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Scattering of an "infraparticle": the one-particle sector of Nelson's massless model

CANDIDATE

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To the memory of my aunt Caterina

Contents

Introduction

The main topic of this thesis is the scattering theory in the one-particle sector of Nelson's massless model. The physical motivations come from the infrared problem in Q.E.D., in particular, from the difficulties arising even in explaining the interaction between the radiation field and a single charged particle.

The infrared problem in Q.E.D. (Feynman-Dyson-Schwinger formulation) arises with the divergence of some diagrams in the S-matrix, as the internal photons' momenta go to zero. Because of such divergences the transition amplitudes between states consisting of electrons (and/or positrons) and a finite number of photons are not defined order by order. The removal of the infrared divergences is achieved by treating the problem in terms of "inclusive" cross sections: it consists in the sum over all possible final states accounting for the unobserved soft photons which escape out of the experimental measures. In fact, the divergences of order n in α (the fine structure constant) due to the internal photons are balanced by the emission of n photons of total energy below an observability threshold $\Delta\epsilon$, as suggested by the so called classical models, starting from the paper by Block and Nordsieck [1]. The classical procedure for removing the infrared divergences provide transition probabilities $P_n^{\Delta\epsilon}$ at the order n, which are finite and $\Delta\epsilon-$dependent in such a way that they are positive only if $\Delta\epsilon$ is not too small. The sum over all the orders provides a total probability which is estimated $O(\Delta\epsilon)$ for $\Delta\epsilon \to 0$.

In the task of defining the correct scattering amplitudes of the theory, the rigorous study of models of quantum mechanical matter interacting with the radiation field clarifies the structure of the soft photon cloud "attached" to each charged particle in the asymptotic states. It is a significant step since the non-Fock representation of the asymptotic electromagnetic field in the Hilbert space of the system is displayed in a rigorous hamiltonian framework. Unfortunately, in the infrared unregularized case, all these models are brought under control at the price of imposing a given (asymptotically free) dynamics for the electron. The removal of the unphysical hypothesis of no radiative reaction is indeed an open problem. It is clearly a prerequisite to understand the asymptotic decoupling in the presence of infinitely many soft photons.

The interest in Nelson's massless model [2] is due to the fact that it is a simplified version of non-relativistic Q.E.D. with a non trivial S-matrix, which retains many features of the infrared problem in electrodynamics as it will be clearer in the next chapters (see also the preliminary discussion in [3]).

The non-relativistic approximation implies that pair production is neglected. However it is reasonably not-significant at low energies and therefore does not interfere with the infrared behavior. The interaction is further simplified in the sense that it is scalar rather than vectorial. Indeed, spinless non-relativistic quantum particles and neutral scalar massless bosons are supposed to interact. Nevertheless, such a difference is suspected to be qualitatively not dramatic for the aspects we are interested in:

- the absence of one-particle states for the electron (which defines an "infraparticle", see [4]), that are states on which the hamiltonian acts as a function of the total momentum;

-the related effects on the asymptotic dynamics.

The thesis is organized in three parts.

Part I is introductory and contains a brief critical review of the infrared problem in Q.E.D., mainly focused on the aspects which enter in the following constructions. Following the developments in this subject, it will help to correctly define the problems faced in Nelson's model afterwards and it should provide a background for the ideas exploited in the model analysis.

Part II is devoted to spectral issues connected with the absence of one-particle states in Nelson's massless model. The results are not completely new, except for the method performed in the construction of the ground state of the hamiltonians at fixed total momentum.
In Part III, starting from the spectral results in Part II and from an assumption on the regularity of the ground energy (as a function of the total momentum), the scattering states are constructed. The convergence of the asymptotic dynamical variables, both for the field and the non relativistic particle, is eventually obtained on the scattering subspaces.
In the appendixes A and B many lengthy proofs are collected, which are related to Part II and Part III respectively.
The mathematical tools employed in this thesis only require some knowledge of the theory of operators in Hilbert space and of techniques in Fock space [30].

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Part I

The infrared problem in Q.E.D..

The following review about the infrared problem in Q.E.D. is not surely exhaustive. We only present a selection of aspects and related investigations that is influenced and restricted by the particular problems that we want to discuss in Nelson's massless model. The relation between infrared aspects and indefinite metric or the role of Gauss law are left out of discussion, and more emphasis is devoted to the rigorous analysis of (non-relativistic) models.

