Scuola Internazionale Superiore di Studi Avanzati

QUANTUM EFFECTIVE ACTIONS IN WEYL-FLAT SPACETIMES AND APPLICATIONS

André Feitosa Benevides

Supervisor: Atish Dabholkar

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Foreword

List of Publications

This thesis contains a partial summary of my PhD research and is heavily based on the following two publications:

- T. Bautista, A. Benevides and A. Dabholkar, Nonlocal Quantum Effective Actions in Weyl-Flat Spacetimes: JHEP 1806 (2018) 055, [arXiv:1711.00135].
- A. Benevides, A. Dabholkar and T. Kobayashi: To B or not to B with QED Weyl anomaly, arXiv:1808.08237.

Another one of my works produced during the PhD, which is not part of this thesis but highly motivated the two papers mentioned before, consists of an analysis of the consequences of a toy model, trying to capture some of the physics of Weyl anomalies in the purely gravitational context.

- T. Bautista, A. Benevides, A. Dabholkar and A. Goel, Quantum Cosmology in Four Dimensions: arXiv:1512.03275.

Abstract: Computing quantum effective actions is paramount to any semi-classical problem in quantum field theory. However the calculation can be extremely challenging in the presence of complicated sources or of a curved spacetime. It is possible to progress when the sources and the gravitational field are weak, however some very relevant physical situations require strong fields and are beyond any such approximations. Half of this thesis is devoted to developing a technique capable of computing effective actions reliably, without assuming a weak gravitational field. The results are applicable to classically Weyl-invariant theories living in Weyl-flat spacetimes. The second half of the thesis is devoted to an application of this particular method. Relatively intense magnetic fields exist in the universe, coherent at Mpc scales. Such fields call for a primordial origin, however to date there is no clear understanding on how they are created. The hypothesis that they are of quantum origin, coming from the Weyl anomaly of the Standard Model itself has been discussed in the past, with some results claiming positively that this is indeed possible. We use our methods developed in the first half of the thesis to settle the issue proving it is actually impossible to generate such fields from the Weyl anomaly of the standard model alone. We conclude that such fields, if they exist, must originate from something beyond the Standard Model.

Contents

Chapter 1

Introduction

General relativity is tremendously successful in describing the classical dynamics of gravity. It does not however, incorporate the quantum properties of matter and of gravity itself in its formulation. One may be interested in going a step further and exploring a semiclassical theory in which quantum fluctuations are studied in a background that is still treated classically. Such an analysis may provide insights and predictions avoiding the complications of a full quantum treatment.

Relativistic quantum fields are the essential building blocks to describe the quantum dynamics of relativistic systems. Although most of the physics of very large scales is perfectly well covered by classical field theory, considerations in the early universe or explorations of AdS/CFT duality naturally leads one to include quantum effects, at least semi-classically. Such an analysis is especially important when massless particles are present, since their interactions can severely modify the infrared dynamics of field theories on curved spaces.

Semiclassical analysis has two sides, the first one consists on the effects of the backgrounds on the quantum fluctuating fields, and the second one on describing the back reaction of the quanta on the background fields. As has been established during the previous decades, gravitational fields can create particles. Black holes emit radiation with a thermal distribution, de Sitter space also generates particles during the rapid expansion, for example. The back reaction of such particles on the backgrounds however, is a much more challenging problem and much about it is still unknown.

Generically quantum loops of massless particles introduce non-localities in the correlation functions. The full spectrum of physical implications of such terms is not yet completely understood, especially when non-trivial background fields are present. In the non-local terms many effects are encoded such as vacuum polarisation and particle creation, for instance. A special subclass of non-local terms in the effective action encode information about the quantum violations of classical symmetries, also called anomalies. They are special in that they can be read directly from the ultraviolet behaviour of the correlators of the infrared degrees of freedom, directly from local quantities.

The aim of this thesis is twofold. We focus on physics of special spacetimes whose metric can be cast in the form $g_{\mu\nu} = \xi(x)\eta_{\mu\nu}$, called Weyl-flat spaces[1]. We

[1]Where $\xi(x)$ is a positive function and η is the metric of Minkowski space.

first develop a technique using anomalies to compute the appropriate quantum effective action relevant for a semi-classical analysis in such backgrounds. Doing so we determine the full dependence of the quantum effective action on the Weyl factor as well as the fate of the remaining non-local terms. Secondly we apply this technique to a cosmological scenario concluding that the quantum fluctuations of the gauge field actually do not get converted into classical fluctuations, contrary to the claims of previous works.

I. Evaluation of the Effective Action The essential information about the correlation functions of a quantum field theory can be compactly encoded in the one-particle-irreducible (1PI) effective action. The same object can be used to compute the semi-classical equations of motion through a variational principle. One can then proceed to solve those equations for the expectation values of the relevant observables. In applications to cosmology or some other scenario involving a causal time evolution, one is primarily interested in computing the matrix elements on a particular state, often the initial state $|in\rangle$. [2]

The main challenge is the evaluation of the quantum effective action itself. The problem is especially hard when light particles are involved, in which case knowledge about the infrared behaviour is hard to obtain. In order to compute the finite contributions one is often forced to rely on weak field approximations, which restrict the range of validity of the analysis to very special circumstances only.

Despite the limitations, weak field perturbation can still be applied to several interesting situations, being used to compute the leading corrections to the classical newtonian potential [1], the quantum corrections to the bending of light [2] and to the geometry around black holes [3], for instance. Such effects however, tend to be tremendously small due to high value of the Plank mass, the leading correction to the perihelion of Mercury being of one part in 10^{90}, for example. One may hope that in situations involving fields and interactions other than gravity and scales other than the Plank mass, the effects may be enhanced.

One powerful approach to simplify the computation of the numerous Feynman diagrams typically involved in such analysis is through the Covariant Heat Kernel expansion, also referred to as the Barvinsky-Vilkovisky (BV) or non-local Heat Kernel expansion. The technique consists of an asymptotic expansion of the Heat Kernel in powers of curvatures and its contractions, such as the Riemann tensor, the gauge curvature, the Ricci scalar and so on. The range of applicability of this expansion is relatively small though. It is only valid for rapidly oscillating background fields, more precisely, background satisfying the condition $\nabla^2\mathcal{R} \gg \mathcal{R}^2$.

Most of the present analytical computations rely on a high degree of symmetry of the relevant background or are based on small perturbations around Minkowski space. Thus developing a technique capable of going beyond those limitations is of paramount relevance. Field theories that are classically Weyl invariant may be more treatable in this regard.

Weyl symmetry may be violated in the quantum level. The breaking of Weyl symmetry leaves imprints on the expectation value of the energy momentum tensor

[2]Such matrix elements $\langle in|\mathcal{O}|in\rangle$, as well as the 1PI generator of such correlation functions are called in-in.

and thus can have important consequences for the semi-classical dynamics in curved spacetimes. Furthermore Weyl anomalies can be computed directly by local methods and used to infer part the quantum action which incorporates them. One must be careful though, since more often than not the information coming from the anomalies alone is not the full answer. Quite generically the 1PI Effective action contains infinitely many Weyl invariant non-local terms. Despite not contributing to the anomaly, those terms may play a role in the semi-classical dynamics.

Information about Weyl invariant non-local terms is very hard to obtain. Unlike the terms encoding the anomalies, there is no shortcut to compute those. However exactly because they are Weyl invariant, they benefit from a tremendous simplification when the theory is put on Weyl-flat spacetime. This means that most of them either evaluate to zero, or can be inferred directly by flat-space Feynman diagrams. One can gain important knowledge about the Weyl invariant terms in such cases. Ultimately, the technique we will discuss combines knowledge about the Weyl anomalies with the special properties of a Weyl-flat background to go beyond the weak field regime. The precise details can be found in chapter 2.

Classically Weyl invariant theories have a vanishing trace of the energy momentum tensor, and thus do not couple to the dynamical conformal factor. However, due to interactions, couplings acquire a dependence on the scale, which in turn contributes to the expectation value of the energy momentum tensor, changing how matter couples to the metric. A concrete scenario in which this technique can be applied is a Robertson-Walker metric in four dimensions with a gauge theory coupled to conformal matter. This simple situation could have been realised in early stages of the universe while the standard model group was still unbroken, making this system physically relevant as well.

II. Applications Cosmology is one appealing set up to study quantum fields in non-trivial backgrounds. In addition to the simplicity of the metric, allowing one to go beyond the weak field approximation at times, inflation can potentially magnify the quantum fluctuations, enhancing what would otherwise be unobservable.

Further motivation to consider semi-classical dynamics on cosmological backgrounds come from experiments. Several cosmological observations still lack a complete theoretical understanding. Magnetic fields, for example, are observed to exist in the universe on various cosmological scales. Magnetic fields of the order of micro Gauss are observed in galaxies and galaxy clusters coherent on scales up to ten kiloparsec. In intergalactic voids, there is evidence from blazar observations for weak magnetic fields of the order of 10^{-15} Gauss that are coherent on magaparsec scales. See [4, 5, 6, 7, 8, 9, 10, 11] for reviews from different perspectives.

The origin of these magnetic fields has been a long-standing mystery. A coherent field on such large scales calls for a primordial origin. However the Maxwell action by itself cannot generate primordial magnetic fields. This is a simple consequence of the Weyl invariance of the action. The dynamics of the electromagnetic field governed by a Weyl invariant action in a Robertson-Walker spacetime is independent of the scale factor and hence unaffected by the expansion of the universe. Primordial magnetic fields therefore require a violation of the Weyl invariance of the Maxwell action.

Most models of primordial magnetogenesis [12, 13, 14] violate the Weyl invariance explicitly at the classical level. A simple class of models starts with the action

$$S = -\frac{1}{4}\int d^4x\sqrt{|g|}\, I^2(\phi)\, F_{\mu\nu}F^{\mu\nu} \tag{1.1}$$

where I^2 is an arbitrary unknown function of various scalar fields denoted collectively by ϕ. In effective field theory, $I^2(\phi)$ is expected to have a Taylor expansion:

$$I^2(\phi) \sim 1 + \alpha_1\frac{\phi}{M} + \alpha_2\frac{\phi^2}{M^2} + \dots . \tag{1.2}$$

where α_i are some couplings and M is a cutoff scale. A scalar field in four dimensions has Weyl weight one (equal to its mass dimension). Hence, the higher dimensional operators coupling the electromagnetic field to the scalar fields must break Weyl invariance. If any of the scalar field has a time-dependent expectation value $\phi(t)$ during cosmological evolution, then it introduces a time dependence in the evolution of the electromagnetic field that can lead to a nontrivial spectrum of fluctuations. Models described by (1.1) provide a useful reference point for exploring the phenomenology of primordial magnetic genesis. However, the conclusions depend on the arbitrary model-dependent function $I^2(\phi)$ which is not determined by known particle physics.

It is important to consider a model-independent source of Weyl non-invariance arising purely from quantum violations of the symmetry and investigate how intense can the magnetic fields generated purely by this effect be. The classical Weyl invariance of the Maxwell action and more generally of the Yang-Mills action is naturally violated in the quantum theory due to the need of regularising the path integral. This is the minimal amount of Weyl invariance one can have without introducing explicitly Weyl-breaking terms or exotic couplings that are extraneous to the Standard Model.

It is natural to ask whether the effects of Weyl symmetry breaking can be computed explicitly and if the resulting effective coupling between the fields and the metric can generate primordial magnetic fields. As we will show, quite generically the standard model prediction is at best 20 orders of magnitude below the observed value. Such a negative answer to these questions indicate that the relatively intense fields we observe come from something beyond the standard model itself.

Another theoretically interesting possibility is that fields of other groups could have been generated while they were still unbroken, which may have left some observable imprints at the relevant scale. Could colour magnetic fields have been generated before the QCD phase transition? We discuss this possibility in the end of chapter 3.

Outline The thesis is organised as follows. In Chapter 2 we discuss nonlocal effective actions in general and further go on to develop a technique to reliably compute them in Weyl-flat spacetimes. We then proceed in Chapter 3 to apply those methods to a cosmological scenario, estimating the amount of gauge fields generated by an inflationary phase. We compare our results to the observed values and conclude with future prospects. Some technical points about Weyl anomalies and Heat Kernel expansions are discussed separately in the appendices.

Chapter 2

Nonlocal quantum effective actions in Weyl-Flat spacetimes

Computing quantum effective actions in curved spaces is an inherently difficult problem, especially in the presence of massless particles, whose presence in quantum loops lead to long range effects, encoded in nonlocal terms. The quantum dynamics of such massless particles coupled to a slowly evolving metric is summarised by the one-particle-irreducible (1PI) quantum effective action for the background fields obtained by integrating out the quantum loops. Unlike the Wilsonian effective action, the 1PI effective action necessarily contains nonlocal terms which are not derivatively suppressed. These nonlocal terms can have interesting consequences, for example, for primordial magnetogenesis in cosmology or for computing finite N corrections in AdS/CFT holography.

The computation of the nonlocal quantum effective action is in principle a well-posed problem in perturbation theory. One can regularise the path integral covariantly using dimensional regularisation or short proper-time regularisation and evaluate the effective action using the background field method. However, explicit evaluation of the path integral is forbiddingly difficult. For instance, to obtain the one-loop effective action it is necessary to compute the heat kernel of a Laplace-like operator in an arbitrary background, which amounts to solving the Schrödinger problem for an arbitrary potential. For short proper time, the heat kernel can be computed using the Schwinger-DeWitt expansion [15, 16] which is analogous to the high temperature expansion. This is adequate for renormalising the local ultraviolet divergences and to obtain the Wilsonian effective action if the proper time is short compared to the typical radius of curvature or the Compton wavelength of the particle being integrated out. However, the nonlocal 1PI effective action receives contributions from the entire range of the proper time integral and the Schwinger-DeWitt expansion is in general not adequate.

To obtain the full nonlocal effective action, one could use the covariant nonlocal expansion of the heat kernel developed by Barvinsky, Vilkovisky, and collaborators [17, 18]. The effective action in this expansion has been worked out to third order in curvatures in a series of important papers [19, 20, 21, 22] and illuminates a number of subtle issues, for example, concerning anomalies and the Riegert action [23, 24, 25, 26, 27]. However, for a general metric the explicit expressions are rather complicated already at the third order. Furthermore, the Barvinsky-Vilkovisky (BV) expansion requires $\mathcal{R}^2 \ll \nabla^2\mathcal{R}$, where $\mathcal{R}$ denotes a *generalised* curvature including both a typical geometric curvature R as well as a typical gauge field strength F. One is often interested though in the regime of slowly varying curvatures, $R^2 \gg \nabla^2 R$, for example during slow-roll inflation. This is beyond the validity of the BV regime.

The aim of the present work is to find practical methods to go beyond these limitations but only for a restricted class of metrics that are Weyl-flat and for classically Weyl invariant actions. In this case, one can exploit the symmetries of the problem. The only dynamical mode of the background metric is the Weyl factor which is a single function. The Weyl anomaly is the Weyl variation of the action which can be viewed as a first order scalar functional equation for the action that can be easily integrated. The initial value of the action functional can often be determined by the flat space results. In this manner, the entire effective action including its anomalous dependence on the Weyl factor can be determined efficiently.

The main advantage of our approach is that one can extract the essential physics with relative ease. Weyl anomalous dimensions of local operators (or equivalently the beta functions) can be computed reliably using *local* computations such as the Schwinger-DeWitt expansion. The resulting actions are necessarily nonlocal much like the Wess-Zumino action for chiral anomalies[1]. Even though we relax the restriction $R^2 \ll \nabla^2 R$, we still need to assume $F^2 \ll \nabla^2 F$ for a typical field strength F. In summary, the Barvinsky-Vilkovisky regime requires *rapidly varying curvature* as well as *rapidly varying field-strength* whereas our regime requires only rapidly varying field-strength. Our method essentially re-sums the BV expansion to all orders in curvatures albeit for a restricted class of Weyl-flat metrics as we discuss in §2.2.6.

These nonlocal actions for Weyl-flat metrics can have a number of interesting applications. In AdS/CFT correspondence, Weyl-flat metrics are relevant for the bulk description of renormalisation group flows in the boundary CFT. Loop effects of massless supergravity fields are important, for example, in the computation of finite N effects in the bulk such as the finite charge corrections to the Bekenstein-Hawking entropy of black holes [28, 29, 30, 31]. In cosmology, the Robertson-Walker metric for an isotropic and homogeneous universe with flat spatial section is Weyl flat. During many epochs in the early universe, various particles can be massless or nearly massless compared to the Hubble scale. Quantum loops of these particles can lead to an anomalous dependence on the Weyl factor which can have interesting consequences. For example, in massless electrodynamics it can contribute to the generation of primordial magnetic fields [32, 12, 13, 33, 34] where one is precisely in the regime of rapidly varying field strengths but slowly varying curvatures. This approach can also be useful for exploring the stability of de Sitter spacetime, and the cosmological evolution of the Weyl factor and other physical parameters in quasi de Sitter spacetimes in four dimensions similar to the two-dimensional models analysed in [35, 36, 37]. Possible implications of nonlocal actions have been explored, for example, in [38, 39, 40, 41, 42, 43, 44, 45].

[1]The chiral anomaly itself can be deduced from local Schwinger-DeWitt expansion. The nonlocal Wess-Zumino action is then obtained by the Wess-Zumino construction which essentially integrates the local anomaly equation. Our method extends this procedure to situations with nontrivial beta functions.

2.1 Effective Actions from Weyl Anomalies

In this section we describe the general method for computing the quantum effective action at the one-loop order for essentially all the standard model fields in Weyl-flat spacetimes by integrating the Weyl anomaly. To simplify the discussion, we ignore Yukawa couplings and work in the conformal massless limit so that all couplings are dimensionless. Dimensionful couplings and non-conformal scalars can possibly be incorporated with some modifications. We first review elements of the background field method and gauge fixing to set up our conventions. We then discuss the anomalies in terms of the Schwinger-DeWitt expansion and a lemma to obtain the effective action by integrating the anomaly.

2.1.1 Classical actions and the Background Field method

Consider the classical action for a conformally coupled real scalar field φ with quartic self-interaction:

$$\mathcal{I}_0[g,\varphi] = -\int d^4x\sqrt{|g|}\left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{12}R\,\varphi^2 + \frac{\lambda_0}{4!}\varphi^4\right], \tag{2.1}$$

where λ_0 is the bare coupling and R the Ricci scalar for the metric g. This can also be viewed as the bare action in the ultraviolet if we regard the fields as bare fields. Even though we are interested in the Lorentzian action, for subsequent computations it is convenient to use the Wick-rotated action on the Euclidean section:

$$\mathcal{S}_0[g,\varphi] = \int d^4x\sqrt{|g|}\left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{12}R\,\varphi^2 + \frac{\lambda_0}{4!}\varphi^4\right]. \tag{2.2}$$

We denote the Lorentzian action by $\mathcal{I}$ and the Euclidean action by $\mathcal{S}$. Wick rotation of Lorentzian time t to Euclidean time t_E can be thought of as a coordinate change $t = -it_E$ in the complexified spacetime. Tensors transform as tensors under this coordinate change and in particular the Lagrangian transforms as a scalar. The path integral is defined with weight $e^{i\mathcal{I}_0}$ in Lorentzian spacetime but with $e^{-\mathcal{S}_0}$ in Euclidean space. Using the fact that the volume element $\sqrt{|g|}$ equals $\sqrt{-g}$ on Lorentzian section but $\sqrt{g}$ on the Euclidean section, the two actions are simply related by $\mathcal{I}[g,\varphi] \to -\mathcal{S}[g,\varphi]$ as above.

