Bihamiltonian Cohomologies and Integrable Hierarchies II: The Tau Structures

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Boris Dubrovin*, Si-Qi Liu†, Youjin Zhang†
* SISSA, via Bonomea 265, Trieste 34136, Italy
† Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China

Abstract

Starting from a so-called flat exact semisimple bihamiltonian structure of hydrodynamic type, we arrive at a Frobenius manifold structure and a tau structure for the associated principal hierarchy. We then classify the deformations of the principal hierarchy which possess tau structures.

Keywords. Bihamiltonian structure; Frobenius manifold; Tau structure; Central invariant; Semi-Hamiltonian system
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1 Introduction

The class of bihamiltonian integrable hierarchies which possess hydrodynamic limits plays an important role in the study of Gromov–Witten invariants, 2D topological field theory, and other research fields of mathematical physics. In [9] the first- and third-named authors of the present paper initiated a program of classifying deformations of bihamiltonian integrable hierarchies of hydrodynamic type under the so-called Miura-type transformations. They introduced the notion of bihamiltonian cohomologies of a bihamiltonian structure and converted the classification problem into the computation of these cohomology groups. The first two bihamiltonian cohomologies for semisimple bihamiltonian structures of hydrodynamic type were calculated in [10, 23], and it was proved that the infinitesimal deformations of a semisimple bihamiltonian structure of hydrodynamic type are parametrized by a set of smooth functions of one variable. For a given deformation of a semisimple bihamiltonian structure of hydrodynamic type these functions $c_1(u^1), \dots, c_n(u^n)$ can be calculated by an explicit formula represented in terms of the canonical coordinates $u^1, \dots, u^n$ of the semisimple

Emails: dubrovin@sissa.it, liusq@tsinghua.edu.cn, youjin@tsinghua.edu.cn

bihamiltonian structure. These functions are invariant under the Miura-type transformations, due to this reason they are called the central invariants of the deformed bihamiltonian structure.

In [25], the second- and third-named authors of the present paper continued the study of the above mentioned classification problem. They reformulated the notion of infinite dimensional Hamiltonian structures in terms of the infinite jet space of a super manifold, and provided a framework of infinite dimensional Hamiltonian structures which is convenient for the study of properties of Hamiltonian and bihamiltonian cohomologies. One of the main results which is crucial for the computation of bihamiltonian cohomologies is given by Lemma 3.7 of [25]. It reduces the computation of the bihamiltonian cohomologies to the computation of cohomology groups of a bicomplex on the space of differential polynomials, instead of on the space of local functionals. Based on this result, they computed the third bihamiltonian cohomology group of the bihamiltonian structure of the dispersionless KdV hierarchy, and showed that any infinitesimal deformation of this bihamiltonian structure can be extended to a full deformation.

In [5], Carlet, Posthuma and Shadrin completed the computation of the third bihamiltonian cohomology group for a general semisimple bihamiltonian structure of hydrodynamic type based on the results of [25]. Their result confirms the validity of the conjecture of [25] that any infinitesimal deformation of a semisimple bihamiltonian structures of hydrodynamic type can be extended to a full deformation, i.e. for any given smooth functions $c_i(u^i)$ $(i = 1, \dots, n)$, there exists a deformation of the corresponding semisimple bihamiltonian structure of hydrodynamic type such that its central invariants are given by $c_i(u^i)$ $(i = 1, \dots, n)$.

This paper is a continuation of [25]. We are to give a detailed study of properties of the integrable hierarchies associated with a special class of semisimple bihamiltonian structures of hydrodynamic type, which are called *flat exact semisimple bihamiltonian structures of hydrodynamic type.* One of their most important properties is the existence of tau structures for the associated integrable hierarchies and their deformations with constant central invariants.

For a hierarchy of Hamiltionian evolutionary PDEs, a tau structure is a suitable choice of the densities of the Hamiltonians satisfying certain conditions which enables one to define a function, called the tau function, for solutions of the hierarchy of evolutionary PDEs, as it is defined in [9]. Such functions were used by R. Hirota at the beginning of 70's of the last century for solving soliton equations [16, 17]. The notion of tau functions was introduced by M. Sato [27] for solutions to the KP equation and by Jimbo, Miwa and Ueno for a class of monodromy preserving deformation equations of linear ODEs with rational coefficients [18, 19, 20] at the beginning of 80's of the last century. It was also adopted to soliton equations that can be represented as equations of isospectral deformations of certain linear spectral problems or as Hamiltonian systems, and has played crucial role in the study of relations of soliton equations with infinite dimensional Lie algebras [6, 21], and with the geometry of infinite dimensional Grassmannians [27, 28, 29]. The importance of the notion of tau functions of soliton equations is manifested by the discovery of the fact that the tau function of a particular solution of the KdV hierarchy is the partition function of 2D topological gravity, see [22, 31] for details. In [9], the first- and the third-named authors of the present paper introduced the notion of tau structures for the class of bihamiltonian integrable hierarchies possessing hydrodynamic limits, and constructed the so-called topological deformations of the principal hierarchy of a semisimple Frobenius manifold and their tau structures. On the other hand, not all bihamiltonian integrable hierarchies possess tau structures.

In this paper we introduce the notion of *flat exact* semisimple bihamiltonian structure of hydrodynamic type, and study the properties of the associated integrable hierarchies. It turns out that this notion is an appropriate generalization of semisimple conformal Frobenius manifolds when considering the associated bihamiltonian integrable hierarchies. For a given flat exact semisimple bihamiltonian structure of hydrodynamic type, we show the existence of a bihamiltonian integrable hierarchy of hydrodynamic type (an analogue of the principal hierarchy) which possesses a tau structure. One can further consider the deformations of a flat exact semisimple bihamiltonian structure of hydrodynamic type which preserve the exactness condition. It is known that such deformations are characterized by the property that the central invariants of the deformed bihamiltonian structure are constant [14]. We show that for such deformed bihamiltonian structures the corresponding deformations of the principal hierarchies also possess tau structures, and we give a classification theorem for these tau structures.

The paper is arranged as follows. In Sec. 2 we introduce the notion of flat exact semisimple bihamiltonian structures of hydrodynamic type and present the main results. In Sec. 3 we study the relations between flat exact semisimple bihamiltonian structures of hydrodynamic type and semisimple Frobenius manifolds, and give a proof of Theorem 2.4. In Sec. 4 we construct the principal hierarchy for a flat exact semisimple bihamiltonian structure of hydrodynamic type and show the existence of a tau structure, and prove Proposition 2.5. In Sec. 5 we consider properties of deformations of the principal hierarchies which possess tau structures and the Galilean symmetry, and then in Sec. 6 we prove the existence of deformations of the principal hierarchy of a flat exact semisimple bihamiltonian structure of hydrodynamic type, which are bihamiltonian integrable hierarchies possessing tau structures and the Galilean symmetry, and we prove Theorem 2.10. Sec. 7 is a conclusion. In the Appendix, we prove some properties of semi-Hamiltonian integrable hierarchies, some of which are used in the proof of certain uniqueness properties given in Sec. 5 and Sec. 6.

2 Some notions and the main results

The class of systems of hydrodynamic type on the infinite jet space of an n-dimensional manifold M consists of systems of n first order quasilinear partial differential equations (PDEs)

$$v_t^\alpha = A_\beta^\alpha(v) v_x^\beta, \quad \alpha = 1, \ldots, n, \quad v = \left(v^1, \ldots, v^n\right) \in M. \tag{2.1}$$

Here and in what follows summation over the repeated upper and lower *Greek* indices (which range from 1 to n) is assumed, and $A_\beta^\alpha(v)$ is a section of the bundle $TM \otimes T^*M$. Note that we will not assume summation over repeated upper and lower *Latin* indices, unless otherwise indicated in the context of the relevant formulae. For the subclass of Hamiltonian systems of hydrodynamic type the r.h.s. of (2.1) admits a representation

$$v_t^\alpha = P^{\alpha\beta} \frac{\partial h(v)}{\partial v^\beta}, \quad \alpha = 1, \ldots, n. \tag{2.2}$$

Here the smooth function $h(v)$ is the density of the Hamiltonian

$$H = \int h(v(x))\, dx, \tag{2.3}$$

and

$$P^{\alpha\beta} = g^{\alpha\beta}(v) \partial_x + \Gamma_\gamma^{\alpha\beta}(v) v_x^\gamma \tag{2.4}$$

is the operator of a *Poisson bracket of hydrodynamic type.* As it was observed in [11] such operators satisfying the *nondegeneracy condition*

$$\det \left(g^{\alpha\beta}(v)\right) \neq 0 \tag{2.5}$$

correspond to flat metrics (Riemannian or pseudo-Riemannian)

$$ds^2 = g_{\alpha\beta}(v) dv^\alpha dv^\beta$$

on the manifold M. Namely,

$$g^{\alpha\beta}(v) = (g_{\alpha\beta}(v))^{-1}$$

is the corresponding inner product on T^*M, the coefficients $\Gamma_\gamma^{\alpha\beta}(v)$ are the contravariant components of the Levi-Civita connection for the metric. In the present paper it will be assumed that all Poisson brackets of hydrodynamic type satisfy the nondegeneracy condition (2.5).

A bihamiltonian structure of hydrodynamic type is a pair (P_1, P_2) of operators of the form (2.4) such that an arbitrary linear combination $\lambda_1 P_1 + \lambda_2 P_2$ is again the operator of a Poisson bracket. They correspond to pairs of flat metrics $g_1^{\alpha\beta}(v)$, $g_2^{\alpha\beta}(v)$ on M satisfying certain compatibility conditions (see below for the details). The bihamiltonian structure of hydrodynamic type is called *semisimple* if the roots $u^1(v), \ldots, u^n(v)$ of the characteristic equation

$$\det\left(g_2^{\alpha\beta}(v) - u\, g_1^{\alpha\beta}(v)\right) = 0 \tag{2.6}$$

are pairwise distinct and are not constant for a generic point $v \in M$. According to Ferapontov's theorem [15], these roots can serve as local coordinates of the manifold M, which are called the canonical coordinates of the bihamiltonian structure (P_1, P_2). We assume in this paper that D is a sufficiently small domain on M such that $(u^1, \ldots, u^n)$ is the local coordinate system on D. In the canonical coordinates the two metrics have diagonal forms

$$g_1^{ij}(u) = f^i(u)\delta^{ij}, \quad g_2^{ij}(u) = u^i f^i(u)\delta^{ij}. \tag{2.7}$$

We will need to use the notion of rotation coefficients of the metric g_1 which are defined by the following formulae:

$$\gamma_{ij}(u) = \frac{1}{2\sqrt{f_i f_j}}\frac{\partial f_i}{\partial u^j}, \quad i \neq j \tag{2.8}$$

with $f_i = \frac{1}{f^i}$. We also define $\gamma_{ii} = 0$.

Definition 2.1 (cf. [9]) *The semisimple bihamiltonian structure (P_1, P_2) is called reducible at $u \in M$ if there exists a partition of the set $\{1, 2, \ldots, n\}$ into the union of two nonempty nonintersecting sets I and J such that*

$$\gamma_{ij}(u) = 0, \quad \forall i \in I,\ \forall j \in J.$$

(P_1, P_2) is called irreducible on a certain domain $D \subset M$, if it is not reducible at any point $u \in D$.

The main goal of the present paper is to introduce tau-functions of bihamiltonian systems of hydrodynamic type and of their dispersive deformations. This will be done under the following additional assumption.

Definition 2.2 *Let (P_1, P_2) be a bihamiltonian structure of hydrodynamic type given by the Hamiltonian operators*

$$P_a^{\alpha\beta} = g_a^{\alpha\beta}(v)\partial_x + \Gamma^{\alpha\beta}_{a,\gamma}(v) v^\gamma_x, \quad a = 1, 2. \tag{2.9}$$

It is called exact *if there exists a vector field*

$$Z = \sum_{\alpha=1}^n Z^\alpha \frac{\partial}{\partial v^\alpha} \in Vect\,(M)$$

such that

$$\mathcal{L}_Z P_1 = 0, \quad \mathcal{L}_Z P_2 = P_1. \tag{2.10}$$

Here the Lie derivatives of the Hamiltonian operators are defined by the Lie derivatives of the metrics and of the affine connections as follows:

$$(\mathcal{L}_Z P_a)^{\alpha\beta} = (\mathcal{L}_Z g_a)^{\alpha\beta}\partial_x + (\mathcal{L}_Z \Gamma_a)^{\alpha\beta}_\gamma v^\gamma_x$$

with

$$(\mathcal{L}_Z g_a)^{\alpha\beta} = Z^\mu \frac{\partial g_a^{\alpha\beta}}{v^\mu} - \frac{\partial Z^\alpha}{\partial v^\mu} g_a^{\mu\beta} - \frac{\partial Z^\beta}{\partial v^\mu} g_a^{\alpha\mu},$$
$$(\mathcal{L}_Z \Gamma_a)^{\alpha\beta}_\gamma = Z^\mu \frac{\partial \Gamma^{\alpha\beta}_{a,\gamma}}{\partial v^\mu} + \frac{\partial Z^\mu}{\partial v^\gamma} \Gamma^{\alpha\beta}_{a,\mu} - \frac{\partial Z^\alpha}{\partial v^\mu} \Gamma^{\mu\beta}_{a,\gamma} - \frac{\partial Z^\beta}{\partial v^\mu} \Gamma^{\alpha\mu}_{a,\gamma} - \frac{\partial^2 Z^\beta}{\partial v^\gamma \partial v^\mu} g_a^{\alpha\mu}.$$

It is called flat exact *if the vector field Z is flat with respect to the metric g_1 associated with the Hamiltonian structure P_1.*

We will denote a flat exact bihamiltonian structure of hydrodynamic type by $(P_1, P_2; Z)$.

Example 2.3 *Let $(M, \cdot, \eta, e, E)$ be a Frobenius manifold. Then the pair of metrics*

$$g_1^{\alpha\beta}(v) = \langle dv^\alpha, dv^\beta \rangle = \eta^{\alpha\beta}, \tag{2.11}$$
$$g_2^{\alpha\beta}(v) = (dv^\alpha, dv^\beta) = i_E \left(dv^\alpha \cdot dv^\beta \right) =: g^{\alpha\beta}(v) \tag{2.12}$$

on T^*M defines a flat exact bihamiltonian structure with $Z = e$ (the unit vector field), see [7] for the details. For a semisimple Frobenius manifold the resulting bihamiltonian structure is semisimple. Roots of the characteristic equation (2.6) *coincide with the canonical coordinates on the Frobenius manifold.*

More bihamiltonian structures can be obtained from those of Example 2.3 by a Legendre-type transformation (see Appendix B of [7] and [33])

$$\hat{v}_\alpha = b^\gamma \frac{\partial^2 F(v)}{\partial v^\gamma \partial v^\alpha}, \quad \hat{v}^\alpha = \eta^{\alpha\beta} \hat{v}_\beta. \tag{2.13}$$

Here $F(v)$ is the potential of the Frobenius manifold and $b = b^\gamma \frac{\partial}{\partial v^\gamma}$ is a flat invertible vector field on it. The new metrics on T^*M by definition have the *same* Gram matrices in the new coordinates

$$\langle d\hat{v}^\alpha, d\hat{v}^\beta \rangle = \eta^{\alpha\beta}, \quad \left(d\hat{v}^\alpha, d\hat{v}^\beta \right) = g^{\alpha\beta}(v). \tag{2.14}$$

Recall that applying the transformation (2.13) to $F(v)$ one obtains a new solution $\hat{F}(\hat{v})$ to the WDVV associativity equations defined from

$$\frac{\partial^2 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} = \frac{\partial^2 F(v)}{\partial v^\alpha \partial v^\beta}. \tag{2.15}$$

The new unit vector field is given by

$$\hat{e} = b^\gamma \frac{\partial}{\partial \hat{v}^\gamma}. \tag{2.16}$$

The new solution to the WDVV associativity equations defines on M another Frobenius manifold structure if the vector $b = b^\gamma \frac{\partial}{\partial v^\gamma}$ satisfies

$$[b, E] = \lambda \cdot b$$

for some $\lambda \in \mathbb{C}$. Otherwise the quasihomogeneity axiom does not hold true.

Theorem 2.4 *For an arbitrary Frobenius manifold M the pair of flat metrics obtained from* (2.11) , (2.12) *by a transformation of the form* (2.13)–(2.14) *defines on M a flat exact bihamiltonian structure of hydrodynamic type. Conversely, any irreducible flat exact semisimple bihamiltonian structure of hydrodynamic type can be obtained in this way.*

Now we can describe the notion of a tau-symmetric bihamiltonian hierarchy associated with a flat exact semisimple bihamiltonian structure $(P_1, P_2; Z)$ of hydrodynamic type. Let us choose a system of flat coordinates $\left(v^1, \dots, v^n\right)$ for the first metric. So the operator P_1 has the form

$$P_1^{\alpha\beta} = \eta^{\alpha\beta} \frac{\partial}{\partial x} \tag{2.17}$$

for a constant symmetric nondegenerate matrix $(\eta^{\alpha\beta}) = (g_1^{\alpha\beta})$. It is convenient to normalize the choice of flat coordinates by the requirement that

$$Z = \frac{\partial}{\partial v^1}.$$

We are looking for an infinite family of systems of first order quasilinear evolutionary PDEs of the form (2.1) satisfying certain additional conditions. The systems of the form (2.1) will be labeled by pairs of indices (α, p), $\alpha = 1, \dots, n$, $p \geq 0$. Same labels will be used for the corresponding time variables $t = t^{\alpha,p}$. The conditions to be imposed are as follows.

1. All the systems under consideration are *bihamiltonian* PDEs w.r.t. (P_1, P_2). This implies pairwise commutativity of the flows [10]

$$\frac{\partial}{\partial t^{\alpha,p}} \frac{\partial v^\gamma}{\partial t^{\beta,q}} = \frac{\partial}{\partial t^{\beta,q}} \frac{\partial v^\gamma}{\partial t^{\alpha,p}}. \tag{2.18}$$

2. Denote

$$H_{\alpha,p} = \int h_{\alpha,p}(v)\, dx \tag{2.19}$$

the Hamiltonian of the (α, p)-flow

$$\frac{\partial v^\gamma}{\partial t^{\alpha,p}} = \eta^{\gamma\lambda} \frac{\partial}{\partial x} \frac{\delta H_{\alpha,p}}{\delta v^\lambda(x)} \equiv \eta^{\gamma\lambda} \frac{\partial}{\partial x} \frac{\partial h_{\alpha,p}(v)}{\partial v^\lambda} \tag{2.20}$$

with respect to the first Poisson bracket. The Hamiltonian densities satisfy the following *recursion:*[1]

$$\frac{\partial}{\partial v^1} h_{\alpha,p}(v) = h_{\alpha,p-1}(v), \quad \alpha = 1, \ldots, n, \quad p \geq 0 \tag{2.21}$$

(recall that $\frac{\partial}{\partial v^1} = Z$) where we denote

$$h_{\alpha,-1}(v) = v_\alpha \equiv \eta_{\alpha\beta} v^\beta, \quad \alpha = 1, \ldots, n. \tag{2.22}$$

Observe that the functionals $H_{\alpha,-1} = \int h_{\alpha,-1}(v)\, dx$ span the space of Casimirs of the first Poisson bracket.

3. Normalization

$$\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}. \tag{2.23}$$

Proposition 2.5 *Integrable hierarchies of the above form satisfy the* tau-symmetry *condition*

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}, \quad \forall\ \alpha,\, \beta = 1, \ldots, n, \quad \forall\ p,\, q \geq 0. \tag{2.24}$$

Moreover, this integrable hierarchy is invariant with respect to the Galilean symmetry

$$\frac{\partial v}{\partial s} = Z(v) + \sum_{p\geq 1} t^{\alpha,p} \frac{\partial v}{\partial t^{\alpha,p-1}}, \tag{2.25}$$

$$\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t^{\alpha,p}}\right] = 0, \quad \forall \alpha = 1, \ldots, n, \quad p \geq 0.$$

Definition 2.6 *A choice of the Hamiltonian densities $h_{\alpha,p}(v)$, $\alpha = 1, \ldots, n$, $p \geq -1$ satisfying the above conditions is called a* calibration *of the flat exact bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$. The integrable hierarchy* (2.20) *is called the* principal hierarchy *of $(P_1, P_2; Z)$ associated with the given calibration.*

Remark 2.7 *The principal hierarchy* (2.20)*, as a bihamiltonian hierarchy, can also be written as a hierarchy of Hamiltonian systems w.r.t. the second Hamiltonian structure P_2, i.e.*

$$\frac{\partial v^\gamma}{\partial t^{\alpha,p}} = P_2^{\gamma\lambda} \frac{\delta K_{\alpha,p}}{\delta v^\lambda} = P_2^{\gamma\lambda} \frac{\partial k_{\alpha,p}}{\partial v^\lambda}$$

for some Hamiltonians $K_{\alpha,p} = \int k_{\alpha,p}(v(x))dx$.

[1] This recursion acts in the opposite direction with respect to the bihamiltonian one - see eq. (2.28) below.

Example 2.8 *Let* $(M, \cdot, \eta, e, E)$ *be a Frobenius manifold. Denote*

$$(\theta_1(v;z), \dots, \theta_n(v;z))\, z^\mu z^R$$

with

$$\theta_\alpha(v;z) = \sum_{p=0}^\infty \theta_{\alpha,p}(v) z^p, \quad \alpha = 1, \dots, n \tag{2.26}$$

a Levelt basis of deformed flat coordinates [7, 8, 9]. Here the matrices

$$\mu = \operatorname{diag}(\mu_1, \dots, \mu_n), \quad R = R_1 + \dots, \quad [\mu, R_k] = k\, R_k$$

constitute a part of the spectrum of the Frobenius manifold, see details in [8, 9]. Then

$$h_{\alpha,p}(v) = \frac{\partial \theta_{\alpha,p+2}(v)}{\partial v^1}, \quad \alpha = 1, \dots, n, \quad p \geq -1 \tag{2.27}$$

is a calibration of the flat exact bihamiltonian structure associated with the metrics (2.11), (2.12) *defined on the Frobenius manifold. In this case the family of pairwise commuting bihamiltonian PDEs* (2.20) *is called the* principle hierarchy *associated with the Frobenius manifold. With this choice of the calibration the Hamiltonians* (2.19), (2.27) *satisfy the bihamilonian recursion relation*

$$\{\,.\,, H_{\beta,q-1}\}_2 = (q + \frac{1}{2} + \mu_\beta)\{\,.\,, H_{\beta,q}\}_1 + \sum_{k=0}^{q-1} (R_{q-k})_\beta^\gamma \{\,.\,, H_{\gamma,k}\}_1. \tag{2.28}$$

All other calibrations can be obtained from $\{h_{\alpha,p} \,|\, \alpha = 1, \dots, n, p \geq -1\}$ *by using the transformation* (4.19) *given below in Remark 4.10 of Sec. 4.*

For the flat exact bihamiltonian structure obtained from (2.11) , (2.12) by a Legendre-type transformation (2.13)–(2.16), one can choose a calibration by introducing functions $\hat\theta_{\alpha,p}\,(\hat v)$ defined by

$$\frac{\partial \hat\theta_\alpha\,(\hat v; z)}{\partial \hat v^\beta} = \frac{\partial \theta_\alpha(v;z)}{\partial v^\beta}, \quad \forall\, \alpha,\, \beta = 1, \dots, n. \tag{2.29}$$

Remarkably in this case the new Hamiltonians satisfy the *same* bihamiltonian recursion (2.28).

