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# Separation of variables for linear Lax algebras and classical $r$-matrices

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**Abstract**

We consider the problem of separation of variables for the algebraically integrable Hamiltonian systems possessing $gl(n)$-valued Lax matrices depending on a spectral parameter that satisfy linear Poisson brackets with some $gl(n)\otimes gl(n)$-valued classical $r$-matrices. We formulate, in terms of the corresponding $r$-matrices, a sufficient condition that guarantees that the "separating polynomials" of E.Sklyanin, *Comm. Math. Phys.* **150**, 181 (1992), D.Scott, *J. Math. Phys.* **35**, 5831 (1994), M.Gekhtman, *Comm. Math. Phys.* **167**, 593 (1995), P.Diener, B.Dubrovin, *Algebraic-geometrical Darboux coordinates in R-matrix formalism*, SISSA preprint 88-94-FM (1994), produce a system of canonical variables. We consider two examples of classical $r$-matrices and separating polynomials. One of these examples, namely, the $n$-parametric family of non-skew-symmetric non-dynamical classical $r$-matrices of T.Skrypnyk, *Phys. Lett. A* **334**, 390, and **347**, 266 (2005) and the corresponding separating polynomials is new. We show that the separating polynomials of P.Diener, B.Dubrovin, *ibid.*, produce in this case a complete set of separated variables for the corresponding generalized Gaudin models with or without external magnetic field.

Keywords: integrable systems, classical $r$-matrices, separation of variables

*Dedicated to the memory of Ludvig Faddeev*

## 1 Introduction

The algebraically integrable Hamiltonian systems admitting Lax representation

$$\dot{L}(u) = [L(u), M(u)] \tag{1.1}$$

have been an object of constant interest during the last forty years. There were a lot of papers (both in the mathematical and physics literature) and results in the field. One of the most important "structural" results in the theory was a discovery of classical $r$-matrices [19, 17, 10] that serve in order to define the Poisson brackets between the elements of the Lax matrices [10] providing a necessary and sufficient condition for Poisson commutativity of the spectral invariants of the Lax matrices [4].

An important problem in the theory of algebraically integrable systems still to be solved in general is a construction of the so-called separated coordinates for the Lax-integrable finite-dimensional Hamiltonian systems. The separated coordinates $x_i, p_j$, $i, j \in \overline{1,d}$ is a set of canonical coordinates that separate the variables in the Hamilton–Jacobi equation or, more generally, such that the following system of equations is satisfied:

$$\Phi_i(x_i, p_i; I_1, ..., I_d) = 0, \; i \in \overline{1,d},$$

where $\Phi_i$ are certain functions, $I_k$ are Poisson-commuting integrals of motion. The important requirement is that the separated coordinates constitute a complete family, i.e. they are independent variables and $d$ is equal to the half of the dimension of symplectic leaves of the phase space

The separated coordinates provide a way to a construction of the action-angle variables from the Liouville theorem and to explicit integration of the Hamiltonian equations of motion. They are also important for solving the corresponding quantum integrable systems.

## 1.1 Separated coordinates of algebro-geometric type

There exists a "magic recipe" for constructing separated coordinates of an algebraically integrable system (1.1). Let the Lax matrix $L(u)$ depending on the dynamical variables of the Hamiltonian system be a rational function of the auxiliary spectral parameter $u$ living on the complex plain or, more generally, on a compact algebraic curve. Consider the *spectral curve* $\mathcal{C}$ on the $(u, \mu)$-plane defined by the characteristic equation

$$\mathcal{C} = \{(u, \mu) \,|\, \det(L(u) - \mu \cdot 1) = 0\}\,. \tag{1.2}$$

Recall that the coefficients of the characteristic polynomial are first integrals of the Hamiltonian system (1.1). Assume that the roots of the characteristic equation (1.2) are pairwise distinct for generic $u \in \mathbb{C}$. Then the eigenvectors[1] $\vec{f} = (f_1, \ldots, f_n)$ of the Lax operator,

$$\vec{f}\, L(u) = \mu\, \vec{f} \tag{1.3}$$

can be considered as sections of a line bundle over the spectral curve (1.2). Then the so-called separated variables of *algebro-geometric type* are obtained as coordinates

$$x_i = u(Q_i), \quad p_i = \mu(Q_i), \quad i = 1, \ldots, D \tag{1.4}$$

of the points of the divisor $Q_1 + \cdots + Q_d$ of poles of a section of the line bundle [2, 11, 16, 20, 1, 7, 14, 18, 21, 23]. For example, for the so-called *standard'* normalization

$$f_n = 1 \tag{1.5}$$

[1]For technical reasons we deal with the eigenvectors of the adjoint operator writing them as row-vectors, i.e., as elements of the dual space.

of the eigenvector $\vec{f} = (f_1, \ldots, f_n)$ the points $Q_1, \ldots, Q_d$ in eq. (1.4) on the spectral curve are determined by the equation $f_n = 0$ together with (1.3). Also a slightly more general normalization

$$\kappa_1 f_1 + \cdots + \kappa_n f_n = 1 \tag{1.6}$$

with an arbitrary constant vector $\vec{\kappa} = (\kappa_1, \kappa_2, ..., \kappa_n)$ will be used below. In this case the poles are such points $(u, \mu)$ of the spectral curve that there exists an eigenvector (1.3) satisfying

$$\kappa_1 f_1 + \cdots + \kappa_n f_n = 0. \tag{1.7}$$

Of course, only the points in the finite part $|u| < \infty$, $|\mu| < \infty$ of the spectral curve are involved in this construction.

### 1.2 Poisson algebra of separated functions

For the coordinates $(x_i, p_i)$ of the poles of a suitably normalized eigenvector $\vec{f}(u, \mu)$, $(u, \mu) \in \mathbb{C}$ one obtains a complicated system of algebraic equations. In many cases one can construct a pair of *separating functions* $A(u)$, $B(u)$ such that the coordinates $x_1, \ldots, x_d$ are determined as zeroes of $B(u)$,

$$B(x_i) = 0, \quad i \in \overline{1, d}$$

and the second half of coordinates is obtained as

$$p_i = A(x_i), \quad i \in \overline{1, d}.$$

The separating functions were constructed in [18], [14], [22] for some $gl(n)$-valued Lax matrices $L(u)$ using the standard normalization (1.5) of the eigenvectors and in [7], [22] for the more general normalization (1.6) (they will be denoted $A_{\vec{\kappa}}(u)$, $B_{\vec{\kappa}}(u)$ below).

The above separating functions $B(u)$, $A(u)$ and $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ formally exist for *all* $gl(n)$-valued Lax matrices $L(u)$ governed by *arbitrary* classical $r$-matrix $r(u, v)$. Nevertheless two important questions are still open. They are:

1. For what $gl(n) \otimes gl(n)$-valued classical $r$-matrices $r(u, v) = \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u, v) X_{ij} \otimes X_{kl}$ ($X_{ij}$, $i, j \in \overline{1, n}$ is a standard basis of $gl(n)$) the functions $B(u)$ and $A(u)$ and/or $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ for some normalizing vector $\vec{k}$ produce the *canonical variables*?

2. For what $gl(n) \otimes gl(n)$-valued classical $r$-matrices $r(u, v) = \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u, v) X_{ij} \otimes X_{kl}$, Lax operators $L(u) = \sum_{i,j=1}^{n} L_{ij}(u) X_{ij}$ and the normalizing vector $\vec{k}$ the functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ produce a *complete set* of canonical coordinates?

In the present paper we give a general answer to the first question, in terms of the corresponding $r$-matrices, in the case of linear Poisson algebras of Lax matrices:

$$\{L(u_1) \otimes 1, 1 \otimes L(u_2)\} = [r^{12}(u_1, u_2), L(u_1) \otimes 1] - [r^{21}(u_2, u_1), 1 \otimes L(u_2)]. \tag{1.8}$$

It will be assumed that the Lax matrix is an analytic function on an open disk $u \in \mathcal{D} \subset \mathbb{C}$. The spectral curve (1.2) is assumed to be irreducible for generic values of the dynamical variables. We will consider only the $r$-matrices $r(u,v)$ that, by re-parametrization, gauge transformations and multiplication by functions of the second spectral parameter admit the following decomposition

$$r(u,v) = \frac{\Omega}{u-v} + \Delta r(u,v), \quad u,\, v \in \mathcal{D} \tag{1.9}$$

where $\Omega = \sum\limits_{i,j=1}^{n} X_{ij} \otimes X_{ji}$ is the tensor Casimir and $\Delta r(u,v)$ is a regular on the diagonal $u = v$ function with values in $gl(n) \otimes gl(n)$, possibly depending also on the dynamical variables. It follows from the results of [6] that the regularity condition (1.9) always holds true for the non-degenerate non-dynamical skew-symmetric classical $r$-matrices.

The answer to the first question is given by the following

**Theorem 1.1** *For a given non-zero vector $\vec{\kappa} = (\kappa_1, \ldots, \kappa_n)^t$, denote $(x_i, p_i) \in \mathcal{C}$, $i = \overline{1,d}$ the coordinates of the finite poles of the eigenvector $\vec{f} = (f_1, \ldots, f_n)$ of the Lax matrix $L(u)$,*

$$\vec{f}\, L(u) = \mu\, \vec{f}, \quad \det(L(u) - \mu \cdot 1) = 0$$

*normalized by the condition*

$$\kappa_1 f_1 + \cdots + \kappa_n f_n = 1 \tag{1.10}$$

*such that $x_i \in \mathcal{D}$. Assume all the poles of $\vec{f}$ to be simple and, moreover, $x_i \neq x_j$ for $i \neq j$. The Poisson brackets* (1.8) *between these variables satisfy*

$$\{x_i, x_j\} = \{p_i, p_j\} = 0 \quad \forall\, i,\, j$$
$$\{x_i, p_j\} = 0, \quad i \neq j$$

*if the $r$-matrix satisfies the following condition:*

$$\sum_{i,\,k,\,s=1}^{n} r_{is,kl}(u,v) \kappa_s \alpha_i \beta_k \equiv 0, \quad l = 1, \ldots, n \tag{1.11}$$
$$\forall\ \alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n) \quad \text{such that} \quad \alpha(\vec{\kappa}) = \beta(\vec{\kappa}) = 0$$

*where $\alpha(\vec{\kappa}) = \sum_{i=1}^{n} \alpha_i \kappa_i$, $\beta(\vec{\kappa}) = \sum_{i=1}^{n} \beta_i \kappa_i$. Moreover, if the $r$-matrix also satisfies the regularity condition* (1.9) *then the Poisson brackets are canonical*

$$\{x_i, p_j\} = \delta_{ij}.$$

Observe that, in the particular case $\vec{\kappa} = (0, 0, ..., 0, 1)^t$ corresponding to the "standard" normalization of the eigenvector of the Lax matrix the condition (1.11) takes the following simple form:

$$r_{in,jk}(u,v) = 0 \quad \forall\, i, j \in \overline{1, n-1}, \quad k \in \overline{1, n}. \tag{1.12}$$

The proof of Theorem 1.1 is based on a certain structure of the Poisson algebra of separating functions (cf. [14] where a particular case of an algebra of separating functions was considered) which is sufficient for them in order to produce the canonical coordinates:

**Lemma 1.1** *Let $A(u)$, $B(u)$ be functions of the dynamical variables depending also on the spectral parameter $u$ satisfying the following Poisson algebra relations*

$$\{B(u), B(v)\} = b(u, v)B(u) - b(v, u)B(v) \tag{1.13a}$$

$$\{A(u), B(v)\} = \alpha(u, v)B(u) - \beta(u, v)B(v) \tag{1.13b}$$

$$\{A(u), A(v)\} = a(u, v)B(u) - a(v, u)B(v). \tag{1.13c}$$

*for some functions $a(u, v)$, $b(u, v)$, $\alpha(u, v)$, $\beta(u, v)$. Define variables $x_i$, $p_i$, $i = \overline{1, d}$ as zeroes of $B(u)$,*

$$B(x_i) = 0 \tag{1.14}$$

*and values of $A(u)$ at these points,*

$$p_i = A(x_i). \tag{1.15}$$

*Then the Poisson brackets among $x_i$ and $p_j$, $\forall i, j \in \overline{1, d}$ are quasi-canonical, i.e:*

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \ \forall i, j \in \overline{1, d}$$

$$\{x_j, p_i\} = 0 \text{ if } i \neq j,$$

*If, moreover, also the condition*

$$\lim_{u\to v}(\alpha(u, v)B(u) - \beta(u, v)B(v)) = \partial_v B(v) + \gamma(v)B(v). \tag{1.16}$$

*holds true for some $\gamma(v)$ then the corresponding Poisson brackets are canonical, i.e.*

$$\{x_i, p_i\} = 1, \quad \forall i \in \overline{1, d}.$$

## 1.3 Constructing separating functions

How to derive the separating functions? Denote $\Delta(u, \mu) = (\Delta_{ij}(u, \mu))$ the $n \times n$ matrix of cofactors of $L(u) - \mu \cdot 1$. As an eigenvector of the matrix $L(u)$ with the eigenvalue $\mu = \mu(u)$ is proportional to

$$(\Delta_{1j}(u, \mu), \dots, \Delta_{nj}(u, \mu))$$

for any $j = 1, \dots, n$, we arrive at the following system of equations for the poles of the eigenvector

$$\sum_{i=1}^{n} \kappa_i \Delta_{ij}(u, \mu) = 0, \quad j = 1, \dots, n. \tag{1.17}$$

Eliminating $\mu$ one arrives at the equaton for the $u$-coordinates of the poles. Solving then the linear system (1.17) for the vector $(1, \mu, \dots, \mu^{n-1})$ we arrive at expressions for the $\mu$-coordinates of the poles.

Explicitely, for the poles $(u = x_i, \mu = p_i)$ of the eigenvector of the Lax operator normalized by the condition (1.10) for an arbitrary non-zero vector $\vec{\kappa}$ the separating functions will be constructed by the following procedure (see, e.g., [7]). Denote

$$B_{\vec{\kappa}}(u) := \vec{\kappa} \wedge L(u)\vec{\kappa} \wedge L^2(u)\vec{\kappa} \wedge \dots \wedge L^{n-1}(u)\vec{\kappa} \in \Lambda^n\mathbb{C}^n \simeq \mathbb{C} \tag{1.18}$$

$$= \det \begin{pmatrix} \kappa_1 & (L\vec{\kappa})_1 & (L^2\vec{\kappa})_1 & \dots & (L^{n-1}\vec{\kappa})_1 \\ \kappa_2 & (L\vec{\kappa})_2 & (L^2\vec{\kappa})_2 & \dots & (L^{n-1}\vec{\kappa})_2 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ \kappa_n & (L\vec{\kappa})_n & (L^2\vec{\kappa})_n & \dots & (L^{n-1}\vec{\kappa})_n \end{pmatrix}$$

where $L = L(u)$. Introduce $n \times n$ matrix $M(u) = (M_{ij}(u))$ defined by

$$M_{ij}(u) = \text{coefficient of } \mu^{n-i} \text{ in } \sum_{l=1}^{n} \Delta_{lj}(u,\mu)\kappa_l, \quad i, j = 1, \dots, n. \tag{1.19}$$

For any $1 \le j_0 \le n$ define the $(n-1) \times (n-1)$ matrix $M^{(j_0)}(u)$ by eliminating from $M(u)$ the last row and the column number $j_0$. Finally, another $(n-1) \times (n-1)$ matrix $\hat{M}^{(j_0)}(u)$ is obtained from $M^{(j_0)}(u)$ by replacing the last row by

$$\left( M_{n,1}(u), \dots, \hat{M}_{n,j_0}(u), \dots, M_{n,n}(u) \right).$$

Here the hat indicates that the term $M_{n,j_0}(u)$ is omitted.

**Lemma 1.2** *Assume that, for a given nonzero vector $\vec{\kappa}$ all finite points $Q_i = (x_i, p_i)$, $i = 1, \dots, d$ of the divisor of poles of the eigenvector $\vec{f}$ normalized by* (1.10) *belong to the smooth part of the spectral curve* (1.2) *and, moreover, the $u$-coordinates of all these points are pairwise distinct, $x_i \neq x_j$ for $i \neq j$. Then*

*1) $x_1, \dots, x_d$ are zeros of the function* (1.18),

$$B_{\vec{\kappa}}(x_i) = 0, \quad i = 1, \dots, d. \tag{1.20}$$

*2) For every $i$ there exists $1 \le j_0 \le n$ such that $\det M^{(j_0)}(x_i) \neq 0$. For any such $j_0$ the $\mu$-coordinate of the point $Q_i$ is given by the formula*

$$p_i = A_{\vec{\kappa}}(x_i) \tag{1.21}$$

*where*

$$A_{\vec{\kappa}}(u) = -\frac{\det \hat{M}^{(j_0)}(u)}{\det M^{(j_0)}(u)} \tag{1.22}$$

**Remark 1.1** *The rhs of eq.* (1.22) *does depend on the choice of $j_0$. Nevertheless, the difference of two such ratios is divisible by $B_{\vec{\kappa}}(u)$. So the value of $A_{\vec{\kappa}}(u)$ at $u = x_i$ does not depend on the choice of $j_0$.*

## 1.4 Formulation of main results

The needed structure of the Poisson algebra of the separating functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ is given in the following Theorem:

**Theorem 1.2** *The Poisson brackets* (1.8) *between the functions* $A(u) = A_{\vec{\kappa}}(u)$, $B(u) = B_{\vec{\kappa}}(u)$ *satisfy the Poisson commutation relations* (1.13) *and* (1.16) *if the* $r$*-matrix satisfies the conditions* (1.11) *and* (1.9) *respectively.*

The proof of this Theorem is obtained by tedious but straightforward calculations.

Let us explain the algebraic meaning of the condition (1.11). For simplicity let us consider here only the case of non-dynamical $r$-matrices with $\vec{\kappa} = (0, 0, \ldots, 0, 1)^t$ (see subsection 2.2 for the general case).

**Proposition 1.1** *The first* $(n-1)$ *rows of the Lax matrix form a closed Poisson subalgebra in the Poisson algebra (1.8) if and only if the condition (1.12) holds true.*

Taking into account that the separating functions $B(u)$ and $A(u)$ for the standard normalization $\vec{\kappa} = (0, 0, \ldots, 0, 1)^t$ are effectively defined on the subalgebra of the Lax matrices generated by the matrix elements of their first $(n-1)$ rows, we obtain that the above proposition gives a simple Lie-algebraic explanation for the necessity of the condition (1.12) in our Theorems. Indeed, if the Poisson algebra of the coordinate functions of the first $(n-1)$ rows of the Lax matrix were not closed then the corresponding Poisson algebra of the functions $p_i$, $x_j$ constructed via the matrix elements of the first $(n-1)$ rows of the Lax matrix would not be a closed Poisson algebra either.

It is necessary to notice that our algebro-geometric approach to the problem of construction of canonical variables works equally well both in the case of skew-symmetric and non-skew-symmetric $r$-matrices, in dynamical and non-dynamical cases.

## 1.5 Examples

We consider briefly also the problem of the completeness of the constructed canonical coordinates. In the general case, without a specification of the Lax operator as a function of the dynamical variables and the spectral parameter it is not possible to give an answer whether the coordinate system obtained with the help of the separating functions is complete. That is why in order to study the problem it is necessary to specify the $r$-matrix and the Lax operator. In the present paper we concentrate our attention on the case of non-dynamical classical $r$-matrices and on two classes of the Lax operators corresponding to them, namely, on the Lax operators of the generalized Gaudin systems with [28] and without [26] external magnetic field. We consider two examples of classical $r$-matrices and their Gaudin-type systems. The first one [18], [14], [1], [7] corresponds to the standard rational classical $r$-matrix and the standard rational Gaudin model [13]. In this case the set of the canonical coordinates constructed with the help of the functions $B_{\vec{\kappa}}(u)$ and $A_{\vec{\kappa}}(u)$ with the standard choice $\vec{\kappa} = (0, 0, \ldots, 0, 1)^t$ is not complete. In order to have a complete set of the separated coordinates one has either to apply the functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ with more general choice of the vector $\vec{\kappa}$ [7] or to make a trick completing the set of the separated coordinates by some additional functions [1]. In both cases the crucial point for the completeness is an introduction of a non-trivial and sufficiently generic external magnetic field in the corresponding rational Gaudin system.

Another example we consider is a new one. This example is connected with the $n$-parametric family of non-dynamical, non-skew-symmetric classical $r$-matrices of the following form [26]:

$$r_a(u,v) = \frac{1}{u-v}\sum_{i,j=1}^{n} \frac{1+a_i u}{1+a_i v} X_{ij}\otimes X_{ji},$$

that can be viewed as an anisotropic deformation of the rational $r$-matrices (the parameters $a_i$ are arbitrary). We consider the corresponding Gaudin-type models [26], [27], [28] and the set of canonical coordinates constructed with the help of the separating functions $B_{\vec{\kappa}}(u)$ and $A_{\vec{\kappa}}(u)$. Remarkably the $r$-matrix $r_a(u,v)$ is the most general representative in the class of the "diagonal" $r$-matrices of the form $r(u,v) = \sum_{i,j=1}^{n} r_{ji}(u,v) X_{ij}\otimes X_{ji}$ for which the functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ define the canonical coordinates for a generic normalization vector $\vec{\kappa}$. We show that, in the case $a_i \neq a_j$, $i,j \in \overline{1,n}$ and generic choice of the vector $\vec{\kappa}$ the canonical coordinates constructed with the help of the separating functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ constitute the complete family. It is necessary to notice that, contrary to the rational case, the set of the separated coordinates is complete even without introducing an external magnetic field in the corresponding generalized Gaudin model.