In the background field method [46], one splits the quantum field as $\hat{\varphi} = \varphi + Q$, a sum of a background field φ and the quantum fluctuations Q around this background. The quantum effective action $\mathcal{S}[\varphi]$ for the background field φ is then given by the path integral

$$\exp\left(-\mathcal{S}[g,\varphi]\right) := \int \mathcal{D}Q\, \exp\left(-\mathcal{S}_0[g, Q+\varphi] - J[\varphi]Q\right), \tag{2.3}$$

where the external current

$$J[\varphi](x) = \frac{\delta\mathcal{S}[\varphi]}{\delta\varphi(x)} = \frac{\delta\mathcal{S}_0[\varphi]}{\delta\varphi(x)} + \dots \tag{2.4}$$

is a function of the background field adjusted so that the tadpoles vanish order by order in perturbation theory. We use short-proper time cutoff as a manifestly covariant regulator in the heat kernel method as described below. In renormalised perturbation theory, the UV divergences are renormalised with appropriately chosen counter-terms and all physical quantities are expressed in terms of the renormalised coupling λ defined at a mass scale M. At one-loop order, the path integral can be approximated by the Gaussian functional integral

$$e^{-\mathcal{S}_1[g,\varphi]} = e^{-\mathcal{S}_0[g,\varphi]} \int \mathcal{D}Q e^{-\frac{1}{2}\langle Q|\mathcal{O}_\varphi|Q\rangle} , \tag{2.5}$$

where $\mathcal{O}_\varphi$ is the quadratic fluctuation operator in the background:

$$\mathcal{O}_\varphi = -\nabla^2 + \frac{\lambda\varphi^2}{2} + \frac{1}{6}R \tag{2.6}$$

with

$$\nabla^2 = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\, g^{\mu\nu}\partial_\nu) . \tag{2.7}$$

The Gaussian integral can be evaluated in terms of the determinant of $\mathcal{O}_\varphi$,

$$\mathcal{S}_1 = \mathcal{S}_0 + \frac{1}{2}\log\det(\mathcal{O}_\varphi) = \mathcal{S}_0 + \frac{1}{2}\mathrm{Tr}\log(\mathcal{O}_\varphi) . \tag{2.8}$$

We use the convention

$$\int d^4x\sqrt{|g|}\,|x\rangle\langle x| = \mathbf{1} ; \qquad \mathrm{Tr}(\mathcal{O}) = \int d^4x\sqrt{|g|}\,\langle x|\,\mathrm{tr}\,\mathcal{O}\,|x\rangle . \tag{2.9}$$

We next consider gauge theory, concretely an $SU(N)$ Yang-Mills field coupled to a massless complex scalar and a massless Dirac fermion transforming in the fundamental representation. The classical Lorentzian action is

$$\mathcal{I}_0[g,A] = -\int d^4x\,\sqrt{|g|}\left[\frac{1}{4\,e_0^2}\,F^a_{\mu\nu}F^{a\mu\nu} + |D\Phi|^2 + \frac{1}{6}R\,|\Phi|^2 + i\,\bar{\Psi}\,\Gamma^\alpha\,e^\mu_\alpha\,D_\mu\Psi\right] , \tag{2.10}$$

where e_0^2 is the bare gauge coupling and a is the adjoint index ($a = 1, 2, \ldots, N^2 - 1$). The covariant derivative is now defined including both the spin and the gauge connection:

$$D_\mu := \partial_\mu + \frac{1}{2}w^{\alpha\beta}_\mu J_{\alpha\beta} + A^a_\mu\,T_a , \qquad (\alpha,\beta = 0,\ldots,3) , \tag{2.11}$$

where $\{J_{\alpha\beta}\}$ are the Lorentz representation matrices and $\{T_a\}$ are the anti-Hermitian $SU(N)$ representation matrices normalised so that $\mathrm{tr}_F(T_aT_b) = -\frac{1}{2}\delta_{ab}$ in the fundamental representation F. The quantum field $\hat{A}_\mu$ is a sum of a background A_μ and a quantum fluctuation a_μ, $\hat{A}_\mu = A_\mu + a_\mu$. To choose the background gauge, the gauge transformation of the quantum gauge field

$$\delta_\epsilon\hat{A}_\mu := \hat{D}_\mu\epsilon = \partial_\mu\epsilon + [\hat{A}_\mu,\epsilon] \tag{2.12}$$

can be split as

$$\delta_\epsilon A_\mu = \partial_\mu\epsilon + [A_\mu,\epsilon] := D_\mu\epsilon , \qquad \delta_\epsilon a_\mu = [a_\mu,\epsilon] . \tag{2.13}$$

It is convenient to choose the background gauge $D_\mu a^\mu = 0$ so that the effective action for the background field is manifestly gauge invariant. We set the background fields for Φ and Ψ to zero. Following the standard Fadeev-Popov procedure we add the gauge fixing term and ghost Euclidean actions which at one-loop are of the form

$$\mathcal{S}_{gf} = \frac{1}{2\, e_0^2\, \xi} \int d^4x \, \sqrt{|g|} \, |D_\mu \, a^\mu|^2 \, , \qquad \mathcal{S}_{gh} = -\int d^4x \, \sqrt{|g|} \, \bar{c} \, D^2 c \tag{2.14}$$

where the covariant derivatives contain only the background connection. We henceforth use the 't Hooft-Feynman gauge $\xi = 1$.

The one-loop quantum effective action is then given by

$$\mathcal{S}_1 = \mathcal{S}_0 + \operatorname{Tr} \log (\mathcal{O}_\Phi) - \frac{1}{2} \operatorname{Tr} \log (\mathcal{O}_\psi) + \frac{1}{2} \operatorname{Tr} \log (\mathcal{O}_A) - \operatorname{Tr} \log (\mathcal{O}_c) \; . \tag{2.15}$$

The operators involved are typically of the second-order Laplace-type

$$\mathcal{O}_f = -g^{\mu\nu} \, D_\mu \, D_\nu \, \mathbf{1} + \mathbf{E} \, , \tag{2.16}$$

where D_μ is the covariant derivative defined above which depends on the representation of the field, $\mathbf{1}$ is the identity in the representation space of the field, and $\mathbf{E}$ is the 'endomorphism matrix' that depends on the background fields.

The regularised functional trace for various operators $\mathcal{O}_f$ can be expressed in terms of the diagonal elements of the corresponding heat kernels $K_f(s) := e^{-s\,\mathcal{O}_f}$ by the standard expression:

$$\begin{aligned} \operatorname{Tr} \log (\mathcal{O}_f) = -\int_\epsilon^\infty \frac{ds}{s} \operatorname{Tr} K_f(s) \;\; &= \;\; -\int_\epsilon^\infty \frac{ds}{s} \int d^4x \sqrt{|g|} \langle x | \operatorname{tr} K_f(s) | x \rangle \\ &= \;\; -\int_\epsilon^\infty \frac{ds}{s} \int d^4x \sqrt{|g|} \operatorname{tr} K_f(x, x; s) \, . \end{aligned} \tag{2.17}$$

Here 'Tr' is a total trace including the spacetime 'index' x as in (2.9) as well as the matrix indices of the Lorentz and $SU(N)$ representations, whereas 'tr' is a trace over only the matrix indices[2]. The short proper time cut-off ϵ has mass dimension -2 and hence we can write $\epsilon = M_0^{-2}$ and regard M_0 as the UV mass cutoff.

In general, it is not possible to evaluate $K_f(x, x; s)$ explicitly for all values of the proper time. However, exploiting Weyl anomalies and the symmetries of Weyl-flat backgrounds, it is possible to compute $\mathcal{S}$ avoiding the proper time integral altogether, as we discuss in the next two sections.

2.1.2 Weyl Anomaly and the Local Renormalisation Group

Since regularisation with a short proper time cutoff ϵ is manifestly covariant, we do not expect any anomalies in the diffeomorphism invariance. On the other hand, the cutoff scale M_0 introduces a mass scale and there is a potential for Weyl anomalies.

The local Weyl transformation of the spacetime metric $g_{\mu\nu}$ is defined by

$$g_{\mu\nu} \to e^{2\xi(x)} g_{\mu\nu} \, , \qquad g^{\mu\nu} \to e^{-2\xi(x)} g^{\mu\nu} \, , \tag{2.18}$$

[2] See for example [47, 48] for notational conventions.

or infinitesimally,

$$g^{\mu\nu}(x) \to g^{\mu\nu}(x) - 2\,\xi(x)\,g^{\mu\nu}(x)\,. \tag{2.19}$$

All other fields we denote collectively as $\{\chi_f\}$ which transform with Weyl weights $\{\Delta_f\}$

$$\chi_f(x) \to e^{-\Delta_f \xi(x)} \chi_f(x)\,. \tag{2.20}$$

In particular, in four dimensions, a conformally coupled scalar field has Weyl weight 1, a fermion field has weight 3/2, a gauge field has weight 0 so that the kinetic terms are scale invariant. The local Weyl group $\mathcal{G}$ is an infinite dimensional abelian group with generators $\{J_x\}$ acting on the space of fields[3]:

$$J_x := -2\,g^{\mu\nu}(x)\frac{\delta}{\delta g^{\mu\nu}(x)} - \Delta_f\,\chi_f(x)\frac{\delta}{\delta \chi_f(x)}\,. \tag{2.21}$$

Treating the coordinate x of the local scaling parameter $\xi(x)$ as a continuous index, we can write an element of this group as

$$e^{\,\xi\cdot J} \tag{2.22}$$

with the 'summation' convention

$$\xi \cdot J := \sum_x \xi_x J_x := \int d^4x\,\xi(x) J_x\,. \tag{2.23}$$

A Weyl-flat metric can be written as

$$g_{\mu\nu} = e^{2\Omega}\eta_{\mu\nu} = e^{\Omega\cdot J}(\eta) \tag{2.24}$$

and is on the Weyl-orbit of the flat Minkowski metric $\eta_{\mu\nu}$.

Weyl invariance of the classical action implies that

$$J_x\left(\mathcal{S}_0[g,\chi_f]\right) = 0\,. \tag{2.25}$$

The cutoff ϵ required for defining the quantum path integral breaks Weyl invariance. Consequently the 1PI quantum effective action $\mathcal{S}$ for the background fields is no longer Weyl invariant. The quantum violation of classical Weyl invariance can be expressed as an anomaly equation:

$$J_x\left(\mathcal{S}[g,\chi_f]\right) := \left(-2\,g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)} - \Delta_f\,\chi_f\frac{\delta}{\delta \chi_f(x)}\right)\left(\mathcal{S}[g,\chi_f]\right) = -\mathcal{A}(x)\sqrt{|g|}\,, \tag{2.26}$$

where $\mathcal{A}(x)$ is the Weyl anomaly scalar[4]. Since the violation of the Weyl symmetry is a result of the short-distance regulator, one expects on general grounds that the

[3] Dimensionful couplings could be treated as additional 'spurion' scalar fields with Weyl dimensions equal to their classical mass dimensions so that the classical action is rendered Weyl invariant. This more general situation will be discussed in [49]. In this case, the background fields $\{\chi_f\}$ will include also the spurion fields.

[4] In conformal field theory, Weyl anomaly is usually understood to mean only the 'conformal anomaly' in curved spacetime at the conformal fixed point, arising from the Weyl non-invariance of the measure. We denote this anomaly by $\mathcal{C}(x)$. More generally, interactions perturb the theory away from the fixed point and the nontrivial beta functions generate a renormalisation group flow. In this case, the Weyl anomaly $\mathcal{A}(x)$ includes the 'beta function anomaly' $\mathcal{B}(x)$ in addition to the conformal anomaly and thus $\mathcal{A}(x) = \mathcal{B}(x) + \mathcal{C}(x)$. This notation should not be confused with the Type-A and Type-B classification of anomalies [50].

anomaly $\mathcal{A}$ must be *local* even though the 1PI action is generically nonlocal. In particular, it must admit a local expansion in terms of the background fields. The locality of the anomaly is of crucial importance. At one-loop, one can prove it explicitly and obtain a formula for the anomaly in terms of the local Schwinger-DeWitt expansion.

We illustrate the general argument for the conformally coupled scalar field φ. The infinitesimal Weyl variation of the quadratic action for the quantum fluctuation vanishes:

$$\delta_\xi \langle Q|\mathcal{O}_\varphi|Q\rangle = \delta_\xi \int d^4x \sqrt{|g|}\, Q(x)\, \mathcal{O}_\varphi\, Q(x) = 0\,. \tag{2.27}$$

Using the Weyl transformations of Q and the background metric $g_{\mu\nu}$ we conclude that

$$\delta_\xi \mathcal{O}_\varphi = -2\,\delta\xi(x)\,\mathcal{O}_\varphi \tag{2.28}$$

up to boundary terms. The quadratic fluctuation operator $\mathcal{O}_\varphi$ is thus covariant under Weyl transformations with weight 2. It then follows that

$$\delta_\xi \mathcal{S}_1[g,\varphi] = -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\ \mathrm{Tr}\,\delta e^{-s\mathcal{O}_\varphi} = \frac{1}{2}\int_\epsilon^\infty ds\ \ \mathrm{Tr}\,(\delta\mathcal{O}_\varphi) e^{-s\mathcal{O}_\varphi} \tag{2.29}$$

$$= -\int_\epsilon^\infty ds\ \int d^4x \sqrt{|g|}\ \delta\xi(x)\ \langle x|\,\mathrm{tr}\,\mathcal{O}_\varphi\, e^{-s\mathcal{O}_\varphi}|x\rangle \tag{2.30}$$

$$= \int_\epsilon^\infty ds\ \int d^4x \sqrt{|g|}\ \delta\xi(x)\ \frac{d}{ds}\,\langle x|\,\mathrm{tr}\, e^{-s\mathcal{O}_\varphi}|x\rangle\,. \tag{2.31}$$

Performing the s integral we obtain[5]

$$\frac{\delta\mathcal{S}_1[g,\varphi]}{\delta\xi(x)} = J_x\left(\mathcal{S}_1[g,\varphi]\right) = -\langle x|\,\mathrm{tr}\, e^{-\epsilon\,\mathcal{O}_\varphi}|x\rangle\sqrt{|g|}\ = -\mathrm{tr}\,K_\varphi(x,x;\epsilon)\sqrt{|g|}\,. \tag{2.32}$$

A similar reasoning can be used for fermions since the Dirac action is Weyl invariant in all dimensions. For gauge fields, there is an additional subtlety because the gauge fixed action and the ghost action are not separately Weyl invariant. However, one obtains an analogous expression for the combined system of gauge and ghost fields [51]. Both for fermions and the gauge-ghosts system, the quadratic operators have Weyl weight two. The action of the Weyl generator on the field space is thus given by

$$J_x\left(\mathcal{S}_1[g,\chi_f]\right) := -\mathcal{A}(x)\sqrt{|g|} = -2\sum_f n_f\,\mathrm{tr}\,K_f(x,x;\epsilon)\sqrt{|g|} \tag{2.33}$$

where n_f is the coefficient of $\mathrm{Tr}\log(\mathcal{O}_f)$ in (2.15) consistent with our convention in (2.17). Thus, $n_\Phi = 1$, $n_\Psi = -\frac{1}{2}$, $n_c = -1$, $n_A = n_\varphi = \frac{1}{2}$.

Equation (2.33) shows the anomaly is determined entirely by the short proper time behavior of the heat kernel. Since the proper time cutoff ϵ effectively provides a covariant short-distance cutoff in spacetime, the resulting anomaly $\mathcal{A}(x)$ is indeed

[5] If the operator $\mathcal{O}_\varphi$ has no zero modes there is no contribution from the upper limit of the integral.

local. Therefore, it must admit an expansion in terms of local fields $V_i(x)$: for the beta function anomaly,

$$\mathcal{B}(x) = \sum_i \beta_i V_i(x)\,; \tag{2.34}$$

the $\mathcal{C}(x)$ anomaly is purely gravitational and has a similar expansion in terms of the local functionals of the metric such as the Euler density [52, 53, 54].

The Weyl anomaly equation is closely related to the local renormalisation group [55, 56, 57] and the coefficients β_i can be simply related to the usual beta functions. We illustrate this connection for Yang-Mills theory. The Weyl variation of the action with respect to the Weyl factor Ω of the metric (2.24) is given by (2.54) at one loop:

$$J_x(\mathcal{S}_1[g,A]) = \frac{\delta \mathcal{S}_1[g,A]}{\delta\Omega(x)} = -\mathcal{B}(x)\sqrt{|g|} = \frac{b}{4}\,F^2(x)\sqrt{|g|}\,, \tag{2.35}$$

where b is given by (2.55) and we have ignored the purely gravitational $\mathcal{C}(x)$ anomaly. To relate it to the local renormalisation group, we note that a Weyl scaling of the metric increases length scales or decreases mass scales. Hence we can regard $M(x) := Me^{\Omega(x)}$ to be the position-dependent local renormalisation scale[6] $M(x)$. Therefore,

$$M(x)\frac{\delta}{\delta M(x)} = \frac{\delta}{\delta\Omega(x)}\,. \tag{2.36}$$

If the scale $M(x)$ is position dependent, then it is natural to regard all renormalised couplings to be also position-dependent expectation values of nondynamical 'spurion' fields. For example, regarding, $1/e^2 = \lambda_e(x)$ as position dependent, and using (2.36) and (2.35) we conclude that

$$\left[M(x)\frac{\delta}{\delta M(x)} + \beta_e\frac{\delta}{\delta\lambda_e(x)}\right]\mathcal{S}_1 = 0\,, \tag{2.37}$$

with

$$\beta_e := M\frac{d\lambda_e}{dM} = M\frac{de^{-2}}{dM} = -b\,. \tag{2.38}$$

For constant $M(x)$, functional derivatives are replaced by ordinary derivatives and one recovers the usual position-independent 'global' homogeneous renormalisation group equation.

More generally, the local renormalisation group equation is best thought of as a Weyl anomaly equation (2.26) with a local expansion for the anomaly $\mathcal{A}$.

2.1.3 Integration of the Weyl Anomaly

Our goal is to deduce the nonlocal quantum effective action $\mathcal{S}[g,\chi_f]$ by integrating the local Weyl anomaly. Towards this end, we consider the following trivial identity[7]

[6]This is true as long as one is dealing with 'primary' fields such as $g_{\mu\nu}$ or $F_{\mu\nu}$ which transform covariantly under Weyl transformation. In general, 'secondary' fields such as $R_{\mu\nu}$ or $\nabla_\mu\varphi$ are also relevant, which contain derivatives of the primary fields. In this case, the Weyl transformations contain terms with derivatives of the Weyl factor Ω and the equality (2.36) holds only up to these derivatives [55, 58, 59, 60, 61, 56, 49].