0.1 Infrared divergences in Feynman diagrams.

Let us consider, for instance, a generic scattering process involving an incoming electron with momentum p and an outgoing electron of momentum p', whose basic diagram is not infrared divergent and has the corresponding matrix element

$$M_0 = \overline{u}(p')\, Q(p,p')\, u(p)\ .$$

The notations are as in [5], chapter 16, $\overline{u}, u$ normalized spinors with $\overline{u_r} u_s = \delta_{rs}\ ,\ r,s = 1,2$.
The radiative corrections to the first order in α are diagrams obtained by inserting an internal photon line in all possible ways. Some of them are infrared divergent, so that the transition probability to the first order in α corresponds to:

$$|M_0|^2 \left(1 + \alpha c(p,p') \cdot \int \frac{d|\mathbf{k}|}{|\mathbf{k}|} + \alpha d(p,p')\right)$$

where $c(p,p')$and $d(p,p')$ are not divergent. Then the transition probability to the first order in α is not defined because of the factor $\int \frac{d|\mathbf{k}|}{|\mathbf{k}|}$. The mechanism, by which the infrared divergences appear, is reproduced at higher orders for proper configurations of the internal photon lines that we add to the basic diagram.

Now, let us quickly have a look at some of the main contributions to solve the infrared problem, which appeared in literature starting from the paper of Block and Nordsieck.

Block and Nordsieck's transformation.

In a famous paper [1], Block and Nordsieck discussed a simplified model of Q.E.D which however captures the nature of the infrared divergences. They consider an electron interacting with the radiation field (described by $\mathbf{E}^{tr}$, $\mathbf{H}$) according to the hamiltonian

$$H = c\left\{\overrightarrow{\alpha}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\left(\mathbf{r}\right)\right)+\beta mc\right\}+\frac{1}{8\pi}\int\left(\mathbf{E}^{tr^2}+\mathbf{H}^2\right)dV$$

where $\overrightarrow{\alpha}$ and β are the usual Dirac matrices, $\mathbf{r},\mathbf{p}$ the electron position and momentum operator respectively; $\mathbf{A}$ is the vector potential (in Coulomb gauge) in a cubic box of volume Ω, with periodic boundary conditions:

$$\mathbf{A}\left(\mathbf{r}\right) = 2c\left(\frac{\pi\hbar}{\Omega}\right)\sum_{s,\lambda}\omega_s^{-\frac{1}{2}}\widehat{\varepsilon}_{s,\lambda}\left(P_{s,\lambda}\cos\left(\mathbf{k}\cdot\mathbf{r}\right)+Q_{s,\lambda}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right)$$

where $\widehat{\varepsilon}_{s,\lambda}$ are polarization vectors, $s\in Z^3$ space of the box momenta, $\omega_s = |\mathbf{k}_s|$, and the following canonical commutation relations hold

$$[P_{s,\lambda},Q_{s',\lambda'}] = -i\delta_{s,s'}\delta_{\lambda,\lambda'}$$
$$[P,P] = [Q,Q] = 0 \quad .$$

Their purpose is to study the scattering in the presence of an external potential $V\left(\mathbf{x}\right)$, by following this procedure:
- the solutions of $H\psi = E\psi$ are found with some approximation on H (see [1]), by considering an electron freely moving with velocity $\mathbf{v}$;
- the transition probabilities between states at different asymptotic velocities are computed in the Born approximation for $V\left(\mathbf{x}\right)$.
The interesting results of their analysis are:
- the approximate solutions of the equation $H\psi = E\psi$ can be easily discussed after having performed a canonical transformation as follows

$P'_{s,\lambda} = P_{s,\lambda}-\sigma_{s,\lambda}\left(\mathbf{v}\right)\cos\left(\mathbf{k}\cdot\mathbf{r}\right)$
$Q'_{s,\lambda} = Q_{s,\lambda}-\sigma_{s,\lambda}\left(\mathbf{v}\right)\sin\left(\mathbf{k}\cdot\mathbf{r}\right)$
$\mathbf{r}' = \mathbf{r}$
$\mathbf{p}'\left(\mathbf{v}\right) = \mathbf{p}+\sigma_{s,\lambda}\left(\mathbf{v}\right)\left(P_{s,\lambda}\cos\left(\mathbf{k}\cdot\mathbf{r}\right)+Q_{s,\lambda}\sin\left(\mathbf{k}\cdot\mathbf{r}\right)\right)$

where $\sigma_{s,\lambda}\left(\mathbf{v}\right) = \frac{\overrightarrow{\mu}\cdot\overrightarrow{f_{s,\lambda}}}{\hbar\left(|\mathbf{k}_s|-\overrightarrow{\mu}\cdot\mathbf{k}_s\right)}$, $\overrightarrow{f_{s,\lambda}} = 2e\left(\frac{\pi\hbar}{\Omega\omega_s}\right)\overrightarrow{\varepsilon_{s,\lambda}}$ and $\overrightarrow{\mu} = \frac{\mathbf{v}}{c}$; the reason is that the approximate solutions are factorized. As far the photon part is concerned, a base for the solutions is given by the eigenvectors of the number operator referred to the transformed photon variables

$$N = \sum_{s,\lambda}\left(\frac{P'_{s,\lambda}+iQ'_{s,\lambda}}{\sqrt{2}}\right)\left(\frac{P'_{s,\lambda}-iQ'_{s,\lambda}}{\sqrt{2}}\right);$$