The described above integrable models associated with "anisotropic" $gl(n)\otimes gl(n)$-valued $r$-matrices provide the fourth example of the models with linear Poisson algebras and $gl(n)$-valued Lax matrices $L(u)$ for which the separating functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ produce a complete set of separated coordinates. The first class of the models is associated with the rational $r$-matrix and it was briefly described above. The second class is associated with a special non-skew-symmetric non-dynamical rational $r$-matrices obtained by (dynamical) gauge transformation from the standard skew-symmetric trigonometric $r$-matrices [25]. Finally, the third class of such models is associated with the dynamical $r$-matrices of the elliptic Calogero models and its various degenerations [22]. The $r$-matrices of all these models satisfy condition (1.12) either immediately or after a suitable transformation.

The structure of the present paper is the following: in the second section we remind the main facts about the classical $r$-matrices and about linear Poisson algebras of the Lax matrices, in the third section we review the main facts about the separated coordinates in the Lax-integrable case, in the fourth section we prove the main theorem and in the fifth section two classes of examples are considered. In the Appendix we develop an alternative technique for studying the necessary and sufficient conditions for the canonicity of the separated coordinates of algebro-geometric type. Such conditions are given by Theorem A.1.

# 2 Hamiltonian systems, Lax algebra and classical $r$-matrices

## 2.1 Definitions and notations

Let us consider a finite-dimensional Hamiltonian system on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\ ,\ \}$ and a Hamiltonian $H$. Let us assume that the corresponding Hamiltonian equations of motion are re-written in the Lax form:

$$\frac{dL(u)}{dt} = [L(u), M_H(u)], \tag{2.1}$$

where $L(u)$ and $M_H(u)$ are some matrices depending on the initial dynamical variables (coordinates on the space $\mathcal{P}$) and the auxiliary complex parameter $u$ which is constant with respect to the Poisson bracket $\{\ ,\ \}$ and the "time" $t$.

In the present paper we will assume that $L(u)$ takes its values in the Lie algebra $gl(n) = gl(n, \mathbb{C})$, i.e $L(u) = \sum\limits_{i,j=1}^{n} L_{ij}(u) X_{ij}$ where $X_{ij}, i, j = \overline{1, n}$ is a standard basis in $gl(n)$ with the commutation relations

$$[X_{ij}, X_{kl}] = \delta_{kj} X_{il} - \delta_{il} X_{kj}. \tag{2.2}$$

The Lax representation (2.1) automatically provides $n$ generating functions of the first integrals of the corresponding Hamiltonian equations, which can be chosen to be the traces of powers $\operatorname{tr} L(u)^k, k \in \overline{1, n}$ of the Lax matrix or, alternatively, the coefficients of the characteristic polynomial of the matrix $L(u)$. From this follows, in particular, that the "spectral curve" of the Lax matrix is preserved under the time evolution:

$$\frac{d}{dt} \det(L(u) - \mu \cdot 1) = 0.$$

For the complete integrability of the Hamiltonian system possessing the Lax representation (2.1) it is necessary to require the Poisson-commutativity of the above generating functions with respect to the Poisson brackets $\{\ ,\ \}$. The necessary and sufficient condition for this [4] is a possibility to represent the initial Poisson brackets re-written on the level of the Lax matrices in the so-called "generalized $r$-matrix form" [12]:

$$\{L(u_1) \otimes 1, 1 \otimes L(u_2)\} = [r^{12}(u_1, u_2), L(u_1) \otimes 1] - [r^{21}(u_2, u_1), 1 \otimes L(u_2)], \tag{2.3}$$

where the function of two complex variables

$$r(u_1, u_2) = \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u_1, u_2) X_{ij} \otimes X_{kl} \tag{2.4}$$

with values in the tensor square of the algebra $\mathfrak{g} = gl(n)$ is called classical $r$-matrix.

The commutation relations (2.3) are written in the component form as follows:

$$\{L_{ij}(u), L_{kl}(v)\} = \sum_{s=1}^{n} (r_{is,kl}(u,v) L_{sj}(u) - r_{sj,kl}(u,v) L_{is}(u)) - \sum_{s=1}^{n} (r_{ks,ij}(v,u) L_{sl}(v) - r_{sl,ij}(v,u) L_{ks}(v)). \tag{2.5}$$

The $r$-matrix (2.4) can depend also on the dynamical variables. In this case it is called "dynamical" and satisfies a complicated equation following from the Jacobi conditions for the bracket (2.3) (see e.g. [15]):

$$\begin{aligned}[L_1(u_1), [r^{12}(u_1, u_2), r^{13}(u_1, u_3)] - [r^{23}(u_2, u_3), r^{12}(u_1, u_2)] + [r^{32}(u_3, u_2), r^{13}(u_1, u_3)] + \\ + \{L_2(u_2), r^{13}(u_1, u_3)\} - \{L_3(u_3), r^{12}(u_1, u_2)\}] + cycl.perm. = 0,\end{aligned} \tag{2.6}$$

where $L_1(u_1) = L(u_1) \otimes 1 \otimes 1$, $L_2(u_2) = 1 \otimes L(u_2) \otimes 1$ etc.

In the case of non-dynamical, i.e. non-depending on the dynamical variables, classical $r$-matrices the equation (2.6) is simplified to the form of the "generalized" or "permuted" classical Yang-Baxter equation: [12],[4],[3]:

$$[r^{12}(u_1,u_2), r^{13}(u_1,u_3)] = [r^{23}(u_2,u_3), r^{12}(u_1,u_2)] - [r^{32}(u_3,u_2), r^{13}(u_1,u_3)], \tag{2.7}$$

where $r^{12}(u_1,u_2) \equiv \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u_1,u_2) X_{ij} \otimes X_{kl} \otimes 1$, $r^{13}(u_1,u_3) \equiv \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u_1,u_3) X_{ij} \otimes 1 \otimes X_{kl}$, etc.

In the case of skew-symmetric $r$-matrices, i.e. when $r^{12}(u_1,u_2) = -r^{21}(u_2,u_1)$, where $r^{21}(u_2,u_1) = P^{12} r^{12}(u_1,u_2) P^{12}$ and $P^{12}$ interchanges the first and second spaces in tensor product, the generalized classical Yang-Baxter equation reduces to the usual classical Yang-Baxter equation [19]:

$$[r^{12}(u_1,u_2), r^{13}(u_1,u_3)] = [r^{23}(u_2,u_3), r^{12}(u_1,u_2) + r^{13}(u_1,u_3)], \tag{2.8}$$

solutions of which have been classified in [6].

## 2.2 Special subalgebras of the Lax algebra

In the sequel we will be interested in certain subalgebras of the Lax algebras. In this section we will consider only non-dynamical $r$-matrices. Let us consider $n \times n$ constant matrix $A = (A_{ij})$ and introduce a linear functional of $L(u)$ by

$$\ell_A(L(u)) = \mathrm{tr}(AL(u)) = \sum_{i,j=1}^{n} A_{ji} L_{ij}(u).$$

Proof of the following lemma is straightforward.

**Lemma 2.1** *For any pair of matrices $A$, $B$ the following formula holds true*

$$\{\ell_A(L(u)), \ell_B(L(v))\} = \ell_{\mathcal{A}}(L(u)) - \ell_{\mathcal{B}}(L(v)),$$

*where*

$$\mathcal{A}_{js} = \sum_{i,k,l=1}^{n} \left(A_{ji} r_{is,kl}(u,v) - r_{ji,kl}(u,v) A_{is}\right) B_{lk},$$

$$\mathcal{B}_{ls} = \sum_{i,j,k=1}^{n} A_{ji} \left(B_{lk} r_{ks,ij}(v,u) - r_{lk,ij}(v,u) B_{ks}\right).$$

Let us consider all the matrices as linear operators on the space $V = \mathbb{C}^n$. Choose a subspace $W \subset V$. Introduce a subspace $\mathcal{L}_W$ in the space of linear functionals of the above form defined by

$$\mathcal{L}_W = \{\ell_A \,|\, A(W) = 0\}.$$

In the component form we have that the matrices $A$ generating $\mathcal{L}_W$ are such that $\sum_{i=1}^{n} A_{ji} w_i = 0$, $\forall \vec{w} \in W$ and $\forall\, j$.

We want to derive a sufficient condition on the $r$-matrix that guarantees that the subspace $\mathcal{L}_W$ is closed with respect to the Poisson bracket. For a given non-zero vector $\vec{w} \in W$ consider a 3-tensor

$$\sum_{s=1}^{n} r_{is,kl}(u,v) w_s \in V \otimes V \otimes V^*. \tag{2.9}$$

It defines a natural map

$$r_{\vec{w}} : V^* \otimes V^* \to V^*. \tag{2.10}$$

Denote $W^* \subset V^*$ the annihilator of the subspace $W$,

$$W^* = \{\alpha \in V^* \,|\, \alpha(\vec{w}) = 0 \quad \forall \vec{w} \in W\}.$$

The following proposition holds true:

**Proposition 2.1** *Let the r-matrix satisfy the following property: for any $\vec{w} \in W$ the map (2.9) acts trivially on $W^* \otimes W^*$:*

$$r_{\vec{w}} \left(W^* \otimes W^*\right) = 0. \tag{2.11}$$

*Then the subspace of linear functionals $\mathcal{L}_W$ is closed with respect to the Poisson bracket. This condition is also a necessary one.*

*Proof.* One has to verify validity of the conditions

$$\sum_{s=1}^{n} \mathcal{A}_{is} w_s = 0, \qquad \sum_{s=1}^{n} \mathcal{B}_{ls} w_s = 0,$$

for any $\vec{w} \in W$. They are spelled as follows

$$\sum_{s,i,k,l=1}^{n} A_{ji} r_{is,kl}(u,v) B_{lk} w_s = 0, \quad \sum_{i,j,k,s=1}^{n} A_{ji} B_{lk} r_{ks,ij}(v,u) w_s = 0.$$

for any $A$, $B$ satisfying $A(\vec{w}) = 0$, $B(\vec{w}) = 0$, and any $\vec{w} \in W$. It is easy to see that these conditions are equivalent to the conditions of the Proposition.

Let us specify the condition (2.11) in the case of the one-dimensional space $W$ spanned by a non-zero vector $\vec{w}$. In this case we have that the map (2.10) has the following explicit component form:

$$M_{ik} \to \sum_{i,k,s=1}^{n} r_{is,kl}(u,v) M_{ik} w_s$$

and the condition (2.11) acquires the form:

$$\sum_{i,k,s=1}^{n} r_{is,kl}(u,v) M_{ik} w_s = 0, \tag{2.12}$$

for any $M_{ij}$ such that $\sum_{i=1}^{n} M_{ij}w_i = \sum_{j=1}^{n} M_{ij}w_j = 0$. Any such matrix is a linear combination of matrices of rank 1 that have the form

$$M_{ij} = \alpha_i\beta_j, \quad \sum \alpha_i w_i = \sum \beta_j w_j = 0.$$

Thus in this case the condition (2.11) coincides with eq. (1.11).

As we will see below in the considered $gl(n)$ case a miracle happens: purely Lie algebraic conditions (2.11) and (2.12) that guarantee that certain subspaces are closed Poisson subalgebras of the $r$-matrix Lax algebra coincide with the sufficient conditions for the variable separation.

# 3 Separation of variables

## 3.1 General integrable case

Let us recall the definitions of Liouville integrability and separation of variables in the general theory of Hamiltonian systems. An integrable Hamiltonian system with $D$ degrees of freedom is determined on a $2D$-dimensional symplectic manifold $\mathcal{M}$ (symplectic leaf in $(\mathcal{P}, \{\ ,\ \})$ and $D$ independent functions (first integrals) $I_j$ commuting with respect to the Poisson bracket

$$\{I_i, I_j\} = 0, \quad i, j \in \overline{1, D}$$

(for the Hamiltonian $H$ of the system it can be taken any first integral $I_j$).

To find separated variables means to find (at least locally) a set of coordinates $x_i, p_j, i, j \in \overline{1, D}$ such that there exist $D$ independent relations:

$$\Phi_i(x_i, p_i; I_1, ..., I_N) = 0, \quad i \in \overline{1, D} \tag{3.1}$$

where the coordinates $x_i, p_j, \ i, j \in \overline{1, D}$ are canonical, i.e:

$$\{x_i, p_j\} = \delta_{ij}, \ \{x_i, x_j\} = 0, \ \{p_i, p_j\} = 0, \quad \forall\, i, j \in \overline{1, D}.$$

The separated variables provide a way to a construction of the action-angle coordinates from the Liouville theorem and half a way to explicit integration of the equations of motion.

Unfortunately, in the general case no algorithm is known to construct a set of separated variables for a given integrable system. Nevertheless, in the case of integrable systems admitting Lax representation with spectral parameter there exists an efficient recipe of their construction.

## 3.2 Lax-integrable case

Let us turn again to the integrable systems admitting Lax representation with the spectral parameter. Let us consider the auxiliary linear problem for the $n \times n$ Lax matrix with the "left" eigenvector $\overrightarrow{f}(u)$ of the Lax matrix:

$$\overrightarrow{f}(u)L(u) = \mu(u)\overrightarrow{f}(u).$$

The eigenvector of the Lax matrix is defined by the above equation up to a normalization. The "magic recipe" to a construction of the canonical variables of *algebro-geometric type* is the following [16]:

*The separated coordinates $x_i$, $i \in \overline{1,d}$ coincide with the u-coordinates of poles of the properly normalized eigenvector of the Lax matrix. The conjugated momenta $p_i$ coincide with the $\mu$-coordinates of the poles, i.e., with the eigenvalues $\mu$ of the Lax operator evaluated at these points: $p_i = \mu(x_i)$. The functions $\Phi_i$ in (3.1) are all the same and they are obtained from the characteristic polynomial of the Lax matrix:*

$$\Phi(x_i, p_i; I_1, ..., I_N) = \det(L(x_i) - p_i \cdot 1_n) = 0.$$

**Remark 3.1** *In certain examples the coordinates of the poles satisfy $\{x_i, x_j\} = \{p_i, p_j\} = 0$, $\{x_i, p_j\} = 0$ for $i \neq j$ but $\{x_i, p_i\}$ can be different from 1. To arrive at canonical coordinates one has to make a change $p_i \mapsto \tilde{p}_i = f(p_i)$ for a suitable function $f(p)$.*

In the general case the "magic recipe" is conjectural. Moreover, in such generality it is difficult to verify its validity. In particular it is difficult to check that the constructed in such a way variables $x_i$, $p_j$, $i, j \in \overline{1,d}$ constitute a complete system of coordinates, i.e. that they are independent and $d = D$. Another problem is that for such a general formulation it is not possible to prove that the Poisson brackets of the constructed coordinates are canonical. That is why in order to prove the last statement we will specify the normalization of the eigenvector of the Lax matrix as follows:

$$\overrightarrow{f}(u)\vec{\kappa} = 1,$$

where $\vec{\kappa} = (\kappa_1, ..., \kappa_n)^t$ is an arbitrary *constant* (i.e. not depending on the dynamical variables or spectral parameters) column vector and the upper "$t$" will hereafter denote the transposition. In this case it is possible to concretize the above general recipe. Indeed, in such a case for any pole $x_j$ of the eigenvector of the Lax matrix the normalization condition yields the following equation:

$$\sum_{i=1}^{n} f_i^{(j)} \kappa_i = 0, \tag{3.2}$$

where $\overrightarrow{f}^{(j)} = (f_1^{(j)}, ..., f_n^{(j)})$ is the residue of the vector function $\overrightarrow{f}(u)$ in the point $x_j$.

The eigenvalue condition yields the following equations

$$\overrightarrow{f}^{(j)} L(x_j) = \mu(x_j) \overrightarrow{f}^{(j)}. \tag{3.3}$$

Combining the equations (3.2) and (3.3) one obtains the following system of the equations for the components of the vector $\overrightarrow{f}^{(j)}$:

$$\overrightarrow{f}^{(j)}(L(x_j))^m \vec{\kappa} = 0, \ m \in \overline{0, n-1}. \tag{3.4}$$

This is a linear homogeneous system of equations for $f_i^{(j)}$. From this it follows that the non-trivial solutions for $f_i^{(j)}$ exist if the following condition holds:

$$B_{\vec{\kappa}}(u)|_{u=x_j} = \det \left( \vec{\kappa} \quad L(u)\vec{\kappa} \quad (L(u))^2\vec{\kappa} \quad ... \quad (L(u))^{n-1}\vec{\kappa} \right)|_{u=x_j} = 0. \tag{3.5}$$

That is, the separated coordinates $x_i$ can be obtained as zeros of the above defined function $B_{\vec{\kappa}}(u)$, which is a polynomial as a function of the elements of Lax matrix but can be more complicated (e.g. meromorphic) as function of $u$. We will call $B_{\vec{\kappa}}(u)$ to be *a separating polynomial*. For an arbitrary vector $\vec{\kappa}$ this polynomial was first written in [7] (see also [22]).

There is a normalization (called "standard") corresponding to the following choice of the vector $\vec{\kappa}$:

$$\vec{\kappa} = \vec{\kappa}_n \equiv (0, 0, ..., 0, 1)^t.$$

Such a choice produces the separating polynomial $B_{\vec{\kappa}}(u)(u)$ considered in [20] for $n = 2, 3$ and [18] for arbitrary $n$. This polynomial is also equivalent to the separating polynomial considered in [14] and later in [25]. Using the properties of the determinant it is possible to show that for the vector $\vec{\kappa}_n$ the polynomial $B_{\vec{\kappa}}(u)$ can be re-written as the determinant of the $(n-1)\times(n-1)$ matrix $\mathcal{B}(u)$ defined as follows:

$$B(u) \equiv B_{\vec{\kappa}_n}(u) = \det\mathcal{B}(u) = \det\left( \overrightarrow{l}(u) \quad \overline{L}(u)\overrightarrow{l}(u) \quad \overline{L}^2(u)\overrightarrow{l}(u) \quad ... \quad \overline{L}^{n-2}(u)\overrightarrow{l}(u) \right), \tag{3.6}$$

where $\overline{L}(u)$ is a submatrix of $L(u)$ constituted by its first $n-1$ rows and columns, i.e $\overline{L}_{ij}(u) = L_{ij}(u)$, $i, j \in \overline{1, n-1}$ and $\overrightarrow{l}(u)$ is the $(n-1)$ component vector constituted by the first $(n-1)$-elements of the last column of $L(u)$, i.e $(\overrightarrow{l}(u))_i = L_{in}(u)$, $i \in \overline{1, n-1}$.

In a similar way it is possible to show [25] that the separated momenta $p_i = \mu(x_i)$ corresponding to a normalization vector $\vec{\kappa}_n$ are given by the values of the "dual" separating function: $p_i = A(x_i)$, where

$$A(u) = \frac{C(u)}{D(u)}, \tag{3.7}$$

and the polynomials $C(u)$ and $D(u)$ are defined as follows:

$$C(u) \equiv C_\xi(u) = \det\left( \overline{L}(u)\overrightarrow{\xi} \quad \overrightarrow{l}(u) \quad \overline{L}^2(u)\overrightarrow{l}(u) \quad ... \quad \overline{L}^{n-3}(u)\overrightarrow{l}(u) \right), \tag{3.8}$$

$$D(u) \equiv D_\xi(u) = \det\left( \overrightarrow{\xi} \quad \overrightarrow{l}(u) \quad \overline{L}^2(u)\overrightarrow{l}(u) \quad ... \quad \overline{L}^{n-3}(u)\overrightarrow{l}(u) \right), \tag{3.9}$$

where $\overrightarrow{\xi}$ is an arbitrary $n-1$ component vector. Following [14] and [18] we will chose it as follows:

$$\overrightarrow{\xi} = \overrightarrow{\xi}_1 \equiv (1, 0, 0, ..., 0)^t, \tag{3.10}$$

i.e. we will put $D(u) = D_{\xi_1}(u)$, $C(u) = C_{\xi_1}(u)$.