[7]We thank Adam Schwimmer for this formulation.

$$e^{\xi\cdot J} = \mathbf{1} + \int_0^1 dt\, e^{t\,\xi\cdot J}\ \xi\cdot J\,. \tag{2.39}$$

We wish to compute $\mathcal{S}[g,\chi_f]$ for (g,χ_f) on the Weyl-orbit of $(\bar{g},\bar{\chi}_f)$ with Weyl factor $\Omega(x)$:

$$(g,\chi_f) = e^{\Omega\cdot J}(\bar{g},\bar{\chi}_f)\,. \tag{2.40}$$

Using the identity above we obtain

$$\begin{aligned}\mathcal{S}[g,\chi_f] \equiv e^{\Omega\cdot J}\left(\mathcal{S}[\bar{g},\bar{\chi}_f]\right) &= \left(\mathbf{1} + \int_0^1 dt\, e^{t\,\Omega\cdot J}\,\Omega\cdot J\right)\left(\mathcal{S}[\bar{g},\bar{\chi}_f]\right) \\ &= \mathcal{S}[\bar{g},\bar{\chi}_f] + \int_0^1 dt\, e^{t\,\Omega\cdot J}\,\Omega\cdot J\left(\mathcal{S}[\bar{g},\bar{\chi}_f]\right) \\ &= \mathcal{S}[\bar{g},\bar{\chi}_f] - \int_0^1 dt\, e^{t\,\Omega\cdot J}\left(\int d^4x\,\Omega(x)\sqrt{|\bar{g}|}\,\mathcal{A}[\bar{g},\bar{\chi}_f](x)\right)\end{aligned} \tag{2.41}$$

where we have used (2.26) in the last line. Using (2.40) we then conclude[8]

$$\mathcal{S}[g,\chi_f] = \mathcal{S}[\bar{g},\bar{\chi}_f] + \mathcal{S}_{\mathcal{A}}[\bar{g},\Omega,\bar{\chi}_f]\,, \tag{2.42}$$

where

$$\mathcal{S}_{\mathcal{A}}[\bar{g},\Omega,\bar{\chi}_f] := -\int_0^1 dt\int d^4x\sqrt{|\bar{g}\,e^{2\,t\,\Omega(x)}|}\ \Omega(x)\ \mathcal{A}[\bar{g}\,e^{2\,t\,\Omega},\bar{\chi}_f\,e^{-\Delta_f\,t\,\Omega}](x) \tag{2.43}$$

is the contribution to the action from the anomaly. Lorentzian continuation of (2.42) gives a similar equation

$$\mathcal{I}[g,\chi_f] = \mathcal{I}[\bar{g},\bar{\chi}_f] + \mathcal{I}_{\mathcal{A}}[\bar{g},\Omega,\bar{\chi}_f] \tag{2.44}$$

but with $\mathcal{I}_{\mathcal{A}}$ given by

$$\mathcal{I}_{\mathcal{A}}[\bar{g},\Omega,\bar{\chi}_f] := \int_0^1 dt\int d^4x\sqrt{|\bar{g}\,e^{2\,t\,\Omega(x)}|}\ \Omega(x)\ \mathcal{A}[\bar{g}\,e^{2\,t\,\Omega},\bar{\chi}_f\,e^{-\Delta_f\,t\,\Omega}](x) \tag{2.45}$$

because the anomaly scalar does not change sign under Wick rotation.

Equation (2.42) is a simple identity that follows essentially from the group structure of Weyl transformations. It is thus applicable to any order in perturbation theory if we can compute the Weyl anomaly to that order. To compute the effective action to the one-loop order, one can use the expression (2.43) with the Weyl anomaly given in terms of the heat kernel as in (2.33). Since the short-time expansion of the heat kernel is determined by the local Schwinger-DeWitt expansion, we see that (2.42) enables us to determine the entire quantum effective action for Weyl-flat background metrics knowing only the local expansion.

Note that the left hand side of (2.44) depends only on the physical metric whereas the right hand side *a priori* depends on the fiducial metric $\bar{g}$ and Ω separately. It

[8] The argument g of the action $\mathcal{S}[g,\chi_f]$ functional here refers to the covariant tensor $g_{\mu\nu}$ and not $g^{\mu\nu}$.

must therefore be true that the action on the right hand side exhibits 'fiducial Weyl *gauge* invariance'

$$\bar{g} \to e^{2\zeta(x)}\bar{g}\,, \qquad \Omega \to \Omega - \zeta(x)\,, \tag{2.46}$$

under which the fiducial metric $\bar{g}$ transforms but the physical metric g is invariant. This gauge invariance reflects the fact that splitting g into $\bar{g}$ and Ω is ambiguous, and all splits related by a fiducial gauge transformation are physically equivalent. The fiducial gauge invariance of the right hand side of (2.44) is necessary to show that it depends only on the physical metric. As we explain in §2.2.6, it is often far from obvious how the answer obtained using our method can be expressed covariantly entirely in terms of the physical metric. However, the procedure guarantees that this must be the case.

2.1.4 Schwinger-DeWitt Expansion of the Heat Kernel

For small values of s (compared to the typical generalised curvatures) the coincident Heat Kernel $K(x, x; s)$ admits an expansion in terms of local quantities. This is immediately relevant for the computation of the divergent terms on the effective action, to compute the one loop integrals of very massive particles or to evaluate the expectation value of the regularised trace of the energy momentum tensor in the absence of dimensionful parameters. In even dimensions we have:

$$\mathrm{tr}K(x, x; s) = \frac{1}{(4\,\pi s)^{d/2}} \sum_{n=0}^{\infty} a_n(x)\, s^n \tag{2.47}$$

The $a_n(x)$ are the Gilkey-Seeley-HaMiDeW [62, 63, 64, 65, 66, 67, 68, 69, 70, 71] coefficients[9] which are local scalar functions of the background fields. A general expression in any spacetime dimension is known explicitly for the first few of them in terms of $\mathbf{E}$ and geometric invariants.

Because the first few terms have negative powers of s, the above short time expansion is divergent when $s \to 0$. The divergences can be renormalized by appropriate local counterterms in the action. For the trace of the energy momentum tensor of massless quanta[10] the $n < d/2$ terms don't contribute in dimensional regularization and in other regularization schemes they are all of trivial cohomology of the Weyl group[11] and thus are eliminated by adding local counterterms in the action. The $a_{d/2}$ term is finite when we take the regulator to zero and $a_{d/2}$ is directly related to the Weyl anomaly (up to some trivial cohomology terms). In any case the nontrivial part of the T is completely contained in $a_{d/2}$:

$$\langle T_\mu{}^\mu \rangle = \frac{1}{(4\pi)^{d/2}}\, a_{d/2} \tag{2.48}$$

When computing the effective action we integrate over the proper time s, in this case the $n = d/2$ term is logarithmically divergent and is related to the running of marginal couplings. In the presence of a heavy mass (compared to the curvatures)

[9] After Hadamard, Minakshisundaram, and DeWitt [72, 73].

[10] In this case T is direclty related to the coincident Heat Kernel as it follows from 2.32.

[11] For further discussion on this see A.

such that $\frac{a_n}{m^{2n}} \ll 1$ the effective action can be computed directly by the Schwinger-DeWitt expansion as the s integral is exponentially suppressed away from $s = 0$. Notice that the a_n in the next expression are the ones coming of the expansion of the massless kernel.

$$\frac{1}{2}\mathrm{Tr}\ln(-D^2+\mathbf{E}+m^2) = \Gamma_{div}+\Gamma_{log}-\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\sum_{n=d/2+1}^{\infty}\frac{\Gamma(n-\frac{d}{2})}{m^{2n}}\int d^dx\sqrt{g}\,\mathrm{tr}\,a_n(x) \tag{2.49}$$

where

$$\Gamma_{log} = \frac{1}{(4\pi)^{d/2}}\int d^dx\sqrt{g}\sum_{n=0}^{d/2}\frac{(-m^2)^{d/2-n}}{(d/2-n)!}\ln\left(\frac{m^2}{\mu^2}\right)\mathrm{tr}\,a_n \tag{2.50}$$

In particular on flat space we can use this expansion to read the Coleman-Weinberg effective potential. For example, in the massless $\lambda\phi^4$ we have $\mathcal{O} = -\partial^2 + \frac{\lambda}{2}\phi^2$, where ϕ is the constant background field. In this case the massless kernel has $a_0 = 1$ and $a_n = 0$ for $n > 0$. Up to divergent terms, the one-loop correction to the action simplifies to

$$\frac{1}{2}\mathrm{Tr}\ln\left(-\partial^2+\frac{\lambda}{2}\phi^2\right) = \frac{3\lambda}{64\pi^2}\int d^4x\sqrt{g}\,\frac{\lambda\phi^4}{4!}\ln\left(\frac{\lambda\phi^2}{2\mu}\right) \tag{2.51}$$

The relevant $a_n(x)$ coefficients for $\mathcal{O} = -D^2 + \mathbf{E}$, up to spacetime dimension $d = 4$, are given by [48]

$$\begin{aligned} a_0 &= \mathrm{tr}\,\mathbf{1} \\ a_1 &= \mathrm{tr}\left(\frac{1}{6}\,R\,\mathbf{1} - \mathbf{E}\right) \\ a_2 &= \mathrm{tr}\Bigg(\frac{1}{2}\,\mathbf{E}^2 - \frac{1}{6}\,\nabla^2\mathbf{E} - \frac{1}{6}\,R\,\mathbf{E} + \frac{1}{12}\,\Omega_{\mu\nu}\,\Omega^{\mu\nu} \\ &\qquad + \frac{1}{180}\left(6\,\nabla^2 R + \frac{5}{2}\,R^2 - \frac{1}{2}E_4 + \frac{3}{2}\,W^2\right)\mathbf{1}\Bigg), \end{aligned} \tag{2.52}$$

where $\mathrm{tr}\,\mathbf{1}$ traces all indices, $\nabla_\mu := \partial_\mu + \omega_\mu$ is the covariant derivative involving only the spin connection, and $\Omega_{\mu\nu} = [D_\mu, D_\nu]$ is the field strength of the full connection. E_4 is the Euler density in four dimensions and W^2 is the square of the Weyl tensor $W_{\mu\nu\rho}{}^{\sigma}$ defined by

$$\begin{aligned} E_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ W^2 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\,. \end{aligned} \tag{2.53}$$

In four dimensions, the anomaly $\mathcal{A} = \mathcal{B} + \mathcal{C}$ is determined by $a_2(x)$. In Table 2.1 we list the anomalies for the operators appearing in the Yang-Mills and conformally-coupled scalar actions. We have dropped the terms proportional to $\nabla^2 R$ and $\nabla^2\varphi^2$. Such operators follow from the Weyl variation of local terms in the action, namely R^2 and $R\,\varphi^2$, hence are not genuine anomalies [74]. The vector potential operator $\mathcal{O}_A$ corresponds to the Feynman gauge $\xi = 1$ and $F^2 := F^a_{\mu\nu}\,F^{a\,\mu\nu}$. Note that the $a_2(x)$ coefficients for the ghost and vector operators individually contain a term proportional to R^2. This is related to the the fact that the operators are not individually Weyl covariant. However, taken together, the R^2 terms cancel from the anomaly as expected from the Wess-Zumino consistency condition[12].

Field	$\mathcal{O}$	$16\pi^2\mathcal{B}$	$16\pi^2\mathcal{C}$
Φ	$-D^2+\frac{1}{6}\,R$	$-\frac{1}{12}\,F^2$	$\frac{N}{180}\,\left(-E_4+3\,W^2\right)$
$c,\bar{c}$	$-D^2$	$\frac{N}{6}\,F^2$	$\frac{N^2-1}{180}\,\left(-5\,R^2+E_4-3\,W^2\right)$
A_μ	$-D^2\,g^{\mu\nu}+R^{\mu\nu}-2\,F^{\mu\nu}$	$\frac{5\,N}{3}\,F^2$	$\frac{N^2-1}{180}\,\left(5\,R^2-32\,E_4+21\,W^2\right)$
Ψ	$-D^2+\frac{1}{4}\,R-\frac{1}{2}F_{\mu\nu}\Gamma^\mu\Gamma^\nu$	$-\frac{1}{3}\,F^2$	$\frac{N}{180}\,\left(-\frac{11}{2}E_4+9\,W^2\right)$
φ	$-\nabla^2+\frac{1}{6}\,R+\frac{1}{2}\lambda\varphi^2$	$\frac{1}{8}\lambda^2\varphi^4$	$\frac{1}{180}\,\left(-E_4+3\,W^2\right)$

Table 2.1: Weyl anomalies in $d=4$. The contributions from the complex scalars Φ and fermions Ψ to the $\mathcal{B}$ anomaly are different for the abelian and non-abelian cases. In the table we have indicated the non-abelian ones relevant for Yang-Mills. For quantum electrodynamics, the contributions are multiplied by a factor of two due to the choice of normalisation of the non-abelian gauge group generators.

Putting these results together, the Weyl anomaly equation for Yang-Mills is

$$J_x(\mathcal{S}_1[g,A]) = \frac{\delta\mathcal{S}_1[g,A]}{\delta\Omega} = \left(\frac{b}{4}\,F^2(x) - \mathcal{C}(x)\right)\sqrt{|g|}\,, \tag{2.54}$$

with

$$b = \frac{1}{48\pi^2}\,(N_S + 4N_F - 22N) \tag{2.55}$$

for an $SU(N)$ theory with N_S scalars and N_F fermions in the fundamental. In quantum electrodynamics integrating out N_F fermions and N_S scalars, one would get a similar result with

$$b = \frac{1}{24\pi^2}\,(N_S + 4N_F)\ . \tag{2.56}$$

For the real scalar field φ with quartic self-interaction, we similarly obtain

$$J_x(\mathcal{S}_1[g,\varphi]) = \frac{\delta\mathcal{S}_1[g,\varphi]}{\delta\Omega} = \left(-\frac{b\,\lambda}{4!}\,\varphi^4(x) - \mathcal{C}(x)\right)\sqrt{|g|} \tag{2.57}$$

with the beta function coefficient given by

$$b = \frac{3\lambda}{16\pi^2}\,. \tag{2.58}$$

[12]See appendix A for a more precise discussion on the Wess-Zumino consistency condition and on the Cohomology of the Weyl group.

2.2 Nonlocal Effective Actions

In this section we derive the one-loop quantum effective actions from the anomalies following the discussion in the previous section. We drop the subscript '1' used earlier to indicate the one-loop results. As a simple illustration, we first derive the two dimensional Polyakov action from the $\mathcal{C}(x)$ anomaly. In four dimensions, we ignore the $\mathcal{C}(x)$ anomaly and focus only on the $\mathcal{B}(x)$ anomaly to derive the effective action for the background fields Ω, A, φ.

2.2.1 The Polyakov action in Two Dimensions

The trace anomaly (2.26) for a massless free scalar in two dimensions is given by

$$\mathcal{A}(x) = \operatorname{tr} K_{\varphi}(x, x, \epsilon) \, . \tag{2.59}$$

The finite contribution to the trace in two dimensions is given by the coefficient $a_1(x)$:

$$\mathcal{A}(x) = \frac{1}{4\pi} a_1(x) = \frac{1}{4\pi} \operatorname{tr} \left(\frac{1}{6} R \, \mathbf{1} \right) = \frac{1}{24\pi} R \, . \tag{2.60}$$

In this case $\mathcal{B} = 0$ and the anomaly is purely gravitational. Using (2.42) and the Weyl transformation for the Ricci scalar

$$R = e^{-2\Omega}(\bar{R} - 2\bar{\nabla}^2\Omega) \quad \text{for} \quad g = e^{2\Omega}\bar{g} \, , \tag{2.61}$$

the effective action is given by $\mathcal{I}[\phi, g] = \mathcal{I}[\phi, \bar{g}] + \mathcal{I}_{\mathcal{C}}[\bar{g}, \Omega]$ with

$$\begin{aligned} \mathcal{I}_{\mathcal{C}}[\bar{g}, \Omega] &= \int_0^1 dt \int d^2x \sqrt{|\bar{g}\, e^{2\,t\,\Omega(x)}|}\, \Omega(x)\, \mathcal{A}[\bar{g}\, e^{2\,t\,\Omega(x)}] \\ &= \frac{1}{24\pi} \int_0^1 dt \int d^2x \sqrt{|\bar{g}|}\, e^{2\,t\,\Omega(x)}\, \Omega(x)\, e^{-2\,t\,\Omega(x)} \left(\bar{R} - 2\,t\,\bar{\nabla}^2\Omega(x)\right) \\ &= \frac{1}{24\pi} \int d^2x \sqrt{|\bar{g}|} \left((\bar{\nabla}\Omega)^2 + \bar{R}\,\Omega(x)\right) \, , \end{aligned} \tag{2.62}$$

which is the Liouville action with the correct normalisation. For $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$, one can solve (2.61) for Ω in terms of g using the fact that $\bar{R} = 0$, and obtain the Polyakov action

$$\mathcal{I}_{\mathcal{C}}[g] = -\frac{1}{96\pi} \int d^2x \sqrt{|g|}\, R\, \frac{1}{\nabla^2}\, R \, . \tag{2.63}$$

Since in two dimensions every metric is Weyl flat, these results are valid for a general metric.

2.2.2 Quantum Effective Action for Yang-Mills Theory

Applying (2.45) to the $\mathcal{B}$ anomaly of the Yang-Mills theory (2.54) in a Weyl-flat spacetime gives

$$\mathcal{I}_{\mathcal{B}}[\eta, \Omega, A] = -\frac{b}{4} \int d^4x \; \eta^{\rho\alpha}\, \eta^{\sigma\beta} \; F^a_{\rho\sigma}(x)\, \Omega(x)\, F^a_{\alpha\beta}(x) \, . \tag{2.64}$$

The flat space action can be easily determined from standard computations and is given by

$$\mathcal{I}[\eta, A] = -\frac{1}{4\, e^2(M)} \int d^4x\, d^4y\; F^a_{\rho\sigma}(x) \,\langle x| \left[1 - \frac{b}{2}\, e^2(M)\, \log\left(\frac{-\partial^2}{M^2}\right)\right] |y\rangle\, F_a^{\rho\sigma}(y) \tag{2.65}$$

where $-\partial^2$ is the flat-space d'Alembertian and M is an arbitrary offshell subtraction point. The kets $|x\rangle$ here are normalised as in (2.9) but now with the flat metric η. The logarithm of an operator is defined by the spectral representation

$$\log\left(\frac{\mathcal{O}}{M^2}\right) = \int_0^\infty d\mu^2 \left(\frac{1}{M^2+\mu^2} - \frac{1}{\mathcal{O}+\mu^2}\right). \tag{2.66}$$

For the flat space d'Alembertian the logarithm can also be defined by a Fourier transform:

$$\langle x| \log\left(\frac{-\partial^2}{M^2}\right)|y\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \log\left(\frac{p^2}{M^2}\right). \tag{2.67}$$

Putting the two things together in (2.44) we conclude

$$\mathcal{I}[g, A] = -\frac{1}{4e^2(M)} \int d^4x\; \eta^{\rho\alpha}\eta^{\sigma\beta}\;\; F^a_{\rho\sigma} \left[1 - \frac{b}{2}\, e^2(M)\, \log\left(\frac{-\partial^2}{M^2}\right) + b\, e^2(M)\, \Omega\right] F^a_{\alpha\beta} \tag{2.68}$$

where the logarithmic operator is to be understood as a bilocal expression integrated over y as in (2.65). There is a gravitational piece coming from the $\mathcal{C}$ anomaly which we do not discuss.

Note that the action (2.64) arising from the anomaly follows from the local Schwinger-DeWitt expansion and does not require any weak-field approximation. Thus, the main limitation in computing (2.68) comes from the evaluation of the flat space action (2.65). In (2.65) we have used the weak gauge field approximation $F^4 \ll \nabla^4 F^2$ as one normally does in flat space quantum field theory. It may be possible to compute the flat space action in other regimes, for example, in the regime of constant field strength. This can extend the range of validity of our results.

It is instructive to deduce this result using dimensional regularisation. Again, the classical action (or the bare action in the UV) is given by

$$\mathcal{I}_0[g, A] = -\frac{1}{4e_0^2} \int d^4x \sqrt{|g|}\, g^{\rho\alpha} g^{\sigma\beta}\, F^a_{\rho\sigma}\, F^a_{\alpha\beta}\,. \tag{2.69}$$

The classical energy momentum tensor

$$T^{cl}_{\mu\nu} = \frac{1}{e_0^2}\left(F^a_{\mu\sigma}F^{a\;\sigma}_{\nu} - \frac{1}{4} g_{\mu\nu} F^2\right) \tag{2.70}$$

is traceless. At the quantum level, the nonzero beta function implies a quantum violation of Weyl invariance. For a manifestly gauge-invariant computation of this Weyl anomaly we use dimensional regularisation. In $4-\varepsilon$ dimensions, the bare coupling e_0 is related to the coupling e renormalised at scale M by

$$\frac{1}{e_0^2} = M^{-\varepsilon}\left(\frac{1}{e^2} - \frac{b}{\varepsilon}\right), \qquad M\frac{de^{-2}}{dM} = -b\,. \tag{2.71}$$

There is the usual pole coming from loop integrations of quantum fluctuations around a background field. For a Weyl-flat metric, the dimensionally regularised background field action depends only on the background gauge field and the Weyl factor Ω:

$$\mathcal{I}^{\varepsilon}[\Omega, A] = -\frac{1}{4}\int d^{4-\varepsilon}x\sqrt{|\eta|}\,\eta^{\rho\alpha}\eta^{\sigma\beta}\,e^{-\varepsilon\Omega}\,M^{-\varepsilon}(\frac{1}{e^2}-\frac{b}{\varepsilon})\,F^a_{\rho\sigma}\,F^a_{\alpha\beta}\,. \tag{2.72}$$

This implies that the Weyl variation of the renormalised effective action for the background field is no longer zero and is given by

$$\delta\mathcal{I}[\eta,\Omega,A] = -\frac{b}{4}\int d^4x\sqrt{|\eta|}\,F^2\,\delta\Omega\,, \tag{2.73}$$

consistent with the results obtained using the proper time regularisation.

