, \tag{3.50}$$

and the (density) scalars

$$X^{ij} = g^{\mu\nu}X^{ij}_{\mu\nu} \quad \text{and} \quad \tilde{X}^{ij} = \sqrt{g}\varepsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\,. \tag{3.51}$$

Note that the scalar densities $\tilde{X}^{ij}$ are independent of the background metric. In fact the entire B-term is metric independent and therefore topological. Furthermore, topological terms are not affected by continuous, non-geometric, parameter deformations of the theory. The upshot is that the topological B-term is unaffected by the deformation and does not enter the analysis below.

The deformed Lagrangian depends on the metric only through the worldsheet scalars X^{ij} and $\tilde{X}^{ij}/\sqrt{g}$. Moreover, the latter only depends on the metric through the factor $\Omega = \sqrt{g}$. Therefore the deformed Lagrangian is expected to be a function of the deformation parameter t, the variables X^{ij} and of Ω, *i.e.*

$$\mathcal{L} = \mathcal{L}(t, X^{ij}, \Omega)\,. \tag{3.52}$$

This allows us to considerably simplify the expression for the deformation operator (3.11)

$$\mathcal{O}_{T\bar{T}} = \frac{2\tilde{X}^{ik}\tilde{X}^{jl}}{\Omega^2}\frac{\partial\mathcal{L}}{\partial X^{ij}}\frac{\partial\mathcal{L}}{\partial X^{kl}} - 2\Omega\frac{\partial\mathcal{L}}{\partial\Omega}X^{ij}\frac{\partial\mathcal{L}}{\partial X^{ij}} + \Omega^2\left(\frac{\partial\mathcal{L}}{\partial\Omega}\right)^2 + \left(1 - X^{ij}\frac{\partial}{\partial X^{ij}} + \Omega\frac{\partial}{\partial\Omega}\right)\mathcal{L}^2\,. \tag{3.53}$$

Note that factors of $\tilde{X}^{ij}$ in this equation should be treated as constant coefficients as they do not depend on the metric.

So far we have only insisted on invariance under worldsheet diffeomorphisms. However, we expect the deformed Lagrangian to also be invariant under target space diffeomorphisms. This further constrains the form of the deformed Lagrangian such that it can only depend on the scalar $X = G_{ij}X^{ij}$, and the above equation simplifies to

$$\partial_t\mathcal{L} = \frac{2\tilde{X}^{ij}\tilde{X}_{ij}}{\Omega^2}\left(\frac{\partial\mathcal{L}}{\partial X}\right)^2 - 2\Omega\frac{\partial\mathcal{L}}{\partial\Omega}X\frac{\partial\mathcal{L}}{\partial X} + \Omega^2\left(\frac{\partial\mathcal{L}}{\partial\Omega}\right)^2 + \left(1 - X\frac{\partial}{\partial X} + \Omega\frac{\partial}{\partial\Omega}\right)\mathcal{L}^2\,. \tag{3.54}$$

The solution to the above equation is given by

$$\mathcal{L} = -\frac{1}{2t} + \frac{1}{2t}\sqrt{1 + 2tX + 2t^2\tilde{X}^{ij}\tilde{X}_{ij}\Omega^{-2}}\,, \tag{3.55}$$

satisfying the boundary condition $\mathcal{L}(t=0) = X/2$. The solution (3.55) is valid for arbitrary target space metric, generalising the case of a flat metric which already appeared in [47]. As was explained the B-term does not enter the analysis and is only introduced through $\mathcal{L}_B(t=0) = X/2 + B_{ij}\tilde{X}^{ij}\Omega^{-1}$ resulting in the deformed action

$$\mathcal{L} = -\frac{1}{2t} + \frac{1}{2t}\sqrt{1 + 2tX + 2t^2\tilde{X}^{ij}\tilde{X}_{ij}\Omega^{-2}} + B_{ij}\tilde{X}^{ij}\Omega^{-1}\,. \tag{3.56}$$

3.6 WZW model

The analysis of σ-models in section 3.5 can readily be applied to Wess-Zumino-Witten (WZW) models. For simplicity we limit the discussion to the case of $SU(N)$ WZW theory described by the action

$$S_\circ = \frac{k}{8\pi}\int_M \mathrm{d}^2x\sqrt{g}g^{\mu\nu}\,\mathrm{Tr}\left(\gamma^{-1}\partial_\mu\gamma\gamma^{-1}\partial_\nu\gamma\right) + \frac{ik}{16\pi^2}\int_{B^3}\left[\mathrm{Tr}\,\gamma^{-1}\mathrm{d}\gamma\right]^3 \tag{3.57}$$

where B^3 is any three manifold whose boundary is the the worldsheet M. The second term in the WZW action above is topological and thus, as explained in section 3.5, does not enter the flow equation and can be treated as a shift in the initial conditions.

In order to use the results of the previous section we first have to express the WZW action in the σ-model variables. To this end we define the $su(N)$-valued vector field

$$A_\mu^a t^a = \gamma^{-1}\partial_\mu \gamma \tag{3.58}$$

where t^a denote the generators of the $su(N)$. In analogy with the σ-model analysis of the previous section we define the scalars

$$X^{ab} = \frac{k}{4\pi} g^{\mu\nu} A_\mu^a A_\nu^b \,, \tag{3.59}$$

and the scalar densities

$$\tilde{X}^{ab} = \frac{k\sqrt{g}}{4\pi} \varepsilon^{\mu\nu} A_\mu^a A_\nu^b \,. \tag{3.60}$$

The deformed Lagrangian satisfies the same equation (3.54) as any σ-model. The resulting deformed action is therefore given by

$$S = \int_M \mathrm{d}^2x\sqrt{g}\left[-\frac{1}{2t} + \frac{1}{2t}\sqrt{1+2tX+2t^2\tilde{X}^{ij}\tilde{X}_{ij}\Omega^{-2}}\right] + \frac{ik}{16\pi^2}\int_{B^3}\left[\mathrm{Tr}\,\gamma^{-1}\mathrm{d}\gamma\right]^3 \,, \tag{3.61}$$

where the terms under the square-root are expressed in terms of the original fields γ as

$$X = \frac{k}{4\pi} g^{\mu\nu}\,\mathrm{Tr}\left(\gamma^{-1}\partial_\mu\gamma\gamma^{-1}\partial_\nu\gamma\right) \,, \tag{3.62}$$

and

$$\frac{\tilde{X}^{ij}\tilde{X}_{ij}}{\Omega^2} = \left(\frac{k}{4\pi}\right)^2 \varepsilon^{\mu\nu}\varepsilon^{\rho\sigma}\,\mathrm{Tr}\left(\gamma^{-1}\partial_\mu\gamma\gamma^{-1}\partial_\rho\gamma\right)\mathrm{Tr}\left(\gamma^{-1}\partial_\nu\gamma\gamma^{-1}\partial_\sigma\gamma\right) \,. \tag{3.63}$$

3.7 Massive Thirring model

We now turn our attention to theories with fermionic fields. Consider a single Dirac fermion with the undeformed action

$$S_\circ = \int \mathrm{d}^2x\sqrt{g}\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi\right) + V\right] \,, \tag{3.64}$$

where the potential is given by

$$V = -m\bar{\psi}\psi + \frac{\lambda}{4}\bar{\psi}\gamma^a\psi\;\bar{\psi}\gamma_a\psi \,. \tag{3.65}$$