**Remark 3.2** *Observe that all the choices of $D_\xi(u)$ and $C_\xi(u)$ give the same answer for $A(u)$ modulo the ideal generated by $B(u)$. We will use the auxiliary polynomials $D_{\xi_j}(u)$, $C_{\xi_j}(u)$, $j \in \overline{2, n-1}$, where $(\overrightarrow{\xi}_j)_k = \delta_{jk}$, $k \in \overline{1, n-1}$ in the intermediate calculations.*

**Remark 3.3** *Observe, that the separating polynomial $B(u)$ and the "dual" separating function $A(u)$ corresponding to the above standard normalization of the eigenvector of the Lax matrix effectively depend only on the first $(n-1)$-rows of the Lax matrix.*

**Remark 3.4** *Observe that it is possible to use throughout this section instead of the left eigenvector of the Lax matrix the right one satisfying the following equations:*

$$L(u)\overrightarrow{f}^t(u) = \mu(u)\overrightarrow{f}^t(u).$$

*Such a choice is used in the papers [7], [25], [22]. Under such a choice all the formulas above should be appropriately modified. For example, the separating polynomial $B(u)$ is written as follows:*

$$B(u) = \det \begin{pmatrix} \vec{\kappa} \\ \vec{\kappa}\, L(u) \\ \dots \\ \vec{\kappa}\, L^{n-1}(u) \end{pmatrix},$$

*the system (1.17) is written as*

$$\sum_{j=1}^{n} \kappa_j \Delta_{ij}(u,\mu) = 0, \quad i = 1,\dots,n$$

*etc.*

## 3.3 Algebra of matrix minors and separating polynomial

In the sequel it will be necessary to rewrite separating functions in another form. For this purpose we will introduce some new notations.

*Definition 1.* We denote by $L\begin{pmatrix} j_1\ j_2 \dots\ j_m \\ i_1\ i_2 \dots\ i_m \end{pmatrix}(u)$ a minor of the matrix $L(u)$, i.e. the determinant of the submatrix of $L(u)$ consisting of the elements standing on the intersection of the $i_1,\ i_2,\dots i_m$-th rows and $j_1,\ j_2,\dots j_m$-th columns. In particular: $L\begin{pmatrix} j \\ i \end{pmatrix}(u) \equiv L_{ij}(u)$.

In what follows we will use the following representation of the polynomial $B(u)$ [14]:

$$B(u) = c_{n-1} \sum_{\alpha_i^j=1;\, i\in\overline{1,n-2};\, j\in\overline{1,i}}^{n-1} B(\{\alpha_i^j\},u), \tag{3.11}$$

where

$$B(\{\alpha_i^j\},u) = L\begin{pmatrix} n \\ \alpha_1^1 \end{pmatrix}(u) L\begin{pmatrix} \alpha_1^1\ n \\ \alpha_2^1\alpha_2^2 \end{pmatrix}(u) \cdots L\begin{pmatrix} \alpha_{n-3}^1 \dots\ \alpha_{n-3}^{n-3}\ n \\ \alpha_{n-2}^1 \dots\ \alpha_{n-2}^{n-2} \end{pmatrix}(u) L\begin{pmatrix} \alpha_{n-2}^1 \dots\ \alpha_{n-2}^{n-2}\ n \\ 1\ 2\ \dots\ n-1 \end{pmatrix}(u), \tag{3.12}$$

We will also use similar representations for the polynomials $C(u) = C_{\xi_1}(u)$ and $D(u) = D_{\xi_1}(u)$ with $\xi_1$ of the form (3.10) (recall that $A(u) = \dfrac{C(u)}{D(u)}$) [14]:

$$C(u) = c_{n-2} \sum_{\alpha_i^j=1;\, i\in\overline{1,n-3},\, j\in\overline{1,i}}^{n-1} C(\{\alpha_i^j\},u),$$

$$D(u) = c_{n-2} \sum_{\alpha_i^j=1;\, i\in\overline{1,n-3},\, j\in\overline{1,i}}^{n-1} D(\{\alpha_i^j\},u),$$

where

$$C(\{\alpha_i^j\},u) = L\begin{pmatrix} n \\ \alpha_1^1 \end{pmatrix}(u) L\begin{pmatrix} \alpha_1^1\ n \\ \alpha_2^1\alpha_2^2 \end{pmatrix}(u) \cdots L\begin{pmatrix} \alpha_{n-4}^1 \ldots\ \alpha_{n-4}^{n-4}\ n \\ \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3} \end{pmatrix}(u) L\begin{pmatrix} 1\ \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3}\ n \\ 1\ \ 2\ \ \ldots\ \ n-1 \end{pmatrix}(u) \tag{3.13}$$

$$D(\{\alpha_i^j\},u) = L\begin{pmatrix} n \\ \alpha_1^1 \end{pmatrix}(u) L\begin{pmatrix} \alpha_1^1\ n \\ \alpha_2^1\alpha_2^2 \end{pmatrix}(u) \cdots L\begin{pmatrix} \alpha_{n-4}^1 \ldots\ \alpha_{n-4}^{n-4}\ n \\ \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3} \end{pmatrix}(u) L\begin{pmatrix} \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3}\ n \\ 2\ \ 3\ \ldots\ \ n-1 \end{pmatrix}(u) \tag{3.14}$$

The normalization coefficients are given by the following formula: $c_k = \dfrac{(-1)^{\frac{k(k-1)}{2}}}{1!2!\ldots(k-1)!}$.

It is also possible to show that [14]

$$C_{\xi_k}(u) = c_{n-2} \sum_{\alpha_i^j=1;i\in\overline{1,n-3},j\in\overline{1,i}}^{n-1} C_{\xi_k}(\{\alpha_i^j\},u),$$

$$D_{\xi_k}(u) = c_{n-2} \sum_{\alpha_i^j=1;i\in\overline{1,n-3},j\in\overline{1,i}}^{n-1} D_{\xi_k}(\{\alpha_i^j\},u),$$

for other choices of the vector $\xi$. Here

$$C_{\xi_k}(\{\alpha_i^j\},u) = L\begin{pmatrix} n \\ \alpha_1^1 \end{pmatrix}(u) L\begin{pmatrix} \alpha_1^1\ n \\ \alpha_2^1\alpha_2^2 \end{pmatrix}(u) \cdots L\begin{pmatrix} \alpha_{n-4}^1 \ldots\ \alpha_{n-4}^{n-4}\ n \\ \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3} \end{pmatrix}(u) L\begin{pmatrix} k\ \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3}\ n \\ 1\ \ 2\ \ \ldots\ \ n-1 \end{pmatrix}(u) \tag{3.15}$$

$$D_{\xi_k}(\{\alpha_i^j\},u) = L\begin{pmatrix} n \\ \alpha_1^1 \end{pmatrix}(u) L\begin{pmatrix} \alpha_1^1\ n \\ \alpha_2^1\alpha_2^2 \end{pmatrix}(u) \cdots L\begin{pmatrix} \alpha_{n-4}^1 \ldots\ \alpha_{n-4}^{n-4}\ n \\ \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3} \end{pmatrix}(u) L\begin{pmatrix} \alpha_{n-3}^1 \ldots\ \alpha_{n-3}^{n-3}\ n \\ 1\ 2 \ldots \check{k} \ldots\ n-1 \end{pmatrix}(u), \tag{3.16}$$

where “check” over $k$ means that this index is omitted.

There is the following usefull identity[14]:

$$(-1)^l D_{\xi_k}(u) C_{\xi_l}(u) - (-1)^k C_{\xi_k}(u) D_{\xi_l}(u) = \mathcal{B}\begin{pmatrix} \widehat{n-2}\widehat{n-1} \\ \widehat{k}\ \ \widehat{l} \end{pmatrix}(u) B(u),\ \ k,l \in \overline{1,n-1}. \tag{3.17}$$

where $\mathcal{B}\begin{pmatrix} \widehat{n-2}\widehat{n-1} \\ \widehat{k}\ \ \widehat{l} \end{pmatrix}(u)$ is the complementary to the $k$-th and $l$-th row and the $(n-2)$ and $(n-1)$-st columns minor of the matrix $\mathcal{B}(u)$. The matrix $\mathcal{B}(u)$ was defined in the eq. (3.6).

**Remark 3.5** *Observe that the above formulas for $C(u)$ and $D(u)$ are well-defined only for $n \geq 3$. Nevertheless, the case $n = 2$ also fits for our construction if we define additionally that for $n = 2$*

$$B(u) = L\begin{pmatrix} 2 \\ 1 \end{pmatrix}(u),\ \ C(u) = L\begin{pmatrix} 1 \\ 1 \end{pmatrix}(u),\ \ D(u) = 1,\ \ A(u) = C(u). \tag{3.18}$$

**Remark 3.6** *Observe that, passing from the left to the right eigenvectors of the Lax matrix discussed in the Remark 3.4 is equivalent to the transposition of the Lax matrix in the formulas above, which leads to exchange between rows and columns in the corresponding minors, i.e:* $L\begin{pmatrix} j_1 \dots j_m \\ i_1 \dots i_m \end{pmatrix}(u) \to L\begin{pmatrix} i_1 \dots i_m \\ j_1 \dots j_m \end{pmatrix}(u)$.

We will also use the following technically important Proposition:

**Proposition 3.1** *Let the entries of the Lax matrix satisfy the Poisson brackets (2.5). Then the following Poisson brackets between minors of the Lax matrix hold true:*

$$\{L\begin{pmatrix} j_1 \dots j_M \\ i_1 \dots i_M \end{pmatrix}(u), L\begin{pmatrix} l_1 \dots l_N \\ k_1 \dots k_N \end{pmatrix}(v)\} =$$
$$= \sum_{p=1}^{M}\sum_{q=1}^{N}\sum_{s,t=1}^{n}\Big(r_{i_q s,k_p t}(u,v)L\begin{pmatrix} j_1 \dots\dots j_M \\ i_1 \,..\check{i}_q s..\, i_M \end{pmatrix}(u)\widetilde{L}\begin{pmatrix} l_1 \dots\dots l_N \\ k_1 \,..\check{k}_p t..\, k_N \end{pmatrix}(v) -$$
$$- r_{s j_q, t l_p}(u,v)L\begin{pmatrix} j_1 \,..\check{j}_q s..\, j_M \\ i_1 \dots\dots i_M \end{pmatrix}(u)\widetilde{L}\begin{pmatrix} l_1 \,..\check{l}_p t..\, l_N \\ k_1 \dots k_N \end{pmatrix}(v) -$$
$$- r_{k_p t, i_q s}(v,u)\widetilde{L}\begin{pmatrix} j_1 \dots\dots j_M \\ i_1 \,..\check{i}_q s..\, i_M \end{pmatrix}(u)L\begin{pmatrix} l_1 \dots\dots l_N \\ k_1 \,..\check{k}_p t..\, k_N \end{pmatrix}(v) +$$
$$+ r_{t l_p, s j_q}(v,u)\widetilde{L}\begin{pmatrix} j_1 \,..\check{j}_q s..\, j_M \\ i_1 \dots\dots i_M \end{pmatrix}(u)L\begin{pmatrix} l_1 \,..\check{l}_p t..\, l_N \\ k_1 \dots k_N \end{pmatrix}(v)\Big), \quad (3.19)$$

*where*

$$\widetilde{L}\begin{pmatrix} l_1 \dots\dots l_N \\ k_1 \,..\check{k}_p t..\, k_N \end{pmatrix}(v) \equiv \sum_{q=1}^{N}(-1)^{p+q}\delta_{t l_q}L\begin{pmatrix} l_1 \,..\check{l}_q..\, l_N \\ k_1 \,..\check{k}_p..\, k_N \end{pmatrix}(v),$$
$$\widetilde{L}\begin{pmatrix} l_1 \,..\check{l}_p t..\, l_N \\ k_1 \dots k_N \end{pmatrix}(v) \equiv \sum_{q=1}^{N}(-1)^{p+q}\delta_{t k_q}L\begin{pmatrix} l_1 \,..\check{l}_p..\, l_N \\ k_1 \,..\check{k}_q..\, k_N \end{pmatrix}(v),$$
$$\widetilde{L}\begin{pmatrix} j_1 \dots\dots j_M \\ i_1 \,..\check{i}_p s..\, i_M \end{pmatrix}(u) \equiv \sum_{q=1}^{M}(-1)^{p+q}\delta_{s j_q}L\begin{pmatrix} j_1 \,..\check{j}_q..\, j_M \\ i_1 \,..\check{i}_p..\, i_M \end{pmatrix}(u),$$
$$\widetilde{L}\begin{pmatrix} j_1 \,..\check{j}_p s..\, j_M \\ i_1 \dots i_M \end{pmatrix}(u) \equiv \sum_{q=1}^{N}(-1)^{p+q}\delta_{s i_q}L\begin{pmatrix} j_1 \,..\check{j}_p..\, j_M \\ i_1 \,..\check{i}_q..\, i_M \end{pmatrix}(u).$$

*Like above check over an index means that this index is omitted in the corresponding minor.*

*Sketch of the Proof.* The Proof is made using double induction on $N$ and $M$. In more details, the base of induction is the case $M = 1$, $N = 1$. Its validity is evident due to the fact that, in this case by the definition of the Poisson bracket (2.5):

$$\{L\begin{pmatrix} j_1 \\ i_1 \end{pmatrix}(u), L\begin{pmatrix} l_1 \\ k_1 \end{pmatrix}(v)\} = \sum_{s,t=1}^{n}\Big(r_{i_1 s, k_1 t}(u,v) L\begin{pmatrix} j_1 \\ s \end{pmatrix}(u)\widetilde{L}\begin{pmatrix} l_1 \\ t \end{pmatrix}(v) - \\ - r_{sj_1, tl_1}(u,v) L\begin{pmatrix} s \\ i_1 \end{pmatrix}(u)\widetilde{L}\begin{pmatrix} t \\ k_1 \end{pmatrix}(v) - r_{k_1 t, i_1 s}(v,u)\widetilde{L}\begin{pmatrix} j_1 \\ s \end{pmatrix}(u) L\begin{pmatrix} l_1 \\ t \end{pmatrix}(v) + \\ + r_{tl_1, sj_1}(v,u)\widetilde{L}\begin{pmatrix} s \\ i_1 \end{pmatrix}(u) L\begin{pmatrix} t \\ k_1 \end{pmatrix}(v)\Big), \quad (3.20)$$

where by our original definition

$$\widetilde{L}\begin{pmatrix} l_1 \\ t \end{pmatrix}(v) = \delta_{tl_1}, \ \widetilde{L}\begin{pmatrix} t \\ k_1 \end{pmatrix}(v) = \delta_{tk_1}, \ \widetilde{L}\begin{pmatrix} s \\ i_1 \end{pmatrix}(u) = \delta_{si_1}, \ \widetilde{L}\begin{pmatrix} j_1 \\ s \end{pmatrix}(u) = \delta_{j_1 s}.$$

The next step is an induction on $N$. It is done decomposing minor of the order $N$ in the row and column, using the Leibnitz rule for the Poisson brackets and regrouping afterwards the summands. To complete the proof one has to do induction on $M$ using similar arguments..

**Remark 3.7** *The introduction of the minors of $L(u)$ with "tilde" seems to have no deep hidden sense in the general case and is used by us for the notational convenience.*

The Proposition (3.1) has the following simple but important Corollary:

**Corollary 3.1** *Let* $D_M(L(u)) = \sum_{i_1,i_2,\ldots,i_M=1}^{n} L\begin{pmatrix} i_1 & \ldots & i_M \\ i_1 & \ldots & i_M \end{pmatrix}(u)$. *Then* $\{D_M(L(u)), D_N(L(v))\} = 0$.

In other words the Proposition (3.1) provides a simple alternative proof of the Poisson commutativity of the coefficients of the characteristic polynomials of the Lax matrix $L(u)$.

### 3.4 Separating functions and canonical coordinates

Let us now remind a method of constructing canonical coordinates using the separating functions. Generally speaking this method can be considered independently from separation of variables. In this subsection we do not assume any special properties of the Poisson manifold $\mathcal{P}$ or Poisson structure $\{\ ,\ \}$. Neither we assume integrability or existence of the Lax representation.

Let $B(u)$ and $A(u)$ be some functions of the dynamical variables and auxiliary complex parameter $u$ [2], which is constant with respect to the bracket $\{\ ,\ \}$. Let the points $x_i$, $i \in \overline{1,d}$ be zeros of the function $B(u)$ and $p_i$, $i \in \overline{1,d}$ be the values of $A(u)$ at these points. We want to compute Poisson brackets between these new coordinates using the Poisson brackets between $B(u)$ and $A(u)$.

The following Proposition holds true:

[2] For the Hamiltonian systems admitting Lax representation with spectral parameter it will naturally coincide with the spectral parameter in separating polynomials.

**Proposition 3.2** *Let $B(x_i)=0$, $p_j=A(x_j)$. Then:*

$$(i)\ \{x_i,x_j\}=\lim_{u\to x_i, v\to x_j}\Big(\frac{\{B(u),B(v)\}}{\partial_u B(u)\partial_v B(v)}\Big),\ \ where\ i\neq j,$$

$$(ii)\ \{x_j,p_i\}=\lim_{u\to x_i, v\to x_j}\Big(\frac{\{A(u),B(v)\}}{\partial_v B(v)}\Big)+\{x_i,x_j\}\lim_{u\to x_i}(\partial_u A(u)),\ \ where\ i\neq j,$$

$$(iii)\ \{p_i,p_j\}=\lim_{u\to x_i, v\to x_j}\Big(\{A(u),A(v)\}\Big)+\{p_i,x_j\}\lim_{v\to x_j}(\partial_v A(v))+\{x_i,p_j\}\lim_{u\to x_i}(\partial_u A(u))-$$
$$-\{x_i,x_j\}\lim_{u\to x_i, v\to x_j}(\partial_u A(u)\partial_v A(v)),\ \ where\ i\neq j.$$

*Sketch of the Proof.* The eqs. (i)-(iii) are obtained by the decomposition of $B(u)$, $A(u)$, $B(v)$, $A(v)$ in Taylor power series in the neighborhood of the points $u=x_i$, $v=x_j$ in the expressions $\{B(u),B(v)\}$, $\{A(u),B(v)\}$, $\{A(u),A(v)\}$ respectively and by considering the limits $u\to x_i, v\to x_j$ after the calculation of the Poisson brackets.

# 4 Main Theorem and its Corollaries

In this section we will formulate our general result about the algebra of separating functions first in the special case of the vector $\vec{\kappa}_n=(0,0,...,0,1)^t$, i.e. for the functions $A(u)$ and $B(u)$ and then we will extend it to the functions $A_{\vec{\kappa}}(u)$, $B_{\vec{\kappa}}(u)$ with general $\vec{\kappa}$ arriving at Theorem 1.1 formulated in the Introduction.

## 4.1 Main Theorem

From the results of the previous subsection it follows that in order to obtain canonical Poisson brackets among the coordinates constructed with the help of polynomials $B(u)$ and $A(u)$ it is sufficient to require that the equations (1.13a–1.16) hold true. The conditions for validity of these equations are given in our main Theorem written in the following component form.

**Theorem 4.1** *Let the Poisson brackets between entries of the Lax matrix $L(u)$ have the form (2.5) with some r-matrix $r(u,v)$. Then the separating functions $B(u)$ and $A(u)$ defined in terms of the Lax matrix $L(u)$ by the formulas (3.6)-(3.9) with the normalization vector $\vec{\kappa}=(0,0,\ldots,0,1)^t$ satisfy the relations (1.13a–1.13c) if the following conditions on the matrix elements of the corresponding r-matrix $r(u,v)$ are satisfied*

$$r_{in,jk}(u,v)=r_{in,jn}(u,v)=0,\quad \forall\, i,j,k\in\overline{1,n-1}. \tag{4.1}$$

*For the non-dynamical r-matrices the condition* (4.1) *is also necessary for validity of the commutation relations (1.13a–1.13c).*

*If, moreover, the r-matrix satisfy the regularity condition (1.9) then the condition (1.16) is also satisfied.*

See below the subsection 4.3 for the detailed proof of the Theorem. Since the function $B_{\vec{\kappa}}(u)$ does not depend on the choice of a basis in $\mathbb{C}^n$, up to multiplication by a constant factor, and also the function $A_{\vec{\kappa}}(u)$ after a change of a basis transforms like

$$A_{\vec{\kappa}}(u) \mapsto a\, A_{\vec{\kappa}}(u) + b\, B_{\vec{\kappa}}(u)$$

for some constants $a \neq 0$ and $b$, hence the Theorem 1.2 follows from the above Theorem.

**Remark 4.1** *Not all classical $gl(n)\otimes gl(n)$-valued $r$-matrices satisfy the conditions (4.1). In particular, skew-symmetric elliptic $r$-matrix of Sklyanin [19] (for $n = 2$) and its generalization of Belavin [5] (for arbitrary $n$) does not satisfy the condition (4.1). Nevertheless, the condition (4.1) is not very rigid. It "kills" only $(n-1)^3 + (n-1)^2$ of $n^4$ components of the $r$-matrix.*

*Example 2.* Let $n = 2$. In this case, as it was already observed, the formulas (3.6)-(3.9) defining the separating functions should be modified. Nevertheless, using the definition (3.18) of the functions $B(u)$ and $A(u)$ and the commutation relations (2.5) it is easy to show that the condition (4.1) with $n = 2$ is necessary, for non-dynamical $r$-matrices, and sufficient to satisfy the relations (1.13a–1.13c) also in this case. The $gl(2) \otimes gl(2)$-valued $r$-matrix possesses $2^4 = 16$ components. The condition (4.1) has the form:

$$r_{12,11}(u,v) = r_{12,12}(u,v) = 0.$$

It "kills" only $1 + 1 = 2$ components of the $r$-matrix.