- the transition probabilities between states at different $\mathbf{v}_{in}$ and $\mathbf{v}_{out}$ and with a "finite number of photons" (with respect to N) are vanishing.

Classical current model.

In this model, the quantized electromagnetic field $A_\mu(x)$ interacts with a current $J_\mu(x) \in R$ which is disconnected from $A_\mu(x)$ and commutes with $A_\mu(x)$ thereby. In the formal treatment of pertubative calculus, once the interaction lagrangian $l_{int} = J_\mu(x) A^\mu(x)$ is given, the scattering matrix $S = T - \exp\left(i \int d^4x l_{int}(x)\right)$ is analytically computable to each order since $[J_\mu(x), A_\mu(x)] = 0$, which implies that the $A_\mu(x)$ couplings only survive in the Wick expansion. The considered currents correspond to asymptotic momenta p_{in} and p'_{out}, for $t \to -\infty$ and for $t \to +\infty$ respectively, i.e.

$$J_\mu(k) = ie\left(\frac{p'^\mu_{out}}{p'_{out} \cdot k} - \frac{p^u_{in}}{p_{in} \cdot k}\right) \qquad for \quad k^\mu \to 0\,.$$

The first result of the computations is that the transition amplitudes between states with a finite asymptotic photons' number vanish. To extract predictive results with respect to the experimental data, it is necessary to consider the "inclusive cross sections" with energy resolution $\Delta\epsilon$: that are the transition probabilities summed over all final states which contain an arbitrary number of photons of total energy below $\Delta\epsilon$. The inclusive cross sections are finite and the unobserved emission appears as a multiplicative constant $b(\Delta\epsilon)$, vanishing for $\Delta\epsilon \to 0$ (see [5]).

Classical regularization.

The results in the previous models suggest that only taking into account the soft emission it is possible to obtain finite transition probabilities. The classical regularization procedure of Q.E.D. diagrams exploit indeed the fact that the infrared divergences due to the radiative corrections are neutralized by the same divergences arising from the soft radiation. In particular, the order n divergences in the transition probabilities are exactly compensated by the emission of n unobserved soft photons.

The structure of the infrared regularization in the Q.E.D. diagrams was initially studied by Jauch and Rohrlich and then it was generalized in many works. Among them, the paper by Yennie, Frautschi and Suura [6] is particularly interesting. In this paper the infrared divergences are factorized in the following way: if M_0 is a not-infrared divergent diagram and if we add n virtual photons to M_0, we obtain the diagrams M_n,

$$M_n = \sum_{r=0}^{n} m_{n-r} \frac{(\alpha \cdot B)^r}{r!}\,,$$

where the m_j are not divergent and are of order α^j, B diverges logarithmically in the photon mass which is the regularization parameter. It formally follows that the complete matrix element is

$$M = \exp(\alpha \cdot B) \sum_{n=0}^{\infty} m_n\,.$$

The standard treatment is affected by computational and conceptual problems:

- if we fix a perturbative order n, the experimental cross section is a polynomial in $\alpha \ln(\Delta\epsilon)$ of order n; therefore it is not positive for small resolution energies any more and it is necessary to consider further orders to obtain meaningful cross sections. Moreover, according to the conjecture by

Schwinger, the total transition probability (sum over all the orders) is estimated $O(\Delta\epsilon)$. Therefore $\Delta\epsilon$ is not a removable cutoff;

- the computation of the inclusive cross sections implies a sum over final states with respect to quantum numbers that we cannot observe. The perturbative calculus is assumed inside the Fock space for the asymptotic particles, so that it implicitly requires that only a finite number of photons escapes from the observation because of the threshold $\Delta\epsilon$. On the other hand, the perturbative results contradict the underlying hypothesis by excluding a description in terms of Fock asymptotic photon states.

A better understanding of the electrodynamic (charged) states' structure was therefore necessary. In this direction, two important contributions were:
- Chung's proposal [7] for the asymptotic states,which was a first attempt towards obtaining finite scattering amplitudes to each order in α;
- the recipe by Kulish and Fadeev [8] to derive the new asymptotic states from the hamiltonian.

Chung's proposal.

