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Two-Dimensional Gauge Theories in BV Formalism and Gluing-Cutting

2D Yang-Mills on Surfaces with Corners in BV-BFV Formalism
and A-Model Observables in the Poisson Sigma Model

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Abstract

The thesis contains the work done during the doctorate programme at SISSA under the supervision of P. Mnëv and A. Tanzini. This resulted in the following publications:

- In [12], reproduced in Chapter 3, we discuss the A-model as a gauge fixing of the Poisson Sigma Model with target a symplectic structure. We complete the discussion in [15], where a gauge fixing defined by a compatible complex structure was introduced, by showing how to recover the A-model hierarchy of observables in terms of the AKSZ observables. Moreover, we discuss the off-shell supersymmetry of the A-model as a residual BV symmetry of the gauge fixed PSM action.

- In [13], reproduced in Chapter 4, we discuss observables of an equivariant extension of the A-model in the framework of the AKSZ construction. We introduce the *A-model observables*, a class of observables that are homotopically equivalent to the canonical *AKSZ observables* but are better behaved in the gauge fixing. We discuss them for two different choices of gauge fixing: the first one is conjectured to compute the correlators of the A-model with target the Marsden-Weinstein reduced space; in the second one we recover the topological Yang-Mills action coupled with A-model so that the A-model observables are closed under supersymmetry.

- In [33], reproduced in Chapter 5, we recover the non-perturbative partition function of 2D Yang-Mills theory from the perturbative path-integral. To achieve this goal, we study the perturbative path-integral quantization for 2D Yang-Mills theory on surfaces with boundaries and corners in the Batalin-Vilkovisky formalism (or, more precisely, in its adaptation to the setting with boundaries, compatible with gluing and cutting – the BV-BFV formalism). We prove that cutting a surface (e.g. a closed one) into simple enough pieces – building blocks – choosing a convenient gauge-fixing on the pieces and assembling back the partition function on the surface, one recovers the known non-perturbative answers for 2D Yang-Mills theory.

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Contents

Chapter 1

Introduction

Locality and functorial quantum field theories

A very important feature of physical theories is *locality.* Classically, in lagrangian field theories, this is encoded in the requirement for the action to be a *local functional* of the fields. At the quantum level, the locality of the theory is expected to imply cutting-gluing compatibility for the partition function. If a manifold Σ is obtained as the gluing of two manifolds (with boundary) $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$, then the partition function on Σ should be given by the gluing of the partition functions on Σ_1 and Σ_2: $Z_\Sigma = \langle Z_{\Sigma_1}, Z_{\Sigma_2}\rangle_\gamma$.

A mathematical formulation of this idea was first introduced by Atiyah and Segal in the context of topological and conformal quantum field theories [3, 47] and led to the *functorial* point of view on quantum field theory (QFT). In this approach, a QFT is a *monoidal functor* from a category of cobordisms of fixed dimension (with additional structure, depending on the particular field theory) called the *spacetime category*, to a given monoidal target category, e.g. the category of vector spaces. In this case, a topological QFT associates, to $(d-1)$ oriented closed manifolds γ, vector spaces

$$Z\colon \gamma \to Z(\gamma)\ .$$

The vector space $Z(\gamma)$ is called the space of *boundary states.* This association should satisfy certain properties, in particular it should agree with orientation reversing map

$$Z(\gamma) \cong Z(\bar{\gamma})^*$$

with $\bar{\gamma}$ the orientation-reversed $(d-1)$-manifold and $Z(\gamma)^*$ the dual vector space, and should preserve the monoidal structure:

$$Z(\gamma \sqcup \gamma') = Z(\gamma) \otimes Z(\gamma')\ .$$

To d-manifolds with boundary, Z associates linear maps between vector spaces associated to the boundaries. If Σ is a cobordism $\Sigma\colon \gamma \longrightarrow \gamma'$, then the linear map

$$Z(\Sigma)\colon Z(\gamma) \longrightarrow Z(\gamma')$$

can be viewed as an element $Z(\Sigma) \in Z(\gamma)^* \otimes Z(\gamma') = Z(\partial\Sigma)$, where $\partial\Sigma = \overline{\gamma} \sqcup \gamma'$ is the boundary of Σ. The vector $Z(\Sigma)$ is called the state or the *partition function* on Σ. The partition function of the composition of two cobordisms $\gamma \xrightarrow{\Sigma} \gamma' \xrightarrow{\Sigma'} \gamma''$ is obtained by

$$Z(\Sigma \cup_{\gamma'} \Sigma') = \langle Z(\Sigma), Z(\Sigma')\rangle_{\gamma'}$$

where $\langle -, -\rangle_{\gamma'}$ is the dual pairing between $Z(\gamma')$ and $Z(\overline{\gamma}') = Z(\gamma')^*$.

In 1 dimension this setting can be used to describe ordinary quantum mechanics: 1-dimensional manifolds with boundaries are interpreted as time intervals, to the end-points is associated the quantum Hilbert space and to the time interval is associated the corresponding evolution operator; the gluing of two time intervals corresponds to the usual composition rule for propagators.

A lagrangian field theory is described classically by a local action functional $S^{\text{cl.}}_{\Sigma}$ on the space of classical fields $\mathfrak{F}^{\text{cl.}}_{\Sigma}$ associated to a d-manifold Σ, e.g. $\mathfrak{F}^{\text{cl.}}_{\Sigma}$ can be the space of sections of a vector bundle over Σ. A relevant question is how to construct a functorial quantum field theory starting from a classical action functional. The BV-BFV formalism aims to give an answer to this question and, as we will see in more detail, to extend this idea to manifolds with higher-codimension boundaries, e.g. corners.

Gauge symmetries and BV formalism

A key feature of many relevant field theories is *gauge symmetry*. For example, all known classical topological field theories have gauge symmetries. This kind of symmetry occurs when there is some field degree of freedom which is not determined by the equations of motion. The evolution of such gauge degrees of freedom is arbitrary and, thus, not physical. As a consequence, physical observables cannot depend on such degrees of freedom or, in other words, physical observables are invariant under gauge transformations. Of course, non-physical degrees of freedom could be eliminated from the theory by assigning to them an arbitrary, but fixed, value. Nevertheless, it is not often convenient to do so as this could spoil some nice properties of the theory or lead to unnecessary complications. Therefore it is typically preferred to use an excess of degrees of freedom in order to obtain a more transparent theory.

The Batalin-Vilkovisky (BV) formalism gives a *cohomological* description of physical observables for field theories defined on *closed manifolds* [7–9]. For simple enough gauge symmetries the BV formalism reproduces, for a certain class of gauge-fixings, the BRST quantization. Though, for more complicated symmetries (i.e. reducible or open gauge symmetries) BRST fails and it is necessary to use the BV construction.

The BV formalism is based on an odd symplectic extension of the space of classical fields. Near to the classical fields $\phi \in \mathfrak{F}^{\text{cl.}}_{\Sigma}$, for each gauge-symmetry (and for each reducibility stage in case of reducible symmetries) there is a ghost field η and for each field there is an antifield $\phi^\ddagger$ and $\eta^\ddagger$. This enriched space of fields, ghosts and antifields $\mathfrak{F}_{\Sigma}$ is a graded manifold and it is endowed with an odd symplectic structure ω of degree $|\omega| = -1$.

The gauge-symmetry of the classical theory is encoded in a differential $\mathcal{Q}$ acting on functions over $\mathfrak{F}_\Sigma$. This is an hamiltonian vector field $\iota_{\mathcal{Q}}\omega = \mathrm{d}\mathcal{S}$, where the *BV action* $\mathcal{S}$ is an extension of the classical action S. The property $\mathcal{Q}^2 = 0$ of the differential, using its hamiltonianity, can be translated in a condition for the BV action, called Classical Master Equation (CME): $\{\mathcal{S}, \mathcal{S}\} = 0$, where $\{-, -\}$ are the Poisson brackets given by the symplectic form ω. The classical observables – i.e. gauge-invariant functionals of the classical fields, evaluated on-shell – are recovered as the degree-zero component of the cohomology of $\mathcal{Q}$: $H^0_{\mathcal{Q}}(\mathfrak{F}_\Sigma)$.

The *path-integral quantization* is defined by integrating the exponential of the BV action over a *lagrangian* submanifold $\mathcal{L} \subseteq \mathfrak{F}_\Sigma$, called gauge-fixing lagrangian:

$$Z_\Sigma := \int_{\mathcal{L}\subseteq\mathfrak{F}_\Sigma} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}[\Phi]} \, [\mathfrak{D}\Phi] \ .$$

Since the space of fields is typically infinite-dimensional, the path-integral is not defined measure-theoretically and has to be understood *perturbatively* as a formal power series in $\hbar$, with coefficients given by amplitudes of Feynman diagrams.

For the partition function Z_Σ to be independent on deformations of the gauge-fixing lagrangian, the BV action has to be a solution of the *Quantum Master Equation* (QME)

$$\Delta \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}[\Phi]} = 0 \ ,$$

where Δ is an odd second order differential operator called *BV laplacian*.[1] Even though the partition function is invariant under deformations of the gauge-fixing, it may happen that two lagrangians are not in the same homology class; this can give rise to *inequivalent gauge-fixings*.

The gauge-fixed action $\mathcal{S}_{\mathcal{L}}$ inherits an odd symmetry $\mathcal{Q}^\pi_{\mathcal{L}}$, which we call *residual symmetry*, from the BV differential. This symmetry depends not only on the lagrangian $\mathcal{L}$ but also on a local symplectomorphism of a neighbourhood of $\mathcal{L}$ with the cotangent bundle $T^*[-1]\mathcal{L}$, with the canonical symplectic structure; we call this structure a *symplectic tubular neighbourhood*. In some examples – where BRST works – and for a certain choice of local symplectomorphism, the residual symmetry coincides with the BRST symmetry. The residual symmetry, in general, squares to zero only on-shell and defines an *on-shell cohomology* which depends only on the homology class of $\mathcal{L}$.

Poisson sigma model and A-model observables

BV formalism may lead to recognize apparently unrelated theories as different gauge-fixings of the same field theory. An example of this relation can be found by comparing

[1] The BV laplacian is actually well-defined only when the space of fields is finite-dimensional, e.g. for residual fields or zero-modes. In the infinite-dimensional setting, Δ needs to be regularized when acting on local functionals.

the *Poisson sigma model* (PSM) with Witten's *A-model*, which are two relevant topological quantum field theories.

The A-model is a sigma model of maps from a Riemann surface to a symplectic manifold and can be constructed, when the target space is Kähler, as a topological twist of the supersymmetric sigma model. It has a supersymmetry $\mathbf{Q}$ which, extending the model by introducing auxiliary fields, squares to zero off-shell. This supersymmetry is responsible for the localization of the A-model on the space of holomorphic maps. A-model observables compute Gromov-Witten invariants of the target symplectic manifold.

The AKSZ construction is a general procedure to lift certain geometrical data from a source – a differential graded manifold with an invariant measure – and a target manifold – a differential graded symplectic manifold with an hamiltonian differential – to produce the BV data on the space of maps between them [2]. The BV formulation of the PSM can be naturally found by the AKSZ construction applied to the space of maps between the shifted tangent bundle of a 2-dimensional surface $T[1]\Sigma$ and the shifted cotangent bundle of a Poisson manifold $T^*[1]M$ (see e.g. [11, 15]). When the target is a general Poisson structure, possibly degenerate, the PSM on the disc reproduces – for a particular gauge-fixing – the Kontsevich formula for deformation quantization of the target Poisson structure as a correlator of boundary observables [18]. For a different gauge-fixing, defined by using complex structures on source and target manifolds in the case of a Kähler target, the gauge-fixed action of PSM reproduces the A-model action after the integration of some fields [15].

In [12] we show how the relevant structures of the A-model can be interpreted in terms of natural constructions in its BV formulation as a complex gauge-fixing of the PSM. In particular, the A-model supersymmetry $\mathbf{Q}$ squaring to zero off-shell is shown to coincide with the residual symmetry $\mathcal{Q}^{\pi}_{\mathcal{L}}$ for a particular choice – depending on additional structure of the target manifold – of the symplectic tubular neighbourhood of the gauge-fixing lagrangian.

As every AKSZ theory, the PSM comes with a hierarchy of BV observables given by the lift to the space of fields of the cohomology for the differential of the target, which for the PSM is the Poisson cohomology for multivector fields. We prove the equivalence between the A-model observables and the AKSZ hierarchy of observables of the PSM by showing their difference to be $(\mathcal{Q} - \mathrm{d})$-exact. This gives a natural interpretation of the independence of the Gromov-Witten invariants on the choice of the compatible complex structure in terms of independence on the choice of the gauge fixing.

Having understood the A-model as a gauge-fixing of an AKSZ theory, in [13] we extend this discussion to the case where the target Poisson manifold is an hamiltonian G-space. In this equivariant version of the PSM, the AKSZ observables are associated to the equivariant cohomology of the target. The homotopy between A-model and AKSZ observables, found in the non-equivariant case, can be extended to the equivariant PSM and leads to the definition of a hierarchy of A-model observables also in the equivariant setting.

Contrary to the AKSZ observables, the A-model observables are well behaved under the gauge-fixings we consider and are naturally closed under the residual symmetry.

We construct two different gauge-fixings for the equivariant PSM. With the first one, we obtain a gauge-fixed action which, if the symplectic reduction of the target manifold is smooth, is conjectured to describe the A-model on the symplectic reduction. The second gauge-fixing considered is constructed to recover the supersymmetric Yang-Mills action and supersymmetry as the Lie algebra sector of the equivariant PSM gauge-fixed action and residual symmetry.

BV-BFV quantization

The BV-BFV formalism, introduced in [22–24, 26], extends the BV quantization of gauge theories to *manifolds with boundary*, in the spirit of Atiyah-Segal functorial approach to quantum field theories. It combines (a deformation of) the BV formulation for the bulk theory with a Batalin-Fradkin-Vilkovisky (BFV) theory on the boundary.

The BFV theory on the boundary, similarly to BV, gives a cohomological description of the gauge symmetries of the boundary. Geometrically, the BFV space of boundary fields is an *even symplectic graded manifold* $(\mathfrak{F}_\partial, \omega_\partial)$ with an *odd cohomological hamiltonian vector field* $\mathcal{Q}_\partial = \{\mathcal{S}_\partial, -\}$, where the odd functional $\mathcal{S}_\partial$ is the *boundary action*.

A classical BV-BFV theory associates to $(d-1)$-closed manifolds exact BFV manifolds $\mathfrak{F}_\partial$, and to d-manifolds with boundary BV manifolds over BFV manifolds, which are (-1)-symplectic manifolds of bulk fields $(\mathfrak{F}_\Sigma, \omega_\Sigma)$ with a differential $\mathcal{Q}_\Sigma$ and a projection $\pi\colon \mathfrak{F}_\Sigma \longrightarrow \mathfrak{F}_{\partial\Sigma}$ from bulk to boundary fields such that $\mathcal{Q}_\Sigma$ projects to $\mathcal{Q}_{\partial\Sigma}$. The main structure relation is the modified Classical Master Equation (mCME)

$$\iota_{\mathcal{Q}_\Sigma}\omega_\Sigma = \mathrm{d}\mathcal{S}_\Sigma + \pi^*\alpha_{\partial\Sigma} \ ,$$

where $\mathcal{S}_\Sigma$ is the bulk action and $\alpha_{\partial\Sigma}$ is a primitive of the boundary symplectic form $\omega_{\partial\Sigma} = \mathrm{d}\alpha_{\partial\Sigma}$. The AKSZ construction can be naturally extended to manifolds with boundary and gives examples of the above BV-BFV structure.

The gluing of two manifolds with boundary $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ corresponds, for classical BV-BFV theories, to the fiber product over the space of boundary fields corresponding to the gluing interface: $\mathfrak{F}_\Sigma = \mathfrak{F}_{\Sigma_1} \times_{\mathfrak{F}_\gamma} \mathfrak{F}_{\Sigma_2}$.

A classical BV-BFV theory can be seen as a functor from the category of d-dimensional cobordisms to the BV-BFV category, with objects given by BFV manifolds and morphisms given by BV manifolds over BFV manifolds.

The quantization of a classical BV-BFV theory depends on the choice of a lagrangian fibration of the space of boundary fields. Different choices give different realizations of the space of quantum states, which is defined as the graded complex $(\mathcal{H}^{\mathcal{P}}_\partial, \Omega_\partial)$ where $\mathcal{H}^{\mathcal{P}}_\partial$ is the space of half-densities $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^{\mathcal{P}}_\partial)$ on the leaf space $\mathcal{B}^{\mathcal{P}}_\partial$ of the fibration, and the coboundary operator Ω_∂ is the quantization of the boundary action $\Omega_\partial = \mathcal{S}_\partial\big(q, -\mathrm{i}\hbar\frac{\partial}{\partial q}\big)$.

The partition function is obtained perturbatively by the path-integral of the bulk action over a lagrangian in the fibers of the projection $\widetilde{\pi}\colon \mathfrak{F}_\Sigma \longrightarrow \mathfrak{F}_{\partial\Sigma} \longrightarrow \mathcal{B}^{\mathcal{P}}_{\partial}$. This perturbative path-integral is actually ill-defined in presence of zero-modes, since in this case the kinetic operator can not be inverted to give the propagator. We have thus to split a (finite-dimensional) space of residual fields $\mathcal{V}_\Sigma$ – containing at least the zero-modes – and path-integrate only on its complement.

The BV-BFV partition function depends, thus, on the boundary fields and on the residual fields: it is an half-density $Z_\Sigma \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}) \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$. For the partition function to be independent on deformations of the gauge-fixing lagrangian, it has to be a solution of the *modified Quantum Master Equation* (mQME)

$$(\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma}) Z_\Sigma = 0 \ ,$$

where $\Delta_{\mathcal{V}_\Sigma}$ is the BV laplacian on the space of residual fields.

The spaces of residual fields form a partially ordered set of different realizations. Passing from bigger to smaller realizations – by a BV pushforward of the partition function – can be interpreted as a version of Wilson's renormalization group flow, and preserves the mQME.

The gluing of partition functions is done by pairing the states in $\mathcal{H}^{\mathcal{P}}_\gamma$, where γ is the gluing interface. The glued partition function is then in a realization with the space of residual fields $\mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2}$. To obtain the partition function on $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ in a realization $\mathcal{V}_\Sigma \subseteq \mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2}$, the gluing formula is thus:

$$Z_\Sigma = \int_{\mathcal{L} \subseteq \mathcal{Y}} \langle Z_{\Sigma_1}, Z_{\Sigma_2} \rangle_{\mathcal{H}^{\mathcal{P}}_\gamma} \ ,$$

where $\mathcal{L}$ is a lagrangian submanifold and we assumed $\mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2} = \mathcal{V}_\Sigma \times \mathcal{Y}$.[2]

To obtain a state in $\mathcal{H}^{\mathcal{P}}_\gamma$ from the partition function, Z_Σ has to be *globalized*. There are different globalization procedures which can be applied to obtain a partition function depending only on the boundary fields; the relation between these procedures is not yet completely understood. In this work, we will obtain global answers by integrating out the finite-dimensional space of residual fields by a BV-pushforward. This is a non-perturbative integration and, depending on the lagrangian gauge-fixing chosen for the BV-pushforward, there may be inequivalent answers for the globalized partition function.

2D Yang-Mills on manifolds with corners

In [33] we apply BV-BFV formalism to the study of *perturbative Yang-Mills* theory on 2-dimensional surfaces with boundary. In doing so, we actually have to construct an extension of this theory to manifolds with corners. The general BV-BFV quantization for manifolds with higher-codimension strata of the boundary is still a work in progress.

[2] The triviality condition for the bundle $\mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2} \longrightarrow \mathcal{V}_\Sigma$ can actually be relaxed: it is sufficient for this bundle to be a "Hedgehog fibration" [26] for the BV-pushforward to make sense.

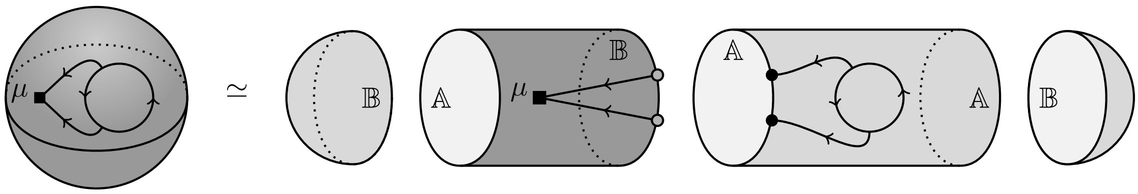
2D Yang-Mills is a gauge field theory described, in the first order formalism, by the classical action functional:

$$S_{\mathrm{YM}}^{\mathrm{cl}} = \int_{\Sigma} \langle B, \mathrm{d}A + A \wedge A \rangle + \frac{1}{2}(B, B)\, \mu \ ,$$

depending on a 1-form A valued in $\mathfrak{g}$ – the Lie algebra of the gauge group G – a $\mathfrak{g}^*$-valued function B and a background area form μ on the worldsheet Σ. This is a perturbation of the non-abelian BF theory obtained in the limit of zero area $\mu \to 0$. The classical BV formulation of 2D YM is obtained by promoting the fields A and B to non-homogeneous Lie algebra valued differential forms, the superfields $\mathsf{A} \in \Omega(\Sigma; \mathfrak{g})[1]$ and $\mathsf{B} \in \Omega(\Sigma; \mathfrak{g}^*)$ composing the BV space of bulk fields $\mathfrak{F}_\Sigma$. The boundary theory is fixed by the projection $\pi \colon \mathfrak{F}_\Sigma \longrightarrow \mathfrak{F}_{\partial\Sigma} = \Omega(\partial\Sigma; \mathfrak{g})[1] \oplus \Omega(\partial\Sigma; \mathfrak{g}^*)$, given by the restrictions $\mathbb{A}$ and $\mathbb{B}$ of the bulk fields to the boundary $\partial\Sigma$.

The goal is to compute explicitly the complete (i.e. to all orders in $\hbar$) perturbative partition function of 2D YM on an arbitrary surface and to compare our perturbative result to the non perturbative solution known in literature [40, 53]. To make this problem approachable, we use the features of BV-BFV quantization.

The main tool comes from the cutting/gluing property of BV-BFV. The perturbative effective action of 2D YM is given by infinitely many Feynman diagrams, which generally are individually very hard to compute. Gluing properties of the propagators allows to decompose this problem in smaller steps, suitably cutting a diagram into simple enough components and then gluing back their contributions. This can be seen as a special gauge-fixing, involving the data of cutting the surface on which the diagram originally lives.



Thus, it is sufficient to compute the partition function on a set of *building blocks* for the surfaces to obtain, by a gluing procedure, the answer for any surface.

Another simplification comes from the symmetries of 2D YM, which allow to shift the area form μ by an exact form to concentrate the area near boundaries. Thus we can compute all the building blocks in the *zero-area limit*, i.e. for non-abelian BF theory, except for a single cylinder which, by gluing, assigns a finite area to other surfaces.

A gauge fixing for 2D YM can be induced by a *weak Hodge decomposition* of the space of fields and defines a choice for the propagator of the gauge-fixed theory. An important class of propagators, which we use extensively and gives a fundamental contribution to the computability of the perturbative series, is given by the *axial gauge*. This is a singular gauge for product manifolds obtained collapsing first one of the two factors and then the other; axial gauge corresponds to certain singular metric on the product manifold such that the ratio of the scales of the two factors tends to zero or to infinity.

The BV-BFV partition function is defined modulo $(\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma})$-exact terms. We exploit this fact to simplify the computations by finding a suitable representative for Z_Σ. This is done by computing the partition function in a *small model* for the space of boundary states, and then lifting the result to the full space of states.

For some surfaces, e.g. the cylinder in $\mathbb{A}$-$\mathbb{A}$ polarization, it was also convenient to compute only the global partition function, obtained by integrating out residual fields. This simplifies the combinatorics of diagrams and reduces the number of relevant diagrams.

Using all the above methods, we compute the complete perturbative partition function for the disk in $\mathbb{B}$-polarization and the cylinders in $\mathbb{A}$-$\mathbb{A}$ and $\mathbb{B}$-$\mathbb{B}$ polarizations in the limit of zero area, and the cylinder in $\mathbb{A}$-$\mathbb{B}$ polarization for finite area. By gluing the above building blocks, we obtain all the cylinders and disks with finite area. The missing building block, needed to be able to glue together all the surfaces with boundary, is given by the pair of pants; this last building block eludes the analysis we are able to do with the above tools and requires the extension of the theory to *manifolds with corners*.

The space of fields which we assign to corners is defined by the restriction of the fields associated to the edges and it carries an odd symplectic structure of degree +1. We now decorate the boundary – both edges and corners – with polarizations, chosen independently for each boundary component.

In this setting, there are a number of new operations involving corners. We can:

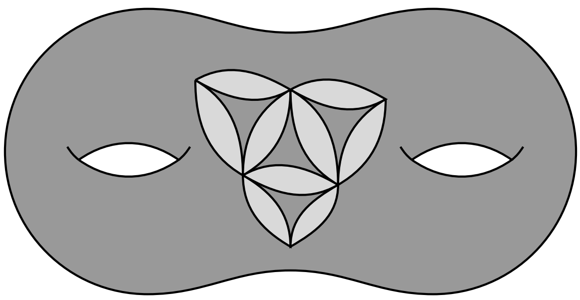
a) *split* an edge by inserting a corner in the same polarization;

b) *merge* two consecutive edges, if they are in the same polarization;

c) *switch* the polarization of a corner separating two edges in opposite polarizations;

Moreover, in some particular gauges, we can *collapse* an edge into a corner carrying the same polarization. Two domains can be glued along edges, provided that the corners where they meet are in the same polarization.

With this operations, we have a set of *building blocks for surfaces with corners*. In particular, the minimal set of building blocks necessary to recover all the surfaces with boundary (but without corners), is given by a triangle with all the edges and corners in the $\mathbb{A}$ polarization, and a "bean"– a disk with zero area and two corners – with edges in $\mathbb{B}$ polarization and corners in α polarization. Indeed, we can obtain any surface from a triangulation by associating beans to thickened edges and using them to glue together the triangles:



In the presence of corners, the operator Ω_∂ gets corrected by *corner contributions*, also depending on interactions between corners and adjacent edges. We construct Ω_∂ explicitly, with corner contributions expressed in terms of generating functions of the Bernoulli numbers. We prove that $\Omega_\partial^2 = 0$, so that the space of states for a stratified boundary is a cochain complex, and we prove that the partition function for manifolds with corners solves the mQME $(\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma}) Z_\Sigma = 0$.

Moreover, we show that the space of states for a stratified circle can be decomposed into contributions of individual corners and edges. The space of states assigned to an edge is a dg-bimodule over the space of states assigned to the corners corresponding to its endpoints. In particular, the space of states for a corner in α-polarization is the supercommutative dg-algebra $\wedge^\bullet \mathfrak{g}^*$ with Chevally-Eilenberg differential. The space of states for a corner in β-polarization is $S^\bullet \mathfrak{g}$ with zero differential; the requirement for the module product to be a chain map, w.r.t. Ω_∂ on the edge, forces a non-commutative $*_\hbar$-product on $S^\bullet \mathfrak{g}$, written in terms of the Baker-Campbell-Hausdorff formula. This structure is connected with the Baez-Dolan-Lurie picture [4, 39] for extended topological QFT, in which codimension 2 strata are mapped to algebras, codimension 1 strata to bimodules and codimension 0 strata to bimodule morphisms.

To summarize, the main results are the following:

i) extension of the BV-BFV formulation of 2D YM to manifolds with corners.

ii) expression of the complete perturbative partition function for manifolds with boundary and corners in terms of explicitly computed building blocks and gluing rules.

iii) comparison of our perturbative result with the non-perturbative solution known in literature.

1.1 Plan of the thesis

This thesis can be divided in two main parts. The first part, containing the results published in [12, 13], is about the Poisson sigma model on closed surfaces in BV formalism and is constituted by Chapters 2,3,4.

In Chapter 2 we will review the BV formalism for closed manifolds, emphasising the main constructions we will use in the following, namely the residual symmetry of the gauge-fixed action and the AKSZ construction of BV theories.

In Chapter 3 we compare the Poisson sigma model with the A-model, giving a BV interpretation to the A-model action – as a complex gauge-fixing of the PSM – to the supersymmetry – as the residual symmetry of the PSM for a particular choice of symplectic tubular neighbourhood of the gauge-fixing lagrangian – and to the A-model observables, homotopically equivalent to the AKSZ observables of the PSM.

In Chapter 4 we construct an equivariant extension of the A-model as a gauge-fixing of an AKSZ theory. We introduce, in particular, a hierarchy of A-model observables

– which are well-behaved under the gauge-fixings we consider – and prove that they are equivalent to the AKSZ observables. We then discuss different gauge-fixings of this theory, establishing a connection with either the A-model on the symplectic reduction of the target or with the topological Yang-Mills action coupled with the A-model on the target Poisson manifold.

Chapter 5 constitutes the second part of this work and contains the results published in [33] for 2D Yang-Mills in BV-BFV formalism on surfaces with boundary and corners. In Section 5.2 we will review, for the reader's convenience, the basics of the BV-BFV formalism introduced in [22–24, 26] emphasizing the constructions that we will use for the analysis in the rest of the thesis.

Sections 5.3,5.4 contain the main original results of this work. We will first, in Section 5.3, compute the perturbative partition function for 2D Yang-Mills on disks and cylinders. Then, in the first part of Section 5.4, we will discuss the extension of the BV-BFV formalism to manifolds with corners for 2D YM. This extension will be used in the second part of Section 5.4 to compute the perturbative 2D YM partition function on surfaces of arbitrary genus, which induces in Ω-cohomology the known non-perturbative answer [40, 53].

The reader who is well-acquainted with BV-BFV formalism and would like to take the shortest route to the proof of the Main Theorem, might want to read the sections in the following order. Sections 5.3.1, 5.3.3, 5.3.4 for the building blocks (I), (III), (IV), which are then assembled into the Yang-Mills on a disk in Sections 5.3.5, 5.3.5. Then in Section 5.4.1 the logic of extension to corners is explained and in Section 5.4.3 the "bean" is computed. Finally, in Sections 5.4.6, 5.4.7 polygons (obtained from the disk) are glued via beans into an arbitrary surface and thus the comparison theorem 5.4.16 is proven.

In Section 5.5 we will discuss how to compute, in this setting, Wilson loop observables for both non-intersecting and intersecting loops, recovering in Ω-cohomology the known non-perturbative result.

Chapter 2

Background: BV formalism

In this chapter we give a brief introduction to the Batalin-Vilkovisky (BV) formulation of gauge field theories in the graded geometry setting.

The BV formalism gives a cohomological description of the algebra of gauge-invariant functions on the stationary surface of a classical action functional. This algebra can be find in two steps: restricting to the stationary surface and quotienting by gauge transformations. The first step is implemented by taking an homological resolution for the functions on the stationary surface; the second step by constructing a differential (modulo the resolution differential) encoding the gauge symmetries, and passing to its cohomology on the stationary surface. These two steps are combined by homological perturbation theory to obtain the relevant algebra of observables as the cohomology of the BV differential [32].

We will not construct here the BV space of fields and action starting from a classical gauge theory as sketched above; instead, we will describe the general geometrical structure of BV theories. The classical gauge-invariant action can be then recovered by setting to zero all the fields with non-zero ghost number in the BV action.

The BV space of fields is a (-1)-symplectic manifold with a cohomological operator $\mathcal{Q}$ and the BV action $\mathcal{S}$ satisfying the classical master equation $\iota_{\mathcal{Q}}\omega = \mathrm{d}\mathcal{S}$. Moreover, there is a second order operator Δ, the BV Laplacian, which squares to zero and generates the Poisson brackets for the symplectic structure. In the quantum theory, the action has to satisfy the quantum master equation $\Delta \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}} = 0$. This gives the gauge invariant properties of the partition function, defined by fixing the gauge on a lagrangian submanifold. We will show that the gauge-fixed action has a residual symmetry, depending on the geometrical data of a symplectic tubular neighbourhood of the lagrangian, and that this symmetry defines an on-shell cohomology characterising the gauge-fixed observables.

We will describe the BV formalism starting from its classical aspects, in Section 2.1, and then passing to the quantum theory in Section 2.2. In Section 2.3 we will discuss the residual symmetry of the gauge-fixed BV action. Finally, in Section 2.4 we describe the AKSZ construction of BV theories on the space of maps between graded manifolds.

In Appendix A we collect the basic notions of graded geometry which we need in the following.

For notational convenience, we will restrict to the case of bosonic classical theories only, so that the parity of fields coincides with the ghost number modulo 2. The $\mathbb{Z}$-grading given by the ghost number, which will be denoted by $|\cdot|$, will thus give the (super)commutative properties of the fields.

2.1 Classical BV formalism

In the BV formalism the space of classical fields ϕ is enriched to contain also families of ghosts η (for each reducibility stage of each gauge symmetry of the classical action functional) and antifields $\phi^\ddagger$ and $\eta^\ddagger$. The fields come with a grading, called *ghost number*, assigning positive degree to ghosts and negative degree to antifields. These fields can be interpreted as coordinates of a graded manifold, or more precisely of a (classical) *BV manifold*.

Definition 2.1.1. A *classical BV manifold* is a differential graded symplectic manifold $(\mathfrak{F}, \omega, \mathcal{Q})$ (see definition A.2.6), where the symplectic form has degree $|\omega| = -1$ and the differential is an hamiltonian vector field $\mathcal{Q} = \{\mathcal{S}, \ \}$ for a solution $\mathcal{S} \in \mathcal{C}^0(\mathfrak{F})$ of the *Classical Master Equation* (CME) $\{\mathcal{S}, \mathcal{S}\} = 0$, where $\{\ ,\ \}$ are the Poisson brackets induced by the symplectic structure. From the algebraic point of view, functions over $\mathfrak{F}$ are endowed, by the symplectic form, with the structure of a Gerstenhaber algebra.

Remark 2.1.2. It can be proved [46] that any graded symplectic manifold $(\mathcal{N}, \omega)$ of degree $|\omega| = -1$ is symplectomorphic to $T^*[-1]\mathcal{M}$ for some graded manifold $\mathcal{M}$. This symplectomorphism is *non canonical* and so it is the identification of fields, corresponding to coordinates on the base $\mathcal{M}$, and antifields, corresponding to fiber coordinates. If we introduce local Darboux coordinates $(x, x^\ddagger)$ the bracket reads

$$\{F, G\} = \frac{\partial_r F}{\partial x^a}\frac{\partial_l G}{\partial x^\ddagger_a} - \frac{\partial_r F}{\partial x^\ddagger_a}\frac{\partial_l G}{\partial x^a} \ , \tag{2.1}$$

where ∂_r and ∂_l denote the right and left derivative, respectively. The CME is expressed in these local coordinates as:

$$\frac{\partial_r \mathcal{S}}{\partial x^\ddagger_a}\frac{\partial_\ell \mathcal{S}}{\partial x^a} = 0 \ . \tag{2.2}$$

Remark 2.1.3. Notice that for the differential, which is symplectic because of the compatibility condition with ω, the condition to be hamiltonian is not a trivial one, since $|\mathcal{Q}| + |\omega| = 0$ (cf. A.2.16).

Remark 2.1.4. A meaningful operation is to set to zero all fields with degree of definite sign, defining the submanifolds $\mathfrak{F}_{\geqslant 0}$, $\mathfrak{F}_{\leqslant 0}$ and $\mathfrak{F}_0 = \mathfrak{F}_{\geqslant 0} \cap \mathfrak{F}_{\leqslant 0}$ in the space of fields $\mathfrak{F}$. Algebraically, this corresponds to quotienting the algebra of functions $\mathcal{C}(\mathfrak{F})$ by the ideals generated by functions of degree respectively negative, positive or different from zero. The classical action is recovered as the restriction $S^{\text{cl.}} = \mathcal{S}|_{\mathfrak{F}_0}$ of the master action to $\mathfrak{F}_0$.

Remark 2.1.5. The BV differential $\mathcal{Q}$ induces by restriction a differential Q on $\mathfrak{F}_{\leqslant 0}$. This differential, in the BV construction, corresponds to the operator δ which gives the homological resolution of the algebra of functions on the stationary surface (see [32]). Therefore in [30] Felder and Kazhdan define a *BV variety* requiring, in addition of the conditions in 2.1.1, also that the cohomology of Q on $\mathfrak{F}_{\leqslant 0}$ is concentrated in zero degree.

2.2 Quantum BV formalism

A key ingredient in quantum BV theories is the BV Laplacian, which is involved in the quantum master equation.

Definition 2.2.1. A *BV algebra* is a Gerstenhaber algebra $(A, \cdot, \{\ ,\ \})$ together with a nilpotent linear operator

$$\Delta\colon A \to A\ , \qquad \Delta^2 = 0\ , \tag{2.3}$$

which generates the Poisson brackets according to:

$$\{f, g\} = (-1)^{|f|}\Delta(f\cdot g) + (-1)^{|f|+1}(\Delta f)\cdot g - f\cdot \Delta g\ . \tag{2.4}$$

Δ is called the *BV Laplacian*.

On the symplectic manifold $T^*[-1]\mathcal{M}$, an invariant definition of the BV Laplacian can be given relating it to the divergence of hamiltonian vector fields:

$$\Delta_v f := \frac{1}{2}(-1)^{|f|}\text{div}_{\mu_v} X_f\ , \tag{2.5}$$

where v is a Berezinian form on $\mathcal{M}$, μ_v is the induced Berezinian on $T^*[-1]\mathcal{M}$ and $X_f = \{f,\ \}$. In local Darboux coordinates the Laplacian can be expressed as:

$$\Delta_v f = \frac{\partial_\ell}{\partial x_a^\ddagger}\frac{\partial_r}{\partial x^a} f + \frac{1}{2}\{f, \ln\rho_v\}\ , \tag{2.6}$$

where ρ_v is the density associated to the Berezinian μ_v. It can be explicitly verified, using the property (A.46) of the divergence operator, that Δ_v generates the antibracket and is nilpotent, so that $\mathcal{C}(T^*[-1]\mathcal{M})$ acquires the structure of a BV algebra [37].

The partition function is defined as the integral

$$Z = \int_{\mathcal{L}} \sqrt{\rho}\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}\ , \tag{2.7}$$

where the gauge-fixing consists in the choice of the lagrangian submanifold $\mathcal{L} \subset \mathfrak{F}$ and $\sqrt{\rho} \in \text{Dens}^{\frac{1}{2}}(\mathfrak{F})$ is a reference half-density such that $\Delta\sqrt{\rho} = 0$.

The role of the master equation is to ensure the path-integral to be independent on the gauge chosen. This is established in the following theorem proved by Schwarz [46].

Theorem 2.2.2. The integral of Δ_μ-closed functions is invariant under deformations of the lagrangian submanifold:

$$\int_{\mathcal{L}} \sqrt{\mu}\, f = \int_{\mathcal{L}'} \sqrt{\mu}\, f \qquad \text{if} \quad \Delta_\mu f = 0 \;, \tag{2.8}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are in the same homology class. Moreover the integral of Δ-exact functions vanishes for any choice of lagrangian submanifold $\mathcal{L}$:

$$\int_{\mathcal{L}} \sqrt{\mu}\, \Delta_\mu f = 0 \;. \tag{2.9}$$

In BV theories we have $\sqrt{\mu} := \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}} \sqrt{\mu_v}$, where $\sqrt{\mu_v}$ is a reference half density, e.g. the one induced by μ_v (see Appendix A.3).

Example 2.2.3. Let us consider the graded symplectic manifold $(T^*[-1]M, \omega)$ with canonical symplectic form and the Berezinian μ_v induced by a volume form $v \in \Omega^m(M)$ on M. On $T^*[-1]M$ take as lagrangian submanifold the conormal bundle $L_C = N^*[-1]C$ of a submanifold $C \subset M$, with induced Berezinian $\sqrt{\mu_v}$ (see example A.3.2). We have:

$$\int_{L_C} \sqrt{\mu_v}\, f = \int_C \phi_v(f) \;, \tag{2.10}$$

where ϕ_v is the isomorphism between $\mathcal{C}^\bullet(T^*[-1]M)$ and $\Omega^{m-\bullet}(M)$ given by the volume form v. Using the definition (2.6) of the BV Laplacian we see that:

$$\int_{L_C} \sqrt{\mu_v}\, \Delta_v f = \int_C \phi_v(\Delta_v f) = \int_C \mathrm{d}\, \phi_v(f) \;. \tag{2.11}$$

In this example, due to the above correspondence between BV Laplacian and de Rham differential, the Schwarz theorems follow from the Stokes theorem: the integral of a d-exact differential form is zero and the integral on two submanifold $C, C' \subset M$ is the same if they are in the same homology class.

In this language, a quantum BV theory consists in a BV manifold, with the BV Laplacian Δ_μ constructed as above, along with a choice of an homology class of lagrangian submanifolds. *Quantum observables* are elements of the cohomology of the BV Laplacian $H_{\Delta_\mu}(\mathcal{M})$. In this way their expectation values

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{L}} \sqrt{\mu_v}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}\, \mathcal{O}}{\int_{\mathcal{L}} \sqrt{\mu_v}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}} \tag{2.12}$$

does not depend on deformations of the lagrangian submanifold $\mathcal{L}$ chosen to fix the gauge.

Notice that it is possible to give a canonical choice for $\mathcal{L}$. The canonical submanifold $\mathfrak{F}_{\geqslant 0}$ defined setting to zero all fields with negative degree is lagrangian indeed. This is because the symplectic form has degree $|\omega| = -1$ and so $\mathfrak{F}_{\geqslant 0}$ is isotropic; moreover $\mathfrak{F}$

is always symplectomorphic to $T^*[-1]\mathcal{M}$ and therefore for each coordinate with negative degree there is one with degree greater than or equal to zero so that $\mathfrak{F}_{\geqslant 0}$ is also lagrangian. Nevertheless also other choices of lagrangian submanifolds generate well defined, but possibly inequivalent, quantum theories. This is an important feature of the BV formalism.

Once a theory is selected by the choice of $\mathcal{L}$, one has still the freedom of picking a nice representative of the homology class of lagrangian submanifolds homologous to $\mathcal{L}$. This can be viewed as a method to regularize the theory.

2.3 Residual symmetry

In this section we review the general structure of the gauge-fixed *residual symmetry* in BV theories [2, 46]. Indeed, for each choice of the gauge-fixing, the gauge-fixed action $\mathcal{S}_{\mathcal{L}}$ has an odd residual symmetry $\mathcal{Q}^{\pi}_{\mathcal{L}}$ induced on $\mathcal{L}$ by the CME. As we will see, this symmetry depends on the choice of a local symplectomorphism between $T^*[-1]\mathcal{L}$, endowed with the canonical symplectic form, and a neighbourhood of $\mathcal{L}$ in $(\mathfrak{F}, \Omega)$.

The BV operator $\mathcal{Q}$ is, because of the classical master equation, a cohomological vector field. On the contrary, the residual symmetry is not necessarily nilpotent, even if it could happen that it squares to zero for some particular choice of the symplectomorphism defining it. However $\mathcal{Q}^{\pi}_{\mathcal{L}}$ is always *on-shell* nilpotent, that is, it squares to zero only *modulo equations of motion*.

Classically, only on-shell quantities are relevant as the equations of motion always hold. If one wants observables to be $\mathcal{Q}^{\pi}_{\mathcal{L}}$-invariant, it is then enough to impose this condition on-shell. For the same reason, $\mathcal{Q}^{\pi}_{\mathcal{L}}$-invariant functions are physically equivalent if they differ by an element of the image of $\mathcal{Q}^{\pi}_{\mathcal{L}}$ *modulo equations of motion*. Whether or not the residual symmetry is nilpotent, the algebra of classical observables is then to be identified with the *on-shell cohomology* of the residual symmetry. Using homological perturbation theory one can actually prove that, when the gauge is fixed, the on-shell cohomology of $\mathcal{Q}^{\pi}_{\mathcal{L}}$ is isomorphic to the cohomology of the BV operator $\mathcal{Q}$. This is true for every choice of $\mathcal{Q}^{\pi}_{\mathcal{L}}$ (with fixed $\mathcal{L}$). In fact, although different choices of the symplectomorphism provide in general different residual symmetries, the on-shell cohomology of $\mathcal{Q}^{\pi}_{\mathcal{L}}$ does not depend on this choice: residual symmetries coming from different choices are always related by a trivial gauge transformation, so that they are on-shell equivalent and define the same algebra of classical observables.

Let us start proving the existence of a residual symmetry for the gauge-fixed action. We begin reminding the definition of tubular neighbourhood and defining a particular class of tubular neighbourhoods in symplectic graded manifolds.

Definition 2.3.1. Let $\mathcal{M}$ be a graded manifold with a submanifold L. A bundle $U \xrightarrow{\pi} L$, where U is an open neighbourhood of L in $\mathcal{M}$, is called *tubular neighbourhood* of L if the zero section $s\colon L \to U$ coincides with the inclusion $\iota\colon L \hookrightarrow U$.

Let now $(\mathcal{M}, \Omega)$ be a symplectic graded manifold of degree k. We call a tubular neighbourhood $U \xrightarrow{\pi} \mathcal{L}$ of a lagrangian manifold $\mathcal{L} \subseteq \mathcal{M}$ a *symplectic tubular neighbourhood* if there exists a symplectomorphism between $T^*[k]\mathcal{L}$, with the canonical symplectic form, and U, with the symplectic form $\Omega|_U$, which is a morphism of fiber bundles.

Remark 2.3.2. More concretely, we can think of this tubular neighbourhood as an atlas of canonical coordinates $\{x, x^\ddagger\}$ adapted to $\mathcal{L}$ (i.e. $\mathcal{L} = \{x^\ddagger = 0\}$) such that the transition functions between $(x, x^\ddagger)$ and $(y, y^\ddagger)$ are $(y = y(x), y^\ddagger = (\partial x/\partial y)x^\ddagger)$ so that the projection $\pi(x, x^\ddagger) = x$ is well defined.

A tubular neighbourhood $U \xrightarrow{\pi} L$ provides a map $V \mapsto V^\pi_L$ from tangent vector fields on M to tangent vector fields on L. This map is defined by

$$V^\pi_L = \iota^* \circ V|_U \circ \pi^* \in \mathfrak{X}(L)\ , \qquad V \in \mathfrak{X}(M), \tag{2.13}$$

where $V|_U$ is the restriction of V on U and $\iota\colon L \hookrightarrow U$ is the inclusion map.

Thus, once we have chosen a tubular neighbourhood of the lagrangian submanifold on which we fixed the gauge, we have a vector field on $\mathcal{L}$ corresponding to the BV cohomological vector field $\mathcal{Q} = \{\mathcal{S},\ \}$. For this tangent vector to be a symmetry of the gauge-fixed action, the tubular neighbourhood has to be compatible with the symplectic structure.

Proposition 2.3.3. Let $U \xrightarrow{\pi} \mathcal{L}$ be a symplectic tubular neighbourhood of a lagrangian submanifold $\mathcal{L} \hookrightarrow \mathcal{M}$. Then the map (2.13) $V \mapsto V^\pi_L$ projects the BV differential $\mathcal{Q}$ into a symmetry of the gauge-fixed action:

$$\mathcal{Q}^\pi_\mathcal{L} \mathcal{S}_\mathcal{L} = 0\ . \tag{2.14}$$

Proof. In the Darboux coordinates of Remark 2.3.2, we have $\pi^* f(x, x^\ddagger) = (f \circ \pi)(x, x^\ddagger) = f(x)$ for any function $f \in \mathcal{C}(\mathcal{L})$. The coordinate expression for $\mathcal{Q}^\pi_\mathcal{L}$ is thus:

$$\mathcal{Q}^\pi_\mathcal{L} f = -\iota^* \left(\frac{\partial_r \mathcal{S}}{\partial x^\ddagger_i} \frac{\partial_\ell \pi^* f}{\partial x^i} \right) = - \frac{\partial_r \mathcal{S}}{\partial x^\ddagger_i}\bigg|_\mathcal{L} \frac{\partial_\ell f}{\partial x^i} \equiv {\mathcal{Q}^\pi_\mathcal{L}}^i \frac{\partial f}{\partial x^i}\ . \tag{2.15}$$

We now expand the BV action in the momenta $x^\ddagger$:

$$\mathcal{S}(x, x^\ddagger) = \mathcal{S}_\mathcal{L}(x) - {\mathcal{Q}^\pi_\mathcal{L}}^i(x) x^\ddagger_i + \frac{1}{2} x^\ddagger_i \sigma^{ij}(x) x^\ddagger_j + O({x^\ddagger}^3)\ , \tag{2.16}$$

where we defined

$$\sigma^{ij}(x) = \left(\frac{\partial_\ell \partial_r \mathcal{S}}{\partial x^\ddagger_i \partial x^\ddagger_j} \right)(x, 0)\ . \tag{2.17}$$

We see that the components of the residual symmetry are simply the linear term in the antifield expansion of the gauge-fixed action. Plugging this expansion in the master equation we get:

$$0 = \frac{1}{2}\{\mathcal{S}, \mathcal{S}\}(x, x^\ddagger) = -\left(\frac{\partial_r \mathcal{S}}{\partial x^\ddagger_i} \frac{\partial_\ell \mathcal{S}}{\partial x^i} \right)(x, x^\ddagger) = {\mathcal{Q}^\pi_\mathcal{L}}^i(x) \frac{\partial \mathcal{S}_\mathcal{L}(x)}{\partial x^i} + O(x^\ddagger)\ ,$$

and so $\mathcal{Q}^\pi_\mathcal{L} \mathcal{S}_\mathcal{L} = 0$ as claimed. □

Remark 2.3.4. The BV construction results in a graded manifold of the form $T^*[-1]\mathcal{N}$. Obviously in this case there is the canonical choice $\mathcal{L} = \mathcal{N}$ of lagrangian submanifold and $U = T^*[-1]\mathcal{N}$ for the symplectic tubular neighbourhood.

More generally, it can be shown that it is possible to find a symplectic tubular neighbourhood for every choice of $\mathcal{L}$. The odd version of Weinstein's theorem on the existence of a local symplectomorphism between a neighbourhood of a lagrangian submanifold and $T^*[-1]\mathcal{L}$ was proved in [46]. It must be pointed out that such a choice is non canonical and non unique: each symplectomorphism of $\mathcal{M}$ into itself which keeps $\mathcal{L}$ fixed defines a new symplectic tubular neighbourhood.

The residual symmetry $\mathcal{Q}^\pi_\mathcal{L}$ depends, through the projection π, on the choice of the symplectic tubular neighbourhood. This ambiguity corresponds to the freedom to combine the residual symmetry with a *trivial gauge transformation*:

$$\mathcal{Q}^{\pi}_{\mathcal{L}}{}^i(x) \to \mathcal{Q}^{\pi}_{\mathcal{L}}{}^i(x) + \frac{\partial_r \mathcal{S}_\mathcal{L}(x)}{\partial x^j}\mu^{ij}(x) \ , \quad \text{with} \quad \mu^{ij} = (-1)^{(|x^i|+1)\,(|x^j|+1)}\mu^{ji} \ . \tag{2.18}$$

Indeed, let us see what happens if we change the symplectomorphism composing it with a canonical transformation that leaves $\mathcal{L}$ fixed. Suppose to have a finite canonical transformation $(x, x^\ddagger) \mapsto (\mathsf{x}, \mathsf{x}^\ddagger)$ generated by the function $F(\mathsf{x}, x^\ddagger)$ [7, 8, 50]. Then:

$$\mathsf{x}^\ddagger_i = \frac{\partial F}{\partial \mathsf{x}^i} \qquad x^i = \frac{\partial F}{\partial x^\ddagger_i} \ . \tag{2.19}$$

Since we want $\mathcal{L}$ to remain the same, we have to impose that $\mathsf{x}^\ddagger = 0$ if $x^\ddagger = 0$. The new residual symmetry $\mathcal{Q}^F_\mathcal{L}$ defined by F is easily found to be:

$$\begin{aligned}\mathcal{Q}^F_\mathcal{L} &= -\frac{\partial_r \mathcal{S}}{\partial \mathsf{x}^\ddagger_i}\bigg|_{\mathsf{x}^\ddagger=0}\frac{\partial}{\partial \mathsf{x}^i} = -\left(\frac{\partial_r \mathcal{S}}{\partial x^\ddagger_j}\frac{\partial_r x^\ddagger_j}{\partial \mathsf{x}^\ddagger_i} + \frac{\partial_r \mathcal{S}}{\partial x^j}\frac{\partial_r x^j}{\partial \mathsf{x}^\ddagger_i}\right)\bigg|_{\mathsf{x}^\ddagger=0}\frac{\partial}{\partial \mathsf{x}^i} = \\ &= \mathcal{Q}^\pi_\mathcal{L} - \frac{\partial_r \mathcal{S}_\mathcal{L}}{\partial x^j}\left(\frac{\partial_r \partial_\ell F}{\partial x^\ddagger_i \partial x^\ddagger_j}\right)\bigg|_{x^\ddagger=0}\frac{\partial}{\partial x^i} \ .\end{aligned} \tag{2.20}$$

We see that the tensor μ^{ij} defined in (2.18) is, in this example:

$$\mu^{ij} = \frac{\partial_r \partial_\ell F}{\partial x^\ddagger_i \partial x^\ddagger_j}\bigg|_\mathcal{L} \ , \tag{2.21}$$

with the correct symmetry properties (remember that $|F| = -1$). Notice that the variation of the residual symmetry depends only on the quadratic terms of the generating function with respect to the antifields.

We anticipated that the residual symmetry, unlike $\mathcal{Q}$, doesn't in general square to zero, as specified in the following

Proposition 2.3.5. The gauge-fixed symmetry $\mathcal{Q}^\pi_\mathcal{L}$ squares to zero *on-shell*:

$$\frac{1}{2}[\mathcal{Q}^\pi_\mathcal{L}, \mathcal{Q}^\pi_\mathcal{L}] = \sigma^{ij}\frac{\partial \mathcal{S}_\mathcal{L}}{\partial x^j}\frac{\partial}{\partial x^i} \ , \tag{2.22}$$

where σ^{ij} is the quadratic term in the antifield expansion of the action (2.16).