## 4.2 Some corollaries

The Theorem (4.1) has several important Corollaries. The first of them is written as follows:

**Corollary 4.1** *Let the Lax matrix $L^g(u)$ be obtained from the Lax matrix $L(u)$ by (possibly dynamical) gauge transformation $L(u) \to L^g(u) = g^{-1}(u)L(u)g(u)$. Then the corresponding separating functions $B^g(u) \equiv B(L^g(u))$ and $A^g(u) \equiv A(L^g(u))$ satisfy the algebra (1.13a–1.13c) if and only if the following conditions hold:*

$$r^g_{in,jk}(u,v) = r^g_{in,jn}(u,v) = 0, \quad \forall\, i,j,k \in \overline{1,n-1}, \tag{4.2}$$

*where $r^g_{ij,kl}(u,v)$ are the matrix elements of the following $r$-matrix:*

$$r^g(u,v) = g^{-1}(u)\otimes g^{-1}(v)\Big(r(u,v) - \{g(u)\otimes 1, 1\otimes L(v)\}g^{-1}(u)\otimes 1 + \\ + \frac{1}{2}[\{g(u)\otimes 1, 1\otimes g(v)\}g^{-1}(u)\otimes g^{-1}(v), 1\otimes L(v)]\Big)g(u)\otimes g(v).$$

*If, moreover, the $r$-matrix $r^g(u,v)$ satisfy the property (1.9) then the condition (1.16) is also satisfied.*

**Remark 4.2** *In the case of non-dynamical transformation matrix $g(u)$ the transformed $r$-matrix is written more simply as follows:*

$$r^g(u,v) = g^{-1}(u)\otimes g^{-1}(v) r(u,v) g(u)\otimes g(v).$$

The analogue of this Corollary holds true also for the constant "external" automorphism of the algebra of the $gl(n)$-valued Lax matrices:

**Corollary 4.2** *Let us consider the automorphism $\sigma(L(u)) = -L^t(u)$. Then the corresponding separating functions $B^\sigma(u) \equiv B(-L^t(u))$ and $A^\sigma(u) \equiv A(-L^t(u))$ satisfy the algebra (1.13a–1.13c) if and only if the following conditions are satisfied:*

$$r^{t_1 t_2}_{in,jk}(u,v) = r^{t_1 t_2}_{in,jn}(u,v) = 0, \quad \forall i,j,k \in \overline{1,n-1}, \tag{4.3}$$

*where $r^{t_1 t_2}_{ij,kl}(u,v)$ are the matrix elements of the $r$-matrix transposed in the first and second component of the tensor product. If, moreover, the $r$-matrix $r(u,v)$ satisfies the property (1.9) then the condition (1.16) is also satisfied.*

**Remark 4.3** *Observe that the condition (4.3) can be rewritten in a more simple way as follows:*

$$r_{ni,kj}(u,v) = r_{ni,nj}(u,v) = 0, \quad \forall i,j,k \in \overline{1,n-1}. \tag{4.4}$$

*It is relevant for the choice of the "right" eigenvector of the Lax matrix and the corresponding separating functions, discussed in the Remark 3.4 and Remark 3.6.*

## 4.3 Proof of the Main Theorem 4.1

Let us now prove the Theorem 4.1. For this purpose we will need (as it follows from the subsection 3.4) to show that the commutation relations (1.13a–1.13c) hold true if and only if the condition (4.1) does. We will do this using the formula of commutation of minors (3.19). In order to simplify the calculation, we will introduce the following notation:

$$\{X(u), Y(v)\} = \{X(u), Y(v)\}_u + \{X(u), Y(v)\}_v,$$

where the first part consists of the summands with the coefficients $r_{ij,kl}(u,v)$ and the second part consists of the summands with the coefficients $r_{ij,kl}(v,u)$. Such a structure is a consequence of the commutation relations (3.19), Leibnitz rule and bilinearity of the Poisson brackets.

In such a way we have $\{X(u), Y(v)\} = \{X(u), Y(v)\}_u - (\{Y(u), X(v)\}_u)_{u\leftrightarrow v}$. In particular, $\{X(u), X(v)\} = \{X(u), X(v)\}_u - (\{X(u), X(v)\}_u)_{u\leftrightarrow v}$. We will use this fact in the subsequent, calculating only the expressions of the type $\{X(u), Y(v)\}_u$ instead of the full Poisson brackets.

### 4.3.1 Proof of the equation (1.13a)

Let us first prove the relation (1.13a) calculating explicitly the Poisson bracket $\{B(u), B(v)\}$. As it follows from the above arguments, for this purpose it is enough to calculate only the first part $\{B(u), B(v)\}_u$.

The following Proposition is true:

**Proposition 4.1** *Let the matrix elements of the Lax matrix $L(u)$ satisfy the commutation relations (2.5) with some $r$-matrix $r(u,v)$. Let the separating functions $B(u)$ and $A(u)$ be defined using Lax matrix $L(u)$ and the formula (3.6). Then the following commutation relations holds:*

$$\{B(u),B(v)\}_u =$$

$$= B(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};c\in\overline{1,n-2},d\in\overline{1,c}} c_{n-1}\sum_{i=0}^{n-2}\sum_{j=1}^{i+1}\sum_{t=1}^{n}(\sum_{l=1}^{n-1} r_{ll,\alpha_{i+1}^j t}(u,v)-(n-1)r_{nn,\alpha_{i+1}^j t}(u,v))\widetilde{B}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)+$$

$$+\sum_{\alpha_c^d\in\overline{1,n-1};c\in\overline{1,n-2},d\in\overline{1,c}} c_{n-1}^2 \sum_{\beta_p^r\in\overline{1,n-1};p\in\overline{1,n-2},r\in\overline{1,p}}\Big(\sum_{t=1}^{n}\sum_{k=1}^{n-2}\sum_{l=1}^{k}\sum_{i=0}^{n-2}\sum_{j=1}^{i+1} r_{\beta_k^l n,\alpha_{i+1}^j t}(u,v)B(\{\beta_p^r\},u)_{\beta_k^l\to n}\times$$

$$\times\widetilde{B}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}+\sum_{t=1}^{n}\sum_{l=1}^{n-1}\sum_{i=0}^{n-2}\sum_{j=1}^{i+1} r_{ln,\alpha_{i+1}^j t}(u,v)B(\{\beta_p^r\},u)_{\beta_{n-1}^l\to n}\widetilde{B}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)-$$

$$\sum_{\alpha_c^d\in\overline{1,n-1};c\in\overline{1,n-2},d\in\overline{1,c}} c_{n-1}^2 \sum_{\beta_p^r\in\overline{1,n-1};p\in\overline{1,n-2},r\in\overline{1,p}}\sum_{s=1}^{n-1}\sum_{t=1}^{n}\sum_{k=0}^{n-2}\sum_{i=0}^{n-2}\sum_{j=1}^{i+1} r_{sn,\alpha_{i+1}^j t}(u,v)B(\{\beta_p^r\},u)^{\beta_k^{k+1}\to s}\widetilde{B}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}$$

*Here the notation $B(\{\alpha_c^d\},v)^{\alpha_i^j\to t}$ means that the upper index $\alpha_i^j$ in the corresponding minor in $B(\{\alpha_c^d\},v)$ is replaced by $t$, $B(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}$ means that the lower index $\alpha_{i+1}^j$ in the corresponding minor in $B(\{\alpha_c^d\},v)$ is replaced by $t$ and "tilde" over the $B(\{\alpha_c^d\},v)$ means tilde over the minor in the product where the corresponding index was replaced by $t$. Besides it is assumed that $\alpha_i^{i+1}=n$, $i\in\overline{0,n-2}$, $\alpha_{n-1}^j=j$, $j\in\overline{1,n-1}$; $\beta_k^{k+1}=n$, $k\in\overline{0,n-2}$, $\beta_{n-1}^l=l$, $l\in\overline{1,n-1}$ and there is no summation over these indices.*

*Proof of the Proposition.* Let us consider the expression $\{B(u),X(v)\}_u$, where $X(v)$ is one of the polynomials $B(u)$, $C(u)$ or $D(u)$. Let us observe that, up to a coefficient they can be represented in the form:

$$X(v)\sim\sum_{\alpha_i^j=\overline{1,n-1};\ j\in\overline{1,i},\ i\in I} X(\{\alpha_i^j\},v),$$

where the set of integers $I$ is defined as follows: $I=\{1,2,...,n-2\}$ in the case of $B(u)$, $I=\{1,2,...,n-3\}$ in the case of $D(u)$ and $C(u)$, $X(\{\alpha_i^j\},v)$ are the monomials in minors of the Lax matrix defined by the formulas (3.12), (3.13) or (3.14).

To simplify the proof we will need one more notation:

$$\{Z(u),Y(v)\}_u=\{Z(u),Y(v)\}_{\tilde{u}}+\{Z(u),Y(v)\}^{\tilde{u}},$$

where $\{Z(u),Y(v)\}_{\tilde{u}}$ is a part of the Poisson bracket given by the first sum in (3.19), $\{Z(u),Y(v)\}^{\tilde{u}}$ is a part of the bracket given by the second sum in (3.19) and $Z(u)$, $Y(v)$ are polynomials or rational functions of the matrix minors of the Lax matrix.

Let us now calculate the expression $\{B(u), X(v)\}_{\widetilde{u}}$. By virtue of the formula (3.19) we obtain:

$$\{B(u), X(\{\alpha_c^d\}, v)\}_{\widetilde{u}} =$$

$$= c_{n-1} \sum_{\beta_p^r \in \overline{1,n-1}; p \in \overline{1,n-2}, r \in \overline{1,p}} \Big( \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{k=1}^{n-2} \sum_{l=1}^{k} \sum_{i=0}^{n-q_X} \sum_{j=1}^{(i+1+\delta_X)} r_{\beta_k^l s, \alpha_{i+1}^j t}(u,v) B(\{\beta_p^r\}, u)_{\beta_k^l \to s} \widetilde{X}(\{\alpha_c^d\}, v)_{\alpha_{i+1}^j \to t} +$$

$$+ \sum_{t=1}^{n} \sum_{s=1}^{n} \sum_{l=1}^{n-1} \sum_{i=0}^{n-q_X} \sum_{j=1}^{(i+1+\delta_X)} r_{\beta_{n-1}^l s, \alpha_{i+1}^j t}(u,v) B(\{\beta_p^r\}, u)_{\beta_{n-1}^l \to s} \widetilde{X}(\{\alpha_c^d\}, v)_{\alpha_{i+1}^j \to t}, \Big) \quad (4.5)$$

where $q_X = 2$ in the case of $X(u) = B(u)$, $q_X = 3$ in the case $X(u) = D(u)$, $q_X = 3$ in the case of $X(u) = C(u)$, and $\alpha_{n-1}^l \equiv l$, $l \in \overline{1, n-1}$ in the first case, $\alpha_{n-2}^l \equiv l+1$, $l \in \overline{1, n-2}$ in the second case, $\alpha_{n-2}^l \equiv l$, $l \in \overline{1, n-1}$ in the third case and $\delta_X \equiv \delta_{i,n-3}\delta_{X,C}$ ($\delta_{X,C}$ is non-zero in the case $X(u) = C(u)$).

On the other hand, the analogous direct calculation using the formula (3.19) shows that

$$\{B(u), X(\{\alpha_c^d\}, v)\}^{\widetilde{u}} =$$

$$= -c_{n-1} \sum_{\beta_p^r \in \overline{1,n-1}; p \in \overline{1,n-2}, r \in \overline{1,p}} \Big( \sum_{s=1}^{n} \sum_{t=1}^{n} \sum_{k=1}^{n-2} \sum_{l=1}^{k} \sum_{i=0}^{n-q_X} \sum_{j=1-\delta_X}^{i+1} r_{s\beta_k^l, t\alpha_i^j}(u,v) B(\{\beta_p^r\}, u)^{\beta_k^l \to s} \widetilde{X}(\{\alpha_i^j\}, v)^{\alpha_i^j \to t} +$$

$$+ \sum_{s=1}^{n} \sum_{t=1}^{n} \sum_{k=0}^{n-2} \sum_{i=0}^{n-q_X} \sum_{j=1-\delta_X}^{i+1} r_{s\beta_k^{k+1}, t\alpha_i^j}(u,v) B(\{\beta_p^r\}, u)^{\beta_k^{k+1} \to s} \widetilde{X}(\{\alpha_i^j\}, v)^{\alpha_i^j \to t} \Big) \quad (4.6)$$

where $\alpha_i^{i+1} = n$, $\beta_k^{k+1} = n$, $q_X$ is defined as above, $\delta_X \equiv \delta_{i,n-3}\delta_{X,C}$ and $\alpha_{n-3}^0 \equiv 1$ if $X(u) = C(u)$.

Here the notation $X(\{\alpha_c^d\}, v)^{\alpha_i^j \to t}$ means that the upper index $\alpha_i^j$ in the corresponding minor in $X(\{\alpha_c^d\}, v)$ is replaced by $t$, $X(\{\alpha_c^d\}, v)_{\alpha_{i+1}^j \to t}$ means that the lower index $\alpha_{i+1}^j$ in the corresponding minor in $X(\{\alpha_c^d\}, v)$ is replaced by $t$ and "tilde" over the $X(\{\alpha_c^d\}, v)$ means tilde over the minor in the product where the corresponding index was replaced by $t$.

To finish the proof of the Proposition we will need the following Lemma:

**Lemma 4.1** *Let $\alpha_i^{i+1} \equiv n$. Then the following identity holds true:*

$$\sum_{t=1}^{n} \sum_{j=1}^{i+1} r_{\gamma\delta, t\alpha_i^j}(u,v) \widetilde{X}(\{\alpha_c^d\}, v)^{\alpha_i^j \to t} = \sum_{t=1}^{n} \sum_{j=1}^{i+1} r_{\gamma\delta, \alpha_{i+1}^j t}(u,v) \widetilde{X}(\{\alpha_c^d\}, v)_{\alpha_{i+1}^j \to t}, \forall \gamma, \delta \in \overline{1, n}.$$

Proof follows from our definition of $\widetilde{L}\begin{pmatrix} \alpha_i^1 .. \check{\alpha}_i^j t .. \alpha_i^i \ \alpha_i^{i+1} \\ \alpha_{i+1}^1 \ \ldots \ \alpha_{i+1}^{i+1} \end{pmatrix}(v)$ and $\widetilde{L}\begin{pmatrix} \alpha_i^1 \ \ldots \ \alpha_i^i \ \alpha_i^{i+1} \\ \alpha_{i+1}^1 .. \check{\alpha}_{i+1}^j t .. \ \alpha_{i+1}^{i+1} \end{pmatrix}(v)$.

Let now $X(u) = B(u)$. Let us consider the first sums in the expressions (4.5)-(4.6). They coincide up to a sign. To see this it is enough to rename in the expression (4.6) the indices of summation $\beta_k^l$ and $s$: $\beta_k^l \leftrightarrow s$ taking into account that the index $s$ in this expression runs effectively from 1 to $n-1$. Indeed, when it is equal to $n$ the corresponding minor in $B(\{\beta_p^r\}, u)^{\beta_k^l \to n}$ has two equal upper indices and, hence, turns zero together with all the expression $B(\{\beta^r_p\},u)^{\beta^l_k\to n}$. From the structure of $B(\{\beta^r_p\},u)$ it follows that $B(\{\beta^r_p\},u)^{\beta^l_k\to s}$ coincides with $B(\{\beta^r_p\},u)_{\beta^l_k\to s}$ after renaming of the indices $\beta^l_k \leftrightarrow s$. Hence, applying the Lemma above, it is easy to see that the first sums in the expressions (4.5)-(4.6) coincide up to the sign and up to the summand corresponding to $s = n$ in the first sum of the right-hand-side of the expression (4.5).

Let us analyze the second sum in the right-hand-side of the expressions (4.5). It is easy to see that $B(\{\beta^r_p\},u)_{\beta^l_{n-1}\to s} = B(\{\beta^r_p\},u)$ if $s = l$ and is zero if $s \neq l$ and $s < n$. Here $\beta^l_{n-1}$ is the lower index in the longest minor in $B(\{\beta^r_p\},u)$. Indeed, if $s < n$ and $s \neq l$ then the longest minor in $B(\{\beta^r_p\},u)_{\beta^l_{n-1}\to s}$ has two equal indices and turns zero together with all the expression $B(\{\beta^r_p\},u)_{\beta^l_{n-1}\to s}$. On the other hand $\beta^l_{n-1} \equiv l$ by the very definition, hence $B(\{\beta^r_p\},u)_{\beta^l_{n-1}\to l} = B(\{\beta^r_p\},u)$. The corresponding summands produce expressions proportional to $B(u)$.

Let us analyze the second sum in the right-hand-side of the expressions (4.6). We have that $B(\{\beta^r_p\},u)^{\beta^{k+1}_k\to s} = B(\{\beta^r_p\},u)$ if $s = n$ due to the fact that by the very definition $\beta^{k+1}_k = n$. We have $n-1$ such summands corresponding to different indices $k \in \overline{0,n-2}$. The corresponding summands produce expressions proportional to $B(u)$. Other values of $s$ produce summands not proportional to $B(u)$.

Substituting all this into the formula $\{B(u),B(v)\}_u = \{B(u),B(v)\}_{\tilde{u}} + \{B(u),B(v)\}^{\tilde{u}}$, using once again the Lemma 4.1 we obtain the statement of the Proposition.

The Proposition is proven.

Now, using the fact that $\{B(u),B(v)\} = \{B(u),B(v)\}_u - (\{B(u),B(v)\}_u)_{u\leftrightarrow v}$ we obtain, analyzing the formula proven in the above Proposition, that the equation (1.13a) holds if and only if the condition (4.1) is satisfied. Indeed, the sufficiency of the condition (4.1) is evident (see below the explicit form of $\{B(u),B(v)\}$ computed under this condition). To prove the necessity, for non-dynamical $r$-matrices we need to show that the sum of the second, third and fourth sums in the proven above formula for $\{B(u),B(v)\}_u$ is proportional to $B(u)$ or zero only if the condition (4.1) is satisfied. On the other hand, analyzing the summands with the coefficients $r_{sn,\alpha^j_{i+1}t}(u,v)$ we obtain that they appear with the following coefficient depending on $v$:

$$\Big(\sum_{\alpha^d_c\in\overline{1,n-1};c\in\overline{1,n-2},d\in\overline{1,c}} \widetilde{B}(\{\alpha^d_c\},v)_{\alpha^j_{i+1}\to t}\Big) \tag{4.7}$$

and with the following coefficient depending on $u$:

$$\sum_{\beta^r_p\in\overline{1,n-1};p\in\overline{1,n-2},r\in\overline{1,p}} \Big(\sum_{k=1}^{n-2}\sum_{l=1}^{k} B(\{\beta^r_p\},u)^{\beta^l_k\to s}_{\beta^l_k\to n} + B(\{\beta^r_p\},u)_{\beta^s_{n-1}\to n} - \sum_{k=0}^{n-2} B(\{\beta^r_p\},u)^{\beta^{k+1}_k\to s}\Big), \tag{4.8}$$

where $s, \alpha^j_{i+1} \in \overline{1,n-1}$, $t \in \overline{1,n}$ and each of $\beta^{k+1}_k$ is equal to $n$.

The coefficient (4.7) is clearly not equal to zero. That is why we have to show that the coefficient (4.8) can not be proportional to $B(u)$. But this follows from the fact that it is a linear function in $L\begin{pmatrix} i \\ n \end{pmatrix}(u) = L_{ni}(u)$, $i \in \overline{1,n}$ with non-trivial coefficients, while $B(u)$ does not depend on the last row of the Lax matrix by the very construction. This shows that the necessary condition for validity of the equation (1.13a) is the condition $r_{sn,\alpha^j_{i+1}t}(u,v)=0$, $s,\alpha^j_{i+1}\in\overline{1,n-1}$, $t\in\overline{1,n}$, i.e. the condition (4.1).