2.2.3 Quantum Effective Action for a Self-interacting Scalar Field

For a conformally coupled scalar field φ, one can similarly determine the one-loop effective action $\mathcal{I}[g,\varphi]$. Integrating the $\mathcal{B}$ anomaly of (2.57) gives

$$\mathcal{I}_{\mathcal{B}}[\eta,\Omega,\bar{\varphi}] = \frac{b\,\lambda}{4!}\int d^4x\,\bar{\varphi}^4(x)\,\Omega(x)\,, \tag{2.74}$$

with b given by (2.58) and λ being the renormalised quartic coupling defined at the scale M. The flat space action obtained from standard computations gives

$$\mathcal{I}[\eta,\bar{\varphi}] = -\int d^4x\left[\frac{1}{2}|\partial\bar{\varphi}|^2 + \frac{\lambda}{4!}\bar{\varphi}^2(x)\left(1+\frac{b}{2}\log\left(\frac{-\partial^2}{M^2}\right)\right)\bar{\varphi}^2(x)\right]. \tag{2.75}$$

Using (2.44), the full effective action is given by

$$\mathcal{I}[g,\varphi] = -\int d^4x\left[\frac{1}{2}|\partial\bar{\varphi}|^2 + \frac{\lambda}{4!}\bar{\varphi}^2(x)\left(1+\frac{b}{2}\log\left(\frac{-\partial^2}{M^2}\right)-b\,\Omega(x)\right)\bar{\varphi}^2(x)\right]. \tag{2.76}$$

As in the case of the Yang-Mills action, the part of the action (2.74) arising from the anomaly does not require any weak-field approximation and is exact. The flat space action (2.75) is valid only assuming rapidly varying field. It could be evaluated though in other regimes of interest using techniques such as the large proper time expansion developed in [75, 76] or the Coleman-Weinberg method. However, note that when the field φ is in the Coleman-Weinberg regime, the field $\bar{\varphi}$ may not be unless the scale factor is also slowly varying.

We see that the net effect of the Weyl anomaly in the combined action is to change the renormalisation scale to an effective *local* renormalisation scale $M(x) := Me^{\Omega(x)}$ consistent with (2.36). One can explain the answer intuitively if the scale factor is varying slowly compared to the typical scale of field variations (for example in a particle physics experiment in an expanding universe). In this case, one can use local momentum expansion to write $-\partial^2 = k^2$. In local experiments (2.76) can be interpreted as a flat space action with momentum-squared k^2 but with a position dependent cutoff $M(x)$. One can equivalently interpret $k^2/M^2(x)$ as $p^2(x)/M^2$ in terms of physical momentum-squared $p^2 = e^{-2\Omega(x)}k^2$ with a fixed RG scale M.

This suggests that we can use the local renormalisation group to define a position-dependent 'running' coupling

$$\begin{aligned}\lambda(p^2(x)) &= \frac{\lambda}{1-\frac{b}{2}\log\left(\frac{p^2(x)}{M^2}\right)} \\ &= \lambda\left[1+\frac{b}{2}\log\left(\frac{p^2(x)}{M^2}\right)+\frac{b^2}{4}\log^2\left(\frac{p^2(x)}{M^2}\right)+\dots\right].\end{aligned} \tag{2.77}$$

Equation (2.77) re-sums the leading logarithms to all orders as with the usual renormalisation group but now locally. The effective coupling decreases as the universe expands because the beta function is positive. Consequently, the renormalisation-group improved answer becomes better and better at late times even though naive perturbation theory would break down. The local renormalisation group thus extends the range of applicability of the perturbative computations.

In more general situations with a rapidly varying scale factor, one cannot use the momentum basis as above but equation (2.76) is still valid. One might be tempted to interpret the full answer in terms of the logarithm of the covariant d'Alembertian in curved spacetime, $\log\left(-\nabla^2/M^2\right)$. However, the full covariantisation is rather nontrivial and requires many more nonlocal covariant terms which combine into a Weyl-invariant piece [22, 77]. We discuss this in detail in §2.2.6.

2.2.4 Equations of Motion

If we are interested in the equations of motion of the fields in a fixed background metric, then the metric does not need to be varied and can be assumed to be Weyl flat. The equations of motion for the background Yang-Mills field follow straightforwardly from the action $\mathcal{I}[g, A]$ (2.68) and are given by

$$\frac{\delta\mathcal{I}[\eta, A]}{\delta A_\mu}+\frac{\delta\mathcal{I}_{\mathcal{B}}[\eta,\Omega, A]}{\delta A_\mu}=0\,. \tag{2.78}$$

The first term gives the logarithmic modifications to the flat space equations of motion arising from integrating out massless charged particles. The second term gives rise to the anomalous coupling to the conformal factor of the metric which breaks the Weyl invariance. Similar considerations extend to the equations of motion for the Weyl-transformed scalar $\bar{\varphi}$.

These actions are thus adequate for studying the equations of motion for the fluctuations of the gauge field or a scalar field in an arbitrary Robertson-Walker background including the full anomalous dependence on the Weyl factor. This is the situation one encounters, for example, in studying the primordial perturbations of a scalar or of the electromagnetic field in a slowly rolling inflationary background.

2.2.5 The Curvature expansion

In the weak curvature limit, we can compare our results with the covariant curvature expansion developed by Barvinsky, Vilkovisky, and collaborators [17, 18, 19, 20, 21, 22]. It provides a useful check on our results obtained using a rather different method which does not rely on the weak curvature approximation.

The curvature expansion benefits from being easily made covariant as well as from not making a small proper time s expansion, thus allowing one to integrate over the proper time to obtain a meaningful finite terms in the effective action in the presence of massless particles. The price to be paid is a restrictive limit of validity regarding the properties of the background fields involved. This perturbative expansion requires rapidly oscillating fields (when compared to their typical values). Much like a gravitational wave travelling on almost flat space and and unlike the typical FRW metric.

The main idea behind the Barvinsky-Vilkovisky (BV) expansion is to decompose the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and treat the fluctuations $h_{\mu\nu}$ as well as interaction terms as perturbations. The heat equation satisfied by the kernel $K(s)$ can be solved perturbatively around flat space using the analog of the Dirac interaction picture in quantum mechanics. Here the flat space laplacian plays the role of the "free Hamiltonian" and corrections to that are accounted by a proper time ordered product. This analogy follows from the fact that the Heat equation is quite similar to the Schröedinger equation.

$$\frac{\partial}{\partial s}K(s) + \mathcal{O}K(s) = 0 \tag{2.79}$$

The perturbative answers then can be 'covariantised' and express in terms of covariant derivatives and polynomials of generalised curvature tensors which will schematically be denoted as $\mathcal{R}$, which includes both terms like $R_{\mu\nu}$ as well as $F_{\mu\nu}$. This expansion is valid for small generalised curvatures but for the entire range of the proper time s.

If we rewrite $\mathcal{O}$ as $-\partial^2 - V$ we can write the Heat equation as:

$$\frac{\partial}{\partial s}K = \left(\partial^2 + V\right) K \tag{2.80}$$

and then proceed to expand K as $K_0 + K_1 + K_2 + \ldots$, where K_n is an operator of order V^n, with $K_0(0) = \mathbf{1}$ and $K_i(0) = 0$ for $i \geq 1$. From that we obtain a system of equations for each K_n.

$$\begin{aligned} \frac{\partial}{\partial s}K_0 &= \partial^2 K_0 \\ \frac{\partial}{\partial s}K_1 &= \partial^2 K_1 + VK_0 \\ &\vdots \end{aligned} \tag{2.81}$$

whose solution can be found iteratively:

$$\begin{aligned} K_0(s) &= e^{s\partial^2} \\ K_1(s) &= \int_0^s dt\ K_0(s-t)VK_0(t) \\ &\vdots \end{aligned} \tag{2.82}$$

Defining $V(s) \equiv K_0(-s)VK_0(s)$ we have

$$K_n(s) = K_0(s)\left[\mathbf{1} + \int_0^s dt\ V(t_1) + \int_0^s \int_0^{t_1} dt_1 dt_2\ V(t_1)V(t_2) + \dots\right] \tag{2.83}$$
$$= K_0(s)\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^s \cdots \int_0^s \mathcal{T}\left\{V(t_1)\dots V(t_n)\right\} dt_1 \dots dt_n$$

The final answer is guaranteed to possess general covariance if one choses the appropriate background field gauge. Although the perturbation theory just employed breaks covariance, one can easily covariantise the obtained expressions at a given order in curvature. The computation of the proper time integrals is complicated but necessary to obtain the non-local form factors on the trace of the Heat Kernel and consequently in the Effective Action. Fortunately those form factors have been computed for a large class of operators that one may encounter in field theory, up to third order in generalised curvatures (including the curvature of gauge connections as well as interaction potentials) in [17, 18, 19, 20, 21, 22].

For $\mathcal{O}$ of the form[13] $-\mathbf{1}D^2 + \mathbf{E}$. we have

$$\begin{aligned}\mathrm{Tr}K(s) = \frac{1}{(4\pi s)^{d/2}}\int d^dx\sqrt{g}\ \mathrm{tr}\Big\{\mathbf{1} + s\left[\frac{1}{6}\mathbf{1}R - \mathbf{E}\right] + \\ +s^2\Big[\mathbf{1}R_{\mu\nu}f_{Ric}(-s\nabla^2)R^{\mu\nu} + \mathbf{1}Rf_{RR}(-s\nabla^2)R + Rf_{RE}(-s\nabla^2)\mathbf{E} + \\ +\mathbf{E}f_{EE}(-s\nabla^2)\mathbf{E} + \Omega_{\mu\nu}f_{\Omega\Omega}(-s\nabla^2)\Omega^{\mu\nu}\Big] + \mathcal{O}(\mathcal{R}^3)\Big\}\end{aligned} \tag{2.84}$$

where f_i are simple functions, all given in terms of the basic function

$$f(x) = \int_0^1 d\xi\ e^{-\xi(1-\xi)x} \tag{2.85}$$

Namely,

$$\begin{aligned}f_{Ric}(x) &= \frac{1}{6x} + \frac{1}{x^2}(f(x)-1) \\ f_{RR}(x) &= \frac{1}{32}f(x) + \frac{1}{8x}f(x) - \frac{7}{48x} - \frac{1}{8x^2}(f(x)-1) \\ f_{RE}(x) &= -\frac{1}{4}f(x) - \frac{1}{2x}(f(x)-1) \\ f_{EE}(x) &= \frac{1}{2}f(x) \\ f_{\Omega\Omega}(x) &= -\frac{1}{2x}(f(x)-1)\end{aligned} \tag{2.86}$$

The effective action can thus be obtained by evaluating the integral (2.17). The final answer can be expressed in terms of non-local 'form factors' and schematically takes the form

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1^{loc} + \int d^dx\sqrt{|g|}\ \sum_{i=1}^{5}\gamma_i(-\nabla_2^2)\mathcal{R}_1\mathcal{R}_2 + \sum_{i=1}^{29}\mathcal{F}_i(-\nabla_1^2, -\nabla_2^2, -\nabla_3^2)\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3(i) + \mathcal{O}(\mathcal{R}^4)\,. \tag{2.87}$$

[13] Where D_μ is defined as in 2.11 to include both the spacetime connection and a generic gauge connection.

The notation is a shorthand for terms containing all possible combinations of curvatures such as $R f_{RR}(-\nabla^2)R$ and $\mathcal{F}_2(-\nabla_1^2, -\nabla_2^2, -\nabla_3^2) F_1{}^{\mu}{}_{\nu} F_2{}^{\nu}{}_{\sigma} R_3{}^{\sigma}{}_{\mu}$, for example. The form factors γ and $\mathcal{F}$ as functions of the covariant Laplacian are generically nonlocal operators, and are to be understood as properly convoluted with the functions they act upon [18]. A similar result has been obtained for massless quantum electrodynamics by a somewhat different method by Donoghue and El-Menoufi [77, 78] by evaluating the one-loop Feynman diagrams for small metric fluctuations around flat space and then covariantising the answers.

An important advantage of this expansion is that it gives all nonlocal terms in the action directly to a given order in perturbation theory. The price to pay though is that these expressions are necessarily perturbative, valid only in the regime of $\mathcal{R}^2 \ll \nabla^2\mathcal{R}$. Note that there are two perturbative expansions at work. The loop expansion parameter is e^2 or λ, while the BV expansion involves a further approximation which treats the field perturbations, such as $h_{\mu\nu}$, A_μ or V'' as small. This *weak field* approximation implies that terms of the form $\partial^2 h \partial^2 h$ are to be regarded as much smaller than terms of the form $\partial^4 h$ even though both have the same number of derivatives. Upon covariantisation, it implies that the BV expansion is valid if $R^2 \ll \nabla^2 R$. By contrast, the local Schwinger-DeWitt expansion is valid for short proper time $\epsilon R \ll 1$ or equivalently for the entire weak gravity regime $R \ll M_0^2$ without any further restrictions on curvatures.

2.2.6 Barvinsky-Vilkovisky Expansion and Conformal Decomposition

To compare (2.87) with our results, it is necessary to go to third order in the BV expansion. Explicit expressions to this order have been worked out in [20] but they are rather complicated going over several pages. It is not immediately obvious how these expressions could reduce to the simple expressions that we obtained earlier. However, one can use the fact that the Weyl variation of the BV effective action must correctly reproduce the *local* Weyl anomaly. This observation suggests a 'conformal decomposition' of the action in terms of a Weyl-invariant piece and a Weyl-variant piece [21, 22, 77]. This conformal decomposition is what is most easily compared with our results.

To illustrate the idea, consider the BV effective action for quantum electrodynamics obtained by integrating out massless charged fields in the presence a background gauge field A. To third order in curvatures it is given by [77]:

$$
\begin{aligned}
\mathcal{I}[g,A] \;=\;& -\frac{1}{4}\int d^4x\sqrt{|g|}\left\{\frac{1}{e^2}F_{\mu\nu}F^{\mu\nu} - \frac{b}{2}\left[F^{\mu\nu}\log\left(\frac{-\nabla^2}{M^2}\right)F_{\mu\nu} + \frac{1}{3}F^2\frac{1}{\nabla^2}R + \right.\right. \\
&+\; 4R^{\mu\nu}\frac{1}{\nabla^2}\left(\log\left(\frac{-\nabla^2}{M^2}\right)\left(F^a_{\mu\sigma}F^{a\ \sigma}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2\right) - F_{\mu\sigma}\log\left(\frac{-\nabla^2}{M^2}\right)F^\sigma_\nu + \right. \\
&+\; \left.\left.\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}\log\left(\frac{-\nabla^2}{M^2}\right)F_{\alpha\beta}\right) - \frac{1}{3}RF^{\mu\nu}\frac{1}{\nabla^2}F_{\mu\nu} + W^{\alpha}_{\ \beta\mu\nu}F_{\alpha}^{\ \beta}\frac{1}{\nabla^2}F^{\mu\nu}\right] + \\
&+\; \left. 4\,\tilde{b}\,F^{\mu\nu}F_{\alpha}^{\ \beta}\frac{1}{\nabla^2}W^{\alpha}_{\ \beta\mu\nu}\right\} + \mathcal{O}(\mathcal{R}^4) \qquad (2.88)
\end{aligned}
$$

where the logarithm of the covariant d'Alembertian $\log(-\nabla^2/M^2)$ is defined as in

(2.66) and

$$b = \frac{1}{24\pi^2}\left(N_S + 4N_F\right)\,, \qquad \tilde{b} = \frac{1}{96\pi^2}(-N_S + 2N_F)\,. \tag{2.89}$$

Note that b is the usual beta function coefficient (2.56) in flat space but $\tilde{b}$ is relevant only in curved backgrounds. We have ignored the purely gravitational terms coming from the $\mathcal{C}$ anomaly that are independent of the background gauge field.

It turns out that except for the second term in the square bracket, all other terms in (2.88) are actually Weyl invariant [22, 77]. This 'conformal decomposition' then implies that the only Weyl-variant term that could contribute to the $\mathcal{B}$ anomaly is precisely this second term:

$$\tilde{\mathcal{I}}_{\mathcal{B}}[g,A] = -\frac{b}{4}\int d^4x\sqrt{|g|}\; F^{\mu\nu}\left(-\frac{1}{6}\frac{1}{\nabla^2}R\right)F_{\mu\nu}\,. \tag{2.90}$$

Since all other terms taken together are Weyl invariant, for a Weyl-flat metric they must reduce to the one-loop effective action on flat space (2.65):

$$\mathcal{I}[\eta,A] = -\frac{1}{4}\int d^4x\; F^{\mu\nu}\left[\frac{1}{e^2(M)} - \frac{b}{2}\log\left(\frac{-\partial^2}{M^2}\right)\right]F_{\mu\nu}\,. \tag{2.91}$$

Hence for a Weyl-flat metric the action (2.88) simplifies dramatically to

$$\mathcal{I}[g,A] = \mathcal{I}[\eta,A] + \tilde{\mathcal{I}}_{\mathcal{B}}[\eta,\Omega,A]\,. \tag{2.92}$$

We would like to compare this result with the one obtained by integrating the anomaly:

$$\mathcal{I}[g,A] = \mathcal{I}[\eta,A] + \mathcal{I}_{\mathcal{B}}[\eta,\Omega,A]\,, \quad \text{with} \quad \mathcal{I}_{\mathcal{B}}[\eta,\Omega,A] = -\frac{b}{4}\int d^4x\;\Omega(x)\,F_{\mu\nu}F^{\mu\nu}. \tag{2.93}$$

To this end, we note that the Weyl factor $\Omega[g](x)$ can be expressed as a nonlocal covariant functional of the metric [79, 80, 23] given by

$$\Omega[g](x) = \frac{1}{4}\int d^4y\,\sqrt{|g|}\,G_4(x,y)\,F_4[g](y)\,, \tag{2.94}$$

where

$$F_4[g] := E_4[g] - \frac{2}{3}\nabla^2 R[g] \;= (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 - \frac{2}{3}\nabla^2 R)[g]\,, \tag{2.95}$$

and the Green function $G_4(x,y)$ defined by

$$\Delta_4^x[g]G_4(x,y) = \delta^{(4)}(x,y) := \frac{\delta^{(4)}(x-y)}{\sqrt{|g|}} \tag{2.96}$$

is the inverse of the Weyl-covariant quartic differential operator

$$\Delta_4[g] = \left(\nabla^2\right)^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}\left(\nabla^\nu R\right)\nabla_\nu - \frac{2}{3}R\,\nabla^2\,. \tag{2.97}$$

The expression (2.94) follows from the fact that for metrics related by a Weyl rescaling $g_{\mu\nu} = e^{2\Omega(x)}\,\eta_{\mu\nu}$, the corresponding F_4 scalars are related by

$$F_4[g] = e^{-4\Omega}\left(F_4[\eta] + 4\,\Delta_4[\eta]\,\Omega\right)\,, \tag{2.98}$$

and the operators Δ_4 by

$$\Delta_4[g] = e^{-4\Omega}\,\Delta_4[\eta]\,. \tag{2.99}$$

Since the Minkowski reference metric satisfies $F_4[\eta] = 0$ the expression (2.94) follows from inverting (2.98). This expression is manifestly covariant but nonlocal, consistent with the fact that the anomalous Ω dependence represents genuine long-distance quantum effects that cannot be removed by counter-terms that are local functionals of the metric.