The covariant derivative acts on the fermions via

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{i}{2}\omega_\mu\gamma^3\psi, \qquad \nabla_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \frac{i}{2}\omega_\mu\bar{\psi}\gamma^3, \qquad \gamma^\mu = e_a^\mu\gamma^a \,, \tag{3.66}$$

and the spin connection (in two dimensions) satisfies

$$\omega_\mu^{ab} = \epsilon^{ab}\left(\frac{1}{2}\epsilon_{cd}\omega_\mu^{cd}\right) = \epsilon^{ab}\omega_\mu\,, \tag{3.67}$$

with $\epsilon^{12} = -\epsilon^{21} = 1$. To study the $T\bar{T}$-flow of this theory we first define the 2×2 matrix X as follows

$$X_{ab} = \frac{i}{2}\left(\bar{\psi}\gamma_a\nabla_b\psi - \nabla_b\bar{\psi}\gamma_a\psi\right)\,. \tag{3.68}$$

Here a and b are flat indices which are raised, lowered and contracted with the flat (Euclidean) metric δ^{ab}. Using (3.66) one can show that

$$X_{a\mu} = e^b_\mu X_{ab} = \frac{i}{2}\left(\bar{\psi}\gamma_a\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_a\psi\right) = \frac{i}{2}\left(\bar{\psi}\gamma_a\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_a\psi\right)\,. \tag{3.69}$$

which is manifestly independent of the metric. Since the undeformed Lagrangian (3.64) is simply $\mathcal{L}_\circ = \operatorname{Tr} X + V = e^{a\mu}X_{a\mu} + V$, we can work out the energy momentum tensor of the undeformed theory

$$T^{(0)}_{ab} = \frac{2}{\sqrt{g}}e^\mu_a e^\nu_b\frac{\delta S^{(0)}}{\delta g^{\mu\nu}} = 2e^\mu_a e^\nu_b\frac{\partial e^\lambda_c}{\partial g^{\mu\nu}}X_{c\lambda} - \delta_{ab}\mathcal{L}^{(0)} = X_{(ab)} - \delta_{ab}\left(\operatorname{Tr} X + V\right)\,. \tag{3.70}$$

It is clear that the deformed Lagrangian is constructed solely from X and V and since these only contain the fermionic fields ψ, $\bar{\psi}$ and their first derivatives we conclude that the deformed Lagrangian can only contain products of up to order X^4, X^2V and V^2 as all higher powers vanish identically. We therefore expect the t-expansion of the deformed Lagrangian to terminate. Consequently we expand the Lagrangian and the energy momentum tensor of the deformed theory as

$$\mathcal{L} = \sum_{n=0}^{N} t^n\mathcal{L}^{(n)} \qquad \text{and} \qquad T_{\mu\nu} = \sum_{n=0}^{N} t^n T^{(n)}_{\mu\nu}\,, \tag{3.71}$$

where

$$T^{(n)}_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^2x\sqrt{g}\mathcal{L}^{(n)}\,. \tag{3.72}$$

Employing this expansion we can solve the flow equation[3]

$$\partial_t\mathcal{L} = \frac{1}{2}\varepsilon^{\mu\rho}\varepsilon^{\nu\sigma}T_{\mu\nu}T_{\rho\sigma} = \frac{1}{2}\left(g^{\mu\nu}T_{\mu\nu}\right)^2 - \frac{1}{2}T^{\mu\nu}T_{\mu\nu}\,, \tag{3.73}$$

order by order. Using (3.71), our flow equation (3.73) at order t^{n-1} reads

$$\mathcal{L}^{(n)} = \frac{1}{2n}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\rho\nu})\sum_{i+j=n-1}(2-\delta_{ij})T^{(i)}_{\mu\nu}T^{(j)}_{\rho\sigma}\,. \tag{3.74}$$

[3]Here we have used the identity $g^{\mu\nu}g^{\rho\sigma} - g^{\rho\nu}g^{\mu\sigma} = \varepsilon^{\mu\rho}\varepsilon^{\nu\sigma}$ which holds true in two dimensions.

Since $\mathcal{L}^{(n)}$ only depends on the metric through X we can apply the chain rule to obtain

$$2e^\mu_a e^\nu_b \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = X_{c(a}\delta_{b)d}\frac{\partial \mathcal{L}}{\partial X_{cd}}\,. \tag{3.75}$$

Furthermore, as we will see below $\mathcal{L}^{(n)}$ only depends on the symmetric part of X which we will denote by $\tilde{X}_{ab} = X_{(ab)}$. We can now solve (3.73) order by order starting from the undeformed energy momentum tensor (3.70). At order t^0 we have

$$\mathcal{L}^{(1)} = \frac{1}{2}\,(\mathrm{Tr}\,X)^2 - \frac{1}{2}\,\mathrm{Tr}(\tilde{X}^2) + V^2 + V\,\mathrm{Tr}\,X\,, \tag{3.76}$$

from which we can evaluate $T^{(1)}_{ab}$ as follows

$$T^{(1)}_{ab} = \mathrm{Tr}\,\tilde{X}\tilde{X}_{ab} - \left(\tilde{X}X\right)_{(ab)} + V\tilde{X}_{ab} - 2\delta_{ab}\mathcal{L}^{(1)}\,. \tag{3.77}$$

Next we analyze the flow equation (3.73) at order t^1 which, after dividing by $2t$, reads

$$\mathcal{L}^{(2)} = \frac{1}{4}\,\mathrm{Tr}\,\tilde{X}^3 - \frac{3}{8}\,\mathrm{Tr}\,\tilde{X}\,\mathrm{Tr}\,\tilde{X}^2 + \frac{1}{8}\left(\mathrm{Tr}\,\tilde{X}\right)^3 + \frac{V}{4}\left((\mathrm{Tr}\,\tilde{X})^2 - \mathrm{Tr}\,\tilde{X}^2\right)\,. \tag{3.78}$$

The corresponding contribution to the energy momentum tensor is

$$\begin{aligned} T^{(2)}_{ab} =& \frac{3}{4}(\tilde{X}^2X)_{(ab)} - \frac{3}{4}(\mathrm{Tr}\,\tilde{X})(\tilde{X}X)_{(ab)} - \frac{3}{8}\left(\mathrm{Tr}\,\tilde{X}^2 - (\mathrm{Tr}\,\tilde{X})^2\right)\tilde{X}_{ab} \\ &+ \frac{V}{2}\left((\tilde{X}X)_{(ab)} - \mathrm{Tr}\,\tilde{X}\,\tilde{X}_{ab}\right) - \delta_{ab}\mathcal{L}^{(2)}\,. \end{aligned} \tag{3.79}$$

The final term in the t-expansion of the Lagrangian is determined by equating the terms at order t^2 in (3.73). At this order we find

$$\mathcal{L}^{(3)} = -\frac{1}{6}\,\mathrm{Tr}\,\tilde{X}^4 + \frac{1}{12}(\mathrm{Tr}\,\tilde{X}^2)^2 + \frac{1}{4}\,\mathrm{Tr}\,\tilde{X}\;\,\mathrm{Tr}\,\tilde{X}^3 - \frac{5}{24}(\mathrm{Tr}\,\tilde{X})^2\,\mathrm{Tr}\,\tilde{X}^2 + \frac{1}{24}(\mathrm{Tr}\,\tilde{X})^4\,. \tag{3.80}$$