Under this condition the formula for $\{B(u),B(v)\}$ acquires a form of the type (1.13a):

$$\{B(u),B(v)\}=$$
$$=B(u)\Big(\sum_{\alpha^d_c\in\overline{1,n-1};c\in\overline{0,n-2},d\in\overline{1,c+1}}c_{n-1}\sum_{i=0}^{n-2}\sum_{j=1}^{i+1}\sum_{t=1}^{n}(\sum_{l=1}^{n-1}r_{ll,\alpha^j_{i+1}t}(u,v)-(n-1)r_{nn,\alpha^j_{i+1}t}(u,v))\widetilde{B}(\{\alpha^d_c\},v)_{\alpha^j_{i+1}\to t}\Big)-$$
$$-B(v)\Big(\sum_{\alpha^d_c\in\overline{1,n-1};c\in\overline{0,n-2},d\in\overline{1,c+1}}c_{n-1}\sum_{i=0}^{n-2}\sum_{j=1}^{i+1}\sum_{t=1}^{n}(\sum_{l=1}^{n-1}r_{ll,\alpha^j_{i+1}t}(v,u)-(n-1)r_{nn,\alpha^j_{i+1}t}(v,u))\widetilde{B}(\{\alpha^d_c\},u)_{\alpha^j_{i+1}\to t}\Big).\tag{4.9}$$

**Remark 4.4** *Observe that we have shown the necessity of the condition (4.1) for the eq. (1.13a) to be true, hence this condition is necessary for all the system (1.13a–1.13c) to be true and, hence the necessity stated in the theorem is proven. We have only to prove further that the condition (4.1) is sufficient in order for the equations (1.13b)-(1.13c) to be true.*

### 4.3.2 Proof of the equation (1.13b)

Let us now consider the Poisson brackets $\{B(u),A(v)\}$. To calculate them we have to calculate the Poisson brackets $\{B(u),C(v)\}$ and $\{B(u),D(v)\}$ and use the fact that $A(v)=\dfrac{C(v)}{D(v)}$, and, hence:

$$\{B(u),A(v)\}=\frac{1}{D^2(v)}(D(v)\{B(u),C(v)\}-C(v)\{B(u),D(v)\}).$$

Using the fact that $\{B(u),X(v)\}=\{B(u),X(v)\}_u-(\{X(u),B(v)\}_u)_{u\leftrightarrow v}$, we have to calculate $\{B(u),C(v)\}_u$, $\{B(u),D(v)\}_u$ and $\{C(u),B(v)\}_u$, $\{D(u),B(v)\}_u$.

On the other hand, using the results obtained in the previous subsection and taking into account the necessary condition (4.1) it is easy to show that:

$$\{B(u),D(v)\}_u=$$
$$=c_{n-2}B(u)\Big(\sum_{\alpha^d_c\in\overline{1,n-1};c\in\overline{1,n-3},d\in\overline{1,c}}\sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n}(\sum_{l=1}^{n-1}r_{ll,\alpha^j_{i+1}t}(u,v)-(n-1)r_{nn,\alpha^j_{i+1}t}(u,v))\widetilde{D}(\{\alpha^d_c\},v)_{\alpha^j_{i+1}\to t}\Big),$$

where it is assumed that $\alpha^j_{n-2}=j+1$, $j\in\overline{1,n-2}$ and there is no summation over these indices.

In the analogous way we obtain:

$$\{B(u),C(v)\}_u=$$
$$=c_{n-2}B(u)\Big(\sum_{\alpha^d_c\in\overline{1,n-1};c\in\overline{1,n-3},d\in\overline{1,c}}\sum_{i=0}^{n-3}\sum_{j=1}^{(i+1+\delta_{i,n-3})}\sum_{t=1}^{n}(\sum_{l=1}^{n-1}r_{ll,\alpha^j_{i+1}t}(u,v)-(n-1)r_{nn,\alpha^j_{i+1}t}(u,v))\widetilde{C}(\{\alpha^d_c\},v)_{\alpha^j_{i+1}\to t}\Big),$$

where $\alpha^l_{n-2} \equiv l$, $l \in \overline{1, n-1}$ and there is no summation over these indices.

Now let us now calculate the expressions $\{D(u), B(v)\}_u$ and $\{C(u), B(v)\}_u$. Like in the case of the previous subsection, it will be useful to calculate more general expressions $\{D(u), X(v)\}_u$, $\{C(u), X(v)\}_u$. Using the arguments analogous to the ones used in the previous subsection and the necessary condition imposed on the matrix elements of the $r$-matrix (4.1) we obtain:

$$\{D(u), X(\{\alpha^d_c\}, v)\}_u = D(u)\Big(\sum_{i=1}^{n-q_X}\sum_{j=1}^{i+1+\delta_X}\sum_{t=1}^{n}\big(\sum_{l=2}^{n-1} r_{ll,\alpha^j_{i+1}t}(u,v) - (n-2) r_{nn,\alpha^j_{i+1}t}(u,v)\big)\widetilde{X}(\{\alpha^d_c\}, v)_{\alpha^j_{i+1}\to t}\Big) +$$
$$+ \sum_{l=2}^{n-1}(-1)^l D_{\xi_l}(u)\Big(\sum_{i=1}^{n-q_X}\sum_{j=1}^{i+1+\delta_X}\sum_{t=1}^{n} r_{l1,\alpha^j_{i+1}t}(u,v)\widetilde{X}(\{\alpha^d_c\}, v)_{\alpha^j_{i+1}\to t}\Big). \quad (4.10)$$

In the analogous way we will have:

$$\{C(u), X(\{\alpha^d_c\}, v)\}_u = C(u)\Big(\sum_{i=1}^{n-q_X}\sum_{j=1}^{i+1+\delta_X}\sum_{t=1}^{n}\big(\sum_{l=2}^{n-1} r_{ll,\alpha^j_{i+1}t}(u,v) - (n-2) r_{nn,\alpha^j_{i+1}t}(u,v)\big)\widetilde{X}(\{\alpha^d_c\}, v)_{\alpha^j_{i+1}\to t}\Big) -$$
$$- \sum_{l=2}^{n-1} C_{\xi_l}(u)\Big(\sum_{i=1}^{n-q_X}\sum_{j=1}^{i+1+\delta_X}\sum_{t=1}^{n} r_{l1,\alpha^j_{i+1}t}(u,v)\widetilde{X}(\{\alpha^d_c\}, v)_{\alpha^j_{i+1}\to t}\Big), \quad (4.11)$$

where $q_X = 2$ in the case of $X(u) = B(u)$, $q_X = 3$ in the case $X(u) = D(u)$, $q_X = 3$ in the case of $X(u) = C(u)$, and $\alpha^l_{n-1} \equiv l$, $l \in \overline{1, n-1}$ in the first case, $\alpha^l_{n-2} \equiv l+1$, $l \in \overline{1, n-2}$ in the second case, $\alpha^l_{n-2} \equiv l$, $l \in \overline{1, n-1}$ in the third case and $\delta_X \equiv \delta_{i,n-3}\delta_{X,C}$ ($\delta_{X,C}$ is non-zero in the case $X(u) = C(u)$).

With the help of the the identity (3.17) with $k = 1$, taking into account that $C(u) = C_{\xi_1}(u)$, $D(u) = D_{\xi_1}(u)$, we obtain from the equations (4.10) and (4.11):

$$\{A(u), X(\{\alpha^d_c\}, v)\}_u = \frac{B(u)}{D^2(u)}\sum_{l=2}^{n-1}(-1)^l \mathcal{B}\begin{pmatrix} \widehat{n-2}\,\widehat{n-1} \\ \hat{1}\ \hat{l} \end{pmatrix}(u)\sum_{i=1}^{n-q_X}\sum_{j=1}^{i+1+\delta_X}\sum_{t=1}^{n} r_{l1,\alpha^j_{i+1}t}(u,v)\widetilde{X}(\{\alpha^d_c\}, v)_{\alpha^j_{i+1}\to t}.$$

Substituting $X(u) = B(u)$ we will have:

$$\{A(u), B(v)\}_u =$$
$$= c_{n-1}\frac{B(u)}{D^2(u)}\sum_{l=2}^{n-1}(-1)^l \mathcal{B}\begin{pmatrix} \widehat{n-2}\,\widehat{n-1} \\ \hat{1}\ \hat{l} \end{pmatrix}(u)\sum_{\alpha^d_c\in\overline{1,n-1};\, c\in\overline{1,n-2};\, d\in\overline{1,c}}\Big(\sum_{i=0}^{n-2}\sum_{j=1}^{i+1}\sum_{t=1}^{n} r_{l1,\alpha^j_{i+1}t}(u,v)\widetilde{B}(\{\alpha^d_c\}, v)_{\alpha^j_{i+1}\to t}\Big).$$

Using this we obtain the following final expression for the bracket $\{B(u), A(v)\}$:

$$\{A(u), B(v)\} = D^{-2}(u)\times$$
$$\Big(c_{n-1}B(u)\sum_{l=2}^{n-1}(-1)^l\mathcal{B}\left(\begin{array}{c}\widehat{n-2}\widehat{n-1}\\ \widehat{1}\ \widehat{l}\end{array}\right)(u)\sum_{\alpha_c^d\in\overline{1,n-1};\,c\in\overline{1,n-2};\,d\in\overline{1,c}}\Big(\sum_{i=0}^{n-2}\sum_{j=1}^{i+1}\sum_{t=1}^{n}r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{B}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)-$$
$$c_{n-2}B(v)\Big(D(u)\sum_{\alpha_c^d\in\overline{1,n-1};\,c\in\overline{1,n-3};\,d\in\overline{1,c}}\sum_{i=0}^{n-3}\sum_{j=1}^{(i+1+\delta_{i,n-3})}\sum_{t=1}^{n}(\sum_{l=1}^{n-1}r_{ll,\alpha_{i+1}^j t}(v,u)-(n-1)r_{nn,\alpha_{i+1}^j t}(v,u)\big)\widetilde{C}(\{\alpha_c^d\},u)_{\alpha_{i+1}^j\to t}$$
$$-C(u)\sum_{\alpha_c^d\in\overline{1,n-1};\,c\in\overline{1,n-3};\,d\in\overline{1,c}}\sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n}(\sum_{l=1}^{n-1}r_{ll,\alpha_{i+1}^j t}(v,u)-(n-1)r_{nn,\alpha_{i+1}^j t}(v,u)\big)\widetilde{D}(\{\alpha_c^d\},u)_{\alpha_{i+1}^j\to t}\Big)\Big),$$

where it is assumed that $\alpha_{n-2}^l\equiv l$, $l\in\overline{1,n-1}$ when this index is relevant to $C(u)$ and $\alpha_{n-2}^l\equiv 1+l$, $l\in\overline{1,n-2}$ when this index is relevant to $D(u)$.

The equation (1.13b) is proven.

### 4.3.3 Proof of the equation (1.13c)

Let us now consider the Poisson brackets $\{A(u), A(v)\}$. To calculate them we have to calculate the Poisson brackets $\{C(u), C(v)\}$, $\{C(u), D(v)\}$, $\{D(u), C(v)\}$ and $\{D(u), D(v)\}$ and use the fact that $A(v) = \dfrac{C(v)}{D(v)}$ and, hence:

$$\{A(u), A(v)\} = \frac{1}{D^2(u)D^2(v)}(D(u)D(v)\{C(u), C(v)\} + C(u)C(v)\{D(u), D(v)\}-$$
$$- C(u)D(v)\{D(u), C(v)\} - C(v)D(u)\{C(u), D(v)\}). \quad (4.12)$$

Using the fact that $\{A(u), A(v)\} = \{A(u), A(v)\}_u - (\{A(u), A(v)\}_u)_{u\leftrightarrow v}$, we have to calculate only $\{C(u), C(v)\}_u$, $\{C(u), D(v)\}_u$, $\{D(u), C(v)\}_u$, $\{D(u), D(v)\}_u$.

We will again assume that the necessary condition (4.1) holds true. We will use the results obtained during the proof of the equation (1.13b). In particular, with the help of equations (4.10) we derive that:

$$\{D(u), D(v)\}_u =$$
$$= c_{n-2}D(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\,c\in\overline{1,n-3};\,d\in\overline{1,c}}\sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n}(\sum_{l=2}^{n-1}r_{ll,\alpha_{i+1}^j t}(u,v)-(n-2)r_{nn,\alpha_{i+1}^j t}(u,v)\big)\widetilde{D}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)+$$
$$+ c_{n-2}\sum_{l=2}^{n-1}(-1)^l D_{\xi_l}(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\,c\in\overline{1,n-3};\,d\in\overline{1,c}}\sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n}r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{D}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big), \quad (4.13)$$

$$\{D(u),C(v)\}_u =$$

$$= c_{n-2}D(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\, c\in\overline{1,n-3};\, d\in\overline{1,c}}\ \sum_{i=0}^{n-3}\ \sum_{j=1}^{(i+1+\delta_{i,n-3})}\ \sum_{t=1}^{n}\Big(\sum_{l=2}^{n-1} r_{ll,\alpha_{i+1}^j t}(u,v) - (n-2)r_{nn,\alpha_{i+1}^j t}(u,v)\Big)\widetilde{C}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)$$

$$+ c_{n-2}\sum_{l=2}^{n-1}(-1)^l D_{\xi_l}(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\, c\in\overline{1,n-3};\, d\in\overline{1,c}}\ \sum_{i=0}^{n-3}\ \sum_{j=1}^{(i+1+\delta_{i,n-3})}\ \sum_{t=1}^{n} r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{C}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big), \quad (4.14)$$

In an analogous way, with the help of equations (4.10) it is easy to derive that:

$$\{C(u),C(v)\}_u =$$

$$= c_{n-2}C(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};c\in\overline{1,n-3},d\in\overline{1,c}}\ \sum_{i=0}^{n-3}\ \sum_{j=1}^{(i+1+\delta_{i,n-3})}\ \sum_{t=1}^{n}\Big(\sum_{l=2}^{n-1} r_{ll,\alpha_{i+1}^j t}(u,v) - (n-2)r_{nn,\alpha_{i+1}^j t}(u,v)\Big)\widetilde{C}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)$$

$$- c_{n-2}\sum_{l=2}^{n-1} C_{\xi_l}(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\, c\in\overline{1,n-3};\, d\in\overline{1,c}}\ \sum_{i=0}^{n-3}\ \sum_{j=1}^{(i+1+\delta_{i,n-3})}\ \sum_{t=1}^{n} r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{C}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big). \quad (4.15)$$

$$\{C(u),D(v)\}_u =$$

$$= c_{n-2}C(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\, c\in\overline{1,n-3};\, d\in\overline{1,c}}\ \sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n}\Big(\sum_{l=2}^{n-1} r_{ll,\alpha_{i+1}^j t}(u,v) - (n-2)r_{nn,\alpha_{i+1}^j t}(u,v)\Big)\widetilde{D}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big) -$$

$$- c_{n-2}\sum_{l=2}^{n-1} C_{\xi_l}(u)\Big(\sum_{\alpha_c^d\in\overline{1,n-1};\, c\in\overline{1,n-3};\, d\in\overline{1,c}}\ \sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n} r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{D}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big). \quad (4.16)$$

Substituting the four expressions obtained above in the equation (4.12) and taking into account the identity (3.17) with $k = 1$ we finally obtain:

$$\{A(u),A(v)\}_u = c_{n-2}B(u)\Big(\sum_{l=2}^{n-1}(-1)^l \mathcal{B}\begin{pmatrix} \widehat{n-2}\,\widehat{n-1} \\ \hat{1}\ \ \hat{l} \end{pmatrix}(u) \sum_{\alpha_c^d\in\overline{1,n-1};c\in\overline{1,n-3},d\in\overline{1,c}}\Big(C(v)\times$$

$$\times\sum_{i=0}^{n-3}\sum_{j=1}^{i+1}\sum_{t=1}^{n} r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{D}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t} - D(v)\sum_{i=0}^{n-3}\ \sum_{j=1}^{(i+1+\delta_{i,n-3})}\ \sum_{t=1}^{n} r_{l1,\alpha_{i+1}^j t}(u,v)\widetilde{C}(\{\alpha_c^d\},v)_{\alpha_{i+1}^j\to t}\Big)\Big), \quad (4.17)$$

where throughout this subsection it is assumed that $\alpha_{n-2}^l \equiv l$, $l \in \overline{1,n-1}$ when this index is relevant to $C(u)$ and $\alpha_{n-2}^l \equiv 1 + l$, $l \in \overline{1,n-2}$ when this index is relevant to $D(u)$.

Keeping in mind that $\{A(u),A(v)\} = \{A(u),A(v)\}_u - (\{A(u),A(v)\}_u)_{u\leftrightarrow v}$ we immediately obtain from the equation (4.17) that the equation (1.13c) holds true.

#### 4.3.4 Proof of the equation (1.16)

Now in order to complete the prooof of the Theorem we have only to prove the equation (1.16). In order to do this let us observe that, the coefficient functions $a(u,v)$, $b(u,v)$, $\alpha(u,v)$, $\beta(u,v)$ entering into the definition of the separation algebra (1.13a–1.13c) have been explicitly calculated in the previous subsection. They are complicated (polynomial or rational) functions of the matrix elements of the Lax operator, but *linear* functions of the matrix elements of the $r$-matrix. We will use this fact proving the equation (1.16). In order to prove this equation we will take into account the regularity property (1.9) of the $r$-matrix:

$$r(u,v) = \frac{\Omega}{u-v} + \Delta r(u,v),$$

which is a key property for this proof. We have:

$$\lim_{u\to v}(\alpha(u,v)B(u)-\beta(u,v)B(v)) = \lim_{u\to v}\left(\left(\frac{\alpha_{-1}(u,v)}{u-v}+\alpha_0(u,v)\right)B(u)-\left(\frac{\beta_{-1}(u,v)}{u-v}+\beta_0(u,v)\right)B(v)\right) =$$

$$= \alpha_{-1}(v,v)\partial_v B(v)+\left((\alpha_0(v,v)-\beta_0(v,v))+\partial_u(\alpha_{-1}(u,v)-\beta_{-1}(u,v))|_{u=v}+\lim_{u\to v}\frac{(\alpha_{-1}(v,v)-\beta_{-1}(v,v))}{u-v}\right)B(v),$$

where the decompositions $\alpha(u,v) = \dfrac{\alpha_{-1}(u,v)}{u-v} + \alpha_0(u,v)$, $\beta(u,v) = \dfrac{\beta_{-1}(u,v)}{u-v} + \beta_0(u,v)$ correspond to the above decomposition of the $r$-matrix and we have used that the functions $\alpha_{-1}(u,v)$, $\beta_{-1}(u,v)$, $\alpha_0(u,v)$, $\beta_0(u,v)$ are regular on the diagonal $u=v$.

Now to prove the equation (1.16) it is necessary to notice that, due to the form of the decomposition (1.9), the functions $\dfrac{\alpha_{-1}(u,v)}{u-v}$ and $\dfrac{\beta_{-1}(u,v)}{u-v}$ coincide with the functions $\alpha(u,v)$ and $\beta(u,v)$ for the case of standard rational $r$-matrix, containing only the pole part. But this case was considered in [14]. It was shown there that $\alpha_{-1}(u,v) = \beta_{-1}(u,v) = 1$. Using this result we obtain:

$$\lim_{u\to v}(\alpha(u,v)B(u) - \beta(u,v)B(v)) = \partial_v B(v) + (\alpha_0(v,v) - \beta_0(v,v))B(v).$$

The equation (1.16) is proven. This finishes the proof of the main Theorem.

# 5 Examples of integrable models and separated variables

## 5.1 The completeness problem

The difficult problem in the theory of separation of variables is the problem of completeness of the constructed canonical coordinates. Let us briefly discuss it.

Till this moment we have been working with an arbitrary Lax matrix satisfying the $r$-matrix Poisson brackets (2.3). Nevertheless, in order to understand whether the separating functions produce a complete family it is necessary to specify not only the $r$-matrix but also the Lax operator as a function of the spectral parameter and dynamical variables, i.e. to specify the Poisson manifold $\mathcal{P}$. Moreover, in the higher rank cases it is necessary also to specify the type of the coadjoint orbit to which the dynamical variables belong, i.e. to specify a symplectic leaf $\mathcal{M} \subset \mathcal{P}$. After having fixed the manifold $\mathcal{M}$ it is possible to compare its dimension with the number of the coordinates produced by separating functions. It turns out that not for all classical $r$-matrices satisfying the condition (4.1) and not for all corresponding Lax operators (which can be different for the same classical $r$-matrices — see [29]) the separating functions produce complete set of separated variables.

In this paper we will consider one of the most general classes of the Lax operators that exist for any non-dynamical classical $r$-matrices, namely the Lax operators of the generalized Gaudin models with [28] and without [26] external magnetic field. We will consider two types of such $r$-matrices satisfying the condition (4.1) for which the separating functions of the generalized Gaudin models produce the complete set of separated coordinates. The first type is the rational $r$-matrix. The separated coordinates for this example are known and were considered previously in [1], [14], [18], [7]. The second class is connected with the so-called "anisotropic" non-skew-symmetric classical $r$-matrix discovered in [26] (see also [27]). The separation of variables for the corresponding generalized Gaudin models is a new result.

## 5.2 The Generalized Gaudin models with or without magnetic field

Let us consider the generalized Gaudin models associated with the general non-dynamical classical $r$-matrix with spectral parameters.