When $R^2 \ll \nabla^2 R$ one can expand the expression for Ω (2.94) in curvatures to obtain

$$\Omega[g](x) = -\frac{1}{6}\frac{1}{\nabla^2}R + \mathcal{O}(R^2) \tag{2.100}$$

which when substituted in (2.93) reproduces the anomaly action obtained in the BV regime (2.90).

To recover the full expression for Ω in the Barvinsky-Vilkovisky formalism one must invert the operator Δ_4 perturbatively, which involves higher and higher orders in curvatures. As a result, the expression (2.94) for Ω will similarly involve terms to arbitrary order in the curvature expansion. This implies that to recover the exact and simple expression (2.68) obtained by integrating the Weyl anomaly it would be necessary to re-sum the covariant perturbation theory (2.88) *to all orders* in curvatures R for the class of Weyl-flat metrics. Since $\mathcal{I}_{\mathcal{B}}$ already contains F^2, the next correction is of order $F^2R^2 \sim \mathcal{R}^4$ in the generalised curvature expansion. In other words,

$$\mathcal{I}_{\mathcal{B}}[\eta, \Omega, A] = \tilde{\mathcal{I}}_{\mathcal{B}}[\eta, \Omega, A] + \mathcal{O}(\mathcal{R}^4)\,. \tag{2.101}$$

Already at order $\mathcal{R}^4$, the expression in the BV expansion becomes unmanageable. It is remarkable that the simple expression (2.44) re-sums this expansion to all orders albeit for a restricted class of Weyl-flat metrics.

Thus, explicit 'covariantisation' of our answer obtained by integrating the anomaly can lead to rather complicated expressions even though the exact answer (2.93) is strikingly simple. As noted earlier, our procedure guarantees that the full answer depends only the physical metric g even though *a priori* the right hand side appears to depend on $\eta_{\mu\nu}$ and Ω separately.

Chapter 3

Primordial Magnetogenesis

It is well known that the cosmological expansion in the early universe can induce particle production. The most well known example is cosmic inflation which freezes the quantum fluctuations of the inflaton field into classical fluctuations, thus sourcing the large-scale structures in the universe. While a rapidly expanding universe can source field fluctuations for scalars with small masses, the generation of gauge fields is a more subtle problem. The classical dynamics of gauge fields is governed by a Weyl invariant action. This means that in a Friedmann-Robertson-Walker spacetime it is independent of the scale factor, and hence unaffected by the expansion of the universe.

However, the classical Weyl invariance of the Yang-Mills action is violated in the quantum theory because of the need to regularise the path integral. These Weyl anomalies, or equivalently the nontrivial beta functions of the theory, imply that the quantum effective action obtained after integrating out massless charged particles is no longer Weyl invariant. This is expected to lead to an anomalous dependence on the scale factor under a fairly mild assumption that the masses of the charged particles that contribute to the quantum loops are negligible compared to the Hubble scale during the cosmological era of interest. For the Maxwell theory, the violation of Weyl invariance can lead to gauge field excitations in the early universe, and thus to the generation of electromagnetic fields.

In our universe, magnetic fields are observed on various scales such as in galaxies and galaxy clusters. Recent gamma ray observations suggest the presence of magnetic fields even in intergalactic voids. In order to explain the origin of the magnetic fields, theories of primordial magnetogenesis have been studied in the literature, where most models violate the Weyl invariance explicitly at the classical level by coupling the gauge field to some degrees of freedom beyond the Standard Model of particle physics [32, 12]. See e.g. [4, 5, 6, 81, 7, 8, 9, 11] for reviews on magnetic fields in the universe from different perspectives.

It was pointed out in [82] that the Weyl anomaly of quantum electrodynamics itself should also induce magnetic field generation. If true, this would be a natural realisation of primordial magnetogenesis within the Standard Model. Moreover, since the anomaly is intrinsic to the Standard Model, its contribution to the magnetic fields, if any, is irreducible[1]. Hence it is important to evaluate this also for the

[1] It is irreducible in the sense that unless the beta function was zero, this contribution would

purpose of identifying the minimum seed magnetic fields of our universe. Since [82], there have indeed been many studies on this topic. However, there is currently little consensus on the effect of the Weyl anomaly on magnetic field generation. One of the main difficulties in proceeding with these computations is that the quantum effective action in curved spacetime is in general very hard to evaluate. In principle, it is a well-posed problem in perturbation theory. One can regularise the path integral covariantly using dimensional regularisation or short proper-time regularisation and evaluate the effective action using the background field method. However, explicit evaluation of the path integral for a generic metric is not feasible. For instance, to obtain the one-loop effective action it is necessary to compute the heat kernel of a Laplace-like operator in an arbitrary background, which amounts to solving the Schrödinger problem for an arbitrary potential.

One could evaluate the effective action perturbatively in the weak field limit using covariant nonlocal expansion of the heat kernel developed by Barvinsky, Vilkovisky, and collaborators [17, 18]. The effective action in this expansion has been worked out to third order in curvatures [19, 20, 21, 22]. Similar results have been obtained independently by Donoghue and El-Menoufi [78, 77] using Feynman diagrams. Some of the earlier works on primordial magnetogenesis from anomalies, e.g. [33], relies on the effective action derived in this weak field approximation. The weak field expansion is valid in the regime $\mathcal{R}^2 \ll \nabla^2\mathcal{R}$, where $\mathcal{R}$ denotes a generalised curvature including both a typical geometric curvature R as well as a typical gauge field strength F. During slow-roll inflation, one is in the regime of slowly varying geometric curvatures, $R^2 \gg \nabla^2 R$, whereas during matter domination, one has $R^2 \sim \nabla^2 R$. Thus, during much of the cosmological evolution, the curvatures are not weak compared to their derivatives. Therefore, to study primordial magnetogenesis reliably over a long range of cosmological evolution, it is essential to overcome the limitations of the weak field approximation.

As we discussed in chapter 2, one can go beyond the weak field approximation for Weyl flat spacetimes. In this case, one can exploit Weyl anomalies and the simplicity of the background metric to completely determine the dependence of the effective action on the scale factor at one-loop even when the changes in the scale factor are large. The main advantage of this approach is that Weyl anomalous dimensions of local operators can be computed reliably using local computations such as the Schwinger-DeWitt expansion without requiring the weak field approximation $\mathcal{R}^2 \ll \nabla^2\mathcal{R}$. The resulting action obtained by integrating the anomaly is necessarily nonlocal and essentially resums the Barvinsky-Vilkovisky expansion to all orders in curvatures albeit for the restricted class of Weyl-flat metrics. A practical advantage is that one can extract the essential physics with relative ease using only the local Schwinger-DeWitt expansion which is computationally much simpler.

In this chapter we use the quantum effective action of 2.68 beyond the weak field limit, and present the first consistent computation of the effect of the Weyl anomaly on cosmological magnetic field generation. We study $U(1)$ gauge fields originating as vacuum fluctuations in the inflationary universe, and analyse their evolution during the inflation and post-inflation epochs. Our main conclusion is that there is no production of coherent magnetic fields from the Weyl anomaly of

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always be there, regardless of the possible extraneous sources of Weyl symmetry breaking such as couplings between the gauge field and inflaton.

quantum electrodynamics, contrary to the claims of previous works. Our results hold independently of the details of the cosmological history, or of the number of massless charged particles in the theory. We show, in particular, that even if there were extra charged particles in addition to those of the Standard Model, the Weyl anomaly with an increased beta function still would not produce any magnetic fields.

Since the time-dependence introduced by the Weyl anomaly is unusually weak, the analysis of the (non)generation of magnetic fields requires careful consideration of the nature of the field fluctuations, in particular whether they are classical or quantum. For this purpose, we introduce general criteria for assessing the quantumness of field fluctuations. Using these criteria, we find that the quantum fluctuations of the gauge field actually do not get converted into classical fluctuations.

This chapter is organised as follows. In §3.1 we specialise the results of chapter 2 regarding the one-loop quantum effective action for a Weyl-flat metric to the specific case of an abelian field in an expanding universe. In §3.2 we canonically quantise the gauge fields using this action and introduce the criteria for quantumness. In §3.3 we analyse the evolution of the gauge field in the early universe and show that there is no production of coherent magnetic fields. In §3.4 we comment on the relation of our work to earlier works and conclude with a discussion of possible extensions.

3.1 Nonlocal Effective Action for Quantum Electrodynamics

In the early universe before the electroweak phase transition, quarks and leptons are massless[2]. Consider the hypercharge $U(1)$ gauge field of the Standard Model coupled to these massless Dirac fermions which we collectively denote by Ψ. Here we follow the same procedure introduced in 2.2.2 specialising to this particular case. The classical Lorentzian action in curved spacetime is

$$S_0[g, A, \Psi] = -\int d^4x \sqrt{|g|} \left[\frac{1}{4\, e_0^2} F_{\mu\nu} F^{\mu\nu} + i\, \bar{\Psi}\, \Gamma^a\, e_a^\mu\, D_\mu \Psi \right], \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and e_0^2 is the bare charge. The covariant derivative is defined including both the gauge connection A_μ and the spin connection in the spinor representation w_μ^{ab}:

$$D_\mu := \partial_\mu - \frac{i}{2} w_\mu^{ab} J_{ab} - iQA_\mu\,, \tag{3.2}$$

where $\{J_{ab}\}$ are the Lorentz representation matrices and Q is the quantised charge of the field in units of e_0.

Classically, this action is invariant under Weyl transformation:

$$g_{\mu\nu} \to e^{2\xi(x)} g_{\mu\nu}\,, \quad g^{\mu\nu} \to e^{-2\xi(x)} g^{\mu\nu}\,, \quad \Psi \to e^{-\frac{3}{2}\xi(x)} \Psi\,, \quad A_\mu \to A_\mu\,. \tag{3.3}$$

[2]The expectation value of the Higgs field could fluctuate during inflation with an amplitude of the order of the inflationary Hubble scale. However, since most of the Yukawa couplings are small, the induced masses for these fermions would still be smaller than the Hubble scale which could be treated as effectively massless.

The Weyl symmetry is anomalous because in the quantum theory one must introduce a mass scale M to renormalise the theory which violates the Weyl invariance. The Weyl anomaly introduces a coupling of the gauge field to the Weyl factor of the metric. To analyse its effects on the fluctuations one can proceed in two steps. One can first perform the path integral over fermions treating both the metric and the gauge field as backgrounds. The resulting effective action for the electromagnetic field will include all quantum effects of fermions in loops. It is necessarily nonlocal because it is obtained by integrating out massless fields. One can then quantise the gauge field using this effective action to study the propagation of photons including all vacuum polarisation effects as well as interactions with the background metric.

In flat spacetime, with $g_{\mu\nu} = \eta_{\mu\nu}$, the quantum effective action can be computed using standard field theory methods. Up to one loop order, the quadratic action for the gauge fields is given by[3]

$$S_{\text{flat}}[\eta, A] = -\frac{1}{4e^2}\int d^4x \left[F_{\mu\nu}(x)F^{\mu\nu}(x) - \tilde{\beta}(e)\int d^4y \; F_{\mu\nu}(x)L(x-y)F^{\mu\nu}(y)\right] \tag{3.4}$$

where $e^2 \equiv e^2(M)$ is the coupling renormalised at a renormalisation scale M, and $\tilde{\beta}(e)$ is the beta function of $\log e$, i.e.,

$$\frac{d\log e}{d\log M} = \tilde{\beta}(e). \tag{3.5}$$

The beta function of quantum electrodynamics takes positive values, which is written as

$$\tilde{\beta}(e) = \frac{be^2}{2}, \quad \text{where} \quad b = \frac{\text{Tr}\,(Q^2)}{6\pi^2}. \tag{3.6}$$

Here the coefficient b is expressed in terms of the trace of the charge operator taken over all massless charged fermions[4]. To keep the discussion general, we will also allow for the possibility of extra massless charged particles beyond the Standard Model in the early universe, and treat the beta function as an arbitrary positive parameter.

As mentioned in section 2.2.2, the bilocal kernel in the second term of the action can be defined by a Fourier transform:

$$L(x-y) \equiv \langle x|\log\left(\frac{-\partial^2}{M^2}\right)|y\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu(x^\mu - y^\mu)}\, \log\left(\frac{p_\nu p^\nu}{M^2}\right). \tag{3.7}$$

The more familiar look of the action, in momentum space, takes the form

$$S_{\text{flat}}[\eta, A] = -\frac{1}{4\,e^2}\int \frac{d^4p}{(2\pi)^4}\,\eta^{\rho\alpha}\eta^{\sigma\beta}\;\tilde{F}_{\rho\sigma}(-p)\left[1 - \tilde{\beta}\log\left(\frac{p_\nu p^\nu}{M^2}\right)\right]\tilde{F}_{\alpha\beta}(p). \tag{3.8}$$

Here one can more easily recognise the first term as the classical action with renormalised coupling and the second term as the usual one-loop logarithmic running of the coupling constant.

[3]The quantum effective action in general contains higher powers of the field strength but the resulting nonlinearities will not be relevant for our purposes.

[4]The photon field is related to the hypercharge gauge field by a number of order unity that depends on the Weinberg angle. This distinction will not be important for our conclusions.

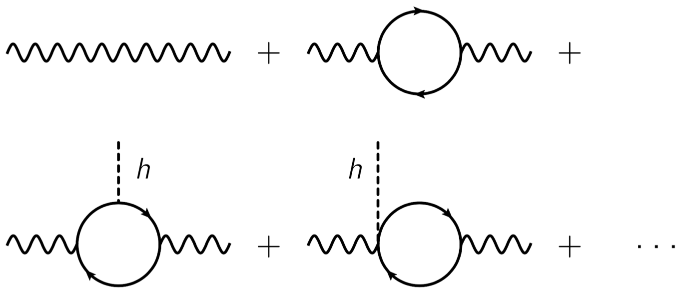

Figure 3.1: The first term in the top line represents the classical propagation of the photon whereas the second term in the top line represents the one-loop correction to the propagator due to vacuum polarisation in flat space. All diagrams in the bottom line represent vacuum polarisation in the presence of a curved metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ treating $h_{\mu\nu}$ as a perturbation. The Barvinsky-Vilkovisky expansion gives the covariantized nonlocal action resumming the specific powers of h required for general covariance. Equation (3.11) obtained by integrating the anomaly resums these diagrams to all orders into a simple expression for Weyl-flat spacetimes.

To get some intuition about the effects of a curved spacetime, it is useful to consider the weak field limit so that the metric is close to being flat, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. If $h_{\mu\nu}$ is very small, then one can treat it as a perturbation to compute the corrections using Feynman diagrams. Various corrections arising from the interactions with the non-flat background metric are shown diagrammatically in Figure 3.1 for the photon propagator. It is clear that even at one loop order, there are an infinite number of diagrams that contribute to the propagator. The Barvinsky-Vilkovisky expansion and related results complete the obtained expressions into non-linear and covariant functions of $h_{\mu\nu}$. Doing so however to a fixed order in $h_{\mu\nu}$ implies one neglects higher curvatures when compared to higher derivatives. More concretely,

$$R^2 \sim (\partial^2 h)^2, \quad \nabla^2 R \sim \partial^4 h. \tag{3.9}$$

As a result this 'curvature expansion' is very different from the usual 'derivative expansion' and is justified only in the limit $\nabla^2 R \gg R^2$. If one is interested in a metric such as the Friedmann-Robertson-Walker metric that differs substantially from the Minkowski metric, a perturbative evaluation in this weak field limit clearly would not be adequate.

As we have discussed in chapter 2, for Weyl-flat metrics, i.e., metrics of the form $g_{\mu\nu} = e^{2\Omega}\eta_{\mu\nu}$, it is indeed possible to obtain the quantum effective action at one-loop as an exact functional of Ω without assuming small h. This is achieved by integrating the Weyl anomaly and matching with the flat space results 2.42. The part of the action that contains the gauge field takes a simple form[5]:

$$S[g, A] = S_{\text{flat}}[\eta, A] + S_{\mathcal{B}}[\eta, \Omega, A]\,, \tag{3.10}$$

[5]In the space of metrics, this action is evaluated in the subspace of Weyl-flat metrics. For this reason it is beyond the reach of this method to compute the equations of motion for the background metric which requires a functional variation with respect to $g_{\mu\nu}$ even in directions orthogonal to the Weyl orbits.

where S_{flat} is the effective action at one-loop as in (3.4), and $S_{\mathcal{B}}$ has the anomalous dependence on the Weyl factor (or the scale factor in a Friedmann-Robertson-Walker spacetime):

$$S_{\mathcal{B}}[\eta, \Omega, A] = -\frac{\tilde{\beta}}{2e^2} \int d^4x \, \Omega(x) \, F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{3.11}$$

where the indices are raised using the Minkowski metric as in (3.4). Thus the total effective action can be written as

$$S = -\frac{1}{4e^2} \int d^4x \, d^4y \, F_{\mu\nu}(x) \, \langle x| \left[1 - \tilde{\beta} \log \left(\frac{-\partial^2}{M^2 \exp(2\Omega(x))} \right) \right] |y\rangle \, F^{\mu\nu}(y). \tag{3.12}$$

Even though the resummed answer of (3.11) is a local functional of Ω, it must come from nonlocal terms when expressed in terms of the original metric $g_{\mu\nu}$. There are non-local functionals that evaluate to the Weyl factor $\Omega(x)$ on Weyl-flat backgrounds [79, 80, 23]. One example is the Riegert functional as we mentioned in 2.94:

$$\Omega[g](x) = \frac{1}{4} \int d^4y \, \sqrt{|g(y)|} \, G_4(x, y) \, F_4[g](y) \, , \tag{3.13}$$

where F_4 is given by 2.95 and the Green function $G_4(x, y)$ defined by

$$\Delta_4^y[g] G_4(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{|g|}} \tag{3.14}$$

is the inverse of the Weyl-covariant quartic differential operator Δ_4 defined in 2.97.

The Riegert functional (2.94) is manifestly covariant but nonlocal, consistent with the fact that the anomalous Ω dependence represents genuine long-distance quantum effects that cannot be removed by counter-terms that are local functionals of the metric. In the perturbative Barvinsky-Vilkovisky regime we have $R^2 \ll \nabla^2 R$ and one can expand the expression for Ω (2.94) in curvatures to obtain to leading order

$$\Omega[g](x) = -\frac{1}{6} \frac{1}{\nabla^2} R + \ldots \, . \tag{3.15}$$

It is clear from (2.94) that this expression receives corrections to all orders in R. The simple expression (2.93) effectively resums these contributions to all orders as explained in chapter 2.

There are other functionals that naturally appear when integrating anomalies. The distinction for us would be particularly important is we were to compute the equations of motion for the metric. Since this is not our case and we are interested solely on Weyl-flat spacetimes, we won't discuss the differences in detail here, but a deeper discussion can be found at [22].

In four dimensions there is one other natural functional that evaluates to Ω. It is interesting to notice that it does coincide with the Riegert functional 2.94 in the first term in weak field expansion. The reason for that is that by definition they must differ by Weyl invariant terms. Since there are no Weyl invariant terms linear in the curvatures in four dimensions, the difference between them can only be seen from the quadratic order and beyond.