Proof. Using the expansion (2.16) of the action to compute the linear term in the antifields $x^\ddagger$ of the master equation, we get:

$$x_i^\ddagger \frac{\partial_r \mathcal{Q}_{\mathcal{L}}^{\pi\, i}}{\partial x_j} \mathcal{Q}_{\mathcal{L}}^{\pi\, j} + \frac{\partial_r \mathcal{S}_{\mathcal{L}}}{\partial x^j} \sigma^{ji} x_i^\ddagger = 0 \ . \tag{2.23}$$

From the definition of σ^{ij} given in (2.17) we have:

$$|\sigma^{ij}| = |x^i| + |x^j| \ ; \quad \sigma^{ij} = (-1)^{|x^i|\,|x^j|+1} \sigma^{ji} \ , \tag{2.24}$$

The claim, then, follows using (2.23) to compute $\left[\mathcal{Q}_{\mathcal{L}}^\pi, \mathcal{Q}_{\mathcal{L}}^\pi\right]$:

$$\frac{1}{2}\left[\mathcal{Q}_{\mathcal{L}}^\pi, \mathcal{Q}_{\mathcal{L}}^\pi\right]^i = \frac{\partial_r \mathcal{Q}_{\mathcal{L}}^{\pi\, i}}{\partial x^j} \mathcal{Q}_{\mathcal{L}}^{\pi\, j} = -\frac{\partial_r \mathcal{S}_{\mathcal{L}}}{\partial x^j} \sigma^{ji} = \sigma^{ij} \frac{\partial_\ell \mathcal{S}_{\mathcal{L}}}{\partial x^j} \ . \qquad \square$$

This, in particular, allows to define the *on-shell cohomology* of $\mathcal{Q}_{\mathcal{L}}^\pi$. In fact, due to the CME, the residual symmetry preserves the space of critical points of $S_{\mathcal{L}}$. We call then on-shell cohomology the cohomology of the restriction of the residual symmetry to the critical points. Moreover, since a change of the tubular neighbourhood only modifies the residual symmetry by a trivial transformation, we have that the on-shell cohomology does not depend on the choice of tubular neighbourhood. We denote it as $H_{\text{on}}(\mathcal{Q}, \mathcal{L})$.

Proposition 2.3.6. The restriction of functions on the lagrangian submanifold gives a map in cohomology:

$$r\colon H(\mathcal{Q}, \mathcal{M}) \longrightarrow H_{\text{on}}(\mathcal{Q}, \mathcal{L}) \ . \tag{2.25}$$

Moreover, if the gauge fixed action has no gauge symmetries, this map is an isomorphism

Proof. The condition of being $\mathcal{Q}$-closed, once restricted to $\mathcal{L}$, reads:

$$Q_{\mathcal{L}}^\pi(f)|_{\mathcal{L}} + V_f(S) = 0 \ , \tag{2.26}$$

where $V_F = \frac{\partial_r F}{\partial x_i^\ddagger}\Big|_{\mathcal{L}} \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathcal{L})$. Therefore $F_{\mathcal{L}}$ is $\mathcal{Q}_{\mathcal{L}}^\pi$-closed modulo equations of motion. For the proof of the fact that, when $\mathcal{S}_{\mathcal{L}}$ has no gauge symmetries, this map is an isomorphism see e.g. [32], Thm. 18.5. $\square$

We will be interested in the *off-shell* residual symmetry. The freedom of changing the symplectic tubular neighbourhood can be used to look for a residual symmetry squaring to zero on all $\mathcal{L}$, not only on-shell. From (2.16), a tubular neighbourhood defines the quadratic part of the BV action $\sigma = \frac{1}{2} x_a^\ddagger \sigma^{ab} x_b^\ddagger \in C_{-2}(T^*[-1]\mathcal{L})$, where the grading is the (opposite) fiber degree. By looking at (2.22), we see that the residual symmetry $\mathcal{Q}_{\mathcal{L}}^\pi$ squares to zero iff $\delta_{\mathcal{S}_{\mathcal{L}}}(\sigma) = 0$, where $\delta_{\mathcal{S}_{\mathcal{L}}} = \iota_{\mathrm{d}\mathcal{S}_{\mathcal{L}}}$. When this happens, the off-shell cohomology is also defined, namely the cohomology of the residual symmetry. It is clear from (2.26) that the restriction of a BV observable to the gauge fixing lagrangian is not in general closed under the residual symmetry.

2.4 AKSZ construction

In the previous sections we outlined the geometrical structure of BV theories. The classical approach to BV, starting from a gauge invariant action functional, requires first to construct the appropriate space of fields and then to find a solution of the master equation with some boundary conditions. This task can be very difficult to pursue for complicated gauge theories. We will now illustrate a method, due to Alexandrov, Kontsevich, Schwarz and Zaboronsky [2], to directly construct (possibly infinite dimensional) classical BV theories. Instead of starting with a gauge invariant classical action, this is eventually recovered from a solution of the classical master equation setting to zero all fields with negative degree.

The AKSZ method is essentially a way to induce a differential graded symplectic structure on the space of maps (cf. Appendix A.4) between two graded manifolds with opportune geometrical structures. A review of this construction can be found in [19] [45]. The starting data are the following:
The Source: A dg manifold $(\mathcal{N}, D)$ with a non degenerate D-invariant volume form μ of degree $|\mu| = -n-1$ for a positive integer n (cf. (A.41) (A.42)).
The Target: A dg symplectic manifold $(\mathcal{M}, \omega, Q)$ with $|\omega| = n$.

From lemma A.2.16 it follows that $Q = \{\Theta, \ \}$ for some function $\Theta \in \mathcal{C}^{n+1}(\mathcal{M})$ which solves the equation $\{\Theta, \Theta\} = 0$.

As we are going to discuss, the differentials and the symplectic structure of these two spaces induce a dg symplectic structure on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ such that the differential is hamiltonian and so we will also have a solution of the CME on the space of maps.

Let us start with the dg structure. The groups of invertible maps $\mathrm{Diff}(\mathcal{N})$ and $\mathrm{Diff}(\mathcal{M})$ act naturally on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ by respectively left and right composition. At the infinitesimal level this means that for each vector field on $\mathcal{N}$ or $\mathcal{M}$ there is a corresponding derivation on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$. For any $V \in \mathfrak{X}(\mathcal{N})$ and $W \in \mathfrak{X}(\mathcal{M})$, we will indicate with $\widehat{V}$ and $\widecheck{W}$ the corresponding vector fields induced on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$. Because left and right group action commutes, then $\widehat{V}$ and $\widecheck{W}$ always commute too. Moreover any cohomological vector field on $\mathcal{N}$ or $\mathcal{M}$ induces a vector field that squares to zero. Therefore, any linear combination

$$\mathcal{Q} = a\widehat{D} + b\widecheck{Q} \tag{2.27}$$

is a differential on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$.

The symplectic structure Ω on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ is induced by the one of $\mathcal{M}$. Consider the product $\mathcal{N} \times \mathrm{Maps}(\mathcal{N}, \mathcal{M})$. The evaluation map $\mathbb{E}\mathrm{v}\colon \mathcal{N} \times \mathrm{Maps}(\mathcal{N}, \mathcal{M}) \to \mathcal{M}$ defined by

$$\mathbb{E}\mathrm{v}\colon (p, \phi) \mapsto \phi(p) \tag{2.28}$$

allows to pull back differential forms. Moreover, the volume form μ on $\mathcal{N}$ defines a fiber integration $\mu_*\colon \Omega^k(\mathcal{N} \times \mathrm{Maps}(\mathcal{N}, \mathcal{M})) \to \Omega^k(\mathrm{Maps}(\mathcal{N}, \mathcal{M}))[-n-1]$ of differential forms:

$$(\mu_*\eta)_x(v_1, \ldots, v_k) := \int_{\mathcal{N}} \mu(y)\, \eta_{(y,x)}(v'_1, \ldots, v'_k) \ , \tag{2.29}$$

where v_i' are the lifts of $v_i \in \mathfrak{X}(\mathrm{Maps}(\mathcal{N},\mathcal{M}))$ to $\mathfrak{X}(\mathcal{N}\times\mathrm{Maps}(\mathcal{N},\mathcal{M}))$. Hence we are now able to transport a differential form from $\mathcal{M}$ to $\mathrm{Maps}(\mathcal{N},\mathcal{M})$:

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{N}\times\mathrm{Maps}(\mathcal{N},\mathcal{M})) & \xleftarrow{\ \mathbb{E}\mathrm{v}^*\ } & \\ \Big\downarrow{\scriptstyle \mu_*} & & \\ \Omega^\bullet(\mathrm{Maps}(\mathcal{N},\mathcal{M}))[-n-1] & \xleftarrow[\ \mu_*\mathbb{E}\mathrm{v}^*\ ]{} & \Omega^\bullet(\mathcal{M}) \end{array} \tag{2.30}$$

In particular $\Omega = \mu_*\mathbb{E}\mathrm{v}^*\omega$, since the volume form μ is non degenerate, is a symplectic form with degree -1:

$$\Omega_\phi = \int_{\mathcal{N}} \mu\, \frac{1}{2}\delta\phi^i\omega_{ij}\delta\phi^j \ . \tag{2.31}$$

We have now to check the compatibility of $\mathcal{Q}$ and Ω. Let us start with $\check{Q}$. Notice that $\imath_{\check{Q}}\mu_*\mathbb{E}\mathrm{v}^* = \mu_*\mathbb{E}\mathrm{v}^*\imath_Q$. Hence:

$$L_{\check{Q}}\Omega = \mu_*\mathbb{E}\mathrm{v}^* L_Q\omega = 0 \ . \tag{2.32}$$

Moreover, we have the following

Proposition 2.4.1. The map $\mu_*\mathbb{E}\mathrm{v}^*$ is an homomorphism of Poisson algebras.

Proof. For an hamiltonian vector field $X = \{f,\ \}$ on $\mathcal{M}$ we have:

$$\imath_{\check{X}}\mu_*\mathbb{E}\mathrm{v}^*\omega = \mu_*\mathbb{E}\mathrm{v}^*\imath_X\omega = \mathrm{d}\mu_*\mathbb{E}\mathrm{v}^* f \ , \tag{2.33}$$

and so $\check{X}_f = \{\mu_*\mathbb{E}\mathrm{v}^* f,\ \}$. Therefore, for $f,g \in \mathcal{C}(\mathcal{M})$:

$$\big\{\mu_*\mathbb{E}\mathrm{v}^* f, \mu_*\mathbb{E}\mathrm{v}^* g\big\} = \imath_{\check{X}}\mathrm{d}\mu_*\mathbb{E}\mathrm{v}^* g = \mu_*\mathbb{E}\mathrm{v}^*\imath_X \mathrm{d}g = \mu_*\mathbb{E}\mathrm{v}^*\big\{f,g\big\} \ . \qquad \square$$

Thus, the differential $\check{Q}$ is not only compatible with Ω but it is also an hamiltonian vector field with hamiltonian $\check{S} = \mu_*\mathbb{E}\mathrm{v}^*\Theta$:

$$\check{S}[\phi] = \int_{\mathcal{N}} \mu\,\Theta(\phi) \ , \qquad \phi \in \mathrm{Maps}(\mathcal{N},\mathcal{M}) \ . \tag{2.34}$$

Also $\widehat{D}$ is compatible with Ω. In fact the Lie derivative with respect to $\widehat{D}$ of any differential form coming from $\mathcal{M}$ does vanish: $L_{\widehat{D}}\mu_*\mathbb{E}\mathrm{v}^* = 0$. To prove this, notice that the evaluation map is invariant with respect to the diagonal (right) action of $\mathrm{Diff}(\mathcal{N})$ on $\mathcal{N}\times\mathrm{Maps}(\mathcal{N},\mathcal{M})$: $\mathbb{E}\mathrm{v}(\phi^{-1}(x), f\circ\phi) = \mathbb{E}\mathrm{v}(x,f)$. This implies that the lifts D' and $\widehat{D}'$ of D and $\widehat{D}$ to $\mathcal{N}\times\mathrm{Maps}(\mathcal{N},\mathcal{M})$ coincides on forms pulled back through the evaluation maps:

$$L_{D'}\mathbb{E}\mathrm{v}^* = L_{\widehat{D}'}\mathbb{E}\mathrm{v}^* \ . \tag{2.35}$$

Moreover, by the definition of fiber integration (2.29) we see that $\imath_{\widehat{D}}\mu_* = \mu_*\imath_{\widehat{D}'}$. Using the D-invariance of the volume form, $\mu_* L_{D'} = 0$, we obtain:

$$L_{\widehat{D}}\mu_*\mathbb{E}\mathrm{v}^* = \mu_* L_{\widehat{D}'}\mathbb{E}\mathrm{v}^* = \mu_* L_{D'}\mathbb{E}\mathrm{v}^* = 0 \ . \tag{2.36}$$

Like $\check{Q}$, also $\widehat{D}$ is an hamiltonian vector field. This is because the symplectic form ω has positive degree and thus from lemma A.2.11 it is exact: $\omega = \mathrm{d}(\imath_\epsilon \omega/n) = \mathrm{d}\vartheta$, where ϵ is the Euler vector field corresponding to the grading in $\mathcal{M}$. Therefore, using (2.36):

$$\imath_{\widehat{D}}\Omega = \imath_{\widehat{D}}\mu_*\mathbb{E}\mathrm{v}^*\,\mathrm{d}\vartheta = \mathrm{d}(\imath_{\widehat{D}}\mu_*\mathbb{E}\mathrm{v}^*\vartheta)\ . \tag{2.37}$$

Hence $\widehat{D}$ has hamiltonian $\widehat{S} = -\imath_{\widehat{D}}\mu_*\mathbb{E}\mathrm{v}^*\vartheta$:

$$\widehat{S}[\phi] = -\imath_{\widehat{D}}\int_{\mathcal{N}}\mu\,\frac{1}{n}\phi^*(\imath_\epsilon\omega) = \int_{\mathcal{N}}\mu\,\frac{1}{2n}\phi^i\omega_{ij}(\phi)D\phi^j\ , \quad \phi\in\mathrm{Maps}(\mathcal{N},\mathcal{M})\ . \tag{2.38}$$

Putting all this together we proved the followingz

Theorem 2.4.2. $(\mathrm{Maps}(\mathcal{N},\mathcal{M}),\Omega,\mathcal{Q})$ is a classical BV manifold (definition 2.1.1). The hamiltonian of $\mathcal{Q}$ is:

$$\mathcal{S}[\phi] = a\widehat{S}[\phi] + b\check{S}[\phi] = \int_{\mathcal{N}}\mu\left(\frac{a}{2n}\phi^i\omega_{ij}(\phi)D\phi^j + b\Theta(\phi)\right)\ . \tag{2.39}$$

This construction also provides a class of classical observables induced by the cohomology of Q on the target space $\mathcal{M}$, the *AKSZ observables* $\mu_*\mathbb{E}\mathrm{v}^*f$, for $[f]\in H_Q(\mathcal{M})$.

Proposition 2.4.3. The map $\mu_*\mathbb{E}\mathrm{v}^*$ induces a map in cohomology:

$$\mu_*\mathbb{E}\mathrm{v}^*\colon H_Q(\mathcal{M})\to H_{\mathcal{Q}}(\mathrm{Maps}(\mathcal{N},\mathcal{M}))\ . \tag{2.40}$$

Proof. Let $f\in H_Q(\mathcal{M})$. Then, using proposition 2.4.1 and equation (2.36), we get:

$$\mathcal{Q}\,\mu_*\mathbb{E}\mathrm{v}^*f = \check{Q}\,\mu_*\mathbb{E}\mathrm{v}^*f = \mu_*\mathbb{E}\mathrm{v}^*\,Qf = 0\ .$$ □

In particular $\mu_*\mathbb{E}\mathrm{v}^*\colon H_Q^{n+1}(\mathcal{M})\to H^0_{\mathcal{Q}}(\mathrm{Maps}(\mathcal{N},\mathcal{M}))$, hence to cohomology classes of degree $n+1$ correspond classical observables:

$$\mu_*\mathbb{E}\mathrm{v}^*f = \int_{\mathcal{N}}\mu\,F(\phi)\ . \tag{2.41}$$

Remark 2.4.4. If the source manifold of the AKSZ construction is the shifted tangent space $T[1]\Sigma$ of a two-dimensional surface, then the symplectic structure of the target must have degree 1. Moreover, if the target is non-negatively graded, it can be proved [46] that it must have the form $T^*[1]M$ for some ordinary manifold M. Example A.2.18 thus implies that M must be a Poisson manifold.

Chapter 3

Poisson Sigma Model

Poisson Sigma Model (PSM) and A-model are relevant examples of two dimensional topological quantum field theories. The A-model is a sigma model of maps from a Riemann surface to a symplectic manifold and computes the Gromov-Witten invariants of the target manifold [52, 54]. The most general formulation of the A-model depends on the choice of a compatible almost complex structure [52], but correlators are independent of it. When the almost complex structure is integrable, in the Kähler case, such a model can be obtained as a *topological twist* of the supersymmetric sigma model. The supersymmetry transformation is responsible for the localization of the model on the space holomorphic maps and eventually for its non perturbative definition. In the twisted formulation, this operator squares to zero only on-shell, but an equivalent formulation with an off-shell cohomological supersymmetry can be obtained introducing auxiliary fields as in [52].

The PSM is a sigma model with target a general Poisson structure, possibly degenerate; when considered on the disc it reproduces the Kontsevich formula for deformation quantization of the target Poisson structure as a correlator of boundary observables (see [18]). It is defined in terms of the AKSZ solution of the classical master equation in the Batalin-Vilkovisky formalism [2, 45]. It must be gauge-fixed by choosing a lagrangian submanifold of the space of fields. The general BV theory assures that a deformation of the lagrangian does not affect the correlators; nevertheless, there can be different homology classes giving rise to inequivalent physical theories. In [15] it was shown that when one considers the PSM with target the inverse of the symplectic form of a Kähler manifold, the complex structure can be used to define the gauge-fixing. The gauge-fixed action coincides then, after a partial integration, with the action of the A-model.

We complete here this comparison by showing that the relevant features of the A-model, observables and supersymmetry, have a natural BV interpretation in the complex gauge-fixing of the PSM.

Let us discuss first observables. Every de Rham cohomology class of the target manifold defines a hierarchy of observables of the A-model, whose mean values compute the Gromov-

Witten invariants. In the AKSZ construction, there is a natural class of observables defined starting from cohomology classes of the odd vector field encoding the geometry of the target [45]. In the case of the PSM, this cohomology is the Lichnerowicz-Poisson (LP) cohomology; in the non degenerate case, this is canonically isomorphic to the de Rham cohomology. It is then natural to think that the observables of the PSM should reproduce the hierarchy of observables of the A-model after gauge fixing. We show that this is true but in a non trivial way.

Indeed, for every Poisson structure the contraction with the Poisson tensor defines a map from forms to multivector fields, intertwining de Rham with LP differential. We prove that for observables associated to multivector fields lying in the image of such a map, there is an equivalent form, up to BV operator $\mathcal{Q}$ and de Rham differential d exact terms, that in the non degenerate case and after gauge fixing reproduces the A-model hierarchy. We call these observables A-model like observables. This fact gives an interpretation of the well known independence of the Gromov-Witten invariants on the choice of the compatible complex structure in terms of independence on the choice of the gauge fixing.

Next, we discuss the residual BV symmetry. This is an odd symmetry of the gauge fixed action, that depends on the choice of a tubular neighbourhood of the gauge fixing lagrangian. It is not true that a BV observable is closed under the residual symmetry when restricted, yet it is closed modulo equations of motion. Moreover, the residual symmetry squares to zero only on-shell. We prove that in the case of the complex gauge fixing of the PSM with symplectic target, under some assumptions, there exists a choice of the tubular neighbourhood such that the residual symmetry squares to zero off-shell and reproduces Witten $\mathbf{Q}$ supersymmetry with the auxiliary field considered in [52]. In particular, the A-model observables are closed under the residual symmetry.

In [55] it has been discussed an approach to the quantization of symplectic manifolds based on the A-model defined on surfaces with boundary. This is a quantum field theoretic approach to quantization that should be compared to the results of [18] and suggests a non trivial relation between the A-model and the PSM with symplectic target on surfaces with boundary that is worth investigating. This requires a comparison of boundary conditions of the two models that we plan to address in a future work.

3.1 PSM and A-model and their observables

We review in this Section the definition and basic properties of PSM and A-model.

3.1.1 A-model

Let us introduce first the A-model following [52]. It is a sigma model of maps from a Riemann surface Σ, with complex structure ε, to a smooth $2n$-dimensional Kähler manifold M, with complex structure J. Let us introduce local coordinates $\{\sigma^\alpha\}$ on Σ and $\{u^\mu\}$ on M. Indices are raised and lowered using the Kähler metric.

The field content of the theory is given by a bosonic map $\phi : \Sigma \to M$ with charge 0; a section χ of $\phi^*(TM)$ with charge 1 and fermionic statistic; a one-form ρ on Σ with values in $\phi^*(TM)$, with charge -1 and fermionic statistics and a one-form H on Σ with values in $\phi^*(TM)$, with charge 0 and bosonic statistics. Finally, both ρ and H satisfy the self duality property:

$$\rho^{\alpha\mu} = \varepsilon^{\alpha}{}_{\beta} J^{\mu}{}_{\nu} \rho^{\beta\nu} \ ; \qquad H^{\alpha\mu} = \varepsilon^{\alpha}{}_{\beta} J^{\mu}{}_{\nu} H^{\beta\nu} \ . \tag{3.1}$$

The action is given by

$$S_{\varepsilon,J} = \int_{\Sigma} \mathrm{d}^2\sigma \Big(-\frac{1}{4} H^{\alpha\mu} H_{\alpha\mu} + H^{\alpha}{}_{\mu} \partial_\alpha u^\mu - \mathrm{i} \rho^{\alpha}{}_{\mu} D_\alpha \chi^\mu - \frac{1}{8} \rho^{\alpha}{}_{\mu} \rho_{\alpha\nu} \chi^\rho \chi^\sigma R_{\rho\sigma}{}^{\mu\nu} \Big) \ , \tag{3.2}$$

where $D_\alpha \chi^\mu := \partial_\alpha \chi^\mu + \Gamma^{\mu}_{\nu\sigma} \chi^\nu \partial_\alpha u^\sigma$ denotes the covariant derivative with respect to the Levi Civita connection (with Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$) induced by the Kähler metric and R is the corresponding Riemann tensor. The action is invariant under the action of the supersymmetry $\mathbf{Q}$:

$$\begin{aligned}
&\mathbf{Q} u^\mu = \mathrm{i} \chi^\mu \ , \\
&\mathbf{Q} \chi^\mu = 0 \ , \\
&\mathbf{Q} \rho_{\alpha}{}^{\mu} = H_{\alpha}{}^{\mu} - \mathrm{i} \Gamma^{\mu}_{\nu\sigma} \chi^\nu \rho_{\alpha}{}^{\sigma} \ , \\
&\mathbf{Q} H^{\alpha\mu} = -\frac{1}{4} \chi^\nu \chi^\sigma \Big(R_{\nu\sigma}{}^{\mu}{}_{\tau} + R_{\nu\sigma\mu'\tau'} J^{\mu'\mu} J^{\tau'}{}_{\tau} \Big) \rho^{\alpha\tau} - \mathrm{i} \Gamma^{\mu}_{\nu\sigma} \chi^\nu H^{\alpha\sigma} \ .
\end{aligned} \tag{3.3}$$

It can be seen that the odd vector field $\mathbf{Q}$ squares to zero. The field H enters quadratically into the action so that it can be integrated out. After this integration, the action is invariant after an odd vector field that squares to zero only on-shell. Moreover, the comparison with the PSM model is more natural including this auxiliary field, so that we will keep it without integrating.

The observables of the A-model are defined by classes of de Rham cohomology of M. For an element $[\omega] \in H^k_{\mathrm{dR}}(M)$ one can define

$$\begin{aligned}
A^{(0)}_{\omega} &= \omega_{\mu_1 \ldots \mu_k} \chi^{\mu_1} \cdots \chi^{\mu_k} \ , \\
A^{(1)}_{\omega} &= \mathrm{i} k \omega_{\mu_1 \ldots \mu_k} \mathrm{d}u^{\mu_1} \chi^{\mu_2} \cdots \chi^{\mu_k} \ , \\
A^{(2)}_{\omega} &= -\frac{k(k-1)}{2} \omega_{\mu_1 \ldots \mu_k} \mathrm{d}u^{\mu_1} \wedge \mathrm{d}u^{\mu_2} \chi^{\mu_3} \cdots \chi^{\mu_k} \ ,
\end{aligned} \tag{3.4}$$

with associated A-model observables:

$$A^{(k)}_{\omega,\gamma_k} = \int_{\gamma_k} A^{(k)}_{\omega} \ , \tag{3.5}$$

where γ_k is a k-cycle on Σ. They satisfy

$$\mathbf{Q} A^{(i)}_{\omega} + \mathrm{i} \mathrm{d} A^{(i-1)}_{\omega} = 0 \ , \tag{3.6}$$

so that $\mathbf{Q} A^{(i)}_{\omega,\gamma_i} = 0$.

3.1.2 Poisson sigma model

Let us introduce now the Poisson Sigma Model (PSM). Let (M, α) be a Poisson manifold with Poisson tensor field α and let Σ be a two dimensional closed surface. The PSM in the AKSZ formalism is a two dimensional topological sigma model whose field content is the space of maps between graded manifolds $\mathfrak{F}_\Sigma = \mathrm{Map}(T[1]\Sigma, T^*[1]M)$. If we introduce local coordinates x^μ on M and u^α on Σ, a point of $\mathfrak{F}_\Sigma$ is given by the superfields

$$\begin{aligned} \mathbf{x}^\mu &= x^\mu + \eta^{+\mu}_\alpha \theta^\alpha + \frac{1}{2} b^{+\mu}_{\alpha\beta}\theta^\alpha\theta^\beta \ , \\ \mathbf{b}_\mu &= b_\mu + \eta_{\mu\alpha}\theta^\alpha + \frac{1}{2} x^+_{\mu\alpha\beta}\theta^\alpha\theta^\beta \ , \end{aligned} \tag{3.7}$$

where θ^α denotes the degree 1 coordinate of $T[1]\Sigma$. If we change coordinates on M as $y^a = y^a(x)$, the superfields transform as:

$$\mathbf{y}^a = y^a(\mathbf{x}) \ , \qquad \mathbf{b}_a = \frac{\partial x^\mu}{\partial y^a}(\mathbf{x})\mathbf{b}_\mu \ . \tag{3.8}$$

The space of fields $\mathfrak{F}_\Sigma$ is a degree -1 symplectic manifold with symplectic structure given by

$$\Omega = \int_{T[1]\Sigma} \mathrm{d}u\mathrm{d}\theta \; \delta\mathbf{x}^\mu \wedge \delta\mathbf{b}_\mu \ , \tag{3.9}$$

where $\mathrm{d}u\mathrm{d}\theta$ is the canonical Berezinian on $T[1]\Sigma$. The action is given by

$$\mathcal{S} = \int_{T[1]\Sigma} \mathrm{d}u\mathrm{d}\theta \left(\mathbf{b}_\mu \mathrm{d}\mathbf{x}^\mu + \frac{1}{2}\alpha^{\mu\nu}(\mathbf{x})\mathbf{b}_\mu\mathbf{b}_\nu \right) . \tag{3.10}$$

The BV vector field $\mathcal{Q} = \{\mathcal{S}, -\}$ reads

$$\begin{aligned} \mathcal{Q}\mathbf{x}^\mu &= \mathrm{d}\mathbf{x}^\mu + \alpha^{\mu\nu}(\mathbf{x})\mathbf{b}_\nu \ , \\ \mathcal{Q}\mathbf{b}_\mu &= \mathrm{d}\mathbf{b}_\mu + \frac{1}{2}\partial_\mu\alpha^{\nu\rho}(\mathbf{x})\mathbf{b}_\nu\mathbf{b}_\rho \ , \end{aligned} \tag{3.11}$$

where d is the de Rham differential on Σ.

We will be interested in the hierarchy of observables defined by Lichnerowicz-Poisson cohomology. We recall that the LP differential on multivector fields of M is defined as $d_\alpha(v) = [\alpha, v]$, for $v \in C^\infty(T^*[1]M) \equiv \mathcal{V}^\bullet(M)$; it squares to zero since α is Poisson and we denote by $H_{LP}(M, \alpha)$ its cohomology. Let $\mathbb{E}\mathrm{v}\colon \mathfrak{F}_\Sigma \times T^*[1]\Sigma \to T^*[1]M$ be the evaluation map, and let us denote $\mathcal{O}_v = \mathbb{E}\mathrm{v}^*(v)$ for any $v \in C^\infty(T^*[1]M)$. We compute

$$\mathcal{Q}(\mathcal{O}_v) = \mathrm{d}\mathcal{O}_v - \frac{1}{2}\mathcal{O}_{d_\alpha(v)} \ . \tag{3.12}$$

Let us expand $\mathcal{O}_v = \mathcal{O}_v^{(0)} + \mathcal{O}_v^{(1)} + \mathcal{O}_v^{(2)}$ in form degree and assume $d_\alpha(v) = 0$; let γ_k a k-cycle in Σ and let $\mathcal{O}^{(k)}_{v,\gamma_k} \equiv \int_{\gamma_k} \mathcal{O}_v^{(k)}$, then:

$$\mathcal{Q}(\mathcal{O}^{(k)}_{v,\gamma_k}) = 0 \ . \tag{3.13}$$

Thus we have a hierarchy of BV observables $[\mathcal{O}^{(k)}_{v,\gamma_k}] \in H^\bullet(\mathcal{Q}, \mathfrak{F}_\Sigma)$ for each $[v] \in H^\bullet_{\mathrm{LP}}(M)$.

Let us discuss now a subclass of these observables. The map $\sharp_\alpha \colon \Omega^\bullet(M) \to \mathcal{V}^\bullet(M)$ defined as

$$\sharp_\alpha \colon \omega_{\mu_1\ldots\mu_k} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k} \mapsto \omega_{\mu_1\ldots\mu_k} \alpha^{\mu_1\nu_1} \cdots \alpha^{\mu_k\nu_k} \partial_{\nu_1} \wedge \ldots \wedge \partial_{\nu_k} \tag{3.14}$$

intertwines de Rham and LP differential $\sharp_\alpha \circ \mathrm{d} = d_\alpha \circ \sharp_\alpha$ (see [51]) so that it descends to $\sharp_\alpha \colon H^\bullet_{\mathrm{dR}}(M) \to H^\bullet_{\mathrm{LP}}(M, \alpha)$. If the Poisson structure is non degenerate, it is an isomorphism between differential forms and multivector fields and induces an isomorphism between LP and de Rham cohomologies.

When the LP cohomology class is in the image of this map (which is always the case when α is non degenerate), there is an alternative expression for the corresponding PSM observable, that we are going to discuss next. A long but straightforward computation shows that the PSM observable $\mathcal{O}_{\sharp_\alpha(\omega)} = \mathbb{E}\mathrm{v}^*(\sharp_\alpha(\omega))$ for a closed $\omega \in \Omega^\bullet(M)$ can be written in the following form

$$\begin{aligned} \mathcal{O}^{(0)}_{\sharp_\alpha(\omega)} &= \frac{\mathrm{i}^k}{k!} \mathcal{A}^{(0)}_\omega \ , \\ \mathcal{O}^{(1)}_{\sharp_\alpha(\omega)} &= \frac{\mathrm{i}^k}{k!} \mathcal{A}^{(1)}_\omega + \mathcal{Q} C^{(1)}_{\sharp_\alpha(\omega)} \ , \\ \mathcal{O}^{(2)}_{\sharp_\alpha(\omega)} &= \frac{\mathrm{i}^k}{k!} \mathcal{A}^{(2)}_\omega + \mathcal{Q} C^{(2)}_{\sharp_\alpha(\omega)} - \mathrm{d} C^{(1)}_{\sharp_\alpha(\omega)} \ , \end{aligned} \tag{3.15}$$

where we have defined

$$\begin{aligned} \mathcal{A}^{(0)}_\omega &= (-\mathrm{i})^k \, \omega_{\mu_1\ldots\mu_k} b^{\mu_1} \cdots b^{\mu_k} \ , \\ \mathcal{A}^{(1)}_\omega &= \mathrm{i}k(-\mathrm{i})^{k-1} \, \omega_{\mu_1\ldots\mu_k} \mathrm{d}x^{\mu_1} b^{\mu_2} \cdots b^{\mu_k} \ , \\ \mathcal{A}^{(2)}_\omega &= \frac{k(k-1)}{2} (-\mathrm{i})^k \, \omega_{\mu_1\ldots\mu_k} \mathrm{d}x^{\mu_1} \mathrm{d}x^{\mu_2} b^{\mu_3} \cdots b^{\mu_k} \ , \\ C^{(1)}_{\sharp_\alpha(\omega)} &= \frac{1}{(k-1)!} \omega_{\mu_1\ldots\mu_k} \eta^{+\mu_1} b^{\mu_2} \cdots b^{\mu_k} \ , \\ C^{(2)}_{\sharp_\alpha(\omega)} &= \omega_{\mu_1\ldots\mu_k} \left(\frac{1}{(k-1)!} b^{+\mu_1} b^{\mu_2} - \frac{1}{(k-2)!} \eta^{+\mu_1} \mathrm{d}x^{\mu_2} \right. \\ &\quad \left. + \frac{1}{2(k-2)!} \eta^{+\mu_1} \mathcal{Q} \eta^{+\mu_2} \right) b^{\mu_3} \cdots b^{\mu_k} \\ &\quad + \frac{1}{2(k-1)!} \partial_\lambda \omega_{\mu_1\ldots\mu_k} \eta^{+\lambda} \eta^{+\mu_1} b^{\mu_2} \cdots b^{\mu_k} \ , \end{aligned} \tag{3.16}$$

and $b^\mu = \alpha^{\mu\nu} b_\nu$. As a consequence of (3.15), for each closed form ω and k-cycle γ_k the observables $\mathcal{O}^{(k)}_{\sharp_\alpha(\omega),\gamma_k}$ and $\mathcal{A}^{(k)}_{\omega,\gamma_k} = \int_{\gamma_k} \mathcal{A}^{(k)}_\omega$ define the same $\mathcal{Q}$-cohomology class.

3.2 Complex gauge fixing

We discuss in this section how the A-model is recovered from the PSM with Kähler target. Let us consider now the PSM with target the inverse of the Kähler form. In [15] a gauge fixing has been introduced such that the gauge fixed PSM action, after a partial integration, coincides with the action of the A-twist of the Supersymmetric sigma model.

Let us introduce complex coordinates z on Σ and x^i on M. Let us consider the lagrangian submanifold $\mathcal{L}_{\varepsilon J} \subset \mathfrak{F}_\Sigma$ defined by

$$X^\ddagger = \{x_i^+, \eta_z^{+i}, \eta_{zi}, b^{+i} \ + \text{c.c.}\} = 0 \ .$$

The coordinates on $\mathcal{L}_{\varepsilon J}$ are collectively called $X = \{x^{\bar{\imath}}, \eta_{z\bar{\imath}}, \eta_z^{+\bar{\imath}}, b_{\bar{\imath}} \ + \text{c.c.}\}$. Let us consider the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection for the Kähler metric $\alpha^{i\bar{\jmath}} = \mathrm{i}g^{i\bar{\jmath}}$ and introduce the coordinates that transform tensorially:

$$p_{\bar{z}i} = \eta_{\bar{z}i} - \Gamma^l_{ij}\eta_{\bar{z}}^{+j} b_l \ .$$

In these coordinates the gauge fixed action reads

$$\begin{aligned}\mathcal{S}_{\mathcal{L}_{\varepsilon J}} = \int_\Sigma \mathrm{d}z\mathrm{d}\bar{z} \Big(&\mathrm{i}p_{z\bar{\jmath}}\partial_{\bar{z}}x^{\bar{\jmath}} - \mathrm{i}p_{\bar{z}i}\partial_z x^i + \mathrm{i}\eta_{\bar{z}}^{+i}D_z b_i - \mathrm{i}\eta_z^{+\bar{\jmath}}D_{\bar{z}}b_{\bar{\jmath}} \\ &+ g^{k\bar{r}}R^l_{k\bar{\jmath}i}\eta_{\bar{z}}^{+i}\eta_z^{+\bar{\jmath}}b_l b_{\bar{r}} + g^{i\bar{\jmath}}p_{\bar{z}i}p_{z\bar{\jmath}}\Big) \ .\end{aligned} \tag{3.17}$$

By using the transformation rules (3.8), one can check that under an holomorphic change of coordinates $y^I(x^i)$ of M, the corresponding transformation of fields on $\mathcal{L}_{\varepsilon J}$ does not depend on momenta $X^\ddagger$. The atlas $\{X, X^\ddagger\}$ of adapted Darboux coordinates then fixes a symplectic tubular neighbourhood of $\mathcal{L}_{\varepsilon J}$ that determines the residual symmetry as

$$\mathcal{Q}_{\mathcal{L}_{\varepsilon J}} = b^i\frac{\delta}{\delta x^i} + \Big(-\partial_{\bar{z}}x^i + \Gamma^i_{kl}\eta_{\bar{z}}^{+l}b^k\Big)\frac{\delta}{\delta\eta_{\bar{z}}^{+i}} + \Big(-\mathrm{i}g_{i\bar{\jmath}}D_{\bar{z}}b^{\bar{\jmath}} + \Gamma^l_{ki}b^k p_{\bar{z}l}\Big)\frac{\delta}{\delta p_{\bar{z}i}} + \text{c.c.} \ ,$$

where $b^i := \alpha^{i\bar{\jmath}}b_{\bar{\jmath}}$. This residual BV transformation does not square to zero off-shell, as one can check by a direct computation.

Let us consider a different tubular neighbourhood and look for conditions under which the corresponding residual symmetry squares to zero also off-shell. We look for a new Darboux atlas of the space of fields adapted to the lagrangian $\mathcal{L}_{\varepsilon J}$. If $\widetilde{X}$ and $\widetilde{X}^\ddagger$ collectively denote the new fields on $\mathcal{L}_{\varepsilon J}$ and their coordinate momenta respectively, then a canonical transformation can be generated by a functional $G[X, \widetilde{X}^\ddagger]$:

$$\widetilde{X} = \frac{\partial G}{\partial \widetilde{X}^\ddagger}, \qquad X^\ddagger = \frac{\partial G}{\partial X} \ . \tag{3.18}$$

This G must have degree -1 (because $|X|+|X^\dagger| = -1$), must be real and local. Moreover, we want that $\widetilde{X}^\ddagger(X,0) = 0$ and we can also ask without loss of generality that the canonical transformation is such that $\widetilde{X}(X,0) = X$. These conditions imply that there are no terms in G depending only on fields and that the linear term in antifields has the form $X\widetilde{X}^\dagger$. This transformation will define a new tubular neighbourhood provided $\partial\widetilde{X}/\partial X^\ddagger(X,0) \neq 0$. We will also assume, for simplicity, that the canonical transformation does not depend on any additional structure on Σ. The most general form of G compatible with all the above conditions is:

$$G[X,\widetilde{X}^\dagger] = \int\limits_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\left(X\widetilde{X}^\dagger + \mathrm{i}\Lambda^{\bar{\imath}}{}_j\tilde{p}_{\bar{z}\bar{\imath}}\tilde{\eta}_z^{+j} - \mathrm{i}\Lambda^i{}_{\bar{\jmath}}\tilde{p}_{zi}\tilde{\eta}_{\bar{z}}^{+\bar{\jmath}} + \mathrm{i}T_{\mu i\bar{\jmath}}b^\mu\tilde{\eta}_z^{+i}\tilde{\eta}_{\bar{z}}^{+\bar{\jmath}}\right) , \tag{3.19}$$

where Λ, T are real tensors on M. In (3.19) Greek letter indices run over all coordinates, holomorphic and antiholomorphic. We collect here the explicit transformations

$$\begin{aligned}
&\tilde{x}^\mu = x^\mu \ , && \tilde{x}^+_\mu = x^+_\mu - \mathrm{i}\partial_\mu\Lambda^{\bar{\jmath}}{}_i\, p_{\bar{z}\bar{\jmath}}\eta_z^{+i} + \mathrm{i}\partial_\mu\Lambda^i{}_{\bar{\jmath}}\, p_{zi}\eta_{\bar{z}}^{+\bar{\jmath}} \\
& && \qquad\quad - \mathrm{i}\partial_\mu T_{\nu i\bar{\jmath}}\, b^\nu\eta_z^{+i}\eta_{\bar{z}}^{+\bar{\jmath}} \ , \\
&\tilde{b}^\mu = b^\mu \ , && \tilde{b}^+_\mu = b^+_\mu - \mathrm{i}T_{\mu i\bar{\jmath}}\,\eta_z^{+i}\eta_{\bar{z}}^{+\bar{\jmath}} \ , \\
&\tilde{p}_{\bar{z}i} = p_{\bar{z}i} + \mathrm{i}\Lambda^{\bar{\jmath}}{}_i\, p_{\bar{z}\bar{\jmath}} - \mathrm{i}T_{\mu i\bar{\jmath}}\, b^\mu\eta_{\bar{z}}^{+\bar{\jmath}} \ , && \tilde{\eta}_z^{+i} = \eta_z^{+i} \ , \\
&\tilde{\eta}_{\bar{z}}^{+i} = \eta_{\bar{z}}^{+i} - \mathrm{i}\Lambda^i{}_{\bar{\jmath}}\,\eta_{\bar{z}}^{+\bar{\jmath}} \ , && \tilde{p}_{zi} = p_{zi} \ .
\end{aligned} \tag{3.20}$$

We see that the new atlas is adapted to $\mathcal{L}_{\varepsilon J}$, and changes the tubular neighbourhood provided Λ and T are non vanishing.

One then finally computes the new residual symmetry as:

$$\begin{aligned}
\mathcal{Q}^G_{\mathcal{L}_{\varepsilon J}} = \mathcal{Q}_{\mathcal{L}_{\varepsilon J}} + \Big(- \mathrm{i}g_{l\bar{\jmath}}\Lambda^{\bar{\jmath}}{}_i D_{\bar{z}} b^l + \mathrm{i}\Lambda^{\bar{\jmath}}{}_i R_{s\bar{r}l\bar{\jmath}}\eta_{\bar{z}}^{+l} b^s b^{\bar{r}} + b^\mu T_{\mu i\bar{\jmath}}\big(\mathrm{i}g^{l\bar{\jmath}}p_{\bar{z}l} - \partial_{\bar{z}}x^{\bar{\jmath}}\big)\Big)\frac{\delta}{\delta p_{\bar{z}i}} \\
-\Lambda^i{}_{\bar{\jmath}}\Big(\partial_{\bar{z}}x^{\bar{\jmath}} - \mathrm{i}g^{l\bar{\jmath}}p_{\bar{z}l}\Big)\frac{\delta}{\delta\eta_{\bar{z}}^{+i}} + \text{c.c.} \ .
\end{aligned}$$

It is easy to check that $(\mathcal{Q}^G_{\mathcal{L}_{\varepsilon J}})^2$ is zero on x and b. Requiring also the vanishing of

$$\begin{aligned}
(\mathcal{Q}^G_{\mathcal{L}_{\varepsilon J}})^2\eta_{\bar{z}}^{+i} =& \big(\Lambda^i{}_{\bar{\jmath}}g^{l\bar{\jmath}}\Lambda^{\bar{r}}{}_l - g^{i\bar{r}}\big)\big(g_{s\bar{r}}D_{\bar{z}}b^s + R_{\bar{r}k\bar{u}s}\eta_{\bar{z}}^{+k}b^s b^{\bar{u}}\big) + \\
&+ \big(\nabla_\mu\Lambda^i{}_{\bar{r}} + \mathrm{i}g^{l\bar{\jmath}}\Lambda^i{}_{\bar{\jmath}}T_{\mu l\bar{r}}\big)\big(\mathrm{i}g^{k\bar{r}}b^\mu p_{\bar{z}k} - b^\mu\partial_{\bar{z}}x^{\bar{r}}\big) \ ,
\end{aligned} \tag{3.21}$$

fixes the following conditions:

$$\Lambda^i{}_{\bar{\jmath}}g^{l\bar{\jmath}}\Lambda^{\bar{r}}{}_l = g^{i\bar{r}} \ , \qquad T_{\mu l\bar{r}} = -\mathrm{i}g_{i\bar{\kappa}}\Lambda^{\bar{\kappa}}{}_l\nabla_\mu\Lambda^i{}_{\bar{r}} \ . \tag{3.22}$$

It can be explicitly shown that these constraints on Λ and T are sufficient to have also $(\mathcal{Q}^G_{\mathcal{L}_{\varepsilon J}})^2 p_{\bar{z}i} = 0$.

The possibility of choosing a tubular neighbourhood, for which the residual symmetry is cohomological, thus depends on the existence of an invertible orthogonal $(1,1)$ tensor Λ satisfying $\Lambda J + J\Lambda = 0$. There are obstructions to the existence of this structure; for instance a direct computation shows that it does not exist on $\mathbb{S}^2$. This choice is possible in many cases, for instance when Λ is a complex structure that defines, together with J, a hyperkähler structure with hyperkähler metric g. We remark that these data define the space filling coisotropic brane discussed in [35] and appear in the quantization scheme through the A-model described in [55]. It is not clear to us if the above condition on Λ is also necessary and so if there is really an obstruction to the existence of the cohomological residual symmetry.

Finally let us compare the gauge fixed action and its residual symmetry with the action of the A-model and its supersymmetry. Let us define

$$H^i_{\bar z} := -\partial_{\bar z}x^i - \Lambda^i{}_{\bar\jmath}(\partial_{\bar z}x^{\bar\jmath} + \alpha^{\bar\jmath k}p_{\bar z k}) \ , \qquad H^{\bar\imath}_z = \overline{H^i_{\bar z}} \ . \tag{3.23}$$

In these new variables the gauge fixed action and residual BV symmetry read

$$\begin{aligned}\mathcal{S}_{\mathcal{L}\varepsilon J} =& \int_\Sigma \mathrm{d}z\mathrm{d}\bar z \Big(- \mathrm{i}\alpha_{i\bar\jmath}H^i_{\bar z}H^{\bar\jmath}_z - \mathrm{i}\alpha_{i\bar\jmath}H^i_{\bar z}\partial_z x^{\bar\jmath} - \mathrm{i}\alpha_{i\bar\jmath}H^{\bar\jmath}_z\partial_{\bar z}x^i + \mathrm{i}\alpha_{i\bar\jmath}\partial_z x^i\partial_{\bar z}x^{\bar\jmath} \\ & - \mathrm{i}\alpha_{i\bar\jmath}\partial_{\bar z}x^i\partial_z x^{\bar\jmath} + \mathrm{i}\alpha_{i\bar\jmath}\eta^{+i}_{\bar z}D_z b^{\bar\jmath} + \mathrm{i}\alpha_{i\bar\jmath}\eta^{+\bar\jmath}_z D_{\bar z}b^i - R_{\bar r l\bar\jmath i}\eta^{+i}_{\bar z}\eta^{+\bar\jmath}_z b^{\bar r}b^l \Big) \ , \\ \mathcal{Q}^G_{\mathcal{L}_\varepsilon J} =& \ b^i\frac{\delta}{\delta x^i} + \Big(H^i_{\bar z} + \Gamma^i_{kl}\eta^{+l}_{\bar z}b^k\Big)\frac{\delta}{\delta\eta^{+i}_{\bar z}} + \Big(- R^i_{k\bar\jmath l}\eta^{+l}_{\bar z}b^k b^{\bar\jmath} - \Gamma^i_{kl}b^l H^k_{\bar z}\Big)\frac{\delta}{\delta H^i_{\bar z}} + \text{c.c.}\end{aligned} \tag{3.24}$$

which coincides, up to the topological term $\mathrm{i}\alpha_{i\bar\jmath}\partial_z x^i\partial_{\bar z}x^{\bar\jmath} - \mathrm{i}\alpha_{i\bar\jmath}\partial_{\bar z}x^i\partial_z x^{\bar\jmath}$, with the extension of the A-model with the auxiliary field H given in (2.16) of [52] after the field identification $x^\mu \equiv u^\mu$, $b^\mu \equiv \mathrm{i}\chi^\mu$, $\eta^{+\bar\imath}_z \equiv \rho^{\bar\imath}_z$. With these field identifications, the restriction of the observables $\mathcal{A}^{(k)}_\omega$ defined in (3.16) on $\mathcal{L}_{\varepsilon J}$ coincides with the A-model observables $A^{(k)}_\omega$ in (3.4); in particular they are closed under the cohomological residual symmetry $\mathcal{Q}^G_{\mathcal{L}_\varepsilon J}$.

3.3 Final remarks

In this chapter we compared the Poisson Sigma Model with complex gauge fixing with the A-model. We proved that the hierarchy of observables of PSM up to $\mathcal{Q}$ and d exact terms coincides after complex gauge fixing with the A-model hierarchy. Moreover, we identified the gauge fixed action of the PSM with the action of the A-model containing the non dynamical field and we determined a symplectic tubular neighbourhood of the gauge-fixing lagrangian such that the residual symmetry coincides with A-model supersymmetry. This analysis shows that the two models are the same when considered on surfaces without boundary; in particular this gives a BV explanation to the fact that Gromov-Witten invariants are independent on the choice of complex structure, as in the BV setting this corresponds to a choice of gauge fixing.

This analysis should be extended to the case with boundary. Both models provide a framework for quantization of the symplectic structure on the target. On the PSM side, the Kontsevich formula [36] for deformation quantization is reproduced in [18] as a correlator of the model on the disk. On the A-model side, in [55] the quantization is provided by the space of coisotropic branes. It will be natural to develop for this case the BV-BFV construction introduced in [23].

CHAPTER 4

Observables in the equivariant A-model

The AKSZ method [2] is a very elegant geometrical construction of solutions of the classical master equation (CME) in the Batalin-Vilkovisky (BV) formalism. It gives solutions in terms of geometrical data that are very compactly formulated in the language of graded geometry. The AKSZ space of fields is the space of maps from the source graded manifold $T[1]\Sigma$ where Σ is a d-dimensional manifold to the target $\mathcal{M}$, which is a degree $(d-1)$ graded symplectic manifold endowed with a degree one hamiltonian vector field $Q = \{\Theta, -\}$ such that $Q^2 = 0$. The solution of the CME, even for classical actions whose gauge invariance is very intricated, can be obtained on the spot directly from these data, without using the tools of homological perturbation theory: see for instance [45] for an introduction to the subject and the discussion of the Courant Sigma Model.

In the BV setting the simplest version of gauge fixing is realized by expressing the antifields as functions of the fields; once that the symplectic interpretation is taken into account and the space of fields is seen as an odd symplectic manifold, the gauge fixing is a choice of a lagrangian submanifold $\mathcal{L}$ of the BV space of fields $\mathfrak{F}$. Even if the BV vector field $\mathcal{Q}$ is not in general parallel to $\mathcal{L}$, the gauge fixed action still has an odd symmetry $\mathcal{Q}^{\pi}_{\mathcal{L}}$ obtained by projecting $\mathcal{Q}$ to $\mathcal{L}$. We call this odd vector field of $\mathcal{L}$ the *residual BV symmetry*. This projection is not unique and depends on an additional geometrical datum, the choice of a symplectic tubular neighbourhood of $\mathcal{L}$, *i.e.* a (local) identification of $\mathfrak{F}$ with $T^*[-1]\mathcal{L}$. This choice can always be done, although in a non unique way; different choices coincide on-shell, *i.e.* when restricted to the surface of solutions of equations of motion. In examples this odd symmetry of the gauge fixed action is an interesting object and so it is worth to take it into account in the full picture. For instance in the BV treatment of ordinary gauge theories it is the BRST differential; in the A-model it is the supersymmetry [12].

A relevant aspect where one can appreciate the beauty of the AKSZ solution is the construction of observables. Indeed, there is a chain map from the complex of the homological vector field Q of the target $\mathcal{M}$ to the complex of $\mathcal{Q}$ that defines the so called *AKSZ observables*. Unfortunately, in general we cannot expect that after gauge fixing a

BV observable is closed under the residual BV symmetry and AKSZ observables are not special in this regard. So in certain cases, it can be useful to introduce an equivalent set of observables that have a better behaviour for the gauge fixing.

This study began in [12] for the case of the A-model, seen as a complex gauge fixing of the Poisson Sigma Model with non degenerate target. In this case, the target graded manifold is just $T[1]M$ with M symplectic and Q the de Rham vector field of M; AKSZ observables are then defined in terms of closed forms on M. In [12] it was shown that one can define an equivalent class of observables, which we called *A-model observables*, related by an explicit homotopy to the AKSZ ones, that are closed under the residual BV symmetry fixed by the complex gauge fixing. The name is due to the fact that they reproduce Witten's hierarchy of observables for the A-model in [52].

In this chapter we extend the analysis to an equivariant version of the Poisson Sigma Model. This is an AKSZ theory that was studied in [11, 48, 58]. The geometrical data of the target encode a hamiltonian G space, *i.e.* a symplectic manifold M with an action of a Lie group G with an equivariant momentum map μ. The target homological vector field encodes the Weil model for equivariant geometry so that the AKSZ observables are associated to equivariant cohomology. In [11] this theory was considered as a model for the PSM with target the symplectic reduction $\mu^{-1}(0)/G$. We introduce the analogue of A-model observables that depend on a minimal set of fields and introduce an explicit homotopy with the AKSZ observables.

We consider two different gauge fixings which are compatible with the A-model observables. The first one is relevant when the symplectic reduction of the target space is smooth; we conjecture that the theory computes the A-model correlator of the reduced symplectic manifold in the spirit of [11]. In the second one, we recover for the Lie algebra sector the supersymmetric Yang Mills action and the residual BV symmetry is the supersymmetry generator.

4.1 A-model and PSM correspondence reconsidered

The correspondence between the AKSZ observables of the PSM and the observables of the A-model established in [12] can be better understood starting from an homotopy between maps of superspaces.