Let $S_{ij}^{(m)}$, $i,j = \overline{1,n}$, $m = \overline{1,N}$ be linear coordinate functions on the dual space to the Lie algebra $gl(n)^{\oplus N}$ with the following Poisson brackets:

$$\{S_{ij}^{(m)}, S_{kl}^{(p)}\} = \delta^{pm}(\delta_{kj} S_{il}^{(m)} - \delta_{il} S_{kj}^{(m)}). \tag{5.1}$$

The linear space $(gl(n)^{\oplus N})^*$ is a Poisson manifold of the dimension $n^2 N$. It is foliated by the symplectic leaves — the coadjoint orbits of the group $GL(n)^N$. We will be interested in symplectic leaves of the maximal dimension — nondegenerate coadjoint orbits. Their dimension is equal to $2D = (n^2 - n) \times N$. They can be described as level surfaces of $n \times N$ Casimir functions:

$$C^{(m),k} = \sum_{i_1,i_2,\dots,i_k=1}^{n} S_{i_1 i_2}^{(m)} S_{i_2 i_3}^{(m)} \dots S_{i_k i_1}^{(m)}, \quad m \in \overline{1,N}, \ k \in \overline{1,n}.$$

Let us fix $N$ distinct points of the complex plane $\nu_m$, $m = 1, 2, \dots, N$ belonging to the open region in the complex plane were the condition (1.9) holds true. For a given $r$-matrix one can construct the following classical Lax operator:

$$L(u) = \sum_{p,q=1}^{n} L_{pq}(u) X_{pq} \equiv \sum_{m=1}^{N} \sum_{i,j,p,q=1}^{n} r_{ij,pq}(\nu_m, u) S_{ij}^{(m)} X_{pq}. \tag{5.2}$$

Using generalized classical Yang-Baxter equation (2.7) it is possible to show (see [26]) that $L(u)$ satisfies a linear $r$-matrix Poisson brackets (2.3). This is the Lax operator of the generalized $gl(n)$-valued classical Gaudin spin chain.

Let, moreover, $c(u) = \sum\limits_{i,j=1}^{n} c_{ij}(u)X_{ij}$ be a solution of the so-called "shift equation" [28]

$$[r^{12}(u,v), c(u)\otimes 1] - [r^{21}(v,u), 1\otimes c(v)] = 0,$$

where $c_{ij}(u)$ are functions of $u$ non-depending on the dynamical variables.

The "shift element" can be added to any Lax operator satisfying linear $r$-matrix Poisson brackets (2.3) (see [28]). In particular, it can be added to the Lax operator (5.2):

$$L(u) = \sum_{m=1}^{N} \sum_{i,j,p,q=1}^{n} r_{ij,pq}(\nu_m, u) S^{(m)}_{ij} X_{pq} + \sum_{i,j=1}^{n} c_{ij}(u) X_{ij}. \quad (5.3)$$

It produces the Lax operator of the classical Gaudin model in an external magnetic field.

Using the spectral invariants of the Lax matrix one obtains $n$ series of generating functions of the integrals of motion. The residues at the points $u = \nu_l$, $l \in \overline{1,N}$ of the second order generating function — trace of the square of the Lax matrix — produces the following quadratic integrals [26]:

$$H_l = \sum_{k=1,k\neq l}^{N} \sum_{i,j,p,q=1}^{n} r_{ij,pq}(\nu_k,\nu_l) S^{(k)}_{ij} S^{(l)}_{pq} + \sum_{i,j,p,q=1}^{n} r^{0}_{ij,pq}(\nu_l,\nu_l) S^{(l)}_{ij} S^{(l)}_{pq}, \quad (5.4)$$

for the case of the Lax operators (5.2) and

$$H_l = \sum_{k=1,k\neq l}^{N} \sum_{i,j,p,q=1}^{n} r_{ij,pq}(\nu_k,\nu_l) S^{(k)}_{ij} S^{(l)}_{pq} + \sum_{i,j,p,q=1}^{n} r^{0}_{ij,pq}(\nu_l,\nu_l) S^{(l)}_{ij} S^{(l)}_{pq} + \sum_{i,j=1}^{n} c_{ij}(\nu_l) S^{(l)}_{ij}, \quad (5.5)$$

for the case of the Lax operators (5.3). Here $r^{0}_{ij,pq}(u,v)$ are the matrix elements of the regular part of the $r$-matrix.

The integrals (5.4) are the generalized classical Gaudin Hamiltonians [26] and the integrals (5.5) are the generalized classical Gaudin Hamiltonians in an external magnetic field [28] corresponding to general non-dynamical $gl(n)\otimes gl(n)$-valued classical $r$-matrix. In the case of skew-symmetric classical $r$-matrices the second term in the Hamiltonians (5.5) and (5.4) vanishes and they coincide with the standard Gaudin Hamiltonians [13] with or without external magnetic field.

## 5.3 Standard rational $r$-matrix and standard rational Gaudin models

Let us consider the simplest possible case, namely the case of the standard rational $r$-matrix and standard rational Gaudin models with or without external magnetic field [13].

### 5.3.1 Standard rational $r$-matrix

The standard rational $r$-matrix is a simplest possible $r$-matrix containing only the pole part

$$r^{12}(u,v) = \frac{\Omega^{12}}{u-v} = \frac{\sum\limits_{i,j=1}^{n} X_{ij}\otimes X_{ji}}{u-v}. \quad (5.6)$$

It is evidently skew-symmetric. The following Proposition holds true:

**Proposition 5.1** *The r-matrix (5.6) satisfies the conditions (1.11) for an arbitrary vector* $\vec{\kappa}$.

*Proof.* The fact that the $r$-matrix (5.6) satisfies the condition (4.1) follows immediately from its definition. Due to its $GL(n)$-invariance it readily follows that also the condition (1.11) is satisfied.

The shift element for the $r$-matrix (5.6) coincides with arbitrary constant element of $gl(n)$:

$$c(u) = \sum_{i,j=1}^{n} c_{ij} X_{ij}.$$

In the present paper we will use only the shift elements of the following form: $c(u) = \sum\limits_{i=1}^{n} c_i X_{ii}$.

### 5.3.2 Standard rational Gaudin models with or without magnetic field

Let us now consider the standard Gaudin models with or without external magnetic field associated with the classical $r$-matrix (5.6). For this purpose we will specify the formulas (5.2), (5.4) and (5.3), (5.5) for the case of the classical $r$-matrix (5.6).

Let us first consider the case of standard rational Gaudin model without external magnetic field. We will have the following Lax operator (5.2) in this case [13]:

$$L(u) = \sum_{k=1}^{N} \frac{1}{\nu_k - u} \sum_{i,j=1}^{n} S_{ij}^{(k)} X_{ji}. \tag{5.7}$$

The corresponding generalized Gaudin Hamiltonians (5.4) are written as follows [13]:

$$H_l = \sum_{k=1,\, k\neq l}^{N} \frac{1}{\nu_k - \nu_l} \sum_{i,j=1}^{n} S_{ij}^{(k)} S_{ji}^{(l)}. \tag{5.8}$$

Using the constant shift element it is possible to define the following "shifted" Lax operator:

$$L(u) = \sum_{k=1}^{N} \frac{1}{\nu_k - u} \sum_{i,j=1}^{n} S_{ij}^{(k)} X_{ji} + \sum_{i=1}^{n} c_i X_{ii} \tag{5.9}$$

and the following standard rational Gaudin Hamiltonians in an external magnetic field:

$$H_l = \sum_{k=1,\, k\neq l}^{N} \frac{1}{\nu_k - \nu_l} \sum_{i,j=1}^{n} S_{ij}^{(k)} S_{ji}^{(l)} + \sum_{i=1}^{n} c_i S_{ii}^{(l)}. \tag{5.10}$$

**Remark 5.1** *Observe that the set of integrals of motion obtained with the help of the rational Lax operator without a magnetic field (5.7) is incomplete. It is obvious for* $N = 1$*; for* $N > 1$ *the incompleteness can be easily checked. It becomes complete only after the introduction in the Lax operator of the external magnetic field* $c(u) = \sum\limits_{i=1}^{n} c_i X_{ii}$ *were* $c_i \neq c_j$, $i \neq j$, $i, j \in \overline{1, n}$.

### 5.3.3 The completeness of separated variables

As it was observed above the Lax operator of rational Gaudin system without magnetic field does not provide by itself a complete family of integrals of motion. That is why all the types of separating polynomials constructed with its help do not provide a complete set of separated coordinates. Motivated by this we will consider in this subsection only the generalized Gaudin system in an external magnetic field with $c_i \neq c_j$, $i \neq j$, $i,j \in \overline{1,n}$. We will be interested in the separated coordinates on the non-degenerate coadjoint orbits. That is why we have to present $n(n-1) \times N$ separated coordinates.

As it was already observed, the $r$-matrix (5.6) satisfies the condition (4.1). That is why the separating functions $A(u)$ and $B(u)$ corresponding to the standard normalization of the eigenvector of the Lax matrix produce the canonical coordinates. Nevertheless, this set of canonical coordinates is not complete.

Indeed, let us consider the set of zeros of the function $B(u)$. It will be enough to consider the case $N=1$. Let us introduce the auxiliary spectral-parameter-independent matrix $S^{(1)} = \sum\limits_{i,j=1}^{n} S^{(1)}_{ij} X_{ij}$. We have that $(u-\nu_1)L(u) = S^{(1)} + (u-\nu_1)c$, where $c$ is a constant diagonal shift element.

Using the fact that the function $B(\{\alpha_i^j\}, u)$ contains the minor $L\begin{pmatrix} \alpha^1_{m-1} \ldots \alpha^{m-1}_{m-1} & n \\ \alpha^1_m \ \ldots \ \alpha^m_m \end{pmatrix}(u)$, where $\alpha_i^j \in \overline{1,n-1}$, we see that, after multiplication by $(u-\nu_1)^m$ this minor becomes a polynomial of the degree at most $(m-1)$ in $u$. The maximal degree is achieved when $(m-1)$ upper indices coincide with $(m-1)$ lower indices in the corresponding minor. Using the definition of the function $B(u)$ written in the form (3.11)–(3.12) we immediately obtain that the function $B(u)$ (after the multiplication by the polynomial $f(u) = (u-\nu_1)^{\frac{n(n-1)}{2}}$) is a product of $(u-\nu_1)^{n-1}$ by a polynomial of the degree $1+2+\ldots+(n-2) = \dfrac{(n-2)(n-1)}{2}$ in $u$ with non-trivial coefficients. This is why zeros of the function $B(u)$ corresponding to the standard normalization of the eigenvector of the Lax matrix give us only $\dfrac{(n-2)(n-1)}{2}$ separated coordinates $x_i$ instead of the necessary $\dfrac{n(n-1)}{2}$ coordinates, i.e $n-1$ coordinates $x_i$ and $n-1$ coordinates $p_j$ are missing.

Let us consider the case $N>1$. It is easy to see from the explicit form of the function $B(\{\alpha_i^j\}, u)$ that, adding to the Lax operator (5.21) poles at $u=\nu_2$, $u=\nu_3$ etc. yields (after the multiplication by $(u-\nu_k)^{\frac{n(n-1)}{2}}$) exactly $\dfrac{n(n-1)}{2}$ new roots of the equations $B(u)=0$. As a result we obtain that the separating function $B(u)$ corresponding to the standard normalization of the eigenvector of the Lax matrix produces $\big(n(n-1)\times N - 2(n-1)\big)$ instead of needed $n(n-1)\times N$ separated coordinates.

There are two ways to resolve the arisen problem of incompleteness. The first one was proposed in [1]. Its idea is to find additionally $2(n-1)$ canonical coordinates commuting with the separated coordinates constructed with the help of the standard separating functions $B(u)$, $A(u)$.

The following Proposition holds true:

**Proposition 5.2** *The complete set of the canonical coordinates for the rational Gaudin system in a diagonal external magnetic field* $c = \sum\limits_{i=1}^{n} c_i X_{ii}$, *where* $c_i \neq c_j$, $i \neq j$, $i,j \in \overline{1,n}$ *consists of the* *separated coordinates constructed with the help of the standard separating functions $B(u)$ and $A(u)$ and the coordinates*

$$p_i^S \equiv -\sum_{k=1}^{N} S_{ii}^{(k)}, \; x_j^S = \ln \sum_{k=1}^{N} S_{jn}^{(k)}, \quad i,j \in \overline{1,n-1}. \tag{5.11}$$

*Proof.* In order to prove this proposition we remind that, as it was shown above, the separated coordinates are invariant with respect to the coadjoint action of the maximal parabolic subgroup. This action can be represented on the infinitesimal level by the action of the generators $M_{ij}$ such that

$$\{M_{ij}, L_{kl}(u)\} = \delta_{kj} L_{il}(u) - \delta_{il} L_{kj}(u), \tag{5.12}$$

where $k \in \overline{1,n-1}$, $l \in \overline{1,n}$, $i \in \overline{1,n-1}$, $j \in \overline{1,n}$. Using the $GL(n)$-invariance of the rational $r$-matrix it is possible to show that, for the Lax operator (5.9) the functions $M_{ii} = \sum\limits_{k=1}^{N} S_{ii}^{(k)}$ and $M_{in} = \sum\limits_{k=1}^{N} S_{in}^{(k)}$ satisfy (5.12). That is why they commute with the separated coordinates constructed with the help of the standard separating functions $B(u)$ and $A(u)$. On the other hand it is easy to show that for such functions $M_{ii}$ and $M_{jn}$ the following commutation relations hold true:

$$\{M_{ii}, M_{jj}\} = \{M_{in}, M_{jn}\} = 0, \; \{M_{ii}, M_{jn}\} = \delta_{ij} M_{in}, \quad i,j \in \overline{1,n-1}.$$

Now, using the parameterization (5.11), we obtain the proof of the Proposition .

The simple way of completing of the set of separating coordinates given in the above Proposition does not work for the case of more general classical $r$-matrices. Indeed, for the $r$-matrices not possessing $GL(n)$ symmetry the functions $M_{ii} = \sum\limits_{k=1}^{N} S_{ii}^{(k)}$, $M_{jn} = \sum\limits_{k=1}^{N} S_{jn}^{(k)}$ do not satisfy the relations (5.12). That is why it is not possible to use them in other cases and it is necessary to find another way for constructing a complete set of separated coordinates. A possible solution was proposed in [7]. Its idea is to consider instead of the functions $B(u)$ and $A(u)$ the separating functions $B_{\vec{\kappa}}(u)$ and $A_{\vec{\kappa}}(u)$ corresponding to arbitrary normalization of the eigenvector of the Lax matrix.

The following proposition holds true:

**Proposition 5.3** *Let $B_{\vec{\kappa}}(u) = B^{g_{\vec{\kappa}}}(u)$, $A_{\vec{\kappa}}(u) = A^{g_{\vec{\kappa}}}(u)$ be the transformed separating functions corresponding to the vector $\vec{\kappa}$ normalizing the eigenvector of the Lax matrix. Let $\kappa_i \neq 0$, $i \in \overline{1,n}$ and $c_i \neq c_j \neq 0$. Then zeros of the function $B_{\vec{\kappa}}(u)$ and the values of the function $A_{\vec{\kappa}}(u)$ in these zeros, where the functions $B_{\vec{\kappa}}(u)$, $A_{\vec{\kappa}}(u)$ are calculated for the Lax operator (5.9), constitute a complete set of the coordinates of separation on the generic coadjoint orbit of the direct product of $N$ copies of $GL(n)$.*

*Proof.* Let us first compute the number of zeros of $B_{\vec{\kappa}}(u)$. From the definition (1.18) it readily follows that it is a rational function of the form

$$B_{\vec{\kappa}}(u) = \frac{\tilde{B}_{\vec{\kappa}}(u)}{\prod_{k=1}^{N} (\nu_k - u)^{\frac{n(n-1)}{2}}}. \tag{5.13}$$

To compute the degree of the polynomial $\tilde{B}_{\vec{\kappa}}(u)$ we use the following expansion

$$L(u) = C + \mathcal{O}\left(\frac{1}{u}\right), \quad u \to \infty$$

where $C = \text{diag}(c_1, \ldots, c_n)$. So the $i$-th column of the matrix (1.18) is equal to

$$L^{i-1}(u)\vec{\kappa} = \begin{pmatrix} c_1^{i-1}\kappa_1 \\ . \\ . \\ . \\ c_n^{i-1}\kappa_n \end{pmatrix} + \mathcal{O}\left(\frac{1}{u}\right)$$

so

$$B_{\vec{\kappa}}(u) = \kappa_1 \ldots \kappa_n \prod_{i>j}(c_i - c_j) + \mathcal{O}\left(\frac{1}{u}\right), \quad u \to \infty.$$

Comparing degrees of the numerator and denominator in (5.13) we conclude that

$$\deg \tilde{B}_{\vec{\kappa}}(u) = N\frac{n(n-1)}{2}.$$

Let us now prove independence of the canonical variables. It suffices to prove independence of the roots $x_1, \ldots, x_d$, $d = N\frac{n(n-1)}{2}$, of the polynomial $B_{\vec{\kappa}}(u)$. We will verify their independence at a particular point of the phase space. As the coefficients of $B_{\vec{\kappa}}(u)$ depend polynomially on the phase variables, such a verification is sufficient for proving independence at a generic point. To simplify the formulae we will do the calculations for the particular case $\vec{\kappa} = (1, 1, \ldots, 1)$.

Consider the case of diagonal matrices $S^{(k)}$,

$$S^{(k)} = \text{diag}\left(\lambda_1^k, \ldots, \lambda_n^k\right), \quad k = 1, \ldots, N.$$

The Lax matrix is diagonal,

$$L(u) = \text{diag}\left(\mu_1(u), \ldots, \mu_n(u)\right), \quad \mu_i(u) = c_i + \sum_{k=1}^{N} \frac{\lambda_i^k}{\nu_k - u}.$$

The definition (1.18) of $B_{\vec{\kappa}}(u)$ reduces to the Vandermonde determinant

$$B(u)_{\vec{\kappa}} = \prod_{i>j}\left(\mu_i(u) - \mu_j(u)\right).$$

So, at this point of the phase space the variables $x_1, \ldots, x_d$ are roots of the equations

$$\mu_i(u) = \mu_j(u), \quad i < j. \tag{5.14}$$

Every such equation has $N$ roots. Denote them $x_{ij}^1, \ldots, x_{ij}^N$. For a generic choice of the constants $\lambda_i^k$ we can assume that all these roots are pairwise distinct.

Let us now consider a small perturbation

$$L(u) \mapsto L(u) + \delta L(u).$$

Since we want to stay on the same symplectic leaf, we have to impose conditions

$$\delta L_{ii}(u) = 0, \quad i = 1, \dots, n.$$

For the variation of the polynomial $B_{\vec{\kappa}}(u)$ we obtain

$$\delta \log B_{\vec{\kappa}}(u) = \sum_{i \neq j} \frac{\delta L_{ij}(u)}{\mu_i(u) - \mu_j(u)}. \tag{5.15}$$

The proof of this formula is identical to the proof of Lemma A.2 in the Appendix; it is even simpler as the basis of eigenvectors of $L(u)$ coincides with the standard basis in $\mathbb{C}^n$. Observe that the symmetric part of the matrix $\delta L(u)$ does not contribute to the rhs. So, without loss of generality we can assume antisymmetry

$$\delta L_{ji}(u) = -\delta L_{ij}(u) \quad \forall\, i,\, j.$$

From eq. (5.15) it readily follows that

$$\delta x_{ij}^p = 2 \frac{\delta L_{ij}\left(x_{ij}^p\right)}{\mu_i'\left(x_{ij}^p\right) - \mu_j'\left(x_{ij}^p\right)}, \quad 1 \leq i < j \leq n, \quad p = 1, \dots, N. \tag{5.16}$$

Hence

$$\frac{\partial x_{ij}^p}{\partial S_{kl}^{(q)}} = \frac{\delta_{ik}\delta_{jl}}{\left(\mu_i'\left(x_{ij}^p\right) - \mu_j'\left(x_{ij}^p\right)\right)\left(\nu_q - x_{ij}^p\right)}, \quad i < j, \quad k < l, \quad p,\, q = 1, \dots, N. \tag{5.17}$$

Using the well known Cauchy determinant

$$\det\left(\frac{1}{a_i - b_j}\right)_{1 \leq i,j \leq N} = \frac{\prod_{i<j}(a_i - a_j)\prod_{k>l}(b_k - b_l)}{\prod_{i,j}(a_i - b_j)}$$

we obtain the determinant of the Jacobi matrix (5.17)

$$\det\left(\frac{\partial x_{ij}^p}{\partial S_{kl}^{(q)}}\right) = 2^{N\frac{n(n-1)}{2}} \prod_{i<j}\left[\prod_{p=1}^{N}\left(\mu_i'\left(x_{ij}^p\right) - \mu_j'\left(x_{ij}^p\right)\right)^{-1} \frac{\prod_{s<t}(\nu_s - \nu_t)\left(x_{ij}^t - x_{ij}^s\right)}{\prod_{s,t}\left(\nu_t - x_{ij}^s\right)}\right] \neq 0. \tag{5.18}$$

The Proposition is proven. □

## 5.4 Anisotropically deformed rational $r$-matrices and Gaudin-type models

### 5.4.1 Anisotropically deformed rational $r$-matrices

Let $X_{ij}$, $i, j \in \overline{1,n}$ be the standard basis of $gl(n)$. Let $a_j$, $j \in \overline{1,n}$ be arbitrary complex numbers. Let us consider the following $gl(n) \otimes gl(n)$-valued function:

$$r_a(u, v) = \frac{1}{u - v} \sum_{i,j=1}^{n} \frac{1 + a_i u}{1 + a_i v} X_{ij} \otimes X_{ji}. \tag{5.19}$$

As it was shown in [28] it satisfies the generalized non-dynamical classical Yang-Baxter equation (2.7), i.e. it is indeed a classical $r$-matrix. This $r$-matrix is gauge equivalent to "irrational" or "hyperelliptic" $r$-matrix discovered in [26]. The $r$-matrix (5.19) can be viewed as an anisotropic deformation of the rational $r$-matrix: when $a_j \to 0$, $j \in \overline{1,n}$ the $r$-matrix (5.19) passes to the standard rational $r$-matrix.