Proposition 2.9 *For a flat exact bihamiltonian structure of hydrodynamic type obtained from a Frobenius manifold by a Legendre-type transformation* (2.13)–(2.16) *the construction* (2.29) *and* (2.27) *defines a calibration. Any other calibration can be obtained in this way up to the transformation* (4.19) *given in Sec. 4.*

We will construct calibrations for flat exact semisimple bihamiltonian structures of hydrodynamic in Sec. 4. The properties of a calibration, in particular the tau-symmetry property (2.24), of such a bihamiltonian structure enable us to define a tau structure and tau functions for the associated principal hierarchy (2.20), see Definitions 4.14 and 4.16 in Sec. 4. One of the main purposes of the present paper is to study the existence and properties of tau structures for deformations of a flat exact semisimple bihamiltonian structure of hydrodynamic $(P_1, P_2; Z)$ and the associated principal hierarchy. Recall that deformations of a Hamiltonian operator P of the form (2.4), in the sense of [9, 10, 23, 25], are Hamiltonian operators of the form

$$\begin{aligned}\tilde{P}^{\alpha\beta} =& P^{\alpha\beta} + \epsilon^2 \sum_{k=0}^{3} A^{\alpha\beta}_{2,k}(v; v_x, \dots, \partial_x^3 v)\partial_x^{3-k} \\ &+ \epsilon^3 \sum_{k=0}^{4} A^{\alpha\beta}_{3,k}(v; v_x, \dots, \partial_x^4 v)\partial_x^{4-k} + \dots,\end{aligned} \tag{2.30}$$

where $A^{\alpha\beta}_{2,k}, A^{\alpha\beta}_{3,k}, \dots$ are polynomials of $v^\alpha_x, v^\alpha_{xx}, \dots$ with coefficients smoothly depend on $v^1, \dots, v^n$, and are homogeneous of degree k with the assignment

$$\deg \partial_x^k v^\alpha = k, \quad \deg \partial_x = 1.$$

We call such functions homogeneous differential polynomial of degree k. By a deformation of the Hamltonian H of the form (2.3), we mean a local Hamiltonian of the form

$$\tilde{H} = \int \Big(h(v(x)) + \epsilon^2 h^{[2]}(v; v_x, v_{xx}) + \epsilon^3 h^{[3]}(v; v_x, \dots, \partial_x^3 v) + \dots \Big)\, dx, \tag{2.31}$$

where $h^{[k]}$ are homogeneous differential polynomial of degree k. Here and in what follows we do not consider deformations with linear in ϵ terms to avoid some subtleties in the presentation of the main theorem below, see Remarks 5.2, 5.8 in Sec. 5. In what follows we will also omit the parameter ϵ accompanying the deformation terms, this parameter can be recovered by using the degrees of the homogeneous terms of the deformations. Given a deformation of the Hamiltonian structure P and the Hamiltonian H, we then have a deformation of the Hamiltonian system of hydrodynamic type (2.2) given by

$$v^\alpha_t = \tilde{P}^{\alpha\beta} \frac{\delta \tilde{H}}{\delta v^\beta}, \quad \alpha = 1, \dots, n. \tag{2.32}$$

Similarly, we have the notions of deformations of bihamiltonian structures and the associated bihamiltonian systems.

For a given flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$ and a collection of arbitrary smooth functions $c_1(u^1)$, $\dots$, $c_n(u^n)$ of the canonical cordinates, we know from [5, 25] the existence of a deformation $(\tilde{P}_1, \tilde{P}_2)$ of (P_1, P_2) such that its central invariants are given by these n functions. By using the triviality of the second bihamiltonian cohomology, one can show that there also exists a unique deformation of the principal hierarchy (2.20) of $(P_1, P_2; Z)$ such that all flows of this hierarchy are bihamiltonian systems w.r.t. the bihamiltonian structure $(\tilde{P}_1, \tilde{P}_2)$ (see Sec. 6). The deformed integrable hierarchy usually does not possess a tau structure unless the central invariants are constant (first observed in [34]). On the other hand, it is shown by Falqui and Lorenzoni in [14] that, if $c_i(u^i)$ $(i = 1, \dots, n)$ are constants, one can choose the representative $(\tilde{P}_1, \tilde{P}_2)$ such that this deformed bihamiltonian structure is also exact. To define the exactness of a deformed bihamiltonian structure, we need to employ the formalism of infinite dimensional Hamiltonian structures in terms of bivector fields on the infinite dimensional jet space $J^\infty(\hat{M})$ of the super manifold $\hat{M}$ (see Sec. 3 and [25]). In this formalism the vector field $Z \in Vect\,(M)$ is lifted to a vector field of $J^\infty(\hat{M})$, and the condition (2.10) can be represented as the vanishing of the Schouten–Nijenhuis brackets (see (3.4) for definition) between Z and the bivector fields on $J^\infty(\hat{M})$ corresponding to P_1, P_2 (which we still denote by P_1, P_2), i.e.

$$[Z, P_1] = 0, \quad [Z, P_2] = P_1. \tag{2.33}$$

We say that the deformed bihamiltonian structure $(\tilde{P}_1, \tilde{P}_2)$ is exact if we have a deformation $\tilde{Z}$, as a vector field on $J^\infty(\hat{M})$, of Z such that

$$[\tilde{Z}, \tilde{P}_1] = 0, \quad [\tilde{Z}, \tilde{P}_2] = \tilde{P}_1. \tag{2.34}$$

We also note that the equation (2.20) of the principal hierarchy associated with $(P_1, P_2; Z)$ corresponds to a vector field on $J^\infty(\hat{M})$. We call it a bihamiltonian vector field.

With such a deformed exact bihamiltonian structure $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ in hand, we can ask the following questions:

1. Does the associated deformed integrable hierarchy have tau structures?
2. If it does, how many of them?

The following theorem is the main result of the present paper, which answers the above questions.

Theorem 2.10 *Let $(P_1, P_2; Z)$ be a flat exact semisimple bihamiltonian structure of hydrodynamic type which satisfies the irreducibility condition. We fix a calibration $\{h_{\alpha,p} \,|\, \alpha = 1, \dots, n;\ p = 0, 1, 2, \dots\}$ of the bihamiltonian structure $(P_1, P_2; Z)$. Then the following statements hold true:*

i) For any deformation $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ of $(P_1, P_2; Z)$ with constant central invariants, there exists a deformation $\{\tilde{h}_{\alpha,p}\}$ of the calibration $\{h_{\alpha,p}\}$ such that the corresponding hierarchy of bihamiltonian integrable systems yields a deformation of the principal hierarchy which possesses a tau structure and the Galilean symmetry.

ii) Let $(\hat{P}_1, \hat{P}_2; \hat{Z})$ be another deformation of $(P_1, P_2; Z)$ with the same central invariants as $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$, and let $\{\hat{h}_{\alpha,p}\}$ be the corresponding deformation of the calibration $\{h_{\alpha,p}\}$, then the logarithm of the tau function of the deformed principal hierarchy associated with $\{\hat{h}_{\alpha,p}\}$ can be obtained from the one that is associated with $\{\tilde{h}_{\alpha,p}\}$ by adding a differential polynomial.

We will give in Sec. 5 the definition of the notions of deformations of a calibration, the tau structures and of the tau functions.

3 Flat exact semisimple bihamitonian structures of hydrodynamic type and Frobenius manifolds

Let M be a smooth manifold of dimension n. Denote by $\hat{M}$ the super manifold of dimension $(n \mid n)$ obtained from the cotangent bundle of M by reversing the parity of the fibers. Suppose U is a local coordinate chart on M with coordinates $(u^1, \dots, u^n)$, then

$$\theta_i = \frac{\partial}{\partial u^i}, \quad i = 1, \dots, n$$

can be regarded as local coordinates on the corresponding local chart $\hat{U}$ on $\hat{M}$. Note that θ_i's are super variables, they satisfy the skew-symmetric commutation law:

$$\theta_i \theta_j + \theta_j \theta_i = 0.$$

Let $J^\infty(M)$ and $J^\infty(\hat{M})$ be the infinite jet space of M and $\hat{M}$, which is just the projective limits of the corresponding finite jet bundles. There is a natural local chart $\hat{U}^\infty$ over $\hat{U}$ with local coordinates

$$\{u^{i,s}, \theta_i^s \mid i = 1, \dots, n; s = 0, 1, 2, \dots\}.$$

See [25] for more details. Denote by $\hat{\mathcal{A}}$ the spaces of differential polynomials on $\hat{M}$. Locally, we can regard $\hat{\mathcal{A}}$ as

$$C^\infty(\hat{U})[[u^{i,s}, \theta_i^s \mid i = 1, \dots, n; s = 1, 2, \dots]].$$

The differential polynomial algebra $\mathcal{A}$ on M can be defined similarly as a subalgebra of $\hat{\mathcal{A}}$. There is a globally defined derivation on $J^\infty(\hat{M})$

$$\partial = \sum_{i=1}^{n} \sum_{s \ge 0} \left(u^{i,s+1} \frac{\partial}{\partial u^{i,s}} + \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s} \right). \tag{3.1}$$

Its cokernel $\hat{\mathcal{F}} = \hat{\mathcal{A}} / \partial \hat{\mathcal{A}}$ is called the space of local functionals. Denote the projection $\hat{\mathcal{A}} \to \hat{\mathcal{F}}$ by $\int$. We can also define $\mathcal{F} = \mathcal{A} / \partial \mathcal{A}$, whose elements are called local functionals on M.

There are two useful degrees on $\hat{\mathcal{A}}$, which are called the standard gradation

$$\deg u^{i,s} = \deg \theta_i^s = s \tag{3.2}$$

and the super gradation

$$\deg_\theta \theta_i^s = 1, \quad \deg_\theta u^{i,s} = 0 \tag{3.3}$$

respectively:

$$\hat{\mathcal{A}} = \bigoplus_{d\geq 0} \hat{\mathcal{A}}_d = \bigoplus_{p\geq 0} \hat{\mathcal{A}}^p.$$

Here

$$\hat{\mathcal{A}}_d = \{f \in \hat{\mathcal{A}} \mid \deg f = d\}, \quad \hat{\mathcal{A}}^p = \{f \in \hat{\mathcal{A}} \mid \deg_\theta f = p\}.$$

We denote $\hat{\mathcal{A}}_d^p = \hat{\mathcal{A}}_d \cap \hat{\mathcal{A}}^p$. In particular, $\mathcal{A} = \hat{\mathcal{A}}^0$, $\mathcal{A}_d = \hat{\mathcal{A}}_d^0$. The derivation ∂ has the property $\partial(\hat{\mathcal{A}}_d^p) \subseteq \hat{\mathcal{A}}_{d+1}^p$, hence it induces the same degrees on $\hat{\mathcal{F}}$, so we also have the homogeneous components $\hat{\mathcal{F}}_d$, $\hat{\mathcal{F}}^p$, $\hat{\mathcal{F}}_d^p$, and the ones for $\mathcal{F} = \hat{\mathcal{F}}^0$. The reader can refer to [25] for details of the definitions of these notations.

There is a graded Lie algebra structure on $\hat{\mathcal{F}}$, whose bracket operation is given by

$$[P, Q] = \int \sum_{i=1}^n \left(\frac{\delta P}{\delta \theta_i} \frac{\delta Q}{\delta u^i} + (-1)^p \frac{\delta P}{\delta u^i} \frac{\delta Q}{\delta \theta_i} \right), \tag{3.4}$$

where $P \in \hat{\mathcal{F}}^p$, $Q \in \hat{\mathcal{F}}^q$. This bracket is called the Schouten–Nijenhuis bracket on $J^\infty(\hat{M})$.

A Hamiltonian structure is defined as an element $P \in \hat{\mathcal{F}}^2$ satisfying $[P, P] = 0$. We also call elements of $\hat{\mathcal{F}}^1$, $\hat{\mathcal{F}}^2$ vector fields and bivector fields on $J^\infty(\hat{M})$ respectively. For example, a Hamiltonian operator P with components

$$\begin{aligned} P^{ij} =& g^{ij}(u)\partial_x + \Gamma_k^{ij}(u)u_x^k + \sum_{k=0}^2 A_{2,k}^{ij}(u; u_x, u_{xx})\partial_x^{2-k} \\ &+ \sum_{k=0}^3 A_{3,k}^{ij}(u; u_x, u_{xx}, u_{xxx})\partial_x^{3-k} + \dots, \end{aligned} \tag{3.5}$$

corresponds to an element $P \in \hat{\mathcal{F}}^2$ of the form

$$\begin{aligned} P =& \frac{1}{2} \int \sum_{i,j=1}^n \Big(g^{ij}(u)\theta_i\theta_j^1 + \Gamma_k^{ij}(u)u^{k,1}\theta_i\theta_j \\ &+ \sum_{k=0}^2 A_{2,k}^{ij}\theta_i\theta_j^{2-k} + \sum_{k=0}^3 A_{3,k}^{ij}\theta_i\theta_j^{3-k} + \dots \Big). \end{aligned} \tag{3.6}$$

In establishing the above correspondence, we identify the variables $u^k_x, u^k_{xx}, \dots$ with the jet coordinates $u^{k,1}$, $u^{k,2}$, ... We will assume such identifications in what follows. The fact that P is a Hamiltonian operator is equivalent to the condition that $[P,P]=0$. A Hamiltonian system

$$\partial_t u^i = \sum_{j=1}^n P^{ij}\frac{\delta H}{\delta u^j} = X^i(u;u_x,\dots), \quad i=1,\dots,n \tag{3.7}$$

corresponds to a vector field $X \in \hat{\mathcal{F}}^1$ on $J^\infty(\hat{M})$ and a local functional $H \in \mathcal{F} = \hat{\mathcal{F}}^0$ on M which have the expressions

$$X = \int \left(\sum_{i=1}^n X^i\theta_i\right) = -[P,H]. \tag{3.8}$$

$$H = \int (h), \quad h \in \mathcal{A}. \tag{3.9}$$

We call such a vector field a Hamiltonian vector field on $J^\infty(\hat{M})$.

Remark 3.1 *For a vector field*

$$Y = \int \left(\sum_{i=1}^n Y^i\theta_i\right) \in \hat{\mathcal{F}}^1,$$

we define in [25] the evolutionary vector field

$$D_Y = \sum_{i=1}^n \sum_{k\ge 0} \partial^k\left(Y^i\right)\frac{\partial}{\partial u^{i,k}} \tag{3.10}$$

which is a derivation on the space $\mathcal{A}$ of differential polynomials. In terms of this notation, a system of evolutionary PDEs

$$\partial_t u^i = Y^i, \quad i=1,\dots,n$$

can be written as

$$\partial_t u^i = D_Y(u^i), \quad i=1,\dots,n.$$

So we will also denote such a system of evolutionary PDEs by $\partial_t = D_Y$.

A bihamiltonian structure of hydrodynamic type is given by a pair of Hamiltonian structures of hydrodynamic type (P_1,P_2) satisfying the additional condition $[P_1,P_2]=0$. Denote by g_1, g_2 the flat metrics associated with the Hamiltonian structures P_1, P_2. In what follows, we will assume that (P_1,P_2) is semisimple with a fixed system of canonical coordinates $u^1,\dots,u^n$, in which the two flat metrics take the diagonal form (2.7), i.e.

$$g_1^{ij}(u) = f^i(u)\delta^{ij}, \quad g_2^{ij}(u) = u^i f^i(u)\delta^{ij}. \tag{3.11}$$

and the contravariant Christoffel coefficients of them have the following expressions respectively:

$$\Gamma^{ij}_k = \frac{1}{2}\frac{\partial f^i}{\partial u^k}\delta^{ij} + \frac{1}{2}\frac{f^i}{f^j}\frac{\partial f^j}{\partial u^i}\delta^{jk} - \frac{1}{2}\frac{f^j}{f^i}\frac{\partial f^i}{\partial u^j}\delta^{ik}, \tag{3.12}$$

$$\hat{\Gamma}^{ij}_k = \frac{1}{2}\frac{\partial(u^i f^i)}{\partial u^k}\delta^{ij} + \frac{1}{2}\frac{u^i f^i}{f^j}\frac{\partial f^j}{\partial u^i}\delta^{jk} - \frac{1}{2}\frac{u^j f^j}{f^i}\frac{\partial f^i}{\partial u^j}\delta^{ik}. \tag{3.13}$$

The diagonal entries f^i satisfy certain non-linear differential equations which are equivalent to the flatness of g_1, g_2 and the condition $[P_1, P_2] = 0$. See the appendix of [10] for details. We denote by ∇, $\hat{\nabla}$ the Levi-Civita connections of the metrics g_1, g_2 respectively.

We also assume henceforth that the semisimple bihamiltonian structure of hydrodynamic type (P_1, P_2) is flat exact (see Definition 2.2), and the corresponding vector field is given by $Z \in \hat{\mathcal{F}}^1_0$ which satisfies the conditions (2.33). We will denote this exact bihamiltonian structure by $(P_1, P_2; Z)$.

From [14] we know that Z must take the form

$$Z = \int \left(\sum_{i=1}^n Z^i \theta_i \right), \quad Z^1 = \cdots = Z^n = 1. \tag{3.14}$$

It corresponds to a derivation on the algebra $\mathcal{A}$ of differential polynomials given by

$$D_Z = \sum_{i=1}^n \sum_{s\geq 0} \partial^s(Z^i)\frac{\partial}{\partial u^{i,s}} = \sum_{i=1}^n \frac{\partial}{\partial u^i},$$

which is also called an evolutionary vector field on M (see Remark 3.1). Note that in the previous section we also denote this vector field D_Z by Z. It is also proved in [14] that if (2.10) holds true then

$$D_Z(f^i) = \sum_{k=1}^n \frac{\partial f^i}{\partial u^k} = 0, \quad i = 1, \ldots, n. \tag{3.15}$$

Note that the flatness of the vector field Z given in Definition 2.2 can be represented as

$$\nabla D_Z = 0. \tag{3.16}$$

Lemma 3.2 *D_Z is flat if and only if $f_i := (f^i)^{-1}$ $(i = 1, \ldots, n)$ satisfy the following Egoroff conditions:*

$$\frac{\partial f_i}{\partial u^j} = \frac{\partial f_j}{\partial u^i}, \quad \forall\, 1 \leq i, j \leq n. \tag{3.17}$$

Proof The components of D_Z read $Z^j = 1$, so we have

$$0 = \nabla^i Z^j = \sum_{k=1}^n \left(g_1^{ik}\frac{\partial Z^j}{\partial u^k} - \Gamma^{ij}_k Z^k \right) = -\sum_{k=1}^n \Gamma^{ij}_k. \tag{3.18}$$

Then the lemma can be easily proved by using (3.12) and (3.15). □

The above lemma implies that if Z is flat then the rotation coefficients of the metric g_1 defined in (2.8) are symmetric, i.e.

$$\gamma_{ij} = \gamma_{ji} = \frac{1}{2\sqrt{f_i f_j}} \frac{\partial f_i}{\partial u^j}, \quad i \neq j.$$

In this case, the conditions that (P_1, P_2) is a bihamiltonian structure are equivalent to the following equations for γ_{ij} (see the appendix of [10]):

$$\frac{\partial \gamma_{ij}}{\partial u^k} = \gamma_{ik}\gamma_{jk}, \quad \text{for distinct } i, j, k, \tag{3.19}$$

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u^k} = 0, \tag{3.20}$$

$$\sum_{k=1}^{n} u^k \frac{\partial \gamma_{ij}}{\partial u^k} = -\gamma_{ij}. \tag{3.21}$$

The condition (3.20) is actually $D_Z(\gamma_{ij}) = 0$. If we introduce the Euler vector field

$$E = \sum_{k=1}^{n} u^k \frac{\partial}{\partial u^k}, \tag{3.22}$$

then the condition (3.21) is $E(\gamma_{ij}) = -\gamma_{ij}$, that is, γ_{ij} has degree -1 if we assign the degree $\deg u^i = 1$, $i = 1, \dots, n$.

Now let us consider the relation of a flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2, ; Z)$ with Frobenius manifold structures. To this end, we first consider the linear system

$$\frac{\partial \psi_j}{\partial u^i} = \gamma_{ji}\psi_i, \quad i \neq j, \tag{3.23}$$

$$\frac{\partial \psi_i}{\partial u^i} = -\sum_{k\neq i} \gamma_{ki}\psi_k. \tag{3.24}$$

The above conditions for γ_{ij} ensure the compatibility of this linear system, so its solution space $\mathcal{S}$ has dimension n, and we can find a fundamental system of solutions

$$\Psi_\alpha = (\psi_{1\alpha}(u), \dots, \psi_{n\alpha}(u))^T, \quad \alpha = 1, \dots, n, \tag{3.25}$$

which form a basis of $\mathcal{S}$.

Lemma 3.3 *Let $\psi = (\psi_1, \dots, \psi_n)$ be a nontrivial solution of the linear system (3.23), (3.24) on the domain D, i.e., there exist $i \in \{1, \dots, n\}$ and $u \in D$ such that $\psi_i(u) \neq 0$. Assume that the rotation coefficients γ_{ij} satisfy the irreducibility condition given in Definition 2.1, then there exists $u_0 \in D$ such that for each $i \in \{1, \dots, n\}$, $\psi_i(u_0) \neq 0$.*

Proof For any subset $S \subseteq \{1,\ldots,n\}$, define $\phi_S = \prod_{i\in S}\psi_i$. We assume $\phi_{\{1,\ldots,n\}} = 0$ on the domain D, then we are to show that ψ is a trivial solution, that is $\phi_{\{i\}} = 0$ on D for each $i = 1,\ldots,n$. To this end, we will prove that for any $S \subseteq \{1,\ldots,n\}$, $\phi_S = 0$ for any $u \in D$ by induction on the size of S. We have known that if $\#S = n$, then $\phi_S = 0$. Assume for some $k \le n$, and any $S \subseteq \{1,\ldots,n\}$ with $\#S = k$, we have $\phi_S(u) = 0$ for any $u \in D$. For $T \subseteq \{1,\ldots,n\}$ with $\#T = k-1$, and any given $u \in D$, we can find $i \in T$, and $j \notin T$ such that $\gamma_{ij}(u) \neq 0$ because of the irreducibility condition. Without loss of generality we can assume that $\psi_i(u)$ does not identically vanish. Take $S = T \cup \{j\}$, then consider $\frac{\partial \phi_S}{\partial u^i}$:

$$0 = \frac{\partial \phi_S}{\partial u^i} = \sum_{k\in S}\phi_{S-\{k\}}\frac{\partial \psi_k}{\partial u^i} = \sum_{k\in S, k\neq i}\phi_{S-\{i,k\}}\gamma_{ik}(\psi_i^2 - \psi_k^2),$$

so we have

$$\phi_T\frac{\partial \phi_S}{\partial u^i} = \gamma_{ij}\phi_T^2\psi_i = 0.$$

Since $\gamma_{ij}(u) \neq 0$, we have $\phi_T^2\psi_i = 0$, which implies $\phi_T = 0$. □

We assume that γ_{ij} is irreducible from now on, and shrink D (if necessary) such that D is contractible, and $\psi_{i1} \neq 0$ on D for each $i = 1,\ldots,n$.

Lemma 3.4 *We have the following facts:*

i) Define

$$\eta_{\alpha\beta} = \sum_{i=1}^n \psi_{i\alpha}\psi_{i\beta},$$

then $(\eta_{\alpha\beta})$ is a constant symmetric non-degenerate matrix. We denote its inverse matrix by $(\eta^{\alpha\beta})$.

ii) For each $\alpha = 1,\ldots,n$, the 1-form

$$\omega_\alpha = \sum_{i=1}^n \psi_{i\alpha}\psi_{i1}du^i$$

is closed, so there exist smooth functions v_α such that $\omega_\alpha = dv_\alpha$. Denote $v^\alpha = \eta^{\alpha\beta}v_\beta$, then $(v^1,\ldots,v^n)$ can serve as a local coordinate system on D. In this local coordinate system we have

$$D_Z = \frac{\partial}{\partial v^1}.$$

iii) Define the functions

$$c_{\alpha\beta\gamma} = \sum_{i=1}^n \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}, \tag{3.26}$$

then $c_{\alpha\beta\gamma}$ are symmetric with respect to the three indices and satisfy the following conditions:

$$c_{1\alpha\beta} = \eta_{\alpha\beta}, \tag{3.27}$$

$$c_{\alpha\beta\xi}\eta^{\xi\zeta}c_{\zeta\gamma\delta} = c_{\delta\beta\xi}\eta^{\xi\zeta}c_{\zeta\gamma\alpha}, \tag{3.28}$$

$$\frac{\partial c_{\alpha\beta\gamma}}{\partial v^\xi} = \frac{\partial c_{\xi\beta\gamma}}{\partial v^\alpha}. \tag{3.29}$$

Proof The items i), ii) and the condition (3.27) are easy, so we omit their proofs. The condition (3.28) follows from the identity $\psi_{i\xi}\eta^{\xi\zeta}\psi_{j\zeta} = \delta_{ij}$. The condition (3.29) can be proved by the chain rule and the following identities

$$\frac{\partial v^\alpha}{\partial u^i} = \psi_i^\alpha \psi_{i1}, \quad \frac{\partial u^i}{\partial v^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}, \tag{3.30}$$

where $\psi_i^\alpha = \eta^{\alpha\beta}\psi_{i\beta}$. □

The above lemma implies immediately the following corollary.