Let M be a symplectic manifold and let us denote with $\alpha = \alpha_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ the symplectic form. The Poisson Sigma Model (PSM) with non degenerate target is the AKSZ construction with target $T^*[1]M$ and hamiltonian $\alpha^{\mu\nu}b_\mu b_\nu$, where $\{x^\mu, b_\mu\}$ are the degree $(0,1)$ coordinates of $T^*[1]M$ and $\alpha^{\mu\nu}$ is the inverse of $\alpha_{\mu\nu}$. The space of AKSZ field is $\mathfrak{F}_\Sigma = \mathrm{Maps}(T[1]\Sigma, T^*[1]M)$. The symplectic form identifies it with $\mathrm{Maps}(T[1]\Sigma, T[1]M)$ and finally with $T[1](\mathrm{Maps}(T[1]\Sigma, M)) \equiv T[1]\mathcal{M}_\Sigma$. With this identification observables are forms on $\mathcal{M}_\Sigma$. Let the superfields $(\mathbf{x}, \mathbf{b}) \in \mathfrak{F}_\Sigma$ be decomposed as

$$\mathbf{x}^\mu = x^\mu + \eta^{+\mu} + b^{+\mu} \ , \quad \mathbf{b}_\mu = b_\mu + \eta_\mu + x^+_\mu \ .$$

The de Rham differential δ of $\mathcal{M}_\Sigma$ acts as $\delta\mathbf{x}^\mu = \mathbf{b}^\mu \equiv \alpha(\mathbf{x})^{\mu\nu}\mathbf{b}_\nu$. It can also be interpreted as the (infinitesimal) diffeomorphism obtained by composing the superfields with the (infinitesimal) diffeomorphism of the target $T[1]M$ defined by the de Rham differential. The BV differential is then defined as

$$\mathcal{Q} = \delta - \mathrm{d}'_\Sigma \ ,$$

where d'_Σ is the vector field of $\mathfrak{F}_\Sigma$ obtained by the action of the de Rham differential of Σ on the superfields. More geometrically, $\mathrm{d}'_\Sigma \in \mathrm{Vect}(\mathfrak{F}_\Sigma)$ is the (infinitesimal) diffeomorphism of $\mathfrak{F}_\Sigma$ obtained by composing maps with the (infinitesimal) diffeomorphism of the source defined by the de Rham differential. Although d'_Σ must not be confused with d_Σ acting on Σ, they coincide on functions of the (evaluated) superfields, *i.e.*

$$(\mathrm{d}'_\Sigma - \mathrm{d}_\Sigma) f(\mathbf{x}(u,\theta), \mathbf{b}(u,\theta)) = 0 \ . \tag{4.1}$$

It is explicitly given by the following formulas:

$$\begin{aligned} \mathrm{d}'_\Sigma x &= 0 \ , & \mathrm{d}'_\Sigma b &= 0 \ , \\ \mathrm{d}'_\Sigma \eta^+ &= \mathrm{d}_\Sigma x \ , & \mathrm{d}'_\Sigma \eta &= \mathrm{d}_\Sigma b \ , \\ \mathrm{d}'_\Sigma b^+ &= \mathrm{d}_\Sigma \eta^+ \ , & \mathrm{d}'_\Sigma x^+ &= \mathrm{d}_\Sigma \eta \ . \end{aligned} \tag{4.2}$$

We are going to define the A-model hierarchy of observables. Let us consider the degree 0 evaluation map $\mathrm{ev}\colon \mathcal{M}_\Sigma \times T[1]\Sigma \longrightarrow M$ defined as:

$$\mathrm{ev}(\mathbf{x}; u, \theta) = x(u) \ . \tag{4.3}$$

Since $\mathfrak{F}_\Sigma$ is a vector bundle over $\mathcal{M}_\Sigma$ we can extend ev to a vector bundle morphism $\widehat{\mathrm{ev}}\colon \mathfrak{F}_\Sigma \times T[1]\Sigma \to T[1]M$ over ev by asking that for each $f \in C^\infty(M)$ we have

$$\widehat{\mathrm{ev}}^* \mathrm{d}f = (\mathcal{Q} + \mathrm{d}_\Sigma)\mathrm{ev}^* f \ .$$

We then compute

$$\widehat{\mathrm{ev}}^* \mathrm{d}x^\mu = (\mathcal{Q} + \mathrm{d}_\Sigma) x^\mu = b^\mu + \mathrm{d}_\Sigma x^\mu \ .$$

For every $\omega \in \Omega^\bullet M$ we can associate a functional A_ω

$$A_\omega \equiv \widehat{\mathrm{ev}}^* \omega = \omega(x, b + \mathrm{d}_\Sigma x) \tag{4.4}$$

satisfying by construction $(\mathcal{Q} + \mathrm{d}_\Sigma) A_\omega = A_{\mathrm{d}\omega}$. If then $\mathrm{d}\omega = 0$ we say that A_ω is the A-model hierarchy of observables of the PSM associated to ω.

The AKSZ hierarchy described in the previous section, after the identification given between $T[1]M$ and $T^*[1]M$ given by α, is defined for each $\omega \in \Omega M$ as

$$\mathcal{O}_\omega = \omega(\mathbf{x}(u,\theta), \mathbf{b}(u,\theta)) = \mathbb{E}\mathrm{v}^* \omega \tag{4.5}$$

where the evaluation map $\mathbb{E}\mathrm{v}\colon \mathfrak{F}_\Sigma \times T[1]\Sigma \to T[1]M$ defined in (2.28) and given by

$$\mathbb{E}\mathrm{v}(\mathbf{x}, \mathbf{b}, u, \theta) = (\mathbf{x}(u,\theta), \mathbf{b}(u,\theta)) \tag{4.6}$$

is a vector bundle morphism over $\mathrm{Ev}\colon \mathcal{M}_\Sigma \times T[1]\Sigma \longrightarrow M$ defined as

$$\mathrm{Ev}(\mathbf{x}; u, \theta) = \mathbf{x}(u, \theta) \ . \tag{4.7}$$

The two morphisms ev and Ev are homotopic with homotopy $k\colon \mathcal{M}_\Sigma \times T[1]\Sigma \times [0,1] \longrightarrow M$ given by

$$k(\mathbf{x}; u, \theta; t) = \mathbf{x}(u, t\theta) = x(u) + t\eta^+(u) + t^2 b^+(u) \ . \tag{4.8}$$

We extend it to the vector bundle morphism $\widehat{k}\colon \mathfrak{F}_\Sigma \times T[1]\Sigma \times T[1]I \to T[1]M$ over k by imposing that for each $f \in C^\infty(M)$ we have

$$\widehat{k}^* \mathrm{d}f = (\mathcal{Q} + \mathrm{d}_\Sigma + \mathrm{d}_I) k^* f \ ,$$

where d_I is the de Rham differential of $I = [0,1]$. We compute

$$\widehat{k}^* \mathrm{d}x^\mu = \delta X^\mu + t\delta\eta^{+\mu} + t^2 \delta b^{+\mu} + (1-t)\mathrm{d}_\Sigma x^\mu + t(1-t)\mathrm{d}_\Sigma \eta^{+\mu} + \mathrm{d}t(\eta^{+\mu} + 2tb^{+\mu}) \ . \tag{4.9}$$

We then define $K(\omega) = \int_{[0,1]} \widehat{k}^*(\omega)$ for each $\omega \in \Omega M$. By construction we have that $\widehat{k}^*\omega|_{t=0} = \mathrm{ev}^*\omega = A_\omega$ and $\widehat{k}^*\omega|_{t=1} = \mathbb{E}\mathrm{v}^*\omega = \mathcal{O}_\omega$ and

$$\mathcal{O}_\omega - A_\omega = K(\mathrm{d}\omega) - (\mathcal{Q} + \mathrm{d}_\Sigma) K(\omega) \ . \tag{4.10}$$

It is now a direct computation to check that the homotopy K coincides with the one defined in [12].

Let us finally discuss the gauge fixing. Let us introduce the complex structures ϵ on Σ and J, compatible with α, on M. We denote the holomorphic coordinates as z and x^i on Σ and M. Let us choose the complex gauge fixing for the superfields $\mathbf{x}$ and $\mathbf{b}$ introduced in [15] and discussed in [12] so that we recover the A-model action for that sector. The gauge fixing lagrangian $\mathcal{L}$ on the A-model sector is defined by

$$x^+ = b^+ = \eta_{zi} = \eta_{\bar{z}\bar{\imath}} = \eta_z^{+i} = \eta_{\bar{z}}^{+\bar{\imath}} = 0 \ . \tag{4.11}$$

The gauge fixed action (3.17) reads

$$\mathcal{S}_{\mathcal{L}_{\epsilon J}} = \int_\Sigma \Big(- \mathrm{i} p_{z\bar{\jmath}} \partial_{\bar{z}} x^{\bar{\jmath}} + \mathrm{i} p_{\bar{z}i} \partial_z x^i - \mathrm{i}\eta_{\bar{z}}^{+i} D_z b_i + \mathrm{i}\eta_z^{+\bar{\jmath}} D_{\bar{z}} b_{\bar{\jmath}} \\ + g^{k\bar{r}} R^l_{k\bar{\jmath}i} \eta_{\bar{z}}^{+i} \eta_z^{+\bar{\jmath}} b_l b_{\bar{r}} + g^{i\bar{\jmath}} p_{\bar{z}i} p_{z\bar{\jmath}} \Big) \ , \tag{4.12}$$

where $p_{\bar{z}i} = \eta_{\bar{z}j} + \Gamma^k_{ij} \eta_{\bar{z}}^{+j} b_k$. Variables appearing in (4.11) are the momenta of a symplectic tubular neighbourhood that determines the BV residual symmetry, as explained in Section 2.3. Contrary to AKSZ observables, the A-model observables do not depend on the momenta so that their restriction to $\mathcal{L}_{\epsilon J}$ is closed under the BV residual symmetry.

4.2 Equivariant A-model from AKSZ

We discuss in this section a BV approach to the equivariant version of the A-model. The geometrical setting consists of a Poisson manifold (M, α) with an action of a Lie group G by Poisson diffeomorphisms. We require the existence of an equivariant momentum map $\mu : M \to \mathfrak{g}^*$, where $\mathfrak{g} = \mathrm{Lie}\, G$. By momentum map we mean that the fundamental vector fields of the G action are hamiltonian vector field. We will be mainly interested in the non degenerate case where this is the usual notion of hamiltonian G-action.

4.2.1 Definition of the model

The model that we are going to discuss was considered in [11, 48, 58]. The graded geometric formulation of the equivariant formulation and its AKSZ theory that we are going to use was discussed in [11]. We briefly recall it.

The equivariant differential can be described by a hamiltonian vector field Q on the symplectic graded manifold $T^*[1]\big(M \times T[1]\mathfrak{g}[1]\big)$. If we take coordinates (x^μ, b_μ) on $T^*[1]M$ and (c^a, ϕ^a) of degree $(1, 2)$ with momenta $(\xi_a, \widetilde{\xi}_a)$ of degree $(0, -1)$ on $T^*[1]T[1]\mathfrak{g}[1]$, we can define the degree 2 hamiltonian

$$\Theta = \frac{1}{2}\alpha^{\mu\nu} b_\mu b_\nu - \xi_a \phi^a - \mu_a \phi^a + v^\mu_a b_\mu c^a + \frac{1}{2}\xi_a [c, c]^a + \widetilde{\xi}_a [c, \phi]^a \ , \tag{4.13}$$

so that $Q(\cdot) = \{\Theta, \cdot\}$ reads:

$$\begin{aligned}
Qx^\mu &= \alpha^{\mu\nu} b_\nu + c^a v^\mu_a \ , \\
Qb_\mu &= \frac{1}{2}\partial_\mu \alpha^{\rho\sigma} b_\rho b_\sigma + \partial_\mu v^\rho_a b_\rho c^a - \phi^a \partial_\mu \mu_a \ , \\
Qc^a &= \phi^a - \frac{1}{2} f^a_{bd} c^b c^d \ , \\
Q\phi^a &= -f^a_{bc} c^b \phi^c \ , \\
Q\xi_a &= v^\mu_a b_\mu - f^c_{ab} \xi_c c^b - f^c_{ab} \widetilde{\xi}_c \phi^b \ , \\
Q\widetilde{\xi}_a &= \xi_a + \mu_a + f^c_{ab} \widetilde{\xi}_c c^b \ .
\end{aligned} \tag{4.14}$$

We recover the Kalkman model for Poisson equivariant cohomology as the differential graded subalgebra $W(M, \pi, \mathfrak{g})$ generated by $\{x, b, c, \phi\}$. We consider here the case where α is non degenerate and let $b^\mu = \alpha^{\mu\nu} b_\nu$. We then compute

$$\begin{aligned}
Qx^\mu &= b^\mu + c^a v^\mu_a \ , \\
Qb^\mu &= -\phi^a v^\mu_a + c^a b^\nu \partial_\nu v^\mu_a \ .
\end{aligned} \tag{4.15}$$

so that $(W(M, \pi, \mathfrak{g}), Q)$ coincides with the Kalkman model for equivariant cohomology (see [34]).

If we look at the target manifold $T^*[1](M \times T[1]\mathfrak{g}[1])$ again as a tangent bundle $T[1](M \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1])$ so that the de Rham differential is defined as $\mathrm{d}x^\mu = b^\mu$, $\mathrm{d}c^a = \phi^a$ and $\mathrm{d}\widetilde{\xi}_a = \xi_a$ we immediately recognize that

$$Q = \mathrm{d} + s \ , \qquad \mathrm{d}s + s\mathrm{d} = s^2 = \mathrm{d}^2 = 0 \ . \tag{4.16}$$

Remark 4.2.1. The identification of the target manifold with $T[1](M \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1])$ can be expressed by defining the tangent fiber degree as $\deg x = \deg c = \deg \widetilde{\xi} = 0$ and $\deg b = \deg \phi = \deg \xi = 1$. Moreover, $\deg \mathrm{d} = 1$ and $\deg s = 0$. Following [11], the antighost degree $\mathrm{ag} = -\mathrm{gh} + \deg$, where gh is the natural degree of the target graded manifold, gives the target manifold the structure of BFV manifold, a model for the symplectic reduction of $T^*[1]M$ with respect to the constraints $\mu = 0$ and $v_a^\nu b_\nu = 0$. We recall that the BFV (Batalin-Fradkin-Vilkovisky) manifolds in general give an homological resolution of constrained system and can be seen as a mathematical formulation of BRST in the hamiltonian setting (see [6, 49]).

Remark 4.2.2. The map $\varphi : W \to W$ defined as $\varphi(x, b, c, \phi) = (x, \widetilde{b}, c, \widetilde{\phi})$ where

$$\widetilde{\phi} = \phi - \frac{1}{2}[c, c] \, , \quad \widetilde{b}^\mu = b^\mu + c^a v_a^\mu \tag{4.17}$$

intertwines $Q|_W$ and the de Rham differential d. Since the Lie algebra part is acyclic, the cohomology of (W, Q) then coincides with $H_{\mathrm{dR}}(M)$. Let us introduce the contraction operator $\iota_a = \frac{\partial}{\partial c^a}$ and Lie derivative L_a on the Lie algebra variables; then we can write the Kalkman differential as

$$Q|_W = \mathrm{d}_M + c^a(L_{v_a} + L_a) - \phi^a(\iota_{v_a} - \iota_a) \ ,$$

where d_M denotes the de Rham differential on M, L_{v_a} and ι_{v_a} are the usual Lie derivative and contraction operators on forms, respectively. We then see that the subcomplex $W' = \bigcap_a (\ker \iota_a \cap \ker(L_a + L_{v_a})) \subset W$ of elements that are independent on c and $\mathfrak{g}$-invariant coincides with the Cartan model for equivariant cohomology.

Let us now consider the AKSZ sigma model with source $\big(T[1]\Sigma, \mathrm{d}_\Sigma\big)$ and target $T^*[1]\big(M \times T[1]\mathfrak{g}[1]\big)$ with differential Q. We can introduce the superfields:

$$\begin{aligned}
\mathbf{x} &= x + \eta^+ + b^+ \ , & \mathbf{b} &= b + \eta + x^+ \ , \\
\mathbf{c} &= c + A + \xi^+ \ , & \mathbf{\Xi} &= \xi + A^+ + c^+ \ , \\
\mathbf{\Phi} &= \phi + \psi + \widetilde{\xi}^+ \ , & \widetilde{\mathbf{\Xi}} &= \widetilde{\xi} + \psi^+ + \phi^+ \ .
\end{aligned} \tag{4.18}$$

The cohomological BV vector field is $\mathcal{Q} = \check{Q} - \mathrm{d}'_\Sigma$, where $\check{Q}$ is the vector field obtained by composing maps with the (infinitesimal) diffeomorphism of the target defined by Q. Recalling that $\mathbf{b}^\mu = \alpha^{\mu\nu}(\mathbf{x})\mathbf{b}_\nu$, it acts on the fields x, b, c, A, ϕ, ψ as

$$\begin{aligned}
\mathcal{Q}x^\mu &= b^\mu + c^a v_a \ , \\
\mathcal{Q}b^\mu &= -\partial_\nu v_a^\mu b^\nu c^a - v_a^\mu \phi^a \ , \\
\mathcal{Q}c^a &= \phi^a - \frac{1}{2}[c, c]^a \ , \\
\mathcal{Q}A^a &= \psi - [c, A]^a - \mathrm{d}_\Sigma c \ , \\
\mathcal{Q}\phi^a &= -[c, \phi]^a \ , \\
\mathcal{Q}\psi^a &= -[c, \psi]^a - [A, \phi]^a - \mathrm{d}_\Sigma \phi^a \ .
\end{aligned} \tag{4.19}$$

We finally write the AKSZ action as

$$\mathcal{S} = \int_{T[1]\Sigma} \frac{1}{2}\alpha^{\mu\nu}(\mathbf{x})\mathbf{b}_\mu\mathbf{b}_\nu - \mathbf{\Xi}_a\mathbf{\Phi}^a - \mu_a(\mathbf{x})\mathbf{\Phi}^a + v_a^\mu(\mathbf{x})\mathbf{b}_\mu\mathbf{c}^a + \frac{1}{2}\mathbf{\Xi}_a[\mathbf{c},\mathbf{c}]^a \\ + \widetilde{\mathbf{\Xi}}_a[\mathbf{c},\mathbf{\Phi}]^a - \mathbf{b}_\mu \mathrm{d}_\Sigma \mathbf{x}^\mu - \mathbf{\Xi}_a \mathrm{d}_\Sigma \mathbf{c}^a - \widetilde{\mathbf{\Xi}}_a \mathrm{d}_\Sigma \mathbf{\Phi}^a \ . \tag{4.20}$$

4.2.2 Equivariant A-model and AKSZ observables

We want to define here the analogue of A-model observables for the equivariant model. Let us look for a map analogue to the partial evaluation map defined in (4.3). Since the target space is the shifted tangent bundle $T[1]\mathcal{M}$ with $\mathcal{M} = M \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1]$ the space $\mathfrak{F}_\Sigma$ of AKSZ fields is $T[1]\mathrm{Map}(T[1]\Sigma, \mathcal{M})$; we then start with a map

$$\mathrm{ev}_0\colon \mathrm{Map}(T[1]\Sigma, \mathcal{M}) \times T[1]\Sigma \to \mathcal{M}$$

defined as

$$\mathrm{ev}_0(\mathbf{x}, \mathbf{c}, \widetilde{\mathbf{\Xi}}; u, \theta) = (x(u), c(u) + A(u,\theta), 0) \ . \tag{4.21}$$

Since the target space differential (4.16) is not simply the de Rham differential, on forms we do not take the pull-back of ev_0, as in the previous section, but we look for a vector bundle morphism $\widehat{\mathrm{ev}}_0\colon \mathfrak{F}_\Sigma \times T[1]\Sigma \to T[1]\mathcal{M}$ over ev_0 that intertwines the differential $\mathcal{Q} + \mathrm{d}_\Sigma$ with the target differential Q, *i.e.*

$$\widehat{\mathrm{ev}}_0^* Q\omega = (\mathcal{Q} + \mathrm{d}_\Sigma)\widehat{\mathrm{ev}}_0^*\omega \ .$$

From the discussion in Remark 4.2.1, we can conclude that $\widehat{\mathrm{ev}}_0$ is completely fixed by ev_0: indeed the equivariant differential decomposes as $Q = \mathrm{d} + s$ with $\deg s = 0$ so that for each $f \in C(\mathcal{M})$ we have

$$\widehat{\mathrm{ev}}_0^* \mathrm{d}f = \widehat{\mathrm{ev}}_0^*(Q - s)f = (\mathcal{Q} + \mathrm{d}_\Sigma)\mathrm{ev}_0^* f - \mathrm{ev}_0^* s f \ .$$

We then compute

$$\begin{aligned}
\widehat{\mathrm{ev}}_0^* b &= \widehat{\mathrm{ev}}_0^* \mathrm{d}x = \widehat{\mathrm{ev}}_0^*(Qx - c^a v_a) = (\mathcal{Q} + \mathrm{d}_\Sigma)x - (c^a + A^a)v_a \\
&= b + \mathrm{d}_\Sigma x - A^a v_a \ , \\
\widehat{\mathrm{ev}}_0^* \phi &= \widehat{\mathrm{ev}}_0^* \mathrm{d}c = \widehat{\mathrm{ev}}_0^*\Big(Qc + \frac{1}{2}[c,c]\Big) = (\mathcal{Q} + \mathrm{d}_\Sigma)(c + A) + \frac{1}{2}[c + A, c + A] \\
&= \phi + \psi + F(A) \ , \\
\widehat{\mathrm{ev}}_0^* \xi_a &= \widehat{\mathrm{ev}}_0^* \mathrm{d}\widetilde{\xi}_a = \widehat{\mathrm{ev}}_0^*(Q\widetilde{\xi}_a - [\widetilde{\xi}, c]_a - \mu_a) = -\mu_a \ .
\end{aligned} \tag{4.22}$$

We then finally define for each $\omega(x, c, \widetilde{\xi}, b, \phi, \xi) \in C(T[1](M \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1]))$ the following functional

$$A_\omega := \widehat{\mathrm{ev}}_0^* \omega = \omega(x, c + A, 0, b + \mathrm{d}_\Sigma x - A^a v_a, \phi + \psi + F(A), -\mu) \ . \tag{4.23}$$

If $Q\omega = 0$ then by construction $(\mathcal{Q} + \mathrm{d}_\Sigma)A_\omega = 0$ and we say that A_ω is the A-model observable associated to ω. In particular we will associate to every equivariantly closed form an observable.

Recall that the AKSZ observable associated to ω is $\mathcal{O}_\omega = \mathbb{E}\mathrm{v}^*(\omega)$, the pullback of ω along the evaluation map $\mathbb{E}\mathrm{v}\colon \mathfrak{F}_\Sigma \times T[1]\Sigma \to T[1]\mathcal{M}$ with

$$\mathbb{E}\mathrm{v}(\mathbf{x}, \mathbf{c}, \mathbf{\Xi}, \mathbf{b}, \mathbf{\Phi}, \widetilde{\mathbf{\Xi}}; u, \theta) = (\mathbf{x}(u,\theta), \mathbf{c}(u,\theta), \mathbf{\Xi}(u,\theta), \mathbf{b}(u,\theta), \mathbf{\Phi}(u,\theta), \widetilde{\mathbf{\Xi}}(u,\theta)) \ .$$

The map $\mathbb{E}\mathrm{v}$ is a bundle map over $\mathrm{Ev}_0\colon \mathrm{Map}(T[1]\Sigma, \mathcal{M}) \times T[1]\Sigma \to \mathcal{M}$ defined as

$$\mathrm{Ev}_0(\mathbf{x}, \mathbf{c}, \widetilde{\mathbf{\Xi}}; u, \theta) = (\mathbf{x}(u,\theta), \mathbf{c}(u,\theta), \widetilde{\mathbf{\Xi}}(u,\theta)) \ .$$

We discuss now a homotopy between the A-model and AKSZ observables generalizing the discussion that we had in the previous section. We start with the following homotopy between ev_0 and (the restriction of) $\mathbb{E}\mathrm{v}$

$$\kappa\colon \mathrm{Map}(T[1]\Sigma, \mathcal{M}) \times T[1]\Sigma \times [0,1] \to \mathcal{M}$$

defined as

$$\kappa(\mathbf{x}, \mathbf{c}, \widetilde{\mathbf{\Xi}}; u, \theta, t) = (x + t\eta^+ + t^2 x^+, c + A + t^2\xi^+, t\widetilde{\mathbf{\Xi}}(u,\theta)) \ . \tag{4.24}$$

We then look for

$$\widehat{\kappa}\colon \mathfrak{F}_\Sigma \times T[1]\Sigma \times T[1][0,1] \to T[1]\mathcal{M}$$

over κ so that

$$(\mathcal{Q} + \mathrm{d}_\Sigma + \mathrm{d}_I)\widehat{\kappa}^* = \widehat{\kappa}^* Q \ , \tag{4.25}$$

where d_I is the de Rham differential of $I = [0,1]$. Again $\widehat{\kappa}$ is completely determined by κ and moreover by construction

$$\widehat{\kappa}^*\omega|_{t=0} = A_\omega \ , \quad \widehat{\kappa}^*\omega|_{t=1} = \mathcal{O}_\omega \ .$$

We then compute

$$\begin{aligned}
\widehat{\kappa}^* b^\mu &= \widehat{\kappa}^* \mathrm{d}x^\mu = (\mathcal{Q} + \mathrm{d}_\Sigma + \mathrm{d}_I)\kappa^* x^\mu - \kappa^*(c^a v_a^\mu) \\
&= b^\mu + (1-t)\mathrm{d}_\Sigma x^\mu - (1-t)A^a v_a^\mu + t(1-t)\partial_\nu v_a^\mu \eta^{+\nu} + t(1-t)\mathrm{d}_\Sigma \eta^{+\mu} + \\
&\quad + \mathrm{d}t(\eta^{+\mu} + 2tb^{+\mu}) + t^2\delta b^{+\mu} + t\delta\eta^{+\mu} \ , \\
\widehat{\kappa}^*\phi &= \widehat{\kappa}^* \mathrm{d}c = (\mathcal{Q} + \mathrm{d}_\Sigma + \mathrm{d}_I)\kappa^* c + \frac{1}{2}[\kappa^* c, \kappa^* c] \\
&= \phi + \psi + (1-t^2)F(A) + t^2\widetilde{\xi}^+ + 2t\xi^+ \mathrm{d}t \ , \\
\widehat{\kappa}^*\xi &= \widehat{\kappa}^* \mathrm{d}\widetilde{\xi} = (\mathcal{Q} + \mathrm{d}_\Sigma + \mathrm{d}_I)(t\widetilde{\mathbf{\Xi}}) - t[\widetilde{\mathbf{\Xi}}, \kappa^* c] - \kappa^*\mu \ .
\end{aligned} \tag{4.26}$$

If we define $\mathcal{K}(\cdot) = \int_I \kappa^*(\cdot)$, we get

$$\mathcal{O}_\omega - A_\omega = \mathcal{K}(Q\omega) - (\mathcal{Q} + \mathrm{d}_\Sigma)\mathcal{K}\omega \ . \tag{4.27}$$

Of course, if we set to zero all the variables associated to the Lie algebra $\mathfrak{g}$, we recover the homotopy between the AKSZ observables of the PSM and the A-model observables described in (4.9).

4.2.3 Cohomology of Q

An interesting consequence of (4.27) is the following characterization of the cohomology of the target differential Q defined in (4.14). Indeed, after the restriction to zero form observables, d_Σ does not appear in (4.27) and $\mathcal{Q}$ acts as $\check{Q}$, the vector field obtained composing the maps of $\mathfrak{F}_\Sigma$ with the infinitesimal diffeomorphism of the target defined by Q. In other terms it does not involve derivatives with respect to the source coordinates, so that it is a pointwise relation that can be read as a relation defined on the target as follows.

Let $i\colon W(M,\pi,\mathfrak{g}) \to \mathcal{A} \equiv (C(T^*[1](M\times\mathfrak{g}[1])), Q)$ be the injection of the Kalkman model described in Subsection 4.2.1 and let $p : \mathcal{A} \to W(M,\pi,\mathfrak{g})$ be the quotient map with kernel generated by $\xi+\mu$ and $\widetilde{\xi}$. It is a direct check to verify that p is a chain map. Clearly we have that $p\circ i = \mathrm{id}_W$. Now it is clear that (4.27) for forms of degree 0 translates into

$$\mathrm{id}_{\mathcal{A}} - i\circ p = \mathcal{K}_0\circ Q - Q\circ\mathcal{K}_0 \tag{4.28}$$

where

$$\mathcal{K}_0(\omega) = \int_I \omega(x,c,t\widetilde{\xi},b,\phi,t\xi-(1-t)\mu+\mathrm{d}_I t\ \widetilde{\xi})$$

for each $\omega\in\mathcal{A}$. We can then conclude that i and p are inverse up to homotopy so that the cohomology of Q is isomorphic to the cohomology of the Kalkman complex (or equivalently of the Weil complex, see [34]) that is de Rham cohomology $H_{\mathrm{dR}}(M)$ (see Remark 4.2.2). It is maybe useful to stress that we are not restricting it to the subcomplex W' giving equivariant cohomology. Finally, if $\varphi\colon W\to W$ is the isomorphism defined in (4.17) and Θ is the degree 2 hamiltonian in (4.13), then $\varphi(p(\Theta)) = -\alpha$, so that we can say that $-\Theta$ represents in $\mathcal{A}$ the class of the symplectic form in the de Rham cohomology of M.

4.3 Gauge fixing

We discuss here two different gauge fixings of the AKSZ theory discussed in the previous section. The A-model sector is always gauge fixed with the complex gauge fixing defined in (4.11).

In both cases the lagrangian gauge fixing $\mathcal{L}$ is given together with an adapted symplectic tubular neighbourhood, *i.e.* a symplectomorphism between the BV space of fields $\mathfrak{F}_\Sigma$ and $T^*[-1]\mathcal{L}$ that fixes also a residual BV symmetry as explained in Section 2.3. The A-model observables do not depend on the momenta so that after the restriction they are invariant under the residual BV symmetry.

4.3.1 Gauge fixing for the symplectic reduction

We assume that $\partial_k v_a^{\bar{\imath}} = 0$, *i.e.* the real G action on M gives rise to an holomorphic action of $G_{\mathbb{C}}$. It can be checked that, once we assume the complex gauge fixing (4.11), the ghost c disappears from the action.

According to the discussion in Remark 4.2.1, the target manifold of the AKSZ construction is a BFV space, *i.e.* a model for the symplectic reduction of $T^*[1]M$ with respect to the graded constraints $\mu = 0$ and $v^\nu b_\nu = 0$. If the G action is free on $\mu = 0$ then $\mu^{-1}(0)/G$ is smooth and the reduced space is $T^*[1](\mu^{-1}(0)/G)$. In this case the BV theory should be regarded as equivalent to the Poisson Sigma Model with the reduced target. According to the discussion in [11], the natural gauge fixing of the Lie algebra sector is defined by putting the antighosts variables to zero; this means:

$$\boldsymbol{\Xi} = \widetilde{\boldsymbol{\Xi}} = 0 \ . \tag{4.29}$$

The residual gauge symmetry $\mathcal{Q}_{\mathcal{L}}$ given by this symplectic tubular neighbourhood is directly read from (4.19) together with

$$\begin{aligned} \mathcal{Q}_{\mathcal{L}}\eta_{\bar{z}i} &= \partial_i\alpha^{k\bar{\jmath}}\eta_{\bar{z}k}b_{\bar{\jmath}} + \partial_i\mu_a\psi^a_{\bar{z}} + \partial_i\partial_j\mu_a\eta^{+j}_{\bar{z}}\phi^a + \\ &\quad + \partial_i v^j_a(\eta_{\bar{z}j}c^a + b_j\psi^a_{\bar{z}}) + \partial_i\partial_j v^k_a\eta^{+j}_{\bar{z}}b_k c^a \ , \\ \mathcal{Q}_{\mathcal{L}}\eta^{+i}_{\bar{z}} &= -\partial_z x^i + \partial_k\alpha^{i\bar{\jmath}}\eta^{+k}_{\bar{z}}b_{\bar{\jmath}} + v^i_a A^a_{\bar{z}} + \partial_k v^i_a\eta^{+k}_{\bar{z}}c^a \ , \\ \mathcal{Q}_{\mathcal{L}}\xi^+ &= \widetilde{\xi}^+ - F(A) - [c,\xi^+] \ , \\ \mathcal{Q}_{\mathcal{L}}\widetilde{\xi}^+ &= -\mathrm{d}_\Sigma\psi - [A,\psi] - [c,\widetilde{\xi}^+] - [\xi^+,\phi] \ . \end{aligned} \tag{4.30}$$

Since the A-model observables defined in (4.23) are independent on the coordinates (4.11) and (4.29), they are also invariant when restricted to the gauge fixing lagrangian under $\mathcal{Q}_{\mathcal{L}}$. This is not true for the AKSZ observables.

After the introduction of an arbitrary affine connection Γ and the definition of $p_{\bar{z}j} = \eta_{\bar{z}j} + \Gamma^k_{ji}\eta^{+i}_{\bar{z}}b_k$, we obtain the gauge-fixed action $\mathcal{S}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}_{\epsilon J}} + \mathcal{S}_{\mathcal{L}_{\mathfrak{g}}}$ where $\mathcal{S}_{\mathcal{L}_{\epsilon J}}$ is computed in (4.12) and

$$\begin{aligned} \mathcal{S}_{\mathcal{L}_{\mathfrak{g}}} = \int_\Sigma \Big(&\mu_a\widetilde{\xi}^{+a} + \partial_i\mu_a\eta^{+i}_{\bar{z}}\psi^a_z + \partial_{\bar{\imath}}\mu_a\eta^{\bar{\imath}}_z\psi^a_{\bar{z}} + \partial_i\partial_{\bar{\jmath}}\mu_a\eta^{+i}_{\bar{z}}\eta^{+\bar{\jmath}}_z\phi^a + v^i_a b_i\xi^{+a} + v^i_a p_{i\bar{z}}A^a_z + \\ &+ \nabla_k v^i_a\eta^{+k}_{\bar{z}}b_i A^a_z + v^{\bar{\imath}}_a b_{\bar{\imath}}\xi^{+a} + v^{\imath}_a p_{\bar{\imath}z}A^a_{\bar{z}} + \nabla_{\bar{k}}v^{\imath}_a\eta^{+\bar{k}}_z b_{\bar{\imath}}A^a_{\bar{z}}\Big) \ . \end{aligned} \tag{4.31}$$

This action is quadratic in the fields p, which can then be integrated out. Their equations of motion are $p_{\bar{z}i} = \alpha_{i\bar{\jmath}}\partial_{\bar{z}}x^{\bar{\jmath}} - \alpha_{i\bar{\jmath}}v^{\bar{\jmath}}_a A^a_{\bar{z}}$ and the effective action obtained with this integration is thus:

$$\begin{aligned} \mathcal{S}_{\mathcal{L}} = \int_\Sigma \Big(&\eta^{+i}_{\bar{z}}D_z b_i + \eta^{+\bar{\jmath}}_z D_{\bar{z}}b_{\bar{\jmath}} + \alpha^{\bar{r}k}R^l_{k\bar{\jmath}i}\eta^{+i}_{\bar{z}}\eta^{+\bar{\jmath}}_z b_l b_{\bar{r}} + \alpha_{i\bar{\jmath}}\partial_A x^i\bar{\partial}_A x^{\bar{\jmath}} + \\ &+ \nabla_l v^k_a\eta^{+l}_{\bar{z}}b_k A^a_z + \nabla_{\bar{\jmath}}v^{\bar{k}}_a\eta^{+\bar{\jmath}}_z b_{\bar{k}}A^a_{\bar{z}} + \big(v^i_a b_i + v^{\bar{\jmath}}_a b_{\bar{\jmath}}\big)\xi^{+a}_{z\bar{z}} + \\ &+ \mu_a\widetilde{\xi}^{+a} + \partial_{\bar{\jmath}}\mu_a\eta^{+\bar{\jmath}}_z\psi^a_{\bar{z}} + \partial_i\mu_a\eta^{+i}_{\bar{z}}\psi^a_z + \partial_i\partial_{\bar{\jmath}}\mu_a\eta^{+\bar{\jmath}}_z\eta^{+i}_{\bar{z}}\phi^a\Big) \ , \end{aligned} \tag{4.32}$$

where $\partial_A x^i = \partial_z x^i + v^i_a A^a_z$. The dependence on the connection A is now at most quadratic and the quadratic term is non degenerate if $\det(v_a, v_b) \neq 0$. This is guaranteed if the G

action is free on $\mu^{-1}(0)$, so that this action is well defined when the symplectic reduction $\mu^{-1}(0)/G$ is smooth.

By construction the Lie algebra fields are Lagrange multipliers that constrain the system to $\mu_a(x+\eta^+) = 0$ and $v_a(x+\eta^+)^i(b+\eta)_i = 0$. We know that in Kähler reduction $T(\mu^{-1}(0))/G$ is realized as the sub-bundle $J(\mathfrak{g}_M)^\perp \cap \mathfrak{g}_M^\perp \subset T(\mu^{-1}(0))$, where $\mathfrak{g}_M$ denotes the bundle spanned by the $\mathfrak{g}$-vectors. The zero and one form component of $\mu(x+\eta^+) = 0$ force then the field x to take values in $\mu^{-1}(0)$ and η^+ in $T(\mu^{-1}(0))/G$. The two form constraint $\partial_i\partial_{\bar{\jmath}}\mu_a\eta^i_{\bar{z}}\eta^{\bar{\jmath}}_z = 0$ is instead just a consequence of the interplay between the constraint on superfields and the complex gauge fixing and it has not a geometric origin. One way to avoid it is to modify (4.29) to

$$c = 0\ , \quad \mathbf{\Xi} = c^+\ , \quad \widetilde{\mathbf{\Xi}} = 0\ . \tag{4.33}$$

This fixes the ghost c and makes c^+ be the multiplier for $\phi = 0$ so that the above undesired constraint disappears. It must be clear that with this choice the A-model observables depending on c are not anymore invariant under the residual gauge fixing.

4.3.2 The gauge multiplet and topological Yang-Mills

We consider here a different gauge fixing of the Lie algebra sector that recovers the so called topological Yang-Mills theory in two dimensions, considered by Witten in [56]. This connection was already established in [58]; here we use a slightly different gauge fixing and emphasize the relation between the residual gauge symmetry and the gauge multiplet of supersymmetry. The basic tool for introducing topological Yang-Mills theory is the gauge multiplet of 2d supersymmetry. In our BV framework it must appear as a residual BV symmetry of the gauge fixed action. We have first to recognize all the fields needed to reconstruct the gauge multiplet. The gauge multiplet consists in the following fields

	ϕ	ψ	A	H	χ	η	λ
ghost #	2	1	0	0	-1	-1	-2
form #	0	1	1	0	0	0	0

(4.34)

where the parity is the ghost modulo 2. It is easy to see that we already have almost all these fields by doing the following matches

AKSZ	ϕ	ψ	A	ξ	$\widetilde{\xi}$	-	-
gauge multiplet	ϕ	ψ	A	H	χ	η	λ

(4.35)

The fields η and λ do not appear in the PSM but can be introduced as a trivial pair. Let us define the trivial pair $\lambda, \rho \in \Omega^0(\Sigma;\mathfrak{g})$ of ghost number -2 and -1 respectively and with momenta $\lambda^+, \rho^+ \in \Omega^2(\Sigma;\mathfrak{g}^*)$ of ghost degree 1 and 0. The BV action will be shifted to

$$\mathcal{S}' := \mathcal{S} + \int_\Sigma \lambda^+_a\rho^a \tag{4.36}$$

and correspondingly we have the following action of the BV symmetry for λ and ρ:

$$\mathcal{Q}\lambda^a = \rho^a \ , \qquad \mathcal{Q}\rho^a = 0 \ . \tag{4.37}$$

If we define $\zeta := \rho + [c, \lambda]$ we get the transformations:

$$\mathcal{Q}\lambda = \zeta - [c, \lambda] \ , \qquad \mathcal{Q}\zeta = [\phi, \lambda] - [c, \zeta] \ . \tag{4.38}$$

The gauge multiplet is reconstructed with $\zeta \sim \eta$.

We can now collect the action of $\mathcal{Q}$ on these fields

$$\begin{aligned}
&\mathcal{Q}\phi = -[c, \phi] \ , && \mathcal{Q}\psi = -[c, \psi] - [A, \phi] - \mathrm{d}_\Sigma \phi \ , \\
&\mathcal{Q}A = \psi - [c, A] - \mathrm{d}_\Sigma c \ , && \mathcal{Q}\xi = -[\xi, c] - [\widetilde{\xi}, \phi] \ , \\
&\mathcal{Q}\widetilde{\xi} = \xi - [\widetilde{\xi}, c] \ , && \mathcal{Q}\lambda = \zeta - [c, \lambda] \ , \\
&\mathcal{Q}\zeta = [\phi, \lambda] - [c, \zeta] && \mathcal{Q}c = \phi - \frac{1}{2}[c, c] \ .
\end{aligned} \tag{4.39}$$

We then see that

$$\mathcal{Q} = \delta_{\mathrm{BRST}} + \delta_{\mathrm{susy}} \ ,$$

i.e. it encodes the superymmetry and the BRST transformation of the gauge multiplet.

The action of topological Yang-Mills is recovered by defining the lagrangian $\mathcal{L}_f$ with the gauge-fixing fermion f defined as

$$f = \int_\Sigma \frac{1}{2}\langle \widetilde{\xi}, \star\xi \rangle + \langle D_A\lambda, \psi \rangle \ , \tag{4.40}$$

where D_A is the covariant derivative of A, $\star$ is the Hodge star for a metric on Σ and $\langle , \rangle$ is a non degenerate invariant bilinear form on $\mathfrak{g}$.

$$\begin{aligned}
\mathcal{S}_{\mathcal{L}_f} &= \mathcal{S}_{\mathcal{L}_0} + \mathcal{Q}f = \mathcal{Q}\Big(f - \int_\Sigma \langle \widetilde{\xi}, F_A \rangle \Big) \\
&= \int_\Sigma \Big(\frac{1}{2}\langle \xi, \star\xi \rangle - \langle \xi, F_A \rangle - \langle \widetilde{\xi}, D_A\psi \rangle + \langle D_A\zeta, \psi \rangle + \\
&\qquad + \langle D_A\lambda, D_A\phi \rangle + \frac{1}{2}\langle [\widetilde{\xi}, \widetilde{\xi}], \star\phi \rangle + \langle [\psi, \lambda], \psi \rangle \Big) \ .
\end{aligned} \tag{4.41}$$

The residual BV symmetry is then given by the same formulas as in (4.39). In particular, the A-model observables are just functions of (c, ϕ, A, ψ) and so are invariant. The full gauge fixed action is then recovered as:

$$\begin{aligned}
\mathcal{S}_{\mathcal{L}} = \int_\Sigma \Big(&\eta_{\bar{z}}^{+i} D_z b_i + \eta_z^{+\bar{\jmath}} D_{\bar{z}} b_{\bar{\jmath}} + \alpha^{\bar{r}k} R^l_{k\bar{\jmath}i} \eta_{\bar{z}}^{+i} \eta_z^{+\bar{\jmath}} b_l b_{\bar{r}} + \alpha_{i\bar{\jmath}} \partial_A x^i \bar{\partial}_A x^{\bar{\jmath}} + \\
&+ \nabla_l v_a^k \eta_{\bar{z}}^{+l} b_k A_z^a - \nabla_{\bar{\jmath}} v_a^{\bar{k}} \eta_z^{+\bar{\jmath}} b_{\bar{k}} A_{\bar{z}}^a + \partial_{\bar{\jmath}} \mu_a \eta_z^{+\bar{\jmath}} \psi_{\bar{z}}^a + \\
&+ \partial_i \mu_a \eta_{\bar{z}}^{+i} \psi_z^a + \partial_i \partial_{\bar{\jmath}} \mu_a \eta_z^{+\bar{\jmath}} \eta_{\bar{z}}^{+i} \phi^a + \\
&+ \frac{1}{2}\langle \xi, \star\xi \rangle - \langle \xi, F_A \rangle - \langle \widetilde{\xi}, D_A\psi \rangle + \langle D_A\zeta, \psi \rangle + \\
&+ \langle D_A\lambda, D_A\phi \rangle + \frac{1}{2}\langle [\widetilde{\xi}, \widetilde{\xi}], \star\phi \rangle + \langle [\psi, \lambda], \psi \rangle \Big) \ .
\end{aligned} \tag{4.42}$$

Up to residual BV transformations this action depends only on equivariant cohomology class of $(\alpha + \mu)$. By construction the BV action (4.20) is defined by the target data through the AKSZ observable associated to the hamiltonian Θ in (4.13). As we proved, this observable is connected to the corresponding A-model observable via the homotopy $\mathcal{K}$. Thus the BV action can be decomposed as: kinetic term + A-model observable + BV exact term. The A-model observables are well-behaved under our gauge-fixing, however BV exact terms become generically $\mathcal{Q}^{\pi}_{\mathcal{L}}$-exact terms only on-shell. However if we choose $\mathcal{Q}^{\pi}_{\mathcal{L}}$-symmetry to close off-shell then the gauge fixed action can be written as A-model observable associated to $(\alpha+\mu)$ plus $\mathcal{Q}^{\pi}_{\mathcal{L}}$-exact terms, similar to the construction presented in [1].

Chapter 5

2D Yang-Mills on surfaces with corners in BV-BFV formalism

5.1 Introduction

In this chapter we study the perturbative path-integral quantization of 2D Yang-Mills theory, defined classically by the first-order action functional

$$S^{\mathrm{cl}}_{\mathrm{YM}} = \int_\Sigma \langle B, \mathrm{d}A + A \wedge A\rangle + \frac{1}{2}\mu\,(B, B)\ , \tag{5.1}$$

with fields A, a 1-form on the surface Σ with coefficients in a semi-simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, for G a compact simply-connected structure group, and B, a 0-form valued in $\mathfrak{g}^*$, and where μ is a fixed background 2-form (the "area form") on Σ. Here $\langle , \rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ and $(,)$ is the inverse Killing form.[1] We study 2D Yang-Mills theory in the Batalin-Vilkovisky formalism on oriented surfaces Σ with boundaries and corners allowed.[2] The quantization is constructed in such a way that it is compatible with gluing and cutting of surfaces.

Our primary motivating goal is to construct explicit partition functions of 2D Yang-Mills theory on arbitrary surfaces via the perturbative path-integral quantization, $Z = \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{YM}}}$, and to compare them with the known non-perturbative answers [40, 53] formulated in terms of the representation-theoretic data of the structure group G. There are two immediate problems to deal with:

Gauge symmetry. To define the Feynman diagrams giving the perturbative expansion of the path-integral, one needs to deal with the gauge symmetry of the action creating

[1] The case $\mu = 0$ of the action functional (5.1) defines the so-called non-abelian BF theory, which is a topological field theory, i.e., is invariant under diffeomorphisms of surfaces.

[2] We assume orientability for convenience of the formalism, but in fact one can define the theory on non-orientable surfaces as well, twisting the field B by the orientation line bundle and defining μ to be a density on Σ, rather than a 2-form. The integral (5.1) is then also understood as an integral of a density.

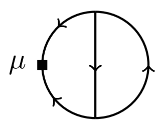
the degeneracy of stationary phase points in the path-integral. To do this, we employ the Batalin-Vilkovisky (BV) formalism. In the BV formalism, the classical fields A, B are promoted to non-homogeneous differential forms on the surface (whose homogeneous components are the original classical fields, the Faddeev-Popov ghost for the gauge symmetry and the anti-fields for those) and the gauge-fixing consists in the choice of a lagrangian submanifold in the BV fields.

Computability of the perturbative answers. Generally, Feynman diagrams are given by certain integrals over configuration spaces of points on the surface, with the integrand given by a product of propagators which depend on the details of the gauge-fixing, and typically these integrals are very hard to compute. The remedy for this comes from two ideas:

i) Firstly, we employ the BV-BFV refinement of the Batalin-Vilkovisky formalism constructed in [23, 26] – a refinement adapted to gauge theories on manifolds with boundaries allowed, compatible with gluing and cutting (thus, it is a version of BV quantization compatible with Atiyah-Segal functorial picture of QFT). In this formalism, we can recover the perturbative partition function on a surface from cutting the surface into pieces – the appropriate "building blocks of surfaces", calculating the perturbative partition function on the pieces and then assembling back into the answer for the whole surface via the gluing formula.

ii) Secondly, to compute the answers on our building blocks, we employ special gauge-fixings which allow for explicit computation of Feynman diagrams on the building block. E.g. we use the axial gauge for cylinders. This procedure is equivalent to imposing a very special gauge-fixing, involving the data of cutting into the building blocks, on the theory on the surface we started with.

For instance, the following Feynman graph for 2D Yang-Mills theory on a sphere



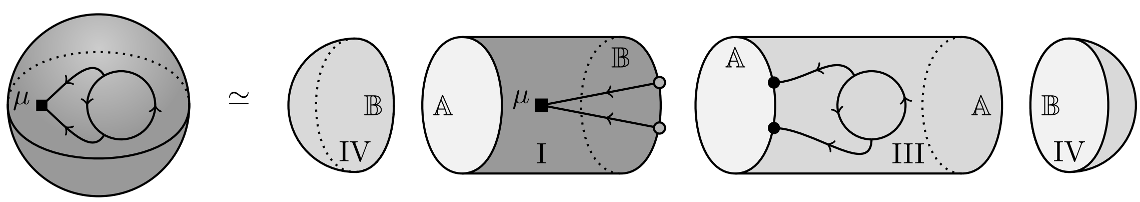


Figure 5.1: A two-loops Feynman diagram for 2D Yang-Mills on the sphere, computed by suitably cutting the surface. The darker regions are the ones where the 2-form μ is allowed to be nonzero.

is given by a complicated integral in Lorenz gauge on the sphere (e.g. the one associated to the standard round metric), but in our approach it is explicitly computable, once we split the sphere into four pieces (figure 5.1).

Remark 5.1.1. Axial gauge, which we often use, corresponds to a singular propagator which can be obtained as a limit of metric propagators (given by smooth forms on the configuration space of two points) corresponding to the degeneration of geometry of the cylinder where the ratio of the circumference to the length tends either to zero or to infinity (corresponding to two versions of the axial gauge). Thus, our computable answers obtained in a convenient gauge arise as a limit (corresponding to a limiting point on a certain curve in the space of metrics on the surface Σ) of perturbative answers computed with smooth propagators.

5.1.1 Surfaces of non-negative Euler characteristic

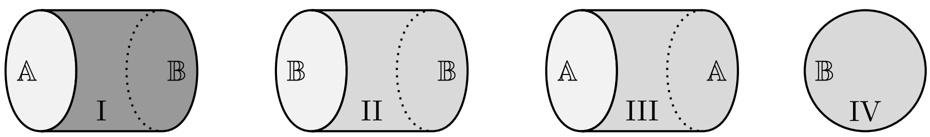
Surfaces of non-negative Euler characteristic (possibly with boundary, but with no corners) can be decomposed into the following building blocks:

(I) Cylinder with polarizations $\mathbb{A}, \mathbb{B}$ fixed on the two boundary circles (i.e. with boundary conditions prescribing the pullback of A to one boundary circle, and the pullback of B to the other circle), with a nonzero area form μ allowed.

(II) Cylinder with polarization $\mathbb{B}$ fixed on both boundaries, with the background 2-form $\mu = 0$.

(III) Cylinder with polarization $\mathbb{A}$ fixed on both boundaries, with $\mu = 0$.

(IV) Disk with polarization $\mathbb{B}$ on the boundary and with $\mu = 0$.



This is the premise of the BV-BFV formalism as in [26], where the connected components of the boundary are decorated with either $\mathbb{A}$- or $\mathbb{B}$-polarization (boundary condition), and one is allowed to glue an $\mathbb{A}$-boundary circle of one surface to a $\mathbb{B}$-boundary circle of another surface.

We manage to compute building blocks (I-III) explicitly using the axial gauge, whereas the building block (IV) can be computed in any gauge due to the vanishing of almost all Feynman diagrams by a degree counting argument. We use the fact that the partition function can only depend on the total area of a surface to concentrate the area form μ

on building blocks (I).[3] Block (III) is the most complicated in this list. We only compute it modulo BV-exact terms: the latter ultimately become irrelevant once we pass from cochain-level answers to the reduced space of states and reduced partition functions (i.e., once we integrate out the bulk residual fields and pass to the cohomology of the boundary BFV differential Ω).[4]

The cohomology in degree 0 of the BFV differential Ω on (the BFV model for) the space of states on a circle $\mathcal{H}^{\mathrm{BFV},\mathbb{A}}_{S^1}$ in the $\mathbb{A}$-polarization yields the standard (reduced) space of states of 2D Yang-Mills theory – the space of class functions on the group G, $\mathcal{H}^{\mathrm{red}}_{S^1} = L^2(G)^G$ (see Section 5.2.4). The following is the central result in this chapter.

Theorem I. The BV-BFV partition function of 2D Yang-Mills theory for Σ any surface with (possibly empty) boundary, with boundary circles decorated with $\mathbb{A}$-polarization, induces, after integrating out the bulk residual fields and passing to the cohomology of the boundary BFV operator Ω, the Migdal-Witten non-perturbative partition function of 2D Yang-Mills:

$$\left[\int_{\text{residual fields}} Z^{\mathrm{BV-BFV}}(\Sigma)\right] = \underbrace{\sum_R (\dim R)^{\chi(\Sigma)}\, \mathrm{e}^{-\frac{i\hbar a}{2}\cdot C_2(R)}\, |R\rangle^{\otimes n}}_{Z^{\mathrm{non-pert}}(\Sigma)} \qquad \in (\mathcal{H}^{\mathrm{red}}_{S^1})^{\otimes n}\ . \quad (5.2)$$

Here on the left, $[\dots]$ stands for passing to the class in zeroth Ω-cohomology. On the right side, non-perturbative partition function is given as the sum over irreducible representations R of the structure group G, $\dim R$ is the dimension of the representation and $C_2(R)$ is the value of the quadratic Casimir; $|R\rangle$ is the class function on G corresponding to the character of the representation R, mapping $g \mapsto \mathrm{tr}_R g$; $\chi(\Sigma)$ is the Euler characteristic; n is the number of boundary circles in Σ; $a = \int_\Sigma \mu$ is the total area of the surface.

We first prove the comparison (5.2) for the case of Σ a disk in Section 5.3.5, by presenting the disk as a gluing of building blocks (I), (III) and (IV) above. We prove the Theorem I for a general surface in Section 5.4.

The gluing property of the r.h.s. of (5.2) is

$$Z(\Sigma_1 \cup_{S^1} \Sigma_2) = \langle Z(\Sigma_1), Z(\Sigma_2)\rangle_{\mathcal{H}^{\mathrm{red}}_{S^1}}\ .$$

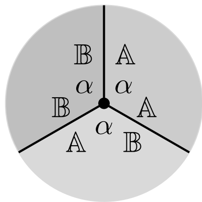
Here on the right side one has the pairing in the space of states for the circle over which the surfaces are being glued. In the BV-BFV framework it corresponds to gluing two $\mathbb{A}$-boundary circles via an "infinitesimally short" $\mathbb{B}-\mathbb{B}$ cylinder – our building block (II).

[3] More precisely, changing the area form μ by an exact 2-form $\mathrm{d}\gamma$ amounts in the BV language to a canonical transformation of the action and the associated change of the partition function by a BV-exact term. Therefore, working modulo BV exact terms, one can concentrate the area term in an arbitrarily small region. We thank Alberto S. Cattaneo for this remark.

[4] In particular, we exploit the gauge-invariance of the answer, known a priori from the quantum master equation, to reduce to the case of constant connections on the boundary.