The $r$-matrix (5.19) admits the decomposition (1.9). Indeed, the direct calculation shows that:

$$r_a(u,v) = \frac{\Omega}{u-v} + \Delta r(v,v) = \frac{\sum\limits_{i,j=1}^{n} X_{ij} \otimes X_{ji}}{u-v} + \sum_{i,j=1}^{n} \frac{a_i X_{ij} \otimes X_{ji}}{1+a_i v}.$$

For the classical $r$-matrix (5.19) exists the following shift element [30]

$$c(u) = \sum_{k=1}^{n} \frac{c_k}{1+a_k u} \sum_{i=1}^{n} \frac{a_i - a_k}{1+a_i u} X_{ii}, \tag{5.20}$$

where $c_k$ are arbitrary complex numbers. Contrary to the case of the standard rational $r$-matrix, the shift element (5.20) nontrivialy depends on the spectral parameter $u$.

The $r$-matrix (5.19) belongs to the class of the so-called "diagonal" with respect to the standard root basis $r$-matrices, i.e. to the $r$-matrices of the following form:

$$r(u,v) = \sum_{i,j=1}^{n} r_{ji}(u,v) X_{ij} \otimes X_{ji},$$

i.e. $r_{ij,kl}(u,v) = r_{ji}(u,v)\delta_{kj}\delta_{il}$. From this we easily obtain that the $r$-matrix (5.19) satisfies the condition (4.1). It occurs, moreover, that the $r$-matrix (5.19) satisfies a stronger condition:

**Proposition 5.4** *The $r$-matrix (5.19) satisfies the condition (1.11) for arbitrary constant vector $\vec{\kappa}$.*

*Proof.* It will be sufficient to prove this statement for a suitably normalized vector $\vec{\kappa}$. Chose $\kappa_n = 1$ as such a normalization. The corresponding group element $g_{\vec{\kappa}}$ is written in this case simply as $g_{\vec{\kappa}} = 1_n + \sum\limits_{i=1}^{n-1} \kappa_i X_{in}$, $g_{\vec{\kappa}}^{-1} = 1_n - \sum\limits_{i=1}^{n-1} \kappa_i X_{in}$. Using this and the direct calculation one shows that for any diagonal in the root basis $r$-matrix the corresponding gauge-transformed $r$-matrix reads as follows:

$$r^{g_{\vec{\kappa}}}(u,v) =$$

$$= \sum_{i,j=1}^{n} r_{ji}(u,v) X_{ij} \otimes X_{ji} + \sum_{i=1}^{n}\sum_{j=1}^{n-1} (r_{ji}(u,v) - r_{ni}(u,v))\kappa_j X_{in} \otimes X_{ji} + \sum_{i=1}^{n-1}\sum_{j=1}^{n} (r_{ji}(u,v) - r_{jn}(u,v))\kappa_i X_{ij} \otimes X_{jn} +$$

$$+ \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} (r_{ji}(u,v) + r_{nn}(u,v) - r_{jn}(u,v) - r_{ni}(u,v))\kappa_j\kappa_i X_{in} \otimes X_{jn}.$$

From the explicit form of the transformed $r$-matrix we immediately obtain that the condition (1.12) is equivalent to the following conditions

$$(i) r_{in,ji} = \kappa_j (r_{ji}(u,v) - r_{ni}(u,v)) = 0, (ii) r_{in,jn} = \kappa_i\kappa_j (r_{ji}(u,v) + r_{nn}(u,v) - r_{jn}(u,v) - r_{ni}(u,v)) = 0,$$

where $i,j \in \overline{1,n-1}$. In the case $\kappa_i \neq 0$, $i \in \overline{1,n-1}$ in order for this condition to be satisfied one has to impose the following requirements on the $r$-matrix:

$$(i)\ r_{ji}(u,v) = r_{ni}(u,v), \quad (ii)\ r_{jn}(u,v) = r_{nn}(u,v),\ \forall\, i,j \in \overline{1,n-1},$$

i.e. one has to require that $r_{ji}(u,v) = r_{ni}(u,v)$, $\forall\, i,j \in \overline{1,n}$. The $r$-matrix (5.19) satisfies this condition.
Proposition is proven.

#### 5.4.2 The generalized Gaudin model with or without external magnetic field

Let us now consider the generalized Gaudin models with or without external magnetic field associated with the classical $r$-matrix (5.19). For this purpose we will specify the formulas (5.2), (5.4) and (5.3), (5.5) for the case of the classical $r$-matrix (5.19).

First we consider the generalized Gaudin model without an external magnetic field. The Lax operator (5.2) is written in this case as follows [26]:

$$L(u) = \sum_{k=1}^{N} \frac{1}{\nu_k - u} \sum_{i,j=1}^{n} \frac{1 + a_i\nu_k}{1 + a_i u} S_{ij}^{(k)} X_{ji}. \tag{5.21}$$

The corresponding generalized Gaudin Hamiltonians (5.4) have the following form [26]:

$$H_l = \sum_{k=1, k\neq l}^{N} \frac{1}{\nu_k - \nu_l} \sum_{i,j=1}^{n} \frac{1 + a_i\nu_k}{1 + a_i\nu_l} S_{ij}^{(k)} S_{ji}^{(l)} + \sum_{i,j=1}^{n} \frac{a_i}{1 + a_i\nu_l} S_{ij}^{(l)} S_{ji}^{(l)}. \tag{5.22}$$

Using the shift element (5.20) we define the following shifted Lax operator (5.3):

$$L(u) = \sum_{k=1}^{N} \frac{1}{\nu_k - u} \sum_{i,j=1}^{n} \frac{1 + a_i\nu_k}{1 + a_i u} S_{ij}^{(k)} X_{ji} + \sum_{k=1}^{n} \frac{c_k}{1 + a_k u} \Big(\sum_{i=1}^{n} \frac{a_i - a_k}{1 + a_i u} X_{ii}\Big) \tag{5.23}$$

and the following generalized Gaudin Hamiltonians in an external magnetic field [28]:

$$H_l = \sum_{k=1, k\neq l}^{N} \frac{1}{\nu_k - \nu_l} \sum_{i,j=1}^{n} \frac{1 + a_i\nu_k}{1 + a_i\nu_l} S_{ij}^{(k)} S_{ji}^{(l)} + \sum_{i,j=1}^{n} \frac{a_i}{1 + a_i\nu_l} S_{ij}^{(l)} S_{ji}^{(l)} + \sum_{k=1}^{n} \frac{c_k}{1 + a_k\nu_l} \left(\sum_{i=1}^{n} \frac{a_i - a_k}{1 + a_i\nu_l} S_{ii}^{(l)}\right). \tag{5.24}$$

**Remark 5.2** *Observe that, when $a_i \neq a_j$, $i,j \in \overline{1,n}$ the introduction of the shift element (contrary to the case of rational $r$-matrices) does not increase the number of integrals obtained using the unshifted Lax matrix.*

### 5.4.3 The completeness of the separated coordinates

Let us now consider the problem of separation of variables for the described in the previous subsection subclass of the generalized Gaudin models. In this subsection we will assume that $a_i \neq a_j$, $i, j \in \overline{1,n}$. It turns out that, in this case introduction of the external magnetic field does not affect completeness of the separated coordinates. That is why, for the purpose of simplicity, we will hereafter consider only the generalized Gaudin models without a magnetic field. Indeed, the set of separated coordinates of the generalized Gaudin models with a magnetic field is complete if and only if the corresponding set of separated coordinates of the generalized Gaudin models without a magnetic field is complete.

The $r$-matrix (5.19) is diagonal in the root basis and obviously satisfies the condition (4.1). That is why the separating functions $A(u)$ and $B(u)$ corresponding to the standard normalization of the eigenvector of the Lax matrix produce the canonical coordinates. Nevertheless, this set of canonical coordinates will not be complete. Indeed, let us consider the set of zeros of the function $B(u)$. It will suffice to consider the case $N = 1$. We omit the upper index of the generalized spin variable in this case.

Let us introduce an auxiliary spectral-parameter-independent matrix

$$S = \sum_{i,j=1}^{n} S_{ij} X_{ij}.$$

We will have the following relation between the minors of the matrices $L(u)$ and $S$:

$$L\begin{pmatrix} j_1 \dots . j_m \\ i_1 \dots . i_m \end{pmatrix}(u) = \frac{1}{(u-\nu_1)^m (1+a_{j_1}u) \dots . (1+a_{j_m}u)} S\begin{pmatrix} i_1 \dots . i_m \\ j_1 \dots . j_m \end{pmatrix}$$

Using the definition of the function $B(u)$ written in the form of minors (3.11)-(3.12), we obtain that $B(u)$ becomes a polynomial of the degree $1 + 2 + \dots + (n-2) = \frac{(n-2)(n-1)}{2}$ in the spectral parameter after multiplication by the following polynomial:

$$f(u) = (u-\nu_1)^{\frac{n(n-1)}{2}} (1+a_n u)^{n-1} \Big( (1+a_1 u)(1+a_2 u) \dots (1+a_{n-1} u) \Big)^{n-2},$$

where we have used that the function $B(\{\alpha_i^j\}, u)$ contains minors $L\begin{pmatrix} \alpha^1_{m-1} \dots \ \alpha^{m-1}_{m-1} \ n \\ \alpha^1_m \ \dots \ \alpha^m_m \end{pmatrix}(u)$ and also the fact that the indices of summation $\alpha_i^j$ used in the function $B(\{\alpha_i^j\}, u)$ run from 1 to $n-1$.

That is why in the case $N = 1$ zeros of the function $B(u)$ corresponding to the standard normalization of the eigenvector of the Lax matrix give us only $\dfrac{(n-2)(n-1)}{2}$ separated coordinates $x_i$ instead of the necessary $\dfrac{n(n-1)}{2}$ coordinates, i.e $n-1$ coordinates $x_i$ are missing.

Fortunately, by virtue of the Proposition (5.4) the $r$-matrix (5.19) satisfies also the condition (1.11). That is why we can use the transformed separating functions instead of the standard ones.

The following proposition is true:

**Proposition 5.5** *Let $B_{\vec\kappa}(u) = B^{g_{\vec\kappa}}(u)$, $A_{\vec\kappa}(u) = A^{g_{\vec\kappa}}(u)$ be the transformed separating functions corresponding to the normalization vector $\vec\kappa$ (see (1.6)) with $\kappa_i \neq 0$, $i \in \overline{1,n}$. Let $a_i \neq a_j$. Then zeros of the function $B_{\vec\kappa}(u)$ and the values of the function $A_{\vec\kappa}(u)$ in these zeros, where the functions $B_{\vec\kappa}(u)$, $A_{\vec\kappa}(u)$ are calculated for the Lax operator (5.21), constitute a complete set of the separated coordinates on the generic coadjoint orbits of the direct product of $N$ copies of $GL(n)$.*

To simplify the calculations we will give the proof for the particular choice

$$\vec\kappa = (1, 1, \dots, 1).$$

*Proof.* For $u \to \infty$ the Lax matrix behaves as follows

$$L(u) = -\frac{1}{u^2} L_\infty + \mathcal{O}\left(\frac{1}{u^3}\right)$$

$$L_\infty = \sum_{k=1}^{N} \sum_{i,j=1}^{n} \frac{1 + a_i \nu_k}{a_i} S_{ij}^{(k)} X_{ji}. \tag{5.25}$$

Let us prove that, assuming that the eigenvalues $\lambda_\infty^1, \dots, \lambda_\infty^n$ of the matrix $L_\infty$ are pairwise distinct that $B_{\vec\kappa}(u)$ can be represented in the form

$$B_{\vec\kappa}(u) = \frac{\tilde B_{\vec\kappa}(u)}{\prod_{k=1}^N (u - \nu_k)^{\frac{n(n-1)}{2}} \prod_{i=1}^n (1 + a_i u)^{n-1}}$$

where $\tilde B_{\vec\kappa}(u)$ is a polynomial in $u$ of degree $\frac{n(n-1)}{2} \times N$. Indeed, from (5.25) it follows, rewriting the determinant (1.18) in the basis of eigenvectors of $L_\infty$, that

$$B_{\vec\kappa}(u) = C(-1)^{\frac{n(n-1)}{2}} u^{-n(n-1)} \prod_{i>j} \left(\lambda_\infty^i - \lambda_\infty^j\right) + \dots, \quad u \to \infty$$

where the constant $C \neq 0$ is the determinant of the transition matrix to the basis of eigenvectors of $L_\infty$ and the periods stand for the terms of higher order. Hence

$$\deg \tilde B_{\vec\kappa}(u) = \deg \left[ \prod_{k=1}^N (u - \nu_k)^{\frac{n(n-1)}{2}} \prod_{i=1}^n (1 + a_i u)^{n-1} \right] - n(n-1) = \frac{n(n-1)}{2} N.$$

So, under the above assumption[3] we have exactly $\frac{n(n-1)}{2} N$ variables $x_i$ defined as roots of the polynomial $\tilde B_{\vec\kappa}(u)$. We will now prove independence of these variables following the scheme used in the proof of Proposition 5.3. Start from a diagonal Lax matrix

$$L(u) = \mathrm{diag}(\mu_1(u), \dots, \mu_n(u)), \quad \mu_i(u) = \sum_{k=1}^N \frac{\lambda_i^k}{\nu_k - u} + \frac{a_i}{1 + a_i u} \lambda_i^0, \quad i = 1, \dots, n$$

[3] Needless to say that the assumption holds true for a generic point in the phase space.

where $\lambda_i^1, \dots, \lambda_i^N$ are arbitrary numbers and we put

$$\lambda_i^0 = \lambda_i^1 + \cdots + \lambda_i^N.$$

For every pair of indices $i < j$ we have $N$ canonical coordinates defined as the roots $u = x_{ij}^1, \dots, u = x_{ij}^N$ of the algebraic equation

$$\mu_i(u) = \mu_j(u).$$

After an infinitesimal variation

$$L(u) \to L(u) + \delta L(u)$$

with $\delta L_{ii}(u) = 0$ for any $i$ we obtain the infinitesimal variation of the roots

$$\delta x_{ij}^p = \frac{1}{\mu_i'\left(x_{ij}^p\right) - \mu_j'\left(x_{ij}^p\right)} \sum_{q=1}^N \left[ \frac{\delta S_{ij}^{(q)} - \delta S_{ji}^{(q)}}{\nu_q - x_{ij}^p} + \frac{a_i}{1 + a_i x_{ij}^p} \delta S_{ij}^{(q)} - \frac{a_j}{1 + a_j x_{ij}^p} \delta S_{ji}^{(q)} \right]$$

(cf. eq. (5.16) above). Using a modification of the Cauchy determinant

$$\det\left( \frac{1}{a_i - b_j} + \frac{1}{b_j - c} \right) = \frac{\prod_{i<j}(a_i - a_j)\prod_{k>l}(b_k - b_l)}{\prod_{i,j}(a_i - b_j)} \prod_i \frac{c - a_i}{c - b_i}$$

obtain the determinant of a minor of the Jacobi matrix $\frac{\partial x_{ij}^p}{\partial S_{kl}^{(q)}}$ selecting the columns with $k < l$

$$\det\left( \frac{\partial x_{ij}^p}{\partial S_{kl}^{(q)}} \right) = \prod_{i<j} \left[ \prod_{p=1}^N \left( \mu_i'\left(x_{ij}^p\right) - \mu_j'\left(x_{ij}^p\right) \right)^{-1} \frac{\prod_{s<t}(\nu_s - \nu_t)\left(x_{ij}^t - x_{ij}^s\right)}{\prod_{s,t}\left(\nu_t - x_{ij}^s\right)} \prod_r \frac{1 + a_i \nu_r}{1 + a_i x_{ij}^r} \right] \neq 0. \quad (5.26)$$

Hence the Jacobi matrix has the maximal rank $N\frac{n(n-1)}{2}$. □

# 6 Conclusion and discussion

In the present paper we have considered the problem of separation of variables for the algebraically integrable Hamiltonian systems possessing $gl(n)$-valued Lax matrices that satisfy linear Poisson brackets with arbitrary $gl(n) \otimes gl(n)$-valued classical $r$-matrices. We have found, in terms of the corresponding $r$-matrices, a sufficient condition that guarantees that the separating polynomials of [20], [18], [14], [7] produce the canonical coordinates. We consider two examples of the non-dynamical classical $r$-matrices, the corresponding Lax operators of Gaudin-type systems along with the separating polynomials and separated coordinates for them. One of these examples is new.

Let us emphasize that our result works also in the case of dynamical classical $r$-matrices. In particular the $r$-matrix of the $gl(n)$ Calogero models does satisfy our separation condition. We have not considered the Calogero models in the present paper redirecting the interested reader to the paper [22]. In this context it is worthwhile to mention existence of the separated coordinates also for the Ruijsenaars systems [22]. However they are related with the *quadratic* $r$-matrix Poisson brackets. This and also other examples of quadratic Poisson algebras will be considered in the subsequent publication [9]. Another challenging problem is to extend the results of the present article to the quantum case. This problem is still open.

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# A Appendix

In this Appendix we obtain a necessary and sufficient condition for validity of the canonical Poisson brackets between the separated variables of algebro-geometric type.

Let $(x_i, p_i)$, $i \in \overline{1,d}$ be the points of the spectral curve specified by the following condition: there exists a (left) eigenvector $f_i \neq 0$

$$f_i\, L(x_i) = p_i f_i$$

of the Lax matrix $L(u)$ with $u = x_i$ satisfying

$$< f_i, \kappa >= 0. \tag{A.1}$$

It will be assumed that the space of such eigenvectors is one-dimensional for every $i$. By $f_i'$ denote the right eigenvector of the same matrix with the same eigenvalue

$$L(x_i) f_i' = p_i f_i'.$$

The $n \times n$ Lax matrix will be considered as a linear operator on the space $V = \mathbb{C}^n$. The right eigenvectors belong to $V$ while the left ones are in the dual space $V^*$. Also $\kappa$ is a fixed nonzero vector in $V$. The angle brackets denote the natural pairing $V^* \otimes V \to \mathbb{C}$,

$$< a, b >= a_s b^s, \quad a = (a_1, \dots, a_n) \in V^*, \quad b = (b^1, \dots, b^n)^T \in V.$$

Let $r(u,v) = \left(r^{ij}_{kl}(u,v)\right)$ be an $r$-matrix depending analytically on the complex variables $u$, $v$ away from the diagonal $u = v$. It defines a linear Poisson bracket on the space of Lax matrices. Our goal is to derive a necessary and sufficient conditions for validity of the commutation relations

$$\{x_i, x_j\} = \{p_i, p_j\} = \{x_i, p_j\} = 0, \quad i,\, j = 1, \dots, d, \quad i \neq j. \tag{A.2}$$

We also add a *regularity assumption* specifying the behavior of $r$-matrix on the diagonal

$$r(u,v) = \frac{\Omega}{u - v} + \Delta r(u,v), \quad u \to v \tag{A.3}$$

where

$$\Omega = \sum_{i,j} X^i_j \otimes X^j_i$$

and $\Delta r(u,v)$ is analytic on the diagonal. This condition turns out to be sufficient for validity of the commutation relations

$$\{x_i, p_i\} = 1, \quad i = 1, \dots, d. \tag{A.4}$$

For every $i \in \overline{1,d}$ let us define a vector $\sigma_i \in V$ using the following

**Lemma A.1** *For every $i \in \overline{1,d}$ there exists a vector $\sigma_i \in V$ satisfying*

$$(L(x_i) - p_i \cdot 1)\,\sigma_i = \kappa. \tag{A.5}$$

*Proof:* The condition of compatibility of such a system coincides with (A.1). □

Denote

$$r_\kappa(u,v) : V^* \otimes V^* \to V^* \tag{A.6}$$

the linear map defined by the (2,1)-tensor

$$(r_\kappa(u,v))^{ij}_k = \kappa^s r^{ij}_{sk}(u,v). \tag{A.7}$$

**Theorem A.1** *1) The variables $x_i$, $p_j$ satisfy the commutation relations* (A.2) iff *the $r$-matrix satifies the following system of equations*

$$\langle (f_i \otimes f_j) r_\kappa(x_i,x_j), f'_j \rangle = 0, \quad i \neq j = 1,\dots,d \tag{A.8}$$

$$\langle (f_i \otimes f_j) r_\kappa(x_i,x_j), \sigma_j \rangle = \langle (f_j \otimes f_i) r_\kappa(x_j,x_i), \sigma_i \rangle\,, \quad i,\, j = 1,\dots,d. \tag{A.9}$$

*2) If moreover the $r$-matrix satisfies the regularity assumption* (A.3) *and the equations*

$$\langle (f_i \otimes f_i) \Delta r_\kappa(x_i,x_i), f'_i \rangle = 0 \tag{A.10}$$

*hold true for any $i$ then the commutation relations* (A.4) *hold true too.*

**Remark A.1** *It is easy to see that the equations* (A.8), (A.9) *follow from the condition* (1.11) *of the paper. Indeed, as the vectors $f_i$, $f_j$ both satisfy the normalization* (A.11), *the condition* (1.11) *implies*

$$(f_i \otimes f_j) r_\kappa(u,v) = 0$$

*for any $u$, $v$. Taking the limit $u \to v$ we also arrive at* (A.10).