Note that the higher order terms in the t-expansion of the flow equation (3.73) vanish identically thanks to the Grassmann nature of the fermionic fields from which X is built. The final form of the $T\bar{T}$-deformed Lagrangian is therefore

$$\begin{aligned} \mathcal{L} =& \,\mathrm{Tr}\,\tilde{X} + V + \frac{t}{2}\left((\mathrm{Tr}\,\tilde{X})^2 - \mathrm{Tr}\,\tilde{X}^2 + 2V(V + \mathrm{Tr}\,\tilde{X})\right) \\ &+ \frac{t^2}{2}\left(\mathrm{Tr}\,\tilde{X}^3 - \frac{3}{2}\,\mathrm{Tr}\,\tilde{X}\;\,\mathrm{Tr}\,\tilde{X}^2 + \frac{1}{2}(\mathrm{Tr}\,\tilde{X})^3 + V(\mathrm{Tr}\,\tilde{X})^2 - V\,\mathrm{Tr}\,\tilde{X}^2\right) \\ &- \frac{t^3}{3}\left(2\,\mathrm{Tr}\,\tilde{X}^4 - (\mathrm{Tr}\,\tilde{X}^2)^2 - 3\,\mathrm{Tr}\,\tilde{X}\;\,\mathrm{Tr}\,\tilde{X}^3 + \frac{5}{2}(\mathrm{Tr}\,\tilde{X})^2\,\mathrm{Tr}\,\tilde{X}^2 - \frac{1}{2}(\mathrm{Tr}\,\tilde{X})^4\right)\,, \end{aligned} \tag{3.81}$$

where $\tilde{X}$ is given by

$$\tilde{X}_{ab} = \frac{i}{2}\left(\bar{\psi}\gamma_{(a}\nabla_{b)}\psi - \nabla_{(a}\bar{\psi}\gamma_{b)}\psi\right)\,. \tag{3.82}$$

By expanding (3.81) and using Fierz identities one can in fact show that the expression for the Lagrangian drastically simplifies to

$$\mathcal{L} = \mathcal{L}^{(0)} + \frac{t}{4}\left((\operatorname{Tr}\tilde{X})^2 - \operatorname{Tr}\tilde{X}^2 + 2m^2(\bar{\psi}\psi)^2 - 2m\bar{\psi}\psi\operatorname{Tr}\tilde{X}\right) - \frac{t^2}{8}m\bar{\psi}\psi\left((\operatorname{Tr}\tilde{X})^2 - \operatorname{Tr}\tilde{X}^2\right). \tag{3.83}$$

The explicit form of the $T\bar{T}$-deformed Lagrangian in (flat) complex coordinates is

$$\begin{aligned}\mathcal{L}(t) = {}& i\left(\bar{\psi}_-\overleftrightarrow{\partial_z}\psi_- - \bar{\psi}_+\overleftrightarrow{\partial_{\bar{z}}}\psi_+\right) - m\left(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-\right) + (\lambda - m^2 t)\bar{\psi}_+\bar{\psi}_-\psi_+\psi_- \\ & - \frac{imt}{2}\left[\bar{\psi}_+\bar{\psi}_-\left(\psi_-\partial_z\psi_- - \psi_+\partial_{\bar{z}}\psi_+\right) + \psi_+\psi_-\left(\bar{\psi}_-\partial_z\bar{\psi}_- - \bar{\psi}_+\partial_{\bar{z}}\bar{\psi}_+\right)\right] \\ & + \frac{t}{4}\left[\left(\bar{\psi}_+\overleftrightarrow{\partial_{\bar{z}}}\psi_+\right)\left(\bar{\psi}_-\overleftrightarrow{\partial_z}\psi_-\right) + \bar{\psi}_-\partial_z\bar{\psi}_-\psi_-\partial_z\psi_- + \bar{\psi}_+\partial_{\bar{z}}\bar{\psi}_+\psi_+\partial_{\bar{z}}\psi_+ - 2\left(\bar{\psi}_+\overleftrightarrow{\partial_z}\psi_+\right)\left(\bar{\psi}_-\overleftrightarrow{\partial_{\bar{z}}}\psi_-\right)\right] \\ & - \frac{mt^2}{8}\bar{\psi}_+\bar{\psi}_-\psi_+\psi_-\left(\partial_z\bar{\psi}_-\partial_{\bar{z}}\psi_+ - \partial_{\bar{z}}\bar{\psi}_+\partial_z\psi_- - 2\partial_{\bar{z}}\bar{\psi}_-\partial_z\psi_+ + 2\partial_z\bar{\psi}_+\partial_{\bar{z}}\psi_-\right).\end{aligned} \tag{3.84}$$

We stress that the expansion in the deformation parameter terminates. This is akin to the observation made in [31] for a Lorentz-breaking irrelevant deformation analogous to $T\bar{T}$. Consequently we anticipate that for the deformed theory to receive an infinite series of corrections, as is the case for the Goldstino [48], we need to turn on an infinite tower of irrelevant deformations.

3.8 Generalisation to higher dimensions

Let us consider higher dimensional generalisations of the $T\bar{T}$-deformations. Such a generalisation was recently proposed by J. Cardy [10] in the form of $|\det T|^{1/\alpha}$ with $\alpha = D-1$ in D dimensions. We will treat this generalisation, for more general values of the parameter α, in some detail later in the section.

Let us remark in passing that there is another possible generalisation of the $T\bar{T}$-deformation which remains quadratic in the energy momentum tensor. Starting in two dimensions we first use the identity $\epsilon^{\mu\nu}\epsilon^{\rho\sigma} = g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}$. This suggests the following D-dimensional generalisation of the flow equation

$$\partial_t S = \frac{1}{2}\int \mathrm{d}^D x\,\sqrt{g}\left[\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}\right)\left(\frac{-2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\rho}}\right)\left(\frac{-2}{\sqrt{g}}\frac{\delta S}{\delta g^{\nu\sigma}}\right)\right]. \tag{3.85}$$

For a single scalar field – without conformal couplings – the flow equation for the Lagrangian takes the form

$$\partial_t\mathcal{L} = (D-1)\left[(D/2)\mathcal{L}^2 - 2X\partial_X\mathcal{L}\mathcal{L}\right], \tag{3.86}$$

where $X = \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi$, which reduces to the Burgers' equation. Although we do not treat this case further in this work, let us note that this could have a more natural AdS dual interpretation compared to the $(\det T)^{1/\alpha}$. Whether either of these generalisations can be defined at the quantum level remains an open question. We remark that the scaling solution, i.e. with $\mathcal{L}_{t=0} = \frac{1}{2}X$, of this equation is given by solving the algebraic equation $\left(1 + \frac{D(D-1)}{2} t\mathcal{L}\right)^{4-D} \mathcal{L}^D = (X/2)^D$. Therefore, the free massless scalar in four dimensions is a fixed point of the flow and one needs to turn on a potential (or a conformal coupling) to have a nontrivial evolution.

We now turn our attention to the flow instigated by an operator of the form $(-\det T)^{1/\alpha}$ in D-dimensions resulting in the flow equation

$$\partial_t S = \frac{1}{\alpha - D} \int \mathrm{d}^D x \sqrt{g} \left[\frac{-1}{D!} \epsilon^{\mu_1 \ldots \mu_D} \epsilon^{\nu_1 \ldots \nu_D} \left(\frac{-2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu_1 \nu_1}} \right) \ldots \left(\frac{-2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu_D \nu_D}} \right) \right]^{1/\alpha} \tag{3.87}$$

where α is a real parameter[4]. For this to be an irrelevant deformation for CFTs we take $0 < \alpha < D$ and we further assume α to be an integer. Let us integrate the above equation in the case of a scalar field ϕ by reducing it to a partial differential equation. Once more we define $X = \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi$ and write the solution as

$$S = \int \mathrm{d}^D x \sqrt{g} \mathcal{L}(X, t) \tag{3.88}$$

with the initial condition

$$\mathcal{L}(X, 0) = \frac{1}{2} X + V \tag{3.89}$$

for a generic local potential $V = V(\phi)$. Since the deformed Lagrangian only depends on the background metric through X the expression for the associated energy momentum tensor simplifies to

$$\frac{-2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = g_{\mu\nu} \mathcal{L} - 2 \partial_X \mathcal{L} \partial_\mu \phi \partial_\nu \phi \,. \tag{3.90}$$