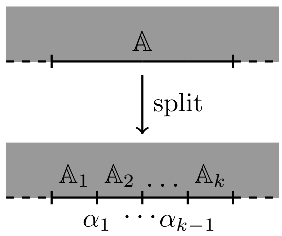
5.1.2 General surfaces, surfaces with corners

To extend the result (5.2) to general surfaces we have to consider gluing and cutting with corners. In this setting we continue to decorate the codimension 1 strata – circles and intervals – with a choice of polarization, $\mathbb{A}$ or $\mathbb{B}$, and we also decorate the codimension 2 corners with a choice of polarization, α or β (corresponding to fixing the value of either field A or field B in the corner).[5] For gluing, we require that if several domains are meeting at a corner, the respective corner polarizations are the same (unlike the situation with gluing over codimension 1 strata – those should have the opposite polarization coming from the two sides of the stratum):[6]



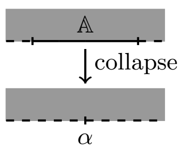
In this setup, one can perform the following moves on codimension 1 strata:

(a) One can *split* an $\mathbb{A}$-interval (or an $\mathbb{A}$-circle) on the boundary of a surface into $k \geq 2$ $\mathbb{A}$-intervals separated by α-corners. Then the partition function for a new surface is obtained by evaluating the partition function for the old surface evaluated on the concatenation of the fields $\mathbb{A}$ on the k intervals.



Similarly, one can split a $\mathbb{B}$-interval (or circle) into $k \geq 2$ $\mathbb{B}$-intervals separated by β-corners.

[5] We think of a corner carrying a polarization as the result of a *collapse* of an interval carrying same polarization, see the discussion of the picture I and picture II for corners in section 5.4.



[6] Actually the BV-BFV formalism does not prescribe, in principle, a particular compatibility between polarizations for the gluing. What we describe here is a choice that simplifies the computations.

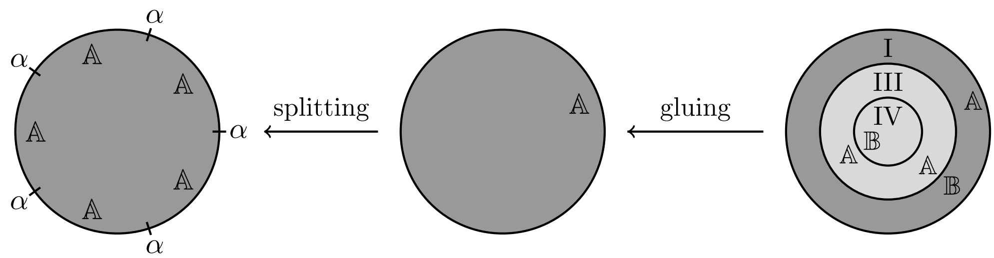
(b) One has the inverse of the move (a): one can *merge* k $\mathbb{A}$-intervals separated by α-corners into a single $\mathbb{A}$-interval – this corresponds to evaluating the partition function on the field $\mathbb{A}$ restricted to the smaller sub-intervals and to the points separating them. One can do the same for the $\mathbb{B}/\beta$ polarizations.

(c) One can *switch* between the polarizations of the corner separating an $\mathbb{A}$-interval and a $\mathbb{B}$-interval.

(d) One can *integrate out* (the field corresponding to) the β-corner separating two $\mathbb{A}$-intervals, merging them together. Likewise, one can integrate out an α-corner separating two $\mathbb{B}$-intervals.

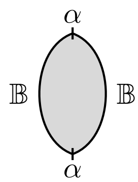
The minimal set of building blocks, sufficient to construct all closed surfaces is the following:

(i) A disk with boundary subdivided into k intervals, all in $\mathbb{A}$-polarization, and all corners taken in α-polarization, with a possibly nonzero 2-form μ.



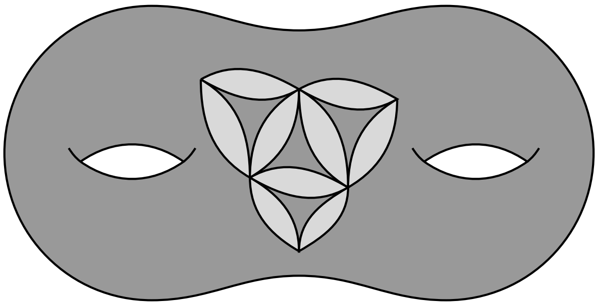
This building block is computed via the $\mathbb{A}$-disk which is expressed in terms of the building blocks (I,III,IV) above; then one applies the splitting move to the boundary. (Recall that in our convention the shaded regions are those that are allowed to carry a nonvanishing 2-form μ.)

(ii) A "bean" – a disk with boundary subdivided into two $\mathbb{B}$-intervals, with the two corners in α-polarization.



This block is computed from considering an axial gauge on a square and collapsing two opposite sides to two points.

Then one can e.g. triangulate any surface, assign the building block (i) with $k = 3$ to each triangle and glue them using building blocks (ii), thickening the edges of the triangulation into "beans".



This way one can construct the partition function for any surface with boundary and corners, as long as corners are all in α-polarization (boundary intervals and circles can be in any polarization). To produce all decorations of both boundary components and corners, one needs the following additional building block:

(iii) Disk in $\mathbb{A}$-polarization, with a single β-corner.

Then one can use the building block (iii), together with the moves on the boundary, to create any combination of polarizations of arcs and corners on the boundary of a surface.[7]

Using the building blocks (i), (ii), we immediately obtain the proof of Theorem I for a general surface Σ (see Sections 5.4.6, 5.4.7).

Remark 5.1.2. Theorem I in fact applies also to non-orientable surfaces, assuming the theory in the non-orientable case is defined as in footnote 2.

Remark 5.1.3. Here we are constructing a "pragmatic" extension of the BV-BFV framework to codimension 2 corners in the case of 2D Yang-Mills theory, motivated by the problem of computing explicit partition functions on surfaces (e.g., closed ones) of arbitrary genus. The general theory of quantization with corners in the BV-BFV formalism is work in progress and will be expanded on in a separate publication.

5.1.3 Main results

- Construction, in terms of explicitly computed building blocks and the gluing rule, of the partition function of 2D Yang-Mills in BV-BFV formalism on any oriented surface with boundary and corners, with any combination of polarizations $\in \{\mathbb{A}, \mathbb{B}\}$ assigned to the codimension 1 strata and polarizations $\in \{\alpha, \beta\}$ assigned to codimension 2 strata.

- Theorem I above, providing the comparison between the perturbative BV-BFV result in the case of a surface with $\mathbb{A}$-polarized boundary and the known non-perturbative result.

[7] One starts with a surface with the corners only in α polarization and creates the desired β-corners surrounded by two $\mathbb{A}$-arcs by gluing in the block (iii) – see Figure 5.12 in Section 5.4.1 and formula (5.107). One creates the β-corners surrounded by two $\mathbb{B}$-arcs by the splitting move and β-corners surrounded by an $\mathbb{A}$-arc and a $\mathbb{B}$-arc by the switch move.

- In Section 5.4.2 we prove that:
  - The BV-BFV partition function on a surface Σ with corners satisfies the modified quantum master equation $(\hbar^2\Delta + \Omega_{\partial\Sigma})Z = 0$, with Δ the BV Laplacian on bulk residual fields (in the minimal realization, they are modelled on de Rham cohomology of the surface), and with $\Omega_{\partial\Sigma}$ the boundary BFV operator. We construct the operator $\Omega_{\partial\Sigma}$ explicitly. In particular, apart from the edge contributions it contains quite nontrivial corner contributions, expressed in terms of the generating function for Bernoulli numbers.
  - We prove that Ω squares to zero, and thus the space of states $\mathcal{H}$ for a stratified boundary is a cochain complex.
  - We show that the space of states for a stratified circle can be disassembled into contributions of edges and corners, as the tensor product of certain differential graded (dg) bimodules – spaces of states assigned to the intervals (depending on the polarization of the interval and of its endponts) – over certain dg algebras – the spaces of states for the corners. In particular, an α-corner gets assigned the supercommutative dg algebra $\wedge^\bullet\mathfrak{g}^*$ with Chevalley-Eilenberg differential. A β-corner gets assigned the algebra $S^\bullet\mathfrak{g}$ endowed with zero differential and a non-commutative star-product, written in terms of Baker-Campbell-Hausdorff formula.

    This picture in particular establishes a link with Baez-Dolan-Lurie setup [4, 39] of extended topological quantum field theory where (in one of the models) one maps strata of the spacetime manifold of codimension $2, 1, 0$, respectively, to algebras, bimodules and bimodule morphisms.[8]

5.2 Background: BV-BFV formalism

We will start this section reviewing the basic constructions of the BV-BFV formalism and fixing the notation. We will then apply this construction to obtain the BV-BFV formulation of the non-abelian BF theory and Yang-Mills theory reviewing some of the known results. For a complete and detailed discussion of this topic we refer to [22–24, 26], where this formalism was first introduced.

5.2.1 Classical BV-BFV

Definition 5.2.1. A BFV manifold is given by the triple $(\mathfrak{F}_\partial, \alpha^\partial, \mathcal{Q}^\partial)$, where: the space of *boundary fields* $\mathfrak{F}_\partial$ is an exact graded symplectic manifold with 0-symplectic form $\omega^\partial =$

[8] A version of extension of Atiyah's axioms accommodating the non-perturbative answers for 2D Yang-Mills with corners was previously suggested in [43]. It can be obtained from our picture by fixing polarizations on all strata to $\mathbb{A}, \alpha$ and passing to the zeroth cohomology of the BFV differential Ω.

$\mathrm{d}\alpha^\partial$ and $\mathcal{Q}^\partial$ is a homological symplectic vector field of degree 1.[9]

In particular the condition $L_{\mathcal{Q}^\partial}\mathrm{d}\alpha^\partial = 0$ for the vector field $\mathcal{Q}^\partial$, since $|\mathcal{Q}^\partial| + |\omega^\partial| \neq 0$, implies that it is also hamiltonian: $\imath_{\mathcal{Q}^\partial}\omega^\partial = \mathrm{d}\mathcal{S}^\partial$. This defines the degree 1 hamiltonian $\mathcal{S}^\partial$, which we will call the *boundary BFV action*.

Definition 5.2.2. A BV-BFV manifold, over a BFV manifold $(\mathfrak{F}_\partial, \alpha^\partial, \mathcal{Q}^\partial)$, is a quintuple $(\mathfrak{F}, \omega, \mathcal{S}, \mathcal{Q}, \pi)$, where the space of *bulk fields* $(\mathfrak{F}, \omega)$ is a (-1)-symplectic manifold, the *bulk action* $\mathcal{S}$ is a function of the fields, the bulk *BRST operator* $\mathcal{Q}$ is a homological vector field of degree 1 and $\pi\colon \mathfrak{F} \to \mathfrak{F}_\partial$ is a surjective submersion, satisfying the following two compatibility conditions:

i) the bulk homological vector field projects on the boundary homological vector field: $\mathrm{d}\pi\, \mathcal{Q} = \mathcal{Q}^\partial$;

ii) the *modified Classical Master Equation* (mCME) holds: $\imath_{\mathcal{Q}}\omega = \mathrm{d}\mathcal{S} + \pi^*\alpha^\partial$.

A *classical BV-BFV theory* is constructed for manifolds with boundaries of some fixed dimension n. It consists of the association to each manifold with boundary Σ of a BV-BFV manifold $\mathfrak{F}_\Sigma$ over the BFV manifold $\mathfrak{F}_{\partial\Sigma}$ associated to the boundary $\partial\Sigma$. This association has to be compatible with disjoint union and "gluing" in the following sense:

i) a disjoint union maps to the direct product: $\mathfrak{F}_{\Sigma_1 \sqcup \Sigma_2} = \mathfrak{F}_{\Sigma_1} \times \mathfrak{F}_{\Sigma_2}$;

ii) a gluing of two manifolds maps to the fiber product over the space of fields associated to the gluing interface γ: $\mathfrak{F}_{\Sigma_1 \cup_\gamma \Sigma_2} = \mathfrak{F}_{\Sigma_1} \times_{\mathfrak{F}_\gamma} \mathfrak{F}_{\Sigma_2}$.

Remark 5.2.3. This can be interpreted as a covariant monoidal functor from the *spacetime category*, with $(n-1)$ closed manifolds as objects and n-manifolds with boundary as morphisms with composition given by gluing,[10] to the *BFV category*, where objects are BFV manifolds, morphisms are BV-BFV manifolds over (products of in- and out-) BFV manifolds, and composition is given by fiber products. The monoidal structure on the spacetime category is given by the disjoint union, while on the BFV category side it is given by the direct product.

Remark 5.2.4. On closed manifolds this construction reduces to a classical BV theory, which gives a homological resolution of the space of classical states for lagrangian gauge field theories and is the classical starting point for the BV quantization of such theories [7, 32].

[9] For simplicity we will consider, here and in the following, all the gradings to be $\mathbb{Z}$ gradings. The parity, determining the commuting/anticommuting properties of coordinates, is given by the degree mod 2.

[10] Depending on the specific theory, the spacetime category could have additional structures: for examples manifolds could be oriented, Riemannian, etc. Also, depending on the specific theory, there may be subtleties to defining the spacetime category as an actual category. This discussion is beyond the scope of (and is not relevant to) this thesis.

5.2.2 Quantum BV-BFV

A *quantum BV-BFV theory* associates to an $(n-1)$ manifold γ a graded cochain complex $\mathcal{H}_\gamma$, the *space of states*, with a coboundary operator Ω_γ called the *quantum BFV charge*. To n-manifolds with boundary Σ the quantum theory assigns a (finite-dimensional) (-1)-symplectic manifold $(\mathcal{V}_\Sigma, \omega_{\mathcal{V}_\Sigma})$, the space of *residual fields*, and the *partition function*, which is an element $Z_\Sigma \in \mathcal{H}_{\partial\Sigma} \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$ in the boundary space of states tensored with the half-densities on the residual fields.[11] The partition function has to satisfy the *modified Quantum Master Equation* (mQME):

$$(\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma}) Z_\Sigma = 0 \,, \tag{5.3}$$

where $\Delta_{\mathcal{V}_\Sigma}$ is the canonical BV Laplacian on the half-densities of the residual fields. The partition function is understood to be defined modulo $(\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma})$-exact terms. Also, the quantum theory satisfies compatibility conditions with respect to the disjoint union and the gluing of spacetime manifolds:

i) To disjoint unions, reflecting the quantum nature of the theory, the BV-BFV theory associates the tensor product of the spaces of states, $\mathcal{H}_{\gamma_1 \sqcup \gamma_2} = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$, the direct product of residual fields $\mathcal{V}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2}$ and the tensor product of partition functions, $Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$.

ii) To the gluing of two manifolds the theory associates the partition function obtained as the pairing, in the space of states of the gluing interface, of the partition functions of the constituent manifolds: $Z_{\Sigma_1 \cup_\gamma \Sigma_2} = \langle Z_{\Sigma_1}, Z_{\Sigma_2} \rangle_\gamma$.[12]

Quantum observables are defined to be cohomology classes of the coboundary operator $Z_\Sigma^{-1}(\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma})(Z_\Sigma \cdot \ldots)$, with expectation value computed by a *BV pushforward* of a representative $\mathcal{O}$ times the partition function, i.e. integrating their product over a lagrangian $\mathcal{L} \subset \mathcal{V}$:

$$\langle \mathcal{O} \rangle_\Sigma := \int_{\mathcal{L}} \mathcal{O}\, Z_\Sigma \,. \tag{5.4}$$

The lagrangian submanifold $\mathcal{L}$ has here the meaning of *gauge-fixing* for the integration over residual fields and the closedness of $\mathcal{O} Z_\Sigma$ with respect to $\Omega_{\partial\Sigma} + \hbar^2 \Delta_{\mathcal{V}_\Sigma}$ ensures that the $\Omega_{\partial\Sigma}$-cohomology class resulting from the integration does not depend on the particular choice of gauge fixing thanks to the following theorem [26, 42].

Theorem 5.2.5. Let $(\mathcal{M}_1, \omega_2)$ and $(\mathcal{M}_2, \omega_2)$ be two graded manifolds with odd symplectic forms ω_i and canonical Laplacians Δ_i. Consider $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with product symplectic

[11] The space of residual fields is not uniquely determined, but comes in a poset of different realizations. The partition function for a smaller realization can be reached with a BV-pushforward (see Subsection 5.2.3 for further discussion).

[12] For oriented spacetime manifolds, this is the dual pairing between $\mathcal{H}_\gamma$ and $\mathcal{H}_\gamma^*$: since the two boundaries that are glued together must have opposite orientations, the associated vector spaces are dual to each other.

form and canonical Laplacian Δ and let $\mathcal{L}, \mathcal{L}' \subset \mathcal{M}_2$ be any two lagrangian submanifolds which can be deformed into each other. For any half-density $f \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$ we have:

i) $\int_{\mathcal{L}} \Delta f = \Delta_1 \int_{\mathcal{L}} f$

ii) $\int_{\mathcal{L}} f - \int_{\mathcal{L}'} f = \Delta_1 \xi \qquad$ for some $\xi \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}_1)$, if $\Delta f = 0$.

In particular, when $\mathcal{M}_1$ is just a point, the r.h.s. of the two equations above vanishes.

5.2.3 Quantization

The quantization procedure is a way to get a quantum BV-BFV theory from the data of a classical BV-BFV theory. The first object to construct is the space of states $\mathcal{H}_\gamma$, which is obtained from the space of boundary fields $\mathfrak{F}_\gamma$ by choosing a lagrangian foliation, or more generally a *polarization* $\mathcal{P}$. We will assume in the following that the 1-form α_γ vanishes along the fibers of $\mathcal{P}$; if this is not the case, we can use a *gauge transformation*:

$$\alpha_\gamma \mapsto \alpha_\gamma - \mathrm{d}f_\gamma \ , \qquad \mathcal{S}_\Sigma \mapsto \mathcal{S}_\Sigma + \pi^* f_\gamma \ , \tag{5.5}$$

which uses an arbitrary function f_γ of the boundary fields to shift α_γ and the bulk action in such a way that the mCME is preserved (cf. def. 5.2.2). The space of states of the quantum theory is defined as the space of complex-valued functions[13] the leaf space $\mathcal{B}_\gamma^{\mathcal{P}} = \mathfrak{F}_\gamma/\mathcal{P}$ (or more generally the space of polarized sections of the trivial "prequantum" $U(1)$-bundle over $\mathfrak{F}_\gamma$).

$$\mathcal{H}_\gamma := \mathrm{Fun}_{\mathbb{C}}(\mathcal{B}_\gamma^{\mathcal{P}}) \ . \tag{5.6}$$

In other words, the space of quantum states is obtained as the *geometric quantization* of the space of boundary fields [26].

The space of quantum states forms a cochain complex. The coboundary operator Ω_γ is constructed as the quantization of the boundary action $\mathcal{S}_\gamma$. Suppose we have Darboux coordinates (q,p) on $\mathfrak{F}_\gamma$, where q are also coordinates of $\mathcal{B}_\gamma^{\mathcal{P}}$. The operator Ω_γ is the standard-ordering quantization of the action:

$$\Omega_\gamma := \mathcal{S}_\gamma\left(q, -\mathrm{i}\hbar\frac{\partial}{\partial q}\right) \ , \tag{5.7}$$

where all the derivatives are positioned to the right. For the theories we will consider in this chapter, with this definition Ω_γ squares to zero; in general it could be needed to add quantum corrections to (5.7) for Ω_γ to actually be a coboundary operator.[14]

Let us consider now the data associated to the bulk n-manifolds. The space of bulk fields has a fibration over $\mathcal{B}_{\partial\Sigma}^{\mathcal{P}}$ defined by composing the projection to the boundary fields

[13] Another possible model for states uses half-densities instead of functions. These two models are isomorphic, with the isomorphism given by multiplication by a fixed reference half-density.

[14] In general there might be cohomological obstructions to do that. Moreover, the partition function might be not compatible with the so constructed Ω, causing the mQME to fail.

with the projection given by the polarization: $\mathfrak{F}_\Sigma \xrightarrow{\pi} \mathfrak{F}_{\partial\Sigma} \longrightarrow \mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$. Suppose for simplicity that this is a trivial bundle: $\mathfrak{F}_\Sigma = \widetilde{\mathcal{B}}^{\mathcal{P}}_{\partial\Sigma} \times \mathcal{Y}$, where $\widetilde{\mathcal{B}}^{\mathcal{P}}_{\partial\Sigma}$ is some bulk extension of $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$ and $\mathcal{Y}$ is some (-1)-symplectic manifold. This assumption will hold in all the theories considered in the following.

The space of residual fields can be taken to be any (finite-dimensional) symplectic subspace $\mathcal{V}$ of the space of fields,[15] separating it as $\mathcal{Y} = \mathcal{V} \times \mathcal{Y}'$, where $\mathcal{Y}'$ is the space of *fluctuations*. The partition function is now defined as a BV pushforward of the exponentiated bulk action:

$$Z_\Sigma(\mathcal{P};\mathcal{V}) := \int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_\Sigma} , \tag{5.8}$$

where $\mathcal{L} \subset \mathcal{Y}'$ is a lagrangian submanifold. If $\Delta_{\mathcal{Y}}\mathcal{S}_\Sigma = 0$, theorem 5.2.5 implies that the partition function is a solution of the mQME (5.3).[16] Moreover, we have that Z_Σ does not depend on (deformations of) the gauge-fixing lagrangian $\mathcal{L}$ used in the BV pushforward, up to $(\Omega + \hbar^2\Delta)$-exact terms.

The discussion, until now, assumes a finite-dimensional situation. This is usually not the setting of quantum field theories; for infinite-dimensional spaces one needs a more delicate analysis to prove the mQME and to prove that the dependence of the partition on the gauge-fixing is BV-exact. A way to make sense of infinite-dimensional integrals is through perturbation theory, as discussed in the following section.

Perturbative expansion

The space of fields $\mathfrak{F}_\Sigma$ is typically infinite-dimensional, for example it can contain the de Rham complex of differential forms over Σ. As a consequence, the integral (5.8) defining the partition function is (almost always) ill-defined as a measure-theoretic integral and has to be understood as a *perturbative series* written in terms of the Feynman diagrams coming from the interactions in the bulk action expanded around a point $\mathsf{x}_0 \in \mathfrak{M}$ in the *Euler-Lagrange moduli space* – the space of solutions of classical equations of motion of $\mathcal{S}_\Sigma$ (modulo gauge symmetries).

In order for the perturbative expansion to be well-defined, the *gauge-fixed action* – the restriction of $\mathcal{S}_\Sigma$ to the gauge-fixing lagrangian $\mathcal{L}$ – needs to have isolated critical points. It is important to remark that this condition does not, in general, hold for every lagrangian.

The existence of such a "good gauge-fixing" depends on the choice of residual fields. In particular the quadratic part of the bulk action can have *zero-modes* $\mathcal{V}^0_\Sigma$, i.e. bulk fields

[15] In the framework of perturbation theory, the requirement that the integral (5.8) below is perturbatively well-defined, imposes restrictions on the possible choices of $\mathcal{V}$. E.g. $\mathcal{V}$ for perturbed BF theories has to be modelled on a deformation retract of the de Rham complex of the bulk manifold.

[16] In the theories considered in the following this condition is verified. However, this is not always the case and there can be theories where the bulk action needs quantum corrections in order for the mQME to hold. This is connected in particular to the presence of quantum anomalies in the theory.

configurations that are annihilated by the kinetic operator.[17] Zero-modes correspond to the tangent directions to the Euler-Lagrange moduli space (cf. [26], appendix F) and therefore their presence in the space of fluctuations indicates non-isolated critical points of the action and obstructs the perturbative expansion. Thus, the space of residual fields has to at least contain the space of zero-modes for a good gauge-fixing lagrangian to exist: $\mathcal{V}^0_\Sigma \subseteq \mathcal{V}_\Sigma$. When the residual fields coincide with zero modes we say that the perturbative partition function is in its *minimal realization.*

Another consequence of the infinite dimensions of $\mathfrak{F}_\Sigma$ is that also the Laplacian is ill-defined. The equations containing it are thus only formal (or require a regularization).[18] In particular, theorem 5.2.5 is proved in a finite-dimensional setting. An important point is thus that even if the action is formally annihilated by the Laplacian, the mQME is only expected to hold and needs to be verified for each particular theory. For perturbed BF theories, including 2D Yang-Mills, the mQME has been proved in the infinite-dimensional perturbative setting in [26] and relies on the Stokes' theorem for integrals over compactified configuration spaces of points.

Renormalization and globalization

A non-minimal realization of a theory is obtained when the zero-modes are a proper subset of the space of residual fields. Of course there are different, inequivalent, non-minimal realizations of any theory. Given a non-minimal realization, one can obtain a smaller one by a BV pushforward. If $\mathcal{V}'_\Sigma = \mathcal{V}''_\Sigma \times \mathcal{Y}'$, with $\mathcal{V}^0_\Sigma \subset \mathcal{V}''_\Sigma$, then:

$$Z_\Sigma(\mathcal{P};\mathcal{V}'') = \int_{\mathcal{L}} Z_\Sigma(\mathcal{P};\mathcal{V}') \tag{5.9}$$

for a lagrangian submanifold $\mathcal{L} \subset \mathcal{Y}'$. The set of all possible realizations forms therefore a partially ordered set, with the final object given by the minimal realization. Passing from bigger to smaller realizations can be interpreted as following the *renormalization group flow.*

Remark 5.2.6. According to the gluing prescription (cf. section 5.2.2), the residual fields of the glued manifold are the direct product of the residual fields of the two manifolds being glued. In particular this means that, generally, if we glue together two partition functions in the minimal realization the result of the gluing will not be in the minimal realization. Let $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$; it typically happens that $\mathcal{V}^0_\Sigma \subset \mathcal{V}^0_{\Sigma_1} \times \mathcal{V}^0_{\Sigma_2}$. The minimal realization for the glued manifold has then to be obtained via a BV pushforward:

$$Z_\Sigma(\mathcal{V}^0_\Sigma) = \int_{\mathcal{L}} \langle Z_{\Sigma_1}(\mathcal{V}^0_{\Sigma_1}), Z_{\Sigma_2}(\mathcal{V}^0_{\Sigma_2})\rangle \ . \tag{5.10}$$

[17] See section 5.2.4 for the definition of zero modes in 2D YM.

[18] The BV Laplacian becomes non-singular within the framework of renormalization theory on the level of residual fields. See also [28] for the discussion of how the RG flow regularizes the BV Laplacian.

Because of its perturbative definition, the partition function depends on the point $\mathsf{x}_0 \in \mathfrak{M}$ around which we are expanding and carries only local information on field configurations infinitesimally close to x_0 (it is defined on a formal neighbourhood of x_0). Nevertheless, at least in the theories considered in this chapter, its minimal realization is the Taylor expansion of a *global* half-density on the tangent bundle of the Euler-Lagrange moduli space (cf. [26], appendix F). Thus, under some assumptions, it can be integrated on the zero section of $T\mathfrak{M}$. This corresponds to setting to zero all the zero-modes $\nu \in \mathcal{V}^0_\Sigma$ and integrating the partition function on the Euler-Lagrange moduli space

$$Z_\Sigma(\mathcal{P}) = \int_{\mathfrak{M}} Z_\Sigma(\mathcal{P}, \mathcal{V}^0_\Sigma; \mathsf{x}_0)\big|_{\nu=0} \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}) \; , \tag{5.11}$$

obtaining a *globalized partition function* depending only on the boundary fields in $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$.

Another way to obtain a partition function which does not depend on $\mathcal{V}^0_\Sigma$ is to integrate its minimal realization over all the zero-modes, again using a BV pushforward. Notice that this cannot be done in a perturbative way – the propagator cannot be defined for zero modes – but since $\mathcal{V}^0_\Sigma$ is a finite-dimensional space, the BV pushforward is well-defined as an ordinary integral on a supermanifold:

$$Z_\Sigma(\mathcal{P}) = \int_{\mathcal{L}\subset\mathcal{V}^0_\Sigma} Z_\Sigma(\mathcal{P}, \mathcal{V}^0_\Sigma) \; . \tag{5.12}$$

This can be viewed as an alternative definition of a globalized partition function and in fact, when both this integral and the one in (5.11) can be computed explicitly, they coincide (cf. section 5.4.7).[19] However, the precise relation between the two globalization procedures is to be understood better.

5.2.4 BV-BFV formulation of 2D YM

We will review in this section the BV-BFV construction for 2D Yang-Mills and non-abelian BF theories; for a deeper discussion and for some of the proofs we refer to [22, 26].

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let A be a connection 1-form on a principal G-bundle over a 2-dimensional surface Σ. In the first order formalism the classical YM action can be written in the following form:

$$S_\Sigma(A, B) = \int_\Sigma \langle B, F_A \rangle + \frac{1}{2}\int_\Sigma (B, B)\, \mu \; , \tag{5.13}$$

where the auxiliary field B is a zero-form valued in $\mathfrak{g}^*$, F_A is the curvature 2-form of A, $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, $(\cdot, \cdot)$ is an invariant non-degenerate pairing on $\mathfrak{g}^*$ and μ is the volume 2-form associated to a metric on Σ. We see that 2D YM can be treated as a perturbation of 2D non-abelian BF theory, which can be obtained in the

[19] One caveat is that one needs to take care to avoid possible overcounting when integrating over zero-modes, cf. Remark 5.3.12 below.

zero-area limit $\mu \to 0$. In the following sections we will generally find it useful to consider BF theory first, introducing the area term only afterwards.

On closed surfaces, the classical BV construction enhances the space of fields by adding differential forms of every degree, usually called ghosts and antifields for positive or negative internal degree respectively. The BV space of fields over Σ is then

$$\mathfrak{F}_\Sigma = \Omega(\Sigma;\mathfrak{g})[1] \oplus \Omega(\Sigma;\mathfrak{g}^*) \ni (\mathsf{A},\mathsf{B}) \ , \tag{5.14}$$

where A and B are the superfields associated to $A = \mathsf{A}_{(1)}$ and $B = \mathsf{B}_{(0)}$ which are their degree-zero components.[20] The BV space of fields is a symplectic graded space, with (-1)-symplectic form given by:

$$\omega_\Sigma = \int_\Sigma \langle \delta\mathsf{B}, \delta\mathsf{A}\rangle \ . \tag{5.15}$$

The BV action on a closed manifold is

$$\mathcal{S}_\Sigma = \int_\Sigma \langle \mathsf{B}, \mathrm{d}\mathsf{A} + \frac{1}{2}[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\int_\Sigma (\mathsf{B},\mathsf{B})\,\mu \tag{5.16}$$

and the corresponding hamiltonian vector field, the homological vector field $\mathcal{Q}_\Sigma$, is:

$$\mathcal{Q}_\Sigma = \int_\Sigma \Big\langle \mathrm{d}\mathsf{A} + \frac{1}{2}[\mathsf{A},\mathsf{A}], \frac{\delta}{\delta\mathsf{A}}\Big\rangle + \int_\Sigma \Big\langle \mathrm{d}\mathsf{B} + \mathrm{ad}^*_{\mathsf{A}}\mathsf{B}, \frac{\delta}{\delta\mathsf{B}}\Big\rangle + \int_\Sigma \Big(\mathsf{B}, \frac{\delta}{\delta\mathsf{B}}\Big)\mu \ , \tag{5.17}$$

In the BV-BFV construction the bulk fields, symplectic structure, action and homological vector field are again the ones described above. Notice that now, when Σ has a non-empty boundary $\partial\Sigma$, the homological vector field (5.17) is not the hamiltonian vector field of the action (5.16). Indeed, it is not even symplectic:

$$\imath_{\mathcal{Q}_\Sigma}\omega_\Sigma = \delta\mathcal{S}_\Sigma + \int_{\partial\Sigma} \langle \mathsf{B}, \delta\mathsf{A}\rangle \ . \tag{5.18}$$

The boundary fields $\mathfrak{F}_{\partial\Sigma}$ are defined analogously to the bulk:

$$\mathfrak{F}_{\partial\Sigma} = \Omega(\partial\Sigma;\mathfrak{g})[1] \oplus \Omega(\partial\Sigma;\mathfrak{g}^*) \ni (\mathbb{A},\mathbb{B}) \ . \tag{5.19}$$

We can thus define the projection $\pi\colon \mathfrak{F}_\Sigma \longrightarrow \mathfrak{F}_{\partial\Sigma}$ to be just the restriction (pullback) to $\partial\Sigma$ of the bulk fields. This, taking into account the compatibility conditions of def. 5.2.2, fixes the remaining boundary data. From (5.18) we get

$$\alpha_{\partial\Sigma} = \int_{\partial\Sigma} \langle \mathbb{B}, \delta\mathbb{A}\rangle \ , \qquad \omega_{\partial\Sigma} = \delta\alpha_{\partial\Sigma} = -\int_{\partial\Sigma}\langle \delta\mathbb{B}, \delta\mathbb{A}\rangle \ , \tag{5.20}$$

the boundary homological vector field is the projection of the bulk homological vector field

$$\mathcal{Q}_{\partial\Sigma} = \mathrm{d}\pi\,\mathcal{Q}_\Sigma = \int_{\partial\Sigma} \Big(\Big\langle \mathrm{d}\mathbb{A} + \frac{1}{2}[\mathbb{A},\mathbb{A}], \frac{\delta}{\delta\mathbb{A}}\Big\rangle + \Big\langle \mathrm{d}\mathbb{B} + \mathrm{ad}^*_{\mathbb{A}}\mathbb{B}, \frac{\delta}{\delta\mathbb{B}}\Big\rangle\Big) \tag{5.21}$$

[20] We will denote by $\mathsf{A}_{(n)}$ the n-form component of a superfield A.

and thus the boundary action is obtained as the hamiltonian of $\mathcal{Q}_{\partial\Sigma}$:

$$\mathcal{S}_{\partial\Sigma} = \int_{\partial\Sigma} \langle \mathbb{B}, \mathrm{d}\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] \rangle \ . \tag{5.22}$$

Notice that, for degree reasons, the area form μ does not appear in the boundary data. The boundary BFV manifold for 2D YM is thus exactly the same as in BF theory; actually, the only difference between the two theories is the area term in the bulk action and, consequently, the state (partition function) defined by the two quantum theories.

To quantize the theory, we need to choose a polarization of the space of boundary fields. From (5.20) we see that there are two simple choices of polarization: the $\mathbb{A}$-polarization $\mathcal{P}_{\mathbb{A}}$,[21] for which the leaf space is decribed by the $\mathbb{A}$ fields

$$\mathcal{B}^{\mathcal{P}_{\mathbb{A}}}_{\partial\Sigma} = \Omega(\partial\Sigma; \mathfrak{g})[1] \tag{5.23}$$

and the $\mathbb{B}$-polarization $\mathcal{P}_{\mathbb{B}}$, for which the leaf space is described by the $\mathbb{B}$ fields

$$\mathcal{B}^{\mathcal{P}_{\mathbb{B}}}_{\partial\Sigma} = \Omega(\partial\Sigma; \mathfrak{g}^*) \ . \tag{5.24}$$

We will, in the rest of this chapter, always use these two transversal polarizations, arbitrarily splitting the boundary of a manifold into the disjoint[22] union of two components, $\partial\Sigma := \partial_{\mathbb{B}}\Sigma \sqcup \partial_{\mathbb{A}}\Sigma$, and choosing the product polarization which assigns the $\mathbb{B}$-polarization to the first and the $\mathbb{A}$-polarization to the latter boundary component:

$$\mathcal{B}^{\mathcal{P}}_{\partial\Sigma} = \Omega(\partial_{\mathbb{A}}\Sigma; \mathfrak{g})[1] \oplus \Omega(\partial_{\mathbb{B}}\Sigma; \mathfrak{g}^*) \ . \tag{5.25}$$

The boundary one-form $\alpha_{\partial\Sigma}$ does not vanish on the fibers of this polarization (cf. (5.20)) but it can be adapted to this choice using a gauge transformation (5.5):

$$\begin{aligned} \alpha^{\mathcal{P}}_{\partial\Sigma} &= \alpha_{\partial\Sigma} + \delta \int_{\partial_{\mathbb{B}}\Sigma} \langle \mathbb{B}, \mathbb{A} \rangle = \int_{\partial_{\mathbb{A}}\Sigma} \langle \mathbb{B}, \delta\mathbb{A} \rangle - \int_{\partial_{\mathbb{B}}\Sigma} \langle \delta\mathbb{B}, \mathbb{A} \rangle \ , \\ \mathcal{S}^{\mathcal{P}}_{\Sigma} &= \mathcal{S}_{\Sigma} - \int_{\partial_{\mathbb{B}}\Sigma} \langle \mathsf{B}, \mathsf{A} \rangle \ . \end{aligned} \tag{5.26}$$

We can now quantize, with the above polarization, the boundary action to obtain the coboundary operator $\Omega^{\mathcal{P}}_{\partial\Sigma}$:

$$\Omega^{\mathcal{P}}_{\partial\Sigma} = \int_{\partial_{\mathbb{A}}\Sigma} \mathrm{i}\hbar \left(\mathrm{d}\mathbb{A}^a + \frac{1}{2} f^a_{bc} \mathbb{A}^b \mathbb{A}^c \right) \frac{\delta}{\delta \mathbb{A}^a} + \int_{\partial_{\mathbb{B}}\Sigma} \left(\mathrm{i}\hbar \mathrm{d}\mathbb{B}_a \frac{\delta}{\delta \mathbb{B}_a} - \frac{\hbar^2}{2} f^a_{bc} \mathbb{B}_a \frac{\delta}{\delta \mathbb{B}_b} \frac{\delta}{\delta \mathbb{B}_c} \right) \ . \tag{5.27}$$

Here f^a_{bc} are the structure constants of the Lie algebra $\mathfrak{g}$.

[21] In the terminology of [26], this is "$\mathbb{A}$-representation", or "$\frac{\delta}{\delta\mathbb{B}}$-polarization" (as those are the vector fields spanning the tangential lagrangian distribution on the phase space).

[22] However, when we start considering corners in section 5.4, the disjointness assumption will fail along codimension 2 strata.

To write the partition function we lift $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$ to $\mathfrak{F}_\Sigma = \widetilde{\mathcal{B}}^{\mathcal{P}}_{\partial\Sigma} \times \mathcal{Y}_\Sigma$ by taking (discontinuous) bulk extensions $(\widetilde{\mathbb{A}}, \widetilde{\mathbb{B}})$ of the boundary fields (cf. the discussion in [26], Section 3.4). We can now split the bulk fields $\mathcal{Y}_\Sigma$ into residual fields $(\mathsf{a}, \mathsf{b}) \in \mathcal{V}_\Sigma$ and fluctuations $(\alpha, \beta) \in \mathcal{Y}'_\Sigma$:

$$\mathsf{A} = \widetilde{\mathbb{A}} + \mathsf{a} + \alpha \ , \qquad \mathsf{B} = \widetilde{\mathbb{B}} + \mathsf{b} + \beta \ . \tag{5.28}$$

We can finally define the partition function as the (perturbative) path-integral:

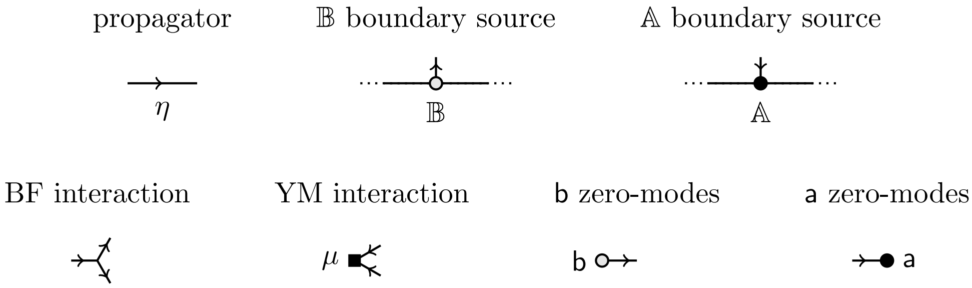
$$Z_\Sigma[\mathbb{A}, \mathbb{B}; \mathsf{a}, \mathsf{b}] = \int_{\mathcal{L} \subset \mathcal{Y}'_\Sigma} \mathfrak{D}[\alpha, \beta] \ \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \mathcal{S}^{\mathcal{P}}_\Sigma (\widetilde{\mathbb{A}} + \mathsf{a} + \alpha, \widetilde{\mathbb{B}} + \mathsf{b} + \beta)} \ . \tag{5.29}$$

Remark 5.2.7. To avoid the appearance of ill-defined derivatives of the discontinuous fields $(\widetilde{\mathbb{A}}, \widetilde{\mathbb{B}})$ in the bulk action $\mathcal{S}^{\mathcal{P}}_\Sigma$, we integrate by parts rewriting it as:

$$\begin{aligned} \mathcal{S}^{\mathcal{P}}_\Sigma (\widetilde{\mathbb{A}} + \mathsf{a} + \alpha, \widetilde{\mathbb{B}} + \mathsf{b} + \beta) = \mathcal{S}^{\mathcal{P}}_\Sigma (\mathsf{a} + \alpha, \mathsf{b} + \beta) + \frac{1}{2} \int_\Sigma (\mathsf{b} + \beta, \mathsf{b} + \beta)\, \mu \\ + \int_{\partial_{\mathbb{A}} \Sigma} \langle \mathsf{b} + \beta, \widetilde{\mathbb{A}} \rangle - \int_{\partial_{\mathbb{B}} \Sigma} \langle \widetilde{\mathbb{B}}, \mathsf{a} + \alpha \rangle \ . \end{aligned} \tag{5.30}$$

The boundary fields thus act as currents in the perturbative expansion of the partition function.

We are now in the position of writing down the diagrammatic elements of the Feynman diagrams expansion of the theory:[23]



(5.31)

With these vertices we can compose a large set of non-trivial Feynman diagrams (e.g. figure 5.2). The general strategy will be to cut the surface, and hence Feynman diagrams, in such a way that there is a simple choice of propagators on each component which allows us to compute the partition function for that surface. Then, using the gluing properties of BV-BFV theories, we can glue back all the pieces to recover the partition function on the original surface we started with. This procedure can be viewed as a method to construct a complicated propagator on the starting surface which, though, allows explicit computations.

[23]Zero-modes are here understood as "loose" half-hedges.

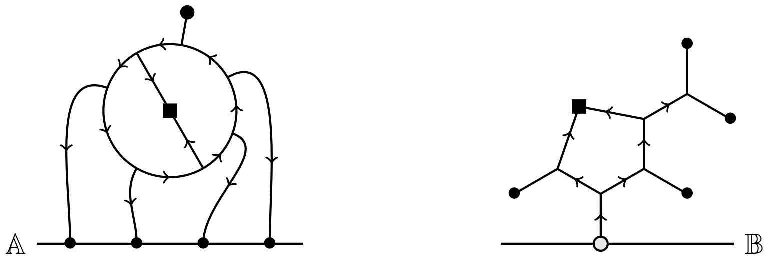


Figure 5.2: Two examples of the many possible Feynman diagrams for 2D YM on a surface with boundary.

Ω-cohomology in $\mathbb{A}$-polarization on a circle

In [26] it was proven that the partition function for 2D YM (and other perturbations of abelian BF) solves the mQME. In the following sections, to simplify some computations, we will exploit this fact by choosing a suitable representative for the cohomology class of the partition function. In particular it will be useful to know the cohomology of Ω, in ghost degree zero, on the space of boundary fields in $\mathbb{A}$ polarization. From equation (5.27) we see that Ω acts as a gauge transformation, thus Ω-closed functionals of ghost degree zero are just gauge-invariant functionals of the connection, which are isomorphic to class functions on the simply connected Lie group G integrating the Lie algebra $\mathfrak{g}$. An alternative approach is to split Ω into the "abelian" part, i.e. the de Rham differential d, plus a perturbation δ containing the structure constants of the Lie algebra:

$$\Omega = \underbrace{\int_{S^1} \mathrm{d}\mathbb{A}^a_{(0)} \frac{\delta}{\delta \mathbb{A}^a_{(1)}}}_{\mathrm{d}} + \underbrace{\int_{S^1} \left(\frac{1}{2} f^a_{bc} \mathbb{A}^b_{(0)} \mathbb{A}^c_{(0)} \frac{\delta}{\delta \mathbb{A}^a_{(0)}} + f^a_{bc} \mathbb{A}^b_{(0)} \mathbb{A}^c_{(1)} \frac{\delta}{\delta \mathbb{A}^a_{(1)}} \right)}_{\delta} . \tag{5.32}$$

We can then compute the cohomology via the homological perturbation lemma [31]. The cohomology of d is given by functions on the de Rham cohomology $H^\bullet_{\mathrm{dR}}(S^1;\mathfrak{g})[1]$; choosing the coordinate t on the circle, these can be represented as functions of the "constant fields" $\underline{\mathbb{A}}_{(0)}$ and $\underline{\mathbb{A}}_{(1)}\mathrm{d}t$, where $\underline{\mathbb{A}}_{(0)} \in \mathfrak{g}[1]$ and $\underline{\mathbb{A}}_{(1)} \in \mathfrak{g}$. Now if we compute the cohomology of the induced differential

$$\underline{\delta} = \frac{1}{2} f^a_{bc} \underline{\mathbb{A}}^b_{(0)} \underline{\mathbb{A}}^c_{(0)} \frac{\delta}{\delta \underline{\mathbb{A}}^a_{(0)}} + f^a_{bc} \underline{\mathbb{A}}^b_{(0)} \underline{\mathbb{A}}^c_{(1)} \frac{\delta}{\delta \underline{\mathbb{A}}^a_{(1)}} \tag{5.33}$$

on $H^\bullet_{\mathrm{dR}}(S^1;\mathfrak{g})[1]$, we get that in ghost degree zero it is given by G-invariant functions on the Lie algebra $\mathfrak{g}$. Comparing with the previous answer, we see that the correct Ω-cohomology corresponds to the subspace of G-invariant functions on $\mathfrak{g}$ coming as the pullback by the exponential map $\exp\colon \mathfrak{g} \to G$ of class functions on G. Such functions on $\mathfrak{g}$ are determined by their values on the fundamental domain B_0 of the exponential map (e.g. for $G = \mathrm{SU}(2)$, B_0 is a ball in $\mathfrak{g}$ centered at the origin).[24] The discrepancy between

[24] Generally, B_0 is the connected component of the origin in $\mathfrak{g} - \phi^{-1}(0)$ where the function $\phi\colon \mathfrak{g} \to \mathbb{R}$, $\phi(x) := \det \frac{\sinh(\mathrm{ad}_x/2)}{\mathrm{ad}_x/2}$, is the Jacobian of the exponential map. In other words, B_0 is the set of

the correct cohomology of Ω and the cohomology of $\underline{\delta}$ is due to a convergence issue arising in homological perturbation theory.[25] The useful remark coming from this discussion is that, modulo Ω-exact terms, the partition function and the physical observables can be represented as a (G-invariant) function of *constant fields* valued in a neighbourhood of zero in $\mathfrak{g}$. Moreover, in ghost degree zero, for Ω-closed objects – depending only on $\mathbb{A}_{(1)}$ – the "reduced wavefunction" $\Psi(\underline{\mathbb{A}}_{(1)})$ can be lifted to an Ω-closed function in the non-reduced space of states by evaluating Ψ on the logarithm of the holonomy of $\mathbb{A}_{(1)}$.

Hodge propagators and axial gauge

The kinetic term in the YM action (5.16) is of the kind $\int_\Sigma \langle \mathsf{B}, D\mathsf{A}\rangle$, where D is a differential on $\mathcal{Y}_\Sigma^\bullet$. Since the propagator is the integral kernel of the inverse of D, we want to find where the differential can actually be inverted.

Let (K, i, p) be a *retraction* of $(\mathcal{Y}_\Sigma^\bullet, D)$ on its cohomology $(\mathcal{V}^\bullet, 0)$, i.e. a triple where $K\colon \mathcal{Y}_\Sigma^\bullet \longrightarrow \mathcal{Y}_\Sigma^{\bullet-1}$ is a chain homotopy, $i\colon \mathcal{V}_\Sigma^\bullet \hookrightarrow \mathcal{Y}_\Sigma^\bullet$ a chain inclusion and $p\colon \mathcal{Y}_\Sigma^\bullet \twoheadrightarrow \mathcal{V}_\Sigma^\bullet$ a chain projection satisfying:

$$K^2 = p \circ K = K \circ i = 0 \;, \qquad i \circ p = \mathrm{id} \;, \qquad DK + KD = \mathrm{id} - i \circ p \;. \tag{5.34}$$

Then the complex $\mathcal{Y}_\Sigma^\bullet$ has a *weak Hodge decomposition*:

$$\mathcal{Y}_\Sigma^\bullet = \underbrace{\Pi\mathcal{Y}_\Sigma^\bullet}_{\simeq \mathcal{V}_\Sigma^\bullet} \oplus \underbrace{K\mathcal{Y}_\Sigma^{\bullet+1} \oplus D\mathcal{Y}_\Sigma^{\bullet-1}}_{=\mathcal{Y}_\Sigma'^\bullet} \;, \tag{5.35}$$

where we have defined $\Pi := i \circ p$. From eq. (5.34), the differential D is invertible as an operator from the image of the chain homotopy K to D-exact cochains $D\colon K\mathcal{Y}_\Sigma^{\bullet+1} \longrightarrow D\mathcal{Y}_\Sigma^{\bullet-1}$ and its inverse is precisely the chain homotopy itself: $K = D^{-1}$.

The gauge can thus be fixed on the lagrangian $\mathcal{L} = K\mathcal{Y}_\Sigma$; the *propagator* $\eta(x'; x)$, with this gauge-fixing, is defined as the integral kernel of the chain homotopy:

$$K\omega(x) = \int_{\Sigma \ni x'} \eta(x; x') \wedge \omega(x') \;, \qquad \omega \in \mathcal{Y}_\Sigma \;. \tag{5.36}$$

When spacetime is a product manifold, $\Sigma = \Sigma_1 \times \Sigma_2$, there is a particular class of propagators which can be induced on Σ from lower-dimensional propagators on the two factors [14]. Since the differential forms on a product manifold are the (closure) of the sum of products of the differential forms on the two factors, we have $\mathcal{Y}_\Sigma = \mathcal{Y}_{\Sigma_1} \otimes \mathcal{Y}_{\Sigma_2}$. For each pair of contractions (K_ℓ, i_ℓ, p_ℓ) on the factors $\mathcal{Y}_{\Sigma_\ell}$ we have an induced weak Hodge

elements $x \in \mathfrak{g}$ such that all eigenvalues of ad_x are contained in the interval $(-2\pi\mathrm{i}, 2\pi\mathrm{i}) \subset \mathrm{i}\mathbb{R}$.

[25] This problem is a version of the Gribov ambiguity (Gribov copies) problem in 4d Yang-Mills theory – the problem of gauge-fixing "section" intersecting the gauge orbits more than once. For that reason, we will refer to B_0 as the "Gribov region".

decomposition on $\mathcal{Y}_\Sigma$:

$$\mathcal{Y}_\Sigma = \overbrace{\left(\Pi_1 \mathcal{Y}_{\Sigma_1} \otimes \Pi_2 \mathcal{Y}_{\Sigma_2}\right)}^{=\Pi\mathcal{Y}_\Sigma \simeq \mathcal{V}_\Sigma} \oplus \overbrace{\left(\Pi_1 \mathcal{Y}_{\Sigma_1} \otimes K_2 \mathcal{Y}_{\Sigma_2}\right) \oplus \left(K_1 \mathcal{Y}_{\Sigma_1} \otimes \mathcal{Y}_{\Sigma_2}\right)}^{K\mathcal{Y}_\Sigma} \\ \oplus \left(\Pi_1 \mathcal{Y}_{\Sigma_1} \otimes D_2 \mathcal{Y}_{\Sigma_2}\right) \oplus \left(D_1 \mathcal{Y}_{\Sigma_1} \otimes \mathcal{Y}_{\Sigma_2}\right) \,. \tag{5.37}$$

The zero modes are the product of the zero modes on the two factors and the induced chain homotopy is $K = \Pi_1 \otimes K_2 \oplus K_1 \otimes \mathrm{id}_{\mathcal{Y}_{\Sigma_2}}$. The associated gauge is called *axial gauge*. If we call π_ℓ the integral kernel of Π_ℓ , the *axial gauge propagator* is:

$$\eta(x_1, x_2; x_1', x_2') = \pi_1(x_1; x_1') \wedge \eta_2(x_2; x_2') + \eta_1(x_1; x_1') \wedge \delta(x_2; x_2') \,. \tag{5.38}$$

5.3 2D YM for surfaces of non-negative Euler characteristic

In this section we will consider 2D YM on manifolds with codimension 1 boundaries. With a good choice of propagators and exploiting the gluing properties of BV-BFV theories, we will be able in this setting to explicitly compute all Feynman diagrams and sum the perturbative series to find the complete partition function of this theory on disks and cylinders. The globalized realization of the partition function on a disk in the $\mathbb{A}$ polarization will coincide with the well-known non-perturbative solution of 2D YM [40, 53].[26]

We consider a set of generators, under gluing, for orientable surfaces of non-negative Euler characteristic: the disk and the cylinder. At the level of the field theory constructed on such surfaces, we have to also consider the data of the polarization associated to the boundaries. The building blocks for 2D YM can be thus chosen to be the disk in the $\mathbb{B}$ polarization, the cylinder in $\mathbb{A}-\mathbb{A}$ polarization and the cylinder in the $\mathbb{B}-\mathbb{B}$ polarization. Moreover, using the invariance of the theory under area-preserving diffeomorphisms, we can concentrate the support of the volume form μ near the boundaries; this allows to use as generators the above surfaces in the limit of zero area, i.e. for BF theory, at the cost of introducing as fourth generator a YM cylinder in $\mathbb{A}-\mathbb{B}$ polarization with finite volume (figure 5.3).

5.3.1 $\mathbb{A}$-$\mathbb{B}$ polarization on the cylinder

Let us start studying the BF theory on the cylinder, $\Sigma = S^1 \times I \ni (\tau, t)$, $I = [0, 1]$. We will firstly choose $\mathbb{B}$ polarization on $S^1 \times \{0\} = \partial_\mathbb{B}\Sigma$ and $\mathbb{A}$ polarization on $S^1 \times \{1\} = \partial_\mathbb{A}\Sigma$. The space of bulk fields, with this polarization, is $\mathcal{Y} = \Omega(\Sigma, \partial_\mathbb{B}\Sigma; \mathfrak{g})[1] \oplus \Omega(\Sigma, \partial_\mathbb{A}\Sigma; \mathfrak{g})$. Since the relative cohomology $H(\Sigma, \partial_i \Sigma)$ is trivial with the above choice of boundaries, we have no zero-modes. Thus the connected diagrams contributing to the effective action of the theory are trees with one root on $\partial_\mathbb{B}\Sigma$ and leafs on $\partial_\mathbb{A}\Sigma$ or 1-loop diagrams with trees rooted on a point of the loop and leafs on $\partial_\mathbb{A}\Sigma$ (figure 5.4).