Corollary 3.5 *There exists a smooth function $F(v)$ on D such that*

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}, \tag{3.31}$$

and it gives the potential of a Frobenius manifold structure (without, in general, the quasi-homogeneity condition) on D with the unity vector field

$$e = D_Z = \frac{\partial}{\partial v^1}.$$

By using (3.30) we have

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = c_{\alpha\beta}^\gamma \frac{\partial v^\alpha}{\partial u^i}\frac{\partial v^\beta}{\partial u^j}\frac{\partial u^k}{\partial v^\gamma}\frac{\partial}{\partial u^k} = \delta_{ij}\frac{\partial}{\partial u^i},$$

so $u^1, \ldots, u^n$ are the canonical coordinates of this Frobenius manifold.

On the other hand, we can also construct a flat exact bihamitonian structure associated with the above defined Frobenius manifold structure in a similar way as we do in Example 2.3. Although the Frobenius manifold structure defined in Lemma 3.4 and Corollary 3.5 is, in general, not quasi-homogeneous with respect to the vector field E given in (3.22), we can still use (2.11) and (2.12) to define the two flat metrics on the Frobenius manifold, where the matrix $(\eta^{\alpha\beta})$ is given in Lemma 3.4, and the multiplication on the cotangent space of the Frobenius manifold is defined by

$$dv^\alpha \cdot dv^\beta = \eta^{\alpha\xi}\eta^{\beta\zeta}c_{\xi\zeta\gamma}dv^\gamma.$$

In the canonical coordinates $u^1, \dots, u^n$ the associated contravariant metric tensors of the two flat metrics have the expressions

$$\langle du^i, du^j \rangle_1 = \delta_{ij} \psi_{i1}^{-2}, \tag{3.32}$$

$$\langle du^i, du^j \rangle_2 = \delta_{ij} u^i \psi_{i1}^{-2}. \tag{3.33}$$

By using the fact that the first flat metric has the same rotation coefficients as the one given in (3.11) associated with the originally given bihamiltonian structure $(P_1, P_2; Z)$, we know that the above pair of flat metrics defines a flat exact bihamiltonian structure of hydrodynamic type with the same unity Z. Since the pair of flat metrics (3.32), (3.33) are in general different from the one given in (3.11), the associated bihamiltonian structures of hydrodynamic type are also different in general. However, if we choose the solution Ψ_1 of the linear system (3.23), (3.24) as follows:

$$\psi_{i1} = f_i^{\frac{1}{2}} = (f^i)^{-\frac{1}{2}}, \quad i = 1, \dots, n,$$

then the two metrics defined in (3.32), (3.33) coincide with g_1, g_2 given in (3.11), thus the bihamiltonian structure that we constructed from the Frobenius manifold also coincides with the original one $(P_1, P_2; Z)$. In particular, the local coordinates $v^1, \dots, v^n$ defined in Lemma 3.4 form a system of local flat coordinates for the metric g_1. In these coordinates the metric g_1 has the constant form $(g_1^{\alpha\beta}) = (\eta^{\alpha\beta})$. We call the corresponding Frobenius manifold structure the *canonical one* associated with $(P_1, P_2; Z)$.

Remark 3.6 *Assume that a system of flat local coordinates $v^1, \dots, v^n$ of the metric g_1 is already given, and in these coordinates $(g_1^{\alpha\beta}) = (\eta^{\alpha\beta})$. For the construction of the canonical Frobenius manifold structure associated with $(P_1, P_2; Z)$, we can choose a fundamental system of solutions* (3.25) *of the linear equations* (3.23), (3.24) *by using the equations* (3.30). *Namely, we can specify $\psi_{i\alpha}$ as follows:*

$$\psi_{i1} = f_i^{\frac{1}{2}}, \quad \psi_{i\alpha} = \psi_{i1} \frac{\partial u^i}{\partial v^\alpha}, \quad i = 1, \alpha = 1, \dots, n.$$

Then the formulae given in Lemma 3.4 hold true, and the potential of the Frobenius manifold is defined as in Corollary 3.5.

We also have choices for Ψ_1 such that the corresponding Frobenius manifolds are quasi-homogeneous. By using the identity (3.21), one can show that Euler vector field E defined by (3.22) acts on the solution space $\mathcal{S}$ as a linear transformation. Suppose we are working in the complex manifold case, then E has at least one eigenvector in $\mathcal{S}$. We fix an eigenvector by Ψ_1, and denote its eigenvalue by μ_1, then choose other basis $\Psi_2, \dots, \Psi_n$ such that the matrix of E takes the Jordan normal form, that is, there exists $\mu_\alpha \in \mathbb{C}$, and $p_\alpha = 0$ or 1, such that

$$E(\Psi_\alpha) = \mu_\alpha \Psi_\alpha + p_{\alpha-1} \Psi_{\alpha-1}.$$

Lemma 3.7 *The Frobenius manifold structure corresponding to the above Ψ_1 is quasi-homogeneous with the Euler vector field E and the charge $d = -2\mu_1$.*

Proof The trivial identity $E(\eta_{\alpha\beta}) = 0$ implies that

$$\left(\mu_\alpha \eta_{\alpha\beta} + p_{\alpha-1}\eta_{(\alpha-1)\beta}\right) + \left(\mu_\beta \eta_{\alpha\beta} + p_{\beta-1}\eta_{\alpha(\beta-1)}\right) = 0.$$

Denote by L_E the Lie derivative with respect to E, then the identity $L_E\omega_\alpha = dE(v_\alpha)$ implies

$$dE(v_\alpha) = (\mu_\alpha + \mu_1 + 1)\, dv_\alpha + p_{\alpha-1} dv_{\alpha-1},$$

so there exist some constants $r_\alpha \in \mathbb{C}$ such that

$$E(v_\alpha) = (\mu_\alpha + \mu_1 + 1)\, v_\alpha + p_{\alpha-1} v_{\alpha-1} + r_\alpha.$$

On the other hand, we have

$$\begin{aligned} E(c_{\alpha\beta\gamma}) = &(\mu_\alpha + \mu_\beta + \mu_\gamma - \mu_1)\, c_{\alpha\beta\gamma} \\ &+ p_{\alpha-1} c_{(\alpha-1)\beta\gamma} + p_{\beta-1} c_{\alpha(\beta-1)\gamma} + p_{\gamma-1} c_{\alpha\beta(\gamma-1)}. \end{aligned}$$

By using the above identities, one can show that

$$\frac{\partial^3}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \left(E(F) - (3 + 2\mu_1) F\right) = 0, \quad \forall\, \alpha, \beta, \gamma$$

that gives the quasi-homogeneity condition for F. □

The above constructed Frobenius manifolds (including the canonical one) are related by Legendre transformations (see [7]). To see this, let us denote by $F(v) = F(v^1, \dots, v^n)$ and $\tilde{F}(\tilde{v}) = \tilde{F}(\tilde{v}^1, \dots, \tilde{v}^n)$ the Frobenius manifold potentials constructed above starting from the fundamental system of solutions

$$(\psi_{i\alpha}) = (\Psi_1, \dots, \Psi_n), \quad (\tilde{\psi}_{i\alpha}) = (\tilde{\Psi}_1, \dots, \tilde{\Psi}_n)$$

of the linear system (3.23), (3.24). These two fundamental systems of solutions are related by a non-degenerate constant matrix $A = (a^\alpha_\beta)$ by the formula

$$(\tilde{\Psi}_1, \dots, \tilde{\Psi}_n) = (\Psi_1, \dots, \Psi_n) A. \tag{3.34}$$

Introduce the new coordinates $\hat{v}^1, \dots \hat{v}^n$ by a linear transformation

$$\begin{pmatrix} \hat{v}^1 \\ \vdots \\ \hat{v}^n \end{pmatrix} = A \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix}, \tag{3.35}$$

and represent the coordinates $\tilde{v}^1, \dots, \tilde{v}^n$ in terms of $\hat{v}^1, \dots, \hat{v}^n$ in the potential $\tilde{F}(\tilde{v})$ we obtain

$$\hat{F}(\hat{v}) = \hat{F}(\hat{v}^1, \dots, \hat{v}^n) := \tilde{F}(\tilde{v}). \tag{3.36}$$

Then the potential $\hat{F}(\hat{v})$ can be determined (up to the addition of a linear function of $\hat{v}^1, \ldots, \hat{v}^n$) by the following Legendre transformation (cf. (2.13), (2.15), (2.16)):

$$\frac{\partial^2 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} = \frac{\partial^2 F(v)}{\partial v^\alpha \partial v^\beta}, \quad \hat{v}^\alpha = \eta^{\alpha\beta} a_1^\gamma \frac{\partial^2 F(v)}{\partial v^\beta \partial v^\gamma}. \tag{3.37}$$

The unit vector field has the expression

$$\tilde{e} = \frac{\partial}{\partial \tilde{v}^1} = a_1^\gamma \frac{\partial}{\partial \hat{v}^\gamma}.$$

To prove this fact, we only need to verify the identities

$$\frac{\partial}{\partial \hat{v}^\xi} \left(\frac{\partial^2 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} \right) = \frac{\partial}{\partial \hat{v}^\xi} \left(\frac{\partial^2 F(v)}{\partial v^\alpha \partial v^\beta} \right), \quad \alpha, \beta, \xi = 1, \ldots, n,$$

which are equivalent to the identities

$$\frac{\partial^3 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta \partial \hat{v}^\gamma} \eta^{\gamma\zeta} a_1^\mu \frac{\partial^2 F(v)}{\partial v^\mu \partial v^\zeta \partial v^\xi} = \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\xi}.$$

They can be easily proved by using the expressions of the third order derivatives of the potential $F(v)$ given in (3.26), (3.31), and similar expressions for the potential $\tilde{F}(\tilde{v})$.

Let us denote the two flat metrics (3.32), (3.33) associated with the Frobenius manifold structure with potential $\tilde{F}(\tilde{v})$ by

$$\langle du^i, du^j \tilde{\rangle}_1 = \delta_{ij} \tilde{\psi}_{i1}^{-2}, \tag{3.38}$$

$$\langle du^i, du^j \tilde{\rangle}_2 = \delta_{ij} u^i \tilde{\psi}_{i1}^{-2}. \tag{3.39}$$

Then these two sets of flat metrics associated with the potentials $F(v)$ and $\tilde{F}(\tilde{v})$ satisfy the relation (2.14), i.e.

$$\langle d\hat{v}^\alpha, d\hat{v}^\beta \tilde{\rangle}_1(\hat{v}) = \langle dv^\alpha, dv^\beta \rangle_1(v), \quad \langle d\hat{v}^\alpha, d\hat{v}^\beta \tilde{\rangle}_2(\hat{v}) = \langle dv^\alpha, dv^\beta \rangle_2(v).$$

Proof of Theorem 2.4 The first part of the theorem follows from the results of [33], and the second part of the theorem is proved by the arguments given above. The theorem is proved. □

Example 3.8 *Let us consider the flat exact bihamiltonian structure* (P_1, P_2, Z) *given by the following flat contravariant metrics*

$$(g_1^{ij}) = \begin{pmatrix} f^1 & 0 \\ 0 & f^2 \end{pmatrix}, \quad (g_2^{ij}) = \begin{pmatrix} u^1 f^1 & 0 \\ 0 & u^2 f^2 \end{pmatrix} \tag{3.40}$$

with

$$f^1 = \frac{u^1 - u^2}{2}, \quad f^2 = -\frac{u^1 - u^2}{2}.$$

The unity Z is given by

$$Z = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^2}.$$

It follows from (2.8) *that the rotation coefficients for the metric g_1 have the expressions*

$$\gamma_{12} = \gamma_{21} = \frac{i}{2(u^1 - u^2)}.$$

We have the following linearly independent solutions of the linear system (3.23), (3.24)*:*

$$\begin{pmatrix} \sqrt{u^1 - u^2} \\ i\sqrt{u^1 - u^2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{u^1 - u^2}} \\ -\frac{i}{\sqrt{u^1 - u^2}} \end{pmatrix},$$

they are eigenvectors of the linear operator

$$E = u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2}$$

with eigenvalues $\mu = \frac{1}{2}$ and $\mu = -\frac{1}{2}$ respectively.

Let us follow the procedure of Lemma 3.4 and Corollary 3.5 to construct quasi-homogeneous Frobenius manifold structures. We have the following two cases.

Case i) We first construct the canonical Frobenius manifold structure. To this end we take the fundamental system of solutions (3.25) *to be*

$$\Psi_1 = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{u^1 - u^2}} \\ -\frac{i}{\sqrt{u^1 - u^2}} \end{pmatrix}, \quad \Psi_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{u^1 - u^2} \\ i\sqrt{u^1 - u^2} \end{pmatrix}.$$

Then the matrix $(\eta_{\alpha\beta})$ and the local coordinates v^1, v^2 defined in Lemma 3.4 are given by

$$(\eta_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$v^1 = \frac{1}{2}(u^1 + u^2), \quad v^2 = 2 \log \left(\frac{u^1 - u^2}{4} \right).$$

The potential F of the Frobenius manifold structure has the expression

$$F = \frac{1}{2}(v^1)^2 v^2 + e^{v^2}.$$

In the local coordinates v^1, v^2, the Euler vector field E has the form

$$E = v^1 \frac{\partial}{\partial v^1} + 2 \frac{\partial}{\partial v^2}.$$

The potential F has the quasi-homogeneity property

$$E(F) = (3 - d)F + (v^1)^2, \quad d = -2\mu_1.$$

Here $\mu_1 = -\frac{1}{2}$ *is the eigenvalue of the eigenvector* Ψ_1 *of* E. *The flat metrics* (3.32), (3.33) *of the Frobenius manifold structure coincide with the ones given in* (3.40). *In the coordinates* v^1, v^2 *(which are called the flat coordinates of the Frobenius manifold) these metrics have expressions*

$$\left(\langle dv^\alpha, dv^\beta\rangle_1\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \left(\langle dv^\alpha, dv^\beta\rangle_2\right) = \begin{pmatrix} 2e^{v^2} & v^1 \\ v^1 & 2 \end{pmatrix}.$$

Case ii) We take the fundamental system of solutions (3.25) *to be*

$$\tilde{\Psi}_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{u^1 - u^2} \\ i\sqrt{u^1 - u^2} \end{pmatrix}, \quad \tilde{\Psi}_2 = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{u^1 - u^2}} \\ -\frac{i}{\sqrt{u^1 - u^2}} \end{pmatrix}.$$

Then the matrix $(\eta_{\alpha\beta})$ *and the local coordinates* v^1, v^2 *defined in Lemma 3.4, which we now denote by* $(\tilde{\eta}_{\alpha\beta})$ *and* $\tilde{v}^1, \tilde{v}^2$ *respectively, are given by*

$$(\tilde{\eta}_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\tilde{v}^1 = \frac{1}{2}(u^1 + u^2), \quad \tilde{v}^2 = \frac{1}{16}\left(u^1 - u^2\right)^2.$$

The potential $\tilde{F}$ *of the Frobenius manifold structure has the expression*

$$\tilde{F} = \frac{1}{2}(\tilde{v}^1)^2 \tilde{v}^2 + \frac{1}{2}(\tilde{v}^2)^2 \left(\log \tilde{v}^2 - \frac{3}{2}\right).$$

In the local coordinates $\tilde{v}^1, \tilde{v}^2$, *the Euler vector field* E *has the form*

$$E = \tilde{v}^1 \frac{\partial}{\partial \tilde{v}^1} + 2\tilde{v}^2 \frac{\partial}{\partial \tilde{v}^2}.$$

The potential $\tilde{F}$ *has the quasi-homogeneity property*

$$E(\tilde{F}) = (3 - \tilde{d})\tilde{F} + (\tilde{v}^2)^2, \quad \tilde{d} = -2\tilde{\mu}_1.$$

Here $\tilde{\mu}_1 = \frac{1}{2}$ *is the eigenvalue of the eigenvector* $\tilde{\Psi}_1$ *of* E. *The flat metrics* (3.32), (3.33) *are given by*

$$\left(\widetilde{\langle du^i, du^j\rangle_1}\right) = \begin{pmatrix} \frac{8}{u^1 - u^2} & 0 \\ 0 & -\frac{8}{u^1 - u^2} \end{pmatrix}, \quad \left(\widetilde{\langle du^i, du^j\rangle_2}\right) = \begin{pmatrix} \frac{8u^1}{u^1 - u^2} & 0 \\ 0 & -\frac{8u^2}{u^1 - u^2} \end{pmatrix}.$$

In the flat coordinates $\tilde{v}^1, \tilde{v}^2$ *these metrics have the expressions*

$$\left(\langle d\tilde{v}^\alpha, d\tilde{v}^\beta\rangle_1\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \left(\langle d\tilde{v}^\alpha, d\tilde{v}^\beta\rangle_2\right) = \begin{pmatrix} 2 & \tilde{v}^1 \\ \tilde{v}^1 & 2\tilde{v}^2 \end{pmatrix}.$$

The above two Frobenius manifold structures are related by the Legendre transformation (3.34)–(3.37) *with the matrix* A *given by*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

More generally, if we choose the fundamental system of solutions (3.25) *to be of the form*

$$\left(\tilde{\Psi}_1, \tilde{\Psi}_2\right) = \begin{pmatrix} \frac{1}{\sqrt{u^1-u^2}} & \sqrt{u^1-u^2} \\ -\frac{i}{\sqrt{u^1-u^2}} & i\sqrt{u^1-u^2} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

with $a_1 \neq 0, a_2 \neq 0$ *and* $a_1 b_2 - a_2 b_1 \neq 0$*, then we get the Frobenius manifold structure with the potential*

$$\begin{aligned} \tilde{F} =& \frac{2}{3} a_1 a_2 v^3 + (a_1 b_2 + a_2 b_1) v^2 w + 2 b_1 b_2 v w^2 \\ & - \frac{b_1^2 (a_2 b_1 - 3 a_1 b_2)}{3 a_1^2} w^3 - \frac{a_1^4}{192 a_2^2} (1 + Q(w))^3 - \frac{a_1^4}{128 a_2^2} (1 + Q(w))^2. \end{aligned}$$

Here we denote $\tilde{v}^1, \tilde{v}^2$ *by* v *and* w *respectively, they are related to the canonical coordinates* u^1, u^2 *by the formulae*

$$v = \frac{1}{2}(u^1 + u^2) + \frac{a_2 b_2}{\beta}(u^1 - u^2)^2 - \frac{2 a_1 b_1}{\beta} \log\left(\frac{u^1 - u^2}{4}\right), \tag{3.41}$$

$$w = -\frac{a_2^2}{\beta}(u^1 - u^2)^2 + \frac{2 a_1^2}{\beta} \log\left(\frac{u^1 - u^2}{4}\right) \tag{3.42}$$

with $\beta = 4(a_1 b_2 - a_2 b_1)$*. The function* $Q(w)$ *is defined by the Lambert-*W *function which is the inverse of the function* $f(z) = z e^z$*, more precisely, we have*

$$Q(w) = W(\alpha e^{\beta w / a_1^2}), \quad \alpha = -16 \frac{a_2^2}{a_1^2}.$$

In terms of the Lambert-W *function, the relation* (3.41), (3.42) *can be represented as*

$$(u^1 - u^2)^2 = -\frac{a_1^2}{a_2^2} Q(w), \tag{3.43}$$

$$u^1 + u^2 = 2v + \frac{2 b_1}{a_1} w + \frac{a_1}{2 a_2} Q(w). \tag{3.44}$$

The two flat metrics that are associated with the Frobenius manifold are given by

$$\left(\langle d\tilde{v}^\alpha, d\tilde{v}^\beta \rangle\tilde{}_1\right) = A^{-1} G_1 \left(A^{-1}\right)^t, \quad \left(\langle d\tilde{v}^\alpha, d\tilde{v}^\beta \rangle\tilde{}_2\right) = A^{-1} G_2 \left(A^{-1}\right)^t.$$

Here

$$A=\frac{1}{\sqrt{2}}\begin{pmatrix}a_1 & b_1\\ 4a_2 & 4b_2\end{pmatrix},\quad G_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$G_2=\begin{pmatrix}-\frac{a_1^2}{8a_2^2}Q(w) & v+\frac{b_1}{a_1}w+\frac{a_1}{4a_2}Q(w)\\ v+\frac{b_1}{a_1}w+\frac{a_1}{4a_2}Q(w) & 2\end{pmatrix}.$$

This Frobenius manifold structure is related to the one given in Case i) by the Legendre transformation (3.34)–(3.37) *with the above matrix* A*.*

One can recover the above two special cases from the general case. By taking

$$\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}=\begin{pmatrix}\sqrt{2} & 0\\ t & \frac{1}{2\sqrt{2}}\end{pmatrix},$$

we get the potential

$$\tilde{F}=-\frac{5}{96}t^{-2}+\left(\frac{1}{2}v^2w+e^w\right)+o(t). \tag{3.45}$$

Note that the potential of a Frobenius manifold is defined up to the addition of quadratic functions of flat coordinates, so we can omit the leading term of (3.45)*. Then by taking the limit* $t\to 0$ *we obtain the Case i).*

Similarly, if we take

$$\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}=\begin{pmatrix}\sqrt{-2t} & \sqrt{2}\\ \frac{1}{2\sqrt{2}} & 0\end{pmatrix},$$

then we have

$$\tilde{F}=\left(\frac{1}{2}v^2w+\frac{1}{2}w^2\left(\log w-\frac{3}{2}\right)\right)+o(t), \tag{3.46}$$

which gives the Case ii) directly. Here we used the following asymptotic behavior of the Lambert W-function:

$$W\left(\frac{1}{t}e^{\frac{w}{t}}\right)=\frac{w}{t}-\log w+o(t).$$

4 The principal hierarchy and its tau structure

In this section, we study the bihamiltonian integrable hierarchies and their tau structures associated with a given flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1,P_2;Z)$, with the flat metrics g_1, g_2 of the manifold M^n. We fix a system of canonical coordinates $\{u^i\,|\,i=1,\dots,n\}$ of the semisimple bihamiltonian structure. In this coordinate system

$$Z=\int\left(\sum_{i=1}^n Z^i\theta_i\right)=\int\left(\sum_{i=1}^n\theta_i\right),\quad D_Z=\sum_{i=1}^n\frac{\partial}{\partial u^i}. \tag{4.1}$$

We also fix a system of flat coordinates $\{v^\alpha \,|\, \alpha = 1, \dots, n\}$ of the metric g_1 so that in this system of coordinates

$$Z = \int \left(Z^\alpha \bar\theta_\alpha \right) = \int (\bar\theta_1), \quad D_Z = \frac{\partial}{\partial v^1} \tag{4.2}$$

and $g_1^{\alpha\beta} = \eta^{\alpha\beta}$. Here and in what follows we use the symbols θ_i and $\bar\theta_i$ to denote the super variables that are dual to the canonical and flat coordinates respectively. We will use Latin and Greek alphabets to index the symbols that are related to the canonical and flat coordinates respectively.

Denote

$$d_a = \mathrm{ad}_{P_a} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}, \quad a = 1, 2, \tag{4.3}$$

$$\delta = ad_Z : \hat{\mathcal{F}} \to \hat{\mathcal{F}}. \tag{4.4}$$

The property $[P_a, P_b] = 0$ $(a, b = 1, 2)$ implies that

$$d_1^2 = 0, \quad d_1 d_2 + d_2 d_1 = 0, \quad d_2^2 = 0,$$

so we can define the following Hamiltonian cohomologies and bihamiltonian cohomologies respectively:

$$H_d^p(\hat{\mathcal{F}}, P_a) = \frac{\hat{\mathcal{F}}_d^p \cap \mathrm{Ker}(d_a)}{\hat{\mathcal{F}}_d^p \cap \mathrm{Im}(d_a)}, \quad a = 1, 2,$$

$$BH_d^p(\hat{\mathcal{F}}, P_1, P_2) = \frac{\hat{\mathcal{F}}_d^p \cap \mathrm{Ker}(d_1) \cap \mathrm{Ker}(d_2)}{\hat{\mathcal{F}}_d^p \cap \mathrm{Im}(d_1 d_2)}.$$

The next two theorems will be frequently used in the following sections.