Using the above expression, equation (3.87) simplifies to

$$\partial_t \mathcal{L} = \frac{1}{\alpha - D} \left[-\mathcal{L}^D + 2\mathcal{L}^{D-1} (X \partial_X) \mathcal{L} \right]^{1/\alpha} . \tag{3.91}$$

This can be further simplified by considering the redefinition

$$Y = (-1)^\alpha X^{\frac{\alpha - D}{2}} \quad \text{and} \quad \mathcal{L} = \sqrt{X} f^{\frac{1}{D-\alpha}} \,, \tag{3.92}$$

yielding

$$(\partial_t f)^\alpha + f^\alpha \partial_Y f = 0 \,, \tag{3.93}$$

[4]The case of [10] is $\alpha = D - 1$, but we keep this parameter free to different examples.

which reduces to the Burgers' equation for $\alpha = 1$. The relevant initial condition is obtained by inverting the above redefinitions in $f = \left(X^{-1/2}\mathcal{L}\right)^{D-\alpha}$ and $X = [(-1)^\alpha Y]^{\frac{2}{\alpha-D}}$. The initial condition therefore takes the form

$$f(0,Y) = (-1)^\alpha Y \left\{\frac{1}{2}\left[(-1)^\alpha Y\right]^{\frac{2}{\alpha-D}} + V\right\}^{D-\alpha} . \tag{3.94}$$

Below we solve (3.93) with this initial condition in a few cases.

The solution of the Burgers' equation, that is (3.93) with $\alpha = 1$, is given by solving the implicit equation,

$$f(t,Y) = f\left(0, Y - tf(t,Y)\right) , \tag{3.95}$$

which for our initial condition reads

$$\frac{f}{tf - Y} = \left[\frac{1}{2}\left(tf - Y\right)^{\frac{2}{1-D}} + V\right]^{D-1} . \tag{3.96}$$

Exact solutions of equation (3.96) can be obtained explicitly for low values of the dimension D and for a generic potential. The solution drastically simplifies in the case of the massless free scalar, *i.e.* $V = 0$, and reads as

$$f(t,Y) = \frac{1}{2t}\left(Y + \sqrt{Y^2 + \frac{t}{2^{D-3}}}\right) . \tag{3.97}$$

This results in the deformed Lagrangian

$$\mathcal{L}_{D,1}(t,X) = \left\{\frac{1}{2t}\left[\sqrt{1 + 4t(X/2)^{D-1}} - 1\right]\right\}^{1/(D-1)} . \tag{3.98}$$

Another interesting case which can be simplified is the scaling solution for the free massless scalar field and arbitrary value of α. In this case the differential equation,

$$(\partial_t f)^\alpha + f^\alpha \partial_Y f = 0 , \tag{3.99}$$

is accompanied by the simple boundary condition $f(0,Y) = \frac{(-1)^\alpha}{2^{D-\alpha}}\frac{1}{Y}$. We can further reduce eq. (3.99) by using scaling symmetry. This enables us to set

$$f(t,Y) = (-\alpha/2)^\alpha t^{-\frac{\alpha}{2}} K\left(Yt^{-\frac{\alpha}{2}}\right) , \tag{3.100}$$

and reduces the PDE above to an ODE in the variable $Z = Yt^{-\frac{\alpha}{2}}$:

$$0 = \left(1 + Z\frac{K'}{K}\right)^\alpha + K' . \tag{3.101}$$

This equation should have a solution such that $K \sim \frac{1}{kZ}$ for large positive Z and with $k = 2^{D-2\alpha}(\alpha)^{\alpha}$. After some thought one can see that such a solution indeed exists. Of course, the case $\alpha = 1$ reproduces the solution (3.97) for the Burgers' equation. The explicit solution for the first few small integer values of α can be computed by reducing to quadratures. For example, at $\alpha = 2$ we find

$$K = \frac{1}{c^2 Z + c} \quad \text{with} \quad c^2 = 2^{D-2}, \tag{3.102}$$

where the two possible signs of c are related by the $t \to -t$ symmetry of the equation. The corresponding Lagrangian (with $c = 2^{(D-2)/2}$) is

$$\mathcal{L}_{D,2} = \frac{X}{2} \frac{1}{\left[1 + t(X/2)^{\frac{D-2}{2}}\right]^{\frac{1}{D-2}}} \,. \tag{3.103}$$

In particular, for $D = 3$, the above Lagrangian is the result of integrating the $(\det T)^{\frac{1}{D-1}}$ deformation proposed in [10] for a free scalar field theory in three dimensions which takes the simple form

$$\mathcal{L}_{3,2} = \frac{X}{2} \frac{1}{1 + t\sqrt{X/2}} \,. \tag{3.104}$$

Chapter 4

Conclusions and discussions

In this thesis we review all the background of $T\bar{T}$ deformation and the applications to holography. Then we focus on studying the flow equation for the $T\bar{T}$ deformed QFTs and its extensions to higher dimensions, both for conformal and for massive theories.

In Chapter 2, we first review the construction of $T\bar{T}$ operator by Zamolodchikov [4]. The non-derivative divergent parts of the operators $T(z)\bar{T}(z')$ and $\Theta(z)\Theta(z')$ cancel each other in the limit $z \to z'$. Thus we can define the compostie operator $T\bar{T} = T(z)\bar{T}(z') - \Theta(z)\Theta(z')$ upto total derivative terms. The $T\bar{T}$ operator turns out to be a special case of a kind of local operators X_s, which are constructed from local conserved currents $(T_{s+1}(z), \Theta_{s-1}(z))$ by $T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')$. Each X_s generate a deformation of the IQFT by equation (1.4). Smirnov and Zamolodchikov [9] proved the above deformation preserves integrability. The idea is under the deformation if we deform the conserved currents in a systematic way the local IM P_σ of the theory are still conserved. The $T\bar{T}$ deformation is special in that the deformed theories are solvable in a certain sense, even when the original theory is not integrable. When the theory is put on a cylinder with circumference R one can derive the closed differential equation for energy levels (2.90) for the $T\bar{T}$ deformed theory from which one can solve for the energy levels. The flow equation for the deformed Lagrangian density (2.80) is also discussed. We then review the deformation of two particle S-matrix under the $T\bar{T}$ deformation of CFT. At last we review the cutoff AdS/$T\bar{T}$-deformed CFT duality following [35]. In this duality the $T\bar{T}$ deformation represents a geometric cutoff that places the QFT on a Dirichlet boundary at finite radial distance. The motivation is the correspondence between quantities computed on the two sides, including the deformed energy spectrum and the perturbative propagation speeds.

In Chapter 3 we propose a simple integration technique for the $T\bar{T}$ flow equation. The flow equation induced by the $T\bar{T}$ deformation can be reformulated as a functional equation. We notice the deformation operator does not generate terms involving derivatives of the metric unless such terms are already present in the undeformed Lagrangian. The flow equation reduces to a partial differential equation in the deformation variable t and invariants formed from the metric and the dynamical fields. In many cases the PDE can be solved exactly giving the deformed Lagrangian in closed form. We study many cases including notably non-linear σ-models and the massive Thirring model. We also try to make some generalisations to higher dimensions.