[26] Although we can present, e.g., the sphere and the torus as assembled from building blocks considered in this section, globalization integrals for them are perturbatively obstructed, see Section 5.3.5. We obtain a non-singular globalized answer in these cases as a part of the general result of Section 5.4.

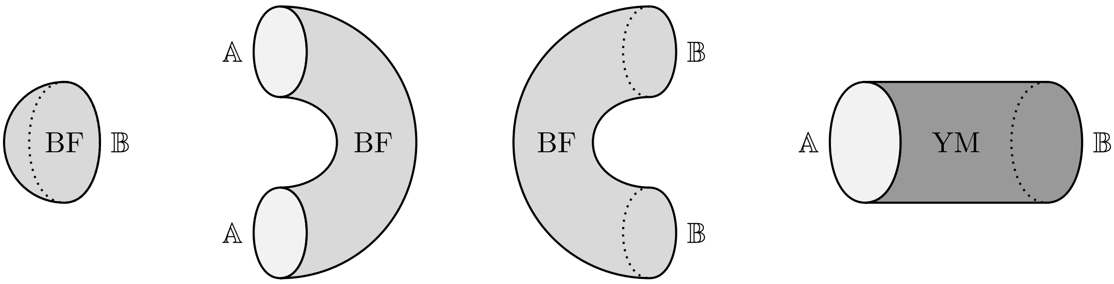


Figure 5.3: Building blocks for 2D YM on surfaces with non-negative Euler characteristic.

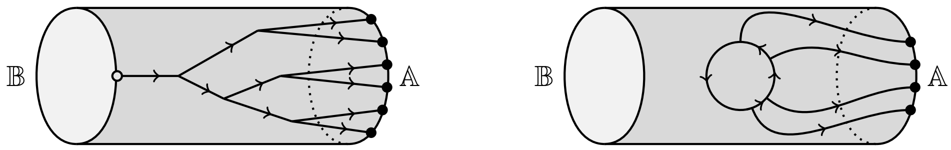
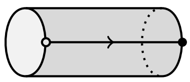


Figure 5.4: Connected diagrams for non-abelian BF on the cylinder in 𝔸-𝔹 polarization.

To compute these diagrams we can use the axial-gauge, with propagator (cf. B.2):

$$(t,\tau) \longrightarrow (t',\tau') \;= \eta(t,\tau;t',\tau') = -\Theta(t'-t)\delta(\tau-\tau')(\mathrm{d}\tau' - \mathrm{d}\tau)\ . \tag{5.39}$$

Looking at this propagator we immediately notice that it is a zero-form on the interval I. Since each bulk vertex carries an integration over $S^1 \times I$, the differential form associated to a diagram has the right form components (that is, s.t. its integral over the configuration space doesn't vanish) only if it doesn't contain any bulk vertex. Thus there is only one non-vanishing diagram contributing to the effective action:

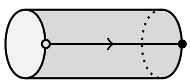
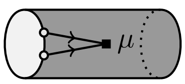
$$\mathcal{S}^{\text{eff}}_{\text{BF}}[\mathbb{B},\mathbb{A}] =$$  $$= \int_{\partial_{\mathbb{A}}\Sigma} \langle p^*\mathbb{B}, \mathbb{A}\rangle\ , \tag{5.40}$$

where $p\colon \Sigma \longrightarrow \partial_{\mathbb{B}}\Sigma$ is a projection to the 𝔹-boundary.

The Yang-Mills action can be rewritten as a perturbation of BF:

$$\mathcal{S}_{\text{YM}} = \mathcal{S}_{\text{BF}} + \frac{1}{2}\int_{\Sigma} \mu\ \mathrm{tr}(\mathsf{B}^2)\ . \tag{5.41}$$

The additional bivalent interaction vertex is proportional to the volume form μ. For degree counting reasons analogous to the one described above, the only additional non-vanishing Feynman diagram is the one containing a single YM vertex:

$$\mathcal{S}^{\text{eff}}_{\text{YM}} =$$  $+$  $$= \int_{\partial_{\mathbb{A}}\Sigma} \langle p^*\mathbb{B}, \mathbb{A}\rangle + \frac{1}{2}\int_{\partial_{\mathbb{B}}\Sigma} p_*\mu\ \mathrm{tr}\mathbb{B}^2\ , \tag{5.42}$$

where the last integral is the integral of a density, with $p_*\mu$ the pushforward of μ, viewed as a density on the cylinder, to the 𝔹-circle. Thus we proved the following:

Proposition 5.3.1 (YM on $\mathbb{A}$-$\mathbb{B}$ cylinder)**.** The partition function for a YM cylinder in the $\mathbb{A}$-$\mathbb{B}$ polarization is:

$$Z[\mathbb{A},\mathbb{B}] = \exp\frac{\mathrm{i}}{\hbar}\Big(\int_{\partial_{\mathbb{A}}\Sigma}\langle p^*\mathbb{B},\mathbb{A}\rangle + \frac{1}{2}\int_{\partial_{\mathbb{B}}\Sigma} p_*\mu \,\mathrm{tr}\mathbb{B}^2\Big)\ . \tag{5.43}$$

A YM $\mathbb{A}$-$\mathbb{B}$ cylinder can be glued to other YM surfaces with boundary to modify their volume. In particular in this way one can convert BF ($\mu = 0$) to YM.

5.3.2 $\mathbb{B}$-$\mathbb{B}$ polarization on the cylinder

Another possible choice is to take the $\mathbb{B}$ polarization on both the boundary components of the cylinder. This time the bulk fields are $\mathcal{Y} = \Omega(M;\mathfrak{g})[1] \oplus \Omega(M,\partial M;\mathfrak{g})$, with zero-modes $\mathcal{V} = H(M;\mathfrak{g})[1] \oplus H(M,\partial M;\mathfrak{g}) \simeq H(S^1;\mathfrak{g})[1] \oplus H(S^1;\mathfrak{g})[-1]$. More explicitly, the zero-modes can be described expanding with respect to a basis $[\chi_i]$ of $H(S^1)$ and its dual $[\chi^i]$:

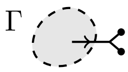
$$\mathsf{a} = \mathsf{a}_i\chi^i \in H(S^1;\mathfrak{g})[1]\ , \qquad \mathsf{b} = \mathsf{b}^i\chi_i \wedge \mathrm{d}t \in H(S^1;\mathfrak{g})[-1]\ . \tag{5.44}$$

We can again fix the gauge using the axial-gauge, obtaining the propagator (B.9):

$$\eta(t,\tau;t',\tau') = \big(t' - \Theta(t'-t)\big)\delta(\tau'-\tau)(\mathrm{d}\tau' - \mathrm{d}\tau) + \mathrm{d}t'\big(\Theta(\tau-\tau') - \tau + \tau' - \frac{1}{2}\big)\ . \tag{5.45}$$

Now the effective action contains trees with root on one of the boundaries and leafs in the bulk or 1-loop diagrams with trees rooted on the loop and leafs in the bulk. Luckily, a lot of these diagrams vanish as it is shown by the following

Lemma 5.3.2. For BF theory on a cylinder with $\mathbb{B}$-$\mathbb{B}$ polarization in the axial gauge, all the diagrams containing a bulk vertex with attached two a zero-modes vanish:



$$= 0\ . \tag{5.46}$$

Proof. Consider any diagram of the kind depicted in formula (5.46). The associated differential form on the configuration space of the diagram will be of the kind

$$\Gamma_c(t,\tau)\ \eta(t,\tau;t',\tau')\ f^c_{ab}\ \mathsf{a}^a(\tau')\ \mathsf{a}^b(\tau')\ .$$

Since a has no form component along $\mathrm{d}t$, using the axial-gauge propagator (5.45) we have for the corresponding amplitude:

$$\begin{aligned}&\int \Gamma_c(t,\tau)\ f^c_{ab}\ \mathsf{a}^a(\tau')\ \mathsf{a}^b(\tau')\ \eta(t,\tau;t',\tau')\\ &= f^c_{ab}\ \mathsf{a}^{ia}\ \mathsf{a}^{jb}\int \Gamma_c(t,\tau)\int_{S^1}\chi^i(\tau')\ \chi_j(\tau')\ \Big(\Theta(\tau-\tau') - \tau + \tau' - \frac{1}{2}\Big) = 0\ .\end{aligned} \tag{5.47}$$

□

In particular this means that contributions to the effective action only come from either one single zero-mode attached to one of the two boundaries or from 1-loop diagrams with $n \geqslant 2$ vertices, each attached to a single a zero-mode (figure 5.5). These diagrams can be explicitly evaluated and the perturbative series can be summed to obtain the effective action.

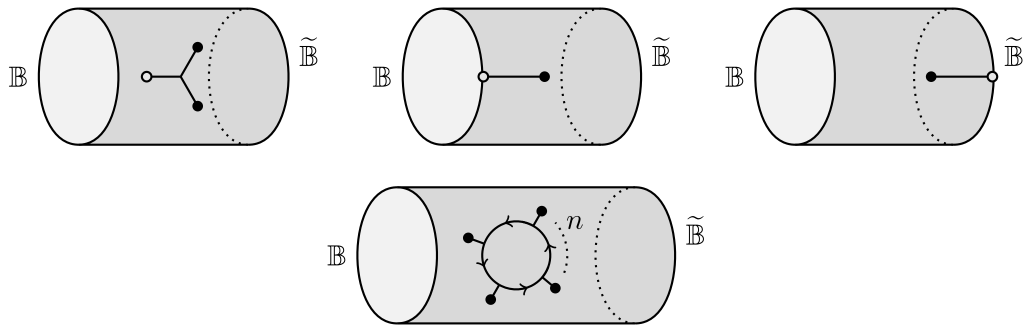


Figure 5.5: Connected diagrams for non-abelian BF on the cylinder in $\mathbb{B}$-$\mathbb{B}$ polarization.

Proposition 5.3.3. The partition function for BF theory on the cylinder in $\mathbb{B}$-$\mathbb{B}$ polarization is:

$$\begin{aligned} Z[\mathbb{B},\widetilde{\mathbb{B}},\mathsf{a},\mathsf{b}] &= \exp\left(\frac{\mathrm{i}}{2\hbar}\int_{I\times S^1}\langle \mathsf{b},[\mathsf{a},\mathsf{a}]\rangle + \frac{\mathrm{i}}{\hbar}\int_{S^1}\langle \mathbb{B}-\widetilde{\mathbb{B}},\mathsf{a}\rangle + \sum_{n\geqslant 2}\frac{1}{n}\mathrm{tr}(\mathrm{ad}_{\mathsf{a}_1})^n\,\frac{B_n}{n!}\right)\cdot \rho_{\mathcal{V}} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\int_{I\times S^1}\langle \mathsf{b},[\mathsf{a},\mathsf{a}]\rangle + \frac{\mathrm{i}}{\hbar}\int_{S^1}\langle \mathbb{B}-\widetilde{\mathbb{B}},\mathsf{a}\rangle}\det\left(\frac{\sinh\left(\mathrm{ad}_{\mathsf{a}_1}/2\right)}{\mathrm{ad}_{\mathsf{a}_1}/2}\right)\cdot \rho_{\mathcal{V}}\ . \end{aligned} \tag{5.48}$$

Here $\rho_{\mathcal{V}} = (-\mathrm{i}\hbar)^{\dim \mathfrak{g}}\, D^{\frac{1}{2}}\mathsf{a}\ D^{\frac{1}{2}}\mathsf{b}$ is the reference half-density on the space of zero-modes.

Proof. We refer for the proof to Appendix C. □

Remark 5.3.4 (Reference half-densities on residual fields)**.** Generally, we choose the following reference half-density on the space of residual fields $\mathcal{V}$:

$$\rho_{\mathcal{V}} = \prod_{k=0}^{2}(\xi_k)^{d_k}\cdot \mathrm{D}^{\frac{1}{2}}\mathsf{a}\ \mathrm{D}^{\frac{1}{2}}\mathsf{b}\ . \tag{5.49}$$

Here $\mathrm{D}^{\frac{1}{2}}\mathsf{a}\ \mathrm{D}^{\frac{1}{2}}\mathsf{b}$ is the standard half-density on $\mathcal{V}$, inducing the standard Berezin-Lebesgue densities $\mathrm{d}\mathsf{a}$, $\mathrm{d}\mathsf{b}$ on the lagrangians $\mathsf{b}=0$ and $\mathsf{a}=0$, respectively. Also, d_k is the dimension of the subspace of $\mathcal{V}$ corresponding to a-fields of de Rham degree $k \in \{0,1,2\}$; in particular, for $\mathcal{V}$ the zero-modes, $d_k = \dim H^k(\Sigma,\partial_{\mathbb{A}}\Sigma;\mathfrak{g})$ are the Betti numbers of relative de Rham cohomology. Factors ξ_k are as follows:

$$\xi_0 = -\mathrm{i}\hbar,\quad \xi_1 = 1,\quad \xi_2 = \frac{1}{2\pi\hbar}\ . \tag{5.50}$$

The logic behind this normalization is that for $\mathcal{V} = W[1] \oplus W^*[-2]$ with W a complex (concentrated in degrees $0, 1, 2$) and $\mathcal{V}' = W'[1] \oplus W'^*[-2]$ with W' a deformation retract of W, we would like the BV pushforward of the half-density $\rho_{\mathcal{V}}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle \mathsf{b}, \mathrm{d}\mathsf{a}\rangle}$ on $\mathcal{V}$ (corresponding to abstract abelian BF theory associated to W) to yield $\rho_{\mathcal{V}'}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle \mathsf{b}', \mathrm{d}\mathsf{a}'\rangle}$ on $\mathcal{V}'$. Thus, we recover the normalization of reference half-densities from the automorphicity with respect to BV pushforwards. Most general normalization satisfying this condition is:

$$\xi_0 = -\mathrm{i}\hbar\phi, \quad \xi_1 = \phi^{-1}, \quad \xi_2 = \frac{\phi}{2\pi\hbar} , \tag{5.51}$$

with $\phi \neq 0$ an arbitrary constant. Our choice is to set $\phi = 1$ which will ultimately lead to the number-valued partition function of 2D Yang-Mills with standard normalization. Choosing any other ϕ would induce a rescaling of partition functions by

$$Z_\Sigma \mapsto \phi^{\chi(\Sigma)\cdot \dim \mathfrak{g}}\, Z_\Sigma , \tag{5.52}$$

which reflects an inherent ambiguity of the normalization of path-integral measure. We refer the reader to [25] for details on the normalization of half-densities compatible with BV pushforwards.

5.3.3 $\mathbb{A}$-$\mathbb{A}$ polarization on the cylinder

The last polarization choice we will consider consists in taking $\mathbb{A}$ polarization for both the boundaries of the cylinder. With this polarization the bulk fields are $\mathcal{Y} = \Omega(M, \partial M; \mathfrak{g})[1] \oplus \Omega(M; \mathfrak{g})$. The zero-modes $\mathcal{V} = H(M, \partial M; \mathfrak{g})[1] \oplus H(M; \mathfrak{g}) \simeq H(S^1; \mathfrak{g}) \oplus H(S^1; \mathfrak{g})$ can be expanded as:

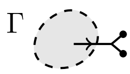
$$\mathsf{a} = \mathsf{a}_i \chi^i \wedge \mathrm{d}t , \qquad \mathsf{b} = \mathsf{b}^i \chi_i . \tag{5.53}$$

The axial-gauge propagator is now (B.9):

$$\eta(t, \tau; t', \tau') = \big(\Theta(t - t') - t\big)\delta(\tau' - \tau)(\mathrm{d}\tau' - \mathrm{d}\tau) - \mathrm{d}t\big(\Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2}\big) . \tag{5.54}$$

The effective action contains trees with the root in the bulk and leafs on one of the boundaries or 1-loop diagrams with trees rooted on the loop and leafs either on the boundary or decorated with the zero mode a^1. Also with this polarization an analogue of lemma 5.3.2 holds:

Lemma 5.3.5. For BF theory on a cylinder with $\mathbb{A}$-$\mathbb{A}$ polarization in the axial gauge, all the diagrams containing a bulk vertex with attached two a zero-modes vanish:



$$= 0 . \tag{5.55}$$

Proof. The proof follows trivially from degree counting, since a zero-modes always have a component along $\mathrm{d}t$. □

The diagrams contributing to the effective action can be restricted further by reducing to the case of *constant* fields $\mathbb{A}, \widetilde{\mathbb{A}}$. Indeed, the gauge-invariance of the partition function (expressed by the mQME) implies that it is sufficient to evaluate it on constant 1-form fields $\mathbb{A} = \mathrm{d}t\, \underline{\mathbb{A}}_{(1)}$, $\widetilde{\mathbb{A}} = \mathrm{d}t\, \underline{\widetilde{\mathbb{A}}}_{(1)}$, with $\underline{\mathbb{A}}_{(1)}, \underline{\widetilde{\mathbb{A}}}_{(1)} \in \mathfrak{g}$ two constants. Then the value of the partition function for generic fields $\mathbb{A}, \widetilde{\mathbb{A}}$ is recovered (modulo a BV-exact term, cf. (5.56) below) by evaluating the constant-field result on the logs of holonomies $\underline{\mathbb{A}}_{(1)} = \log U(\mathbb{A})$, $\underline{\widetilde{\mathbb{A}}}_{(1)} = \log U(\widetilde{\mathbb{A}})$.[27] Here $U(\cdots)$ stands for the holonomy of a connection 1-form around a circle. In other words, using the language of homological perturbation theory, we have a quasi-isomorphism between the two models for the space of states for an $\mathbb{A}$-circle:

(i) The full BFV model $\mathcal{H}^{\mathbb{A}} = \mathrm{Fun}_{\mathbb{C}}(\Omega^\bullet(S^1, \mathfrak{g})[1])$ given by functions of a general differential form $\mathbb{A}$ on the circle, with differential Ω defined by (5.32).

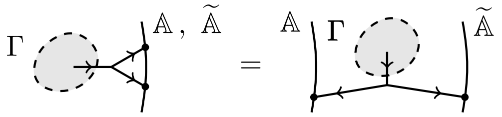
(ii) The constant-field model $\mathcal{H}^{\mathbb{A},\mathrm{const}} = \mathrm{Fun}_{\mathbb{C}}(H^\bullet(S^1, \mathfrak{g})[1])$ – functions of a constant form $\underline{\mathbb{A}}_{(0)} + \mathrm{d}t\, \underline{\mathbb{A}}_{(1)}$, with differential $\underline{\delta}$ defined by (5.33).

We have two chain maps: first, the projection $p_{\mathcal{H}} \colon \mathcal{H}^{\mathbb{A}} \to \mathcal{H}^{\mathbb{A},\mathrm{const}}$ – evaluation of a wavefunction on constant forms or equivalently the pullback $p_{\mathcal{H}} = \iota^*$ by the inclusion of the cohomology by as constant forms $H^\bullet(S^1, \mathfrak{g}) \mapsto \Omega^\bullet(S^1, \mathfrak{g})$. Second, the inclusion $i_{\mathcal{H}} \colon \mathcal{H}^{\mathbb{A},\mathrm{const}} \to \mathcal{H}^{\mathbb{A}}$ sending $\underline{\Psi} \mapsto \Big(\Psi \colon \mathbb{A} \mapsto \underline{\Psi}\big(\, \mathbb{A}_{(0)}\big|_p, \log U(\mathbb{A})\big)\Big)$ where p is the base point on the circle used to define the holonomy. Denoting $K_{\mathcal{H}}$ the chain homotopy for the retraction of chain complexes $(\mathcal{H}^{\mathbb{A}}, \Omega) \rightsquigarrow (\mathcal{H}^{\mathbb{A},\mathrm{const}}, \underline{\delta})$, we have the following (cf. the discussion of the reduced partition function in [25], section 7.4):

$$i_{\mathcal{H}} \circ p_{\mathcal{H}}\, Z = (\mathrm{id} - K_{\mathcal{H}}\Omega - \Omega K_{\mathcal{H}})Z = Z + (\Omega + \hbar^2 \Delta)(\cdots)\ , \tag{5.56}$$

where $\cdots = -K_{\mathcal{H}} Z$. The left hand side in (5.56) is exactly the partition function evaluated on constant 1-form fields having the same holonomy as the original non-constant ones.

Lemma 5.3.6. For BF theory on a cylinder with $\mathbb{A}$-$\mathbb{A}$ polarization in the axial gauge, all the diagrams containing a bulk vertex with attached two boundary fields vanish, assuming that $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ are constant 1-forms.



$$= 0\ . \tag{5.57}$$

Proof. Using the assumption of constancy of boundary fields and the axial-gauge propagator (5.54), when we have two boundary fields connected to the same bulk vertex we find the amplitude:

$$\begin{aligned} &\int \Gamma_c(\tilde{t}, \tilde{\tau})\, f^c_{ab}\, \underline{\mathbb{A}}^a\, \underline{\mathbb{A}}^b\, \eta(\tilde{t}, \tilde{\tau}; t, \tau)\, \eta(t, \tau; 0, \tau')\, \eta(t, \tau; 0, \tau'') \\ &= \frac{1}{2} f^c_{ab}\, \underline{\mathbb{A}}^a\, \underline{\mathbb{A}}^b \int \Gamma_c(\tilde{t}, \tilde{\tau}) \eta(\tilde{t}, \tilde{\tau}; t, \tau) \int_{S^1} \mathrm{d}\tau' \Big(\Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2}\Big) = 0\ . \end{aligned} \tag{5.58}$$

[27] For simplicity of notations we are omitting the subscript of the 1-form $\mathbb{A}_{(1)}$ when it appears in the holonomy $U(\mathbb{A})$.

Similar amplitudes are found also in the case one or both the boundary fields live on the boundary at $t = 1$. □

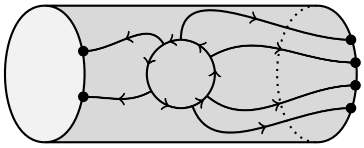


Figure 5.6: Relevant loop diagrams for the globalized effective action of BF theory on a cylinder with $\mathbb{A}$-$\mathbb{A}$ polarization

Like in the case of non-abelian BF theory on closed surfaces, the computation of the effective action greatly simplifies if we look at the *globalized* answer. In this case we can define the globalized partition function by integrating the perturbative partition function over a lagrangian submanifold of the space of residual fields. If $\mathcal{S}_{\text{eff}}[\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}]$ is the perturbative effective action on the space of boundary fields and zero-modes and $\mathcal{L} \subset \mathcal{V}$ is a lagrangian submanifold, the globalized partition function can be defined as:

$$Z = \int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\text{eff}}[\mathbb{A},\mathbb{B},\mathsf{a},\mathsf{b}]} \,. \tag{5.59}$$

A possible choice for the lagrangian is $\mathcal{L} := \{\mathsf{a} = 0\}$, which in particular implies that all diagrams containing a zero-modes will not contribute to the globalized effective action. The effective action of BF theory is always linear in the b zero-modes. Moreover, for the $\mathbb{A}$-$\mathbb{A}$ polarization on the cylinder in the axial gauge, lemma 5.3.6 implies that there are no tree diagrams with this $\mathcal{L}$. Thus:

$$Z = \int \frac{\mathrm{d}\mathsf{b}}{(2\pi\hbar)^{\dim\mathfrak{g}}}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle \mathsf{b}, \mathbb{A}-\widetilde{\mathbb{A}}\rangle + \frac{\mathrm{i}}{\hbar}\mathcal{S}_{\text{eff}}(\mathbb{A},\widetilde{\mathbb{A}},\mathsf{a}=0,\mathsf{b}=0)} = \left(\frac{\mathrm{i}}{\hbar}\right)^{\dim\mathfrak{g}} \delta(\mathbb{A},\widetilde{\mathbb{A}})\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\text{eff}}(\mathbb{A},\widetilde{\mathbb{A}}=\mathbb{A},\mathsf{a}=0,\mathsf{b}=0)} \,. \tag{5.60}$$

The loop diagrams contributing to the globalized effective action are now only those where each loop vertex is connected to a boundary field with a single propagator and the fields on the two boundary components coincide (figure 5.6). The amplitude of such a diagram with n boundary fields is:

$$-\frac{1}{n}\mathrm{tr}(\mathrm{ad}^n_{\underline{\mathbb{A}}_{(1)}}) \int_{(S^1)^n} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \; \eta_{S^1}(\tau_1;\tau_n)\eta_{S^1}(\tau_n;\tau_{n-1})\cdots\eta_{S^1}(\tau_2;\tau_1) \,. \tag{5.61}$$

The integrals involved are exactly the same as the ones of the case of $\mathbb{B}$-$\mathbb{B}$ polarization (C.1). Thus we have:

Proposition 5.3.7. The globalized partition function for BF theory on the cylinder in the $\mathbb{A}$-$\mathbb{A}$ polarization is

$$Z[\mathbb{A},\widetilde{\mathbb{A}}] = \left(\frac{\mathrm{i}}{\hbar}\right)^{\dim\mathfrak{g}} \delta(\log U(\mathbb{A}), \log U(\widetilde{\mathbb{A}}))\cdot\delta(\mathbb{A}_p, \widetilde{\mathbb{A}}_{\tilde{p}}) \,\det\left(\frac{\sinh\left(\mathrm{ad}_{\log U(\mathbb{A})}/2\right)}{\mathrm{ad}_{\log U(\mathbb{A})}/2}\right)^{-1} \tag{5.62}$$

where $U(\cdots)$ is the holonomy of the connection around a circle and $\mathbb{A}_p, \widetilde{\mathbb{A}}_{\widetilde{p}}$ are the zero-form components of boundary fields $\mathbb{A}, \widetilde{\mathbb{A}}$ evaluated at the base points $p, \widetilde{p}$ on the two boundary circles.

Remark 5.3.8. Since $\det\left(\frac{\sinh\left(\mathrm{ad}_x/2\right)}{\mathrm{ad}_x/2}\right)$ is the determinant of the Jacobian of the exponential map $\exp\colon \mathfrak{g} \longrightarrow G$, we can rewrite (5.62) in terms of the delta function on the Lie group:

$$Z[\mathbb{A}, \widetilde{\mathbb{A}}] = \left(\frac{\mathrm{i}}{\hbar}\right)^{\dim \mathfrak{g}} \delta(\mathbb{A}_p, \widetilde{\mathbb{A}}_{\tilde{p}}) \cdot \delta_G(U(\mathbb{A}), U(\widetilde{\mathbb{A}})) \ . \tag{5.63}$$

5.3.4 $\mathbb{B}$ polarization on the disk

Let us consider now non-abelian BF theory on the disk D. Using the $\mathbb{B}$ polarization on the boundary, the bulk fields are $\mathcal{Y} = \Omega(D; \mathfrak{g})[1] \oplus \Omega(D, S^1; \mathfrak{g})$. The zero-modes are $\mathcal{V} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$ with generators the constant zero-form $[1 \cdot t_a]$ and an area 2-form $[\mu \cdot t^a]$, where t_a and t^a are dual basis of $\mathfrak{g}$ and $\mathfrak{g}^*$.

The Feynman graphs appearing in the effective action are trees, with root either in a boundary $\mathbb{B}$-field or in a b zero-mode in the bulk, or 1-loop diagrams (figure 5.7).

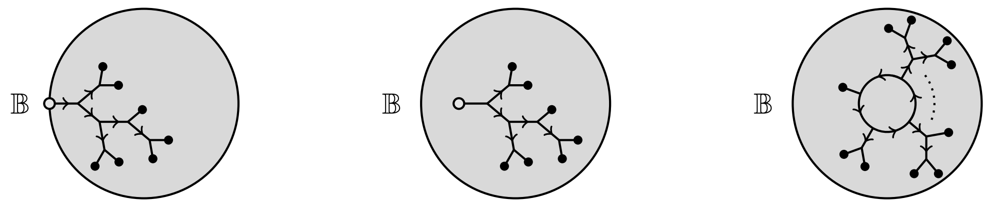


Figure 5.7: Connected diagrams for non-abelian BF on the disk in $\mathbb{B}$ polarization.

Proposition 5.3.9. In the effective action for BF theory on the disk in $\mathbb{B}$-polarization, all the diagrams containing at least one propagator are vanishing. In particular the partition function reads:

$$Z[\mathbb{B}, \mathsf{a}, \mathsf{b}] = \exp \frac{\mathrm{i}}{\hbar}\left(-\int_{S^1} \langle \mathbb{B}, \mathsf{a}\rangle + \int_D \frac{1}{2}\langle \mathsf{b}, [\mathsf{a}, \mathsf{a}]\rangle\right) \cdot \rho_{\mathcal{V}} \ , \tag{5.64}$$

with $\rho_{\mathcal{V}} = (-\mathrm{i}\hbar)^{\dim \mathfrak{g}} D^{\frac{1}{2}}\mathsf{a}\ D^{\frac{1}{2}}\mathsf{b}$ the reference half-density on residual fields.

Proof. The result follows from degree counting. Let us consider first a tree diagram rooted in the bulk. If n is the number of bulk vertices, then we have $n-1$ propagators, $n+1$ a zero-modes and one b zero-mode. Since propagators are 1-forms, a only has the zero-form component and b is a 2-form, then the differential form associated to the diagram is a $(n+1)$-form. This has to be integrated on the configuration space of the diagram, which is of dimension $2n$. Thus the only possibly non-vanishing diagram is for $n = 1$. Its contribution is:

$$\frac{1}{2}\int_D \langle \mathsf{b}, [\mathsf{a}, \mathsf{a}]\rangle \ . \tag{5.65}$$

Consider now a tree diagram rooted on the boundary with n bulk vertices. We have n propagators, $n+1$ a zero-modes and one boundary field $\mathbb{B}$. Thus the differential form associated to the diagram is a n-form or a $(n+1)$-form, depending on the form degree of $\mathbb{B}$, and has to be integrated again on a $(2n+1)$-dimensional configuration space. In this case we only have a contribution with $n=0$:

$$-\int_{S^1} \langle \mathbb{B}, \mathsf{a} \rangle \ . \tag{5.66}$$

Last, for a 1-loop diagram with $n \geqslant 1$ vertices in the loop and l vertices in the trees rooted on the n loop vertices, we have $n+l$ propagators and $n+l$ a zero-modes. Thus we have to integrate a differential form of degree $n+l$ on a $2(n+l)$-dimensional configuration space and, since $n \geqslant 1$, we have no non-vanishing contributions. □

5.3.5 Gluing

We computed the YM partition function on the $\mathbb{A}$-$\mathbb{B}$ cylinder and the BF partition function on the $\mathbb{B}$-disk and the $\mathbb{A}$-$\mathbb{A}$ cylinder. As we will show in this section, using the gluing property of BV-BFV theories, this is sufficient to prove a gluing formula between $\mathbb{A}$-polarized boundaries and to find the YM state on any surface with non-negative Euler characteristic.

BF disk in $\mathbb{A}$ polarization

The BF disk in $\mathbb{A}$ polarization can be obtained changing polarization to the disk in $\mathbb{B}$ polarization by gluing to it an $\mathbb{A}$-$\mathbb{A}$ BF cylinder (figure 5.8).

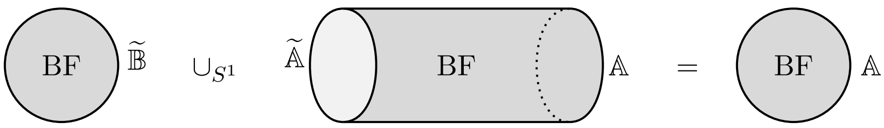


Figure 5.8: $\mathbb{A}$ disk as the gluing of a $\mathbb{B}$ disk with an $\mathbb{A}$-$\mathbb{A}$ cylinder.

For the $\mathbb{A}$-$\mathbb{A}$ cylinder we only know the projection in Ω cohomology of the globalized answer; since both globalization and projection to cohomology commute with gluing, we are still able to compute the partition function for the disk. The glued partition function is:

$$\begin{aligned} Z[\mathbb{A}] = \int & \mathrm{d}\mathsf{a}\, \mathrm{d}\widetilde{\mathbb{B}}\, \mathrm{d}\widetilde{\mathbb{A}}\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle \widetilde{\mathbb{B}}, \mathsf{a}-\widetilde{\mathbb{A}}\rangle}\ \delta(\mathbb{A}_p, \widetilde{\mathbb{A}}_{\widetilde{p}}) \\ & \cdot \det\left(\frac{\sinh\left(\mathrm{ad}_{\log U(\mathbb{A})}/2\right)}{\mathrm{ad}_{\log U(\mathbb{A})}/2}\right)^{-1} \delta(\log U(\widetilde{\mathbb{A}}), \log U(\mathbb{A})) \\ = \det & \left(\frac{\sinh\left(\mathrm{ad}_{\log U(\mathbb{A})}/2\right)}{\mathrm{ad}_{\log U(\mathbb{A})}/2}\right)^{-1} \delta(\log U(\mathbb{A}), 0) = \delta_G(\mathrm{e}^{\log U(\mathbb{A})}, \mathbb{I}) \\ = \delta_G & (U(\mathbb{A}), \mathbb{I}) \ . \end{aligned} \tag{5.67}$$

Remark 5.3.10. To have consistency with gluing, we assume that the integration measure over the boundary fields is normalized in such a way that

$$\int \mathrm{d}\widetilde{\mathbb{B}}\, \mathrm{d}\widetilde{\mathbb{A}}\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle\widetilde{\mathbb{B}},\widetilde{\mathbb{A}}\rangle} = 1\ . \tag{5.68}$$

As a matter of convenience, we moreover distribute the normalization between $\mathrm{d}\widetilde{\mathbb{A}}$ and $\mathrm{d}\widetilde{\mathbb{B}}$ in such a way that

$$\int \mathrm{d}\widetilde{\mathbb{B}}\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle\widetilde{\mathbb{B}},\widetilde{\mathbb{A}}\rangle} = \delta(\widetilde{\mathbb{A}})\ . \tag{5.69}$$

YM disk in $\mathbb{A}$ polarization

We can obtain the partition function for the YM disk in $\mathbb{A}$ polarization gluing to the BF disk a YM cylinder in $\mathbb{A}$-$\mathbb{B}$ polarization (figure 5.9).

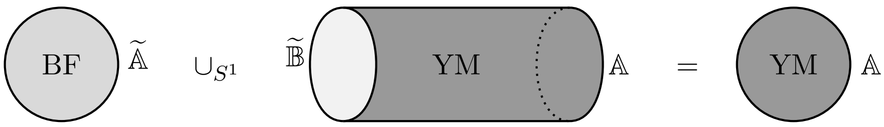


Figure 5.9: YM $\mathbb{A}$ disk as the gluing of a BF $\mathbb{A}$ disk with a YM $\mathbb{A}$-$\mathbb{B}$ cylinder.

As the partition function for the BF disk, also the YM partition function coincides with the non-perturbative answer.

Proposition 5.3.11. The globalized partition function for 2D YM on the disk in $\mathbb{A}$-polarization is:

$$Z_{\mathrm{YM}}[\mathbb{A}] = \sum_R (\dim R)\ \chi_R(U(\mathbb{A}))\ \mathrm{e}^{-\frac{\mathrm{i}\hbar a}{2}C_2(R)}\ , \tag{5.70}$$

where $a = \int \mu$ is the area of the disk, χ_R the character and $C_2(R)$ the quadratic Casimir of the representation R.

Proof. Gluing a BF disk to a YM cylinder in $\mathbb{A}$-$\mathbb{B}$ polarization we get:

$$\begin{aligned} Z_{\mathrm{YM}}[\mathbb{A}] &= \int \mathrm{d}\widetilde{\mathbb{B}}\, \mathrm{d}\widetilde{\mathbb{A}}\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle\widetilde{\mathbb{B}},(\widetilde{\mathbb{A}}-\mathbb{A})\rangle - \frac{\mathrm{i}}{2\hbar}\int_{S^1\times I}\mu\ \mathrm{tr}(\widetilde{\mathbb{B}}^2)}\delta(U(\widetilde{\mathbb{A}}),\mathbb{I}) \\ &= \mathrm{e}^{\frac{\mathrm{i}\hbar a}{2}\left(\frac{\partial}{\partial\mathbb{A}},\frac{\partial}{\partial\mathbb{A}}\right)}\ \delta(U(\mathbb{A}),\mathbb{I}) = \langle\mathbb{I}|\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_{\mathrm{YM}}}\ |U(\mathbb{A})\rangle \\ &= \sum_R (\dim R)\ \chi_R(U(\mathbb{A}))\ \mathrm{e}^{-\frac{\mathrm{i}\hbar a}{2}C_2(R)}\ . \end{aligned} \tag{5.71}$$

□

Gluing circles in $\mathbb{A}$ polarization

Two boundaries in $\mathbb{A}$ polarization can be glued together using a BF cylinder in $\mathbb{B}$-$\mathbb{B}$ polarization. If $Z_{\Sigma_i}[\mathbb{A}_i]$ is the globalized partition function on a surface Σ_i , $i = 1, 2$, and Σ is the gluing $\Sigma_1 \cup_{S^1} \Sigma_2$ along a common boundary in $\mathbb{A}$ polarization, we get:

$$\begin{aligned} Z_\Sigma = &\int \mathrm{d}\widetilde{\mathbb{B}}\, \mathrm{d}\widetilde{\mathbb{A}}\, \mathrm{d}\mathbb{B}\, \mathrm{d}\mathbb{A}\, \mathrm{d}\mathsf{a}_1\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle\widetilde{\mathbb{B}},(\widetilde{\mathbb{A}}-\mathsf{a})\rangle - \frac{\mathrm{i}}{\hbar}\int_{S^1}\langle\mathbb{B},(\mathbb{A}-\mathsf{a})\rangle + \frac{\mathrm{i}}{2\hbar}\langle\mathsf{b}^{(2)},[\mathsf{a}^{(0)},\mathsf{a}^{(0)}]\rangle} \\ &\cdot \det\left(\frac{\sinh\left(\mathrm{ad}_{\mathsf{a}_1}/2\right)}{\mathrm{ad}_{\mathsf{a}_1}/2}\right) Z_{\Sigma_1}[\mathbb{A}]\ Z_{\Sigma_2}[\widetilde{\mathbb{A}}]\cdot\rho_{\mathcal{V}} \\ = &\ \rho_{\mathcal{V}}\cdot\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\langle\mathsf{b}^{(2)},[\mathsf{a}^{(0)},\mathsf{a}^{(0)}]\rangle}\int_G \mathrm{d}U\ Z_{\Sigma_1}[U]\ Z_{\Sigma_2}[U]\ , \end{aligned} \tag{5.72}$$

which coincides with the gluing formula for YM known in literature [40, 53] up to a zero-mode dependent factor. Here, instead of integrating out the zero-modes on the $\mathbb{B}$-$\mathbb{B}$ cylinder completely (which would yield an ill-defined integral), we performed a partial integration (BV pushforward), retaining the zero-modes $\mathsf{a}^{(0)}, \mathsf{b}^{(2)}$. Here the index in brackets stands for the form degree of a zero-mode and $\rho_{\mathcal{V}} = (-\mathrm{i}\hbar)^{\dim\mathfrak{g}}\, D^{\frac{1}{2}}\mathsf{a}^{(0)}\ D^{\frac{1}{2}}\mathsf{b}^{(2)}$ is the reference half-density on the remaining zero-modes.

Remark 5.3.12. In (5.72), the domain of integration over a_1 is the "Gribov region" $B_0 \subset \mathfrak{g}$ (cf. subsection 5.2.4) – the preimage of an open dense subset of the group G under the exponential map $\exp\colon \mathfrak{g} \to G$. On one hand, this is the domain corresponding to values of a for which the sum of Feynman diagrams converges. On the other hand, this corresponds to avoiding overcounting when performing the globalization via integrating over zero-modes as opposed to integrating over the moduli space of solutions of Euler-Lagrange equations.

Other surfaces of non-negative Euler characteristic

To obtain the YM cylinder in $\mathbb{A}$-$\mathbb{A}$ polarization we can simply change polarization to the YM $\mathbb{A}$-$\mathbb{B}$ cylinder by gluing a BF cylinder (figure 5.10):

$$\begin{aligned} Z_{\mathrm{YM}}[\mathbb{A},\mathbb{A}'] &= \left(\frac{\mathrm{i}}{\hbar}\right)^{\dim\mathfrak{g}}\int \mathrm{d}\widetilde{\mathbb{B}}\,\mathrm{d}\widetilde{\mathbb{A}}\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{S^1}\langle\widetilde{\mathbb{B}},(\widetilde{\mathbb{A}}-\mathbb{A})\rangle - \frac{\mathrm{i}}{2\hbar}\int\mu\,\mathrm{tr}(\widetilde{\mathbb{B}}^2)}\delta_G(U(\widetilde{\mathbb{A}}),U(\mathbb{A}'))\ \delta(\mathbb{A}'_{p'},\widetilde{\mathbb{A}}_{\widetilde{p}}) \\ &= \left(\frac{\mathrm{i}}{\hbar}\right)^{\dim\mathfrak{g}}\delta(\mathbb{A}'_{p'},\mathbb{A}_p)\ \sum_R(\dim R)\ \chi_R\big(U^{-1}(\mathbb{A}')\cdot U(\mathbb{A})\big)\ \mathrm{e}^{-\frac{\mathrm{i}\hbar a}{2}C_2(R)}\ . \end{aligned} \tag{5.73}$$

Remark 5.3.13. The answer (5.73) *does not* coincide with the non-perturbative answer, which will be recovered perturbatively in Section 5.4 using manifolds with corners. This discrepancy is due to the presence of inequivalent gauge-fixings in the globalization process.

Let us now compute the YM partition function for a sphere S^2 obtained by the gluing of two disks: one with area a and in the $\mathbb{A}$ polarization, the other with zero area and in $\mathbb{B}$ polarization.

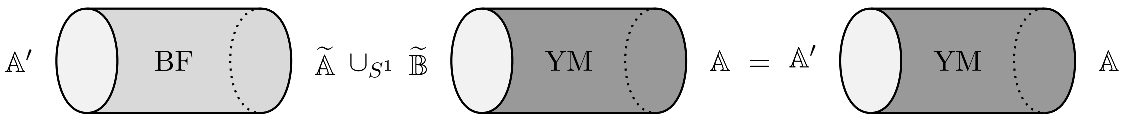
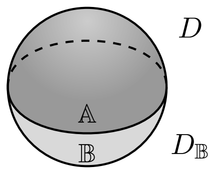


Figure 5.10: YM $\mathbb{A}$-$\mathbb{A}$ cylinder as the gluing of a BF $\mathbb{A}$-$\mathbb{A}$ cylinder with a YM $\mathbb{A}$-$\mathbb{B}$ cylinder.



Using the *globalized* partition function (5.70) for the $\mathbb{A}$-disk and the *non-globalized* answer (5.64) for the $\mathbb{B}$-disk we get:

$$\begin{aligned} Z_{\mathrm{YM}}^{S^2}[\mathsf{a},\mathsf{b}] &= \rho_{\mathcal{V}} \int d\mathbb{A}\, d\mathbb{B}\; e^{-\frac{i}{\hbar}\int_{S^1}\langle \mathbb{B},\mathbb{A}-\mathsf{a}\rangle + \frac{i}{2\hbar}\int_{D_{\mathbb{B}}}\langle \mathsf{b},[\mathsf{a},\mathsf{a}]\rangle}\; Z_{\mathrm{YM}}^{D}[\mathbb{A}] \\ &= \rho_{\mathcal{V}} \cdot e^{\frac{i}{2\hbar}\int_{D_{\mathbb{B}}}\langle \mathsf{b},[\mathsf{a},\mathsf{a}]\rangle}\; Z_{\mathrm{YM}}^{D}[U(\mathbb{A}) = \mathbb{I}] \\ &= \rho_{\mathcal{V}} \cdot e^{\frac{i}{2\hbar}\int\langle \mathsf{b},[\mathsf{a},\mathsf{a}]\rangle} \sum_R (\dim R)^2\, e^{-\frac{i\hbar a}{2}C_2(R)}\;. \end{aligned} \tag{5.74}$$

Here $\rho_{\mathcal{V}} = (-i\hbar)^{\dim\mathfrak{g}}\, D^{\frac{1}{2}}\mathsf{a}\; D^{\frac{1}{2}}\mathsf{b}$ is the reference half-density on zero-modes. We immediately notice that this *non-globalized* answer consists of the product of a function of the zero-modes times the non-perturbative Migdal-Witten partition function for the sphere. Moreover, the partition function (5.74) *does not* produce well-defined global answers by integrating out the zero modes.

Similarly, trying to calculate the globalized partition function for the torus by gluing a YM cylinder in $\mathbb{A}$-$\mathbb{A}$ polarization (5.73) with a cylinder in $\mathbb{B}$-$\mathbb{B}$ polarization (5.48), one obtains an ill-defined answer.

Remark 5.3.14. We remark that the form of the perturbative answer here – as the non-perturbative (number-valued) answer times the exponential of a cubic term in zero-modes – is similar to the form of the perturbative result for Chern-Simons theory in BV formalism on a rational homology 3-sphere [21]:

$$Z_{\mathrm{CS}} = \rho_{\mathcal{V}} \cdot e^{\frac{i}{2\hbar}\langle \mathsf{a}^{(3)},[\mathsf{a}^{(0)},\mathsf{a}^{(0)}]\rangle} \cdot e^{\frac{i}{\hbar}\zeta(\hbar)}\;.$$

Here $\zeta(\hbar)$ is the sum of contributions of connected 3-valent graphs without leaves.

Remark 5.3.15. The gluing construction of Section 5.4 (gluing along edges rather than circles) produces a well-defined globalized answer for all surfaces – including the cylinder, the sphere and the torus – coinciding with the non-perturbative answer in case of surfaces with boundary in $\mathbb{A}$-polarization (5.2). In particular, the gluing construction of Section 5.4 produces the answer for the sphere as in (5.74) but without the zero-mode factor. This discrepancy is due to inequivalence of gauge-fixings used in the two approaches.

5.4 2D Yang-Mills for general surfaces with boundaries and corners

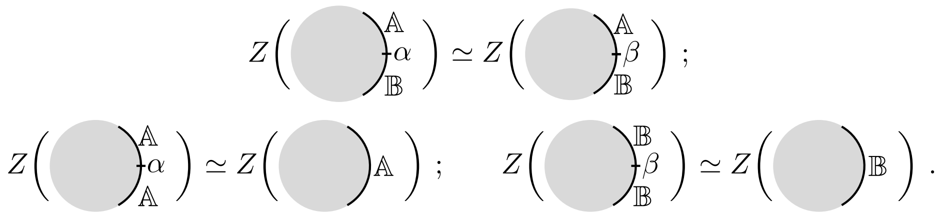
To be able to compute the partition function of 2D YM for general surfaces we need to also consider corners, i.e. codimension 2 strata – marked points on the boundary. In topology surfaces can be described as collections of polygons modulo an equivalence relation which identifies pairs of edges. The idea is to transport this description to the level of field theory: if we can compute the partition function on polygons with arbitrary combinations of polarizations associated to the edges, then we can recover the partition function on surfaces with boundary by gluing pairs of edges with transversal polarizations.

In this section we will formulate a set of rules for corners dictated by the logic of the path-integral and find a set of building blocks that generates under gluing 2D YM on all manifolds with boundaries and corners. We will then discuss the mQME in presence of corners and compute the partition function of the various building blocks. Finally, we will use the results of this analysis to prove a gluing formula in presence of corners and compute the 2D YM partition function on a generic surface with boundary, recovering the well known non-perturbative solution.

5.4.1 Corners and building blocks for 2D YM

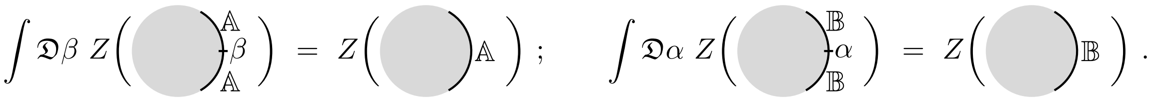
The partition function is an element of the space of boundary states, which are defined by the data of a choice of polarization on the boundary; this choice reflects on the (fluctuations of the) bulk fields by imposing boundary conditions. In the presence of corners dividing two arcs with different polarizations, we have to consider mixed boundary conditions for the bulk fields. More generally we can associate a polarization also to corners, inducing boundary conditions for all adjacent bulk or boundary fields. In this case, corners can be considered as collapsed arcs, with associated polarization the same as the corner they represent, but carrying only some of the boundary fields, namely the ones pulled back from the corner (i.e. constant zero-forms).

Notice that the presence of a corner with the same polarization as one of the adjacent edges has no effect on the partition function (but could require modifications of Ω: cf. section 5.4.2). For example taking a corner with the same polarization of both adjacent edges simply means that we are formally splitting the boundary field into two concatenating fields, but this doesn't change the boundary conditions for the bulk fields:



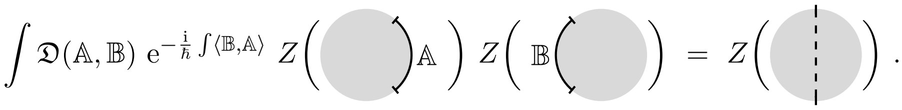
(5.75)

Moreover, by "freeing" the bulk fields from the boundary conditions imposed by a corner, i.e., upon integrating over all possible values of corner fields, the partition function of the surface without that corner is recovered:



(5.76)

The gluing of two arcs is analogous to the case without corners, with the only additional requirement that the fields of the corners that will be identified by the gluing have to coincide:

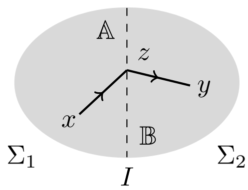


(5.77)

All these statements will be tested with explicit computations in the following sections.

Remark 5.4.1. Let the surface Σ be the result of gluing of surfaces Σ_1 and Σ_2 along an interval I, as above. Assume that the partition functions for Σ_1, Σ_2 are computed perturbatively, using the propagators η_1, η_2. Then the gluing formula (5.77) above yields the partition function for the glued surface Σ computed using the "glued propagator" $\eta = \eta_1 * \eta_2$ on Σ, constructed as follows:

- For $x, y \in \Sigma_1$, $\eta(x, y) = \eta_1(x, y)$.
- For $x, y \in \Sigma_2$, $\eta(x, y) = \eta_2(x, y)$.
- For $x \in \Sigma_2$, $y \in \Sigma_1$, $\eta(x, y) = 0$.
- For $x \in \Sigma_1$, $y \in \Sigma_2$, we have:



$$\eta(x, y) = \int_{I \ni z} \eta_1(x, z)\eta_2(z, y) \ . \tag{5.78}$$

This is precisely the gluing construction for propagators from [26], which turns out to work also in the setting with corners.

Assuming this set of rules for the corners, we have the following set of building blocks for 2D YM, as illustrated in figure 5.11. The disk in the $\mathbb{A}$ polarization was already computed in section 5.3.5 and, using equation (5.75), it is equivalent to a polygon with an arbitrary number of edges where all the edges and the corners are in $\mathbb{A}$-polarization. To change polarization of one of its edges, we can glue to it the BF disk with two corners in the α polarization and two edges in $\mathbb{B}$ polarization. The last BF disk of figure 5.11, with only one $\mathbb{A}$-edge and one corner in the opposite polarization, can be then used in combination with the other building blocks to change the polarization of one corner (figure 5.12). In this way we can obtain a polygon with any number of edges and with any combination of polarizations associated to edges and corners; thus we can also obtain the partition function for any given surface with boundary (and corners).

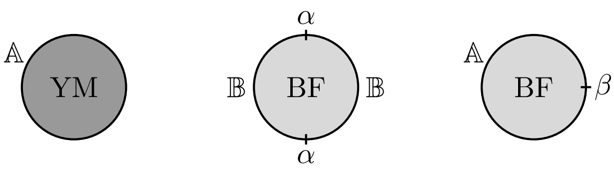


Figure 5.11: Building blocks for 2D YM with corners.

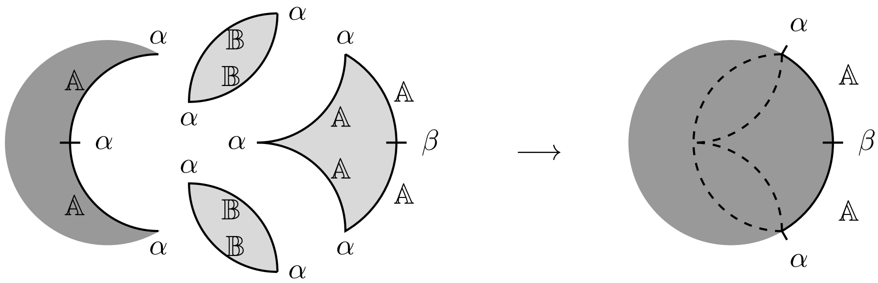


Figure 5.12: The polarization on a corner can be changed by gluing. In this picture it is illustrated how to convert a corner in α polarization to a corner in β polarization using the building blocks of figure 5.11.

5.4.2 Corners, spaces of states and the modified quantum master equation

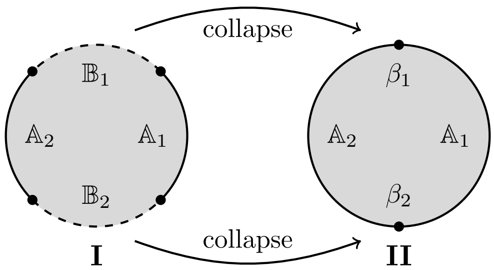
We have two pictures for surfaces with boundary and corners.

I. (**Non-polarized corners.**) Boundary circles are split into intervals by vertices (corners). Each interval carries a polarization $\mathbb{A}$ or $\mathbb{B}$, corresponding to imposing the boundary condition on the pullback to the interval of the bulk field A or B. Corners do not carry a polarization.

II. (**Polarized corners.**) In addition to the intervals carrying a polarization $\mathbb{A}$ or $\mathbb{B}$, each corner is also equipped with a polarization α or β corresponding to prescribing the pullback of either A or B field to the corner.

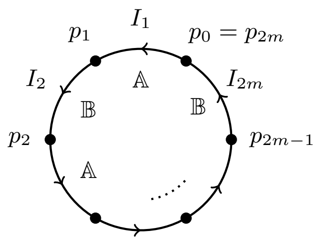
Picture II is our main framework in this thesis. One can transition from picture I to picture II by *collapsing* every other arc on a circle (assuming that initially the number of arcs was even) into a vertex with the corresponding polarization, by the rule $\mathbb{A} \to \alpha$, $\mathbb{B} \to \beta$. One obtains the partition function Z_{II} in the picture II by evaluating the partition function Z_{I} of picture I on constant 0-form fields on the arcs that are being collapsed – pullbacks of the corner fields to the arc. E.g., for a disk with the boundary split into 4 arcs of alternating polarizations in picture I, collapsing the $\mathbb{B}$-arcs into β-corners corresponds to the following:

$$Z_{\mathrm{II}}(\mathbb{A}_1, \beta_1, \mathbb{A}_2, \beta_2;\ \text{zero-modes}) = \quad Z_{\mathrm{I}}(\mathbb{A}_1, \mathbb{B}_1 = \beta_1, \mathbb{A}_2, \mathbb{B}_2 = \beta_2;\ \text{zero-modes})\ . \tag{5.79}$$



Picture I: non-polarized corners. Modified quantum master equation

Consider a circle (thought of as a boundary component of a surface Σ) split by n points $p_1, p_2, \ldots, p_{2m} = p_0$ ("corners") into intervals $I_1, I_2, \ldots, I_{2m}$ with $I_k = [p_{k-1}, p_k]$.