3.2 Quantisation of the Gauge Field

We now quantise the gauge field in a flat Friedmann-Robertson-Walker background,

$$ds^2 = a(\tau)^2 \left(-d\tau^2 + d\boldsymbol{x}^2\right). \tag{3.16}$$

Here the Weyl factor $\Omega = \log a$ now depends only on time. For later convenience we rewrite the effective action (3.12) in the form

$$S = -\frac{1}{4}\int d^4x_1 d^4x_2\, \mathcal{I}^2(x_1,x_2) F_{\mu\nu}(x_1) F^{\mu\nu}(x_2), \tag{3.17}$$

with

$$\mathcal{I}^2(x_1,x_2) = \frac{1}{e^2}\int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x_1^\mu - x_2^\mu)} \left[1 - \tilde{\beta}\log\left(\frac{a_\star^2}{a(\tau_1)^2}\right) - \tilde{\beta}\log\left(\frac{k_\nu k^\nu}{M^2 a_\star^2}\right)\right], \tag{3.18}$$

where x^μ and k^μ are comoving coordinates and wave number, respectively, and the indices are raised and lowered with the Minkowski metric. We have introduced a reference "scale factor" $a_\star$ in order to split the action into $k_\nu k^\nu$-dependent and independent parts; this splitting is completely arbitrary, and hence $a_\star$ can also be chosen arbitrarily.

3.2.1 Simplified Effective Action

Let us decompose the spatial components of the gauge field into irrotational and incompressible parts,

$$A_\mu = (A_0, \partial_i S + V_i) \quad \text{with} \quad \partial_i V_i = 0, \tag{3.19}$$

where we use Latin letters to denote spatial indices ($i = 1,2,3$), and the sum over repeated spatial indices is implied irrespective of their positions. One can check that A_0 is a Lagrange multiplier, whose constraint equation can be used to eliminate both A_0 and S from the action to yield

$$S = \frac{1}{2}\int d^4x_1 d^4x_2\, \mathcal{I}^2(x_1,x_2)\left\{V_i'(x_1)V_i'(x_2) - \partial_i V_j(x_1)\partial_i V_j(x_2)\right\}, \tag{3.20}$$

where we drop surface terms, and a prime denotes a derivative with respect to the conformal time τ. We now go to momentum space,

$$V_i(\tau,\boldsymbol{x}) = \sum_{p=1,2}\int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \epsilon_i^{(p)}(\boldsymbol{k})\, u_{\boldsymbol{k}}^{(p)}(\tau), \tag{3.21}$$

where $\epsilon_i^{(p)}(\boldsymbol{k})$ ($p = 1,2$) are two orthonormal polarisation vectors that satisfy

$$\epsilon_i^{(p)}(\boldsymbol{k})\, k_i = 0, \quad \epsilon_i^{(p)}(\boldsymbol{k})\, \epsilon_i^{(q)}(\boldsymbol{k}) = \delta_{pq}. \tag{3.22}$$

From these conditions, it follows that

$$\sum_{p=1,2} \epsilon_i^{(p)}(\boldsymbol{k})\, \epsilon_j^{(p)}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{3.23}$$

where we use k to denote the amplitude of the spatial wave number, i.e. $k = |\boldsymbol{k}|$. Unlike the spacetime indices, we do not assume implicit summation over the polarisation index (p).

The equation of motion of V_i requires the mode function $u_{\boldsymbol{k}}^{(p)}(\tau)$ to obey

$$\begin{aligned} 0 = &\left\{1 + 2\tilde{\beta} \log\left(\frac{a(\tau)}{a_\star}\right)\right\} \left\{u_{\boldsymbol{k}}^{\prime\prime(p)}(\tau) + k^2 u_{\boldsymbol{k}}^{(p)}(\tau)\right\} + 2\tilde{\beta} \frac{a'(\tau)}{a(\tau)} u_{\boldsymbol{k}}^{\prime(p)}(\tau) \\ &- \tilde{\beta} \int d\tilde{\tau} \left\{u_{\boldsymbol{k}}^{\prime\prime(p)}(\tilde{\tau}) + k^2 u_{\boldsymbol{k}}^{(p)}(\tilde{\tau})\right\} \int \frac{dk^0}{2\pi} e^{-ik^0(\tau-\tilde{\tau})} \log\left(\frac{k_\mu k^\mu}{M^2 a_\star^2}\right). \end{aligned} \tag{3.24}$$

In order to estimate the second line, let us make the crude assumption that the k^0 integral amounts to the replacement

$$\int \frac{dk^0}{2\pi} e^{-ik^0(\tau-\tilde{\tau})} \log\left(\frac{k_\mu k^\mu}{M^2 a_\star^2}\right) \to \delta(\tau - \tilde{\tau}) \log\left(\frac{k^2}{M^2 a_\star^2}\right), \tag{3.25}$$

where the coefficient of $\delta(\tau - \tilde{\tau})$ is obtained by integrating both sides over τ. Then comparing with the terms in the $\{\}$ parentheses in the first line of (3.24), one sees that the second line is negligible when

$$\left|1 + 2\tilde{\beta} \log\left(\frac{a}{a_\star}\right)\right| \gg \left|\tilde{\beta} \log\left(\frac{k^2}{M^2 a_\star^2}\right)\right|. \tag{3.26}$$

The second line of the equation of motion follows from the $\log(k_\nu k^\nu)$ term of (3.18) in the action. Hence as long as the wave modes of interest satisfy the condition (3.26), we can ignore this term and use a simplified effective action of

$$S_{\text{loc}} = -\frac{1}{4} \int d^4x \, I(\tau)^2 \, F_{\mu\nu}(x) F^{\mu\nu}(x), \tag{3.27}$$

where

$$I(\tau)^2 = \frac{1}{e^2} \left[1 + 2\tilde{\beta} \log\left(\frac{a(\tau)}{a_\star}\right)\right]. \tag{3.28}$$

The equation of motion (3.24) reduces to

$$u_{\boldsymbol{k}}^{\prime\prime(p)} + 2\frac{I'}{I} u_{\boldsymbol{k}}^{\prime(p)} + k^2 u_{\boldsymbol{k}}^{(p)} = 0. \tag{3.29}$$

The action of the form (3.27) with various time-dependent functions I^2 has been studied in the context of primordial magnetogenesis since the seminal work of [12]. However we stress that, unlike many models of magnetogenesis whose time dependences are attributed to couplings to scalar fields extraneous to the Standard Model, here, the function I^2 of (3.28) arises from the Weyl anomaly of quantum electrodynamics and thus is intrinsic to the Standard Model. It should also be noted that, due to the positivity of the beta function of quantum electrodynamics, I^2 monotonically increases in time.

As is indicated by the equation of motion, there is no mixing between different wave modes under the simplified action. This allows us to take the parameter $a_\star$ differently for each wave mode upon carrying out computations. For convenience we will choose

$$a_\star = \frac{k}{M}, \tag{3.30}$$

so that the simplifying condition (3.26) can be satisfied for a sufficiently long period of time for every wave mode. However we should also remark that even with this choice, the $\log(k_\nu k^\nu)$ term does not drop out completely. This is because we have used the approximation (3.25), and thus in the right hand side of the condition (3.26), the argument of the log should be considered to have some width around $k^2/M^2 a_\star^2$. Hence we rewrite the simplifying condition for the choice of (3.30), by combining with the further assumption of $I^2 > 0$, as

$$1 + 2\tilde{\beta} \log \left(\frac{aM}{k} \right) \gg \tilde{\beta}. \tag{3.31}$$

If, on the other hand, the $\log(k_\nu k^\nu)$ term cannot be ignored, this signals that the theory is strongly coupled[6]. The Landau pole at which the coupling e blows up can be read off from the running of the coupling (3.5) as

$$\Lambda_{\max} = M \exp \left(\frac{1}{2\tilde{\beta}} \right). \tag{3.32}$$

In terms of this, (3.31) is rewritten as $k/a < \Lambda_{\max} \exp(-1/2)$. Hence the simplifying condition can be understood as the requirement that the physical momentum should be below the Landau pole during the times when one wishes to carry out computations.

The function $\mathcal{I}^2$ (3.18) in the full effective action is independent of the renormalisation scale M, since the coupling runs as (3.5). We note that with the choice (3.30) for $a_\star$, the function I^2 (3.28) in the simplified action also becomes independent of M.

3.2.2 Canonical Quantisation

In order to quantise the gauge field, we promote V_i to an operator,

$$V_i(\tau, \boldsymbol{x}) = \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} \epsilon_i^{(p)}(\boldsymbol{k}) \left\{ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}}^{(p)} u_{\boldsymbol{k}}^{(p)}(\tau) + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}}^{\dagger(p)} u_{\boldsymbol{k}}^{*(p)}(\tau) \right\}, \tag{3.33}$$

where $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$ are annihilation and creation operators satisfying the commutation relations,

$$[a_{\boldsymbol{k}}^{(p)}, a_{\boldsymbol{l}}^{(q)}] = [a_{\boldsymbol{k}}^{\dagger(p)}, a_{\boldsymbol{l}}^{\dagger(q)}] = 0, \quad [a_{\boldsymbol{k}}^{(p)}, a_{\boldsymbol{l}}^{\dagger(q)}] = (2\pi)^3 \delta^{pq} \delta^{(3)}(\boldsymbol{k} - \boldsymbol{l}). \tag{3.34}$$

For V_i and its conjugate momentum which follows from the Lagrangian $\mathcal{L} = (I^2/2)(V_i' V_i' - \partial_i V_j \partial_i V_j)$ (cf. (3.20)) as

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial V_i'} = I^2 V_i', \tag{3.35}$$

we further impose the commutation relations

$$\begin{aligned} &[V_i(\tau, \boldsymbol{x}), V_j(\tau, \boldsymbol{y})] = [\Pi_i(\tau, \boldsymbol{x}), \Pi_j(\tau, \boldsymbol{y})] = 0, \\ &[V_i(\tau, \boldsymbol{x}), \Pi_j(\tau, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial_l \partial_l} \right). \end{aligned} \tag{3.36}$$

[6] The condition (3.31) is rewritten as $I^2 \gg \tilde{\beta}/e^2 = b/2$. Violating this condition provides an explicit example of what is often referred to in the literature as the "strong coupling problem" of magnetogenesis with a tiny I [13].

The second line can be rewritten using (3.23) as

$$[V_i(\tau,\boldsymbol{x}),\, \Pi_j(\tau,\boldsymbol{y})] = i \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \epsilon_i^{(p)}(\boldsymbol{k})\, \epsilon_j^{(p)}(\boldsymbol{k}). \tag{3.37}$$

Choosing the polarisation vectors such that

$$\epsilon_i^{(p)}(\boldsymbol{k}) = \epsilon_i^{(p)}(-\boldsymbol{k}), \tag{3.38}$$

one can check that the commutation relations (3.34) are equivalent to (3.36) when the mode function is independent of the direction of $\boldsymbol{k}$, i.e.,

$$u_{\boldsymbol{k}}^{(p)} = u_k^{(p)}, \tag{3.39}$$

and also obeys

$$I^2 \left(u_k^{(p)} u_k^{\prime *(p)} - u_k^{*(p)} u_k^{\prime (p)} \right) = i. \tag{3.40}$$

It follows from the equation of motion (3.29) that the left hand side of this condition is time-independent, and thus this sets the normalisation of the mode function.

3.2.3 Photon Number and Quantumness Measure

Before proceeding to compute the cosmological evolution of the gauge field fluctuations, we introduce two measures of 'quantumness' to determine when the field fluctuations can be regarded as classical. See also [83, 84] for discussions along similar lines.

In order to separately discuss each wave mode, we focus on the Fourier components of the operator V_i (3.33) and its conjugate momentum:

$$\begin{aligned} V_i(\tau,\boldsymbol{x}) &= \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \epsilon_i^{(p)}(\boldsymbol{k})\, v_{\boldsymbol{k}}^{(p)}(\tau), \\ \Pi_i(\tau,\boldsymbol{x}) &= \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \epsilon_i^{(p)}(\boldsymbol{k})\, \pi_{\boldsymbol{k}}^{(p)}(\tau)\,. \end{aligned} \tag{3.41}$$

The Fourier modes can be expressed in terms of the annihilation and creation operators as

$$v_{\boldsymbol{k}}^{(p)}(\tau) = a_{\boldsymbol{k}}^{(p)} u_k^{(p)}(\tau) + a_{-\boldsymbol{k}}^{\dagger(p)} u_k^{*(p)}(\tau), \quad \pi_{\boldsymbol{k}}^{(p)}(\tau) = I(\tau)^2 \left(a_{\boldsymbol{k}}^{(p)} u_k^{\prime(p)}(\tau) + a_{-\boldsymbol{k}}^{\dagger(p)} u_k^{\prime *(p)}(\tau) \right). \tag{3.42}$$

The commutation relations (3.34) or (3.36) entail

$$[v_{\boldsymbol{k}}^{(p)}(\tau),\, v_{\boldsymbol{l}}^{(q)}(\tau)] = [\pi_{\boldsymbol{k}}^{(p)}(\tau),\, \pi_{\boldsymbol{l}}^{(q)}(\tau)] = 0, \quad [v_{\boldsymbol{k}}^{(p)}(\tau),\, \pi_{\boldsymbol{l}}^{(q)}(\tau)] = i(2\pi)^3\, \delta^{pq}\, \delta^{(3)}(\boldsymbol{k}+\boldsymbol{l}). \tag{3.43}$$

We now introduce time-dependent annihilation and creation operators as

$$b_{\boldsymbol{k}}^{(p)}(\tau) \equiv \sqrt{\frac{k}{2}}\, I(\tau)\, v_{\boldsymbol{k}}^{(p)}(\tau) + \frac{i}{\sqrt{2k}} \frac{\pi_{\boldsymbol{k}}^{(p)}(\tau)}{I(\tau)}, \quad b_{\boldsymbol{k}}^{\dagger(p)}(\tau) \equiv \sqrt{\frac{k}{2}}\, I(\tau)\, v_{-\boldsymbol{k}}^{(p)}(\tau) - \frac{i}{\sqrt{2k}} \frac{\pi_{-\boldsymbol{k}}^{(p)}(\tau)}{I(\tau)}, \tag{3.44}$$

so that $b_{\boldsymbol{k}}^{(p)}$ and $b_{\boldsymbol{k}}^{\dagger(p)}$ satisfy equal-time commutation relations similar to (3.34) of $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$, as well as diagonalise the Hamiltonian,

$$\tilde{H} = \int d^3x \left(\Pi_i V_i' - \mathcal{L}\right) = \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3}\, k \left(b_{\boldsymbol{k}}^{\dagger(p)} b_{\boldsymbol{k}}^{(p)} + \frac{1}{2}[b_{\boldsymbol{k}}^{(p)}, b_{\boldsymbol{k}}^{\dagger(p)}] \right). \tag{3.45}$$

The two sets of annihilation and creation operators are related by

$$b_{\boldsymbol{k}}^{(p)}(\tau) = \alpha_k^{(p)}(\tau)\, a_{\boldsymbol{k}}^{(p)} + \beta_k^{*(p)}(\tau)\, a_{-\boldsymbol{k}}^{\dagger(p)}, \quad b_{\boldsymbol{k}}^{\dagger(p)}(\tau) = \alpha_k^{*(p)}(\tau)\, a_{\boldsymbol{k}}^{\dagger(p)} + \beta_k^{(p)}(\tau)\, a_{-\boldsymbol{k}}^{(p)}, \tag{3.46}$$

through time-dependent Bogoliubov coefficients:

$$\alpha_k^{(p)} = I \left(\sqrt{\frac{k}{2}}\, u_k^{(p)} + \frac{i}{\sqrt{2k}}\, u_k'^{(p)} \right), \quad \beta_k^{(p)} = I \left(\sqrt{\frac{k}{2}}\, u_k^{(p)} - \frac{i}{\sqrt{2k}}\, u_k'^{(p)} \right). \tag{3.47}$$

Using the normalisation condition (3.40), one can check that the amplitudes of the coefficients obey

$$|\alpha_k^{(p)}|^2 - |\beta_k^{(p)}|^2 = 1, \tag{3.48}$$

$$|\beta_k^{(p)}|^2 = \frac{I^2}{2} \left(k\, |u_k^{(p)}|^2 + \frac{|u_k'^{(p)}|^2}{k} \right) - \frac{1}{2}. \tag{3.49}$$

When an adiabatic vacuum exists, $b_{\boldsymbol{k}}^{\dagger(p)} b_{\boldsymbol{k}}^{(p)}$ counts the numbers of photons with polarisation p and comoving momentum $\boldsymbol{k}$. However this operator itself is defined at all times, and it can be interpreted as an instantaneous photon number. Now let us suppose $a_{\boldsymbol{k}}^{(p)}$ and $a_{\boldsymbol{k}}^{\dagger(p)}$ to have initially diagonalised the Hamiltonian, i.e. $\beta_k^{(p)} = 0$ in the distant past, and that the system was initially in a vacuum state defined by $a_{\boldsymbol{k}}^{(p)}|0\rangle = 0$ for $p = 1, 2$ and for all $\boldsymbol{k}$. Then at some later time, the number of created photons per comoving volume is written as

$$\frac{1}{V} \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} \langle 0| b_{\boldsymbol{k}}^{\dagger(p)} b_{\boldsymbol{k}}^{(p)} |0\rangle = \sum_{p=1,2} \int \frac{dk}{k} \left(\frac{4\pi}{3k^3} \right)^{-1} \frac{2}{3\pi} |\beta_k^{(p)}|^2, \tag{3.50}$$

where $V \equiv \int d^3x = (2\pi)^3 \delta^{(3)}(\mathbf{0})$. Thus one sees that $\frac{2}{3\pi}|\beta_k^{(p)}|^2$ represents the number of photons with polarisation p and comoving momentum of order[7] k, within a comoving sphere of radius k^{-1}. The photon number $|\beta_k^{(p)}|^2$ (we will omit the coefficient $\frac{2}{3\pi}$ as we are interested in order-of-magnitude estimates) is useful for judging whether magnetic field generation takes place: A successful magnetogenesis model that gives rise to magnetic fields with correlation length of k^{-1} does so by creating a large number of photons with momentum k, thus is characterised by $|\beta_k^{(p)}|^2 \gg 1$. On the other hand, if the photon number is as small as $|\beta_k^{(p)}|^2 = \mathcal{O}(1)$, then it is clearly not enough to support coherent magnetic fields in the universe.

[7] We assume that $|\beta_k^{(p)}|^2$ is smooth in k so that it does not have sharp features in any narrow range of $\Delta k \ll k$.

One can define another measure of quantumness introduced in [84] (see also discussions in [85]), by the product of the standard deviations of $v_{\boldsymbol{k}}^{(p)}$ and $\pi_{\boldsymbol{k}}^{(p)}$ in units of their commutator:

$$\kappa_k^{(p)}(\tau) \equiv \left| \frac{\langle 0|v_{\boldsymbol{k}}^{(p)}(\tau)\, v_{-\boldsymbol{k}}^{(p)}(\tau)|0\rangle\, \langle 0|\pi_{\boldsymbol{k}}^{(p)}(\tau)\, \pi_{-\boldsymbol{k}}^{(p)}(\tau)|0\rangle}{[v_{\boldsymbol{k}}^{(p)}(\tau),\, \pi_{-\boldsymbol{k}}^{(p)}(\tau)]^2} \right|^{1/2} = I(\tau)^2 \left| u_k^{(p)}(\tau)\, u_k'^{(p)}(\tau) \right|. \tag{3.51}$$

This quantity also corresponds to the classical volume of the space spanned by $v_{\boldsymbol{k}}$ and $\pi_{-\boldsymbol{k}}$, divided by their quantum uncertainty[8]. It takes a value of $\kappa_k^{(p)} \sim 1$ if the gauge field fluctuation with wave number k is quantum mechanical. On the other hand, if the fluctuations are effectively classical and large compared to the quantum uncertainty, then $\kappa_k^{(p)} \gg 1$.