Theorem 4.1 *(see Theorem 3.9 of [25] and references therein) Suppose P is a Hamiltonian structure of hydrodynamic type, then we have*

$$H_{d\ge 1}^p(\hat{\mathcal{F}}, P) \cong 0, \quad \textit{for } p \ge 0.$$

Theorem 4.2 *(see [5, 10, 23]) Suppose (P_1, P_2) is a semisimple bihamiltonian structure of hydrodynamic type, then we have*

$$BH_{d\ge 2}^1(\hat{\mathcal{F}}, P_1, P_2) \cong 0,$$
$$BH_{d\ge 4}^2(\hat{\mathcal{F}}, P_1, P_2) \cong 0,$$
$$BH_{d\ge 5}^3(\hat{\mathcal{F}}, P_1, P_2) \cong 0.$$

Definition 4.3

i) Define $\mathcal{H} := \mathrm{Ker}(d_2 \circ d_1) \cap \hat{\mathcal{F}}^0$, whose elements are called bihamiltonian conserved quantities.

ii) Define $\mathcal{X} := \mathrm{Ker}(d_1) \cap \mathrm{Ker}(d_2) \cap \hat{\mathcal{F}}^1$*, whose elements are called bihamiltonian vector fields.*

Let $H_1 \in \mathcal{H}$ be a bihamiltonian conserved quantity, then the Hamiltonian vector field (w.r.t. the first Hamitonian structure P_1)

$$X = -d_1(H_1) = -[P_1, H_1]$$

defines a Hamiltonian system. The correspondence between a Hamiltonian vector field and a Hamiltonian system is given in (3.7), (3.8). By using the condition

$$[P_2, X] = -[P_2, [P_1, H_1]] = 0$$

and triviality of the Hamiltonian cohomology $H^1_{d\geq 1}(\hat{\mathcal{F}}, P_2)$, we know the existence of a Hamiltonian $H_2 \in \hat{\mathcal{F}}^0$ such that

$$X = -d_2(H_2) = -[P_2, H_2],$$

i.e., X is also a Hamiltonian vector field w.r.t. the second Hamitonian structure P_2. So X is a bihamiltonian vector field, and the associated system of evolutionary PDEs is a bihamiltonian system. The following lemma shows that such a bihamiltonian system is also of hydrodynamic type

Lemma 4.4 $\mathcal{H} \subset \hat{\mathcal{F}}^0_0 = \mathcal{F}_0$*, and* $\mathcal{X} \subset \hat{\mathcal{F}}^1_1$*.*

Proof If $H_1 \in \mathcal{H}$ then, as we show above, there exists $H_2 \in \hat{\mathcal{F}}^0 = \mathcal{F}$ such that $[P_1, H_1] = [P_2, H_2]$. By using Lemma 4.1 of [10], we know that $H_1 \in \hat{\mathcal{F}}^0_0$, so we proved that $\mathcal{H} \subset \hat{\mathcal{F}}^0_0$.

To prove that $\mathcal{X} \subset \hat{\mathcal{F}}^1_1$, let us note that the space $\mathcal{X}$ is actually the bihamiltonian cohomology $BH^1(\hat{\mathcal{F}}, P_1, P_2)$. If $X = \int(X^i\theta_i) \in \hat{\mathcal{F}}^1_0$ satisfies $[P_1, X] = [P_2, X] = 0$, then we have

$$\nabla_j X^i = 0, \quad \hat{\nabla}_j X^i = 0.$$

Recall that ∇, $\hat{\nabla}$ are the Levi-Civita connections of the metrics g_1, g_2 associated with P_1, P_2 respectively,

$$\nabla_i = \nabla_{\frac{\partial}{\partial u^i}}, \quad \hat{\nabla}_i = \hat{\nabla}_{\frac{\partial}{\partial u^i}} \tag{4.5}$$

and $u^1, \ldots, u^n$ are the canonical coordinates of (P_1, P_2). It follows from the explicit expressions of g^{ij}_a, $\Gamma^{ij}_{a,k}$ that $X^i = 0$ and so we have $BH^1_0(\hat{\mathcal{F}}, P_1, P_2) \cong 0$.

On the other hand, if $X \in BH^1_{\geq 2}(\hat{\mathcal{F}}, P_1, P_2)$, then there exist $I, J \in \mathcal{F}_{\geq 1}$ such that

$$X = [P_1, I] = [P_2, J].$$

According to Lemma 4.1 of [10], we can choose densities of the functionals I, J such that $I, J \in \mathcal{F}_0$, so we have $X \in BH^1_1(\hat{\mathcal{F}}, P_1, P_2)$, from which it follows that $X = 0$.

The lemma is proved. □

Corollary 4.5 *i) For any $X, Y \in \mathcal{X}$, we have $[X, Y] = 0$;*

ii) For any $X \in \mathcal{X}$, $H \in \mathcal{H}$, we have $[X, H] = 0$;

iii) For any $H, K \in \mathcal{H}$, we have $\{H, K\}_{P_a} := [[P_a, H], K] = 0, a = 1, 2$. Here $\{\cdot\,,\,\cdot\}_{P_a}$ is the Poisson bracket defined by the Hamiltonian operator P_a for local functionals.

Proof i) If $X, Y \in \mathcal{X}$, then the above lemma shows that $\deg X = \deg Y = 1$, so $\deg[X, Y] = 2$. Here we use the standard gradation defined in (3.2). But we also have $[X, Y] \in \mathcal{X}$, so $[X, Y] = 0$.

ii) If $X \in \mathcal{X}$, $H \in \mathcal{H}$, then $K = [X, H] \in \mathcal{H}$. But $\deg X = 1$, $\deg H = 0$, so $\deg K = 1$, which implies $K = 0$.

iii) Take $X = [P_a, H]$, then by applying ii) we obtain $\{H, K\}_{P_a} = 0$. □

Lemma 4.6 *We have the following isomorphism*

$$\mathcal{X} \cong \mathcal{H}/\mathcal{V}, \tag{4.6}$$

where $\mathcal{V} = \mathrm{Ker}(d_1) \cap \hat{\mathcal{F}}^0$ is the space of Casimirs of P_1. Moreover, we have

i) A local functional $H \in \hat{\mathcal{F}}^0$ is a bihamiltonian conserved quantity if and only if one can choose its density h so that $h \in \mathcal{A}_0$ and satisfies the condition

$$\nabla_i \nabla_j h = 0, \quad i \neq j, \tag{4.7}$$

where $\nabla_i = \nabla_{\frac{\partial}{\partial u^i}}$ are defined as in (4.5).

ii) A vector field $X \in \hat{\mathcal{F}}^1$ is a bihamiltonian vector field if and only if it has the form

$$X = \int \left(\sum_{i=1}^{n} A^i(u) u^{i,1} \theta_i \right),$$

where $A^i(u)$ satisfy the following equations:

$$\frac{\partial A^i}{\partial u^j} = \Gamma^i_{ij} \left(A^j - A^i \right), \quad \text{for } j \neq i, \tag{4.8}$$

and Γ^i_{ij} are the Christoffel coefficients of the Levi-Civita connection of g_1.

Proof Consider the map $\phi = d_1|_{\mathcal{H}} : \mathcal{H} \to \mathcal{X}$. It is easy to see that ϕ is well-defined, and $\mathrm{Ker}(\phi) = \mathcal{V}$. It follows from Theorem 4.1 that

$$H^1_{\geq 1}(\hat{\mathcal{F}}, P_a) \cong 0, \quad a = 1, 2,$$

so for a given $X \in BH^1_{\geq 1}(\hat{\mathcal{F}}, P_1, P_2)$, there exists $H, G \in \mathcal{F}$ such that

$$X = [P_1, H] = [P_2, G].$$

From the second equality we also know that $H \in \mathcal{H}$. So the map ϕ is surjective and we proved that the map ϕ induces the isomorphism (4.6).

Let $H \in \mathcal{H}$, then it yields a bihamiltonian vector field $X = -[P_1, H]$. According to Lemma 4.4, $H \in \mathcal{F}_0$, $X \in \hat{\mathcal{F}}_1^1$. So we can choose the density of $H = \int(h)$ such that $h \in \mathcal{A}_0$, and

$$X = \int \left(\sum_{i,j=1}^{n} X_j^i u^{j,1} \theta_i \right),$$

where $X_j^i = \nabla^i \nabla_j h$, and $\nabla^i = \sum_{k=1}^n g_1^{ik} \nabla_k$. The conditions $[P_1, X] = 0$ and $[P_2, X] = 0$ read

$$\sum_{j=1}^{n} g_1^{ij} X_j^k = \sum_{j=1}^{n} g_1^{kj} X_j^i, \quad \nabla_k X_j^i = \nabla_j X_k^i, \tag{4.9}$$

$$\sum_{j=1}^{n} g_2^{ij} X_j^k = \sum_{j=1}^{n} g_2^{kj} X_j^i, \quad \hat{\nabla}_k X_j^i = \hat{\nabla}_j X_k^i, \tag{4.10}$$

The diagonal form (2.7) of g_1 and g_2 and the first equations of (4.9) and (4.10) imply that

$$(u^i - u^j) f^j X_j^i = 0,$$

so X_j^i is diagonal. Then the second equation of (4.9) gives the desired equation (4.8). Let $\hat{\Gamma}_{ij}^i$ be the Christoffel coefficients of the Levi-Civita connection of g_2, then one can show that for $i \neq j$

$$\hat{\Gamma}_{ij}^i = \Gamma_{ij}^i = \frac{1}{2f_i} \frac{\partial f_i}{\partial u^j},$$

so the second equation of (4.10) also gives (4.8). The lemma is proved. □

Lemma 4.7 *We have $\delta(\mathcal{H}) \subseteq \mathcal{H}$. Denote $\varphi = \delta|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$, then φ is surjective and* $\dim \operatorname{Ker}(\varphi) = n$.

Proof Let $H \in \mathcal{H}$, then we have $[P_2, [P_1, H]] = 0$. From the graded Jacobi identity it follows that

$$\begin{aligned}[P_2, [P_1, [Z, H]]] =& [P_2, -[[H, P_1], Z] - [[P_1, Z], H]] \\ =& [[P_2, Z], [P_1, H]] + [[P_2, [P_1, H]], Z] = [P_1, [P_1, H]] = 0,\end{aligned}$$

so we have $\delta(\mathcal{H}) \subseteq \mathcal{H}$.

Suppose $H = \int(h) \in \mathcal{H}$, then from Lemma 4.6 it follows that the density h can be chosen to belong to $\mathcal{A}_0$ and $\nabla_i \nabla_j h = 0$ for $i \neq j$. If $\varphi(H) = 0$, then

$$\sum_{i=1}^{n} \nabla_i h = 0,$$

so we have $\nabla_i\nabla_j h = 0$ for any i, j, i.e. $h \in \mathcal{V}$. Thus h can be represented as

$$h = \sum_{\alpha=1}^{n} c_\alpha v^\alpha + c_0,$$

where $c_0, c_1, \ldots, c_n$ are some constants, and v^α are the flat coordinates of g_1. From the condition $D_Z = \frac{\partial}{\partial v^1}$ it follows that $c_1 = 0$, so $\dim \mathrm{Ker}(\varphi) = n$.

To prove that φ is surjective, we need to show that for any $g \in \mathcal{A}_0$ satisfying $\nabla_i\nabla_j g = 0$ $(i \neq j)$, there exists $h \in \mathcal{A}_0$ such that

$$\nabla_i\nabla_j h = 0 \ (i \neq j), \quad \sum_{i=1}^{n} \nabla_i h = g. \tag{4.11}$$

Denote $\xi_j = \nabla_j h$, then by using the identity (see (3.18))

$$\sum_{j=1}^{n} \Gamma^k_{ij} = 0$$

we know that the above equations imply that

$$\nabla_i \xi_j = \begin{cases} 0, & i \neq j; \\ \nabla_i g, & i = j. \end{cases} \tag{4.12}$$

Here, as in the previous lemma, we denote by Γ^k_{ij} the Christoffel coefficients of the Levi-Civita connection of g_1.

Let us first prove that the functions ζ_{ij} defined by the l.h.s. of (4.12) satisfy the equations

$$\nabla_k \zeta_{ij} = \nabla_i \zeta_{kj}. \tag{4.13}$$

The left hand side of this equation can be written as

$$\begin{aligned} \nabla_k \zeta_{ij} &= \zeta_{ij,k} - \sum_{l=1}^{n} \left(\Gamma^l_{ki}\zeta_{lj} + \Gamma^l_{kj}\zeta_{il} \right) \\ &= \zeta_{ij,k} - \Gamma^j_{ki}\zeta_{jj} - \Gamma^i_{kj}\zeta_{ii}. \end{aligned}$$

Since $\Gamma^j_{ki} = \Gamma^j_{ik}$, in order to prove the identity (4.13) we only need to show that

$$\zeta_{ij,k} - \Gamma^i_{kj}\zeta_{ii} = \zeta_{kj,i} - \Gamma^k_{ij}\zeta_{kk}.$$

When $i = j = k$ the validity of the equalities is trivial. In the case when i, j, k are distinct, the above equations also hold true, since from (3.12) and (4.12) we have

$$\Gamma^i_{kj} = \Gamma^k_{ij} = 0, \quad \zeta_{ij} = \zeta_{kj} = 0.$$

So we only need to consider the case when $i = j$ and $i \neq k$. In this case, the above equations become

$$(\nabla_i g)_{,k} - \Gamma^i_{ki}\nabla_i g + \Gamma^k_{ii}\nabla_k g = 0.$$

On the other hand, the function g satisfies $\nabla_k \nabla_i g = 0$ $(k \neq i)$, which implies

$$(\nabla_i g)_{,k} = \Gamma^k_{ki} \nabla_k g + \Gamma^i_{ki} \nabla_i g,$$

here we use again the fact that $\Gamma^k_{ij} = 0$ if i, j, k are distinct. So we only need to show

$$\Gamma^k_{ki} + \Gamma^k_{ii} = 0, \quad i \neq k,$$

which is equivalent to the flatness condition (3.17), (3.18).

The equations (4.13) imply that there exist solutions $\xi_1, \dots, \xi_n$ of the equations (4.12). Since ζ_{ij} are symmetric with respect to the indices i, j, we can find a function $h \in \mathcal{A}_0$ so that $\xi_i = \nabla_i h$. It follows from (3.18) and (4.12) that $\sum_{i=1}^n \nabla_i h - g$ is a constant, thus by adjusting the function h by adding $c\, v^1$ for a certain constant c we prove the existence of $h \in \mathcal{A}_0$ satisfying the equations given in (4.11). The lemma is proved. $\square$

The space $\mathcal{H}$ is too big, so we restrict our interest to a "dense" (in a certain sense) subspace of $\mathcal{H}$.

Definition 4.8 *Define $\mathcal{H}^{(-1)} = \mathcal{V}$, $\mathcal{H}^{(p)} = \varphi^{-1}\left(\mathcal{H}^{(p-1)}\right)$, and*

$$\mathcal{H}^{(\infty)} = \bigcup_{p \geq -1} \mathcal{H}^{(p)}.$$

Remark 4.9 *The action of φ is just $\frac{\partial}{\partial v^1}$, so the space $\mathcal{H}^{(\infty)}$ is a polynomial ring in the indeterminate v^1. It is indeed dense in the space of smooth functions in v^1 with respect to an appropriate topology.*

It is easy to see that $\delta(\mathcal{V}) \subseteq \mathcal{V}$, so

$$\mathcal{V} = \mathcal{H}^{(-1)} \subseteq \mathcal{H}^{(0)} \subseteq \cdots \subseteq \mathcal{H}^{(\infty)}.$$

Note that $\dim \mathcal{H}^{(-1)} = n + 1$, and

$$\dim \mathcal{H}^{(p)} = \dim \mathcal{H}^{(p-1)} + \dim \operatorname{Ker}(\varphi) = \dim \mathcal{H}^{(p-1)} + n,$$

so we have $\dim \mathcal{H}^{(p)} = n(p+2) + 1$.

Suppose the collection of functions

$$\{h_{\alpha,p} \in \mathcal{A}_0 \mid \alpha = 1, \dots, n;\ p = 0, 1, 2, \dots\}$$

is a calibration of $(P_1, P_2; Z)$ (see Definition 2.6). Then it is easy to see that $h_{\alpha,p} \in \mathcal{H}^{(p)}$, and when $p \geq 0$, they form a basis of $\mathcal{H}^{(p)}/\mathcal{H}^{(p-1)}$. When $p = -1$, $\mathcal{H}^{(-1)} = \mathcal{V}$ contains not only $h_{\alpha,0} = v_\alpha$ but also a trivial functional $\int(1)$, they form a basis of $\mathcal{H}^{(-1)}$. Let us rephrase the conditions that must be satisfied by the functions $h_{\alpha,p}$ of a calibration as follows:

1. $H_{\alpha,p} = \displaystyle\int (h_{\alpha,p}) \in \mathcal{H},$ (4.14)
2. $h_{\alpha,-1} = v_\alpha, \quad D_Z(h_{\alpha,p}) = h_{\alpha,p-1}\ (p \geq 0),$ (4.15)
3. The normalization condition (2.23). (4.16)

Remark 4.10 *Let us describe the ambiguity of the choice of calibration of $(P_1, P_2; Z)$. Note that*

$$1, h_{\alpha,q}(\alpha = 1, \dots, n; q = -1, \dots, p)$$

form a basis of $\mathcal{H}^{(p)}$, so if there is another calibration $\{\tilde{h}_{\alpha,q}\}$, then we have

$$\tilde{h}_{\alpha,q} = \sum_{r=-1}^{q} c_{\alpha,q}^{\beta,r} h_{\beta,r} + b_{\alpha,q}.$$

The conditions (4.14)-(4.16) imply that

$$\begin{aligned} &c_{\alpha,-1}^{\beta,-1} = \delta_\alpha^\beta, \quad c_{\alpha,0}^{\beta,-1}\eta_{\beta 1} = 0, \quad b_{\alpha,-1} = 0, \\ &c_{\alpha,q}^{\beta,r} = c_{\alpha,q+1}^{\beta,r+1}, \quad b_{\alpha,q} = c_{\alpha,q+1}^{\beta,-1}\eta_{\beta,1}. \end{aligned}$$

Denote $(C_p)_\alpha^\beta = c_{\alpha,p-1}^{\beta,-1}$ for $p \geq 0$, then we have

$$c_{\alpha,q}^{\beta,r} = (C_{q-r})_\alpha^\beta, \quad b_{\alpha,q} = (C_{q+2})_\alpha^\beta \eta_{\beta 1}, \tag{4.17}$$

and

$$(C_0)_\alpha^\beta = \delta_\alpha^\beta, \quad (C_1)_\alpha^\beta \eta_{\beta 1} = 0. \tag{4.18}$$

Introduce the generating functions

$$h_\alpha(z) = \sum_{q \geq -1} h_{\alpha,q} z^{q+1}, \quad C_\alpha^\beta(z) = \sum_{p \geq 0} (C_p)_\alpha^\beta z^p,$$

then we have

$$\tilde{h}_\alpha(z) = C_\alpha^\beta(z) h_\beta(z) + \frac{C_\alpha^\beta(z) - \delta_\alpha^\beta}{z} \eta_{\beta 1}. \tag{4.19}$$

The principal hierarchy (2.20) of the calibration $\{h_{\alpha,p}\}$ corresponds to the following bihamiltonian vector fields

$$X_{\alpha,p} = -[P_1, H_{\alpha,p}] = \int \left(\eta^{\gamma\lambda} \partial \left(\frac{\delta H_{\alpha,p}}{\delta v^\lambda} \right) \bar{\theta}_\gamma \right), \quad p \geq 0. \tag{4.20}$$

It can be represented in terms of the associated evolutionary vector field $D_{X_{\alpha,p}}$ (see Remark 3.1) as follows:

$$\frac{\partial v^\gamma}{\partial t^{\alpha,p}} = D_{X_{\alpha,p}}(v^\gamma), \quad \alpha = 1, \dots, n, \ p \geq 0. \tag{4.21}$$

Now let us proceed to construct a calibration for the canonical Frobenius manifold structure $F(v)$ of $(P_1, P_2; Z)$ defined in Sec. 3. Following the construction of [7], we first define the functions

$$\theta_{\alpha,0}(v) = v_\alpha = \eta_{\alpha\gamma} v^\gamma, \quad \theta_{\alpha,1}(v) = \frac{\partial F(v)}{\partial v^\alpha}, \quad \alpha = 1, \dots, n,$$

where F is introduced in Corollary 3.5. By adding to the function $F(v)$ a certain quadratic term in $v^1, \dots, v^n$, if needed, we can assume that

$$\frac{\partial^2 F(v)}{\partial v^1 \partial v^\alpha} = \eta_{\alpha\gamma} v^\gamma = v_\alpha.$$

Thus we have the following relation:

$$D_Z \theta_{\alpha,1} = \frac{\partial \theta_{\alpha,1}}{\partial v^1} = \theta_{\alpha,0}.$$

The functions $\theta_{\alpha,p}(v)$ for $p \geq 2$ can be defined recursively by using the following relations:

$$\frac{\partial^2 \theta_{\gamma,p+1}(v)}{\partial v^\alpha \partial v^\beta} = c_{\alpha\beta\xi} \eta^{\xi\zeta} \frac{\partial \theta_{\gamma,p}(v)}{\partial v^\zeta}, \quad \alpha, \beta, \gamma = 1, \dots, n. \tag{4.22}$$

The existence of solutions of these recursion relations is ensured by the associativity equations (3.28). We can require, as it is done in [7], that these functions also satisfy the following normalization conditions

$$\frac{\partial \theta_\alpha(v;z)}{\partial v^\xi} \eta^{\xi\zeta} \frac{\partial \theta_\beta(v;-z)}{\partial v^\zeta} = \eta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n.$$

Here $\theta_\alpha(v;z) = \sum_{p\geq 0} \theta_{\alpha,p}(v) z^p$. Now we define the functions $h_{\alpha,p}(v)$ so that their generating functions $h_\alpha(v;z) = \sum_{p\geq -1} h_{\alpha,p}(v) z^{p+1}$ satisfy the following defining relations

$$h_\alpha(v;z) = \frac{1}{z} \frac{\partial \theta_\alpha(v;z)}{\partial v^1} - \frac{1}{z} \eta_{\alpha 1}. \tag{4.23}$$

Then these functions also satisfy the normalization condition

$$\frac{\partial h_\alpha(v;z)}{\partial v^\xi} \eta^{\xi\zeta} \frac{\partial h_\beta(v;-z)}{\partial v^\zeta} = \eta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n.$$

By adding, if needed, a certain linear in $v^1, \dots v^n$ term to the functions $F(v)$ we also have the relations

$$h_{\alpha,0}(v) = \frac{\partial F(v)}{\partial v^\alpha}, \quad \alpha = 1, \dots, n. \tag{4.24}$$

For the above constructed functions $\{h_{\alpha,p}\}$, we have the Hamiltonians $H_{\alpha,p} = \int h_{\alpha,p}$, the Hamiltonian vector fields $X_{\alpha,p}$ defined in (4.20) and the associated hierarchy of first the order quasilinear evolutionary PDEs (4.21).

Lemma 4.11 *The functions $h_{\alpha,p}$ and the associated local functionals that we constructed above have the following properties:*

i) $H_{\alpha,p} = \int (h_{\alpha,p}) \in \mathcal{H}^{(p)}$, *ii)* $h_{\alpha,-1} = v_\alpha$, $D_Z(h_{\alpha,p}) = h_{\alpha,p-1}$ $(p \geq 0)$.