Let us proceed to the *proof* of the Theorem. We begin with proving eq. (A.9) deriving it from vanishing of the brackets $\{x_i, x_j\}$.

Let us label the sheets of the spectral curve over $u \in \mathbb{C}$ in a sufficiently small neighborhood of $u = x_i$ by numbers 1, 2, …, $n$ in such a way that the point $(x_i, p_i)$ belongs to the $n$-th sheet. In a similar way we label the sheets over a small neighborhood of $v = x_j$. Denote $(f^a_k(u))$ the $n \times n$ matrix of (left) eigenvectors of $L(u)$

$$f^a_k(u) L^k_l(u) = \mu_a(u) f^a_l(u), \quad a,\, l = 1,\dots,n.$$

Here $\mu_a(u)$ is the value of the algebraic function $\mu(u)$ on the $a$-th sheet. The eigenvectors

$$f^a(u) = (f^a_1(u), \dots, f^a_n(u))$$

coincide with the rows of the matrix. They will be normalized by the following conditions

$$\langle f^a(u), \kappa \rangle = 1, \quad a = 1,\dots,n-1, \quad \langle f^n(u), \kappa \rangle = u - x_i. \tag{A.11}$$

With such a normalization the matrix $(f^a_k(u))$ depends analytically on $u$. Denote $\left(g^k_a(u)\right) = (f^a_k(u))^{-1}$ the inverse matrix. It also depends analytically on $u$. Its columns $g_a(u) = \left(g^1_a(u), \dots, g^n_a(u)\right)^T$ are right eigenvectors of $L(u)$. Choose

$$f_i = f^n(u = x_i), \quad f'_i = g_n(u = x_i). \tag{A.12}$$

In a similar way we define the matrices $(f^a_k(v))$ and $\left(g^k_a(v)\right)$ analytic near $v = x_j$ and choose the eigenvectors $f_j$ and $f'_j$ like in (A.12).

In the subsequent calculations we will use indices $a$, $b$, ... for labelling the eigenvectors and the corresponding eigenvalues of the matrix $L(u)$ while the indices $i$, $j$, ... will be used to label their coordinates in the original basis in $\mathbb{C}^n$. It will be always assumed summation over repeated indices $i$, $j$, ... but not in $a$, $b$, ....

Recall that $x_1, \dots, x_d$ are zeros of the determinant

$$B(u) = \kappa \wedge L(u)\kappa \wedge L^2(u)\kappa \wedge \dots \wedge L^{n-1}(u)\kappa.$$

We will first compute derivatives of $B(u)$ as a function of $L(u)$.

Denote

$$\kappa^a(u) = f^a_i(u)\kappa^i,$$
$$\delta L^a_b(u) = f^a_i(u) g^j_b(u) \delta L^i_j(u)$$

the components of the vector $\kappa$ and the entries of the matrix $\delta L(u)$ in the basis of eigenvectors. Notice that, due to the normalization (A.11) one has

$$\kappa^1(u) = \dots = \kappa^{n-1}(u) \equiv 1, \quad \kappa^n(u) = u - x_i. \tag{A.13}$$

Similar expressions hold true for $\kappa^a(v) = f^a_i(v)\kappa^i$

$$\kappa^1(v) = \dots = \kappa^{n-1}(v) \equiv 1, \quad \kappa^n(v) = v - x_j. \tag{A.14}$$

**Lemma A.2** *The following formula holds true*

$$\delta \log B(u) = \sum_{a,b} \frac{\kappa^b(u)}{\kappa^a(u)} Y^b_a(u) \delta L^a_b(u) \tag{A.15}$$

*where*

$$Y^b_a(u) = \begin{cases} \frac{1}{\mu_a(u) - \mu_b(u)}, & a \neq b \\ \sum_{c \neq a} \frac{1}{\mu_a(u) - \mu_c(u)}, & a = b. \end{cases} \tag{A.16}$$

The formula (A.15) can be derived from eq. (30) in [7]. For convenience of the reader we will sketch the proof here.

*Proof:* The following wellknown formula for derivatives of determinants will be used.

**Lemma A.3** *Let $X_1, \ldots, X_n$ be $n$ linearly independent vectors in $\mathbb{C}^n$ depending on parameters. Then for an arbitrary variation*

$$X_i \mapsto X_i + \delta X_i, \quad \delta X_i = A_i^j X_j$$

*one has*

$$\delta \left( X_1 \wedge \cdots \wedge X_n \right) = \mathrm{tr}\, A \cdot X_1 \wedge \cdots \wedge X_n \tag{A.17}$$

*in the linear approximation wrt the variation.*

In our case

$$X_i = L^{i-1}(u)\kappa$$

so $\delta X_1 = 0$,

$$\delta X_i(u) = \sum_{p+q=i-1} L^p(u)\delta L(u) L^q(u)\kappa, \quad i = 2, \ldots, n.$$

Rewriting in the basis of eigenvectors obtain

$$\delta X_i(u) = \sum_{p+q=i-1} \sum_{a,b} \mu_a^p(u)\mu_b^q(u)\delta L_b^a(u)\kappa^b(u) f_a(u).$$

We are now to decompose this vector with respect to the basis

$$X_j(u) = L^{j-1}(u)\kappa = \sum_c \mu_c^{j-1}(u)\kappa^c(u) f_c(u).$$

To this end we first decompose the eigenvectors with respect to the basis $X_1, \ldots, X_n$. The following linear system is to be solved

$$h_c^j(u) X_j(u) = f_c(u) \quad \Leftrightarrow \quad \sum_j h_c^j(u)\mu_c^{j-1}(u)\kappa^c(u) = \delta_c^j.$$

Write $\tilde{h}_c^j(u) = h_c^j(u)\kappa^c(u)$. Thus the entries of the matrix $\tilde{h}_c^j(u)$ are coefficients of the Lagrange interpolation polynomials depending on the parameter $u$

$$P_c(z) = \sum_{j=1}^n \tilde{h}_c^j(u) z^{j-1}, \quad P_c\left(z = \mu_a(u)\right) = \delta_{ac}.$$

We arrive at the following decomposition

$$\delta X_i(u) = A_i^j(u) X_j(u)$$

where

$$A_i^j(u) = \sum_{a,b} \sum_{p+q=i-2} \frac{\kappa^b(u)}{\kappa^a(u)} \tilde{h}_a^j(u)\mu_a^p(u)\mu_b^q(u)\delta L_b^a(u).$$

To compute the trace $A^i_i(u)$ we first simplify the following sum

$$\sum_i \sum_{p+q=i-2} \tilde{h}^i_a \mu^p_a \mu^q_b$$

(here and below we omit the dependence on $u$ in order to simplify the notations). For $a \neq b$ we have

$$\sum_{p+q=i-2} \mu^p_a \mu^q_b = \frac{\mu^{i-1}_a - \mu^{i-1}_b}{\mu_a - \mu_b}.$$

So

$$\sum_i \sum_{p+q=i-2} \tilde{h}^i_a \mu^p_a \mu^q_b = \frac{1}{\mu_a - \mu_b}.$$

For $a = b$

$$\sum_{p+q=i-2} \mu^p_a \mu^q_a = (i-1)\mu^{i-2}_a.$$

Hence

$$\sum_i \sum_{p+q=i-2} \tilde{h}^i_a \mu^p_a \mu^q_a = \sum_{i=1}^n (i-1)\tilde{h}^i_a \mu^{i-2}_a = \left(\frac{d}{dz} P_a(z)\right)_{z=\mu_a} = \sum_c \frac{1}{\mu_a - \mu_c}.$$

Here we have used the explicit expression for the Lagrange polynomials

$$P_a(z) = \frac{\prod_{c\neq a}(z - \mu_a)}{\prod_{c\neq a}(\mu_a - \mu_c)}.$$

This completes the proof of Lemma. □

**Lemma A.4** *The Poisson brackets between the logarithms of the separating functions $B(u)$ are given by the following formula*

$$\{\log B(u), \log B(v)\} = \sum_{a\neq b} \sum_{c,d} \frac{\kappa^b(u)}{\kappa^a(u)} \frac{\kappa^d(v)}{\kappa^c(v)} Y^d_c(v) r^{ac}_{bd}(u,v) - \sum_{a,b} \sum_{c\neq d} \frac{\kappa^b(u)}{\kappa^a(u)} \frac{\kappa^d(v)}{\kappa^c(v)} Y^b_a(u) r^{ca}_{db}(v,u) \quad \text{(A.18)}$$

*where $r^{ac}_{bd}(u,v)$ are the entries of the $r$-matrix in the basis of eigenvectors*

$$r^{ac}_{bd}(u,v) = f^a_i(u) f^c_j(v) g^k_b(u) g^l_d(v) r^{ij}_{kl}(u,v).$$

*The matrix $r^{ca}_{db}(v,u)$ is given by a similar expression interchanging $u$ and $v$.*

*Proof:* Using eq. (A.15) obtain

$$\{\log B(u), \log B(v)\} = \sum_{a,b} \frac{\kappa^b(u)}{\kappa^a(u)} Y^b_a(u) f^a_j(u) g^i_b(u) \left\{L^j_i(u), L^l_k(v\right\} \sum_{c,d} \frac{\kappa^d(v)}{\kappa^c(v)} Y^d_c(v) f^c_l(v) g^k_d(v).$$

Rewriting the formula for the Poisson bracket

$$\{L_i^j(u), L_k^l(v\} = r_{sk}^{jl}(u,v)L_i^s(u) - r_{ik}^{sl}(u,v)L_s^j(u) - r_{si}^{lj}(v,u)L_k^s(v) + r_{ki}^{sj}(v,u)L_s^l(v)$$

in the basis of eigenvectors we obtain, after simple calculations

$$\{\log B(u), \log B(v)\} = \sum_{a,b,c,d} \frac{\kappa^b(u)}{\kappa^a(u)} \frac{\kappa^d(v)}{\kappa^c(v)} Y_a^b(u) Y_c^d(v) \left[ (\mu_b(u) - \mu_a(u))\, r_{bd}^{ac}(u,v) - (\mu_d(v) - \mu_c(v))\, r_{db}^{ca}(v,u) \right].$$

For the first term in the square brackets it suffices to sum over $a \neq b$. Then the factor $(\mu_b(u) - \mu_a(u))$ cancels with $Y_a^b(u)$. In a similar way we deal with the second term. This gives (A.18). □

**Corollary A.1** *The following formula holds true for the Poisson brackets between the variables* $x_i$

$$\{x_i, x_j\} = \sum_{a,b=1}^{n-1} \left( \frac{r_{ab}^{nn}(x_i, x_j)}{p_j - \mu_b(x_j)} - \frac{r_{ab}^{nn}(x_j, x_i)}{p_i - \mu_b(x_i)} \right). \tag{A.19}$$

*Proof:* For $u \to x_i$, $v \to x_j$ the lhs of (A.18) behaves as

$$\{\log B(u), \log B(v)\} = \frac{\{x_i, x_j\}}{(u - x_i)(v - x_j)} + \text{regular terms}.$$

In the rhs the terms with the same poles are obtained only for $a = c = n$ and $d \neq n$, due to the normalization (A.11). Due to the same normalization $\kappa^b(u) = \kappa^d(v) \equiv 1$ for $b$, $d$ different from $n$. This proves the Corollary. □

Let us now consider the brackets $\{x_i, p_j\}$ for $i \neq j$. Denote

$$P(u, \mu) = \det\left(L(u) - \mu \cdot 1\right)$$

the characteristic polynomial of the Lax matrix.

**Lemma A.5** *For any* $i$ *and any* $a = 1, \ldots, n$ *and an arbitrary* $v$ *the following formula holds true*

$$\{x_i, \mu_a(v)\} = \sum_{b=1}^{n-1} r_{ba}^{na}(x_i, v). \tag{A.20}$$

*Proof:* Let us compute the Poisson bracket

$$\{\log B(u), \log P(v, \mu)\} = \{\log B(u), L_k^l(v)\} M_l^k(v, \mu)$$

where

$$M(v, \mu) = \left(L(v) - \mu \cdot 1\right)^{-1}.$$

Using eq. (A.15) after a straightforward calculation we arrive at

$$\{\log B(u), \log P(v,\mu)\} = -\sum_{a\neq b} \frac{\kappa^b(u)}{\kappa^a(u)} M_l^k(v,\mu) f_j^a(u) g_b^i(u) r_{ik}^{jl}(u,v).$$

Representing the inverse matrix in the basis of eigenvectors

$$M_l^k(v,\mu) = \sum_c g_c^k(v) \frac{1}{\mu_c(v)-\mu} f_l^c(v)$$

we finally obtain the following expression

$$\{\log B(u), \log P(v,\mu)\} = \sum_{a\neq b} \frac{\kappa^b(u)}{\kappa^a(u)} \frac{r_{bc}^{ac}(u,v)}{\mu-\mu_c(v)}.$$

Considering the limit $u \to x_i$ and using the normalization (A.11) like in the proof of the previous lemma we arrive at the following bracket

$$\{x_i, \log P(v,\mu)\} = \sum_{b=1}^{n-1} \sum_c \frac{r_{bc}^{nc}(x_i,v)}{\mu_c(v)-\mu}.$$

Finally, applying similar arguments to the above equation at the limit $\mu \to \mu_c(v)$ we complete the proof of Lemma. □

**Corollary A.2** *If the condition $\{x_i, x_j\} = 0$ is fulfilled then*

$$\{x_i, p_j\} = \sum_{b=1}^{n-1} r_{bn}^{nn}(x_i, x_j), \quad i \neq j. \tag{A.21}$$

*Proof:* Specializing eq. (A.20) for $a = n$ and $v = x_j$ we obtain

$$\{x_i, p_j\} = \{x_i, \mu_n(v)\}_{v=x_j} + \left(\frac{d\mu_n(v)}{dv}\right)_{v=x_j} \{x_i, x_j\} = \sum_{b=1}^{n-1} r_{bn}^{nn}(x_i, x_j).$$

□

We now proceed to considering the brackets of momenta.

**Lemma A.6** *Vanishing of the Poisson brackets $\{x_i, x_j\}$ and $\{x_i, p_j\}$, $\{x_j, p_i\}$ implies that also $\{p_i, p_j\} =$* 0.

*Proof:* Denote

$$P(u,\mu) = \det\left(L(u) - \mu \cdot 1\right)$$

the characteristic polynomial of the Lax matrix. It is well known that the $r$-matrix nature of the Poisson bracket implies commutativity of the characteristic polynomials

$$\{P(u,\mu), P(v,\nu)\} = 0$$

for arbitrary $u$, $v$, $\mu$, $\nu$ (see, e.g. the book [10]). Hence the eigenvalues of the Lax matrix also commute

$$\{\mu_a(u), \mu_b(v)\} = 0 \quad \forall\, u,\, v, \quad \forall\, a,\, b = 1, \dots, n.$$

Choose $a = b = n$ and set $u = x_i$, $v = x_j$ to obtain

$$\{p_i, p_j\} = \{\mu_n(u), \mu_n(v)\}_{u=x_i,\, v=x_j} + \frac{d\mu_n(x_i)}{du}\{x_i, p_j\} + \{p_i, x_j\}\frac{d\mu_n(x_j)}{dv} + \frac{d\mu_n(x_i)}{du}\frac{d\mu_n(x_j)}{dv}\{x_i, x_j\} = 0.$$

□

As it follows from the statements proven above vanishing of the brackets $\{x_i, x_j\} = \{x_i, p_j\} = \{p_i, p_j\} = 0$ for $i \neq j$ is equivalent to the following system of equations written in terms of certain components of the $r$-matrix in the basis of eigenvectors of the Lax matrix

$$\sum_{b=1}^{n-1} r_{bn}^{nn}(x_i, x_j) = 0 \tag{A.22}$$

and

$$\sum_{a,\,b=1}^{n-1} \frac{r_{ab}^{nn}(x_i, x_j)}{p_j - \mu_b(x_j)} = \sum_{a,\,b=1}^{n-1} \frac{r_{ab}^{nn}(x_j, x_i)}{p_i - \mu_b(x_i)} \tag{A.23}$$

evaluated at $u = x_i$, $v = x_j$ for any $i \neq j$. Our goal now is to "translate" these conditions in terms of the entries of the $r$-matrix in the original coordinates. The following elementary statement will be useful.

**Lemma A.7** *For any $i = 1, \dots, d$ the following identity holds true for the right eigenvectors $g_1(x_i)$, $\dots$, $g_n(x_i)$ of the Lax matrix $L(x_i)$*

$$\sum_{k=1}^{n-1} g_k(x_i) = \kappa. \tag{A.24}$$

*Proof:* Denote $F = (f_k^a(x_i))$ the matrix of (left) eigenvectors evaluated at the point $u = x_i$ and $G = \left(g_a^k(x_i)\right)$ its inverse matrix. Due to the normalization (A.1) one has

$$F\kappa = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}.$$

Multiplying both sides by $G$ on the left we obtain (A.24). □

Let us verify that eq. (A.22) coincides with (A.8). The lhs of (A.22) is nothing but the following multiple sum

$$\sum_{b=1}^{n-1} r_{bn}^{nn}(x_i,x_j) = \sum_{b=1}^{n-1} f_k^n(x_i)g_b^s(x_i)f_l^n(x_j)g_n^t(x_j)r_{st}^{kl}(x_i,x_j).$$

We have

$$(f_1^n(x_i),\ldots,f_n^n(x_i)) = f_i, \quad (f_1^n(x_j),\ldots,f_n^n(x_j)) = f_j, \quad \left(g_n^1(x_j),\ldots,g_n^n(x_j)\right)^T = f_j'$$

and

$$\sum_{b=1}^{n-1} \left(g_n^1(x_i),\ldots,g_n^n(x_i)\right)^T = \kappa$$

due to the above Lemma. So we arrive at eq. (A.9).

Let us now look at eq. (A.23). Proceeding like in the previous calculation we see that the following vector appears in the lhs

$$\sum_{b=1}^{n-1} \frac{g_b(x_j)}{p_j - \mu_b(x_j)}.$$

On the rhs a similar vector appears with $i \leftrightarrow j$. Clearly this vector satisfies

$$(L(x_j) - p_j \cdot 1)\sum_{b=1}^{n-1} \frac{g_b(x_j)}{p_j - \mu_b(x_j)} = -\sum_{b=1}^{n-1} g_b(x_j) = -\kappa.$$

This implies eq. (A.9) with $\sigma_j = -\sum_{b=1}^{n-1} \frac{g_b(x_j)}{p_j-\mu_b(x_j)}$. Notice that validity of this equation does not depend on the choice of a solution to (A.5). Indeed, the freedom in choosing $\sigma_j$ is in adding a vector proportional to $g_n(x_j) = f_j'$. Such an addition does not contribute to (A.9) due to eq. (A.8).

It remains to verify that $\{x_i, p_i\} = 1$ under the assumptions (A.3), (A.10). Indeed, eq. (A.10) implies that the regular part $\Delta r(u,v)$ of the $r$-matrix does not contribute to this bracket. And for the standard rational $r$-matrix

$$r(u,v) = \frac{\Omega}{u-v}$$

it was proven in various ways [7, 1, 14, 18] that $\{x_i, p_i\} = 1$. The Theorem is proved. □

**Remark A.2** *The necessary and sufficient conditions* (A.8), (A.9) *can be certainly simplified when working with concrete examples of Lax algebras. In the analysis of these examples one has first to prove that the $u$-projections of the poles of the eigenvectors of the Lax matrix normalized by the condition* (A.1) *can be arbitrary complex numbers. If this is the case then the next step would be to check that, choosing appropriately an element $L(u)$ with poles over prescribed values $u = x_i$, $v = x_j$ one can obtain arbitrary (left) eigenvectors $f_i$ and $f_j$ satisfying the normalization* (A.1) *and, moreover, the right eigenvector $f_j'$ also can be arbitrary. If it happens like this then the equation* (A.8) *immediately implies the condition* (1.11) *derived in the main part of this paper from Poisson-closedness of the algebra of separating functions. The equation* (A.9) *in this case holds true automatically. We plan to consider more examples of Lax algebras in subsequent publications.*

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