This measure can also be used to quantify the conversion of quantum fluctuations into classical ones. As an example, consider a (nearly) massless scalar field in a de Sitter background (such as the inflaton), for which the measure κ_k can similarly be defined in terms of the scalar fluctuation and its conjugate momentum. Given that the fluctuation starts in a Bunch-Davies vacuum when the wave mode k is deep inside the Hubble horizon, one can check that κ_k grows from ~ 1 when the wave mode is inside the horizon, to $\kappa_k \gg 1$ outside the horizon, suggesting that the quantum fluctuations "become classical" upon horizon exit.

The quantumness measure can also be expressed in terms of the Bogoliubov coefficients as

$$(\kappa_k^{(p)})^2 = \frac{1}{4} \left| (\alpha_k^{(p)})^2 - (\beta_k^{(p)})^2 \right|^2 = \frac{1}{4} + |\beta_k^{(p)}|^2 \left(1 + |\beta_k^{(p)}|^2 \right) \sin^2 \left\{ \arg(\alpha_k^{(p)} \beta_k^{*(p)}) \right\}, \tag{3.52}$$

where we have used (3.48) upon moving to the far right hand side. This clearly shows that $\kappa_k^{(p)}$ takes its minimum value $1/2$ when there is no photon production, i.e., for $\beta_k^{(p)} = 0$. Is is also useful to note that the instantaneous photon number $|\beta_k^{(p)}|^2$ corresponds to the sum of squares of the standard deviations of $v_{\boldsymbol{k}}^{(p)}$ and $\pi_{\boldsymbol{k}}^{(p)}$ with weights $(kI^2)^{\pm 1}$, cf. (3.49). An inequality relation of

$$|\beta_k^{(p)}|^2 \geq \kappa_k^{(p)} - \frac{1}{2} \tag{3.53}$$

is satisfied.

The classical Maxwell theory is described by setting $I^2 = 1/e^2$, with which the mode function is a linear combination of plane waves. Then $|\beta_k^{(p)}|$ simply corresponds to the amplitude of the coefficient of the negative frequency wave, and thus is time-independent. It can also be checked in this case that $\arg(\alpha_k^{(p)} \beta_k^{*(p)}) = -2k\tau + \text{const.}$, and hence one sees from (3.52) that $\kappa_k^{(p)}$ for plane waves oscillates in time within the range $1/2 \leq \kappa_k^{(p)} \leq 1/2 + |\beta_k^{(p)}|^2$.

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[8] Here $\kappa_k^{(p)}$ is defined slightly differently from the κ introduced in Appendix B.3 of [84]: $\kappa = (2\kappa_k^{(p)})^{-2}$.

3.3 Cosmological Evolution of the Gauge Field

The anomalous dependence of the effective action for quantum electrodynamics on the scale factor couples the gauge field to the cosmological expansion. Here, in order to study the time evolution, the initial condition of the gauge field needs to be specified. A natural option is to start as quantum fluctuations when the wave modes were once deep inside the Hubble horizon of the inflationary universe. This Bunch-Davies vacuum during the early stage of the inflationary epoch will be the starting point of our computation.

It should also be noted that the scale factor dependence does not "switch off", as long as there are massless particles around, and thus the cosmological background continues to affect the gauge field equation of motion even after inflation. (In this respect, the effect of the Weyl anomaly serves as a subclass of the inflationary plus post-inflationary magnetogenesis scenario proposed in [14].) After inflation ends, the universe typically enters an epoch dominated by a harmonically oscillating inflaton field, whose kinetic and potential energies averaged over the oscillation are equal and thus behaves as pressureless matter. Then eventually the inflaton decays and heats up the universe; during this reheating phase, the universe is expected to become filled with charged particles and thus the gauge field evolution can no longer be described by the source-free equation of motion (3.29). We also note that after the electroweak phase transition, the charged particles in the Standard Model will obtain masses and therefore our effective action (2.93) becomes invalid. Hence the gauge field evolution will be followed up until the time of reheating or electroweak phase transition, whichever happens earlier.

3.3.1 Bunch-Davies Vacuum

For the purpose of obtaining a gauge field solution that corresponds to the vacuum fluctuations, it is convenient to rewrite the equation of motion (3.29) into the following form:

$$(Iu_k)'' + \omega_k^2\, I u_k = 0, \quad \text{where} \quad \omega_k = \left(k^2 - \frac{I''}{I}\right)^{1/2}. \tag{3.54}$$

We have dropped the polarisation index (p) because the action is symmetric between the two polarisations. This equation admits an approximate solution of the WKB-type

$$u_k^{\mathrm{WKB}}(\tau) = \frac{1}{\sqrt{2\omega_k(\tau)\, I(\tau)}} \exp\left(-i \int^{\tau} d\tilde{\tau}\, \omega_k(\tilde{\tau})\right), \tag{3.55}$$

given that the time-dependent frequency ω_k satisfies the adiabatic conditions,

$$\left|\frac{\omega_k'}{\omega_k^2}\right|^2, \left|\frac{\omega_k''}{\omega_k^3}\right| \ll 1. \tag{3.56}$$

When further

$$\omega_k^2 > 0, \tag{3.57}$$

then ω_k is real and positive, and the WKB solution (3.55) describes a positive frequency solution that satisfies the normalisation condition (3.40). The period when the above conditions are satisfied can be understood by noting that

$$\frac{I''}{k^2 I} = \frac{b}{2I^2}\left(\frac{aH}{k}\right)^2\left(1 + \frac{H'}{aH^2} - \frac{b}{2I^2}\right). \tag{3.58}$$

Here $H = a'/a^2$ is the Hubble rate. The simplifying condition (3.31) imposes $b \ll 2I^2$, and $|H'/aH^2| \lesssim 1$ is usually satisfied in a cosmological background. Hence for wave modes that are inside the Hubble horizon, i.e. $k > aH$, it follows that $k^2 \gg |I''/I|$. This yields $\omega_k^2 \simeq k^2$, satisfying the conditions (3.56) and (3.57).

In an inflationary universe, if one traces fluctuations with a fixed comoving wave number k back in time, then its physical wavelength becomes smaller than the Hubble radius. Therefore we adopt the solution (3.55) when each wave mode was sub-horizon during inflation, and take as the initial state the Bunch-Davies vacuum $|0\rangle$ annihilated by $a_{\boldsymbol{k}}$. Starting from this initial condition, we will see in the following sections how the vacuum fluctuations evolve as the universe expands.

3.3.2 Landau Pole Bound

If we go back in time sufficiently far, the physical momentum of a comoving mode k hits the Landau pole (3.32) and we enter the strong coupling regime. Here the simplifying condition (3.31) also breaks down. Hence in order to be able to set the Bunch-Davies initial condition while maintaining perturbative control, there needs to be a period during inflation when $k/a < \Lambda_{\text{max}}$ as well as the adiabaticity (3.56) and stability (3.57) conditions hold simultaneously. We just saw that the conditions (3.56) and (3.57) hold when the mode is sub-horizon, i.e. $k/a > H_{\text{inf}}$, where H_{inf} is the Hubble rate during inflation. Therefore we infer a bound for the inflationary Hubble rate

$$H_{\text{inf}} < \Lambda_{\text{max}}, \tag{3.59}$$

so that the Bunch-Davies vacuum can be adopted during the period of $H_{\text{inf}} < k/a < \Lambda_{\text{max}}$. We also see that this Landau pole bound on inflation collectively describes the various conditions imposed in the previous sections, namely, adiabaticity (3.56) and stability (3.57) during the early stage of inflation, as well as the simplifying condition (3.31) throughout the times of interest.

The current observational limit on primordial gravitational waves sets an upper bound on the inflation scale as $H_{\text{inf}} \lesssim 10^{14}\,\text{GeV}$ [?]. The Landau pole Λ_{max} can be smaller than this observational bound if there were sufficiently many massless charged particles in the early universe. Taking for example the coupling to run through $e^2(M_Z) \approx 4\pi/128$ at $M_Z \approx 91.2\,\text{GeV}$ [86], a beta function coefficient as large as $b \gtrsim 0.4$ would lead to $\Lambda_{\text{max}} \lesssim 10^{14}\,\text{GeV}$. For such a large beta function, the adiabatic and perturbative regimes cannot coexist for the gauge field during high-scale inflation.

3.3.3 Slowly Running Coupling

Before analysing the gauge field evolution in full generality, let us first focus on cases with tiny beta functions. Such cases can be treated analytically, by approximating the I^2 function (3.28) for small $\tilde{\beta}$ as

$$I^2 \simeq \frac{1}{e^2}\left(\frac{a}{a_\star}\right)^{2\tilde{\beta}}. \tag{3.60}$$

We will later verify the validity of this approximation by comparing with the results obtained from the original logarithmic I^2.

In a flat Friedmann-Robertson-Walker universe with a constant equation of state w, the equation of motion (3.29) under the power-law I^2 admits solutions in terms of Hankel functions as [14],

$$u_k = z^\nu \left\{ c_1 H_\nu^{(1)}(z) + c_2 H_\nu^{(2)}(z) \right\}, \quad \text{where} \quad z = \frac{2}{|1+3w|}\frac{k}{aH}, \quad \nu = \frac{1}{2} - \frac{2\tilde{\beta}}{1+3w}, \tag{3.61}$$

and the coefficients c_1, c_2 are independent of time. Here, the equation of state parameter w can take any value except for $-1/3$, and the variable z scales with the scale factor as $z \propto a^{(1+3w)/2}$. The time derivative of the mode function is written as

$$u_k' = \operatorname{sign}(1+3w)\, k z^\nu \left\{ c_1 H_{\nu-1}^{(1)}(z) + c_2 H_{\nu-1}^{(2)}(z) \right\}. \tag{3.62}$$

The behaviours of u_k and u_k' in the super-horizon limit, i.e. $z \to 0$, can be read off from the asymptotic forms of the Hankel function:

$$\begin{aligned} H_\nu^{(1)}(z) &= \left(H_\nu^{(2)}(z)\right)^* \sim -\frac{i}{\pi}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \\ H_{\nu-1}^{(1)}(z) &= \left(H_{\nu-1}^{(2)}(z)\right)^* \sim -e^{(1-\nu)\pi i}\,\frac{i}{\pi}\Gamma(1-\nu)\left(\frac{z}{2}\right)^{\nu-1}, \end{aligned} \tag{3.63}$$

which are valid when $\tilde{\beta}$ is small such that $0 < \nu < 1$ is satisfied.

During Inflation

The inflationary epoch is characterised by the equation of state $w = -1$ and a time-independent Hubble rate $H_{\rm inf}$. The solution that asymptotes to a positive frequency solution in the past is

$$u_k = \frac{1}{2I}\left(\frac{\pi}{aH_{\rm inf}}\right)^{\frac{1}{2}} H^{(1)}_{\frac{1}{2}+\tilde{\beta}}\left(\frac{k}{aH_{\rm inf}}\right), \tag{3.64}$$

whose normalisation is set by (3.40) up to an unphysical phase. Therefore the amplitudes of the mode function and its time derivative are obtained as

$$\begin{aligned} kI^2|u_k|^2 &= \frac{\pi k}{4aH_{\rm inf}}\left|H^{(1)}_{\frac{1}{2}+\tilde{\beta}}\left(\frac{k}{aH_{\rm inf}}\right)\right|^2 \sim \frac{(\Gamma(\frac{1}{2}+\tilde{\beta}))^2}{2\pi}\left(\frac{2aH_{\rm inf}}{k}\right)^{2\tilde{\beta}}, \\ \frac{1}{k}I^2|u_k'|^2 &= \frac{\pi k}{4aH_{\rm inf}}\left|H^{(1)}_{-\frac{1}{2}+\tilde{\beta}}\left(\frac{k}{aH_{\rm inf}}\right)\right|^2 \sim \frac{(\Gamma(\frac{1}{2}-\tilde{\beta}))^2}{2\pi}\left(\frac{2aH_{\rm inf}}{k}\right)^{-2\tilde{\beta}}, \end{aligned} \tag{3.65}$$

where the far right hand sides show the asymptotic forms in the super-horizon limit obtained by using (3.63). The geometric mean of these amplitudes yields the quantumness measure (3.51),

$$\kappa_k = \frac{\pi k}{4aH_{\rm inf}}\left|H^{(1)}_{\frac{1}{2}+\tilde{\beta}}\left(\frac{k}{aH_{\rm inf}}\right) H^{(1)}_{-\frac{1}{2}+\tilde{\beta}}\left(\frac{k}{aH_{\rm inf}}\right)\right|. \tag{3.66}$$

In the sub-horizon limit $k/aH_{\text{inf}} \to \infty$, this parameter approaches $\kappa_k \sim 1/2$ and thus the gauge field fluctuations are quantum mechanical, which should be the case since we have started in the Bunch-Davies vacuum.

The important question is whether the fluctuations become classical upon horizon exit, as in the case for light scalar fields during inflation. Using the reflection relation $\Gamma(\varpi)\Gamma(1-\varpi) = \pi/\sin(\pi\varpi)$ for $\varpi \notin \mathbb{Z}$, the asymptotic value of the quantumness parameter in the super-horizon limit $k/aH_{\text{inf}} \to 0$ is obtained as

$$\kappa_k \sim \frac{1}{2\cos(\pi\tilde{\beta})}. \tag{3.67}$$

Thus we find that κ_k becomes time-independent outside the horizon, and its asymptotic value depends[9] only on $\tilde{\beta}$. Most importantly, κ_k is of order unity for $\tilde{\beta} \ll 1$. This implies that if the beta function is small in the early universe, the time-dependence induced by the Weyl anomaly is not sufficient for converting vacuum fluctuations of the gauge field into classical ones. Therefore no classical magnetic fields would arise.

We also estimate the instantaneous photon number (3.49) outside the horizon by summing the asymptotic expressions of (3.65), yielding

$$|\beta_k|^2 \sim \frac{(\Gamma(\frac{1}{2}+\tilde{\beta}))^2}{4\pi}\left(\frac{2aH_{\text{inf}}}{k}\right)^{2\tilde{\beta}} + \frac{(\Gamma(\frac{1}{2}-\tilde{\beta}))^2}{4\pi}\left(\frac{2aH_{\text{inf}}}{k}\right)^{-2\tilde{\beta}} - \frac{1}{2}. \tag{3.68}$$

The first term grows in time as $\propto a^{2\tilde{\beta}}$, hence it will eventually dominate the right hand side if we wait long enough. In a realistic cosmology, however, this term does not become much larger than unity. We will see this explicitly in the following sections.

After Inflation

One can evaluate the mode function also in the effectively matter-dominated epoch after inflation by matching solutions for $w = -1$ and $w = 0$ at the end of inflation. However let us take a simplified approach: From the solutions (3.61) and (3.62) for generic w, and the asymptotic forms of the Hankel function (3.63), one can infer the time-dependences of the mode function outside the horizon in a generic cosmological background as

$$I^2|u_k|^2 \propto a^{2\tilde{\beta}}, \quad I^2|u_k'|^2 \propto a^{-2\tilde{\beta}}. \tag{3.69}$$

These super-horizon evolutions are determined only by the beta function $\tilde{\beta}$. Hence we find that for wave modes that exit the horizon during inflation, the super-horizon expressions in (3.65) continue to hold even after inflation, until the mode re-enters the horizon.[10] In particular, the super-horizon expressions (3.67) for κ_k

[9] Since the approximate expression (3.60) for I^2 explicitly depends on the renormalisation scale M, so does the asymptotic value (3.67). However this M-dependence is tiny for slowly running couplings.

[10] This kind of argument breaks down when the leading order approximations for the two Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ cancel each other in the mode function. Such cases are presented in [14]. However in the current case where the power $\tilde{\beta}$ of the I^2 function is tiny, the cancellation does not happen as we will see in the next section by comparing with numerical results that the scaling (3.69) indeed holds until horizon re-entry.

and (3.68) for $|\beta_k|^2$ also hold while the wave mode is outside the horizon; these expressions are the main results of the small-$\tilde{\beta}$ analysis. Thus we find that κ_k stays constant, while $|\beta_k|^2$ basically continues to grow until horizon re-entry.

If the mode re-enters the horizon before reheating and electroweak phase transition, then we can continually use our effective action for analysing the gauge field dynamics. Inside the horizon, where adiabaticity is recovered, the photon number $|\beta_k|^2$ becomes constant. On the other hand, the quantumness parameter κ_k oscillates in time between $1/2$ and $1/2 + |\beta_k|^2$, as described below (3.53).

We see from (3.68) that in order to have substantial photon production, i.e. $|\beta_k|^2 \gg 1$, the quantity $(aH_{\text{inf}}/k)^{2\tilde{\beta}}$ needs to become large while the mode is outside the horizon. A larger H_{inf} and $\tilde{\beta}$, as well as a smaller k are favourable for this purpose. Here, for example, the magnitude of aH_{inf}/k upon the electroweak phase transition at $T_{\text{EW}} \sim 100\,\text{GeV}$ is, given that the universe has thermalised by then,

$$\frac{a_{\text{EW}} H_{\text{inf}}}{k} \sim 10^{41} \left(\frac{H_{\text{inf}}}{10^{14}\,\text{GeV}} \right) \left(\frac{k}{a_0} \cdot 10\,\text{Gpc} \right)^{-1}, \tag{3.70}$$

where a_0 is the scale factor today. The detailed value can be modified for different cosmological histories, but what is relevant here is that even with the observably allowed highest inflation scale $H_{\text{inf}} \sim 10^{14}\,\text{GeV}$, and with the size of the observable universe $a_0/k \sim 10\,\text{Gpc}$ (or even on scales tens of orders of magnitude beyond that), if the beta function is $\tilde{\beta} = \mathcal{O}(0.01)$, then $(a_{\text{EW}} H_{\text{inf}}/k)^{2\tilde{\beta}} = \mathcal{O}(10)$. Hence the number of photons created over the cosmological history would only be of $|\beta_k|^2 = \mathcal{O}(10)$, which is too small to support coherent magnetic fields.

On the other hand, the power-law I^2 (3.60) with $\tilde{\beta} = 1$ yields an equation of motion equivalent to that of a minimally coupled massless scalar field. Indeed, if one were to use the power-law I^2 with $\tilde{\beta} \gtrsim 1$, then $|\beta_k|^2$ and κ_k are found to significantly grow outside the horizon; thus one would conclude that gauge fluctuations do become classical and give rise to cosmological magnetic fields for a large beta function. However, in reality the power-law approximation breaks down when $\tilde{\beta}$ is not tiny, and we will explicitly see in the next section that the fluctuations of the gauge field actually never become classical, independently of the value of $\tilde{\beta}$.

3.3.4 General Coupling

In order to analyse quantum electrodynamics with generic beta functions, we have numerically solved the equation of motion (3.29) for the original logarithmic I^2 function (3.28), with $a_\star$ chosen as (3.30). Starting from the WKB initial condition (3.55) during inflation when $H_{\text{inf}} < k/a < \Lambda_{\text{max}}$ is satisfied, the mode function is computed in an inflationary as well as the post-inflation matter-dominated backgrounds. For the coupling we used $e^2(M_Z \approx 91.2\,\text{GeV}) \approx 4\pi/128$ [86], and considered it to run with a constant beta function coefficient b in (3.6) which is of order 0.1 for three light generations[11].

[11] In reality, b is not a constant since the number of effectively massless particles depends on the energy scale. Moreover, the hypercharge is related to the physical electric charge through the Weinberg angle. However, these do not change the orders of magnitude of β_k and κ_k for the electromagnetic field.