There are many other theories of interest in two dimensions to which our approach can be applied, notably Yang-Mills theories and gauged linear σ-models as well as their supersymmetric counterparts. Moreover, the method presented here can be extended to flows instigated by analogs of the $T\bar{T}$-operator involving higher spin currents proposed in [9] or symmetry breaking currents such as the one discussed in [31]. One of the most pressing questions left unanswered is the issue of extending the exact integration method to theories whose undeformed action includes curvature couplings, such as the conformal coupling term for scalars in higher dimensions. Another crucial issue yet to be addressed with regards to the higher dimensional generalisations of the $T\bar{T}$ operator is to analyze the existence or absence of contact terms, along the lines of [4], for the composite operator $(\det T)^{1/\alpha}$ in $D > 2$ and for different values of α. The holographic interpretation of $T\bar{T}$-like deformations in higher dimensions is also of great interest and needs to be addressed.

Appendix A

Method of characteristic curves

The flow equation can actually be solved independently through the method of characteristic curves. Lets first rewrite the equation in a more familiar form by defining

$$y := -2X^{-1/2}, \qquad f(y,t) := -\frac{1}{2} y\, \mathcal{L}(4/y^2, t) \tag{A.1}$$

The equation translates to the following equation for our new function f as a function of the new variable y and t:

$$\partial_t f + f \partial_y f = 0 \tag{A.2}$$

This equation is known as the Burger's equation and can be solved as follows. Consider the three dimensional space $\mathbb{R}^3$ spanned by three real variables $\{t, y, f\}$. The above equation defines a (hyper)surface Σ in this space whose normal vector field is

$$n = (\partial_t f,\, \partial_y f,\, -1) \tag{A.3}$$

since the one for $\partial_t f\, \mathrm{d}t + \partial_y f\, \mathrm{d}y - \mathrm{d}f$ vanishes identically when projected onto the surface $f = f(t, y)$. Equation (A.2) has a simple geometrical interpretation as it simply states that the vector field

$$v = (1,\, f,\, 0) \tag{A.4}$$

is tangent to Σ and defines a characteristic curve. We can parameterize such a curve with a parameter s and set

$$v = \left(\frac{\mathrm{d}t}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}f}{\mathrm{d}s}\right) \quad \Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = f, \quad \frac{\mathrm{d}f}{\mathrm{d}s} = 0 \tag{A.5}$$

The first equation $\frac{\mathrm{d}t}{\mathrm{d}s} = 0$ enables us choose $s = t$ to parameterize the curve. The second and third equation together then yield

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0 \quad \Rightarrow \quad y(t) = y_\circ + tf \quad \Rightarrow \quad y_\circ = y(t) - tf \tag{A.6}$$

Note that the third equation as an equation for $f = f(y(t), t)$ reads

$$\frac{\mathrm{d}f(y(t), t)}{\mathrm{d}t} = \partial_t f + \frac{\mathrm{d}y}{\mathrm{d}t}\partial_y f = \partial_t f + f\partial_y f = 0 \tag{A.7}$$

which is simply (A.2). However, this equation is also constraining $f(y(t), t)$ to be constant along the curve. This implies

$$f(y(t), t) = f(y(0), 0) = f(y_\circ, 0) = f(y(t) - tf, 0) \tag{A.8}$$

where $f(u, 0)$ is related to the *undeformed* Lagrangian

$$f(u, 0) = -\frac{1}{2}u\,\mathcal{L}(4/u^2, 0) \tag{A.9}$$

Appendix B

Details of integrating $T\bar{T}$ deformed theories with curvature couplings

Consider the undeformed Lagrangian

$$\mathcal{L}(t=0) = \frac{1}{2}X + \alpha_\circ \phi R \tag{B.1}$$

where R denotes the Ricci scalar associated with the background metric $g_{\mu\nu}$. This Lagrangian describes a theory with central charge $c = 1 + 6Q^2$, where $\alpha_\circ = \sqrt{2\pi}Q$. let us use the notation $\mathcal{L}_i^{(j)}$ to denote the expansion of term order (t^j, α_0^i). The expansion can be written as

$$\mathcal{L} = \sum_i \sum_j \alpha_0^i t^j \mathcal{L}_i^{(j)} \tag{B.2}$$

$$\partial_t \mathcal{L} = \sum_i \sum_{j=1} \alpha_0^i j t^{j-1} \mathcal{L}_i^{(j)}. \tag{B.3}$$

The flow equation is expanded as

$$\sum_i \sum_{j=0} (j+1)\, \alpha_0^i t^j \mathcal{L}_i^{(j+1)} = \sum_{q=0}^{i} \sum_{k=0}^{j} \alpha_0^i t^j \varepsilon^{\mu\rho} \varepsilon^{\upsilon\sigma} T_{q\mu\nu}^{(k)} T_{(i-q)\rho\sigma}^{(j-k)} \tag{B.4}$$

where

$$T_{i\mu\nu}^{(j)} = -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^2x \sqrt{g} \mathcal{L}_i^{(j)}. \tag{B.5}$$

The initial condition is now

$$\mathcal{L}_0^{(0)} = \frac{1}{2}X \tag{B.6}$$

$$\mathcal{L}_1^{(0)} = \alpha_0 \phi R. \tag{B.7}$$

Also we know that at order α_0^0 we recover the result of free massless scalar

$$\mathcal{L}_0 = \frac{-1+\sqrt{1+2tX}}{2t}. \tag{B.8}$$

To calculate the energy-momentum tensor we need to vary the Ricci scalar R against the metric $g_{\mu\nu}$. For example let us compute the variation of $\phi\sqrt{g}R$

$$\begin{aligned}
&\delta\left(\phi\sqrt{g}R\right) \\
&= \delta\left(\phi\sqrt{g}g^{\mu\nu}R_{\mu\nu}\right) \\
&= \phi\sqrt{g}\left(g^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}R\right).
\end{aligned} \tag{B.9}$$

The first term is

$$\begin{aligned}
&\phi\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} \\
&= \phi\sqrt{g}\nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\lambda{}_{\mu\lambda}\right) \\
&= \phi\partial_\rho\left(\sqrt{g}g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - \sqrt{g}g^{\mu\rho}\delta\Gamma^\lambda{}_{\mu\lambda}\right).
\end{aligned} \tag{B.10}$$

Variation of Christoffel symbol is

$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left[\nabla_\upsilon\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\upsilon\lambda} - \nabla_\lambda\delta g_{\mu\upsilon}\right] \tag{B.11}$$

$$\delta\Gamma^\beta{}_{\mu\beta} = \frac{1}{2}g^{\beta\lambda}\left[\nabla_\beta\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\beta\lambda} - \nabla_\lambda\delta g_{\mu\beta}\right]. \tag{B.12}$$

So the first term is computed as

$$\begin{aligned}
&\sqrt{g}\phi g^{\mu\nu}\delta R_{\mu\nu} \\
&= -\frac{1}{2}\sqrt{g}\partial_\rho\phi\left(g^{\mu\nu}g^{\rho\lambda}\left[\nabla_\upsilon\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\upsilon\lambda} - \nabla_\lambda\delta g_{\mu\upsilon}\right] - g^{\mu\rho}g^{\beta\lambda}\left[\nabla_\beta\delta g_{\mu\lambda} + \nabla_\mu\delta g_{\beta\lambda} - \nabla_\lambda\delta g_{\mu\beta}\right]\right) \\
&= -\frac{1}{2}\sqrt{g}\partial_\rho\phi\left(-\nabla_\upsilon\delta g^{\rho\upsilon} - \nabla_\mu\delta g^{\rho\mu} + \nabla_\lambda\left(\delta g^{\mu\nu}g^{\rho\lambda}g_{\mu\upsilon}\right) - \left(-\nabla_\beta\delta g^{\beta\rho} - \nabla_\mu\left(g^{\mu\rho}g_{\beta\lambda}\delta g^{\beta\lambda}\right) + \nabla_\lambda\delta g^{\rho\lambda}\right)\right) \\
&= -\sqrt{g}\nabla_\mu\nabla_\rho\phi\delta g^{\rho\mu} + \frac{1}{2}\sqrt{g}\left(\nabla_\lambda\nabla_\rho\phi\left(\delta g^{\mu\nu}g^{\rho\lambda}g_{\mu\upsilon}\right) + \frac{1}{2}\sqrt{g}\nabla_\mu\nabla_\rho\phi\left(g^{\mu\rho}g_{\beta\lambda}\delta g^{\beta\lambda}\right)\right) \\
&= \frac{1}{2}\sqrt{g}\left(\Box\phi g_{\mu\upsilon}\delta g^{\mu\nu} + \Box\phi g_{\beta\lambda}\delta g^{\beta\lambda}\right) - \sqrt{g}\nabla_\mu\nabla_\rho\phi\delta g^{\rho\mu} \\
&= \sqrt{g}\Box\phi g_{\mu\upsilon}\delta g^{\mu\nu} - \sqrt{g}\nabla_\mu\nabla_\upsilon\phi\delta g^{\mu\upsilon}
\end{aligned} \tag{B.13}$$