Assume that we fix the $\mathbb{A}$-polarization on the intervals I_k with k odd and the $\mathbb{B}$-polarization for k even. We understand that we can, by a tautological transformation, further subdivide each $\mathbb{A}$- or $\mathbb{B}$-interval into several intervals carrying the same polarization. No polarization data is assigned to the corners p_k (this is our "picture I" for corners).

The BFV space of states $\mathcal{H}$, associated to the circle with such a stratification and a choice of polarizations, is the space of complex-valued functions of the fields on the intervals:

$$\mathcal{H} = \Big\{ \text{functions } \Psi(\mathbb{A}|_{I_1}, \mathbb{B}|_{I_2}, \ldots, \mathbb{B}|_{I_{2m}}) \Big\}\ . \tag{5.80}$$

The space of states is equipped with the BFV operator (which with an appropriate refinement becomes a differential, see Remark 5.4.3 below)

$$\Omega = \underbrace{\sum_{k\ \text{odd}} \Omega^{\mathbb{A}}_{I_k} + \sum_{k\ \text{even}} \Omega^{\mathbb{B}}_{I_k}}_{\text{edge contributions}} + \underbrace{\sum_{k\ \text{odd}} \Omega^{\mathbb{AB}}_{p_k} + \sum_{k\ \text{even}} \Omega^{\mathbb{BA}}_{p_k}}_{\text{corner contributions}} \ . \tag{5.81}$$

Here the edge contributions from the intervals, depending on the polarization, are:

$$\Omega^{\mathbb{A}}_I = \mathrm{i}\hbar \int_I \left\langle \mathrm{d}\mathbb{A} + \frac{1}{2}[\mathbb{A},\mathbb{A}], \frac{\delta}{\delta\mathbb{A}} \right\rangle , \tag{5.82}$$

$$\Omega^{\mathbb{B}}_I = \int_I \mathrm{i}\hbar \left\langle \mathrm{d}\mathbb{B}, \frac{\delta}{\delta\mathbb{B}} \right\rangle + (\mathrm{i}\hbar)^2 \left\langle \mathbb{B}, \frac{1}{2}\left[\frac{\delta}{\delta\mathbb{B}}, \frac{\delta}{\delta\mathbb{B}}\right] \right\rangle . \tag{5.83}$$

The corner contributions from the vertices p_k are the multiplication operators by the product of the limiting values of the $\mathbb{A}$-field and the $\mathbb{B}$-field coming from the incident arcs, with a sign depending on the order of the arcs relative to the orientation:

$$\Omega^{\mathbb{AB}}_p = -\langle \mathbb{B}_p, \mathbb{A}_p \rangle \ , \qquad \Omega^{\mathbb{BA}}_p = \langle \mathbb{B}_p, \mathbb{A}_p \rangle \ . \tag{5.84}$$

These corner contributions to the boundary BFV operator Ω and their necessity for the modified quantum master equation were observed by Alberto S. Cattaneo [16].

The following is a refinement of Lemma 4.11 in [26] for a surface with boundary, with non-polarized corners allowed, in the case of 2D Yang-Mills theory.

Proposition 5.4.2 (mQME in picture I)**.** The BV-BFV partition function Z of 2D Yang-Mills theory on a surface with boundary consisting of stratified circles decorated with a choice of $\mathbb{A}, \mathbb{B}$ polarizations on the codimension 1 strata (and no polarization data on codimension 2 strata) satisfies the mQME

$$(\hbar^2\Delta + \Omega) Z = 0 \ , \tag{5.85}$$

where Ω is the sum of expressions (5.81) for the stratified boundary circles.

Sketch of proof. The proof follows the proof of Lemma 4.11 in [26] where we need to take care of collapses of point near a corner. Let Γ be a Feynman graph for the partition function; its contribution to Z is $\int_{C_\Gamma} \omega_\Gamma$: the integral over the configuration space C_Γ – where vertices of Γ are restricted to the respective strata of Σ (bulk, boundary arcs or corners) – of ω_Γ, the differential form on C_Γ, which is the product of propagators, boundary fields and zero-modes, as prescribed by the combinatorics of Γ.[28] One considers the Stokes' theorem for configuration space integrals:

$$\mathrm{i}\hbar \sum_\Gamma \int_{C_\Gamma} \mathrm{d}\omega_\Gamma = \mathrm{i}\hbar \sum_\Gamma \int_{\partial C_\Gamma} \omega_\Gamma \ . \tag{5.86}$$

[28] A tacit assumption in this proof is that the propagator is a *smooth* 1-form on the configuration space of two points. E.g, the "metric propagator" arising from Hodge theory satisfies this property. Singular propagators considered in this thesis arise as limits of such smooth propagators.

On the left hand side, the terms with d acting on the propagators assemble into $\hbar^2\Delta Z$ and the terms with d acting on $\mathbb{A},\mathbb{B}$ fields assemble into $\Omega_0 Z$ where $\Omega_0 = \mathrm{i}\hbar\int_{\partial_{\mathbb{A}}\Sigma}\langle \mathrm{d}\mathbb{A}\,,\,\frac{\delta}{\delta\mathbb{A}}\rangle + \mathrm{i}\hbar\int_{\partial_{\mathbb{B}}\Sigma}\langle \mathrm{d}\mathbb{B}\,,\,\frac{\delta}{\delta\mathbb{B}}\rangle$. Here $\partial_{\mathbb{A}}\Sigma$ and $\partial_{\mathbb{B}}\Sigma$ are the parts of the boundary equipped with polarizations $\mathbb{A}$ and $\mathbb{B}$, respectively. Thus, the l.h.s. of (5.86) is $(\hbar^2\Delta+\Omega_0)Z$. The r.h.s. contains several types of terms, corresponding to types of boundary strata of C_Γ:

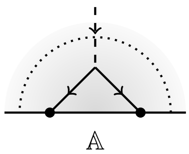
i) Collapses of 2 points in the bulk – cancel out when summed over graphs, due to the classical master equation satisfied by the BV action.

ii) Collapses of ≥ 3 points in the bulk – vanish by the standard vanishing arguments for hidden strata of the configuration spaces [36].

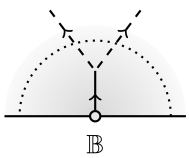
iii) Collapses of one or more points at a point on a boundary arc. These contributions assemble into $-\Omega_1 Z$, where contributions to the differential operator Ω_1 are given by the collapsed subgraphs.

iv) Collapses of several points at a corner – they assemble into $-\Omega_2 Z$.

Thus, one obtains the modified quantum master equation (5.85) with $\Omega = \Omega_0+\Omega_1+\Omega_2$. Analyzing the possible contributing collapses at an arc yields two graphs contributing to Ω_1:

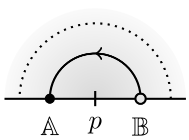


$$\to \mathrm{i}\hbar\int_{\partial_{\mathbb{A}}\Sigma}\left\langle \frac{1}{2}[\mathbb{A},\mathbb{A}]\,,\,\frac{\delta}{\delta\mathbb{A}}\right\rangle ,$$

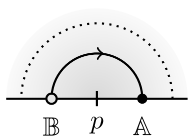


$$\to (\mathrm{i}\hbar)^2\int_{\partial_{\mathbb{B}}\Sigma}\left\langle \mathbb{B}\,,\,\frac{1}{2}\left[\frac{\delta}{\delta\mathbb{B}},\frac{\delta}{\delta\mathbb{B}}\right]\right\rangle . \qquad (5.87)$$

For Ω_2, the only contributing graphs are



$$\to -\langle \mathbb{B}_p,\mathbb{A}_p\rangle ,$$



$$\to \langle \mathbb{B}_p,\mathbb{A}_p\rangle . \qquad (5.88)$$

□

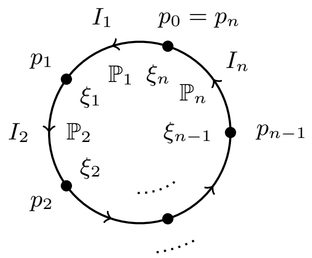
Remark 5.4.3. It was found out in [26] that, in order to have the property $\Omega^2 = 0$ for the BFV operator, generally one should consider a certain refinement of the space of states, allowing the states to depend on the so-called "composite fields" on the boundary, which correspond in Feynman diagrams to boundary vertices of valency ≥ 2.[29] We are

[29] In fact, the operator Ω constructed above (5.81) with edge contributions (5.82,5.83) and corner contributions (5.84) does not satisfy $\Omega^2 = 0$ on the nose, whenever corners are present. In the setting

not considering composite fields in this thesis: below, in Section 5.4.2, we manage to construct Ω for the setting of polarized corners, which squares to zero on the nose, without having to introduce composite fields.

Picture II: polarized corners

Now consider a circle split by n points $p_1, \ldots, p_n = p_0$ ("corners") into intervals $I_1, \ldots, I_n$. Assume that for each k we fix on the interval I_k the polarization $\mathbb{P}_k \in \{\mathbb{A}, \mathbb{B}\}$ – i.e. we prescribe either the the pullback of $\mathbb{A}_k$ field A or the pullback $\mathbb{B}_k$ of the field B on I_k (by an abuse of notations, we denote the differential form $\mathbb{A}_k$ or $\mathbb{B}_k$ also by $\mathbb{P}_k$). Likewise, we fix a polarization $\xi_k \in \{\alpha, \beta\}$ on the corners p_k.



(5.89)

The BFV space of states $\mathcal{H}$, associated to the circle with such a stratification and a choice of polarizations, is the space of complex-valued functions of the fields on the intervals and the corners, subject to the natural corner value conditions:

$$\mathcal{H} = \left\{ \text{functions } \Psi(\mathbb{P}_1, \xi_1, \mathbb{P}_2, \xi_n, \ldots, \mathbb{P}_n, \xi_n) \;\middle|\; \begin{array}{ll} \mathbb{P}_k|_{p_k} = \xi_k & \text{if polarizations } \mathbb{P}_k \text{ and } \xi_k \text{ agree} \\ \mathbb{P}_k|_{p_{k-1}} = \xi_{k-1} & \text{if polarizations } \mathbb{P}_k \text{ and } \xi_{k-1} \text{ agree} \end{array} \right\}. \tag{5.90}$$

Here we say that the polarization of an interval "agrees" with the polarization of the incident corner if this pair of polarizations is either $(\mathbb{A}, \alpha)$ or $(\mathbb{B}, \beta)$. The space of states is a cochain complex with the differential

$$\Omega = \sum_k \underbrace{\Omega^{\mathbb{P}_k}_{I_k}}_{\text{edge contribution from } I_k} + \sum_k \underbrace{\Omega^{\mathbb{P}_k \xi_k \mathbb{P}_{k+1}}_{p_k}}_{\text{corner contribution from } p_k}, \tag{5.91}$$

where the edge contributions are given by (5.82,5.83).

The corner contributions to Ω depend on the polarization ξ_k at the corner and polarizations of the incident edges $\mathbb{P}_k, \mathbb{P}_{k+1}$ and are assembled from the contribution of the corner itself and the contributions of the corner interacting with the incident edges:

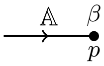
$$\Omega^{\mathbb{P}_k \xi_k \mathbb{P}_{k+1}}_{p_k} = \Omega^{\mathbb{P}_k \xi_k}_{p_k} + \Omega^{\xi_k}_{p_k} + \Omega^{\xi_k \mathbb{P}_{k+1}}_{p_k}. \tag{5.92}$$

of [26] this is remedied by adding corrections to Ω, depending on composite boundary fields. Then in addition to the diagrams (5.88) at a corner one should consider other diagrams, involving boundary vertices of valency ≥ 2.

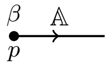
Here the pure corner contributions are:

$$\Omega_p^{\alpha} = \mathrm{i}\hbar\Big\langle \frac{1}{2}\big[\alpha,\alpha\big], \frac{\partial}{\partial\alpha}\Big\rangle \ , \qquad \Omega_p^{\beta} = 0 \ . \tag{5.93}$$

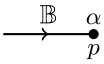
The corner-edge contributions $\Omega_p^{\mathbb{P}\xi}, \Omega_p^{\xi\mathbb{P}}$ vanish if the polarization ξ at the corner matches the polarization $\mathbb{P}$ of the incident edge. For mismatching corner-edge polarizations, we have nontrivial contributions to Ω:



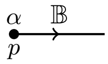
$$\longrightarrow \ \Big\langle \beta, \mathsf{F}_-\Big(\mathrm{ad}_{\mathrm{i}\hbar\frac{\partial}{\partial\beta}}\Big)\mathbb{A}_p\Big\rangle \ ,$$



$$\longrightarrow \ \Big\langle \beta, \mathsf{F}_+\Big(\mathrm{ad}_{\mathrm{i}\hbar\frac{\partial}{\partial\beta}}\Big)\mathbb{A}_p\Big\rangle \ , \tag{5.94}$$



$$\longrightarrow \ \Big\langle \mathbb{B}_p, \mathsf{F}_+\Big(\mathrm{ad}_{\mathrm{i}\hbar\frac{\partial}{\partial\mathbb{B}_p}}\Big)\alpha\Big\rangle \ ,$$



$$\longrightarrow \ \Big\langle \mathbb{B}_p, \mathsf{F}_-\Big(\mathrm{ad}_{\mathrm{i}\hbar\frac{\partial}{\partial\mathbb{B}_p}}\Big)\alpha\Big\rangle \ .$$

Here we have introduced the following functions:

$$\begin{aligned} \mathsf{F}_+(x) &= \frac{x}{1-\mathrm{e}^{-x}} &&= \sum_{j=0}^{\infty}(-1)^j\frac{B_j}{j!}x^j &&= 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \ , \\ \mathsf{F}_-(x) &= \frac{x}{1-\mathrm{e}^{x}} &&= -\sum_{j=0}^{\infty}\frac{B_j}{j!}x^j &&= -1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^4}{720} + \cdots \ , \end{aligned} \tag{5.95}$$

where B_j are the Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, ... In (5.94), functions $\mathsf{F}_\pm$ are evaluated on $x = \mathrm{ad}_{\mathrm{i}\hbar\frac{\partial}{\partial\beta}}$, producing $\mathrm{End}(\mathfrak{g})$-valued derivations (of infinite order) of the space of functions of β. Further in this section we will also need the following two functions, related to the generating functions for Bernoulli polynomials:

$$\mathsf{G}_+(t,x) = \frac{1-\mathrm{e}^{-tx}}{1-\mathrm{e}^{-x}} \ , \qquad \mathsf{G}_-(t,x) = \frac{1-\mathrm{e}^{(1-t)x}}{1-\mathrm{e}^{x}} \ . \tag{5.96}$$

Note that, when acting on the partition function, the complicated operators $\Omega_p^{\mathbb{B}\alpha}, \Omega_p^{\alpha\mathbb{B}}$ from (5.94) act simply as multiplication operators

$$\Omega_p^{\mathbb{B}\alpha} \sim \langle\mathbb{B}_p,\alpha\rangle \ , \qquad \Omega_p^{\alpha\mathbb{B}} \sim -\langle\mathbb{B}_p,\alpha\rangle \ , \tag{5.97}$$

since the derivative in the corner value of the field $\mathbb{B}$ acts by zero. Thus, we have:

$$\Omega_p^{\mathbb{B}\alpha} = \langle\mathbb{B}_p,\alpha\rangle + \cdots \ , \qquad \Omega_p^{\alpha\mathbb{B}} = -\langle\mathbb{B}_p,\alpha\rangle + \cdots \ , \tag{5.98}$$

where we have added the terms $\cdots$ (irrelevant for the master equation) so as to have the property $\Omega^2 = 0$. To be precise, we impose the following mild restriction on the states.

Assumption 5.4.4 (Admissible states)**.** We assume that the states do not depend explicitly on the limiting values of 1-form components of fields $\mathbb{A}, \mathbb{B}$ at corners. I.e., for p a corner, the derivatives $\frac{\partial}{\partial\mathbb{A}_p^{(1)}}, \frac{\partial}{\partial\mathbb{B}_p^{(1)}}$ act by zero on on admissible states.

Then, by a direct computation, one verifies the following (we give the explicit proof in Appendix D.1).

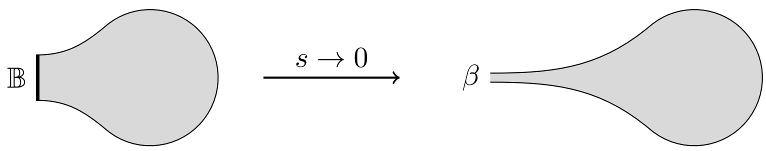
Proposition 5.4.5. For a stratified circle, with any choice of polarizations on the strata, the operator Ω as defined by (5.91,5.93,5.94) satisfies $\Omega^2 = 0$ on admissible states in the sense of Assumption 5.4.4.

Let us introduce the following terminology. For a product of intervals (or circles) $I \times J$ with I parameterized by coordinate t and J parameterized by τ, we call the axial gauge propagator containing $\delta(\tau - \tau')$ *parallel* to I (and *perpendicular* to J), since the intervals on which the δ-term is supported are parallel to I. Note that in all the computations of Section 5.3, the axial gauge was always chosen to be perpendicular to the boundary.

Consider a surface Σ with stratified boundary circles in picture II, as in (5.89), i.e., with arcs and corners carrying polarization data. We can view such a surface as a limit at $s \to 0$ of a family of surfaces Σ_s, with corners of Σ expanded into arcs of corresponding polarization (thus, surfaces Σ_s for $s > 0$ are in picture I). Let η_s be a family of propagators (corresponding to a family of gauge-fixings) on the surfaces Σ_s, converging to a propagator η on Σ. We make the following assumption.

Assumption 5.4.6 (Collapsible gauge condition)**.** The contraction of the propagator η_s with a 1-form $\mathbb{B}^{(1)}$ on the $\mathbb{B}$-interval $I_s \subset \partial\Sigma_s$ that is being collapsed into a β-corner p of Σ, becomes supported at p in the limit $s \to 0$.

This assumption can be realized by considering an s-dependent family of metric gauge-fixings associated to equipping Σ_s with a metric g_s in which the $\mathbb{B}$-arc undergoing the collapse is placed at the end of a long "tentacle". Thus, at $s \to 0$, the β-corner is placed infinitely far from the rest of the surface.



Put another way, if both arcs adjacent to I_s are in $\mathbb{A}$-polarization, the assumption requires that η_s asymptotically approaches the axial gauge propagator $\eta(t,\tau;t',\tau') = (\Theta(t-t')-t) \cdot\delta(\tau-\tau')\,(\mathrm{d}\tau' - \mathrm{d}\tau) - \mathrm{d}t\,\Theta(\tau'-\tau)$ (the axial propagator *parallel* to I_s) near the collapsing interval I_s, as $s \to 0$, with t the coordinate along I_s and τ the coordinate along the "tentacle".

Proposition 5.4.7 (mQME in picture II)**.** Under the assumption above, the partition function Z for the surface Σ with boundary and corners equipped with polarization data, satisfies the modified quantum master equation

$$(\hbar^2\Delta + \Omega)Z = 0\ ,$$

where Ω is given as the sum of expressions (5.91) over the boundary circles, with edge contributions given by (5.82,5.83) and corner contributions defined by (5.92,5.93,5.94).

Here Z is understood as the limit $s \to 0$ of the evaluation of partition function of picture I on Σ_s on the fields pulled back from edges and corners of Σ along the collapse map $\Sigma_s \to \Sigma$. See Remark 5.4.8 below for an explicit example of the mQME with corners in the picture II. Also, in Remark 5.4.14 we will have a non-example showing that the mQME does indeed fail without the Assumption 5.4.6.

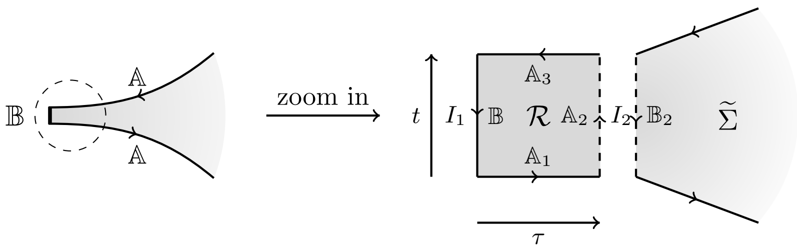
Sketch of proof. Proposition 5.4.7 arises as a corollary of Proposition 5.4.2, since Z is understood as a limit $s \to 0$, in the sense explained above, of partition functions Z_s on surfaces Σ_s in picture I, which do satisfy the modified quantum master equation by Proposition 5.4.2. Contribution $\mathrm{i}\hbar\big\langle \frac{1}{2}\big[\alpha,\alpha\big], \frac{\partial}{\partial\alpha}\big\rangle$ to Ω in picture II at a $\mathbb{B}-\alpha-\mathbb{B}$ corner arises as an $s \to 0$ limit of the $\mathbb{A}$-edge contribution (5.82) from the $\mathbb{A}$-edge of Σ_s collapsing to the corner.

Next, consider an $\mathbb{A}-\beta-\mathbb{A}$ corner where, in addition to Assumption 5.4.6, we assume for the moment that the 0-form component of the $\mathbb{A}$ field is continuous through the corner. In this case, the corner contribution to Ω given by (5.92,5.94) simplifies to $\Omega^{\mathbb{A}\beta\mathbb{A}} = \mathrm{i}\hbar\big\langle[\mathbb{A}_p,\beta], \frac{\partial}{\partial\beta}\big\rangle$ and it arises from the fact that Z_s depends on the 1-form field $\mathbb{B}^{(1)}$ at the edge I_s collapsing into the β-corner p, and this dependence is important for the mQME in picture I. Using the Assumption 5.4.6 and the continuity of $\mathbb{A}^{(0)}$ through the corner, the dependence of Z_s on $\mathbb{B}^{(1)}$ for small s is: $Z_s \sim \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{I_s}\langle\mathbb{B}^{(1)},\mathbb{A}_p\rangle}$, and one has:

$$\Omega^{\mathbb{B}}_{I_s} Z_s \sim \Big(\mathrm{i}\hbar\int_{I_s}\big\langle[\mathbb{A}_p,\mathbb{B}^{(0)}], \frac{\partial}{\partial\mathbb{B}^{(0)}}\big\rangle\Big) Z_s \to \mathrm{i}\hbar\big\langle[\mathbb{A}_p,\beta], \frac{\partial}{\partial\beta}\big\rangle Z\ .$$

Therefore, one can compensate for the loss of dependence on $\mathbb{B}^{(1)}$ during the collapse by inclusion of the term $\mathrm{i}\hbar\big\langle[\mathbb{A}_p,\beta], \frac{\partial}{\partial\beta}\big\rangle$ in Ω.

Finally, consider the $\mathbb{A}-\beta-\mathbb{A}$ corner without assuming the continuity of $\mathbb{A}^{(0)}$ through the corner. To analyze the dependence of Z_s on $\mathbb{B}^{(1)}$, we cut a rectangle $\mathcal{R}$ out of Σ_s at the collapsing edge:



(5.99)

Thus, we present the surface Σ_s as $\mathcal{R} \cup_{I_2} \widetilde{\Sigma}$. Computing the partition function on the rectangle in the axial gauge,[30] setting $\mathbb{A}_1 = \mathbb{A}_{p+0}$, $\mathbb{A}_3 = \mathbb{A}_{p-0}$ – constant zero-forms, the

[30] We are using the axial gauge propagator parallel to I_s, i.e. $\eta(t,\tau\,;t',\tau') = (\Theta(t-t')-t)\,\delta(\tau-\tau')\,(\mathrm{d}\tau'-\mathrm{d}\tau) - \mathrm{d}t\,\Theta(\tau'-\tau)$ with t,τ the vertical and horizontal coordinate on the rectangle. This choice is the one consistent with the Assumption 5.4.6.

limiting values of $\mathbb{A}^{(0)}$ to the right and left of the corner p on Σ, and setting $\mathbb{A}_2 = \mathrm{d}t\,\underline{\mathbb{A}}$ – a constant 1-form, we find the following:

$$Z_{\mathcal{R}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\bar{I}_s} \mathrm{d}t\,\langle \mathbb{B}^{(0)},\underline{\mathbb{A}}\rangle + \langle \mathbb{B}^{(1)},\mathsf{G}_-(t,\mathrm{ad}_{\underline{\mathbb{A}}})\,\mathbb{A}_{p+0} + \mathsf{G}_+(t,\mathrm{ad}_{\underline{\mathbb{A}}})\,\mathbb{A}_{p-0}\rangle}\ , \tag{5.100}$$

with $\mathsf{G}_\pm$ as in (5.96). Here the integral is over $t \in [0,1]$, or equivalently over I_s with reversed orientation. This implies

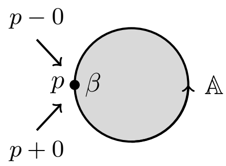
$$(\Omega^{\mathbb{B}}_{I_s} + \langle \beta, \mathbb{A}_{p+0}\rangle - \langle \beta, \mathbb{A}_{p-0}\rangle) Z_{\mathcal{R}}\Big|_{\mathbb{B}=\beta} = -\langle \beta, \mathsf{F}_-(\mathrm{ad}_{\underline{\mathbb{A}}})\,\mathbb{A}_{p+0} + \mathsf{F}_+(\mathrm{ad}_{\underline{\mathbb{A}}})\,\mathbb{A}_{p-0}\rangle\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle \beta, \underline{\mathbb{A}}\rangle}\ . \tag{5.101}$$

The operator acting on $Z_{\mathcal{R}}$ on the left hand side is the part of the Ω in picture I corresponding to the collapsing interval I_s and its two endpoints. Thus, combining with the gluing formula for partition functions we have:

$$\begin{aligned}(\Omega^{\mathbb{B}}_{I_s} + \langle \beta, \mathbb{A}_{p+0}\rangle - \langle \beta, \mathbb{A}_{p-0}\rangle)\ \underbrace{\int \mathrm{d}\underline{\mathbb{A}}\,\mathrm{d}\beta'\ Z_{\mathcal{R}} \cdot \mathrm{e}^{-\frac{i}{\hbar}\langle \beta', \underline{\mathbb{A}}\rangle} \cdot Z_{\widetilde{\Sigma}}(\beta', \cdots)}_{Z_{\Sigma_s}} = \\ = \langle \beta, \mathsf{F}_+(\mathrm{ad}_{i\hbar\frac{\partial}{\partial\beta}})\,\mathbb{A}_{p+0} + \mathsf{F}_-(\mathrm{ad}_{i\hbar\frac{\partial}{\partial\beta}})\,\mathbb{A}_{p-0}\rangle\ Z_{\widetilde{\Sigma}}(\beta, \cdots)\ .\end{aligned} \tag{5.102}$$

Thus, the action on Z_{Σ_s} of the part of Ω in picture I corresponding to the collapsing interval (with its endpoints) is compensated by the action on the partition function in picture II of the operator appearing on the right hand side – which is precisely our anticipated corner contribution in picture II, $\Omega^{\mathbb{A}\beta\mathbb{A}} = \Omega^{\mathbb{A}\beta} + \Omega^{\beta\mathbb{A}}$, see (5.94). □

Remark 5.4.8. Another argument for the contribution to Ω from a β-corner is as follows. In Section 5.4.5 we will obtain the explicit partition function for an $\mathbb{A}$-disk D with a single β-corner



in the form

$$Z_D = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle \beta, \log U(\mathbb{A})\rangle}\ , \tag{5.103}$$

with $U(\mathbb{A})$ the holonomy of the 1-form field $\mathbb{A}^{(1)}$ along the boundary circle. From Baker-Campbell-Hausdorff formula, one finds

$$\Omega^{\mathbb{A}} \log U(\mathbb{A}) = -\mathrm{i}\hbar\,(\mathsf{F}_+(\mathrm{ad}_{\log U(\mathbb{A})})\,\mathbb{A}_{p-0} + \mathsf{F}_-(\mathrm{ad}_{\log U(\mathbb{A})})\,\mathbb{A}_{p+0})\ , \tag{5.104}$$

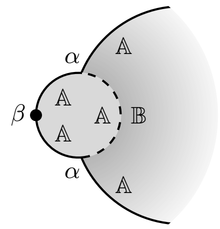
which implies

$$\Omega^{\mathbb{A}} Z_D = \langle \beta, \mathsf{F}_+(\mathrm{ad}_{\log U(\mathbb{A})})\,\mathbb{A}_{p-0} + \mathsf{F}_-(\mathrm{ad}_{\log U(\mathbb{A})})\,\mathbb{A}_{p+0}\rangle \cdot Z_D\ . \tag{5.105}$$

From this one immediately sees that

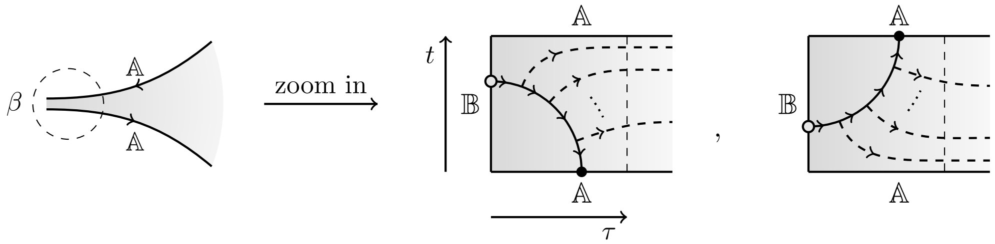
$$(\Omega^{\mathbb{A}} + \Omega_p^{\beta\mathbb{A}} + \Omega_p^{\mathbb{A}\beta}) Z_D = 0 \;, \tag{5.106}$$

with the corner contributions as prescribed by (5.94). Thus, the mQME works by a direct computation. For a general surface Σ containing a β-corner, surrounded by $\mathbb{A}$-edges, one can cut out a disk around the corner and the mQME will follow from the one we just checked for the disk and from the one for the remaining part of the surface (thus by induction one can reduce to the case of surfaces without $\mathbb{A}-\beta-\mathbb{A}$-corners).



(5.107)

Yet another approach to the proof of Proposition 5.4.7, explaining the corner contributions (5.94), is in the vein of the proof of Proposition 5.4.2, with Ω given by Feynman subgraphs collapsing at the boundary/corners. Consider e.g. a collapse at a $\mathbb{A}-\beta-\mathbb{A}$ corner. The following subgraphs are contributing:



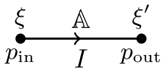
(5.108)

One computes these contributions to Ω using the propagator $\eta = (\Theta(t-t') - t)\delta(\tau - \tau')(\mathrm{d}\tau - \mathrm{d}\tau') - \mathrm{d}t\,\Theta(\tau' - \tau)$ in the rectangle that we see when zooming into the corner. In the zoomed-in picture we are considering configurations of points modulo the horizontal rescalings $\tau \mapsto c\cdot\tau$. We fix a representative of the quotient by fixing the horizontal position of one marked vertex. Edges leaving the collapsing subgraph are assigned the expression $\mathrm{d}t \cdot \mathrm{i}\hbar\frac{\partial}{\partial\beta}$ (the factor $\mathrm{d}t$ comes from the propagator associated to the external edge). The graphs in (5.108) are easily computed and yield, when summed over the number of external edges, $\Omega^{\mathbb{A}\beta\mathbb{A}} = \Omega^{\mathbb{A}\beta} + \Omega^{\beta\mathbb{A}}$ with $\Omega^{\mathbb{A}\beta}, \Omega^{\beta\mathbb{A}}$ given by the formulae (5.94).

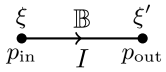
Space of states for the stratified circle as assembled from spaces of states for edges and corners

One can regard the space of states 5.90 for the stratified circle as constructed from the spaces of states for individual edges. One assigns to an interval (with chosen polarization $\mathbb{P}$

in the bulk and ξ,ξ' on the endpoints) a space of states – a cochain complex – constructed as the space of functions on the $\mathbb{P}$-field at the edge and fields at the corners (understood as independent fields if the corner and edge polarizations disagree; if the polarizations agree, the corner field is the limiting value of the edge field):



$$\longrightarrow \mathcal{H}_I^{\xi,\mathbb{A},\xi'} = \mathrm{Fun}_{\mathbb{C}}\left(\left\{\begin{array}{c}\mathfrak{g}[1]\\ \mathfrak{g}^*\end{array}\right\} \times_{\mathfrak{g}[1]} \Omega^\bullet(I,\mathfrak{g})[1] \times_{\mathfrak{g}[1]} \left\{\begin{array}{c}\mathfrak{g}[1]\\ \mathfrak{g}^*\end{array}\right\}\right) ,$$
$$\Omega_I^{\xi,\mathbb{A},\xi'} = \Omega^{\xi}_{\mathrm{in}} + \Omega^{\xi\mathbb{A}}_{\mathrm{in}} + \Omega^{\mathbb{A}}_I + \Omega^{\mathbb{A}\xi'}_{\mathrm{out}} + \Omega^{\xi'}_{\mathrm{out}} ,$$



$$\longrightarrow \mathcal{H}_I^{\xi,\mathbb{B},\xi'} = \mathrm{Fun}_{\mathbb{C}}\left(\left\{\begin{array}{c}\mathfrak{g}[1]\\ \mathfrak{g}^*\end{array}\right\} \times_{\mathfrak{g}^*} \Omega^\bullet(I,\mathfrak{g}^*) \times_{\mathfrak{g}^*} \left\{\begin{array}{c}\mathfrak{g}[1]\\ \mathfrak{g}^*\end{array}\right\}\right) ,$$
$$\Omega_I^{\xi,\mathbb{A},\xi'} = \Omega^{\xi}_{\mathrm{in}} + \Omega^{\xi\mathbb{B}}_{\mathrm{in}} + \Omega^{\mathbb{B}}_I + \Omega^{\mathbb{B}\xi'}_{\mathrm{out}} + \Omega^{\xi'}_{\mathrm{out}} . \tag{5.109}$$

Here top/bottom choice for the fiber product factors on the left/right corresponds to α or β polarization on the left/right endpoint. Note that the polarizations of the endpoints affect the BFV differential, which is given by the edge term defined by (5.82,5.83) plus the two endpoint-edge terms defined by (5.94), plus two pure endpoint terms defined by (5.93).

One can also assign a space of states to a corner p in α- or β-polarization as follows:

$$\begin{aligned} \mathcal{H}_p^{\alpha} &= \mathrm{Fun}_{\mathbb{C}}(\mathfrak{g}[1]) = \mathbb{C}\otimes\wedge^\bullet\mathfrak{g}^* , \qquad & \Omega_p^{\alpha} &= \frac{\mathrm{i}\hbar}{2}\left\langle[\alpha,\alpha],\frac{\partial}{\partial\alpha}\right\rangle , \\ \mathcal{H}_p^{\beta} &= \mathrm{Fun}_{\mathbb{C}}(\mathfrak{g}^*) = \mathbb{C}\otimes S^\bullet\mathfrak{g} , \qquad & \Omega_p^{\beta} &= 0 . \end{aligned} \tag{5.110}$$

Note that, as a cochain complex, $\mathcal{H}_I^{\alpha,\mathbb{A},\alpha}$ is quasi-isomorphic to $\mathcal{H}_p^{\alpha}$ – the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$. Geometrically, this corresponds to the collapse of an $\mathbb{A}$-interval with endpoints in α-polarization into a single α-point. Likewise, the cochain complex $\mathcal{H}_I^{\beta,\mathbb{B},\beta}$ is quasi-isomorphic to $\mathcal{H}_p^{\beta}$:

$$\mathcal{H}_I^{\alpha,\mathbb{A},\alpha} \rightsquigarrow \mathcal{H}_p^{\alpha} , \qquad \mathcal{H}_I^{\beta,\mathbb{B},\beta} \rightsquigarrow \mathcal{H}_p^{\beta} . \tag{5.111}$$

One can regard $\mathcal{H}_p^{\alpha}$ and $\mathcal{H}_p^{\beta}$ as differential graded algebras. The algebra structure on $\mathcal{H}_p^{\alpha}$ is the standard supercommutative multiplication in the exterior algebra, while for $\mathcal{H}_p^{\beta}$ we need to deform the naïve commutative product in the symmetric algebra into a star-product $*_\hbar$ – the deformation quantization of the Kirillov-Kostant-Souriaux Poisson structure on $\mathfrak{g}^*$, as we explain below.

One can regard the space of states for the interval as a bimodule over the spaces of states associated to the end-points. The action of the end-point algebra $\mathcal{H}_p^{\xi}$ on the space of states $\mathcal{H}_I^{\xi,\mathbb{P},\xi'}$ for the edge is via multiplication in the algebra, e.g. $\psi(\alpha)\otimes\Psi(\alpha,\mathbb{P},\xi')\mapsto \psi(\alpha)\Psi(\alpha,\mathbb{P},\xi')$, $\psi(\beta)\otimes\Psi(\beta,\mathbb{P},\xi')\mapsto\psi(\beta)*_\hbar\Psi(\beta,\mathbb{P},\xi')$. The reason we need to deform the product in $\mathcal{H}^{\beta}$ from the commutative one is that we want the edge to give a *differential graded* bimodule over the corner spaces. In particular, the module structure map $\mathcal{H}_p^{\beta}\otimes\mathcal{H}_I^{\beta,\mathbb{A},\xi'}\to\mathcal{H}_I^{\beta,\mathbb{A},\xi'}$ should be a chain map with respect to the differential $\Omega_p^{\beta\mathbb{A}}+\Omega_I^{\mathbb{A}}+\Omega_{p'}^{\mathbb{A}\xi'}$.

This requirement is incompatible with the commutative product on $\mathcal{H}_p^\beta$ and forces the following associative non-commutative deformation $*_\hbar : \mathcal{H}^\beta \otimes \mathcal{H}^\beta \to \mathcal{H}^\beta$.

Proposition 5.4.9. The associative product structure $*_\hbar$ on $\mathcal{H}^\beta$ is fixed uniquely by the two properties:

i) The module structure map $m : \mathcal{H}_p^\beta \otimes \mathcal{H}_I^{\beta,\mathbb{A},\xi'} \to \mathcal{H}_I^{\beta,\mathbb{A},\xi'}$, obtained by extending $*_\hbar$ by linearity in the second factor, is a chain map.

ii) $*_\hbar$ is unital with $\psi(\beta) = 1$ the unit.

The product $*_\hbar$ is explicitly described as follows:

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,x\rangle} *_\hbar \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,y\rangle} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\mathrm{BCH}(x,y)\rangle} \tag{5.112}$$

Here $x, y \in \mathfrak{g}$ are arbitrary parameters in the Lie algebra and $\mathrm{BCH}(x,y) = \log(\mathrm{e}^x\mathrm{e}^y)$ is the Baker-Campbell-Hausdorff group law.

Proof. Let us check that the star-product (5.112) does indeed make the module structure map a chain map. Note that, for $\psi(\beta) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,x\rangle}$, the action of $\Omega_p^{\beta\mathbb{A}}$ on ψ can be written as

$$\Omega_p^{\beta\mathbb{A}}\psi = \langle\beta, \mathsf{F}_+(\mathrm{ad}_x)\mathbb{A}_p\rangle\,\psi = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\mathrm{BCH}(x,\epsilon\,\mathbb{A}_p)\rangle} = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\psi *_\hbar \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\epsilon\,\mathbb{A}_p\rangle}\ , \tag{5.113}$$

with ϵ an odd, ghost degree -1 parameter and $*_\hbar$ defined by (5.112). Here we have used the identity $\mathrm{BCH}(x,y) = x + \mathsf{F}_+(\mathrm{ad}_x)y + \mathcal{O}(y^2)$. This implies that for any $\Psi \in \mathcal{H}_I^{\beta,\mathbb{A},\xi'}$ we have:

$$\Omega_p^{\beta\mathbb{A}}\Psi = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\Psi *_\hbar \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\epsilon\,\mathbb{A}_p\rangle}\ . \tag{5.114}$$

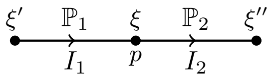
Therefore, for any $\widetilde{\psi} \in \mathcal{H}_p^\beta$ we have:

$$\begin{aligned} m\circ(\mathrm{id}\otimes\Omega^{\beta\mathbb{A}})(\widetilde{\psi}\otimes\Psi) &= \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\widetilde{\psi} *_\hbar \left(\Psi *_\hbar \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\epsilon\,\mathbb{A}_p\rangle}\right) \\ &= \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\left(\widetilde{\psi} *_\hbar \Psi\right) *_\hbar \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\epsilon\,\mathbb{A}_p\rangle} = \Omega^{\beta\mathbb{A}}\circ m(\widetilde{\psi}\otimes\Psi)\ . \end{aligned} \tag{5.115}$$

Here we used the associativity of the star-product (5.112). Note that the other pieces of the differential, $\Omega_I^{\mathbb{A}}$ and $\Omega_{p'}^{\xi'}$, clearly commute with the module structure map m. Thus we have proven that m, defined by (5.112) and extended by $\mathrm{Fun}(\mathbb{A},\xi')$-linearity in the second factor, is indeed a chain map.

Moreover, assume that $\bullet$ is some unital associative product on $\mathcal{H}^\beta$ with $\psi(\beta) = 1$ the unit. Then the argument above shows that the module structure map m defined using $\bullet$ is a chain map if and only if $\psi_1 \bullet (\psi_2 *_\hbar \psi_3) = (\psi_1 \bullet \psi_2) *_\hbar \psi_3$ for any $\psi_{1,2,3} \in \mathcal{H}^\beta$. Choosing $\psi_2 = 1$, we obtain $\psi_1 \bullet \psi_3 = \psi_1 *_\hbar \psi_3$. This proves uniqueness of the star-product (5.112). □

Gluing two intervals over a point corresponds to taking the tensor product of the spaces of states for the intervals over the algebra associated to the point:[31]



$$\longrightarrow \quad \mathcal{H} = \mathcal{H}_{I_1}^{\xi',\mathbb{P}_1,\xi} \otimes_{\mathcal{H}_p^{\xi}} \mathcal{H}_{I_2}^{\xi,\mathbb{P}_2,\xi''} \ . \tag{5.116}$$

The space of states (5.90) for the stratified circle can then be written, in terms of the spaces of states for intervals and corners introduced above, as:

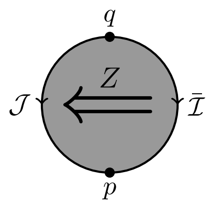
$$\mathcal{H} = \left(\mathcal{H}_{I_1}^{\xi_n,\mathbb{P}_1,\xi_1} \otimes_{\mathcal{H}_{p_1}^{\xi_1}} \mathcal{H}_{I_2}^{\xi_1,\mathbb{P}_2,\xi_2} \cdots \otimes_{\mathcal{H}_{p_{n-1}}^{\xi_{n-1}}} \mathcal{H}_{I_n}^{\xi_{n-1},\mathbb{P}_n,\xi_n}\right) \otimes_{\mathcal{H}_{p_n}^{\xi_n} \otimes \left(\mathcal{H}_{p_n}^{\xi_n}\right)^{\mathrm{op}}} \mathcal{H}_{p_n}^{\xi_n} \ . \tag{5.117}$$

Here the superscript op stands for the opposite algebra.

Remark 5.4.10. Denote $\mathcal{I} = \cup_{k=1}^{l} I_k$ be the union of l consecutive intervals on the stratified circle (5.89) and $\mathcal{J} = \cup_{k=l+1}^{n} I_k$ the union of the remaining intervals, and let $p = p_0$, $q = p_l$ be the points separating $\mathcal{I}$ and $\mathcal{J}$. The globalized partition function Z for the disk D filling the stratified circle is, by the mQME, an Ω-closed element of the space of states

$$\mathcal{H}_{S^1} = \mathcal{H}_{\mathcal{I}} \otimes_{\mathcal{H}_q} \mathcal{H}_{\mathcal{J}} \otimes_{\mathcal{H}_p \otimes \mathcal{H}_p^{\mathrm{op}}} \mathcal{H}_p \cong \mathrm{Hom}_{(\mathcal{H}_q,\mathcal{H}_p)-\mathrm{bimod}}(\mathcal{H}_{\bar{\mathcal{I}}}, \mathcal{H}_{\mathcal{J}}) \ . \tag{5.118}$$

Here on the right hand side we have the space of morphisms of dg bimodules over $\mathcal{H}_q$ on the left and $\mathcal{H}_p$ on the right; bar on $\bar{\mathcal{I}}$ stands for orientation reversal. Thus, the partition function for a disk can be seen as a bimodule morphism between two bimodules associated to the two arcs constituting the boundary.



Note that the picture for 2D Yang-Mills we just described, mapping points to algebras, intervals to bimodules and disks to morphisms of bimodules, is in agreement with Baez-Dolan-Lurie setting of extended topological quantum field theory [4, 39], with the correction that our spaces of states depend on the choice of polarization and partition functions depend on the area of the surface (and pre-globalization partition functions additionally depend on residual fields).

[31] Note that, in dg setting, when taking the tensor product $M_1 \otimes_{\mathcal{A}} M_2$ of a right $\mathcal{A}$-module M_1 and a left $\mathcal{A}$-module M_2 over a dg algebra $\mathcal{A}$, the total differential is the sum of the differentials on M_1, M_2 minus the differential on $\mathcal{A}$.

Towards quantization of codimension 2 corners in more general BV-BFV theories

The algebra $\mathcal{H}^\beta, *_\hbar$ is isomorphic to $U_\hbar(\mathfrak{g})$ – the enveloping algebra of $\mathfrak{g}$ with the normalized Lie bracket $\mathrm{i}\hbar[-,-]$. This algebra arises as Kontsevich's deformation quantization [18, 36] of the algebra of functions on $\mathfrak{g}^*$ equipped with the Kirillov-Kostant-Souriaux linear Poisson structure. This observation fits well into the following expected picture of quantization of corners of codimension 2.

In a general gauge theory, a codimension 2 stratum γ is classically associated a BFV "corner phase space" [23] Φ_γ equipped with a degree $+1$ symplectic form ω_γ and a BFV charge S_γ of degree $+2$. On the level of quantization, we impose a polarization $\Phi_\gamma \simeq T^*[1]\mathcal{B}_\gamma$. The BFV charge S_γ generates a P_∞ (Poisson up-to-homotopy) algebra structure on $C^\infty(\mathcal{B}_\gamma)$, coming from interpreting S_γ as a self-commuting polyvector Π on $\mathcal{B}_\gamma$.[32] Then the quantum space of states $\mathcal{H}_\gamma$ is expected to be the A_∞ algebra obtained as Kontsevich's deformation quantization of the P_∞ algebra $C^\infty(\mathcal{B}_\gamma)$. In particular, the A_∞ structure maps arise from Feynman diagrams on a thickening of γ to $\gamma \times D$, with D a 2-disk, for a field theory coming from the AKSZ construction on the mapping space $\mathrm{Map}(T[1]D, \Phi_\gamma)$. We plan to revisit this construction in more detail in a future work on corners in BV-BFV formalism.

Note that, in the case of 2D Yang-Mills theory, the corner phase space is $\Phi_p = \mathfrak{g}[1] \oplus \mathfrak{g}^*$, with $\omega_p = \langle \mathrm{d}\beta, \mathrm{d}\alpha \rangle$, $S_p = \frac{1}{2}\langle \beta, [\alpha, \alpha] \rangle$. Deformation quantization of $C^\infty(\mathfrak{g}^*)$ with Poisson bivector $\Pi = \frac{1}{2}\langle \beta, [\frac{\partial}{\partial \beta} \overset{\wedge}{,} \frac{\partial}{\partial \beta}] \rangle$ yields the algebra $\mathcal{H}^\beta$. Taking the opposite polarization, one gets the deformation quantization of $C^\infty(\mathfrak{g}[1]) = \wedge \mathfrak{g}^*$ with 1-vector $\Pi = \frac{1}{2}\langle [\alpha, \alpha], \frac{\partial}{\partial \alpha} \rangle$, which is the dg algebra $\mathcal{H}^\alpha$.

In general, one expects all the structure maps on (and between) the spaces of states associated to various strata to come from Feynman diagrams.

A related picture was obtained in [20] in the context of Poisson sigma model on a disk with intervals on the boundary decorated with coisotropic submanifolds C_i of the Poisson target M. In this setting the quantization yields algebras assigned to intervals (deformation quantization of the rings of functions on C_i) and bimodules assigned to the corners separating the intervals. In particular, the algebra $\mathcal{H}^\beta$ arises in this context as a quantization of the space-filling coisotropic in $M = \mathfrak{g}^*$. This picture can be thought of as Poincaré dual to our picture on the boundary of a disk.

Gluing regions along an interval and the Fourier transform property of BFV differentials

Recall that the BFV differentials for an $\mathbb{A}$-circle and a $\mathbb{B}$-circle are related by Fourier transform. This property in particular implies that mQME is compatible with gluing: if

[32] Equivalently, the P_∞ structure arises from S_γ via the derived bracket construction, $\{\psi_1, \ldots, \psi_n\}_\Pi := (\cdots(S_\gamma, \psi_1), \cdots, \psi_n)$ with $\psi_1, \ldots, \psi_n \in C^\infty(\mathcal{B}_\gamma)$. The brackets $(-,-)$ on the r.h.s. are the Poisson brackets on functions on the phase space Φ_γ defined by ω_γ.

$\Sigma = \Sigma_1 \cup_{S^1} \Sigma_2$ a union of surfaces over a circle and if partition functions Z_{Σ_1}, Z_{Σ_2} are known to satisfy mQME, then the glued partition function $Z_\Sigma = \langle Z_{\Sigma_1}, Z_{\Sigma_2} \rangle_{\mathcal{H}_{S^1}}$ automatically satisfies mQME on the glued surface.

One has an analogous property in the setting with corners. Consider e.g. an $\mathbb{A}$-interval I parameterized by $t \in [0,1]$ with endpoints in polarizations ξ, ξ' and consider a $\mathbb{B}$-interval $\widetilde{I}$ parameterized by $\widetilde{t} \in [0,1]$, with endpoints in ξ', ξ. Let $r\colon I \to \widetilde{I}$ be an orientation-reversing diffeomorphism $t \mapsto \widetilde{t} = 1 - t$. Gluing along r corresponds to the following pairing of states on I and $\widetilde{I}$:

$$\begin{array}{cccc} \langle -, - \rangle_I\colon & \mathcal{H}_I^{\xi,\mathbb{A},\xi'} \otimes \mathcal{H}_{\widetilde{I}}^{\xi',\mathbb{B},\xi} & \longrightarrow & \mathbb{C} \\ & \psi_1 \otimes \psi_2 & \mapsto & \displaystyle\int \mathcal{D}\mathbb{A}\,\mathcal{D}\mathbb{B}\; \psi_1(\xi,\mathbb{A},\xi') \cdot \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_I \langle r^*\mathbb{B}, \mathbb{A}\rangle} \cdot \psi_2(\xi',\mathbb{B},\xi) \end{array} \tag{5.119}$$

One easily verifies the following:

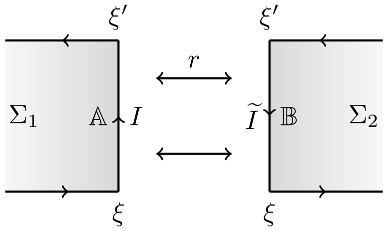
$$\langle (\Omega_{\mathrm{in}}^{\xi\mathbb{A}} + \Omega_I^{\mathbb{A}} + \Omega_{\mathrm{out}}^{\mathbb{A}\xi'})\,\psi_1, \psi_2 \rangle_I = -\langle \psi_1, (\Omega_{\mathrm{in}}^{\xi'\mathbb{B}} + \Omega_{\widetilde{I}}^{\mathbb{B}} + \Omega_{\mathrm{out}}^{\mathbb{B}\xi})\,\psi_2 \rangle_I \,. \tag{5.120}$$

Here we are making the Assumption 5.4.4 on states ψ_1, ψ_2. In other words, the operators $\Omega_{\mathrm{in}}^{\xi\mathbb{A}} + \Omega_I^{\mathbb{A}} + \Omega_{\mathrm{out}}^{\mathbb{A}\xi'}$ and $\Omega_{\mathrm{in}}^{\xi'\mathbb{B}} + \Omega_{\widetilde{I}}^{\mathbb{B}} + \Omega_{\mathrm{out}}^{\mathbb{B}\xi}$ are, up to sign, the Fourier transform of each other (when acting on admissible states).

This immediately implies the following. Assume that Σ is a result of gluing of surfaces Σ_1 and Σ_2 via attaching an interval $I \subset \partial\Sigma_1$ to $\widetilde{I} \subset \partial\Sigma_2$ along the diffeomorphism r. Then for $\Psi_1 \in \mathcal{H}_{\partial\Sigma_1}$, $\Psi_2 \in \mathcal{H}_{\partial\Sigma_2}$ any two states on the boundary of Σ_1, Σ_2, we have

$$\Omega_{\partial\Sigma} \langle \Psi_1, \Psi_2 \rangle_I = \langle \Omega_{\partial\Sigma_1}\Psi_1, \Psi_2 \rangle_I + \langle \Psi_1, \Omega_{\partial\Sigma_2}\Psi_2 \rangle_I \,, \tag{5.121}$$

where $\langle \Psi_1, \Psi_2 \rangle_I \in \mathcal{H}_{\partial\Sigma}$ is understood as the "gluing" of states Ψ_1, Ψ_2 along I.

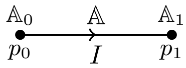


In particular, if the partition functions on Σ_1, Σ_2 are known to satisfy the mQME, the glued partition function $Z_\Sigma = \langle Z_{\Sigma_1}, Z_{\Sigma_2} \rangle_I$ automatically satisfies the mQME on Σ.

Small model for states on an $\mathbb{A}$-interval

In preparation for the calculations of section 5.4.5, we want to present a "small model" for the space of states on an $\mathbb{A}$-interval, corresponding to the passage to a constant 1-form field $\mathbb{A}^{(1)}$ on the interval. This is an extension of the discussion of section 5.3.3 (and in particular, formula (5.56)), and of section 5.2.4.