Proof According to the definition (4.23) of $h_{\gamma,p}$, we have

$$h_{\gamma,p} = \frac{\partial \theta_{\gamma,p+2}}{\partial v^1} = \sum_{k=1}^{n} \frac{\partial \theta_{\gamma,p+2}}{\partial u^k}.$$

We only need to prove that $H_{\gamma,p} = \int (h_{\gamma,p}) \in \mathcal{H}$, that is $\nabla_i \nabla_j h_{\gamma,p} = 0$ for $i \neq j$. The other properties are easy to verify.

The condition $\nabla_i \nabla_j h_{\gamma,p} = 0$ for $i \neq j$ reads

$$\frac{\partial^2 h_{\gamma,p}}{\partial u^i \partial u^j} = \sum_{l=1}^{n} \Gamma^l_{ij} \frac{\partial h_{\gamma,p}}{\partial u^l},$$

which is equivalent to

$$\frac{\partial^2 \theta_{\gamma,p+1}}{\partial u^i \partial u^j} = \sum_{k,l=1}^{n} \Gamma^l_{ij} \frac{\partial^2 \theta_{\gamma,p+2}}{\partial u^k \partial u^l}. \tag{4.25}$$

The recursion relation (4.22) of $\theta_{\alpha,p}$ has the following form in the canonical coordinates:

$$\frac{\partial^2 \theta_{\gamma,p+1}}{\partial u^i \partial u^j} = \delta_{ij} \frac{\partial \theta_{\gamma,p}}{\partial u^i} + \frac{\partial (\psi_i^\alpha \psi_{i1})}{\partial u^j} \frac{\partial \theta_{\gamma,p+1}}{\partial v^\alpha}.$$

Note that $i \neq j$ in the equations (4.25), so its left hand side reads

$$\frac{\partial (\psi_i^\alpha \psi_{i1})}{\partial u^j} \frac{\partial \theta_{\gamma,p+1}}{\partial v^\alpha} = \gamma_{ij} \left(\psi_{i1} \psi_j^\alpha + \psi_{j1} \psi_i^\alpha \right) \frac{\partial \theta_{\gamma,p+1}}{\partial v^\alpha}.$$

The right hand side of (4.25) then reads

$$\sum_{k,l=1}^{n} \Gamma^l_{ij} \frac{\partial^2 \theta_{\gamma,p+2}}{\partial u^k \partial u^l} = \sum_{k,l=1}^{n} \Gamma^l_{ij} \left(\delta_{kl} \frac{\partial \theta_{\gamma,p+2}}{\partial u^k} + \frac{\partial (\psi_k^\alpha \psi_{k1})}{\partial u^l} \frac{\partial \theta_{\gamma,p+2}}{\partial v^\alpha} \right). \tag{4.26}$$

Note that

$$\sum_{k=1}^{n} \psi_k^\alpha \psi_{k1} = \delta_1^\alpha$$

is a constant, so the second summation in the r.h.s. of (4.26) vanishes. In the first summation, we have

$$\Gamma^l_{ij} = \gamma_{ij} \left(\delta_{il} \frac{\psi_{j1}}{\psi_{i1}} + \delta_{jl} \frac{\psi_{i1}}{\psi_{j1}} \right), \quad \text{for } i \neq j,$$

and

$$\frac{\partial \theta_{\gamma,p+2}}{\partial u^k} = \frac{\partial v^\alpha}{\partial u^k} \frac{\partial \theta_{\gamma,p+2}}{\partial v^\alpha} = \psi_k^\alpha \psi_{k1} \frac{\partial \theta_{\gamma,p+2}}{\partial v^\alpha},$$

which leads to the validity of the equations (4.25). The lemma is proved. □

Lemma 4.12 *The first flow* $\frac{\partial}{\partial t^{1,0}}$ *is given by the translation along the spatial variable* x*, i.e.*

$$\frac{\partial}{\partial t^{1,0}} = \partial.$$

Proof From our definition (4.20), (4.21) of the evolutionary vector fields we have

$$\begin{aligned}\frac{\partial v^\alpha}{\partial t^{1,0}} &= \eta^{\alpha\beta}\partial\frac{\partial h_{1,0}}{\partial v^\beta} = \eta^{\alpha\beta}\partial\frac{\partial^2\theta_{1,2}}{\partial v^\beta\partial v^1}\\ &=\eta^{\alpha\beta}\partial\frac{\partial\theta_{1,1}}{\partial v^\beta} = \eta^{\alpha\beta}\frac{\partial^2\theta_{1,1}}{\partial v^\beta\partial v^\gamma}v_x^\gamma = v_x^\alpha.\end{aligned}$$

Here we use the recursion relation (4.22). The lemma is proved. □

From Lemma 4.11 and Lemma 4.12 we have the following proposition.

Proposition 4.13 *The collection of functions*

$$\{h_{\alpha,p}(v)\,|\,\alpha = 1,\dots,n; p = 0,1,2,\dots\}$$

that we constructed above is a calibration of the flat exact bihamiltonian structure $(P_1, P_2; Z)$*.*

Let us proceed to prove Proposition 2.5 which shows that the functions $h_{\alpha,p}$ of a calibration of $(P_1, P_2; Z)$ satisfy the tau symmetry condition, and the associated principal hierarchy (2.20) possesses Galilean symmetry.

Proof of Proposition 2.5 We first note that the principal hierarchy (2.20) associated with a calibration $\{h_{\alpha,p}\}$ can be written, in the canonical coordinates, as follows:

$$\frac{\partial u^i}{\partial t^{\beta,q}} = \sum_{j=1}^n \nabla^i\nabla_j h_{\beta,q}u^{j,1} = f^i\nabla_i\nabla_i h_{\beta,q}u^{i,1}.$$

So by using the property (4.15) of a calibration we have

$$\begin{aligned}\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} &= \sum_{i=1}^n \frac{\partial u^i}{\partial t^{\beta,q}}\frac{\partial h_{\alpha,p-1}}{\partial u^i}\\ &=\sum_{i=1}^n\left(\sum_{j=1}^n \nabla^i\nabla_j h_{\beta,q}u^{j,1}\right)\left(\sum_{k=1}^n\frac{\partial^2 h_{\alpha,p}}{\partial u^i\partial u^k}\right)\\ &=\sum_{i=1}^n\left(f^i\nabla_i\nabla_i h_{\beta,q}u^{i,1}\right)\left(\sum_{k=1}^n\frac{\partial^2 h_{\alpha,p}}{\partial u^i\partial u^k}\right).\end{aligned}$$

Note that the flatness of Z implies the identity (3.18), so by using the equalities (4.7) we have

$$\sum_{k=1}^{n}\frac{\partial^2 h_{\alpha,p}}{\partial u^i\partial u^k}=\sum_{k=1}^{n}\nabla_i\nabla_k h_{\alpha,p}+\sum_{j,k=1}^{n}\Gamma^j_{ik}\frac{\partial h_{\alpha,p}}{\partial u^j}$$
$$=\sum_{k=1}^{n}\nabla_i\nabla_k h_{\alpha,p}=\nabla_i\nabla_i h_{\alpha,p}.$$

Therefore,

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}}=\sum_{i=1}^{n} f^i\left(\nabla_i\nabla_i h_{\beta,q}\right)\left(\nabla_i\nabla_i h_{\alpha,p}\right)u^{i,1}=\frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}.$$

Next we show that for any $\gamma=1,\dots,n$,

$$\frac{\partial}{\partial t^{\alpha,p}}\frac{\partial v^\gamma}{\partial s}=\frac{\partial}{\partial s}\frac{\partial v^\gamma}{\partial t^{\alpha,p}}.$$

The left hand side reads

$$\frac{\partial}{\partial t^{\alpha,p}}\frac{\partial v^\gamma}{\partial s}=\frac{\partial v^\gamma}{\partial t^{\alpha,p-1}}+\sum_{q\geq 1}t^{\beta,q}\frac{\partial^2 v^\gamma}{\partial t^{\alpha,p}\partial t^{\beta,q-1}}.$$

Note that $\frac{\partial v^\gamma}{\partial t^{\alpha,p}}$ only depends on v^μ and v^μ_x, so we have

$$\frac{\partial}{\partial s}\frac{\partial v^\gamma}{\partial t^{\alpha,p}}=\frac{\partial v^\mu}{\partial s}\frac{\partial}{\partial v^\mu}\left(\frac{\partial v^\gamma}{\partial t^{\alpha,p}}\right)+\frac{\partial v^\mu_x}{\partial s}\frac{\partial}{\partial v^\mu_x}\left(\frac{\partial v^\gamma}{\partial t^{\alpha,p}}\right)$$
$$=\left(\delta^\mu_1+\sum_{q\geq 1}t^{\beta,q}\frac{\partial v^\mu}{\partial t^{\beta,q-1}}\right)\frac{\partial}{\partial v^\mu}\left(\frac{\partial v^\gamma}{\partial t^{\alpha,p}}\right)$$
$$+\left(\sum_{q\geq 1}t^{\beta,q}\frac{\partial v^\mu_x}{\partial t^{\beta,q-1}}\right)\frac{\partial}{\partial v^\mu_x}\left(\frac{\partial v^\gamma}{\partial t^{\alpha,p}}\right)$$
$$=\frac{\partial}{\partial v^1}\frac{\partial v^\gamma}{\partial t^{\alpha,p}}+\sum_{q\geq 1}t^{\beta,q}\frac{\partial^2 v^\gamma}{\partial t^{\beta,q-1}\partial t^{\alpha,p}},$$

so we only need to show that $\frac{\partial v^\gamma}{\partial t^{\alpha,p-1}}=\frac{\partial}{\partial v^1}\frac{\partial v^\gamma}{\partial t^{\alpha,p}}$, which can be easily obtained from the fact that

$$[Z,X_{\alpha,p}]=-[Z,[P_1,H_{\alpha,p}]]=-[P_1,[H_{\alpha,p},Z]]-[H_{\alpha,p},[Z,P_1]]$$
$$=-[P_1,H_{\alpha,p-1}]=X_{\alpha,p-1},\tag{4.27}$$

where we use the property (4.15) of a calibration. The Proposition is proved. □

For a given calibration $\{h_{\alpha,p}\}$ of the flat exact semsimple bihamiltonian structure $(P_1, P_2; Z)$, let us now give the definition of tau structures and tau functions of the associated principal hierarchy. From the property of a bihamiltonian vector field given in Corollary 4.5 and the definition (4.20) of $X_{\alpha,p}$ we have

$$0 = [X_{\beta,q}, H_{\alpha,p-1}] = \int \left(\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} \right),$$

so the differential polynomials $\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}}$ must be total x-derivatives, thus there exist functions $\Omega_{\alpha,p;\beta,q} \in \mathcal{A}_0$ such that

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}} = \partial \Omega_{\alpha,p;\beta,q}. \tag{4.28}$$

The functions $\Omega_{\alpha,p;\beta,q}$ are determined up to the addition of constants, so one can adjust the constants such that these functions satisfy some other properties which we describe below.

Definition 4.14 *A collection of functions*

$$\{\Omega_{\alpha,p;\beta,q} \in \mathcal{A}_0 \mid \alpha, \beta = 1, \dots, n;\ p, q = 0, 1, 2, \dots\}$$

is called a tau structure of the flat exact bihamiltonian structure $(P_1, P_2; Z)$ *with a fixed calibration* $\{h_{\alpha,p}\}$ *(we also call it a tau structure of the principal hierarchy* (2.20)*) if the following conditions are satisfied:*

i) $\partial \Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}$.

ii) $\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}$.

iii) $\Omega_{\alpha,p;1,0} = h_{\alpha,p-1}$.

Lemma 4.15 *A tau structure* $\{\Omega_{\alpha,p;\beta,q}\}$ *satisfies the following equations:*

$$\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\gamma,r}} = \frac{\partial \Omega_{\alpha,p;\gamma,r}}{\partial t^{\beta,q}}. \tag{4.29}$$

Proof By using Definition 4.14 of tau structures we have

$$\partial \left(\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\gamma,r}} - \frac{\partial \Omega_{\alpha,p;\gamma,r}}{\partial t^{\beta,q}} \right) = \frac{\partial}{\partial t^{\gamma,r}} \frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} - \frac{\partial}{\partial t^{\beta,q}} \frac{\partial h_{\alpha,p-1}}{\partial t^{\gamma,r}} = 0,$$

so the difference between the left hand side and the right hand side of (4.29) is a constant. However, both sides can be represented as differential polynomials of degree 1, so the constant must be zero. The lemma is proved. □

Definition 4.16 (cf. [9]) *Let $\{\Omega_{\alpha,p;\beta,q}\}$ be a tau structure of $(P_1, P_2; Z)$ with the calibration $\{h_{\alpha,p}\}$. The family of partial differential equations*

$$\frac{\partial f}{\partial t^{\alpha,p}} = f_{\alpha,p}, \tag{4.30}$$

$$\frac{\partial f_{\beta,q}}{\partial t^{\alpha,p}} = \Omega_{\alpha,p;\beta,q}(v), \tag{4.31}$$

$$\frac{\partial v^\gamma}{\partial t^{\alpha,p}} = \eta^{\gamma\xi}\partial\Omega_{\alpha,p;\xi,0}(v) \tag{4.32}$$

with unknown functions $(f, \{f_{\beta,q}\}, \{v^\gamma\})$ is called the tau cover of the principal hierarchy (2.20) *with respect to the tau structure $\{\Omega_{\alpha,p;\beta,q}\}$, and the function $\tau = e^f$ is called the tau function of the principal hierarchy. Here $\alpha, \beta, \gamma = 1, \dots, n, p \geq 0$.*

By using Lemma 4.15, one can easily show that members of the tau cover commute with each other. It is obvious that the covering map

$$(f, \{f_{\beta,q}\}, \{v^\gamma\}) \mapsto (\{v^\gamma\})$$

pushes forward the tau cover to the principal hierarchy. This is the reason why it is named "tau cover".

Now let us consider the tau structure of the calibration $\{h_{\alpha,p}\}$ that is constructed from $\{\theta_{\alpha,p}\}$ as above, see Proposition 4.13. We can construct, following [7], the functions $\Omega_{\alpha,p;\beta,q}(v)$ by

$$\frac{\partial h_\alpha(v; z_1)}{\partial v^\xi}\eta^{\xi\zeta}\frac{\partial h_\beta(v; z_2)}{\partial v^\zeta} - \eta_{\alpha\beta} = (z_1 + z_2)\sum_{p,q\geq 0}\Omega_{\alpha,p;\beta,q}(v)z_1^p z_2^q. \tag{4.33}$$

It is easy to prove the following proposition.

Proposition 4.17 *The collection of functions*

$$\{\Omega_{\alpha,p;\beta,q}(v) \,|\, \alpha, \beta = 1, \dots, ; p, q = 0, 1, 2, \dots\}$$

is a tau structure of the exact bihamiltonian structure $(P_1, P_2; Z)$ with the given calibration $\{h_{\alpha,p}\}$.

Lemma 4.18 *The functions $\{\Omega_{\alpha,p;\beta,q}\}$ constructed in* (4.33) *satisfy the identities*

$$\frac{\partial\Omega_{\alpha,p;\beta,q}}{\partial v^1} = \Omega_{\alpha,p-1;\beta,q} + \Omega_{\alpha,p;\beta,q-1} + \eta_{\alpha\beta}\delta_{p0}\delta_{q0}. \tag{4.34}$$

Proof For a fixed pair of indices $\{\alpha, \beta\}$, the above identities are equivalent to the identity

$$\frac{\partial\Omega_{\alpha;\beta}(v; z_1, z_2)}{\partial v^1} = (z_1 + z_2)\Omega_{\alpha;\beta}(v; z_1, z_2) + \eta_{\alpha\beta} \tag{4.35}$$

for the generating function

$$\Omega_{\alpha;\beta}(v; z_1, z_2) = \sum_{p,q\geq 0} \Omega_{\alpha,p;\beta,q}(v) z_1^p z_2^q.$$

Note that the generation function

$$h_\alpha(v;\, z) = \sum_{p\geq 0} h_{\alpha,p}(v) z^p$$

satisfies

$$\frac{\partial h_\alpha(v; z)}{\partial v^1} = z\, h_\alpha(v; z) + \eta_{\alpha 1},$$

then the identity (4.35) can be easily proved by using the definition (4.33). The lemma is proved. □

Theorem 4.19 *The tau cover admits the following Galilean symmetry:*

$$\frac{\partial f}{\partial s} = \frac{1}{2}\eta_{\alpha\beta} t^{\alpha,0} t^{\beta,0} + \sum_{p\geq 0} t^{\alpha,p+1} f_{\alpha,p}, \tag{4.36}$$

$$\frac{\partial f_{\beta,q}}{\partial s} = \eta_{\alpha\beta} t^{\alpha,0}\delta_{q0} + f_{\beta,q-1} + \sum_{p\geq 0} t^{\alpha,p+1}\Omega_{\alpha,p;\beta,q}, \tag{4.37}$$

$$\frac{\partial v^\gamma}{\partial s} = \delta_1^\gamma + \sum_{p\geq 0} t^{\alpha,p+1}\frac{\partial v^\gamma}{\partial t^{\alpha,p}}. \tag{4.38}$$

Proof To prove $\frac{\partial}{\partial s}$ is a symmetry of the tau cover, we only need to show:

$$\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t^{\alpha,p}}\right] K = 0, \tag{4.39}$$

where $K = f,\ f_{\beta,q}$, or v^γ. Denote the right hand side of (4.36) by W, then (4.37), (4.38) can be written as

$$\frac{\partial f_{\beta,q}}{\partial s} = \frac{\partial W}{\partial t^{\beta,q}}, \quad \frac{\partial v^\gamma}{\partial s} = \eta^{\gamma\beta}\frac{\partial^2 W}{\partial t^{1,0}\partial t^{\beta,0}},$$

so the identity (4.39) is equivalent to the following one:

$$\frac{\partial}{\partial s}\Omega_{\alpha,p;\beta,q} = \frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}} W.$$

By using the chain rule, we have

$$\frac{\partial}{\partial s}\Omega_{\alpha,p;\beta,q} = \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial v^\gamma}\frac{\partial v^\gamma}{\partial s} = \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial v^\gamma}\left(\delta_1^\gamma + \sum_{s\geq 0} t^{\xi,s+1}\frac{\partial v^\gamma}{\partial t^{\xi,s}}\right)$$

$$= \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial v^1} + \sum_{s\geq 0} t^{\xi,s+1}\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\xi,s}}.$$

On the other hand,

$$\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}}W = \frac{\partial}{\partial t^{\alpha,p}}\left(\eta_{\xi\beta}t^{\xi,0}\delta_{q0} + f_{\beta,q-1} + \sum_{s\geq 0} t^{\xi,s+1}\Omega_{\xi,s;\beta,q}\right)$$
$$=\eta_{\alpha\beta}\delta_{p0}\delta_{q0} + \Omega_{\alpha,p-1;\beta,q} + \Omega_{\alpha,p;\beta,q-1} + \sum_{s\geq 0} t^{\xi,s+1}\frac{\partial\Omega_{\xi,s;\beta,q}}{\partial t^{\alpha,p}}.$$

The theorem then follows from Lemma 4.15 and 4.18. □

To end this section, let us give some results about the nondegeneracy of bihamiltonian vector fields. In the next section, we will use some results proved in the Appendix, which requires existence of a bihamiltonian vector field

$$X = \int\left(\sum_{i=1}^{n} A^i(u)u^{i,1}\theta_i\right) \in \mathcal{X}$$

such that for all $i = 1, \ldots, n$, and for some $u \in D$,

$$\frac{\partial}{\partial u^i}A^i(u) \neq 0.$$

In this case, X is called *nondegenerate*.

Lemma 4.20 *If the bihamiltonian vector field X is nondegenerate, then $A^i(u) \neq A^j(u)$ for all $i \neq j$ and for some $u \in D$.*

Proof According to (4.8), if, for any $u \in D$, $A^i(u) = A^j(u)$ for $i \neq j$ then

$$\frac{\partial A^i}{\partial u^i} = \frac{\partial A^j}{\partial u^i} = \Gamma^j_{ji}\left(A^i - A^j\right) = 0.$$

The lemma is proved. □

By shrinking the domain D, the nondegeneracy condition for X and the result of the above lemma can be modified to "for all $u \in D$" instead of "for some $u \in D$".

Lemma 4.21

i) When $n = 1$, the bihamiltonian vector fields $X_{1,p}$ $(p > 0)$ defined in (4.20) *are always nondegenerate.*

ii) When $n \geq 2$, suppose the bihamiltonian structure (P_1, P_2) is irreducible, then there exists a nondegenerate bihamiltonian vector field X satisfying $[Z, X] = 0$.

Proof The bihamiltonian vector fields $X_{1,p}$ defined in (4.20) can be written in the form

$$X_{\alpha,p} = \int \left(\sum_{i=1}^{n} A^i_{\alpha,p}(u) u^{i,1} \theta_i \right) \in \mathcal{X}$$

By using the relations (4.27) between the bihamiltonian vector fields we have

$$\frac{\partial A^i_{\alpha,p}}{\partial u^i} = -\sum_{j \neq i} \frac{\partial A^i_{\alpha,p}}{\partial u^j} + A^i_{\alpha,p-1}.$$

From Lemma 4.12 we also have $A^i_{1,0} = 1$. Thus in the case when $n = 1$ we know that

$$A^1_{1,p} = \frac{(u^1)^p}{p!} + a_{p,1}(u^1)^{p-1} + \cdots + a_{p,p}$$

for some constants $a_{p,1}, \ldots, a_{p,p}$, so $X_{1,p}$ $(p > 0)$ are always nondegenerate.

When $n \geq 2$, a bihamiltonian vector field $X = \int \left(\sum_{i=1}^{n} A^i(u) u^{i,1} \theta_i \right)$ satisfying $[Z, X] = 0$ is characterized by the following equations

$$\frac{\partial A^i}{\partial u^j} = \Gamma^i_{ij} \left(A^j - A^i \right), \quad \text{for } j \neq i, \tag{4.40}$$

$$\frac{\partial A^i}{\partial u^i} = -\sum_{j \neq i} \frac{\partial A^i}{\partial u^j}. \tag{4.41}$$

The solution space of this system has dimension n. If X is degenerate, that is, there exists $i_0 \in \{1, \ldots, n\}$ such that

$$0 \equiv \frac{\partial A^{i_0}}{\partial u^{i_0}} = -\sum_{j \neq i_0} \Gamma^{i_0}_{i_0 j} \left(A^j - A^{i_0} \right) = -\sum_{j=1}^{n} \Gamma^{i_0}_{i_0 j} A^j.$$

Since (P_1, P_2) is irreducible, from Definition 2.1 it follows the existence of $j_0 \in \{1, \ldots, n\}$ with $j_0 \neq i_0$ such that $\gamma_{i_0 j_0}(u) \neq 0$ for some $u \in D$. By using the formulae (2.8) and (3.12) we know that

$$\Gamma^{i_0}_{i_0 j_0}(u) = \frac{\sqrt{f_{j_0}(u)}}{\sqrt{f_{i_0}(u)}} \gamma_{i_0 j_0}(u) \neq 0.$$

So from the above equation we have

$$A^{j_0}(u) = -\frac{1}{\Gamma^{i_0}_{i_0 j_0}(u)} \sum_{k \neq j_0} \Gamma^{i_0}_{i_0 k}(u) A^k(u).$$

Substituting this expression of A^{j_0} into (4.40) and (4.41), we obtain a new homogeneous linear system with unknowns $A^k (k \neq j_0)$ which does not depend on $A^{j_0}(u)$ and only depends on the pair of indices (i_0, j_0). The dimension of the solution space of this new system is at most $n - 1$, so any degenerate bihamiltonian vector field X satisfying the condition $[Z,X]=0$ belongs to an $(n-1)$-dimensional subspace of the solution space of (4.40), (4.41) which depends only on a pair of indices (i_0,j_0). Since the total number of such pairs is finite, we conclude that not all solutions of (4.40) and (4.41) are degenerate. The lemma is proved. □

5 Tau-symmetric integrable Hamiltonian deformations of the principal hierarchy

Let $(P_1,P_2;Z)$ be a flat exact semisimple bihamiltonian structure of hydrodynamic type. In this and the next section we consider properties of deformations of the principal hierarchy (2.20) and its tau structure. To this end, we fix a calibration $\{h_{\alpha,p}\}$ and a tau structure $\{\Omega_{\alpha,p;\beta,q}\}$ as in the previous section, and we assume that $(P_1,P_2;Z)$ is also irreducible.