where

$$\Box\phi = g^{\rho\lambda}\nabla_\lambda\nabla_\rho\phi. \tag{B.14}$$

So we get

$$\begin{aligned}
&\delta\left(\phi\sqrt{g}R\right) \\
&= \sqrt{g}\phi\left(g^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}R\right) \\
&= \sqrt{g}\phi g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{g}\phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}R\right)\delta g^{\mu\nu} \\
&= \sqrt{g}\Box\phi g_{\mu\upsilon}\delta g^{\mu\nu} - \sqrt{g}\nabla_\mu\nabla_\upsilon\phi\delta g^{\mu\upsilon} + \sqrt{g}\phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} \\
&= \sqrt{g}\Box\phi g_{\mu\upsilon}\delta g^{\mu\nu} - \sqrt{g}\nabla_\mu\nabla_\upsilon\phi\delta g^{\mu\upsilon}.
\end{aligned} \tag{B.15}$$

In two dimensions we have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{B.16}$$

The flow equation at order α_0 is

$$\begin{aligned}
\alpha_0\mathcal{L}_1^{(1)} &= \sum_{q=0}^{1}\alpha_0\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(0)}_{q\mu\nu}T^{(0)}_{(i-q)\mu\nu} \\
&= 2\alpha_0\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(0)}_{0\mu\nu}T^{(0)}_{1\rho\sigma}.
\end{aligned} \tag{B.17}$$

We need to compute $T^{(0)}_{0\mu\nu}$ and $T^{(0)}_{1\mu\nu}$

$$T^{(0)}_{0\mu\nu} = \frac{1}{2}X_{\mu\nu} - \frac{1}{4}g_{\mu\nu}X \tag{B.18}$$

$$T^{(0)}_{1\mu\nu} = \Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi \tag{B.19}$$

and the corresponding trace

$$TrT^{(0)}_0 = 0 \tag{B.20}$$

$$TrT^{(0)}_1 = 2\Box\phi - \Box\phi + R\phi - \phi R = \Box\phi \tag{B.21}$$

where we have used

$$\begin{aligned}
&\phi\delta R \\
&= \phi\left(g^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}R_{\mu\nu}\right) \\
&= \phi g^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu}R_{\mu\nu}\phi \\
&= \Box\phi g_{\mu\upsilon}\delta g^{\mu\nu} - \nabla_\mu\nabla_\upsilon\phi\delta g^{\mu\upsilon} + \delta g^{\mu\nu}R_{\mu\nu}\phi.
\end{aligned} \tag{B.22}$$

Since

$$\begin{aligned}&\phi g^{\mu\nu}\delta R_{\mu\nu} \\ &= \Box\phi g_{\mu\upsilon}\delta g^{\mu\nu} - \nabla_\mu\nabla_\upsilon\phi\delta g^{\mu\upsilon},\end{aligned} \tag{B.23}$$

$\mathcal{L}_1^{(1)}$ is easily computed

$$\begin{aligned}&\mathcal{L}_1^{(1)} \\ &= -4\left(\frac{1}{2}X^{\mu\nu} - \frac{1}{4}g^{\mu\nu}X\right)(\Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi) \\ &= -2\Box\phi X.\end{aligned} \tag{B.24}$$

We go on to compute $\mathcal{L}_1^{(2)}$

$$2\alpha_0 t\mathcal{L}_1^{(2)} = 2\alpha_0 t\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{0\mu\nu}^{(0)}T_{1\mu\nu}^{(1)} + 2\alpha_0 t\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{0\mu\nu}^{(1)}T_{1\mu\nu}^{(0)}. \tag{B.25}$$

The first term is computed as

$$T_{0\mu\nu}^{(0)} = \frac{1}{2}X_{\mu\nu} - \frac{1}{4}g_{\mu\nu}X \tag{B.26}$$

$$TrT_0^{(0)} = 0 \tag{B.27}$$

$$\begin{aligned}T_{1\mu\nu}^{(1)} &= -\Box\phi X_{\mu\nu} + \frac{1}{2}\nabla_\mu X\partial_\nu\phi + \frac{1}{2}\nabla_\nu X\partial_\mu\phi - \frac{1}{2}\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu} - \frac{1}{2}X\Box\phi g_{\mu\nu} + \frac{1}{2}\Box\phi X g_{\mu\nu} \\ &= -\Box\phi X_{\mu\nu} + \frac{1}{2}\nabla_\mu X\partial_\nu\phi + \frac{1}{2}\nabla_\nu X\partial_\mu\phi - \frac{1}{2}\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu}.\end{aligned} \tag{B.28}$$

So

$$\begin{aligned}&\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{0\mu\nu}^{(0)}T_{1\mu\nu}^{(1)} \\ &= 4\left(-\frac{1}{2}X^{\mu\nu} + \frac{1}{4}g^{\mu\nu}X\right)\left(-\Box\phi X_{\mu\nu} + \frac{1}{2}\nabla_\mu X\partial_\nu\phi + \frac{1}{2}\nabla_\nu X\partial_\mu\phi - \frac{1}{2}\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu}\right) \\ &= \frac{3}{2}X^2\Box\phi.\end{aligned} \tag{B.29}$$

To compute the second term we need

$$T_{0\mu\nu}^{(1)} = -\frac{1}{4}XX_{\mu\nu} + \frac{1}{16}X^2 g_{\mu\upsilon} \tag{B.30}$$

$$TrT_0^{(1)} = -\frac{1}{8}X^2 \tag{B.31}$$

$$T_{1\mu\nu}^{(0)} = \Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi \tag{B.32}$$

$$TrT_1^{(0)} = \Box\phi. \tag{B.33}$$

Thus

$$\begin{aligned}
&\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(1)}_{0\mu\nu}T^{(0)}_{1\rho\sigma} \\
&= -\frac{1}{2}X^2\Box\phi + 4\left(\frac{1}{4}XX^{\mu\nu} - \frac{1}{16}X^2g^{\mu\nu}\right)(\Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi) \\
&= \frac{1}{2}X^2\Box\phi.
\end{aligned} \tag{B.34}$$

Together we have

$$\begin{aligned}
\mathcal{L}^{(2)}_1 &= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(0)}_{0\mu\nu}T^{(1)}_{1\mu\nu} + \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(1)}_{0\mu\nu}T^{(0)}_{1\mu\nu} \\
&= \frac{3}{2}X^2\Box\phi + \frac{1}{2}X^2\Box\phi \\
&= 2X^2\Box\phi.
\end{aligned} \tag{B.35}$$