In Figure 3.2 we plot the evolution of $|\beta_k|^2$ and $\kappa_k - 1/2$ as functions of the scale factor a/a_0. Here the inflation scale is fixed to $H_{\rm inf} = 10^{14}\,\text{GeV}$, and the reheating temperature to $T_{\rm reh} = 100\,\text{GeV}$ such that it coincides with the scale of electroweak phase transition. The beta function coefficient is taken as $b = 0.01$ (thus $\tilde{\beta}(M_Z) \approx 5 \times 10^{-4}$), and the gauge field parameters are shown for two wave numbers: $k/a_0 = (10\,\text{Gpc})^{-1}$ (red lines) which corresponds to the size of the observable universe today, and $k/a_0 = (10^{-6}\,\text{pc})^{-1}$ (blue lines) which re-enters the Hubble horizon before reheating. The figure displays the time evolution from when both modes are inside the horizon during inflation, until the time of reheating. The vertical dotted line indicates the end of inflation, and the dot-dashed lines for the moments of Hubble horizon exit/re-entry. With the beta function being tiny, the analytic expressions (3.67) and (3.68) derived in the previous section well describe the behaviours of κ_k and $|\beta_k|^2$ outside the horizon. After the mode $k/a_0 = (10^{-6}\,\text{pc})^{-1}$ re-enters the horizon (after the blue dot-dashed line on the right), $|\beta_k|^2$ becomes constant whereas κ_k oscillates within the range of (3.53). $|\beta_k|^2$ is larger for smaller k as there is more time for super-horizon evolution, however even with $k/a_0 = (10\,\text{Gpc})^{-1}$, $|\beta_k|^2$ does not exceed unity by the time of reheating.

Figure 3.3 shows $|\beta_k|^2$ and $\kappa_k - 1/2$ as functions of the beta function coefficient b. Here the wave number is fixed to $k/a_0 = (10\,\text{Gpc})^{-1}$, and the reheating temperature to $T_{\rm reh} = 100\,\text{GeV}$. The gauge field parameters $|\beta_k|^2$ and κ_k in the figure are evaluated at the electroweak phase transition, which coincides with the time of reheating. The solid curves with different colours correspond to different inflation scales, which are chosen as $H_{\rm inf} = 10^{14}\,\text{GeV}$ (blue), $10^6\,\text{GeV}$ (orange), and $1\,\text{GeV}$ (red). The Landau pole bound on the inflation scale (3.59) imposes an upper bound on the beta function coefficient as $b_{\rm max} \approx 0.4$ for $H_{\rm inf} = 10^{14}\,\text{GeV}$, and $b_{\rm max} \approx 1.1$ for $H_{\rm inf} = 10^6\,\text{GeV}$. The computations have been performed for values of b up to $0.7 \times b_{\rm max}$, which are shown as the endpoints of the blue and orange curves. On the other hand, if the inflation scale is as low as $H_{\rm inf}/2\pi \lesssim 100\,\text{GeV}$, the electroweak symmetry would already be broken during inflation. However even in such cases, there might still be massless charged particles in the early universe for some reason. Hence for completeness, we have also carried out computations with $H_{\rm inf} = 1\,\text{GeV}$. There is no Landau pole bound on b with such a low-scale inflation, as is obvious from $H_{\rm inf}$ being smaller than the scale M_Z where we set the coupling. Hence this extreme case allows us to assess the implications of large beta functions, although it should also be noted that as one increases b, perturbation theory will eventually break down.

The wave mode $k/a_0 = (10\,\text{Gpc})^{-1}$ for which the parameters are evaluated is way outside the horizon at the electroweak phase transition, thus the super-horizon approximations (3.67) and (3.68) should be valid for small beta functions. These are shown as the dashed lines in the plots: In the left panel, (3.68) is plotted using (3.70), with the colours of the dashed lines corresponding to the different inflation scales. In the right panel there is just one black dashed line, because (3.67) for κ_k only depends on the beta function.[12] The analytic approximations indeed agree well with the numerical results at $b \lesssim 0.1$. With larger b, the numerical results for $H_{\rm inf} = 10^{14}\,\text{GeV}$ and $10^6\,\text{GeV}$ show that even when approaching the Landau pole

[12] The expressions (3.67) and (3.68) assume a tiny beta function, hence upon plotting the dashed lines, the running is neglected and the coupling is fixed to $e^2 = 4\pi/128$.

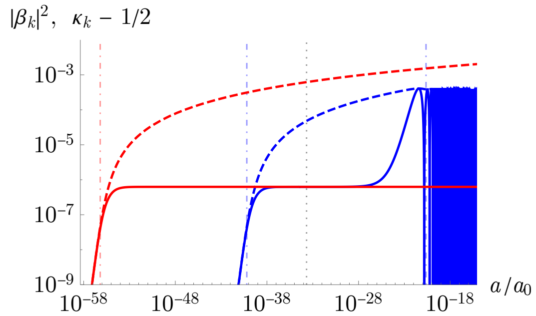

Figure 3.2: Time evolution of the instantaneous photon number $|\beta_k|^2$ (dashed lines) and quantumness parameter κ_k subtracted by 1/2 (solid lines), for wave numbers $k/a_0 = (10\,\mathrm{Gpc})^{-1}$ (red lines) and $(10^{-6}\,\mathrm{pc})^{-1}$ (blue lines). The beta function coefficient is set to $b = 0.01$. The background cosmology is fixed as $H_{\mathrm{inf}} = 10^{14}\,\mathrm{GeV}$ where inflation ends at the vertical dotted line, and reheating with $T_{\mathrm{reh}} = 100\,\mathrm{GeV}$ taking place at the right edge of the plot. The wave mode $k/a_0 = (10\,\mathrm{Gpc})^{-1}$ exits the Hubble horizon at the vertical red dot-dashed line, whereas $k/a_0 = (10^{-6}\,\mathrm{pc})^{-1}$ exits and then re-enters the horizon at the blue dot-dashed lines.

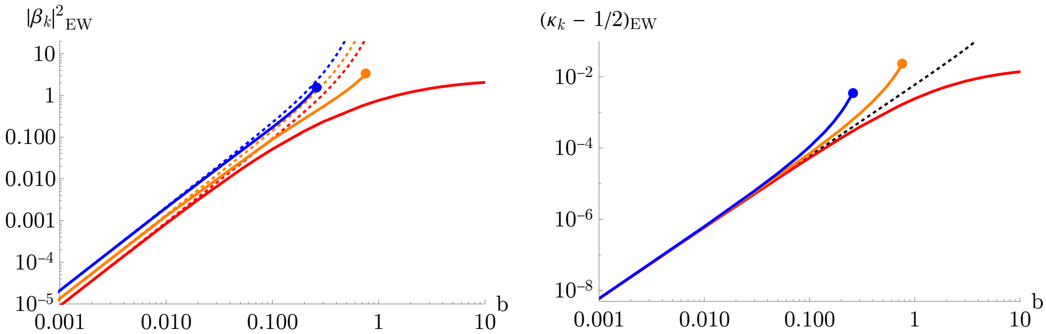

Figure 3.3: Instantaneous photon number $|\beta_k|^2$ (left) and quantumness parameter κ_k subtracted by 1/2 (right) at the electroweak phase transition, as functions of the beta function coefficient b. The results are shown for a wave number $k/a_0 = (10\,\mathrm{Gpc})^{-1}$. The reheating temperature is fixed to $T_{\mathrm{reh}} = 100\,\mathrm{GeV}$, while the inflation scale is varied as $H_{\mathrm{inf}} = 10^{14}\,\mathrm{GeV}$ (blue solid lines), $10^{6}\,\mathrm{GeV}$ (orange solid), and 1 GeV (red solid). The endpoints of the curves show where the Landau pole bound is saturated (see text for details). The dashed lines show the analytic approximations derived for small beta functions: (3.68) for $|\beta_k|^2$, and (3.67) for κ_k.

bound, the parameters $|\beta_k|^2$ and κ_k are at most of order unity. For $H_{\rm inf} = 1\,{\rm GeV}$ (assuming the existence of massless charged particles), $|\beta_k|^2$ and κ_k become less sensitive to b at $b \gtrsim 1$ and thus turns out not to exceed order unity even with large b. Here we have focused on a rather small wave number $k/a_0 = (10\,{\rm Gpc})^{-1}$ and a low reheating temperature $T_{\rm reh} = 100\,{\rm GeV}$; however for larger k and $T_{\rm reh}$, the value of $|\beta_k|^2$ upon reheating becomes even smaller as there is less time for the super-horizon evolution.

A heuristic argument for why the logarithmic I^2 function (3.28) never leads to substantial photon production goes as follows: Even if the beta function is as large as $\tilde{\beta} = 1$, while the universe expands by, say, 100 e-foldings starting from $a_\star$, the logarithmic I^2 grows only by a factor of 200. On the other hand, if one were to obtain the same growth rate with a power-law I^2 (3.60), the power would have to be as small as $\tilde{\beta} \approx 0.03$; then it is clear from the expressions (3.67) and (3.68) that the effect on photon production is tiny.

Thus we find that, for generic values of the beta function, inflation/reheating scales, and wave number, the instantaneous photon number $|\beta_k|^2$ and quantumness measure κ_k do not become much greater than unity. Here, the physical meaning of $|\beta_k|^2$ may seem ambiguous when the wave mode is outside the horizon and thus an adiabatic vacuum is absent. However as was discussed above (3.70), the quantity $|\beta_k|^2$ needs to become large while outside the horizon in order to have a large number of photons to support coherent magnetic fields. Moreover, the quantumness measure κ_k is bounded from above by $|\beta_k|^2 + 1/2$, cf. (3.53). Therefore we can conclude that, unless some additional process significantly excites the gauge field after the electroweak phase transition or reheating (namely, after our effective action becomes invalid), the Weyl anomaly does not convert vacuum fluctuations of the gauge field into classical fluctuations, let alone coherent magnetic fields in the universe.

3.4 Conclusions and Discussion

We have analysed cosmological excitation of magnetic fields due to the Weyl anomaly of quantum electrodynamics. Despite the anomalous dependence of the quantum effective action on the scale factor of the metric, we showed that the vacuum fluctuations of the gauge field do not get converted into classical fluctuations, as long as inflation happens at scales below the Landau pole. In particular, the number of photons with a comoving momentum k produced within a comoving volume k^{-3} was found to be at most of order unity, for generic k. With such a small number of created photons, we conclude that the Weyl anomaly does not give rise to coherent magnetic fields in the universe. Our conclusion is independent of the details of the cosmological history, or the number of massless charged particles in the theory.

For obtaining this result, which disproves the claims of many previous works, there were two key ingredients. The first was the quantum effective action beyond the weak gravitational field limit. We saw that, especially for cases where the beta function of quantum electrodynamics was large in the early universe, one could draw dramatically incorrect conclusions from inappropriate assumptions about the effective action. The essential point is that the anomalous dependence of the effective action on the metric is associated to the renormalisation group flow of the gauge coupling, and therefore the dependence is only logarithmic in the scale factor, cf. (3.17) and (3.18); this is in contrast with the case of massless scalar fields having power-law dependences on the scale factor at the classical level. The second element was a proper evaluation of the nature of the gauge field fluctuations, which we discussed quantitatively in terms of the photon number (3.49) and the quantumness parameter (3.51). Focusing on these quantities, we explicitly showed that the logarithmic dependence on the background metric induced by the Weyl anomaly does not lead to any generation of coherent classical magnetic fields.

We now briefly comment on some of the earlier works on Weyl anomaly-driven magnetogenesis. The original works [87, 82] approximated the effect from the Weyl anomaly as a power-law I^2 for a generic beta function, and thus arrived at the incorrect conclusion that a large beta function gives rise to observably large magnetic fields. On the other hand, the recent work [33] relies on the effective action derived in the weak gravitational field limit. The Weyl factor in an inflationary background is computed using the curvature expansion of (3.15), which yields $\Omega \sim (2/3) \log a$ in the asymptotic future, instead of the exact answer of $\log a$. At any rate, a logarithmic I^2 is obtained with a form similar to (3.28) up to numerical coefficients. However, the fact that a logarithmic I^2 cannot produce enough photons to support coherent magnetic fields was overlooked.

Our considerations can also be applied to quantum chromodynamics. The effective action is analogous to (2.93) with $\tilde{\beta}$ given by the beta function of quantum chromodynamics coupled to massless quarks. One main difference from electrodynamics is that the beta function is negative, yielding asymptotic freedom; hence the theory goes into the strongly coupled regime in the late universe. The time evolution of the mode function can further be altered by the nonlinearities of the Yang-Mills action. Here, since the dependence of the effective action on the scale factor is anyway logarithmic, it may turn out that colour magnetic fields are also not generated by the Weyl anomaly; however, it would be worthwhile to analyse systematically the range of possibilities that can arise for $SU(N)$ Yang-Mills fields. With such analyses, one should also be able to evaluate the effect of the possible mixing of the $SU(2)$ gauge field fluctuations into the photons upon the electroweak phase transition, which we did not consider in this chapter. The study of the effects of the Weyl anomaly in the strongly coupled regime, for instance electrodynamics with inflation scales higher than the Landau pole (thus with a very large beta function), or chromodynamics near the confinement transition is very interesting but would require nonperturbative methods.

Chapter 4

Conclusion

In this thesis we have analysed the semi-classical physics of quantum fields in Weyl-flat spacetimes. We have shown that non-localities coming from virtual massless particles can be handled beyond the weak field approximation. The complete gamut of physical consequences is still open to exploration, we have focused on the simplest application: primordial magnetogenesis in cosmology. It is conceivable that such ideas may find use in computing finite N corrections in holography or in other cosmological scenarios.

We described how quantum effective actions encoding information about the RG-flow can be computed efficiently for Weyl-flat metrics. In essence our method works by simplifying the non-local Weyl invariant terms by using Weyl-flatness of the background. Then, by integrating the Weyl anomaly we compute the remaining terms. This relies only on the local Schwinger-DeWitt expansion of the heat kernel avoiding the weak field approximation for the background metric.

As an illustration, we obtained the quantum effective action for the Yang-Mills field coupled to conformal matter, and the self-interacting massless scalar field. It was shown that our action reduces to the nonlocal action obtained using the Barvinsky-Vilkovisky covariant perturbation theory in the regime $R^2 \ll \nabla^2 R$ for a typical curvature scale R, but has a greater range of validity effectively re-summing the covariant perturbation theory to all orders in curvatures (but only for Weyl-flat spacetimes). In particular, it is applicable also in the opposite regime $R^2 \gg \nabla^2 R$, which is often of interest in cosmology.

One of the main results of the thesis is determining precisely how the gauge field couples to the scale factor of the metric. We investigated the possibility of the Weyl anomaly of the standard model itself sourcing the cosmological magnetic fields in the early universe using the effective action developed in chapter 2. We concluded that the Weyl anomaly is cannot convert vacuum fluctuations of the gauge field into classical fluctuations, independently of the number of massless charged particles in the theory. Our results reinforce the conclusion that the physics behind such intense magnetic fields on huge scales must come from beyond the Standard Model.

On-going explorations involve generalising the results to theories that are not classically Weyl invariant, allowing for mass terms and non-conformal couplings as well as applications of those results to some subtle problems in Liouville theory.

Appendix A

Cohomology of Weyl Anomalies

Weyl anomalies can be analysed in the context of cohomology theory. Studying the cohomology of the Weyl group on field space can be useful for classifying the consistent terms that can appear as a Weyl anomaly as well as to separate the non-trivial cohomological terms from the ones that are simply a variation of a local term in the action [74]. For definiteness consider a self-interacting scalar field coupled to a dynamical gauge boson and to a non-trivial metric, such that the action is Weyl invariant in four dimensions.

$$S[\phi, A_\mu, g_{\mu\nu}] = \int_{\mathcal{M}} d^4x\, \sqrt{g} \left[-\frac{1}{4} F^2 + |D\phi|^2 + \frac{1}{6} R + \frac{\lambda}{4!} \phi^4 \right] \tag{A.1}$$

This action is invariant under the transformation

$$\phi(x) \longrightarrow e^{-\omega(x)} \phi(x) \tag{A.2}$$

$$A_\mu(x) \longrightarrow A_\mu(x) \tag{A.3}$$

$$g_{\mu\nu}(x) \longrightarrow e^{2\omega(x)} g_{\mu\nu}(x) \tag{A.4}$$

where $\omega(x)$ is any function.

One way we can study the cohomology of the Weyl group action is by introducing an anticommuting parameter in the infinitesimal Weyl transformation, defined by:

$$\mathcal{W} \equiv \int d^4x \left(2\,\xi\, g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \xi\, \phi \frac{\delta}{\delta \phi} \right) \tag{A.5}$$

The statement of Weyl invariance can then be written as:

$$\mathcal{W} S[\phi, A_\mu, g_{\mu\nu}] = 0 \tag{A.6}$$

The quantum effective action defined from S need not be invariant under Weyl transformations. That is to say

$$\mathcal{W}\Gamma = \mathcal{C} \tag{A.7}$$

Since ξ is an anticommuting variable, $\mathcal{W}$ is automatically nilpotent. Nilpotency of $\mathcal{W}$ implies that $\mathcal{C}$ is of the form $\mathcal{C} = \mathcal{W}\gamma + \mathcal{A}$ with $\mathcal{W}\mathcal{A} = 0$ and $\mathcal{A} \neq \mathcal{W}\bar{\gamma}$, where γ and $\bar{\gamma}$ are local actions. Here $\mathcal{A}$ is the true anomaly while the remaining terms can be affected by different renormalisation choices.

i	$\mathcal{C}_i$	$\mathcal{W}\mathcal{C}_i$	γ_i	$\mathcal{W}\gamma_i$
1	$\int \xi R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$	$-4\int R\,\xi\nabla^2\xi$	$\int R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$	$-4\int \xi\nabla^2 R$
2	$\int \xi R_{\mu\nu}R^{\mu\nu}$	$-4\int R\,\xi\nabla^2\xi$	$\int R_{\mu\nu}R^{\mu\nu}$	$-4\int \xi\nabla^2 R$
3	$\int \xi R^2$	$-12\int R\,\xi\nabla^2\xi$	$\int R^2$	$-12\int \xi\nabla^2 R$
4	$\int \xi\nabla^2 R$	0		
5	$\int \xi\phi\nabla^2\phi$	$-\int \phi^2\,\xi\nabla^2\xi$	$\int \phi\nabla^2\phi$	$-\int \xi\nabla^2\phi^2$
6	$\int \xi R\phi^2$	$-6\int \phi^2\,\xi\nabla^2\xi$	$\int R\phi^2$	$-6\int \xi\nabla^2\phi^2$
7	$\int \xi\phi^4$	0	$\int \phi^4$	0
9	$-\int \xi\nabla^2\phi^2$	0		
8	$\int \xi F_{\mu\nu}F^{\mu\nu}$	0	$\int F_{\mu\nu}F^{\mu\nu}$	0

Here we included in the same table a list of all possible terms that can appear in $\mathcal{C}$ and γ and their respective variations. In all the expressions $\int$ is a shorthand for $\int \sqrt{g}\, d^4x$. Certain terms were not included because of parity.

Notice that $\mathcal{C}_4 = -\frac{1}{12}\mathcal{W}\gamma_3$, $\mathcal{C}_9 = \frac{1}{6}\gamma_6$, which means such terms are simply the Weyl variation of a local term in the action and thus are not anomalies. The condition $\mathcal{W}\mathcal{C} = 0$ forces some of the $\mathcal{C}_i$ to appear only in specific linear combinations, that is:

$$\begin{aligned} W^2 &= R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \\ E_4 &= R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ K &= -\phi\nabla^2\phi + \frac{1}{6}R\phi^2 \end{aligned} \tag{A.8}$$

We then conclude that for this particular example the Weyl anomaly is generally given by

$$\mathcal{A} = \int d^4x\sqrt{g}\xi\left(a_1 W^2 + a_2 E_4 + a_3 K + a_4\phi^4 + a_5 F_{\mu\nu}F^{\mu\nu}\right) \tag{A.9}$$

where the coefficients can be fixed by a direct computation.

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