Note that the principal hierarchy is determined by the first Hamiltonian structure P_1 and the calibration $\{h_{\alpha,p}\}$, so we first consider their deformations.

Definition 5.1 *The pair $(\tilde P_1,\{\tilde h_{\alpha,p}\})$ is called a tau-symmetric integrable deformation, or simply a deformation for short, of $(P_1,\{h_{\alpha,p}\})$ if it satisfies the following conditions:*

i) *$\tilde P_1\in\hat{\mathcal F}^2$ has the form*

$$\tilde P_1=P_1+P_1^{[2]}+P_1^{[3]}+\dots,$$

where $P_1^{[k]}\in\hat{\mathcal F}^2_{k+1}$, and it is a Hamiltonian structure.

ii) *$\tilde h_{\alpha,p}$ has the form*

$$\tilde h_{\alpha,p}=h_{\alpha,p}+h_{\alpha,p}^{[2]}+h_{\alpha,p}^{[3]}+\cdots,$$

where $h^{[k]}_{\alpha,p}\in\mathcal A_k$. Define $\tilde H_{\alpha,p}=\int(\tilde h_{\alpha,p})$, then for any pair of indices (α,p), (β,q) we must have

$$\{\tilde H_{\alpha,p},\tilde H_{\beta,q}\}_{\tilde P_1}=0. \tag{5.1}$$

Here $\{F,G\}_{\tilde P_1}=[[\tilde P_1,F],G]$ for $F,G\in\mathcal F$.

iii) *Define $\tilde X_{\alpha,p}=-[\tilde P_1,\tilde H_{\alpha,p}]$, and denote the associated evolutionary vector field by $\tilde\partial_{\alpha,p}=D_{\tilde X_{\alpha,p}}$, then $\{\tilde h_{\alpha,p}\}$ satisfy the tau-symmetry condition*

$$\tilde\partial_{\alpha,p}\left(\tilde h_{\beta,q-1}\right)=\tilde\partial_{\beta,q}\left(\tilde h_{\alpha,p-1}\right). \tag{5.2}$$

Remark 5.2 *We assume that there is no* $P_1^{[1]} \in \hat{\mathcal{F}}_2^2$ *and* $h_{\alpha,p}^{[1]} \in \mathcal{A}_1$ *terms in the deformations* $\tilde{P}_1$ *and* $\tilde{h}_{\alpha,p}$*. Without this condition we can also prove the next lemma, and then define the tau cover. We add it to avoid some subtle problems in Theorem 5.7 (see Remark 5.8 for more details). Note that for integrable hierarchies that arise in the study of semisimple cohomological field theories, there are no even degree terms in the deformed Hamiltonian structure and no odd degree terms in the deformed densities of the Hamiltonians.*

Let $v^1, \dots, v^n$ be the flat coordinates of the metric g_1 of the Hamiltonian structure P_1 which has the expression (2.17) in these coordinates. The deformed Hamiltonian structure $\tilde{P}_1$ can be uniquely represented in the form

$$\tilde{P}_1 = \frac{1}{2}\int \left(\eta^{\alpha\beta}(v)\bar{\theta}_\alpha\bar{\theta}_\beta^1 + \sum_{k=0}^{3} A_{2,k}^{\alpha\beta}\bar{\theta}_\alpha\bar{\theta}_\beta^{3-k} + \sum_{k=0}^{4} A_{3,k}^{\alpha\beta}\bar{\theta}_\alpha\bar{\theta}_\beta^{4-k} + \dots\right), \quad (5.3)$$

where the coefficients $A_{m,k}^{\alpha\beta} \in \mathcal{A}$ are homogeneous differential polynomials of degree k satisfying the anti-symmetry conditions

$$\sum_{k=0}^{m+1} A_{m,k}^{\alpha\beta}\partial_x^{m+1-k} = -\sum_{k=0}^{m+1}(-1)^{m+1-k}\partial_x^{m+1-k}A_{m,k}^{\beta\alpha}, \quad m \geq 2.$$

The corresponding Hamiltonian operator is then given by

$$\tilde{P}_1^{\alpha\beta} = \eta^{\alpha\beta}(v)\partial_x + \sum_{k=0}^{3} A_{2,k}^{\alpha\beta}\partial_x^{3-k} + \sum_{k=0}^{4} A_{3,k}^{\alpha\beta}\partial_x^{4-k} + \dots. \quad (5.4)$$

The flows given by the evolutionary vector field $\tilde{\partial}_{\alpha,p}$ yield a deformation of the principal hierarchy (2.20) which has the Hamiltonian representation

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = D_{\tilde{X}_{\beta,q}}(v^\alpha) = \tilde{P}_1^{\alpha\gamma}\frac{\delta \tilde{H}_{\alpha,p}}{\delta v^\gamma}, \quad 1 \leq \alpha, \beta \leq n,\ q \geq 0. \quad (5.5)$$

From the property ii), iii) of Definition 5.1 we know that the flows of this hierarchy of evolutionary PDEs are mutually commutative, and satisfy an analogue of the tau symmetry condition (2.24). For this reason we call the above deformed hierarchy (5.5) a tau-symmetric integrable Hamiltonian deformation of the principal hierarchy. We will show below that the deformed hierarchy also possesses a tau structure. We note that the notion of *tau-symmetric integrable Hamiltonian deformation* of the principal hierarchy associated with a Frobenius manifold was introduced in [12]. In the definition given there the following additional conditions are required:

1. $\tilde{\partial}_{1,0} = \partial$.

2. $\tilde{H}_{\alpha,-1}$ are Casimirs of $\tilde{P}_1$.

These two conditions are consequences of the Definition 5.1. In fact, since the evolutionary vector field X corresponding to the flow $\tilde{\partial}_{1,0} - \partial$ is a symmetry of the deformed integrable hierarchy and it belongs to $\hat{\mathcal{F}}^1_{\geq 2}$, by using the existence of a non-degenerate bihamiltonian vector field proved in Lemma 4.21 and the property ii) of Corollary A.4 we know that X must vanishes. Thus we have

$$\tilde{\partial}_{1,0} = \partial. \tag{5.6}$$

Similarly, from the fact that $[P_1, H_{\alpha,-1}] = 0$ we know that the vector field $X = -[\tilde{P}_1, \tilde{H}_{\alpha,-1}] \in \hat{\mathcal{F}}^1_{\geq 2}$. Since it is a symmetry of the deformed integrable hierarchy (5.5) we know that it also vanishes. Thus the second condition also holds true.

Lemma 5.3 *For any deformation* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *of* $(P_1, h_{\alpha,p}\})$*, there exists a unique collection of differential polynomials* $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ *satisfying the following conditions:*

i) $\tilde{\Omega}_{\alpha,p;\beta,q} = \Omega_{\alpha,p;\beta,q} + \Omega^{[2]}_{\alpha,p;\beta,q} + \Omega^{[3]}_{\alpha,p;\beta,q} + \cdots$, *where* $\Omega^{[k]}_{\alpha,p;\beta,q} \in \mathcal{A}_k$.

ii) $\partial\tilde{\Omega}_{\alpha,p;\beta,q} = \tilde{\partial}_{\alpha,p}\left(\tilde{h}_{\beta,q-1}\right)$.

iii) $\tilde{\Omega}_{\alpha,p;\beta,q} = \tilde{\Omega}_{\beta,q;\alpha,p}$, *and* $\tilde{\Omega}_{\alpha,p;1,0} = \tilde{h}_{\alpha,p-1}$.

iv) $\tilde{\partial}_{\gamma,r}\tilde{\Omega}_{\alpha,p;\beta,q} = \tilde{\partial}_{\beta,q}\tilde{\Omega}_{\alpha,p;\gamma,r}$.

Here $\alpha, \beta, \gamma = 1, \ldots, n$ *and* $p, q, r \geq 0$. *This collection of differential polynomials* $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ *is called a* tau structure *of* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$.

Proof According to the definition of $\{\tilde{h}_{\alpha,p}\}$,

$$\int \left(\tilde{\partial}_{\alpha,p}\left(\tilde{h}_{\beta,q}\right)\right) = [\tilde{X}_{\alpha,p}, \tilde{H}_{\beta,q}] = -\{\tilde{H}_{\alpha,p}, \tilde{H}_{\beta,q}\}_{\tilde{P}_1} = 0,$$

so there exists $\tilde{\Omega}_{\alpha,p;\beta,q}$ satisfying the conditions i), ii). These conditions determine $\tilde{\Omega}_{\alpha,p;\beta,q}$ up to a constant, which has degree zero. Note that the condition i) fixes the degree zero part of $\tilde{\Omega}_{\alpha,p;\beta,q}$, so it is unique. The conditions iii) and iv) can be verified by considering the action of ∂ on both sides of the equalities, as we did in the proof of Lemma 4.15. □

Definition 5.4 ([9]) *The differential polynomials*

$$w^\alpha = \eta^{\alpha\beta}\tilde{h}_{\beta,-1} = v^\alpha + F^\alpha_2 + F^\alpha_3 + \cdots, \quad F^\alpha_k \in \mathcal{A}_k \tag{5.7}$$

are called the normal coordinates *of* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *and of the deformed principal hierarchy* (5.5).

The properties of the differential polynomials $\tilde{\Omega}_{\alpha,p;\beta,q}$ enable us to define the tau cover for $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ and the deformed principal hierarchy (5.5), just as we did for the principal hierarchy given in Definition 4.16. From (5.7) we know that we can also represent v^α in the form

$$v^\alpha = w^\alpha + \tilde{F}_2^\alpha + \tilde{F}_3^\alpha + \cdots , \tag{5.8}$$

where $\tilde{F}_k^\alpha$ are differential polynomials of $w^1, \dots, w^n$ of degree k. So the functions $\tilde{\Omega}_{\alpha,p;\beta,q}(v, v_x, \dots)$ can also be represented as differential polynomials in $w^1, \dots, w^n$ by the change of coordinates formulae given in (5.8).

Definition 5.5 (c.f. [9]) *The family of partial differential equations*

$$\frac{\partial \tilde{f}}{\partial t^{\alpha,p}} = \tilde{f}_{\alpha,p}, \tag{5.9}$$

$$\frac{\partial \tilde{f}_{\beta,q}}{\partial t^{\alpha,p}} = \tilde{\Omega}_{\alpha,p;\beta,q}, \tag{5.10}$$

$$\frac{\partial w^\gamma}{\partial t^{\alpha,p}} = \eta^{\gamma\xi} \partial \tilde{\Omega}_{\alpha,p;\xi,0} \tag{5.11}$$

with the unknowns functions $(\{w^\alpha\}, \{\tilde{f}_{\alpha,p}\}, \tilde{f})$ *is called the tau cover of the deformed principal hierarchy* (5.5) *with respect to the tau structure* $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$, *and the function* $\tilde{\tau} = e^{\tilde{f}}$ *is called the tau function of the deformed principal hierarchy.*

Now let us proceed to consider the relationship between tau structures for two different deformations of $(P_1, \{h_{\alpha,p}\})$. We first recall the notion of equivalent deformations given by Definition 3.6 in [25].

Definition 5.6 *Suppose* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *and* $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ *are two deformations of* $(P_1, \{h_{\alpha,p}\})$. *Define* $\tilde{H}_{\alpha,p} = \int \left(\tilde{h}_{\alpha,p}\right)$ *and* $\hat{H}_{\alpha,p} = \int \left(\hat{h}_{\alpha,p}\right)$. *If there exists* $Y \in \hat{\mathcal{F}}^1_{\geq 2}$ *such that*

$$\hat{P}_1 = e^{\mathrm{ad}_Y}\left(\tilde{P}_1\right), \quad \hat{H}_{\alpha,p} = e^{\mathrm{ad}_Y}\left(\tilde{H}_{\alpha,p}\right), \tag{5.12}$$

then we say that $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *and* $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ *are equivalent.*

If $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ and $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ are equivalent, then the corresponding Hamiltonian vector fields and the evolutionary vector fields have the relation

$$\hat{X}_{\alpha,p} = -[\hat{P}_1, \hat{H}_{\alpha,p}] = -e^{\mathrm{ad}_Y}\left([\tilde{P}_1, \tilde{H}_{\alpha,p}]\right) = e^{\mathrm{ad}_Y}\left(\tilde{X}_{\alpha,p}\right), \tag{5.13}$$

$$\hat{\partial}_{\alpha,p} = e^{D_Y} \tilde{\partial}_{\alpha,p} e^{-D_Y}. \tag{5.14}$$

We note that the associated deformed principal hierarchy (c.f. (5.5))

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = D_{\hat{X}_{\beta,q}}(v^\alpha), \quad 1 \leq \alpha, \beta \leq n,\ q \geq 0 \tag{5.15}$$

is obtained from (5.5) by representing the equations of the hierarchy in terms of the new unknown functions

$$\hat{v}^\alpha = e^{-D_Y}\left(v^\alpha\right) \tag{5.16}$$

and re-denoting $\hat{v}^\alpha, \hat{v}^\alpha_x, \dots$ by $v^\alpha, v^\alpha_x, \dots$. We call transformations of the form (5.12)–(5.14) and (5.16) Miura-type transformations.

Theorem 5.7 *Suppose $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ and $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ are two equivalent deformations related by a Miura-type transformation e^{ad_Y}, and they have tau structures $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ and $\{\hat{\Omega}_{\alpha,p;\beta,q}\}$ respectively. Then there exists a differential polynomial G such that*

$$\hat{h}_{\alpha,p} = e^{D_Y}\left(\tilde{h}_{\alpha,p}\right) + \partial\hat{\partial}_{\alpha,p}G,$$
$$\hat{\Omega}_{\alpha,p;\beta,q} = e^{D_Y}\left(\tilde{\Omega}_{\alpha,p;\beta,q}\right) + \hat{\partial}_{\alpha,p}\hat{\partial}_{\beta,q}G.$$

Moreover, suppose $\{\tilde{f}(t), \{\tilde{f}_{\alpha,p}(t)\}, \{\tilde{w}^\alpha(t)\}\}$ is a solution to the tau cover corresponding to the tau structure $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$, then

$$\hat{f}(t) = \tilde{f}(t) + G(t), \quad \hat{f}_{\alpha,p}(t) = \tilde{f}_{\alpha,p} + \frac{\partial G(t)}{\partial t^{\alpha,p}}, \quad \hat{w}^\alpha(t) = \tilde{w}^\alpha(t) + \eta^{\alpha\beta}\frac{\partial^2 G(t)}{\partial x \partial t^{\beta,0}}$$

give a solution $\{\hat{f}(t), \{\hat{f}_{\alpha,p}(t)\}, \{\hat{w}^\alpha(t)\}\}$ to the tau cover corresponding to the tau structure $\{\hat{\Omega}_{\alpha,p;\beta,q}\}$ and the associated deformed principal hierarchy. Here $G(t)$ is defined from the differential polynomial $G = G(v, v_x, \dots)$ by

$$G(t) = \left(e^{-D_Y}G(v, v_x, \dots)\right)|_{v^\alpha = v^\alpha(\tilde{w}(t)), \tilde{w}_x(t), \dots)},$$

and $v^\alpha = v^\alpha(\tilde{w}, \tilde{w}_x, \dots)$ are defined by the relation $\tilde{w}^\alpha = \eta^{\alpha,\gamma}\tilde{h}_{\gamma,0}(v, v_x, \dots)$ just as we did in (5.8).

Proof The condition $\hat{H}_{\alpha,p} = e^{\mathrm{ad}_Y}\left(\tilde{H}_{\alpha,p}\right)$ implies that there exists $g_{\alpha,p} \in \mathcal{A}_{\geq 1}$ such that

$$\hat{h}_{\alpha,p} = e^{D_Y}\left(\tilde{h}_{\alpha,p}\right) + \partial g_{\alpha,p}.$$

The tau-symmetry condition $\hat{\partial}_{\alpha,p}\hat{h}_{\beta,q-1} = \hat{\partial}_{\beta,q}\hat{h}_{\alpha,p-1}$ for $\{\hat{h}_{\alpha,p}\}$ and the one for $\{\tilde{h}_{\alpha,p}\}$ implies that

$$\partial\left(\hat{\partial}_{\alpha,p}g_{\beta,q-1} - \hat{\partial}_{\beta,q}g_{\alpha,p-1}\right) = 0,$$

so we have $\hat{\partial}_{\alpha,p}g_{\beta,q-1} = \hat{\partial}_{\beta,q}g_{\alpha,p-1}$. In particular, by taking $(\beta,q) = (1,0)$, we have

$$\hat{\partial}_{\alpha,p}g_{1,-1} = \partial g_{\alpha,p-1},$$

so $\int (g_{1,-1})$ gives a conserved quantity for $\hat\partial_{\alpha,p}$ with a positive degree. According to Theorem A.3, there exists $G \in \mathcal{A}$ such that

$$g_{1,-1} = \partial G, \tag{5.17}$$

then we have

$$\partial\left(\hat\partial_{\alpha,p} G - g_{\alpha,p-1}\right) = 0,$$

so $g_{\alpha,p-1} = \hat\partial_{\alpha,p} G$ for $\alpha = 1,\dots,n, p \ge 0$. Thus we have

$$\begin{aligned}\partial\hat\Omega_{\alpha,p;\beta,q} &= \hat\partial_{\alpha,p}\hat h_{\beta,q-1} = \hat\partial_{\alpha,p}\left(e^{D_Y}\left(\tilde h_{\beta,q-1}\right) + \partial\hat\partial_{\beta,q}G\right)\\ &= e^{D_Y}\tilde\partial_{\alpha,p}\left(\tilde h_{\beta,q-1}\right) + \partial\hat\partial_{\alpha,p}\hat\partial_{\beta,q}G = \partial\left(e^{D_Y}\left(\tilde\Omega_{\alpha,p;\beta,q}\right) + \hat\partial_{\alpha,p}\hat\partial_{\beta,q}G\right),\end{aligned}$$

so the difference between $\hat\Omega_{\alpha,p;\beta,q}$ and $e^{D_Y}\left(\tilde\Omega_{\alpha,p;\beta,q}\right) + \hat\partial_{\alpha,p}\hat\partial_{\beta,q}G$ is a constant. However, they have the same leading terms, so the constant must be zero.

The remaining assertions of the theorem follow from our definition of the tau covers of the deformed principal hierarchies. The theorem is proved. □

Remark 5.8 *If in Definition 5.1 we permit the appearance of $P_1^{[1]}$ and $h_{\alpha,p}^{[1]}$, the first identity of the above theorem should be replaced by*

$$\hat h_{\alpha,p} = e^{D_Y}\left(\tilde h_{\alpha,p}\right) + \hat\partial_{\alpha,p}\sigma,$$

where σ is a conserved density of $\hat\partial_{\alpha,p}$, and the solutions $\hat f = \log\hat\tau$ and $\tilde f = \log\tilde\tau$ of the tau covers of $\hat\Omega_{\alpha,p;\beta,q}$ and $\tilde\Omega_{\alpha,p;\beta,q}$ satisfy the relation

$$\partial\left(\log\hat\tau - \log\tilde\tau\right) = \sigma.$$

The different tau functions defined in [13, 26, 32] for the Drinfeld–Sokolov hierarchies have such a relationship.

Next let us consider the Galilean symmetry of the deformed principal hierarchy.

Definition 5.9 *The triple $(\tilde P_1, \{\tilde h_{\alpha,p}\}, \tilde Z)$ is a deformation of $(P_1, \{h_{\alpha,p}\}, Z)$ if*

i) The pair $(\tilde P_1, \{\tilde h_{\alpha,p}\})$ is a deformation of $(P_1, \{h_{\alpha,p}\})$.

ii) The vector field $\tilde Z$ has the form

$$\tilde Z = Z + Z^{[2]} + Z^{[3]} + \dots, \quad Z^{[k]} \in \hat{\mathcal{F}}^1_k,$$

and satisfies the conditions $[\tilde Z, \tilde P_1] = 0$ and

$$D_{\tilde Z}\tilde h_{\alpha,-1} = \eta_{\alpha,1}, \quad D_{\tilde Z}\tilde h_{\alpha,p} = \tilde h_{\alpha,p-1}, \quad \alpha = 1,\dots,n,\ p \ge 0. \tag{5.18}$$

Lemma 5.10 *Let* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\}, \tilde{Z})$ *be a deformation of* $(P_1, \{h_{\alpha,p}\}, Z)$, $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ *be a tau structure of* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *given in Lemma 5.3, and* $w^1, \ldots, w^n$ *are the normal coordinates defined in Definition 5.4. Assume that the identity* (4.34) *holds true, then we have:*

$$\frac{\partial \tilde{\Omega}_{\alpha,p;\beta,q}}{\partial w^1} = \tilde{\Omega}_{\alpha,p-1;\beta,q} + \tilde{\Omega}_{\alpha,p;\beta,q-1} + \eta_{\alpha\beta}\delta_{p0}\delta_{q0}.$$

Proof According to Lemma 4.18, we only need to show that

$$\partial \frac{\partial \tilde{\Omega}_{\alpha,p;\beta,q}}{\partial w^1} = \partial \tilde{\Omega}_{\alpha,p-1;\beta,q} + \partial \tilde{\Omega}_{\alpha,p;\beta,q-1},$$

that is,

$$\frac{\partial}{\partial w^1}\left(\tilde{\partial}_{\beta,q}\left(\tilde{h}_{\alpha,p-1}\right)\right) = \tilde{\partial}_{\beta,q}\left(\tilde{h}_{\alpha,p-2}\right) + \tilde{\partial}_{\beta,q-1}\left(\tilde{h}_{\alpha,p-1}\right). \tag{5.19}$$

By using the Definition 5.4 of normal coordinates and the conditions given in (5.18) we have

$$D_{\tilde{Z}} = \partial^s\left(D_{\tilde{Z}}\left(w^\gamma\right)\right)\frac{\partial}{\partial w^{\gamma,s}} = \partial^s\left(\delta^\gamma_1\right)\frac{\partial}{\partial w^{\gamma,s}} = \frac{\partial}{\partial w^1}.$$

So by using (5.18) again we see that the identity (5.19) is equivalent to $[D_{\tilde{Z}}, \tilde{\partial}_{\beta,q}] = \tilde{\partial}_{\beta,q-1}$, which follows from the identities $\tilde{\partial}_{\beta,q} = -D_{[\tilde{P}_1, \tilde{H}_{\beta,q}]}$, and

$$[D_{\tilde{Z}}, D_{[\tilde{P}_1,\tilde{H}_{\beta,q}]}] = D_{[\tilde{Z},[\tilde{P}_1,\tilde{H}_{\beta,q}]]} = D_{[\tilde{P}_1,\tilde{H}_{\beta,q-1}]}.$$

Here we use the property ii) of Theorem 2.4 given in [25]. The lemma is proved. □

Similar to Theorem 4.19, we have the following theorem on the Galilean symmetry of the deformed principal hierarchy $\{\tilde{\partial}_{\alpha,p}\}$.