Consider a single interval in $\mathbb{A}$-polarization:



(5.122)

We view its endpoints as corners in picture I (non-polarized), with $\mathbb{A}_0$, $\mathbb{A}_1$ the limiting values of the 0-form field $\mathbb{A}$ at the endpoints p_0, p_1. Equivalently, we can treat the endpoint in picture II, putting α-polarization on them, with corner fields $\alpha_{0,1}$ identified with $\mathbb{A}_{0,1}$.

The space of states for the interval (5.122) is a cochain complex

$$\mathcal{H} = \mathrm{Fun}_{\mathbb{C}}\big(\Omega^\bullet(I,\mathfrak{g})[1]\big) = \big\{\Psi(\mathbb{A})\big\} \tag{5.123}$$

with differential

$$\Omega = \mathrm{i}\hbar\Big(\int\Big\langle \mathrm{d}\mathbb{A}^{(0)} + [\mathbb{A}^{(0)},\mathbb{A}^{(1)}]\ ,\ \frac{\delta}{\delta\mathbb{A}^{(1)}}\Big\rangle + \int\Big\langle \frac{1}{2}\ [\mathbb{A}^{(0)},\mathbb{A}^{(0)}]\ ,\ \frac{\delta}{\delta\mathbb{A}^{(0)}}\Big\rangle\Big)\ . \tag{5.124}$$

One has the following "small" quasi-isomorphic model for the space of states – the cochain complex

$$\mathcal{H}^{\mathrm{small}} = \mathrm{Fun}_{\mathbb{C}}\Big(C^\bullet(I,\mathfrak{g})[1]\Big) = \Big\{\underline{\Psi}\big(\underline{\mathbb{A}}_0,\, \underline{\mathbb{A}},\, \underline{\mathbb{A}}_1\big)\Big\}\ . \tag{5.125}$$

Here $C^\bullet(I,\mathfrak{g}) = \mathfrak{g}\oplus\mathfrak{g}[-1]\oplus\mathfrak{g}$ is the complex of $\mathfrak{g}$-valued cellular cochains on the interval I endowed with the standard CW complex structure, with two 0-cells p_0, p_1 and a single 1-cell I. Variables $\underline{\mathbb{A}}_0$, $\underline{\mathbb{A}}_1 \in \mathfrak{g}[1]$ are the values of the cochain on the 0-cells p_0 and p_1 (endpoints), respectively, and $\underline{\mathbb{A}} \in \mathfrak{g}$ is the value of the cochain on the 1-cell I itself. The differential on $\mathcal{H}^{\mathrm{small}}$ is given by:

$$\begin{aligned}\Omega^{\mathrm{small}} = \mathrm{i}\hbar\Big(\Big\langle\frac{1}{2}[\underline{\mathbb{A}}_0,\, \underline{\mathbb{A}}_0]\ ,\ \frac{\partial}{\partial\underline{\mathbb{A}}_0}\Big\rangle + \Big\langle\frac{1}{2}[\underline{\mathbb{A}}_1,\, \underline{\mathbb{A}}_1]\ ,\ \frac{\partial}{\partial\underline{\mathbb{A}}_1}\Big\rangle + \\ -\Big\langle \mathsf{F}_-(\mathrm{ad}_{\underline{\mathbb{A}}})\circ\underline{\mathbb{A}}_0 + \mathsf{F}_+(\mathrm{ad}_{\underline{\mathbb{A}}})\circ\underline{\mathbb{A}}_1\ ,\ \frac{\partial}{\partial\underline{\mathbb{A}}}\Big\rangle\Big)\ .\end{aligned} \tag{5.126}$$

The chain projection $p_{\mathcal{H}}\colon \mathcal{H}\to\mathcal{H}^{\mathrm{small}}$ is the following map:

$$\Psi \mapsto \Big(\underline{\Psi}\colon \{\underline{\mathbb{A}}_0,\, \underline{\mathbb{A}},\, \underline{\mathbb{A}}_1\} \mapsto \Psi\Big(\mathbb{A}^{(0)} = \mathsf{G}_-(t,\mathrm{ad}_{\underline{\mathbb{A}}})\circ\underline{\mathbb{A}}_0 + \mathsf{G}_+(t,\mathrm{ad}_{\underline{\mathbb{A}}})\circ\underline{\mathbb{A}}_1\ ,\ \mathbb{A}^{(1)} = \mathrm{d}t\cdot\underline{\mathbb{A}}\Big)\Big)\ . \tag{5.127}$$

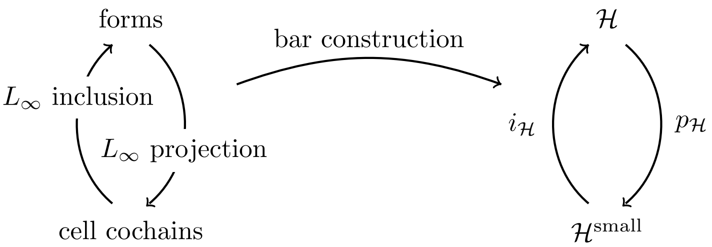
Here we parameterize the interval by the coordinate $t\in[0,1]$ and $\mathsf{G}_\pm$ are the generating functions for Bernoulli polynomials (5.96).

The chain inclusion $i_{\mathcal{H}}\colon \mathcal{H}^{\mathrm{small}}\to\mathcal{H}$ is given as follows:

$$i_{\mathcal{H}}\colon \underline{\Psi} \mapsto \Big(\Psi:\ \ \mathbb{A}\mapsto\underline{\Psi}\Big(\ \underline{\mathbb{A}}_0 = \mathbb{A}_0\ ,\ \underline{\mathbb{A}}_1 = \mathbb{A}_1\ ,\ \underline{\mathbb{A}} = \log U(\mathbb{A})\Big)\Big) \tag{5.128}$$

Here the group element $U(\cdots)\in G$ is the holonomy of the connection 1-form along the interval I.

Remark 5.4.11. The space of states $\mathcal{H}, \Omega$ is the Chevalley-Eilenberg complex (or the dual of the bar complex) of the differential graded Lie algebra of $\mathfrak{g}$-valued differential forms on the interval, $\Omega^\bullet(I, \mathfrak{g}), \mathrm{d}, [-,-]$. Likewise, $\mathcal{H}^{\mathrm{small}}, \Omega^{\mathrm{small}}$ is the Chevalley-Eilenberg complex for the L_∞ algebra structure on $\mathfrak{g}$-valued cellular cochains on an interval, constructed in [41, 42, see also [38, 57]]. This L_∞ algebra arises as the homotopy transfer of the "big" algebra $\Omega^\bullet(I, \mathfrak{g})$ onto the deformation retract $C^\bullet(I, \mathfrak{g})$ – cochains, realized as Whitney forms on the interval. Chain map (5.127) corresponds to the L_∞ morphism from $C^\bullet(I, \mathfrak{g})$ to $\Omega^\bullet(I, \mathfrak{g})$ constructed explicitly in [41, 42, – Statement 14]; it is a non-abelian deformation of the inclusion of cochains as Whitney forms. The map (5.128), constructed via holonomies, corresponds to the L_∞ morphism from forms to cochains - the non-abelian version of the integration-over-cells map, cf. [5]. We give a proof of the chain map property of(5.128) in Appendix D.2.



One has similar small models for the space of states on the $\mathbb{A}$-interval with endpoints in any combination of polarizations ξ_0, ξ_1. E.g. for both endpoints in β-polarization, we have the small model (5.125,5.126) and the maps (5.127,5.128), where we adjoin the corner variables β_0, β_1 on which the wavefunctions Ψ, $\underline{\Psi}$ are allowed to depend, and we add corner-edge terms $\Omega^{\beta\mathbb{A}}$, $\Omega^{\mathbb{A}\beta}$ (5.94) to the respective differentials Ω and Ω^{small}.

Finally, consider a surface Σ with stratified boundary circles decorated with an arbitrary combination of polarizations of arcs and corners. By the discussion above, we have a small quasi-isomorphic model $\mathcal{H}'$ for the space of states $\mathcal{H}$ corresponding to replacing the states on some (or all) $\mathbb{A}$-arcs with respective small models for $\mathbb{A}$-arcs in the formula (5.117), and we have chain maps $p_\mathcal{H} : \mathcal{H} \to \mathcal{H}'$, $i_\mathcal{H} : \mathcal{H}' \to \mathcal{H}$. They correspond to a quasi-isomorphism of complexes and thus there exists a chain homotopy $K_\mathcal{H} : \mathcal{H} \to \mathcal{H}$ between the identity and the projection $i_\mathcal{H} \circ p_\mathcal{H}$. Therefore, we can apply the argument (5.56) to the partition function Z on Σ:

$$i_\mathcal{H} \circ p_\mathcal{H}\ Z = Z + (\Omega + \hbar^2 \Delta)(\cdots)\ . \tag{5.129}$$

In particular, one can recover Z, modulo BV exact terms, by evaluating it on constant 1-forms $\mathrm{d}t \cdot \underline{\mathbb{A}}_k$ on the boundary arcs, provided that their holonomy coincides with the holonomy of the original $\mathbb{A}^{(1)}$ field along the respective intervals, i.e. $\underline{\mathbb{A}}_k = \log U_{I_k}(\mathbb{A})$.

5.4.3 BF $\mathbb{B}$-disk with two α corners

Let us consider now the case of a BF disk with the boundary split into two arcs $\gamma_i\colon [0,1] \to S^1$, $i = 1, 2$, with $\gamma_i(0) = v_0$ and $\gamma_i(1) = v_1$, both in $\mathbb{B}$ polarization. On both vertices of the arcs we fix the value α_i for the restriction of the bulk A fields. Expanding the vertices v_i into two edges in $\mathbb{A}$ polarization we can think of this disk as a square (Figure 5.13).

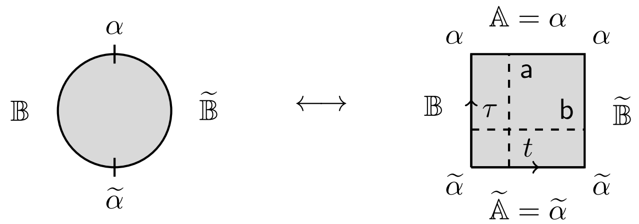


Figure 5.13: $\mathbb{B}$ disk with the boundary split into two arcs separated by points in A-polarization.

The square can be viewed as the product of two intervals, with $\mathbb{A}$ or $\mathbb{B}$ polarization on both endpoints respectively. The zero-modes now contain 1-form components for the a and b fields: $\mathsf{a} = \mathsf{a}_1 \mathrm{d}\tau$, $\mathsf{b} = \mathsf{b}^1 \mathrm{d}t$. A possible choice for axial-gauge propagator is (cf. appendix (B.1)):

$$\eta(t,\tau;t'\tau') = (\Theta(\tau-\tau') - \tau)\mathrm{d}t' - (\Theta(t'-t) - t')\delta(\tau'-\tau)(\mathrm{d}\tau' - \mathrm{d}\tau) \ . \tag{5.130}$$

The contributing Feynman diagrams to the effective action are wheels with n a zero-modes and trees, rooted either on the $\mathbb{B}_{(1)}$ boundary field or on the b zero-mode and ending on one $\mathbb{A}_{(0)}$ boundary field, with no bifurcations and the insertion of n leafs decorated with a zero-modes (Figure 5.14).

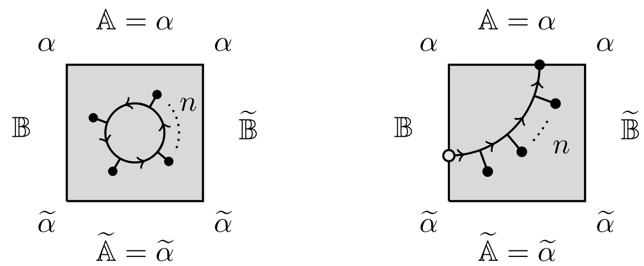


Figure 5.14: Examples of the tree and 1-loop Feynman diagrams contributing to the effective action for the BF disk in $\mathbb{B}$ polarization with two α corners.

Proposition 5.4.12. The partition function for the BF disk in $\mathbb{B}$ polarization with two

α corners is:

$$Z = \exp\Big(\frac{\mathrm{i}}{\hbar}\int_{\gamma}\langle \mathbb{B}, \mathsf{a} + \mathsf{G}_{+}(\tau, \mathrm{ad}_{\mathsf{a}_1})\alpha + \mathsf{G}_{-}(\tau, \mathrm{ad}_{\mathsf{a}_1})\widetilde{\alpha}\rangle + \\ -\frac{\mathrm{i}}{\hbar}\int_{\widetilde{\gamma}}\langle \widetilde{\mathbb{B}}, \mathsf{a} + \mathsf{G}_{+}(\tau, \mathrm{ad}_{\mathsf{a}_1})\alpha + \mathsf{G}_{-}(\tau, \mathrm{ad}_{\mathsf{a}_1})\widetilde{\alpha}\rangle + \\ +\frac{\mathrm{i}}{\hbar}\langle \mathsf{b}^1, \mathsf{F}_{+}(\mathrm{ad}_{\mathsf{a}_1})\alpha + \mathsf{F}_{-}(\mathrm{ad}_{\mathsf{a}_1})\widetilde{\alpha}\rangle\Big) \det\left(\frac{\sinh\left(\mathrm{ad}_{\mathsf{a}_1}/2\right)}{\mathrm{ad}_{\mathsf{a}_1}/2}\right)\cdot \rho_{\mathcal{V}} \ , \tag{5.131}$$

where $\rho_{\mathcal{V}} = D^{\frac{1}{2}}\mathsf{a}\ D^{\frac{1}{2}}\mathsf{b}$ is the reference half-density on zero-modes.

Proof. See Appendix C. □

5.4.4 BF $\mathbb{B}$-disk with one α corner

Let us consider a disk in $\mathbb{B}$-polarization with a single corner in α-polarization. We will denote by α the value of the zero-form component of the A-fields on the corner. Notice that the space of zero-modes is now empty, in contrast to the $\mathbb{B}$-disk without corners or with two α corners.

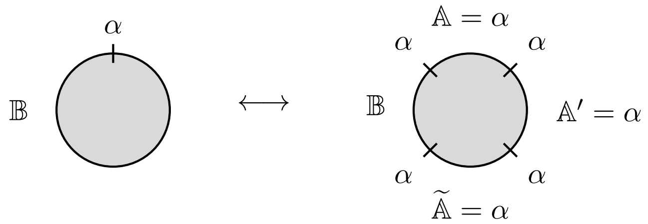


Figure 5.15: $\mathbb{B}$ disk with one α corner as the "collapse" of three edges in a square.

The corner can be expanded to an $\mathbb{A}$-polarized edge with $\mathbb{A} = \alpha$, which can be then split in three consecutive edges. We then get a square, which is the product of an $\mathbb{A}$-$\mathbb{A}$ interval times an $\mathbb{A}$-$\mathbb{B}$ interval (Figure 5.15). We can thus choose the axial gauge propagator to compute the effective action. If we denote with t the coordinate on the $\mathbb{A}$-$\mathbb{A}$ interval and with τ the coordinate on the $\mathbb{A}$-$\mathbb{B}$ interval we have:

$$\eta(t, \tau; t', \tau') = -\Theta(\tau' - \tau)\delta(t' - t)(\mathrm{d}t' - \mathrm{d}t) \ . \tag{5.132}$$

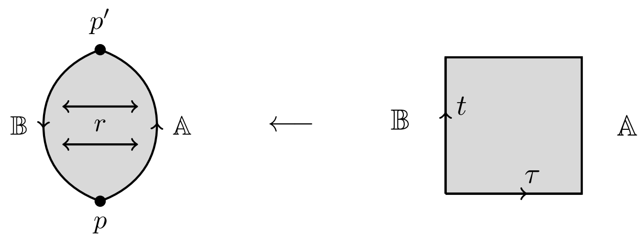
Since there are no zero-modes and the boundary $\mathbb{A}$-field has only the zero-form component α, from degree counting we get that the only non-vanishing diagrams contributing to the partition function are the ones containing no interaction vertices:

$$Z[\mathbb{B}, \alpha] = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_I \langle \mathbb{B}, \alpha\rangle} \ . \tag{5.133}$$

Remark 5.4.13. If we compare this effective action with the one of the $\mathbb{B}$-disk without corners (5.64) we notice that the corner field α plays here the role of the a zero-mode (the other term for action (5.64), containing only the zero-modes, is vanishing when restricted to the globalizing lagrangian $\mathcal{L} = \{\mathsf{b} = 0\}$). Thus, integrating over the fields on the corner reproduces the globalized effective action for the $\mathbb{B}$-disk without corners.
We can also compare (5.133) with the partition function of the $\mathbb{B}$-disk with two corners computed in (5.131). We recover the partition function for the disk with one corner globalizing (5.131) over $\mathcal{L} = \{\mathsf{a} = 0\}$ and then integrating out one corner field α.

5.4.5 BF $\mathbb{A}$-disk with one β-corner

In order to calculate the one remaining building block of the theory, the partition function for an $\mathbb{A}$-disk with a single β-corner, we do the following. We first consider a disk D with boundary split into two intervals in $\mathbb{A}$ and $\mathbb{B}$-polarization with the two corners not decorated by polarization data (i.e. in the setting of the "picture I" for corners, cf. subsection 5.4.2).[33]



(5.134)

The partition function is easily computed by expanding the corners into two intervals (with arbitrary polarization) and putting the axial gauge on the square.[34] This yields the answer

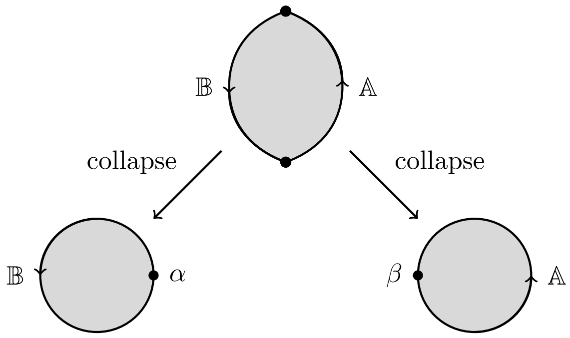
$$Z(\mathbb{A}, \mathbb{B}) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\partial_{\mathbb{A}} D}\langle r^*\mathbb{B}, \mathbb{A}\rangle}\ , \tag{5.135}$$

where r is an orientation-reversing involution on the boundary of the disk, mapping the $\mathbb{A}$-arc diffeomorphically onto the $\mathbb{B}$-arc, having the two corners as fixed points (in terms of the square, r is the involution $(t, 0) \leftrightarrow (t, 1)$). This partition function satisfies the mQME, $\Omega Z = 0$, with $\Omega = \Omega^{\mathbb{A}}_{\partial_{\mathbb{A}}} + \Omega^{\mathbb{B}}_{\partial_{\mathbb{B}}} + \langle \mathbb{B}_p, \mathbb{A}_p\rangle - \langle \mathbb{B}_{p'}, \mathbb{A}_{p'}\rangle$, as per Proposition 5.4.2, and as one can easily check explicitly.

Remark 5.4.14. One can consider collapsing the $\mathbb{A}$- or $\mathbb{B}$-arc on the boundary of the

[33] In fact, we can decorate the two corners with an arbitrary choice of polarizations ξ, ξ'. The partition function does not depend on this choice.
[34] Here we use the axial gauge with the propagator $\eta(t, \tau; t', \tau') = \delta(t' - t)\,(\mathrm{d}t' - \mathrm{d}t)\,\Theta(\tau' - \tau)$.

disk (5.134):



- Collapsing the $\mathbb{A}$-arc into an α-corner, we obtain a $\mathbb{B}$-disk with a single α-corner (in the picture II). Moreover, evaluating the partition function (5.135) on $\mathbb{B} = \beta$, we obtain the partition function $Z(\mathbb{B}, \alpha) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{\partial D}\langle \mathbb{B}, \alpha\rangle}$, which agrees with our result (5.133) from Section 5.4.4 and, indeed, satisfies the mQME with $\Omega = \Omega^{\mathbb{B}}_{\partial D} - \langle \mathbb{B}\big|^{+0}_{-0}, \alpha\rangle + \mathrm{i}\hbar\langle \frac{1}{2}[\alpha, \alpha], \frac{\partial}{\partial \alpha}\rangle$. Here $\mathbb{B}\big|^{+0}_{-0}$ is the jump of the field $\mathbb{B}$ when passing through the α-corner in positive direction.
- Collapsing the $\mathbb{B}$-arc of the disk (5.134) into a β-corner, we obtain a $\mathbb{A}$-disk with a single β-corner. However, evaluating the partition function (5.135) on $\mathbb{B} = \beta$ yields $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\partial D}\langle \beta, \mathbb{A}\rangle}$ which *does not* satisfy the mQME! The reason for this is that the gauge-fixing on the disk (5.134) which was used to compute the partition function (5.135), which in turn came from the axial gauge on a square, is not "collapsible", i.e. fails Assumption 5.4.6, and therefore Proposition 5.4.7 does not apply and we obtained a nonsensical answer after the collapse of the $\mathbb{B}$-arc.

Using the construction of Section 5.4.2, we can consider the projection $p_{\mathcal{H}}$ to the "small model" for the states on the $\mathbb{A}$-arc followed by respective inclusion $i_{\mathcal{H}}$, cf. (5.127,5.128). Thus we obtain a version of the partition function, factored through the small model for $\mathbb{A}$-states:

$$\begin{aligned}\widetilde{Z}(\mathbb{A}, \mathbb{B}) &= i_{\mathcal{H}} \circ p_{\mathcal{H}}\, Z \\ &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\partial_{\mathbb{A}} D}\left\langle r^*\mathbb{B}^{(0)}, dt \log U(\mathbb{A})\right\rangle + \left\langle r^*\mathbb{B}^{(1)}, \mathsf{G}_-(t, \mathrm{ad}_{\log U(\mathbb{A})})\,\mathbb{A}_p + \mathsf{G}_+(t, \mathrm{ad}_{\log U(\mathbb{A})})\,\mathbb{A}_{p'}\right\rangle}\ .\end{aligned} \tag{5.136}$$

Note that, by (5.129), $\widetilde{Z} = Z + \Omega(\cdots)$ – a modification of the answer (5.135) by an Ω-exact term; this deformation can be interpreted as corresponding to a computation in a different gauge.[35] Also, observe that in (5.136), the field $\mathbb{B}^{(1)}$ only interacts with the corner values of $\mathbb{A}^{(0)}$, and thus the gauge corresponding to the answer (5.136) is "collapsible", i.e., satisfies the Assumption 5.4.6. Therefore, we can collapse the $\mathbb{B}$-arc into a β-corner, as in Section 5.4.2, by setting $\mathbb{B}^{(0)} = \beta$, $\mathbb{B}^{(1)} = 0$ in (5.136). Thus we finally arrive to the following result.

[35] We also remark that the answer (5.136) can be obtained directly, by starting with an $\mathbb{A}$-disk with two α-corners, and gluing it along one of the boundary arcs to the "bean" (5.131).

Proposition 5.4.15. The partition function for an $\mathbb{A}$-disk with a single β-corner is:

$$Z(\mathbb{A},\beta)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle\beta,\log U(\mathbb{A})\rangle}\ . \tag{5.137}$$

Note that this answer has a rigidity property: it cannot be changed by a BV-exact term $\Omega(\cdots)$ for a degree reason – there no boundary/corner fields of negative degree needed to construct a degree -1 primitive. The answer (5.137) does indeed satisfy the mQME, i.e. is Ω-closed, as we have verified explicitly in Remark 5.4.8 above.

5.4.6 Gluing arcs in $\mathbb{A}$ polarization

We want now to recover the YM gluing law of two arcs in $\mathbb{A}$ polarization. To compute this gluing law we can use an intermediate BF disk with the boundary split in two arcs with $\mathbb{B}$ polarization, separated by points in α polarization (figure 5.16). Thus, gluing together two $\mathbb{A}$-arcs with endpoints in α-polarization, via the "bean" (5.131), for the partition function of the glued surface we obtain the following:

$$\begin{aligned}Z_\Sigma=&\int \mathrm{d}\mathbb{B}\,\mathrm{d}\widetilde{\mathbb{B}}\,\mathrm{d}\widetilde{\mathbb{A}}\,\mathrm{d}\mathbb{A}\,\mathrm{d}\mathsf{a}_1\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_\gamma\langle\mathbb{B},\mathsf{a}-\mathbb{A}+\mathsf{G}_-(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_0+\mathsf{G}_+(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_1\rangle}\\ &\qquad\cdot\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{\widetilde{\gamma}}\langle\widetilde{\mathbb{B}},\mathsf{a}-\widetilde{\mathbb{A}}+\mathsf{G}_-(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_0+\mathsf{G}_+(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_1\rangle}\ \det\left(\frac{\sinh\left(\mathrm{ad}_{\mathsf{a}_1}/2\right)}{\mathrm{ad}_{\mathsf{a}_1}/2}\right)Z_{\Sigma_1}[\mathbb{A}]\ Z_{\Sigma_2}[\widetilde{\mathbb{A}}]\\ =&\int_{\mathsf{a}_1\in B_0}\mathrm{d}\mathsf{a}_1\ \ \det\left(\frac{\sinh\left(\mathrm{ad}_{\mathsf{a}_1}/2\right)}{\mathrm{ad}_{\mathsf{a}_1}/2}\right)Z_{\Sigma_1}\left[\mathbb{A}=\mathsf{a}+\mathsf{G}_-(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_0+\mathsf{G}_+(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_1\right]\\ &\qquad\cdot\ Z_{\Sigma_2}\left[\widetilde{\mathbb{A}}=\mathsf{a}+\mathsf{G}_-(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_0+\mathsf{G}_+(\tau,\mathrm{ad}_{\mathsf{a}_1})\alpha_1\right]\ .\end{aligned} \tag{5.138}$$

Here the integration domain for the zero-mode a_1 is the "Gribov region" $B_0\subset\mathfrak{g}$. Notice that in this gluing formula the states on the $\mathbb{A}$-arcs factor through the "small model" for the space of states introduced in Section 5.4.2.

If we assume also that the all boundary strata of Σ_1, Σ_2 are in $\mathbb{A}$-polarization and that partition functions Z_{Σ_1}, Z_{Σ_2} are globalized, then the partition functions of Σ_1, Σ_2 does not depend on the ghost fields $\mathbb{A}_{(0)}$,$\widetilde{\mathbb{A}}_{(0)}$[36] and so the gluing formula reduces to

$$Z_\Sigma=\int_G \mathrm{d}U(\mathbb{A})\ Z_{\Sigma_1}[U(\mathbb{A})]\ Z_{\Sigma_2}[U(\mathbb{A})]\ , \tag{5.139}$$

which coincides with the gluing formula for YM known in literature [40, 53].

[36] Independence on the ghosts can be seen by assembling the surface with $\mathbb{A}$-boundary by gluing $\mathbb{A}$-polygons using beans as above. Partition functions for polygons do not depend on the ghosts and the gluing formula (5.138) does not generate ghost dependence. A curious point is that the answer for $\mathbb{A}$-$\mathbb{A}$ cylinder in Section 5.3.3 did contain ghost dependence which seems to contradict what we are saying here. In fact, there is no contradiction, rather there are inequivalent gauge-fixings: one can obtain an $\mathbb{A}$-$\mathbb{A}$ cylinder from an $\mathbb{A}$-square, gluing two opposite sides using the bean (5.131). Choosing the gauge-fixing for the globalization on the bean as in (5.138) – integrating over a_1 – we get the answer for the cylinder without the ghost delta-function. If instead we use the opposite globalization on the bean – integrating over b^1 – we obtain the answer of Section 5.3.3, involving the ghost delta-function.

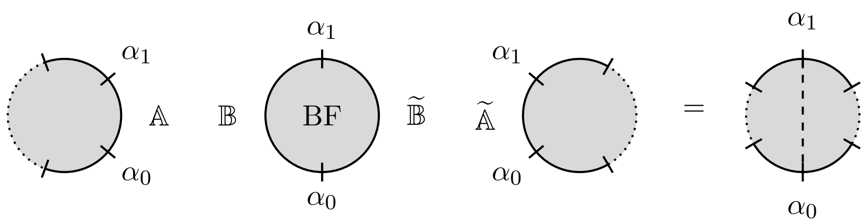


Figure 5.16: Two $\mathbb{A}$-polarized boundaries glued together using an intermediate $\mathbb{B}$-$\mathbb{B}$ BF disk.

5.4.7 2D YM partition function on surfaces with boundaries

We can now compute the partition function on a general surface with boundaries. Indeed, any surface with boundary can be obtained by gluing edges of some polygon (or a collection of polygons – any triangulation or a cellular decomposition gives a presentation of the surface of this kind). Thus using the gluing properties of BV-BFV theories we can compute the YM partition function on a general surface with boundary starting from the partition function on the disk with the boundary split in several arcs γ_i :

$$Z_{D^2}[\mathbb{A}_1, \ldots, \mathbb{A}_n] = \sum_R (\dim\, R)\; \chi_R\big(U_{\gamma_1}(\mathbb{A}_1) \cdots U_{\gamma_n}(\mathbb{A}_n)\big)\; \mathrm{e}^{-\frac{\mathrm{i}\hbar a}{2} C_2(R)}\,. \tag{5.140}$$

Each time we glue together two arcs, using the property (5.139), we have an integral of the kind:

$$\begin{aligned} \int_G \mathrm{d}U\; \chi_R(VUWU^\dagger) &= \frac{\chi_R(V)\chi_R(W)}{\dim\, R}\,, \\ \int_G \mathrm{d}U\; \chi_R(VU)\chi_R(WU^\dagger) &= \frac{\chi_R(VW)}{\dim\, R}\,. \end{aligned} \tag{5.141}$$

This way we get the following result.

Theorem 5.4.16. The globalized YM partition function on a surface with genus g and b boundaries in the $\mathbb{A}$ polarization is:

$$Z_{\Sigma_{g,b}}[\mathbb{A}_1, \ldots, \mathbb{A}_b] = \sum_R (\dim\, R)^{2-2g-b} \mathrm{e}^{-\frac{\mathrm{i}\hbar a}{2} C_2(R)} \prod_{i=1}^{b} \chi_R\big(U_{b_i}(\mathbb{A}_i)\big)\,. \tag{5.142}$$

5.5 Wilson loop observables

Let us consider now observables in 2D YM. These are operators on the Hilbert space $\mathcal{H}_\Gamma$ associated to some boundary Γ which, in $\mathbb{A}$ polarization, is the space of functions of the holonomy $U_\Gamma(\mathbb{A})$.

Let us consider for example the multiplication operator for the factor $\chi_R(U_\Gamma(\mathbb{A}))$ for some representation R. We can compute the matrix element of this operator between two states defined by the partition functions on two surfaces Σ_1 and Σ_2 with the same boundary $\Gamma = S^1$. Using the gluing rule (5.72) for boundaries in $\mathbb{A}$ polarization we get:

$$\begin{aligned}
&\langle Z_{\Sigma_1}|\chi_R(U_\Gamma(\mathbb{A}))|Z_{\Sigma_2}\rangle = \int_G \mathrm{d}U \; Z_{\Sigma_1}(U^\dagger)\chi_R(U)Z_{\Sigma_2}(U) \\
&= \sum_{R_1,R_2} (\dim R_1)^{1-2g_1}(\dim R_2)^{1-2g_2}\mathrm{e}^{-\frac{\mathrm{i}\hbar a_1}{2}C_2(R_1)-\frac{\mathrm{i}\hbar a_2}{2}C_2(R_2)}\int_G \mathrm{d}U \; \chi_{R_1}(U^\dagger)\chi_R(U)\chi_{R_2}(U) \\
&= \sum_{R_1,R_2} (\dim R_1)^{1-2g_1}(\dim R_2)^{1-2g_2}\mathrm{e}^{-\frac{\mathrm{i}\hbar a_1}{2}C_2(R_1)-\frac{\mathrm{i}\hbar a_2}{2}C_2(R_2)} N^{R_1}_{R,R_2} \,,
\end{aligned} \tag{5.143}$$

where we used the expression (5.142) for the partition functions of the surfaces Σ_i, with genus g_i, and where $N^{R_1}_{R,R_2}$ are the fusion numbers defined by the decomposition of the product of irreducible representations: $R \otimes R_2 = \oplus_{R_1} N^{R_1}_{R,R_2} R_1$. This quite obviously corresponds to the computation of the expectation value of a non self-intersecting Wilson loop $W_R(\Gamma)$ on the surface $\Sigma_1 \bigcup_\Gamma \Sigma_2$.

More generally, we can consider operators going from some space of "inbound" states to some "outbound" states: $\mathcal{O}\colon \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}$. Such an operator can be represented by a surface (possibly with corners) with the appropriate boundary components, i.e. such that the boundary Hilbert space is $\mathcal{H}^*_{\text{in}} \otimes \mathcal{H}_{\text{out}}$, and a particular state corresponding to $\mathcal{O}$. The operator now acts on the inbound states by gluing. For example to the (non self-intersecting) Wilson loop we computed above we can associate a cylinder in $\mathbb{A}$-$\mathbb{A}$ polarization and the state $\chi_R(U(\mathbb{A}))\delta(U(\mathbb{A}), U(\mathbb{A}'))$.

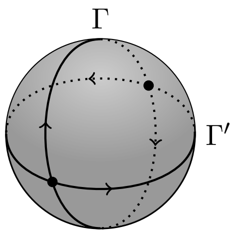


Figure 5.17: Two intersecting Wilson loops Γ and Γ' on the 2-sphere.

Consider now the case of two Wilson loops $W_\Gamma(R)W_{\Gamma'}(R')$ intersecting in 2 points: $\Gamma \cap \Gamma' = v_1 \cup v_2$. We can view them as 4 separate arcs γ_i, γ'_i, $i = 1, 2$, joining the two intersection points. These intersecting Wilson loops can be thought as a multiplication operator on the space of states of the 4 arcs – multiplication by the factor $\chi_R\big(U_{\gamma_1}(\mathbb{A}_1)U_{\gamma_2}(\mathbb{A}_2)\big)\chi_{R'}\big(U_{\gamma'_1}(\mathbb{A}'_1)U_{\gamma'_2}(\mathbb{A}'_2)\big)$. We can thus compute matrix elements between states defined by surfaces with opportune boundary components. Let us consider for example four disks, each with the boundary circle separated into two arcs, glued into a sphere with two intersecting Wilson loops as in figure 5.17:

$$\langle W_\Gamma(R) W_{\Gamma'}(R')\rangle_{S^2} = \sum_{\substack{R_1,R_2,\\R_3,R_4}} \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\sum_i a_i C_2(R_i)} \prod_i (\dim R_i) \int_G \mathrm{d}U_1 \mathrm{d}U_2 \mathrm{d}U_3 \mathrm{d}U_4\ \chi_R(U_1U_3)$$
$$\cdot \chi_{R'}(U_2U_4)\chi_{R_1}(U_4U_1)\chi_{R_2}(U_1^\dagger U_2)\chi_{R_3}(U_2^\dagger U_3^\dagger)\chi_{R_4}(U_3U_4^\dagger)\ . \qquad (5.144)$$

This integral can be evaluated using the Peter-Weyl theorem (part 3) which implies:

$$\int_G \mathrm{d}U\ R_1(U)^i_{i'} R_2(U)^j_{j'} R_3(U^\dagger)^{k'}_k = \frac{1}{\dim R_3}\sum_\mu C_\mu(R_1,R_2;R_3)^{ij}_k C^*_\mu(R_1,R_2;R_3)^{k'}_{i'j'}\ , \qquad (5.145)$$

where $C_\mu(R_1,R_2;R_3)^{ij}_k$ are Clebsch-Gordan coefficients.[37] We get:

$$\langle W_\Gamma(R) W_{\Gamma'}(R')\rangle_{S^2} = \sum_{\substack{R_1,R_2,\\R_3,R_4}} \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\sum_i a_i C_2(R_i)} \frac{\dim R_1}{\dim R_3} \sum_{\substack{\mu_1,\mu_2\\\mu_3,\mu_4}} C_{\mu_1}(R,R_1;R_2)^{ij}_k C^*_{\mu_1}(R,R_1;R_2)^{k'}_{i'j'}$$
$$\cdot C_{\mu_2}(R',R_2;R_3)^{lk}_m C^*_{\mu_2}(R',R_2;R_3)^{m'}_{l'k'} C_{\mu_3}(R,R_4;R_3)^{i'n'}_{m'} C^*_{\mu_3}(R,R_4;R_3)^m_{in} C_{\mu_4}(R',R_1;R_4)^{l'j'}_{n'}$$
$$\cdot C^*_{\mu_4}(R',R_1;R_4)^n_{lj} = \sum_{\substack{R_1,R_2,\\R_3,R_4}} \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\sum_i a_i C_2(R_i)} \prod_{i=1,\ldots,4} (\dim R_i) \sum_{\substack{\mu_1,\mu_2\\\mu_3,\mu_4}} \left\{ \begin{smallmatrix} R & R_1 & R_2 \\ R' & R_3 & R_4 \end{smallmatrix} \right\}^{\mu_1\mu_2}_{\mu_3\mu_4} \left\{ \begin{smallmatrix} R' & R_1 & R_4 \\ R & R_3 & R_2 \end{smallmatrix} \right\}^{\mu_3\mu_4}_{\mu_1\mu_2}\ , \qquad (5.146)$$

where $\left\{ \begin{smallmatrix} R & R_1 & R_2 \\ R' & R_3 & R_4 \end{smallmatrix} \right\}^{\mu_1\mu_2}_{\mu_3\mu_4}$ are Wigner 6-j symbols.[38]

We can generalize this to compute the value of any number of (possibly intersecting) Wilson loops over any surface with boundary. Given a set of Wilson loops we can consider separately the various Wilson lines connecting intersection points.[39] Each line carries a group variable and and contributes with the integral (5.145), where R_1 is the representation of the Wilson loop containing that line and R_2, R_3 are the representations carried by the two regions adjacent to that line. The main observation is that this integral factorises into the product of two Clebsch-Gordan coefficents, each depending only on indices living

[37] If we have representations R_1, R_2 we can decompose their product into the sum of irreducible representations. Let $\{e^i_1\}$ and $\{e^j_2\}$ be two basis of the representation spaces of R_1 and R_2 respectively, and let $\{e^k_{\mu_a}\}$ be a basis of their tensor product such that the product representation is in the block-diagonal form, where a denotes the irreps and μ_a labels the various copies of the representation R_a appearing in the product $R_1 \otimes R_2$. The Clebsh-Gordan coefficients are defined as the basis changing coefficients:

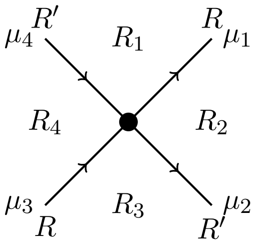
$$e^i_1 \otimes e^j_2 = \sum_a \sum_{\mu_a} C_{\mu_a}(1,2;a)^{ij}_k e^k_{\mu_a}\ .$$

[38] 6-j symbols are defined by:

$$\begin{Bmatrix} R_1 & R_2 & R_3 \\ R_4 & R_5 & R_6 \end{Bmatrix}^{\mu_1\mu_2}_{\mu_3\mu_4} = \frac{C_{\mu_1}(R_1,R_2;R_3)^{ij}_k}{(\dim R_3)^{\frac12}} \frac{C_{\mu_2}(R_4,R_3;R_5)^{lk}_m}{(\dim R_5)^{\frac12}} \frac{C^*_{\mu_3}(R_1,R_6;R_5)^m_{in}}{(\dim R_5)^{\frac12}} \frac{C^*_{\mu_4}(R_4,R_2;R_6)^n_{lj}}{(\dim R_6)^{\frac12}}\ .$$

[39] If a loop has no intersections, then its contribution will be $N^{R_3}_{R_1,R_2}$ as in equation (5.143).

on one of the two edges of the line.[40] Thus when 4 lines meet at an intersection point, the factors associated to that intersection combine to give a 6-j symbol, as in the previous example:



$$\rightsquigarrow \quad \begin{Bmatrix} R & R_1 & R_2 \\ R' & R_3 & R_4 \end{Bmatrix}^{\mu_1\mu_2}_{\mu_3\mu_4} \equiv G(R, R'; \dots) \ . \tag{5.147}$$

Finally, the expectation value of a set of Wilson loops $\{\Gamma_l\}$ on a surface Σ is given by the following formula:

$$\begin{aligned} \langle \prod_l W_{\Gamma_l}(R_l) \rangle_\Sigma = \sum_{R_\lambda} \prod_\lambda \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2} a_\lambda C_2(R_\lambda)} (\dim\, R_\lambda)^{2-2g_\lambda - b_\lambda} \prod_{b_\lambda} \chi_{R_\lambda}(U_{b_\lambda}) \\ \cdot \prod_v \sum_{\mu_i} G_v(R_l; R_\lambda; \mu_i) \prod_{l_0} N(R_{l_0}; R_\lambda) \ . \end{aligned} \tag{5.148}$$

where the index λ runs over connected components Σ_λ of the surface obtained by cutting Σ along the Wilson loops, b_λ labels the boundaries of Σ contained in Σ_λ , v labels the intersections between loops, G_v indicates the 6-j symbol at the vertex v evaluated on the surrounding representations according to (5.147) and $N(R_{l_0}; R_\lambda)$ denotes the fusion numbers for the non-intersecting Wilson loops labelled by l_0 .

[40] Each oriented boundary, or Wilson loop, carries the character of the holonomy of $\mathbb{A}$. If we split the circle into various arcs, then it will carry the character of the products of the holonomies over different arcs, multiplied according to the orientation of the loop. This defines inbound and outbound indices for the holonomy over each oriented arc.

Appendix A

Elements of graded geometry

A.1 Graded manifolds

Definition A.1.1. A *graded manifold* $\mathcal{M}$ over the body M is a sheaf of $\mathbb{Z}$-graded commutative algebras $\mathcal{C}(\mathcal{M})$, over a smooth manifold M, locally isomorphic to the free graded commutative algebra $C^\infty(U) \otimes S(V)$, where $U \subseteq M$ is an open subset and V is a $\mathbb{Z}$-graded vector space with symmetric graded algebra $S(V)$. Such a local isomorphism is called an *affine coordinate chart* over $\mathcal{M}$ and the sheaf $\mathcal{C}(\mathcal{M})$ is called the *sheaf of polynomial functions* over $\mathcal{M}$. The sheaf of polynomial function can be decomposed according to the grading $\mathcal{C}(\mathcal{M}) = \bigoplus_{k\in\mathbb{Z}} \mathcal{C}^k(\mathcal{M})$, where $\mathcal{C}^k(\mathcal{M})$ is locally isomorphic to $C^\infty(M) \otimes S^k(V)$.

Graded manifolds form the category *GrMflds*, where morphisms are homomorphisms of graded algebras: $\mathrm{Mor}(\mathcal{M},\mathcal{N}) = \mathrm{Hom}(\mathcal{C}(\mathcal{N}),\mathcal{C}(\mathcal{M}))$. Smooth manifolds are graded manifolds with zero degree, that is with $\mathcal{C}(M) = \mathcal{C}^0(M) = C^\infty(M)$, and form a full subcategory of *GrMflds*.

A real *supermanifold* can be defined in a similar way, simply substituting the $\mathbb{Z}$-graded vector space V in the definition A.1.1 with $\Pi\mathbb{R}^m$.

Some intuition on the definition of graded manifolds can be acquired thinking about generators of the algebra $C^\infty(U) \otimes S(V)$ as local coordinates on $\mathcal{M}$. Roughly speaking this is similar to smooth manifolds: locally there are graded-coordinates which are patched together with degree-preserving morphisms.

Let us see now some simple but relevant examples of graded manifolds.

Example A.1.2. Let $TM \xrightarrow{\pi} M$ be the tangent bundle to a smooth manifold M with dimension $\dim(M) = m$. Given local coordinates x^μ over an open subset $U \subseteq M$ and called θ^μ the fiber coordinates relative to the basis $\{\partial_\mu = \partial/\partial x^\mu\}$, we can define local coordinates (x^μ, θ^ν) of TM over $\pi^{-1}(U)$. The graded manifold obtained assigning degree 1 to fiber coordinates is denoted by $T[1]M$:

$$|x^\mu| = 0 \ ; \qquad |\theta^\mu| = 1 \ . \tag{A.1}$$

The patching between different coordinate systems is defined by:

$$\tilde{x}^\mu = \tilde{x}^\mu(x) \; ; \qquad \tilde{\theta}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\,\theta^\nu \; . \tag{A.2}$$

A polynomial function $f \in \mathcal{C}(T[1]M)$ takes locally the form:

$$f(x,\theta) = \sum_{k=0,\dots,m} \frac{1}{k!} f_{\mu_1,\dots,\mu_k}(x)\,\theta^{\mu_1}\cdots\theta^{\mu_k} \; . \tag{A.3}$$

By definition, fiber coordinates θ are anticommuting and their transformation rules (A.2) are identical with those of differentials $\mathrm{d}x^\mu$. The sheaf of polynomial functions over $T[1]M$ is then naturally identified with differential forms over M :

$$\mathcal{C}^\bullet(T[1]M) = \Omega^\bullet(M) \; . \tag{A.4}$$

Example A.1.3. In the same way as the previous example, the graded manifold $T^*[-1]M$ is defined by giving degree -1 to fiber coordinates of the cotangent bundle T^*M of a smooth manifold. Polynomial functions on $T^*[-1]M$ are then identified with *multi-vector fields* , i.e. sections of the exterior product of tangent bundles: $\mathcal{C}^\bullet(T^*[-1]M) = Sec(\Lambda^{-\bullet}TM)$.

In *GrMflds* tangent vector fields are defined like in smooth manifolds.

Definition A.1.4. A tangent vector field $X \in \mathfrak{X}(\mathcal{M})$ on a graded manifold $\mathcal{M}$ is a derivation of the graded algebra $\mathcal{C}(\mathcal{M})$. Using affine local homogeneous coordinates x^a for $\mathcal{M}$, tangent vector fields can be written in the form:[1]

$$X = X^a \frac{\partial}{\partial x^a} \; . \tag{A.5}$$

The vector field X has degree $|X|$ if its components have degree $|X^a| = |X| + |x^a|$.

Remark A.1.5. On vector fields are defined the brackets $[\ ,\ ]\colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$

$$[X,Y] = X \circ Y - (-1)^{|X|\,|Y|} Y \circ X \tag{A.6}$$

or, using homogeneous coordinates:

$$[X,Y] = X^a \frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b} - (-1)^{|X|\,|Y|}\, Y^b \frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a} \; . \tag{A.7}$$

Since $[\ ,\ ]$ are Lie brackets with zero degree, vector fields are a sheaf of graded Lie algebras.

[1] When dealing with graded quantities, it is important to distinguish left and right derivatives. Unless noted otherwise, we assume derivatives acts from the left.

Definition A.1.6. Let $\mathcal{M}$ be a graded manifold. The *Euler vector field* is the derivation ϵ of the polynomial functions $\mathcal{C}(\mathcal{M})$ defined by:

$$\epsilon f = kf \qquad \forall\, f \in \mathcal{C}^k(\mathcal{M}) \ . \tag{A.8}$$

If x^a are homogeneous local affine coordinates of $\mathcal{M}$, then the Euler vector field has the form:

$$\epsilon = |x^a|\, x^a \frac{\partial}{\partial x^a} \ . \tag{A.9}$$

In particular the body M is recovered as the set of fixed points of ϵ.

Remark A.1.7. The Euler vector field gives the degree of a vector field X through the *Lie derivative* $L_\epsilon X = [\epsilon, X]$. Indeed, in homogeneous coordinates x^a we have:

$$\begin{aligned} L_\epsilon X &= |x^b|\, x^b \frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial x^a} - X^a \frac{\partial |x^b|\, x^b}{\partial x^a}\frac{\partial}{\partial x^b} = \\ &= (|X^a| - |x^a|) X^a \frac{\partial}{\partial x^a} = |X| X \ . \end{aligned} \tag{A.10}$$

Using the Euler vector field we can give an intuitive definition of vector bundle in *GrMflds*.

Definition A.1.8. Let $\mathcal{A}$ and $\mathcal{M}$ be graded manifolds. A *vector bundle* in *GrMflds* is a surjective immersion $\mathcal{A} \longrightarrow \mathcal{M}$ and a linear structure given by an additional Euler vector field ϵ_{vect} which assigns degree 1 to fiber coordinates and degree 0 to functions pulled back from $\mathcal{M}$.

We can now generalize the notation used in example A.1.2 to vector bundles. We will call *shifted vector bundle* the graded manifold $\mathcal{A}[n]$ corresponding to the Euler vector field $\epsilon_{\mathcal{A}} + n\epsilon_{\text{vect}}$. Hence, all fiber coordinates have degree shifted by n with respect to the grading in $\mathcal{A}$.

Example A.1.9. Let $\mathcal{M}$ be a graded manifold with homogeneous local coordinates x^a. The tangent bundle $T\mathcal{M}$ is the graded manifold described by local coordinates (x^a, θ^a) with grading and gluing rules given by:

$$|\theta^a| = |x^a| \ ; \qquad \tilde{\theta}^a = \theta^b \frac{\partial \tilde{x}^a}{\partial x^b} \ . \tag{A.11}$$

Clearly $T\mathcal{M}$ is a vector bundle on $\mathcal{M}$ with projection $(x, \theta) \mapsto x$ and Euler vector field:

$$\epsilon_{T\mathcal{M}} = |x^a| \Big(\theta^a \frac{\partial}{\partial \theta^a} + x^a \frac{\partial}{\partial x^a}\Big) \ , \qquad \epsilon_{\text{vect}} = \theta^a \frac{\partial}{\partial \theta^a} \ . \tag{A.12}$$

The shifted tangent bundle $T[1]\mathcal{M}$ is simply defined by the new Euler vector field $\epsilon_{T[1]\mathcal{M}} = \epsilon_{T\mathcal{M}} + \epsilon_{\text{vect}}$ Coordinates θ in $T[1]\mathcal{M}$ have degree $|\theta^a| = |x^a| + 1$, while the degree of x and the gluing rules remain the same.

A.2 Differentials and symplectic structure

Definition A.2.1. A *differential* Q, or cohomological vector field, is an anticommuting vector field with degree 1:

$$[\epsilon, Q] = 1 \; ; \qquad [Q, Q] = 0 \; , \tag{A.13}$$

where ϵ is the Euler vector field. Graded manifolds endowed with a differential are called *differential graded manifolds* and form the category *dgMflds*.

Example A.2.2. Let $T[1]M$ be the shifted tangent bundle of an ordinary smooth manifold M as in the example A.1.2. On this graded manifold it is defined the cohomological vector field d:

$$\mathrm{d} = \theta^\mu \frac{\partial}{\partial x^\mu} \; . \tag{A.14}$$

We already noticed that the algebra of polynomial functions on $T[1]M$ coincides with the exterior algebra of the differential forms on M. The cohomological vector field d then corresponds to the de Rham differential on $\Omega(M)$.

This example motivates the following definition of differential forms on graded manifolds.

Definition A.2.3. *Differential forms* on a graded manifold $\mathcal{M}$ are the polynomial functions on its shifted tangent bundle:

$$\Omega(\mathcal{M}) = \mathcal{C}(T[1]\mathcal{M}) \, . \tag{A.15}$$

On $T[1]\mathcal{M}$ there is the cohomological vector field

$$\mathrm{d} = \theta^a \frac{\partial}{\partial x^a} \; , \qquad \mathrm{d} \colon \mathcal{C}^\bullet(T[1]\mathcal{M}) \to \mathcal{C}^{\bullet+1}(T[1]\mathcal{M}) \; , \tag{A.16}$$

where (x^a, θ^a) are homogeneous local affine coordinates like in example A.1.9. $\mathcal{C}(T[1]\mathcal{M})$ endowed with the differential d is the de Rham complex $(\Omega(\mathcal{M}), \mathrm{d})$ of $\mathcal{M}$.

Remark A.2.4. To each vector field X on a graded manifold $\mathcal{M}$ corresponds a vector field $\imath_X$ on $T[1]\mathcal{M}$ defined by:

$$\imath_X = (-1)^{|X|} X^a \frac{\partial}{\partial \theta^a} \; , \qquad \imath_X \colon \mathcal{C}^\bullet(T[1]\mathcal{M}) \to \mathcal{C}^{\bullet-1}(T[1]\mathcal{M}) \; . \tag{A.17}$$

The *interior product* $\imath_X$ has degree $|\imath_X| = |X| - 1$. *Lie derivative* on differential forms $\Omega(\mathcal{M})$ is then defined through the Cartan formula:

$$L_X = [\imath_X, \mathrm{d}] = \imath_X \mathrm{d} + (-1)^{|X|} \mathrm{d}\, \imath_X \; . \tag{A.18}$$

With a direct calculation, it can be shown that:

$$|L_X| = |X| \; , \quad [L_X, L_Y] = L_{[X,Y]} \; , \quad [L_X, \mathrm{d}] = 0 \; . \tag{A.19}$$

We remarked in A.1.7 that Lie derivative with respect to the Euler vector field returns the degree of tangent vectors. Indeed, the grading on tangent and cotangent bundles is given by the Euler vector field of $\mathcal{M}$ lifted canonically to $T\mathcal{M}$ or $T^*\mathcal{M}$ via Lie derivative. Consider for example the tangent bundle $T\mathcal{M}$. If ϵ is the Euler vector field on $\mathcal{M}$ we have:

$$L_\epsilon \mathrm{d}x^a = \mathrm{d}L_\epsilon x^a = |x^a|\mathrm{d}x^a \ , \tag{A.20}$$

i.e. the grading of $T\mathcal{M}$ is given by the Lie derivative L_ϵ .

Definition A.2.5. A p-form of degree k is a form $\omega \in \Omega(\mathcal{M})$ with $\epsilon_{\text{vect}}\,\omega = p\,\omega$ and $L_\epsilon\,\omega = k\,\omega$.

With this definition, the structure of symplectic manifold can be extended to graded manifolds in a natural way.

Definition A.2.6. A *symplectic form* of degree n in *GrMflds* is a two-form $\omega \in \Omega^2(\mathcal{M})$

i) homogeneous of degree n: $L_\epsilon\,\omega = n\,\omega$;

ii) closed with respect to the de Rham differential: $\mathrm{d}\omega = 0$;

iii) non singular, i.e. such that the vector bundle morphism $\omega\colon T\mathcal{M} \to T^*[n]\mathcal{M}$ is an isomorphism.

The pair $(\mathcal{M},\omega)$ is called *symplectic graded manifold of degree* n . If on $(\mathcal{M},\omega)$ there is a differential Q preserving the symplectic form, that is such that $L_Q\omega = 0$, then $(\mathcal{M},\omega,Q)$ is a *differential graded symplectic manifold.*

Example A.2.7. Every ordinary symplectic manifold (M,ω) can be viewed as a graded symplectic manifold with $\epsilon_M = 0$ and so $|\omega| = 0$.