Similarly for $\mathcal{L}^{(3)}_1$

$$3\mathcal{L}^{(2+1)}_1 = 2\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(1)}_{0\mu\nu}T^{(1)}_{1\rho\sigma} + 2\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(2)}_{0\mu\nu}T^{(0)}_{1\rho\sigma} + 2\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T^{(0)}_{0\mu\nu}T^{(2)}_{1\rho\sigma}. \tag{B.36}$$

Let us collect the energy-momentum tensors

$$T^{(1)}_{0\mu\nu} = -\frac{1}{4}XX_{\mu\nu} + \frac{1}{16}X^2g_{\mu\upsilon} \tag{B.37}$$

$$T^{(1)}_{1\mu\nu} = -\Box\phi X_{\mu\nu} + \frac{1}{2}\nabla_\mu X\partial_\nu\phi + \frac{1}{2}\nabla_\nu X\partial_\mu\phi - \frac{1}{2}\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu} \tag{B.38}$$

$$T^{(2)}_{0\mu\nu} = \frac{3}{16}X^2X_{\mu\nu} - \frac{1}{32}g_{\mu\upsilon}X^3 \tag{B.39}$$

$$T^{(0)}_{1\mu\nu} = \Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi \tag{B.40}$$

$$T^{(0)}_{0\mu\nu} = \frac{1}{2}X_{\mu\nu} - \frac{1}{4}g_{\mu\nu}X \tag{B.41}$$

$$T^{(2)}_{1\mu\nu} = X\Box\phi X_{\mu\nu} - \frac{1}{2}X\nabla_\mu X\partial_\nu\phi - \frac{1}{2}X\nabla_\nu X\partial_\mu\phi + \frac{1}{2}X\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu} - \frac{1}{4}X^2\Box\phi g_{\mu\nu} \tag{B.42}$$

where we have used

$$\begin{aligned}
&\frac{1}{2}\alpha_0X^2\delta(\Box\phi) \\
&= -\alpha_0X\nabla_\alpha X\partial_\lambda\phi\delta g^{\alpha\lambda} + \frac{1}{2}\alpha_0X\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\alpha\beta}\delta g^{\alpha\beta} + \frac{1}{4}\alpha_0X^2\Box\phi g_{\alpha\beta}\delta g^{\alpha\beta}
\end{aligned} \tag{B.43}$$

and the trace

$$TrT_0^{(1)} = -\frac{1}{8}X^2 \tag{B.44}$$
$$TrT_1^{(1)} = -\Box\phi X \tag{B.45}$$
$$TrT_0^{(2)} = \frac{1}{8}X^3 \tag{B.46}$$
$$TrT_1^{(0)} = \Box\phi \tag{B.47}$$
$$TrT_0^{(0)} = 0 \tag{B.48}$$
$$TrT_1^{(2)} = \frac{1}{2}X^2\Box\phi. \tag{B.49}$$

Together we get

$$\mathcal{L}_1^{(3)} = -\frac{8}{3}X^3\Box\phi. \tag{B.50}$$

Put the first terms together

$$\mathcal{L}_1 = \phi R - 2tX\Box\phi + 2t^2X^2\Box\phi - \frac{8}{3}t^3X^3\Box\phi + \mathcal{O}\left(t^3\right). \tag{B.51}$$

This leads us to consider the following ansatz

$$\mathcal{L}_1 = \phi R + f\left(tX\right)\Box\phi. \tag{B.52}$$

Then

$$\begin{aligned} T_{1\mu\nu} &= \Box\phi g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi + f'\left(tX\right)tX_{\mu\nu}\Box\phi - \frac{1}{2}\partial_\mu f\partial_\nu\phi - \frac{1}{2}\partial_\nu f\partial_\mu\phi \\ &\quad + \frac{1}{2}\partial_\rho f\partial^\rho\phi g_{\mu\nu} + \frac{1}{2}f\Box\phi g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f\left(tX\right)\Box\phi \\ &= \Box\phi g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi + f'\left(tX\right)tX_{\mu\nu}\Box\phi - \frac{1}{2}\partial_\mu f\partial_\nu\phi - \frac{1}{2}\partial_\nu f\partial_\mu\phi + \frac{1}{2}\partial_\rho f\partial^\rho\phi g_{\mu\nu} \end{aligned} \tag{B.53}$$

and

$$TrT_1 = \Box\phi + f'\left(tX\right)tX\Box\phi. \tag{B.54}$$

Our flow equation is

$$\partial_t\mathcal{L} = \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\rho\sigma}}\right). \tag{B.55}$$

Expand it on α_0

$$\sum_i \alpha_0^i\partial_t\mathcal{L}_i = \alpha_0^i\sum_{k=0}^{i}\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{k\mu\nu}T_{(i-k)\rho\sigma}. \tag{B.56}$$

Now for $i = 1$

$$\partial_t \mathcal{L}_1 = 2\varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}T_{0\mu\nu}T_{1\rho\sigma}. \tag{B.57}$$

Since

$$\mathcal{L}_0 = -\frac{1}{2t} + \frac{1}{2t}\sqrt{1+2tX}, \tag{B.58}$$

we have

$$T_{0\mu\nu} = \frac{X_{\mu\nu}}{2\sqrt{1+2tX}} + \frac{1}{4t}g_{\mu\nu} - \frac{\sqrt{1+2tX}}{4t}g_{\mu\nu} \tag{B.59}$$

$$TrT_0 = \frac{X}{2\sqrt{1+2tX}} + \frac{1}{2t} - \frac{\sqrt{1+2tX}}{t}. \tag{B.60}$$

The flow equation

$$\begin{aligned} & f'X\Box\phi \\ = 2\Box\phi & \left(\frac{1}{4t} - \frac{1}{4t}\sqrt{1+2tX} + \frac{1}{2}f'X - \frac{1}{2}f'\sqrt{1+2tX}X - \frac{1}{2t}q\left(2tX\right) - \frac{\sqrt{1+2tX}}{4t}\right) \end{aligned} \tag{B.61}$$

where

$$q\left(y\right) = 1 + \int \frac{y}{2\sqrt{1+2y}}f'dy. \tag{B.62}$$

Multiply t on both sides we get

$$2\sqrt{1+2y} + f'y\sqrt{1+y} + 2q\left(y\right) - 2 = 0 \tag{B.63}$$

where

$$y = tX. \tag{B.64}$$

solve for f'

$$f'\left(y\right) = -\frac{1}{1+2y} + \frac{C_1}{2y\left(1+2y\right)} \tag{B.65}$$

$$f\left(y\right) = C_1 \log 2y - \left(1+C_1\right)\log\left(1+2y\right) + C_2 \tag{B.66}$$

with initial condition

$$\lim_{y\to 0} f\left(y\right) = 0. \tag{B.67}$$

we get

$$f\left(y\right) = -\log\left(1+2y\right). \tag{B.68}$$

So

$$\mathcal{L}_1 = \phi R - \log\left(1+2tX\right)\Box\phi \tag{B.69}$$

$$\mathcal{L} = -\frac{1}{2t} + \frac{1}{2t}\sqrt{1+2tX} + \alpha_0 \phi R - \alpha_0 \log\left(1+2tX\right)\Box\phi + \mathcal{O}\left(\alpha_0^2\right). \tag{B.70}$$

We can go on to investigate order α_0^2 terms. We can easily compute

$$\mathcal{L}_2^{(0)} = 0 \tag{B.71}$$

$$\mathcal{L}_2^{(1)} = \frac{1}{2}RX \tag{B.72}$$

for $\mathcal{L}_2^{(2)}$ we have

$$\begin{aligned}\mathcal{L}_2^{(2)} &= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(T^{(0)}_{2\mu\nu}T^{(1)}_{0\rho\sigma} + T^{(0)}_{1\mu\nu}T^{(1)}_{1\rho\sigma} + T^{(0)}_{0\mu\nu}T^{(1)}_{2\rho\sigma}\right) \\ &= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(T^{(0)}_{1\mu\nu}T^{(1)}_{1\rho\sigma} + T^{(0)}_{0\mu\nu}T^{(1)}_{2\rho\sigma}\right).\end{aligned} \tag{B.73}$$