Theorem 5.11 *Under the assumption of Lemma 5.10, the above defined tau cover* (5.9)–(5.11) *admits the following Galilean symmetry:*

$$\frac{\partial \tilde{f}}{\partial s} = \frac{1}{2}\eta_{\alpha\beta}t^{\alpha,0}t^{\beta,0} + \sum_{\alpha,p} t^{\alpha,p+1}\tilde{f}_{\alpha,p}, \tag{5.20}$$

$$\frac{\partial \tilde{f}_{\beta,q}}{\partial s} = \eta_{\alpha\beta}t^{\alpha,0}\delta_{q0} + \tilde{f}_{\beta,q-1} + \sum_{\alpha,p} t^{\alpha,p+1}\tilde{\Omega}_{\alpha,p;\beta,q}, \tag{5.21}$$

$$\frac{\partial w^\gamma}{\partial s} = \delta^\gamma_1 + \sum_{\alpha,p} t^{\alpha,p+1}\frac{\partial w^\gamma}{\partial t^{\alpha,p}}. \tag{5.22}$$

Proof We can prove the theorem by using the same argument as the one given in the proof of Theorem 4.19, and by using Lemma 5.10. □

Example 5.12 *Let $c = \{c_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})\}$ be a semisimple cohomological field theory. Its genus zero part defines a semisimple Frobenius manifold, which corresponds to a flat exact semisimple bihamiltonian structure of hydrodynamic type. Its principal hierarchy has a special deformation, called topological deformation, such that the partition function of c is a tau function of this deformed hierarchy [9, 3, 4]. On the other hand, Buryak constructed another deformation, called double ramification deformation, from the same data, and conjectured that these deformations are actually equivalent [1]. This conjecture is refined in [2] as follow:*

Suppose $\mathcal{F}$ is the free energy of the topological deformation. Buryak et al *show that there exists a unique differential polynomial P such that $\mathcal{F}^{\text{red}} = \mathcal{F} + P$ satisfies the following condition:*

$$\operatorname{Coef}_{\epsilon^{2g}} \frac{\partial^n \mathcal{F}^{\text{red}}}{\partial t^{\alpha_1,p_1} \cdots \partial t^{\alpha_n,p_n}}\bigg|_{t^{*,*}=0} = 0, \quad p_1 + \cdots + p_n \le 2g - 2.$$

It is conjectured that $\mathcal{F}^{\text{red}}$ is just the free energy of the double ramification deformation.

Buryak's et al *refined conjecture is compatible with Theorem 5.7. They also show that the double ramification deformation satisfies the string equation, which can also be derived from Theorem 5.11.*

6 Tau-symmetric bihamiltonian deformations of the principal hierarchy

In this section, we construct a class of tau-symmetric integrable Hamiltonian deformations of the principal hierarchy associated with a semisimple flat exact bihamiltonian structure $(P_1, P_2; Z)$ of hydrodynamic type. These deformations of the principal hierarchies are in fact bihamiltonian integrable hierarchies.

We denote by $u^1, \dots, u^n$ and $v^1, \dots, v^n$ the canonical coordinates of (P_1, P_2) and the flat coordinates of P_1 respectively. In these coordinates we have

$$\sum_{i=1}^n \frac{\partial}{\partial u^i} = \frac{\partial}{\partial v^1}. \tag{6.1}$$

We also fix a calibration

$$\{h_{\alpha,p}(v) \in \mathcal{A}_0 \mid \alpha = 1, \dots, n;\ p = 0, 1, 2, \dots\}$$

and a tau structure

$$\{\Omega_{\alpha,p;\beta,q}(v) \in \mathcal{A}_0 \mid \alpha, \beta = 1, \dots, n;\ p, q = 0, 1, 2, \dots\}$$

of the flat exact semisimple bihamiltonian structure $(P_1, P_2; Z)$ (see their construction given in Propositions 4.13, 4.17).

From [5, 10, 23, 25] we know that the bihamiltonian structure (P_1, P_2) possesses deformations of the form

$$\tilde{P}_1 = P_1 + \sum_{k\geq 1} Q_{1,k}, \quad \tilde{P}_2 = P_2 + \sum_{k\geq 1} Q_{2,k}, \quad Q_{1,k}, Q_{2,k} \in \hat{\mathcal{F}}^2_{k+1}$$

such that $(\tilde{P}_1, \tilde{P}_2)$ is still a bihamiltonian structure, i.e.

$$[\tilde{P}_a, \tilde{P}_b] = 0, \quad a, b = 1, 2.$$

The space of deformations of the bihamiltonian structure (P_1, P_2) is characterized by the central invariants $c_1(u^1), \ldots, c_n(u^n)$ of $(\tilde{P}_1, \tilde{P}_2)$. The following theorem of Falqui and Lorenzoni gives a condition under which the deformed bihamiltonian structure inherits the exactness property. This means that there exists a vector field $\tilde{Z} \in \hat{\mathcal{F}}^1$ such that

$$[\tilde{Z}, \tilde{P}_1] = 0, \quad [\tilde{Z}, \tilde{P}_2] = \tilde{P}_1.$$

Theorem 6.1 ([14]) *The deformed bihamiltonian structure $(\tilde{P}_1, \tilde{P}_2)$ is exact if and only if its central invariants $c_1, \ldots, c_n$ are constant functions. Moreover, there exists a Miura-type transformation $g = e^{\mathrm{ad}_Y}, Y \in \hat{\mathcal{F}}^1$ such that*

$$g(\tilde{P}_1) = P_1, \quad g(\tilde{P}_2) = P_2 + \sum_{k\geq 1} Q_k, \quad Q_k \in \hat{\mathcal{F}}^2_{2k+1} \tag{6.2}$$

and $g(\tilde{Z}) = Z$, where Z is represented in the canonical coordinates $u^1, \ldots, u^n$ as follows:

$$Z = \int \left(\sum_{i=1}^{n} Z^i \theta_i \right), \quad Z^1 = \cdots = Z^n = 1.$$

In what follows, we assume that $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ is a deformation of the flat exact bihamiltonian structure $(P_1, P_2; Z)$ with constant central invariants $c_1, \ldots, c_n$, and $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ has the form

$$\tilde{P}_1 = P_1, \quad \tilde{P}_2 = P_2 + \sum_{k\geq 1} Q_k, \quad Q_k \in \hat{\mathcal{F}}^2_{2k+1}, \quad \tilde{Z} = Z. \tag{6.3}$$

We define the space of Casimirs of $\tilde{P}_1$, the space of bihamiltonian conserved quantities and the space of bihamiltonian vector fields respectively, just like we did for (P_1, P_2) in Sec. 4, i.e.

$$\begin{aligned}
\tilde{\mathcal{V}} &:= \mathrm{Ker}([\tilde{P}_1, \cdot]) \cap \mathcal{F}, \\
\tilde{\mathcal{H}} &:= \mathrm{Ker}([\tilde{P}_2, [\tilde{P}_1, \cdot]]) \cap \mathcal{F}, \\
\tilde{\mathcal{X}} &:= \mathrm{Ker}([\tilde{P}_1, \cdot]) \cap \mathrm{Ker}([\tilde{P}_2, \cdot]) \cap \hat{\mathcal{F}}^1.
\end{aligned}$$

Theorem 6.2 *We have the following isomorphisms:*

$$\mathcal{V} \cong \tilde{\mathcal{V}}, \quad \mathcal{H} \cong \tilde{\mathcal{H}}, \quad \mathcal{X} \cong \tilde{\mathcal{X}}. \tag{6.4}$$

In particular, $\tilde{\mathcal{X}} \cong \tilde{\mathcal{H}}/\tilde{\mathcal{V}}$.

Proof Since $\tilde{P}_1 = P_1$, we only need to prove that $\mathcal{H} \cong \tilde{\mathcal{H}}$, $\mathcal{X} \cong \tilde{\mathcal{X}}$. Suppose $H \in \tilde{\mathcal{H}}$ is a bihamiltonian conserved quantity of $(\tilde{P}_1, \tilde{P}_2)$. Expand H as the sum of homogeneous components

$$H = H_0 + H_1 + H_2 + \cdots, \quad H_k \in \mathcal{F}_k,$$

then H_0 is a bihamiltonian conserved quantity of (P_1, P_2), so we have a map $\pi : \tilde{\mathcal{H}} \to \mathcal{H}$, $H \mapsto H_0$. The fact that $\mathcal{H}$ is concentrated in degree zero (see Lemma 4.4) implies that π is injective. To prove the isomorphism $\mathcal{H} \cong \tilde{\mathcal{H}}$, we only need to show that π is surjective, that is, for any bihamiltonian conserved quantity H_0 of (P_1, P_2) there exists a bihamiltonian conserved quantity H of $(\tilde{P}_1, \tilde{P}_2)$ with H_0 as its leading term.

Recall that $(\tilde{P}_1, \tilde{P}_2; Z)$ takes the form (6.3). Denote as in (4.3)

$$d_a = \mathrm{ad}_{P_a} = [P_a, \cdot], \quad a = 1, 2,$$

then Q_k satisfy the following equations:

$$d_1 Q_k = 0, \quad d_2 Q_k + \frac{1}{2} \sum_{i=1}^{k-1} [Q_i, Q_{k-i}] = 0.$$

We assert that, for any bihamiltonian conserved quantity $H_0 \in \mathcal{H}$ of (P_1, P_2), there exists $H_{2k} \in \mathcal{F}_{2k}$ such that

$$H = H_0 + H_2 + H_4 + \cdots$$

is a bihamiltonian conserved quantity of $(\tilde{P}_1, \tilde{P}_2)$. This assertion is equivalent to the solvability of the following equations for H_{2k} to be solved recursively:

$$d_1 d_2 H_{2k} = \sum_{i=1}^{k} [Q_i, d_1 H_{2k-2i}], \quad k = 1, 2, \ldots.$$

Assume that we have already solved the above equations for $H_2, \ldots, H_{2k-2}$ starting from H_0. Denote by W_k the right hand side of the above equation.

Then it is easy to see that $d_1 W_k = 0$, and

$$\begin{aligned}
d_2 W_k &= \left[P_2, \sum_{i=1}^{k}[Q_i, d_1 H_{2k-2i}]\right] \\
=& -\sum_{i=1}^{k}\left([[d_1 H_{2k-2i}, P_2], Q_i] + [[P_2, Q_i], d_1 H_{2k-2i}]\right) \\
=& \sum_{i=1}^{k}\left([d_1 d_2 H_{2k-2i}, Q_i] + [-d_2 Q_i, d_1 H_{2k-2i}]\right) \\
=& \sum_{i=1}^{k}\sum_{j=1}^{k-i}[[Q_j, d_1 H_{2k-2i-2j}], Q_i] + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i-1}[[Q_j, Q_{i-j}], d_1 H_{2k-2i}] \\
=& \frac{1}{2}\sum_{i,j\geq 1, l\geq 0, i+j+l=k}\left([[Q_j, d_1 H_{2l}], Q_i] + [[Q_i, d_1 H_{2l}], Q_j] + [[Q_i, Q_j], d_1 H_{2l}]\right) \\
=& 0,
\end{aligned}$$

so $W_k \in \mathrm{Ker}(d_1) \cap \mathrm{Ker}(d_2) \cap \hat{\mathcal{F}}^2_{\geq 4}$. Since $BH^2_{\geq 4}(\hat{\mathcal{F}}, P_1, P_2) \cong 0$, there exists $H_{2k} \in \mathcal{F}$ such that $W_k = d_1 d_2 H_{2k}$. Thus the isomorphism $\mathcal{H} \cong \tilde{\mathcal{H}}$ is proved.

It is easy to see that the map

$$d_1 : \tilde{\mathcal{H}}/\tilde{\mathcal{V}} \to \tilde{\mathcal{X}}, \quad H \mapsto X = -[\tilde{P}_1, H]$$

gives the isomorphism $\tilde{\mathcal{H}}/\tilde{\mathcal{V}} \cong \tilde{\mathcal{X}}$, which also induces the isomorphism $\mathcal{X} \cong \tilde{\mathcal{X}}$. The theorem is proved. □

It follows from the above theorem that, for every bihamiltonian conserved quantity $H_{\alpha,p} = \int(h_{\alpha,p}) \in \mathcal{H}$ there exists a unique deformation

$$\tilde{H}_{\alpha,p} = H_{\alpha,p} + \tilde{H}^{[1]}_{\alpha,p} + \tilde{H}^{[2]}_{\alpha,p} + \dots, \quad \tilde{H}^{[k]}_{\alpha,p} \in \mathcal{F}_{2k}$$

such that these deformations, together with the constant local functional $\int(1)$, form a basis of the subspace

$$\tilde{\mathcal{H}}^{\infty} = \bigcup_{p\geq 0} \tilde{\mathcal{H}}^{(p)}$$

of $\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}^{(p)}$ is the image of $\mathcal{H}^{(p)}$ in $\tilde{\mathcal{H}}$ of the isomorphism given in the above theorem. For any pair of indices (α, p), (β, q), it is easy to see that the local functional $H = \{\tilde{H}_{\alpha,p}, \tilde{H}_{\beta,q}\}_{\tilde{P}_1} := [[\tilde{P}_1, \tilde{H}_{\alpha,p}], \tilde{H}_{\beta,q}]$ is a bihamiltonian conserved quantity w.r.t. $(\tilde{P}_1, \tilde{P}_2)$. Since $H \in \mathcal{F}_{\geq 1}$ we obtain

$$\{\tilde{H}_{\alpha,p}, \tilde{H}_{\beta,q}\}_{\tilde{P}_1} = 0 \tag{6.5}$$

by using Lemma 4.21 and the property i) of Corollary A.4.

Define the operator δ_Z (c.f. the definition of δ given in (4.4)) by

$$\delta_Z : \hat{\mathcal{F}} \to \hat{\mathcal{A}}, \quad Q \mapsto \sum_{i=1}^n \frac{\delta Q}{\delta u^i} = \frac{\delta Q}{\delta v^1}. \tag{6.6}$$

Here we have used the relation between the canonical and flat coordinates given by (6.1), and the equation (2.30) of [24] relating the operations of the variational derivatives in different coordinates. Then for a local functional $H \in \mathcal{F}$ we have

$$[Z, H] = \int (\delta_Z(H)).$$

Now let us define

$$\tilde{h}_{\alpha,p} = \delta_Z \tilde{H}_{\alpha,p+1} = \frac{\delta \tilde{H}_{\alpha,p+1}}{\delta v^1}, \quad \alpha = 1, \dots, n, \ p = -1, 0, 1, \dots. \tag{6.7}$$

Theorem 6.3 *The triple* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\}, \tilde{Z})$ *gives a deformation of* $(P_1, \{h_{\alpha,p}\}, Z)$.

Proof We denote

$$\tilde{H}'_{\alpha,p} = \int \left(\tilde{h}_{\alpha,p}\right).$$

From the definition of $\tilde{h}_{\alpha,p}$ we see that $\tilde{H}'_{\alpha,p} = [Z, \tilde{H}_{\alpha,p+1}]$, so it belongs to $\tilde{\mathcal{H}}$. From the property $D_Z h_{\alpha,p+1} = h_{\alpha,p}$ (see (4.15)) we know that $\tilde{H}'_{\alpha,p}$ and $\tilde{H}_{\alpha,p}$ have the same leading term $\int(h_{\alpha,p})$. From Lemma 4.4 it follows that the bihamiltonian conserved quantities of $(\tilde{P}_1, \tilde{P}_2)$ are uniquely determined by their leading terms, so we obtain

$$\tilde{H}'_{\alpha,p} = \tilde{H}_{\alpha,p}.$$

In particular, we know from (6.5) that $\{\tilde{H}'_{\alpha,p}, \tilde{H}'_{\beta,q}\}_{\tilde{P}_1} = 0$, and

$$\tilde{X}'_{\alpha,p} = -[\tilde{P}_1, \tilde{H}'_{\alpha,p}] = -[\tilde{P}_1, \tilde{H}_{\alpha,p}] = \tilde{X}_{\alpha,p}.$$

Recall that

$$\tilde{P}_1 = P_1 = \frac{1}{2} \int \left(\eta^{\alpha\beta} \bar{\theta}_\alpha \bar{\theta}^1_\beta\right), \quad \text{where } \eta^{\alpha\beta} = \langle dv^\alpha, dv^\beta \rangle_{g_1},$$

so the evolutionary vector field $\tilde{\partial}_{\alpha,p}$ associated with $\tilde{X}_{\alpha,p}$ is given by

$$\frac{\partial v^\beta}{\partial t^{\alpha,p}} = \eta^{\beta\gamma} \frac{\partial}{\partial x} \left(\frac{\delta \tilde{H}_{\alpha,p}}{\delta v^\gamma}\right), \quad \alpha, \beta = 1, \dots, n, p \geq 0.$$

Denote $V = \eta_{1\beta} v^\beta$, then

$$\frac{\partial V}{\partial t^{\alpha,p}} = \eta_{1\beta} \eta^{\beta\gamma} \frac{\partial}{\partial x} \left(\frac{\delta \tilde{H}_{\alpha,p}}{\delta v^\gamma}\right) = \frac{\partial}{\partial x} \left(\frac{\delta \tilde{H}_{\alpha,p}}{\delta v^1}\right) = \frac{\partial \tilde{h}_{\alpha,p-1}}{\partial x},$$

from which it follows that

$$\frac{\partial}{\partial x}\left(\frac{\partial \tilde{h}_{\beta,q-1}}{\partial t^{\alpha,p}}-\frac{\partial \tilde{h}_{\alpha,p-1}}{\partial t^{\beta,q}}\right)=\frac{\partial}{\partial t^{\alpha,p}}\left(\frac{\partial V}{\partial t^{\beta,q}}\right)-\frac{\partial}{\partial t^{\beta,q}}\left(\frac{\partial V}{\partial t^{\alpha,p}}\right)=0.$$

Since the difference $\tilde{\partial}_{\alpha,p}\left(\tilde{h}_{\beta,q}\right)-\tilde{\partial}_{\beta,q}\left(\tilde{h}_{\alpha,p}\right)$ is a differential polynomial with terms of degree greater or equal to one, so it must be zero. The above computation shows that $(\tilde{P}_1,\{\tilde{h}_{\alpha,p}\})$ is a deformation of $(P_1,\{h_{\alpha,p}\})$, see Definition 5.1.

Next let us consider the action of D_Z on $\tilde{h}_{\alpha,p}$. We have

$$D_Z(\tilde{h}_{\alpha,p+1})=\frac{\partial}{\partial v^1}\frac{\delta}{\delta v^1}\tilde{H}_{\alpha,p+2}=\frac{\delta}{\delta v^1}\frac{\delta}{\delta v^1}\tilde{H}_{\alpha,p+2}=\frac{\delta}{\delta v^1}\tilde{H}_{\alpha,p+1}=\tilde{h}_{\alpha,p}. \quad (6.8)$$

Here we used the following identity for variational derivatives:

$$\frac{\partial}{\partial v^1}\frac{\delta}{\delta v^1}=\frac{\delta}{\delta v^1}\frac{\delta}{\delta v^1},$$

which is a particular case of the identity (i) of Lemma 2.1.5 in [24].

We still need to check the identities $D_Z(\tilde{h}_{\alpha,-1})=\eta_{\alpha,1}$. By using the identities given in (6.8) we know that it is equivalent to prove the identity $\delta_Z\tilde{H}_{\alpha,-1}=\eta_{\alpha,1}$. Note that the leading term $\tilde{H}^{[0]}_{\alpha,-1}=\int\left(\eta_{\alpha\beta}v^\beta\right)$ of $\tilde{H}_{\alpha,-1}$ is a Casimir of $P_1=\tilde{P}_1$, so it also belongs to $\tilde{\mathcal{H}}$. On the other hand, elements of $\tilde{\mathcal{H}}$ are determined by their leading terms, so we have $\tilde{H}_{\alpha,-1}=\tilde{H}^{[0]}_{\alpha,-1}$, which implies the desired identity. The theorem is proved. □

Remark 6.4 *Our construction* (6.7) *of the Hamiltonian densities that satisfy the tau symmetry property follows the approach given in [9] for the construction of the tau structure of the KdV hierarchy. Note that this approach was also employed in [2] to construct tau structures for the double ramification hierarchies associated with cohomological field theories.*

The deformation $(\tilde{P}_1,\{\tilde{h}_{\alpha,p}\},\tilde{Z})$ constructed in the above theorem depends on the choice of $\tilde{P}_2$. It is natural to ask: if we start from another deformation $(\hat{P}_1,\hat{P}_2;\hat{Z})$ which has the same central invariants as $(\tilde{P}_1,\tilde{P}_2;\tilde{Z})$ does, how does the result on the deformation $(\tilde{P}_1,\{\tilde{h}_{\alpha,p}\},\tilde{Z})$ change?

Without loss of generality, we can assume that both $(\hat{P}_1,\hat{P}_2;\hat{Z})$ and $(\tilde{P}_1,\tilde{P}_2;\tilde{Z})$ have been transformed to the form (6.3). If $(\hat{P}_1,\hat{P}_2)$ has the same central invariants as $(\tilde{P}_1,\tilde{P}_2)$, then there exists a Miura-type transformation e^{ad_Y} with $Y\in\hat{\mathcal{F}}^1_{\geq 2}$ such that

$$\hat{P}_a=e^{\mathrm{ad}_Y}\left(\tilde{P}_a\right),\quad a=1,2. \quad (6.9)$$

Let $\{\hat{H}_{\alpha,p}\}$, $\{\tilde{H}_{\alpha,p}\}$ be the unique deformations of the bihamiltonian conserved quantities $\{H_{\alpha,p}\}$ of (P_1,P_2) associated with $(\hat{P}_1,\hat{P}_2)$ and $(\tilde{P}_1,\tilde{P}_2)$

respectively, and let $\{\hat{h}_{\alpha,p}\}$, $\{\tilde{h}_{\alpha,p}\}$ be their densities defined as in (6.7). It follows from the uniqueness of deformations of bihamiltonian conserved quantities and the relations given in (6.9) that

$$\hat{H}_{\alpha,p} = e^{\mathrm{ad}_Y}\left(\tilde{H}_{\alpha,p}\right). \tag{6.10}$$

The corresponding bihamiltonian vector fields

$$\hat{X}_{\alpha,p} = -[P_1, \hat{H}_{\alpha,p}], \quad \tilde{X}_{\alpha,p} = -[P_1, \tilde{H}_{\alpha,p}].$$

are then related by

$$\hat{X}_{\alpha,p} = e^{\mathrm{ad}_Y}\left(\tilde{X}_{\alpha,p}\right). \tag{6.11}$$

We denote by $\{\hat{\partial}_{\alpha,p}\}$ and $\{\tilde{\partial}_{\alpha,p}\}$ the evolutionary vector fields corresponding to $\hat{X}_{\alpha,p}$ and $\tilde{X}_{\alpha,p}$ respectively. We also have the associated triples $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\}, \tilde{Z})$ and $(\hat{P}_1, \{\hat{h}_{\alpha,p}\}, \hat{Z})$ which are constructed in Theorem 6.3. Let $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ and $\{\hat{\Omega}_{\alpha,p;\beta,q}\}$ be the corresponding tau structures. Then the relation between these tau structures and the solutions of the associated tau covers of the deformed principal hierarchies is given by Theorem 5.7, and the theorem that we are to present below gives the explicit expression of the differential polynomial G.

We first note that from the condition $\hat{P}_1 = \tilde{P}_1 = P_1$ and the relation (6.9) it follows that $[P_1, Y] = 0$. Then by using the triviality of the first Poisson cohomology of P_1 we know the existence of $K \in \mathcal{F}_{\geq 1}$ such that $Y = [P_1, K]$.

Lemma 6.5 *The vector field Y and the local functional K satisfy the equations*

$$[Y, Z] = 0, \quad [K, Z] = 0.$$

Proof Denote $Z' = e^{\mathrm{ad}_Y}(Z)$, then we have

$$[Z', \hat{P}_1] = e^{\mathrm{ad}_Y}\left([Z, \tilde{P}_1]\right) = 0,$$
$$[Z', \hat{P}_2] = e^{\mathrm{ad}_Y}\left([Z, \tilde{P}_2]\right) = \hat{P}_1,$$

so $W = Z' - Z$ is a bihamiltonian vector field of $(\hat{P}_1, \hat{P}_2)$. On the other hand, $W \in \hat{\mathcal{F}}^1_{\geq 2}$, so it follows from Lemma 4.4 that $W = 0$ and, consequently, we have $[Y, Z] = 0$.