Example A.2.8. Consider the graded manifold $\mathbb{R}[1]$ with coordinate x and Euler vector field $\epsilon = x\frac{\partial}{\partial x}$. The tangent bundle $T[1](\mathbb{R}[1])$ has coordinates $(x, \mathrm{d}x)$, with $|\mathrm{d}x| = 2$ and $L_\epsilon \mathrm{d}x = \mathrm{d}x$. The two-form $\omega = \mathrm{d}x\,\mathrm{d}x$ is then a symplectic form on $\mathbb{R}[1]$ of degree $L_\epsilon\omega = 2\omega$.

Example A.2.9. Let M be an ordinary smooth manifold, i.e. with $\epsilon_M = 0$, and $T^*[1]M$ its shifted cotangent bundle with homogeneous coordinates (x^μ, p_ν) like in example A.1.2. The Euler vector field $\epsilon = \epsilon_{\text{vect}}$ for $T^*[1]M$ is:

$$\epsilon = p_\mu \frac{\partial}{\partial p_\mu} \ . \tag{A.21}$$

Consider now the vector bundle $T^*[2](T^*[1]M)$. Local coordinates on $T^*[1]M$ define the basis $\{\mathrm{d}x^\mu, \mathrm{d}p_\nu\}$ for the fibers and the corresponding fiber coordinates (θ_μ, ψ^ν) . The grading on $T^*[2](T^*[1]M)$ is given by the Euler vector field $\tilde{\epsilon} = L_\epsilon + 2\epsilon_{\text{vect}}$, so that

(x, p, θ, ψ) have degree respectively 0, 1, 2 and 1. Coordinates transformation rules are easily found to be:

$$\begin{aligned} &\tilde{x}^\mu = \tilde{x}^\mu(x) \; ; \qquad \tilde{p}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \, p_\nu \; ; \qquad \tilde{\psi}^\mu = \frac{\partial p_\nu}{\partial \tilde{p}_\mu} \psi^\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \psi^\nu \; ; \\ &\tilde{\theta}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \, \theta_\nu + \psi^\nu \frac{\partial p_\nu}{\partial \tilde{x}^\mu} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \, \theta_\nu + \frac{\partial^2 \tilde{x}^\rho}{\partial x^\gamma \partial x^\nu} \, \frac{\partial x^\gamma}{\partial \tilde{x}^\mu} \, \frac{\partial x^\sigma}{\partial \tilde{x}^\rho} \, \psi^\nu \, p_\sigma = \\ &\quad = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \, \theta_\nu + \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\rho \partial \tilde{x}^\mu} \, \frac{\partial \tilde{x}^\rho}{\partial x^\nu} \, p_\sigma \, \psi^\nu \; ; \end{aligned} \tag{A.22}$$

Notice that, as it should be, all these transformations are degree-preserving. Moreover θ undergoes an affine transformation, whence the word affine in the definition A.1.1 of coordinates on a graded manifold. There is a canonical symplectic form given by:

$$\omega = \mathrm{d}x^\mu \mathrm{d}\theta_\mu + \mathrm{d}p_\mu \mathrm{d}\psi^\mu \; ; \qquad |\omega| = 2 \; . \tag{A.23}$$

Also, we can define the differential Q:

$$Q = \psi^\mu \frac{\partial}{\partial x^\mu} + \theta_\mu \frac{\partial}{\partial p_\mu} \; ; \qquad |Q| = 1 \; ; \qquad Q^2 = 0 \; . \tag{A.24}$$

As this differential is compatible with the symplectic structure, $L_Q \omega = 0$, $T^*[2](T^*[1]M)$ is a differential graded symplectic manifold of degree 2.

Example A.2.10. In general $T^*[n]\mathcal{M}$ is a symplectic graded manifold with canonical symplectic form:

$$\omega = \mathrm{d}x^a \mathrm{d}\theta_a \; ; \qquad |\omega| = n \; ; \tag{A.25}$$

where x^a are homogeneous local affine coordinates of $\mathcal{M}$ and θ_a are the fiber coordinates relative to the basis $\mathrm{d}x^a$.

Lemma A.2.11. Let ω be a symplectic form. If ω has degree $n \neq 0$, then it is exact.

Proof. By definition ω is closed and of degree n. Then:

$$n\omega = L_\epsilon \omega = \mathrm{d}(\imath_\epsilon \omega) \; . \tag{A.26}$$

If $n \neq 0$ we have:

$$\omega = \mathrm{d} \, \frac{\imath_\epsilon \omega}{n} \; . \tag{A.27}$$

□

Definition A.2.12. Let $(\mathcal{M}, \omega)$ be a symplectic graded manifold of degree n. A tangent vector field X over $\mathcal{M}$ is called:

i) *Symplectic* if the 1-form $\imath_X \omega$ is closed, or equivalently if $L_X \omega = 0$;

ii) *hamiltonian* if the 1-form $\imath_X \omega$ is exact, i.e. if $\exists \, f \in \mathcal{C}(\mathcal{M})$ such that:

$$\imath_X \omega = (-1)^{n+1} \mathrm{d}f \; . \tag{A.28}$$

An hamiltonian vector field with hamiltonian function f is denoted by X^f.

Remark A.2.13. Like in ordinary symplectic geometry, symplectic vector fields generates *canonical transformations* [7, 8, 50], which preserve the symplectic structure.

Definition A.2.14. Let $(\mathcal{M}, \omega)$ be a symplectic graded manifold of degree n. On $\mathcal{C}(\mathcal{M})$ are defined the Poisson brackets $\{\ ,\ \}\colon \mathcal{C}^r(\mathcal{M}) \times \mathcal{C}^s(\mathcal{M}) \to \mathcal{C}^{r+s-n}(\mathcal{M})$:

$$\{f, g\} = X^f g = (-1)^{|X^f|}\, \imath_{X^f}\, \mathrm{d}g = (-1)^{|f|+1} \imath_{X^f}\, \imath_{X^g}\, \omega \ ; \qquad f, g \in \mathcal{C}(\mathcal{M}) \ . \tag{A.29}$$

Remark A.2.15. Poisson brackets give to $\mathcal{C}(\mathcal{M})$ the structure of a graded Lie algebra. Moreover, the map $f \mapsto X^f$ is an homomorphisms of graded Lie algebras:

$$X^{\{f,g\}} = [X^f, X^g] \ . \tag{A.30}$$

Lemma A.2.16. Let $(\mathcal{M}, \omega)$ be a symplectic graded manifold of degree n and X a symplectic vector field of degree k. If $n + k \neq 0$, then X is hamiltonian.

Proof. We have $L_\epsilon X = k$, $L_\epsilon \omega = n$ and $\mathrm{d}(\imath_X \omega) = 0$. Then

$$(n + k)\imath_X \omega = L_\epsilon(\imath_X \omega) = \mathrm{d}(\imath_\epsilon \imath_X \omega) \tag{A.31}$$

and so $X = X^f$, with

$$f = (-1)^{n+1} \frac{\imath_\epsilon \imath_X \omega}{n + k} \ . \tag{A.32}$$

□

Remark A.2.17. Let $(\mathcal{M}, \omega, Q)$ be a differential graded symplectic manifold, with $|\omega| = n$. Suppose Q to be hamiltonian, i.e. $Q = \{S,\ \}$ for some hamiltonian $S \in \mathcal{C}(\mathcal{M})$. Reminding that $|Q| = 1$ and $|\{S,\ \}| = |S| - n$, it must be $|S| = n + 1$. Thus, we have:

$$0 = [Q, Q]f = \{\{S, S\}, f\} \qquad \forall\, f \in \mathcal{C}(\mathcal{M}) \ . \tag{A.33}$$

This implies $\{S, S\}$ is a constant of degree $|\{S, S\}| = n + 2$. As a non zero constant has to be of degree zero, S satisfies the *classical master equation* $\{S, S\} = 0$.

Example A.2.18. Let us consider $T^*[1]M$ with the canonical symplectic form ω of degree 1 and a solution S of the classical master equation $\{S, S\}$. Let $\{x, \theta\}$ be local coordinates of degree respectively 0 and 1. As the action is of degree 2, it must have the form:

$$S = \frac{1}{2}\alpha^{\mu\nu}(x)\theta_\mu \theta_\nu \ . \tag{A.34}$$

Therefore the action corresponds to a bivector field on M. The master equation reads:

$$\{S, S\} = 2\alpha^{\mu\nu}\partial_\nu \alpha^{\rho\sigma}\theta_\mu \theta_\rho \theta_\sigma \tag{A.35}$$

and so $\{S, S\} = 0$ implies that $\alpha^{\mu\nu}$ is a *Poisson bivector field* on M.

A.3 Integration on supermanifolds

Before to define the integration on supermanifolds, let us remind how to integrate over odd variables. Let $\{\theta_\alpha\}$ be a set of generators of a Grassmann algebra $\mathfrak{G}$ (for concreteness one could take $\mathfrak{G} = S(\mathbb{R}^{0|n})$). Then:

$$\int \mathrm{d}\theta_\alpha = 0\ ; \qquad \int \mathrm{d}\theta_\alpha\, \theta_\beta = \delta_{\alpha\beta}\ . \tag{A.36}$$

Consider now the algebra of functions $\mathcal{C}(\mathbb{R}^{m|n})$ with even generators x^μ. A function $f \in \mathcal{C}(\mathbb{R}^{m|n})$ has the form $f(x,\theta) = f_{(0)}(x) + \theta_\alpha f^\alpha_{(1)}(x) + \ldots + \theta_1 \cdots \theta_n f_{(n)}(x)$. The integral over $\mathbb{R}^{m|n}$ is:

$$\int\limits_{\mathbb{R}^{m|n}} [\mathrm{d}x^1 \cdots \mathrm{d}x^m \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1]\, f(x,\theta) = \int\limits_{\mathbb{R}^m} \mathrm{d}x^1 \cdots \mathrm{d}x^m\, f_{(n)}(x) \tag{A.37}$$

For the integral to be independent of the particular choice of coordinates on $\mathbb{R}^{m|n}$, the measure has to transform according to the following rule [29]:

$$[\mathrm{d}x^1 \cdots \mathrm{d}x^m \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1] = [\mathrm{d}\tilde{x}^1 \cdots \mathrm{d}\tilde{x}^m \mathrm{d}\tilde{\theta}_n \cdots \mathrm{d}\tilde{\theta}_1]\, \mathrm{Ber}\left(\frac{\partial X}{\partial \widetilde{X}}\right)\ , \tag{A.38}$$

where $X = (x,\theta) \to \widetilde{X} = \big(\tilde{x}(x,\theta), \tilde{\theta}(x,\theta)\big)$ is a general change of coordinates and the Berezinian is defined by:

$$\begin{aligned} \frac{\partial X}{\partial \widetilde{X}} &= \left(\begin{array}{c|c} \partial x/\partial \tilde{x} & \partial x/\partial \tilde{\theta} \\ \hline \partial \theta/\partial \tilde{x} & \partial \theta/\partial \tilde{\theta} \end{array}\right) \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\ , \\ \mathrm{Ber}\left(\frac{\partial X}{\partial \widetilde{X}}\right) &= \det(A - BD^{-1}C)\det(D)^{-1}\ . \end{aligned} \tag{A.39}$$

Notice that the Berezinian is defined only if D is invertible, but this is always true for a change of coordinates. In the particular case in which the change of coordinates does not mix odd and even generators ($B = C = 0$) the Berezinian gives simply a determinant of the Jacobian matrix for the even coordinates and the inverse of the Jacobian determinant for the odd generators.

Integration on supermanifolds is defined from the above Berezin integration alike integration on smooth manifold is constructed from the integral over $\mathbb{R}^m$ [10]. Let us consider a supermanifold $\mathcal{M}$ with dimension $\dim(\mathcal{M}) = (m,n)$ over the base space M and take an atlas $\{U_a\}$ of M. By definition, the algebra $\mathcal{C}(\mathcal{M})$ is locally $C^\infty(U_a) \otimes S(\Pi\mathbb{R}^n)$. Let $\Sigma_a = \{x^i, \theta_\alpha\}$ be a set of generators of $C^\infty(U_a) \otimes S(\Pi\mathbb{R}^n)$. Moreover, for each U_a choose a *local density* $\rho_a(X_a)$ such that:

i) $\rho_a(\widetilde{X}_a) = \mathrm{Ber}(\partial X_a/\partial \widetilde{X}_a)\, \rho_a(X_a(\widetilde{X}_a))$;

ii) on the intersection $U_a \cap U_b$: $\rho_a(X_a) = \mathrm{Ber}(\partial X_b/\partial X_a)\, \rho_b(X_b(X_a))$.

The objects which can be integrated on a supermanifold are the *Berezinian forms* ρ defined by:

$$\int_{\mathcal{M}} \rho = \sum_a \int [\mathrm{d}x^1 \cdots \mathrm{d}x^m \mathrm{d}\theta_n \cdots \mathrm{d}\theta_1]\ \rho_a s_a\ , \tag{A.40}$$

where $\{s_a(x)\}$ is a partition of the unity subordinated to the open cover $\{U_a\}$. Notice that thanks to the transformation rules of ρ_a, this definition depends neither on the atlas nor on the partition of unity chosen. Obviously, for every function $f \in \mathcal{C}(\mathcal{M})$, ρf is again a Berezinian form and so the choice of a Berezinian form allows to integrate any function as $\int_{\mathcal{M}} \rho\, f$.

Definition A.3.1. A Berezinian form ρ is called *non degenerate* if the bilinear form

$$\langle f, g\rangle = \int_{\mathcal{M}} \rho\, f g\ , \qquad f, g \in \mathcal{C}(\mathcal{M})\ , \tag{A.41}$$

is non degenerate. Moreover, given a vector field $v \in \mathfrak{X}(\mathcal{M})$, the Berezinian form ρ is *v-invariant* if

$$\int_{\mathcal{M}} \rho\, v(f) = 0 \tag{A.42}$$

for every function $f \in \mathcal{C}(\mathcal{M})$.

For every odd vector bundle $E \longrightarrow M$, with M an ordinary smooth manifold, a Berezinian form can be viewed as a section of the *Berezinian bundle* $\wedge E \otimes \wedge^{\mathrm{top}} T^*M$: indeed it associates to functions $\mathcal{C}(E) \simeq \Gamma(\wedge E^*)$ top forms to be integrated on M [17]. A non degenerate Berezinian is then obtained as a nowhere-vanishing section concentrated in top degree on E: $\mu \in \Gamma(\wedge^{\mathrm{top}} E \otimes \wedge^{\mathrm{top}} T^*M)$.

Example A.3.2. Consider the odd cotangent bundle ΠT^*M.[2] The Berezinian bundle is $\mathrm{Ber}(\Pi T^*M) = (\wedge^m T^*M)^{\otimes 2}$, where $\dim(M) = m$. Hence, to every volume form $v \in \Omega^m(M)$ there is an associated Berezinian. As we already remarked, functions on ΠT^*M can be identified with multivector fields, therefore a volume form on M defines the isomorphism $\phi_v\colon \mathcal{C}^\bullet(\Pi T^*M) \to \Omega^{m-\bullet}(M)$, $\phi_v(f) = \imath_f v$. The Berezinian form μ_v associated to v is:

$$\int_{\Pi T^*M} \mu_v\, f = \int_M v \wedge \phi_v(f)\ . \tag{A.43}$$

Let now C be a submanifold of M. A volume form on M also induces a Berezinian on the *odd conormal bundle* $\Pi N^*C \subset \Pi T^*M$.[3] Indeed the Berezinian bundle is $\mathrm{Ber}(\Pi N^*C) =$

[2] We recall that the parity reversion functor Π exchanges even and odd fiber coordinates.

[3] The normal bundle NC of $C \subset M$, $\dim(C) = k$, is defined by the short exact sequence of bundles:

$$0 \to TC \to T_C M \to NC \to 0\ .$$

The conormal bundle N^*C is its dual, namely: $N^*_p C = \{\eta \in T^*_p M |\ \eta(w) = 0\ \forall\, w \in T_p C\}$, $p \in C$. Moreover N^*C is a lagrangian submanifold of T^*M with the canonical symplectic structure.

$\wedge^{m-k}N^*C\otimes\wedge^k T^*C = \wedge^m T^*_C M$ and a volume form v determines a section $\sqrt{\mu_v}\in\Gamma(\mathrm{Ber}(\Pi N^*C))$ by restriction:

$$\int_{\Pi N^*C}\sqrt{\mu_v}\,f = \int_C \phi_v(\tilde f)\ , \tag{A.44}$$

where $\tilde f$ is any function on $\Pi T^*_C M$ such that $\tilde f|_{\Pi N^*C} = f$.

The constructions contained in the previous example can be generalized in the graded setting [46]: to every Berezinian form v on a graded manifold $\mathcal{M}$ is canonically associated a Berezinian on $T^*[k]\mathcal{M}$; moreover every Berezinian μ on $T^*[k]\mathcal{M}$, induces canonically an half density $\sqrt{\mu}$ which can be integrated on lagrangian submanifolds, where it induces a Berezinian.

In the graded setting there is a definition of divergence analogous to the one in ordinary differential geometry [37].

Lemma A.3.3. For every Berezinian form μ there is a map $\mathrm{div}_\mu\colon \mathfrak{X}(\mathcal{M})\to\mathcal{C}(\mathcal{M})$ such that:

$$\int_{\mathcal{M}}\mu\,X(f) = -\int_{\mathcal{M}}\mu\,(\mathrm{div}_\mu X)f\ , \qquad \forall\, f\in\mathcal{C}(\mathcal{M})\ . \tag{A.45}$$

Moreover, the *divergence operator* div_μ satisfies:

$$\mathrm{div}_\mu(fX) = f\mathrm{div}_\mu X + (-1)^{|f|\,|X|}X(f)\ . \tag{A.46}$$

A.4 Mapping spaces between graded manifolds

We recall here some basic definitions and facts about mapping space of graded manifolds. A more detailed discussion can be found, for example, in [27, 44]. In physics literature a map between two graded manifolds is what is called a *superfield*. If $\mathcal{M}$ and $\mathcal{N}$ are two supermanifolds with coordinates respectively (x,θ) and (y,η) – where x and y are even and θ and η are odd variables – then superfields from $\mathcal{M}$ to $\mathcal{N}$ can be expressed by an expansion in the odd variables:

$$\begin{aligned}\mathbf{y}(x,\theta) &= y(x) + y'(x)\theta\ ,\\ \boldsymbol{\eta}(x,\theta) &= \eta(x) + \eta'(x)\theta\ .\end{aligned} \tag{A.47}$$

Before showing how superfields looks in general, we are going to give the formal definition of mapping space. For smooth manifolds the space of maps, that is the space of smooth morphisms between two manifolds, satisfies the important property:

$$\mathrm{Mor}(P\times N, M) = \mathrm{Mor}(P,\mathrm{Mor}(N,M))\ . \tag{A.48}$$

The problem is that this property does not hold for morphisms in *GrMflds* and so it is useful to change the definition of the mapping space between two graded manifolds. It can be shown that there exists a graded manifold $\text{Maps}(\mathcal{X},\mathcal{Y})$, canonically associated to a pair $(\mathcal{X},\mathcal{Y})$, such that:

$$\text{Mor}(\mathcal{Z}\times\mathcal{X},\mathcal{Y}) \cong \text{Mor}(\mathcal{Z},\text{Maps}(\mathcal{X},\mathcal{Y})) \tag{A.49}$$

for any graded manifold $\mathcal{Z}$ (here Mor is in *GrMflds*, see definition A.1.1). With this definition of mapping space, one recovers the property (A.48) also for graded manifolds: $\text{Maps}(\mathcal{Z}\times\mathcal{X},\mathcal{Y}) \cong \text{Maps}(\mathcal{Z},\text{Maps}(\mathcal{X},\mathcal{Y}))$.

Let us take for example $\mathcal{X}$ and $\mathcal{Y}$ to be non negatively graded, with coordinates respectively $x=\{x_0^\mu; x_1^\nu\}$ and $y=\{y_a^i\}$, where the subscript indicates the degree. A morphism between the graded manifolds $\mathcal{Z}\times\mathcal{X}$ and $\mathcal{Y}$ can be expanded according to the degree in $\mathcal{X}$ as:

$$\begin{aligned}
&y_0^i(x,z) = y_{0,0}^i(x_0,z) + y_{(0,-1)\,\mu}^i(x_0,z)x_1^\mu + y_{(0,-2)\,\mu\nu}^i(x_0,z)x_1^\mu x_1^\nu + \dots\ ;\\
&y_1^i(x,z) = y_{1,1}^i(x_0,z) + y_{(1,0)\,\mu}^i(x_0,z)x_1^\mu + y_{(1,-1)\,\mu\nu}^i(x_0,z)x_1^\mu x_1^\nu + \dots\ ;\\
&y_2^i(x,z) = y_{2,2}^i(x_0,z) + y_{(2,1)\,\mu}^i(x_0,z)x_1^\mu + y_{(2,0)\,\mu\nu}^i(x_0,z)x_1^\mu x_1^\nu + \dots\ ;\\
&\vdots
\end{aligned} \tag{A.50}$$

where the functions $y_{(a,b)}(x_0;z)$ are of total degree $|y_{(a,b)}|=b$. The relation (A.49) tells that the functions $y_{(a,b)}(x_0)$ are to be considered as coordinates of degree b of the graded manifold $\text{Maps}(\mathcal{X},\mathcal{Y})$. The transformation rules for $y_{(a,b)}(x_0)$ are induced by the ones on $\mathcal{X}$ and $\mathcal{Y}$.

Notice that $y_{a,0}(x_0)$ corresponds to degree-preserving maps $\text{Mor}(\mathcal{X},\mathcal{Y})$, which are then naturally included in $\text{Maps}(\mathcal{X},\mathcal{Y})$ as the submanifold of degree zero, but there are also the coordinates $y_{(a,b>0)}(x_0)$ corresponding to maps with non-zero degree. Moreover we remark that, in spite of the fact that we took both $\mathcal{X}$ and $\mathcal{Y}$ to be non-negatively graded, $\text{Maps}(\mathcal{X},\mathcal{Y})$ has in general both positive and negative degrees.

Although in general $\text{Maps}(\mathcal{X},\mathcal{Y})$ is infinite-dimensional, this is not always the case as shown in the following example.

Example A.4.1. Let $\mathcal{M}$ be any graded manifold and consider $\text{Maps}(\mathbb{R}[-1],\mathcal{M})$. If e is the coordinate of $\mathbb{R}[-1]$ and x_a^μ are local coordinates of $\mathcal{M}$ with degree a, we have the superfields:

$$\mathbf{x}_a^\mu(e) = x_{(a,0)}^\mu + x_{(a,a+1)}^\mu e\ . \tag{A.51}$$

Under a change of coordinate $y(x)$ on $\mathcal{M}$ the superfields becomes:

$$\mathbf{y}_a^\mu(e) = y_{(a,0)}^\mu(x_{(\cdot,0)}) + \frac{\partial y_a^\mu}{\partial x_b^\nu}(x_{(\cdot,0)})\, x_{(b,b+1)}^\nu e\ . \tag{A.52}$$

We see then that $x_{(a,a+1)}$ transform as fiber coordinates of the tangent bundle to $\mathcal{M}$ and so we have the isomorphism $\text{Maps}(\mathbb{R}[-1],\mathcal{M}) = T[1]\mathcal{M}$.

Appendix B

Propagators

We collect in this appendix the computations of the propagators used in Chapter 5.4. We will firstly consider one-dimensional BF propagator on the circle and on the interval with the various possible polarizations on the two end-points. Then we will use these to compute the axial-gauge propagator on 2D surfaces, in particular on the cylinder $S^1 \times I$ and on the square $I \times I$.

B.1 One-dimensional propagators

Propagator on the circle

Let us consider non-abelian BF theory on the circle S^1 . We are looking for the propagator when we expand the action with respect to the trivial connection. In this case the kinetic term is $\int_{S^1} \langle B, \mathrm{d}A\rangle$. The space of zero-modes is thus given by the de Rham cohomology: $\mathcal{V} = H_{\mathrm{dR}}(S^1; \mathfrak{g})[1] \oplus H_{\mathrm{dR}}(S^1; \mathfrak{g}^*)$. If $\tau \in [0, 1]$ is the coordinate of the circle, we have the corresponding basis $[1], [\mathrm{d}\tau]$ for the cohomology of the circle and the following coordinate expression for the zero modes:

$$\mathsf{a} = \mathsf{a}_0 + \mathsf{a}_1 \mathrm{d}\tau \ , \qquad \mathsf{b} = \mathsf{b}^0 + \mathsf{b}^1 \mathrm{d}\tau \ , \tag{B.1}$$

where $\mathsf{a}_{(i)} \in \mathfrak{g}$ and $\mathsf{b}_{(i)} \in \mathfrak{g}^*$. A Hodge decomposition for the de Rham complex of the circle is given by the following induction data:

$$\begin{aligned} \Pi\omega(\tau) &= \int_{S^1} (\mathrm{d}\tau' - \mathrm{d}\tau)\omega(\tau') \ , \\ K\omega(\tau) &= \int_{S^1} \Big(\Theta(\tau - \tau') - \tau + \tau' - \frac{1}{2}\Big)\omega(\tau') \ . \end{aligned} \tag{B.2}$$

The extension to the space of fields –Lie-algebra valued differential forms– is immediate and the resulting propagator is:[1]

$$b,\tau \xrightarrow{\eta_{S_1}} a,\tau' \quad = \eta_{S^1}(\tau, b; \tau', a) = \Big(\Theta(\tau-\tau') - \tau + \tau' - \frac{1}{2}\Big)\delta^a_b \ . \tag{B.3}$$

Propagators on the interval

Interval in $\mathbb{A}$-$\mathbb{B}$ polarization

Let us consider now BF theory on the unit interval $I = [0,1]$ with $\mathbb{B}$ polarization at $\{0\}$ and $\mathbb{A}$ polarization at $\{1\}$. The space of bulk fields is now given by differential forms with Dirichlet boundary conditions on one of the two endpoints. The cohomology of the differential d on this space of differential forms is vanishing, thus the space of zero-modes is empty $\mathcal{V} = 0$. The chain homotopy K is now

$$K\omega(t) = \int_t^1 \omega(t') \tag{B.4}$$

and we have the corresponding propagator

$$t \xrightarrow{\eta} t' \quad = \eta(t;t') = -\Theta(t'-t) \ . \tag{B.5}$$

Notice that propagation can only occur if $t < t'$, i.e. moving away from the $\mathbb{B}$ endpoint and toward the $\mathbb{A}$ endpoint of the interval.

Interval in $\mathbb{A}$-$\mathbb{A}$ polarization

If we take the $\mathbb{A}$ polarization on both endpoints of the interval, the A fields will have Dirichlet boundary conditions at the endpoints while the B fields will have free boundary conditions: $\mathcal{Y} = \Omega(I, \partial I; \mathfrak{g})[1] \oplus \Omega(I; \mathfrak{g}^*)$. The cohomology is concentrated in form-degree 1 for the A fields and in form-degree 0 for the B fields

$$\mathcal{V} = H(I, \partial I; \mathfrak{g})[1] \oplus H(I; \mathfrak{g}^*) \simeq \mathfrak{g}[1] \oplus \mathfrak{g}^* \ , \tag{B.6}$$

so that the form-degree expansion of the zero modes is $\mathsf{a} = \mathsf{a}_1 \mathrm{d}t$, $\mathsf{b} = \mathsf{b}^0$. The chain retraction is given by the following data:

$$\eta(t,t') = \Theta(t-t') - t \ , \qquad \pi(t,t') = -\mathrm{d}t \ . \tag{B.7}$$

Interval in $\mathbb{B}$-$\mathbb{B}$ polarization

The interval with $\mathbb{B}$ polarization on both endpoints has the role of A and B fields reversed with respect to the previous case. The space of bulk fields is $\mathcal{Y} = \Omega(I; \mathfrak{g})[1] \oplus \Omega(I, \partial I; \mathfrak{g}^*)$ and the zero-modes are: $\mathsf{a} = \mathsf{a}_0$, $\mathsf{b} = \mathsf{b}^1 \mathrm{d}t$. The propagator and the projection to cohomology are:

$$\eta(t,t') = -\Theta(t'-t) + t' \ , \qquad \pi(t,t') = \mathrm{d}t' \ . \tag{B.8}$$

[1] The Lie-algebra part of the propagator in this thesis is always the identity δ^a_b and will be often omitted.

B.2 Axial gauge propagators on the cylinder

Consider now a cylinder $S^1 \times I$ and let t denote the coordinate of the interval, τ the coordinate along the circle, χ_i a basis for the cohomology of S^1 and χ^i its dual basis. Using the 1-dimensional propagators of appendix B.1, from the axial-gauge formula (5.38) we get the following propagators on the cylinder.

Polarization		zero-modes	$\tau, t \xrightarrow{\eta(\tau,t;\tau',t')} \tau', t'$
	$\mathbb{A} - \mathbb{B}$	0	$-\Theta(t'-t)\delta(\tau-\tau')(\mathrm{d}\tau' - \mathrm{d}\tau)$
	$\mathbb{A} - \mathbb{A}$	$\mathsf{a} = \mathsf{a}_i \, \mathrm{d}t \wedge \chi^i$ $\mathsf{b} = \mathsf{b}^i \, \chi_i$	$(\Theta(t-t') - t)\delta(\tau'-\tau)(\mathrm{d}\tau' - \mathrm{d}\tau) - \mathrm{d}t\big(\Theta(\tau-\tau') - \tau - \tau' - \frac{1}{2}\big)$
	$\mathbb{B} - \mathbb{B}$	$\mathsf{a} = \mathsf{a}_i \, \chi^i$ $\mathsf{b} = \mathsf{b}^i \, \mathrm{d}t \wedge \chi_i$	$(t' - \Theta(t'-t))\delta(\tau'-\tau)(\mathrm{d}\tau' - \mathrm{d}\tau) + \mathrm{d}t'\big(\Theta(\tau-\tau') - \tau - \tau' - \frac{1}{2}\big)$

(B.9)

Reversing the role of the circle and the interval in formula (5.38) we would obtain different expressions for the propagator, called for the cylinder *horizontal gauge*, but we don't need this choice in this thesis.

Appendix C

Computations of some Feynman diagrams

We present here the proofs of Propositions 5.3.3, 5.4.12, consisting in the evaluation of tree and loop diagrams in the axial gauge. These computations are variations of the ones contained in [42], Lemma 3 and 4, obtained in the 1-dimensional setting.

Proof of Proposition 5.3.3. We have to evaluate the 1-loop diagrams of figure 5.5. The amplitude for a diagram with $n \geqslant 2$ vertices is:

$$\begin{aligned} &\frac{1}{n}\mathrm{tr}(\mathrm{ad}^n_{\mathsf{a}_1}) \int_{(S^1)^n} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \; \eta_{S^1}(\tau_1;\tau_2)\cdots\eta_{S^1}(\tau_{n-1};\tau_n)\eta_{S^1}(\tau_n;\tau_1) \\ &= \frac{1}{n}\mathrm{tr}(\mathrm{ad}_{\mathsf{a}_1})^n \, \mathrm{tr}\big(K(\chi_1\wedge\bullet)\big)^n \; . \end{aligned} \tag{C.1}$$

where we choose the basis $\chi_0 = 1, \chi_1 = \mathrm{d}\tau$ for $H^\bullet(S^1)$ and K is the chain homotopy with integral kernel $\eta_{S^1}(\tau;\tau') = \Theta(\tau-\tau') - \tau + \tau' - \frac{1}{2}$. We will compute $\mathrm{tr}\big(K(\chi_1\wedge\bullet)\big)^n$ in the monomial basis $1, \tau, \tau^2, \dots$. Let us define the generating function:

$$f_m(x,\tau) = \sum_{n=0}^{\infty} x^n \big(K(\chi_1\wedge\bullet)\big)^n \tau^m \; . \tag{C.2}$$

Applying $xK(\chi_1 \wedge \bullet)$ on both sides we get

$$xK(\chi_1 f_m)(x,\tau) = f_m(x,\tau) - \tau^m \tag{C.3}$$

and, differentiating w.r.t. τ, we obtain the differential equation:

$$\frac{\partial}{\partial\tau} f_m = x f_m + m\tau^{m-1} - x\int_0^1 \mathrm{d}\tau \; f_m \; . \tag{C.4}$$

Solutions to the above equation are of the form

$$f_m(x,\tau) = A(x)\mathrm{e}^{x\tau} + B(x) + \mathrm{e}^{x\tau}\int_0^\tau \mathrm{d}\tilde\tau \; m\tilde\tau^{m-1}\mathrm{e}^{-x\tilde\tau} \; , \tag{C.5}$$

where $A(x) = \frac{1}{e^x-1}(1 - e^x \int_0^1 d\tau \; m\tau^{m-1} e^{-x\tau})$ and $B(x)$ is to be determined from the boundary conditions. Since $K(\chi_1 f_m)(x,0) = K(\chi_1 f_m)(x,1)$, from (C.3) we have:

$$\begin{aligned} f_m(x,1) - 1 = f_m(x,0) &= xK(\chi_1 f_m)(x,0) \\ &= x \int_0^1 d\tau \left(A(x)e^{x\tau} + B(x) + e^{x\tau} \int_0^\tau d\tilde{\tau} \; m\tilde{\tau}^{m-1} e^{-x\tilde{\tau}} \right)(\tau - \frac{1}{2}) \\ &= A(x)g(x) + C(x) \; , \end{aligned} \tag{C.6}$$

where $g(x) = x \int_0^1 d\tilde{\tau} \; (\tilde{\tau} - \frac{1}{2})e^{x\tilde{\tau}}$ and $C(x) = x \int_0^1 d\tau \; e^{x\tau}(\tau - \frac{1}{2}) \int_0^\tau d\tilde{\tau} \; m\tilde{\tau}^{m-1} e^{-x\tilde{\tau}}$. Moreover from (C.5) we have $f_m(x,0) = A(x) + B(x)$ and thus:

$$f_m(x,\tau) = \frac{e^{x\tau} - 1}{e^x - 1} \left(1 - e^x \int_0^1 d\tilde{\tau} \; m\tilde{\tau}^{m-1} e^{-x\tilde{\tau}} \right) + e^{x\tau} \int_0^\tau d\tilde{\tau} \; m\tilde{\tau}^{m-1} e^{-x\tilde{\tau}} + f_m(x,0) \; . \tag{C.7}$$

We can now extract the trace of powers of $\mathcal{M} := K(\chi_1 \wedge \bullet)$ from the series of the coefficients of τ^m in the expansion of f_m:

$$\begin{aligned} f_{mm}(x) &:= \sum_{n=0}^\infty \langle \tau^m | x^n \mathcal{M}^n | \tau^m \rangle \; , \\ \Rightarrow \quad \sum_{m=1}^\infty (f_{mm}(x) - 1) &= \sum_{n=1}^\infty x^n \mathrm{tr} \mathcal{M}^n \; . \end{aligned} \tag{C.8}$$

The coefficients $f_{mm}(x)$ can be read from (C.7):

$$\begin{aligned} f_m(x,\tau) &= \frac{e^{x\tau} - 1}{e^x - 1} \sum_{k=0}^{m-1} \frac{m!}{(m-k)!} x^{-k} - \sum_{k=0}^{m-1} \frac{m!}{(m-k)!} \tau^{m-k} x^{-k} + f_m(x,0), \\ \Rightarrow \quad f_{mm}(x) &= 1 - \frac{1}{e^x - 1} \sum_{k=m+1}^\infty \frac{x^k}{k!} \; . \end{aligned} \tag{C.9}$$

Thus we get:

$$\begin{aligned} \sum_{n=1}^\infty x^n \mathrm{tr} \mathcal{M}^n &= -\frac{1}{e^x - 1} \sum_{m=1}^\infty \sum_{k=m+1}^\infty \frac{x^k}{k!} = -\frac{1}{e^x - 1} \sum_{k=2}^\infty \frac{k-1}{k!} x^k \\ &= 1 - x - \frac{x}{e^x - 1} = -\frac{1}{2}x - \sum_{n=2}^\infty \frac{B_n}{n!} x^n \; , \\ \Rightarrow \quad \mathrm{tr} \mathcal{M}^n &= -\frac{B_n}{n!} \qquad \text{for } n \geqslant 2 \; , \end{aligned} \tag{C.10}$$

where B_n are the Bernoulli numbers. □

Proof of Proposition 5.4.12. We have to evaluate the diagrams of the kind depicted in figure 5.14. The amplitude I_n for a tree rooted on a $\mathbb{B}$ boundary field and ending on α is:

$$I_n = (-1)^{n+1} \int_{I^{\times(n+1)}} \langle \mathbb{B}_{(1)}(\tau_0), \eta_I(\tau_0,\tau_1) \cdots \eta_I(\tau_{n-1},\tau_n) \, \eta_I(\tau_n,1) \, \mathrm{ad}^n_{\mathsf{a}_1} \alpha \rangle d\tau_0 \cdots d\tau_n \; , \tag{C.11}$$

where $\eta_I(\tau,\tau') = \Theta(\tau-\tau') - \tau$. The result of this integral can be expressed in terms of the Bernoulli polynomials:

$$I_n = (-1)^n \int_I \langle \mathbb{B}_{(1)}(\tau), \frac{B_{n+1}(\tau) - B_{n+1}}{(n+1)!} \mathrm{ad}^n_{\mathsf{a}_1} \alpha \rangle \ . \tag{C.12}$$

To prove (C.12), let us define the operator $Kg(\tau) := \int_I \eta_I(\tau,\tau')g(\tau')\mathrm{d}\tau'$ and the generating function

$$f(x;\tau) := \sum_{n=0}^{\infty} x^n K^n(t) \ . \tag{C.13}$$

This function satisfies the differential equation:

$$\frac{\partial}{\partial\tau} f(x;\tau) = xf(x;\tau) + 1 - \int_I f(x;\tau')\mathrm{d}\tau' = xf(x;\tau) + C(x) \ , \tag{C.14}$$

where $C(x)$ does not depend on τ. Since only the term $n=0$ contributes to f evaluated on the endpoints $\tau = 0,1$, f satisfies $f(x;0)=0$, $f(x;1)=1$. Solving the differential equation with this boundary conditions we get:

$$f(x;\tau) = \frac{1-\mathrm{e}^{x\tau}}{1-\mathrm{e}^{x}} = \frac{1}{x}\Big(\frac{x}{1-\mathrm{e}^{x}} - \frac{x\mathrm{e}^{x\tau}}{1-\mathrm{e}^{x}}\Big) = \sum_{n=0}^{\infty} \frac{B_{n+1}(\tau)-B_{n+1}}{(n+1)!} x^n \ . \tag{C.15}$$

Since $K(1) = 0$, we have $K^n(\eta_I(\tau;1)) = -K^n(\eta_I(\tau;0))$. Thus, similar contributions to C.12 come from trees ending on $\widetilde{\alpha}$ (the main difference being in the term for $n=0$) or rooted on $\widetilde{\mathbb{B}}$ or on b. By summing over n we get, for the tree part of the effective action:

$$\begin{aligned}\mathcal{S}^{\text{eff.}}_{\text{tree}} = \int_0^1 \langle \mathbb{B}_{(1)}(\tau) - \widetilde{\mathbb{B}}_{(1)}(\tau), \mathsf{G}_+(\tau, \mathrm{ad}_{\mathsf{a}_1})\alpha + \mathsf{G}_-(\tau,\mathrm{ad}_{\mathsf{a}_1})\widetilde{\alpha}\rangle \mathrm{d}\tau + \\ + \langle \mathsf{b}^1, \mathsf{F}_+(\mathrm{ad}_{\mathsf{a}_1})\alpha + \mathsf{F}_-(\mathrm{ad}_{\mathsf{a}_1})\widetilde{\alpha}\rangle \ .\end{aligned} \tag{C.16}$$

The amplitude for a wheel diagram is:

$$-\frac{\mathrm{i}\hbar}{n}\mathrm{tr}(\mathrm{ad}^n_{\mathsf{a}_1}) \int_{I^n} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \ \eta_I(\tau_1;\tau_2)\cdots\eta_I(\tau_{n-1};\tau_n)\eta_I(\tau_n;\tau_1) \ . \tag{C.17}$$

This integral is the same as the one appearing in [42] and can be computed with the technique used to prove equation C.1. The result for the loop contribution to the effective action is thus:

$$\mathcal{S}^{\text{eff.}}_{\text{loop}} = -\mathrm{i}\hbar \sum_{n\geqslant 2} \frac{1}{n!}\mathrm{tr}(\mathrm{ad}^n_{\mathsf{a}_1})\frac{B_n}{n} = -\mathrm{i}\hbar \ \mathrm{tr}\left(\log\left(\frac{\sinh(\mathrm{ad}_{\mathsf{a}_1}/2)}{\mathrm{ad}_{\mathsf{a}_1}/2}\right)\right) \ . \tag{C.18}$$

□

Appendix D

Two technical proofs

D.1 Proof of Proposition 5.4.5

Here we present a direct computational proof that $\Omega^2 = 0$ for a stratified circle, with any choice of polarizations on the edges and corners.

First note that edge contributions $\Omega_I^{\mathbb{A}}$, $\Omega_I^{\mathbb{B}}$ and pure corner contributions Ω_p^{α}, Ω_p^{β} all square to zero. Also, edge contributions and pure corner contributions commute. In particular, we have

$$\Omega^2 = \sum_k \left(\Omega_{p_k}^{\mathbb{P}_k \xi_k \mathbb{P}_{k+1}}\right)^2 + \left[\Omega_{p_k}^{\mathbb{P}_k \xi_k \mathbb{P}_{k+1}}, \Omega_{I_k}^{\mathbb{P}_k} + \Omega_{I_{k+1}}^{\mathbb{P}_{k+1}}\right] . \tag{D.1}$$

Denote $\mathrm{BCH}(x,y) = \log(\mathrm{e}^x \mathrm{e}^y)$ for $x, y \in \mathfrak{g}$ – the Baker-Campbell-Hausdorff group law. We will need the following identities

$$\mathrm{BCH}(x,y) = x + \mathsf{F}_+(\mathrm{ad}_x)y + \mathcal{O}(y^2) \ , \qquad \mathrm{BCH}(x,y) = y - \mathsf{F}_-(\mathrm{ad}_y)x + \mathcal{O}(x^2) \tag{D.2}$$

which are the cases of the BCH formula when either first or second argument is infinitesimal.

Let us study e.g. a β-corner p surrounded by a $\mathbb{B}$-edge I' on the left and an $\mathbb{A}$-edge I on the right. We have $\Omega_p^{\mathbb{B}\beta\mathbb{A}} = \Omega_p^{\beta\mathbb{A}}$ given by (5.94). Applying this operator to a wavefunction of form $\psi(\beta) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, x\rangle}$, with $x \in \mathfrak{g}$ a parameter, yields

$$\Omega_p^{\beta\mathbb{A}}\psi = \langle \beta, \mathsf{F}_+(\mathrm{ad}_x)\mathbb{A}_p\rangle\, \psi = \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}\epsilon}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, \mathrm{BCH}(x, \epsilon\, \mathbb{A}_p)\rangle} \ , \tag{D.3}$$

with ϵ an odd parameter. Here we have used the first identity in (D.2). Note that similarly one can write $\Omega_p^{\mathbb{A}\beta}\psi = -\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, \mathrm{BCH}(\epsilon\, \mathbb{A}_p, x)\rangle}$, using the second identity in (D.2).

This implies:

$$
\begin{aligned}
\left(\Omega_p^{\beta\mathbb{A}}\right)^2\psi &= (\mathrm{i}\hbar)^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon_2}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, \mathrm{BCH}(\mathrm{BCH}(x,\epsilon_1\,\mathbb{A}_p),\epsilon_2\,\mathbb{A}_p)\rangle} \\
&= (\mathrm{i}\hbar)^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon_2}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, \mathrm{BCH}(x,\mathrm{BCH}(\epsilon_1\,\mathbb{A}_p,\epsilon_2\,\mathbb{A}_p))\rangle} \\
&= (\mathrm{i}\hbar)^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon_2}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, \mathrm{BCH}(x,(\epsilon_1+\epsilon_2)\,\mathbb{A}_p-\frac{1}{2}\epsilon_1\epsilon_2\,[\mathbb{A}_p,\mathbb{A}_p])\rangle} \\
&= (\mathrm{i}\hbar)^2 \frac{\mathrm{d}}{\mathrm{d}\epsilon_2}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \beta, \mathrm{BCH}(x,-\frac{1}{2}\epsilon_1\epsilon_2\,[\mathbb{A}_p,\mathbb{A}_p])\rangle} \\
&= -\frac{\mathrm{i}\hbar}{2}\left\langle [\mathbb{A}_p,\mathbb{A}_p], \frac{\partial}{\partial \mathbb{A}_p}\right\rangle \Omega_p^{\beta\mathbb{A}}\psi = -[\Omega_I^{\mathbb{A}},\Omega_p^{\beta\mathbb{A}}]\psi\ .
\end{aligned} \tag{D.4}
$$

Note that the main trick of this computation is the use of associativity of the BCH formula. Operators $\left(\Omega_p^{\beta\mathbb{A}}\right)^2$, $[\Omega_I^{\mathbb{A}},\Omega_p^{\beta\mathbb{A}}]$ are multiplication operators in the variable $\mathbb{A}_p$, thus the computation above, for ψ independent of $\mathbb{A}_p$ is sufficient to ascertain that $\left(\Omega_p^{\beta\mathbb{A}}\right)^2+[\Omega_p^{\beta\mathbb{A}},\Omega_I^{\mathbb{A}}]=0$ as operators. Further, note that $[\Omega_p^{\beta\mathbb{A}},\Omega_{I'}^{\mathbb{B}}]$ contains derivatives in $\mathbb{B}_p^{(1)}$ and therefore vanishes on *admissible* states, in the sense of Assumption 5.4.4. This proves that the contribution of a $\mathbb{B}\beta\mathbb{A}$ corner to Ω^2 (cf. the right hand side of (D.1)) vanishes. The case $\mathbb{A}\beta\mathbb{B}$ is an orientation reversal of the case we just studied; it is treated analogously and also yields a zero contribution to the r.h.s. of (D.1).

Case of an $\mathbb{A}\beta\mathbb{A}$ corner is treated similarly. Here $\Omega_p^{\mathbb{A}\beta\mathbb{A}}=\Omega_p^{\mathbb{A}\beta}+\Omega_p^{\beta\mathbb{A}}$. We have $\left(\Omega_p^{\beta\mathbb{A}}\right)^2+[\Omega_p^{\beta\mathbb{A}},\Omega_I^{\mathbb{A}}]=0$ as above, and similarly $\left(\Omega_p^{\mathbb{A}\beta}\right)^2+[\Omega_p^{\mathbb{A}\beta},\Omega_{I'}^{\mathbb{A}}]=0$. We also need to understand the term $[\Omega_p^{\mathbb{A}\beta},\Omega_p^{\beta\mathbb{A}}]$, which is done similarly to (D.4):

$$
\begin{aligned}
\Omega_p^{\mathbb{A}\beta}\Omega_p^{\beta\mathbb{A}}\psi &= -(\mathrm{i}\hbar)^2\frac{\mathrm{d}}{\mathrm{d}\epsilon_2}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\mathrm{BCH}(\epsilon_2\,\mathbb{A}_{p-0},\mathrm{BCH}(x,\epsilon_1\,\mathbb{A}_{p+0}))\rangle} \\
&= -(\mathrm{i}\hbar)^2\frac{\mathrm{d}}{\mathrm{d}\epsilon_2}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\beta,\mathrm{BCH}(\mathrm{BCH}(\epsilon_2\,\mathbb{A}_{p-0},x),\epsilon_1\,\mathbb{A}_{p+0}))\rangle} = -\Omega_p^{\beta\mathbb{A}}\Omega_p^{\mathbb{A}\beta}\psi\ .
\end{aligned} \tag{D.5}
$$

Hence, $[\Omega_p^{\mathbb{A}\beta},\Omega_p^{\beta\mathbb{A}}]=0$ and the contribution of an $\mathbb{A}\beta\mathbb{A}$ corner to the r.h.s. of (D.1) vanishes.

In the case of an $\mathbb{A}\alpha\mathbb{B}$ corner, we have $\Omega_p^{\mathbb{A}\alpha\mathbb{B}}=\Omega_p^{\alpha}+\Omega_p^{\alpha\mathbb{B}}$. By a computation similar to (D.4), one shows that $(\left(\Omega_p^{\alpha\mathbb{B}}\right)^2+[\Omega_p^{\alpha},\Omega_p^{\alpha\mathbb{B}}])\psi=0$ for $\psi=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\mathbb{B}_p,x\rangle}$. Together with $\left(\Omega_p^{\alpha}\right)^2=0$, this shows that $\left(\Omega_p^{\mathbb{A}\alpha\mathbb{B}}\right)^2=0$. Furthermore, $[\Omega_p^{\mathbb{A}\alpha\mathbb{B}},\Omega_{I'}^{\mathbb{A}}]=0$ and $[\Omega_p^{\mathbb{A}\alpha\mathbb{B}},\Omega_I^{\mathbb{B}}]=0$ on admissible states. Thus, the contribution an $\mathbb{A}\alpha\mathbb{B}$ to the r.h.s. of (D.1) also vanishes. Orientation-reversed case $\mathbb{B}\alpha\mathbb{A}$ is similar.

Case of a $\mathbb{B}\alpha\mathbb{B}$ corner is similar to the above: we have $\Omega_p^{\mathbb{B}\alpha\mathbb{B}}=\Omega_p^{\mathbb{B}\alpha}+\Omega_p^{\alpha}+\Omega_p^{\alpha\mathbb{B}}$. As above, we have $\left(\Omega_p^{\alpha\mathbb{B}}\right)^2+[\Omega_p^{\alpha},\Omega_p^{\alpha\mathbb{B}}]=0$ and similarly $\left(\Omega_p^{\mathbb{B}\alpha}\right)^2+[\Omega_p^{\alpha},\Omega_p^{\mathbb{B}\alpha}]=0$. One also trivially has $[\Omega_p^{\mathbb{B}\alpha},\Omega_p^{\alpha\mathbb{B}}]=0$. Thus, $\left(\Omega_p^{\mathbb{B}\alpha\mathbb{B}}\right)^2=0$. Also, the corner contribution to Ω commutes with the edge terms on admissible states. This proves that the contribution of a $\mathbb{B}\alpha\mathbb{B}$ corner to the r.h.s. of (D.1) vanishes, too.

Cases of $\mathbb{A}\alpha\mathbb{A}$ and $\mathbb{B}\beta\mathbb{B}$ corners are trivial. This finishes the proof that all terms in the sum (D.1) over the corners vanish, and thus $\Omega^2 = 0$ for an arbitrarily stratified and polarized circle.

D.2 A check of the chain map property of the inclusion of the small model for $\mathbb{A}$-states on an interval into the full model

One can check directly that (5.128) is indeed a chain map. First, it is clearly an algebra morphism (w.r.t. the standard supercommutative pointwise product on $\mathrm{Fun}(\cdots)$), so it is enough to check the chain map property on a set of generators of $\mathcal{H}^{\mathrm{small}}$. Assume for simplicity that $\mathfrak{g} \subset \mathrm{Mat}_N$ is a matrix Lie algebra and choose as generators

$$f_{k,\rho} := \mathrm{tr}\,\rho\,\underline{\mathbb{A}}_k\ , \qquad g_\rho := \mathrm{tr}\,\rho\,\mathrm{e}^{\underline{\mathbb{A}}}\ , \tag{D.6}$$

with $\rho \in \mathrm{Mat}_N$ arbitrary parameter and $k = 0, 1$. From (5.126) and (5.128), we immediately obtain that $i_\mathcal{H} \circ \Omega^{\mathrm{small}} = \Omega \circ i_\mathcal{H}$ when applied to the generators $f_{k,\rho}$. For g_ρ, from (5.126,5.128) and from the rule for the deformation of holonomy under an infinitesimal gauge transformation, we obtain:

$$(i_\mathcal{H} \circ \Omega^{\mathrm{small}})g_\rho = (\Omega \circ i_\mathcal{H})g_\rho = -\mathrm{i}\hbar\,\mathrm{tr}\,\rho\,\big(U(\mathbb{A}) \cdot \mathbb{A}_1 - \mathbb{A}_0 \cdot U(\mathbb{A})\big)\ . \tag{D.7}$$

Here we used the observation that $\Omega^{\mathrm{small}} g_\rho = -\mathrm{i}\hbar\,\mathrm{tr}\,\rho(\mathrm{e}^X c_1 - c_0 \mathrm{e}^X)$ with shorthand notation $c_0 = \underline{\mathbb{A}}_0$, $c_1 = \underline{\mathbb{A}}_1$, $X = \underline{\mathbb{A}}$. Indeed, we have

$$\begin{aligned} \frac{\mathrm{i}}{\hbar}\Omega^{\mathrm{small}} g_\rho &= \mathrm{tr}\,\rho \sum_{p,q\geq 0} \frac{1}{(p+q+1)!} X^p (\mathsf{F}_+(\mathrm{ad}_X)c_1 + \mathsf{F}_-(\mathrm{ad}_X)c_0) X^q \\ &= \mathrm{tr}\,\rho \sum_{p,q,j,l\geq 0} \frac{1}{(p+q+1)!} \left(\frac{(-1)^l B^+_{j+l}}{j!l!} X^{j+p} c_1 X^{l+q} - \frac{(-1)^l B^-_{j+l}}{j!l!} X^{j+p} c_0 X^{l+q} \right)\ , \end{aligned} \tag{D.8}$$

where B_i^+ and B_i^- are the Taylor coefficients of $\mathsf{F}_+(x)$ and $-\mathsf{F}_-(x)$, respectively. Note that, for x, y scalars, we have

$$\begin{aligned} &\sum_{p,q,j,l\geq 0} \frac{1}{(p+q+1)!} \frac{(-1)^l B^+_{j+l}}{j!l!} x^{j+p} y^{l+q} = \left(\sum_{p,q\geq 0} \frac{x^p y^q}{(p+q+1)!} \right) \left(\sum_{j,l\geq 0} \frac{(-1)^l B^+_{j+l}}{j!l!} x^j y^l \right) \\ &= \frac{\mathrm{e}^x - \mathrm{e}^y}{x-y} \cdot \mathsf{F}_+(x-y) = \mathrm{e}^x \end{aligned} \tag{D.9}$$

and, similarly, $\sum_{p,q,j,l\geq 0} \frac{1}{(p+q+1)!} \frac{(-1)^l B^-_{j+l}}{j!l!} x^{j+p} y^{l+q} = \mathrm{e}^y$. Thus:

$$\frac{\mathrm{i}}{\hbar}\Omega^{\mathrm{small}} g_\rho = \mathrm{tr}\,\rho\big(\mathrm{e}^X c_1 - c_0 \mathrm{e}^X\big) \tag{D.10}$$

as claimed.

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