We need

$$T^{(0)}_{1\mu\nu} = \Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi \tag{B.74}$$

$$T^{(1)}_{1\mu\nu} = -\Box\phi X_{\mu\nu} + \frac{1}{2}\nabla_\mu X\partial_\nu\phi + \frac{1}{2}\nabla_\nu X\partial_\mu\phi - \frac{1}{2}\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu} \tag{B.75}$$

$$T^{(0)}_{0\mu\nu} = \frac{1}{2}X_{\mu\nu} - \frac{1}{4}g_{\mu\nu}X \tag{B.76}$$

$$T^{(1)}_{2\mu\nu} = \frac{1}{2}X_{\mu\nu}R + \frac{1}{2}\Box X g_{\mu\upsilon} - \frac{1}{2}\nabla_\mu\nabla_\upsilon X \tag{B.77}$$

and the trace

$$TrT_1^{(0)} = \Box\phi \tag{B.78}$$

$$TrT_1^{(1)} = -\Box\phi X \tag{B.79}$$

$$TrT_0^{(0)} = 0 \tag{B.80}$$

$$TrT_2^{(1)} = \frac{1}{2}XR + \frac{1}{2}\Box X. \tag{B.81}$$

We get

$$\begin{aligned}\mathcal{L}_2^{(2)} &= T^{(0)}_{1\mu\nu}T^{(1)}_{1\rho\sigma} + T^{(0)}_{0\mu\nu}T^{(1)}_{2\rho\sigma} \\ &= -\frac{1}{8}RX^2 - \frac{5}{4}Y\Box\phi - \frac{1}{2}X\Box X.\end{aligned} \tag{B.82}$$

To compute $\mathcal{L}_2^{(3)}$

$$3\alpha_0^2 t^3 \mathcal{L}_2^{(2+1)} = \sum_{q=0}^{2}\sum_{k=0}^{2} \alpha_0^i t^j \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma} T_{q\mu\nu}^{(k)} T_{(i-q)\rho\sigma}^{(j-k)} \tag{B.83}$$
$$= \varepsilon^{\mu\rho}\varepsilon^{\upsilon\sigma}\left(T_{1\mu\nu}^{(1)}T_{1\rho\sigma}^{(1)} + 2T_{0\mu\nu}^{(1)}T_{2\rho\sigma}^{(1)} + 2T_{1\mu\nu}^{(0)}T_{1\rho\sigma}^{(2)} + 2T_{0\mu\nu}^{(0)}T_{2\rho\sigma}^{(2)}\right),$$

we need

$$T_{1\mu\nu}^{(1)} = -\Box\phi X_{\mu\nu} + \frac{1}{2}\nabla_\mu X\partial_\nu\phi + \frac{1}{2}\nabla_\nu X\partial_\mu\phi - \frac{1}{2}\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu} \tag{B.84}$$
$$T_{0\mu\nu}^{(1)} = -\frac{1}{4}XX_{\mu\nu} + \frac{1}{16}X^2 g_{\mu\upsilon} \tag{B.85}$$
$$T_{2\mu\nu}^{(1)} = \frac{1}{2}X_{\mu\nu}R + \frac{1}{2}\Box X g_{\mu\upsilon} - \frac{1}{2}\nabla_\mu\nabla_\upsilon X \tag{B.86}$$
$$T_{1\mu\nu}^{(0)} = \Box\phi g_{\mu\upsilon} - \nabla_\mu\nabla_\upsilon\phi \tag{B.87}$$
$$T_{1\mu\nu}^{(2)} = X\Box\phi X_{\mu\nu} - \frac{1}{2}X\nabla_\mu X\partial_\nu\phi - \frac{1}{2}X\nabla_\nu X\partial_\mu\phi + \frac{1}{2}X\partial_\gamma X\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu} - \frac{1}{4}X^2\Box\phi g_{\mu\nu} \tag{B.88}$$
$$T_{0\mu\nu}^{(0)} = \frac{1}{2}X_{\mu\nu} - \frac{1}{4}g_{\mu\nu}X. \tag{B.89}$$

Also we need to compute $T_{2\mu\nu}^{(2)}$, since

$$\mathcal{L}_2^{(2)} = -\frac{1}{8}RX^2 - \frac{5}{4}X_3\Box\phi - \frac{1}{2}X\Box X \tag{B.90}$$

$$T_{2\mu\nu}^{(2)} = -\frac{1}{4}RXX_{\mu\nu} - \frac{1}{8}\Box X^2 g_{\mu\upsilon} + \frac{1}{8}\nabla_\mu\nabla_\upsilon X^2 \tag{B.91}$$
$$+\frac{5}{8}\nabla_\mu X_3\partial_\nu\phi + \frac{5}{8}\nabla_\nu X_3\partial_\mu\phi - \frac{5}{8}\partial_\gamma X_3\partial_\lambda\phi g^{\lambda\gamma}g_{\mu\nu}$$
$$-\frac{5}{8}\partial_\mu X\partial_\nu\phi\Box\phi - \frac{5}{8}\partial_\nu X\partial_\mu\phi\Box\phi + \frac{5}{4}X_{\mu\nu}(\Box\phi)^2 + \frac{5}{4}X_{\mu\nu}\nabla_\rho\phi\nabla^\rho(\Box\phi)$$
$$+\frac{1}{2}X_{\mu\nu}\Box X + \frac{1}{2}\partial_\mu X\partial_\nu X - \frac{1}{4}Yg_{\mu\nu} - \frac{1}{4}X\Box Xg_{\mu\nu}$$
$$+g_{\mu\upsilon}\left(\frac{1}{16}RX^2 + \frac{5}{8}X_3\Box\phi + \frac{1}{4}X\Box X\right)$$

$$
\begin{aligned}
& TrT_2^{(2)} \\
= & -\frac{1}{4}RX^2 - \frac{1}{4}\Box X^2 + \frac{1}{8}\Box X^2 \\
& -\frac{5}{4}X_3\Box\phi + \frac{5}{4}X\left(\Box\phi\right)^2 + \frac{5}{4}X\nabla_\rho\phi\nabla^\rho\left(\Box\phi\right) \\
& +\frac{1}{8}RX^2 + \frac{5}{4}X_3\Box\phi + \frac{1}{2}X\Box X \\
= & -\frac{1}{8}RX^2 - \frac{1}{8}\Box X^2 + \frac{5}{4}X\left(\Box\phi\right)^2 + \frac{5}{4}X\nabla_\rho\phi\nabla^\rho\left(\Box\phi\right) + \frac{1}{2}X\Box X \\
= & -\frac{1}{8}RX^2 - \frac{1}{4}Y + \frac{1}{4}X\Box X + \frac{5}{4}X\left(\Box\phi\right)^2 + \frac{5}{4}X\nabla_\rho\phi\nabla^\rho\left(\Box\phi\right)
\end{aligned}
\tag{B.92}
$$

where

$$X_3 = \partial^\rho X\partial_\rho\phi \tag{B.93}$$

$$Y = \partial^\rho X\partial_\rho X. \tag{B.94}$$

We get

$$\mathcal{L}_2^{(3)} = \frac{1}{16}\left(3RX^3 + 6X^2\Box X - 8X^2\left(\Box\phi\right)^2 + 96XX_3\Box\phi + 6X_3^2\right). \tag{B.95}$$

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