From the identity $[Y, Z] = 0$ it follows that $[P_1, [K, Z]] = 0$, so $C = [K, Z]$ is a Casimir of P_1, and thus it is also a bihamiltonian conserved quantity of (P_1, P_2). Since $C \in \mathcal{F}_{\geq 1}$, by using Lemma 4.4 again we obtain $C = 0$. The lemma is proved. □

From the above lemma it follows that

$$\int (\delta_Z K) = [Z, K] = 0,$$

so there exists $g \in \mathcal{A}$ such that

$$\delta_Z K = \partial g. \tag{6.12}$$

Lemma 6.6 *The operator*

$$D_P = \sum_{s\geq 0} \left(\partial^s \left(\frac{\delta P}{\delta \theta_\alpha} \right) \frac{\partial}{\partial u^{\alpha,s}} + (-1)^p \partial^s \left(\frac{\delta P}{\delta u^\alpha} \right) \frac{\partial}{\partial \theta_\alpha^s} \right), \quad P \in \hat{\mathcal{F}}^p$$

and the bracket

$$[P, Q] = \int \left(\frac{\delta P}{\delta \theta_\alpha} \frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha} \frac{\delta Q}{\delta \theta_\alpha} \right), \quad P \in \hat{\mathcal{F}}^p,\ Q \in \hat{\mathcal{F}}^q$$

satisfy the following identities:

$$\begin{aligned} &[\partial, D_P] = 0; \\ &\frac{\delta}{\delta u^\alpha}[P, Q] = D_P \left(\frac{\delta Q}{\delta u^\alpha} \right) + (-1)^{pq} D_Q \left(\frac{\delta P}{\delta u^\alpha} \right); \\ &(-1)^{p-1} D_{[P,Q]} = D_P \circ D_Q - (-1)^{(p-1)(q-1)} D_Q \circ D_P. \end{aligned}$$

Proof The first identity can be obtained from the definition of D_P. The second one is a corollary of the identity (iii) of Lemma 2.1.3 and the identity (i) of Lemma 2.1.5 given in [24]. The third identity is a corollary of the second one. The lemma is proved. □

Theorem 6.7 *The differential polynomial G of Theorem 5.7 is given by the formula*

$$G = \sum_{i=1}^{\infty} \frac{1}{i!} D_Y^{i-1}(g), \tag{6.13}$$

where the function g is defined in (6.12).

Proof From our construction of the densities of the Hamiltonians we have

$$\hat{h}_{\alpha,p} = \delta_Z \hat{H}_{\alpha,p+1}, \quad \tilde{h}_{\alpha,p} = \delta_Z \tilde{H}_{\alpha,p+1},$$

then from (6.10) it follows that

$$\hat{h}_{\alpha,p} = \delta_Z \left(e^{\mathrm{ad}_Y} \left(\tilde{H}_{\alpha,p+1} \right) \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \delta_Z \left(\mathrm{ad}_Y^k \left(\tilde{H}_{\alpha,p+1} \right) \right).$$

By using the the definition (6.6) of δ_Z and the identities given in Lemma 6.6 we can show that

$$\delta_Z \left(\mathrm{ad}_Y^k \left(\tilde{H}_{\alpha,p+1} \right) \right) = D_Y^k \left(\tilde{h}_{\alpha,p} \right) + \sum_{i=1}^{k} \binom{k}{i} D_{\mathrm{ad}_Y^{k-i}\left(\tilde{H}_{\alpha,p+1}\right)} D_Y^{i-1} \left(\delta_Z Y \right),$$

so we have

$$\begin{aligned}\hat{h}_{\alpha,p} =&\sum_{k=0}^{\infty}\frac{1}{k!}\left(D_Y^k\left(\tilde{h}_{\alpha,p}\right)+\sum_{i=1}^{k}\binom{k}{i}D_{\mathrm{ad}_Y^{k-i}(\tilde{H}_{\alpha,p+1})}D_Y^{i-1}\left(\delta_Z Y\right)\right)\\ =&e^{D_Y}\left(\tilde{h}_{\alpha,p}\right)+D_{\hat{H}_{\alpha,p+1}}\left(\sum_{i=1}^{\infty}\frac{1}{i!}D_Y^{i-1}\left(\delta_Z Y\right)\right)\end{aligned}$$

By using the fact that

$$\delta_Z Y=\delta_Z[P_1,K]=D_{P_1}(\delta_Z K)=\partial D_{P_1}(g),$$

and $[D_Y,D_{P_1}]=0$, $[\partial,D_Q]=0$ for $Q\in\hat{\mathcal{F}}$ (see Lemma 6.6), we obtain

$$\hat{h}_{\alpha,p}=e^{D_Y}\left(\tilde{h}_{\alpha,p}\right)+\partial D_{\hat{H}_{\alpha,p+1}}D_{P_1}G,$$

where

$$G=\sum_{i=1}^{\infty}\frac{1}{i!}D_Y^{i-1}\left(g\right).$$

Then by using the identity (see Lemma 6.6)

$$D_H D_{P_1}=D_{-[P_1,H]}-D_{P_1}D_H,$$

and the fact that $D_H\left(G\right)=0$, we have

$$\hat{h}_{\alpha,p}=e^{D_Y}\left(\tilde{h}_{\alpha,p}\right)+\partial D_{\hat{X}_{\alpha,p+1}}G=e^{D_Y}\left(\tilde{h}_{\alpha,p}\right)+\partial\hat{\partial}_{\alpha,p}G.$$

The theorem is proved. □

Theorem 6.3 gives the existence part of Theorem 2.10, and Theorem 6.7 (combining with Theorem 5.7) gives the uniqueness part.

There are two important examples of such deformations when the flat exact semisimple bihamiltonian structures is provided by a semisimple cohomological field theory. In [9] the first- and the third-named authors constructed, for any semisimple Frobenius manifold, the so-called topological deformation of the associated principal hierarchy and its tau structure. They conjectured that the bihamiltonian structure of the principal hierarchy also has an associated deformation. As we mentioned in Example 5.12, in [1] Buryak constructed a Hamiltonian integrable hierarchy associated with any cohomological field theory, and in [2] he and his collaborators showed that this integrable hierarchy also possesses a tau structure. Buryak conjectured in [1] that the above two integrable hierarchies are equivalent via a Miura-type transformation. According to the above two conjectures, Buryak's integrable hierarchy should also possess a bihamiltonian structure. Buryak and his collaborators further refined their conjecture in [2] as an equivalence between tau-symmetric Hamiltonian deformations via a normal Miura-type transformation. The notion of normal Miura-type transformation was introduced in [12], our Definition 5.6 (see also Theorem 5.7) is a kind of its generalization. We hope that our results could be useful to solve the above mentioned conjectures.

7 Conclusion

We consider in this paper the integrable hierarchies associated with a class of flat exact semisimple bihamiltonian structures of hydrodynamic type. This property of flat exactness enables us to associate to any semisimple bihamiltonian structure of hydrodynamic type a Frobenius manifold structure (without the Euler vector field), and a bihamiltonian integrable hierarchy which is called the principal hierarchy. We show that this principal hierarchy possesses a tau structure and also the Galilean symmetry. For any deformation of the flat exact semisimple bihamiltonian structures of hydrodynamic type which has constant central invariants, we construct the deformation of the principal hierarchy and show the existence of tau structure and Galilean symmetry for this deformed integrable hierarchy. We also describe the ambiguity of the choice of tau structure for the deformed integrable hierarchy. Our next step is to study properties of the Virasoro symmetries that are inherited from the Galilean symmetry of the deformed integrable hierarchy in order to fix an appropriate representative of the tau structures which, in the case associated with a cohomological field theory, corresponds to the partition function. We will do it in a subsequent publication.

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A On semi-Hamiltonian hierarchies

In this appendix, we prove a classification theorem for conserved quantities and symmetries of a semi-Hamiltonian system satisfying certain nondegeneracy conditions.

Definition A.1 ([30]) *i) A system of evolutionary partial differential equations of the form*

$$\frac{\partial u^i}{\partial t} = A^i(u)u^{i,1}, \quad i = 1, \dots, n \tag{A.1}$$

is called semi-Hamiltonian, if $A^i \neq A^j$ for $i \neq j$ and

$$\partial_k \left(\frac{\partial_j A^i}{A^j - A^i} \right) = \partial_j \left(\frac{\partial_k A^i}{A^k - A^i} \right), \quad \text{for distinct } i, j, k,$$

where $\partial_k = \frac{\partial}{\partial u^k}$.

ii) A semi-Hamiltonian system of evolutionary partial differential equations is called nondegenerate if $\partial_i A^i \neq 0$ for all $i = 1, \ldots, n$. We denote $A^i_i = \partial_i A^i$ henceforth.

Remark A.2 *Suppose $(P_1, P_2; Z)$ is an irreducible flat exact semisimple bihamiltonian structure of hydrodynamic type. We have shown in Lemma 4.21 that there exists a nondegenerate bihamiltonian vector field X, which has the following form*

$$X = \int \left(\sum_{i=1}^{n} A^i(u) u^{i,1} \theta_i \right) \in \hat{\mathcal{F}}^1_1.$$

Since X is a bihamiltonian vector field, the associated system of evolutionary partial differential equations (A.1) *is semi-Hamiltonian [30]. Thus the results of Theorem A.3 and Corollary A.4 can be applied to such a bihamiltonian vector field. In fact, we used the results of Corollary A.4 in the proof of the equations* (5.6) *and* (6.5)*.*

According to Tsarev's results [30], a semi-Hamiltonian system has infinitely many conserved quantities of the form

$$H = \int h(u) \in \hat{\mathcal{F}}^0_0,$$

and infinitely many symmetries of the form

$$Y = \int \left(\sum_{i=1}^{n} B^i(u) u^{i,1} \theta_i \right) \in \hat{\mathcal{F}}^1_1.$$

An important question is: Are there conserved quantities and symmetries with higher degrees, which belong to $\hat{\mathcal{F}}^0_{\geq 1}$ and $\hat{\mathcal{F}}^1_{\geq 2}$? The following Theorem and Corollary give us the answer.

Theorem A.3 *Let*

$$X = \int \left(\sum_{i=1}^{n} A^i(u) u^{i,1} \theta_i \right) \in \hat{\mathcal{F}}^1_1$$

be the vector field associated with a nondegenerate semi-Hamiltonian system of evolutionary partial differential equations of the form (A.1)*. The following assertions hold true:*

i) If $H \in \hat{\mathcal{F}}^0_{\geq 1}$ satisfies $[X, H] = 0$, then $H = 0$.

ii) If $Y \in \hat{\mathcal{F}}^1_{\geq 2}$ satisfies $[X, Y] = 0$, then $Y = 0$.

Proof i) Denote by $\mathcal{A}^{(N)}$ the space of differential polynomials that do not depend on $u^{i,s}$ with $s > N$. Suppose $H = \int h$, where $h \in \mathcal{A}_d \cap \mathcal{A}^{(N)}$ with $d \geq 1$, $N \geq 0$. It is obvious that if $N = 0$, then $H = 0$. The following proof is based on an induction on N. We denote $\delta_i = \frac{\delta}{\delta u^i}$ and define $Z_i = \delta_i[X, H]$. Since $[X, H] = 0$, we know that $Z_i = 0$. On the other hand, by using Lemma 6.6 we have

$$\begin{aligned} Z_i &= D_X\left(\delta_i H\right) + D_H\left(\delta_i X\right) \\ &= \sum_{k\geq 0}\sum_{j=1}^{n} \partial^k\left(A^j u^{j,1}\right)\frac{\partial\left(\delta_i H\right)}{\partial u^{j,k}} + \sum_{j=1}^{n} \partial_i A^j u^{j,1}\delta_j H - \partial\left(A^i \delta_i H\right), \end{aligned}$$

from which it follows that

$$0 = (-1)^N \left(Z_i\right)_{(k,2N+1)} = \left(A^k - A^i\right) h_{(k,N)(i,N)},$$

where $(f)_{(i,t)} = \frac{\partial f}{\partial u^{i,t}}$ for $f \in \mathcal{A}$. The above equation implies that

$$h_{(k,N)(i,N)} = 0, \quad i \neq k,$$

so we can assume that

$$h = \sum_i h_i(u, \dots, u^{(N-1)}; u^{i,N}).$$

If we can prove that h_i depends on $u^{i,N}$ linearly, that is

$$h = \sum_i g_i(u, \dots, u^{(N-1)})u^{i,N} + R(u, \dots, u^{(N-1)}),$$

then for $k \neq i$ we have

$$0 = (-1)^N \left(Z_i\right)_{(k,2N)} = \left(A^k - A^i\right)\left(\left(g_i\right)_{(k,N-1)} - \left(g_k\right)_{(i,N-1)}\right).$$

So there exists $g \in \mathcal{A}_{d-1} \cap \mathcal{A}^{(N-1)}$ such that $g_{(i,N-1)} = g_i$. In particular, $h - \partial g \in \mathcal{A}_d \cap \mathcal{A}^{(N-1)}$. Then the first part of the theorem can be proved by induction on N.

Now let us proceed to prove the linear dependence of h_i on $u^{i,N}$. Define

$$Y_i = \left(h_i\right)_{(i,N)(i,N)}, \quad i = 1, \dots, n,$$

then we have the following identity:

$$\begin{aligned} 0 =& (-1)^N \left(Z_i\right)_{(i,2N)} \\ =& \sum_{k\geq 0}\sum_{j=1}^{n} \partial^k\left(X^j\right)\left(Y_i\right)_{(j,k)} - A^i \partial Y_i + \left(2A^i_i u^{i,1} + (2N-1)\partial A^i\right) Y^i, \quad \text{(A.2)} \end{aligned}$$

where $X^j = A^j u^{j,1}$. We need to show that $Y_i = 0$.

Let us fix an index i and consider Y_i as a polynomial in $u^{i,1}, \dots, u^{i,N}$, and expand it as

$$Y_i = \sum_\beta c_\beta \left(u^{i,1}\right)^{\beta_1} \cdots \left(u^{i,N}\right)^{\beta_N},$$

where $\beta = (\beta_1, \dots, \beta_N)$, and c_β do not depend on $u^{i,1}, \dots, u^{i,N}$. We define a lexicographical order on the set of monomials recursively:

$$\left(u^{i,1}\right)^{\beta_1} \cdots \left(u^{i,N}\right)^{\beta_N} \preceq \left(u^{i,1}\right)^{\gamma_1} \cdots \left(u^{i,N}\right)^{\gamma_N}$$

iff $\beta_N < \gamma_N$ or $\beta_N = \gamma_N$, and

$$\left(u^{i,1}\right)^{\beta_1} \cdots \left(u^{i,N-1}\right)^{\beta_{N-1}} \preceq \left(u^{i,1}\right)^{\gamma_1} \cdots \left(u^{i,N-1}\right)^{\gamma_{N-1}}.$$

Then Y_i can be written as the sum of its leading term and the remainder

$$Y_i = c\, w + R,$$

where $w = \left(u^{i,1}\right)^{\beta_1} \cdots \left(u^{i,N}\right)^{\beta_N}$.

The equation (A.2) now reads

$$0 = \left(\sum_{k\geq 0}\sum_{j=1}^n \partial^k X^j (c)_{(j,k)} - A^i \partial c\right) w + \left(\sum_{k\geq 0}\sum_{j=1}^n \partial^k X^j (w)_{(j,k)} - A^i \partial w\right) c$$
$$+ c\left(2A^i_i u^{i,1} + (2N-1)\partial A^i\right) w + \tilde{R},$$

where $\tilde{R}$ is the sum of terms coming from R. Consider the quotient of the above equation modulo w:

$$0 = \sum_{k\geq 0}\sum_{j=1}^n \partial^k X^j (c)_{(j,k)} - A^i \partial c + \big((2 + \beta_2 + \cdots + \beta_N)\, A^i_i u^{i,1}$$
$$+ (2N - 1 + \beta_1 + 2\beta_2 + \cdots + N\beta_N)\, \partial A^i\big)\, c. \tag{A.3}$$

Suppose $c \in \mathcal{A}^{(m)}$ $(m \geq 1)$, the derivative with respect to $u^{k,m+1}$ of the above equation gives

$$0 = \left(A^k - A^i\right)(c)_{(k,m)} \quad \Rightarrow \quad (c)_{(k,m)} = 0 \text{ for } k \neq i.$$

On the other hand $(c)_{(i,m)} = 0$ by definition, so we have $c \in \mathcal{A}^{(m-1)}$. An induction on m implies that $c \in \mathcal{A}_0$.

Finally, take the leading term of (A.3) with respect to the lexicographical order, we obtain

$$(\beta_1 + 3\beta_2 + \cdots + (N+1)\beta_N + 2N + 1)\, A^i_i u^{i,1}\, c = 0.$$

By using the nondegeneracy condition $A^i_i \neq 0$ we obtain $c = 0$, and consequently we deduce that $Y_i = 0$. The first part of the theorem is proved.

ii) Suppose $Y = \int \left(\sum_{i=1}^n B^i\theta_i\right)$, where $B^i \in \mathcal{A}_d \cap \mathcal{A}^{(N)}$ with $d \geq 2, N \geq 1$, and define $Z^i = \frac{\delta}{\delta\theta_i}[X, Y]$, then we have

$$0 = Z^i = \sum_{k\geq 0}\sum_{j=1}^n \partial^k X^j B^i_{(j,k)} - \sum_{j=1}^n B^j X^i_{(j,0)} - A^i \partial B^i.$$

The equation $Z^i_{(k,N+1)} = 0$ implies that $B^i_{(k,N)} = 0$ for $k \neq i$. Denote $Y^i = B^i_{(i,N)}$, then we have

$$0 = Z^i_{i,N} = \sum_{k\geq 0}\sum_{j=1}^n \partial^k X^j Y^i_{(j,k)} - A^i \partial Y^i + N\partial A^i\, Y^i - \sum_{j=1}^n B^j A^i_{(j,0)}\delta_{N,1}.$$

When $N \geq 2$, by using a similar argument as the one that we used in the proof of the first part of the theorem, we can show that the above equations have the only solution $Y^i = 0$. When $N = 1$, suppose $B^i = c\left(u^{i,1}\right)^\beta + R$, then the leading term of the above equation reads

$$\left(\beta^2 - 1\right) A^i_i \left(u^{i,1}\right)^\beta c = 0,$$

so we have $\beta = 1$ or $c = 0$. The second part of the theorem is proved. □

Corollary A.4 *Given a vector field*

$$X = X_1 + X_2 + \cdots \in \hat{\mathcal{F}}^1_{\geq 1},$$

where $X_d \in \hat{\mathcal{F}}^1_d$, and X_1 is nondegenerate semi-Hamiltonian. Then the following statements hold true:

i) If $H \in \mathcal{F}$ is a conserved quantity of X, i.e. $[X, H] = 0$, then H can be represented as

$$H = H_0 + H_1 + H_2 + \cdots,$$

where H_0 is a conserved quantity of X_1, and $H_d \in \mathcal{F}_d\,(d \geq 1)$ are uniquely determined by H_0.

ii) If $Y \in \hat{\mathcal{F}}^1_{\geq 1}$ is a symmetry of X, i.e. $[X, Y] = 0$, then Y can be represented as

$$Y = Y_1 + Y_2 + \cdots$$

where Y_1 is a symmetry of X_1, and $Y_d \in \hat{\mathcal{F}}^1_d\,(d \geq 2)$ are uniquely determined by Y_1.

The proof follows easily from Theorem A.3, so we omit it.

References

[1] Buryak A.: Double ramification cycles and integrable hierarchies, Comm. Math. Phys. **336**, 1085–1107 (2015)

[2] Buryak A., Dubrovin B., Guéré J., Rossi P.: Tau-structure for the Double Ramification Hierarchies, eprint arXiv: 1602.05423.

[3] Buryak, A., Posthuma, H., Shadrin, S.: On deformations of quasi-Miura transformations and the Dubrovin-Zhang bracket. J. Geom. Phys. **62**, 1639–1651 (2012)

[4] Buryak, A., Posthuma, H., Shadrin, S.: A polynomial bracket for the Dubrovin-Zhang hierarchies. J. Diff. Geom. **92**, 153–185 (2012)

[5] Carlet, G., Posthuma, H., Shadrin, S.: Deformations of semisimple Poisson pencils of hydrodynamic type are unobstructed. J. Diff. Geom. **108**, 63–89 (2018)

[6] Date, E., Kashiwara, M., Jimbo, M., Miwa, T.: Transformation groups for soliton equations. Nonlinear integrable systems–classical theory and quantum theory (Kyoto, 1981), 39–119, World Sci. Publishing, Singapore, 1983.

[7] Dubrovin, B.: Geometry of 2D topological field theories. Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math. **1620**, Springer, Berlin, 1996.

[8] Dubrovin, B.: Painlevé transcendents in two-dimensional topological field theory. The Painlevé property, 287–412, CRM Ser. Math. Phys., Springer, New York, 1999.

[9] Dubrovin, B., Zhang, Y.: Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants. eprint arXiv: math/0108160.

[10] Dubrovin, B., Liu, S.-Q., Zhang, Y.: On Hamiltonian perturbations of hyperbolic systems of conservation laws I: Quasi-triviality of bi-Hamiltonian perturbations. Comm. Pure Appl. Math. **59**, 559–615 (2006)

[11] Dubrovin, B., Novikov, S. P.: Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method. (Russian) Dokl. Akad. Nauk SSSR **270**, 781–785 (1983)

[12] Dubrovin, B., Liu, S.-Q., Yang D., Zhang, Y. : Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs. Adv. Math. **293**, 382–435 (2016)

[13] Enriquez, B., Frenkel, E.: Equivalence of two approaches to integrable hierarchies of KdV type. Comm. Math. Phys. **185**, 211–230 (1997)

[14] Falqui, G., Lorenzoni, P.: Exact Poisson pencils, τ-structures and topological hierarchies. Phys. D **241**, 2178–2187 (2012)

[15] Ferapontov, E. V.: Compatible Poisson brackets of hydrodynamic type. J. Phys. A **34**, 2377–2388 (2001)

[16] Hirota, R.: Exact envelope-soliton solutions of a nonlinear wave equation. J. Math. Phys. **14**, 805–809 (1973)

[17] Hirota, R.: The direct method in soliton theory. Cambridge Tracts in Mathematics, **155**. Cambridge University Press, Cambridge, 2004.

[18] Jimbo, M., Miwa, T., Ueno, K.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function. Phys. D **2**, 306–352 (1981)

[19] Jimbo, M., Miwa, T.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. Phys. D **2**, 407–448 (1981)

[20] Jimbo, M., Miwa, T.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III. Phys. D **4**, 26–46 (1981/82)

[21] Kac, V., Wakimoto, M.: Exceptional hierarchies of soliton equations. Theta functions–Bowdoin 1987, Part 1 (Brunswick, ME, 1987), 191–237, Proc. Sympos. Pure Math., 49, Part 1, Amer. Math. Soc., Providence, RI, 1989.

[22] Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. Comm. Math. Phys **147**, 1–23 (1992)

[23] Liu, S.-Q., Zhang, Y.: Deformations of semisimple bihamiltonian structures of hydrodynamic type. J. Geom. Phys. **54**, 427–453 (2005)

[24] Liu, S.-Q., Zhang, Y.: Jacobi structures of evolutionary partial differential equations. Adv. Math. **227**, 73–130 (2011)

[25] Liu, S.-Q., Zhang Y.: Bihamiltonian cohomologies and integrable hierarchies I: a special case. Comm. Math. Phys. **324**, 897–935 (2013)

[26] Miramontes, J. L.: Tau-functions generating the conservation laws for generalized integrable hierarchies of KdV and affine Toda type. Nuclear Phys. **B 547**, 623–663 (1999)

[27] Sato, M.: Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. RIMS Kokyuroku **439**, 30–46 (1981)

[28] Sato, M., Sato, Y.: Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. In: Nonlinear Partial Differential Equations in Applied Science (Tokyo, 1982), 259–271, North-Holland Math. Stud., 81, North-Holland, Amsterdam, 1983.

[29] Segal, G., Wilson, G.: Loop groups and equations of KdV type. Publ. Math. IHES 61, 5–65 (1985).

[30] Tsarev, S.: Geometry of Hamiltonian systems of hydrodynamic type. Generalized hodograph method. Izv. Akad. Nauk SSSR, Ser. Mat., 1990.

[31] Witten, E.: Two-dimensional gravity and intersection theory on moduli space. Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.

[32] Wu, C.-Z.: Tau Functions and Virasoro Symmetries for Drinfeld-Sokolov Hierarchies. Adv. Math. 306, 603–652 (2017)

[33] Xue, T., Zhang, Y.: Bihamiltonian systems of hydrodynamic type and reciprocal transformations. Lett. Math. Phys. **75**, 79–92 (2006)

[34] Zhang, Y.: Deformations of the bihamiltonian structures on the loop space of Frobenius manifolds. Recent advances in integrable systems (Kowloon, 2000). J. Nonlinear Math. Phys. **9** suppl. 1, 243–257 (2002)