Chung successfully applies the results by Bloch and Nordsieck to the perturbative series of Q.E.D. He considers the following asymptotic states, for one electron, depending on the infrared cut-off ϵ

$$\psi_{in}^{\epsilon} = \exp\left(-\frac{1}{2}\sum_{l=1}^{2}\int_{|\mathbf{k}|>\epsilon}\left|S_{in}^{l}(\mathbf{k})\right|^{2}d^{3}k\right)\cdot\exp\left(\sum_{l=1}^{2}\int_{|\mathbf{k}|>\epsilon}^{2}S_{in}^{l}(\mathbf{k})\cdot\left(e_{l}(\mathbf{k})\cdot a_{l}^{\dagger}(\mathbf{k})\right)d^{3}k\right)\psi_{p_{in}}$$

(analogous recipe for ψ_{out}^{ϵ}) where

- $k,p\in R^{4}$, $e_{l}(k)$ transverse polarization vectors, $a_{l},a_{l}^{\dagger}$ annihilation and creation operator of the electromagnetic field;

- $S_{in}^{l}(\mathbf{k})\approx\frac{e}{\left(2(2\pi)^{3}\cdot k_{0}\right)^{\frac{1}{2}}}\frac{p_{in}\cdot e_{l}(\mathbf{k})}{k\cdot p_{in}}$ for $k\to 0$, and $L^{2}\left(d^{3}k\right)$ −integrable outside $k=0$;
- $\psi_{p_{in}}$ wave function of an electron with momentum p_{in}.

Chung's computations prove that the new S-matrix elements formally exist to each order, in the limit $\epsilon\to 0$, in the case of scattering from an external potential. Because of the states' dependence on p_{in} and p_{out}, scattering implies the emission of an infinite number of photons for $p_{in}\neq p_{out}$. These results provide a strong evidence for the conjecture that the asymptotic states of Q.E.D. are generalized coherent states in the infrared region; nevertheless, there are reasons of uneasiness:
- it is not clear if the state's dependence on the asymptotic electron momentum introduces some superselection rule; the choice of asymptotic states is arbitrary to some extent, a physically relevant space should be specified (see also [9]);
- in the approach by Chung, the coherent states are disconnected from the rest of theory, that is they cannot be reconcile with the *ansatz* of the perturbative theory which assumes the free theory states as the asymptotic states.

Kulish and Fadeev's asymptotic dynamics.

A different point of view inspires the work by Kulish and Fadeev. Their aim is not only to reformulate the perturbative treatment by the definition of generalized transition amplitudes not plagued by infrared divergences; they want to convince us that the asymptotic space of non-Fock coherent states (in the infrared region) is the direct result of a corrected asymptotic dynamics, which is necessary in the presence of massless particles. In this sense, they adapt Dollard's treatment of Coulomb scattering [10] to Q.E.D.: the claim is that the asymptotic hamiltonian is not necessarily the free hamiltonian but it depends on the interaction if it is a long range one. The proposed asymptotic hamiltonian is:

$$H_{as}(t) = H_0 + V_{as}(t)$$

where

- $V_{as}(t) = (2\pi)^{-\frac{3}{2}} \int \left(a_\mu(\mathbf{k}) + a_\mu^\dagger(-\mathbf{k})\right) J_{as}^\mu(\mathbf{k},t) \frac{d^3k}{\sqrt{2k_0}}$
-$J_{as}^\mu(\mathbf{k},t) = -e\int p^\mu \rho(\mathbf{p}) \exp\left(i\frac{\mathbf{p}\cdot\mathbf{k}}{p_0}t\right)\frac{d^3p}{p_0}$, and $\rho(\mathbf{p}) = \sum_n \left(b_n^\dagger(\mathbf{p})\, b_n(\mathbf{p}) - d_n^\dagger(\mathbf{p})\, d_n(\mathbf{p})\right)$ (for further details about definitions see [8]).

The so obtained wave operators lead to an S-matrix which is free of infrared divergences when it acts on new asymptotic state which are a generalization of Chung's states for more than one fermion. By the new S-matrix, the infinite Coulomb phase is balanced. In this context all the derivations are formal and the perturbative theory supports the consistency of the construction.

The analysis by Kulish and Fadeev points out some issues.
In their treatment the convergence of the asymptotic limits seems to be solved by the "Dollard's dynamical correction", the new problem is localized in the characterization of the asymptotic states to which that dynamics leads. To satisfy the Gupta condition on the asymptotic states, a problem arises in the Lorentz invariance of the asymptotic spaces. A sharp mass-shell for the electron seems to be missing.
There is no mathematical control of the existence of the fields asymptotic limits in the Heisenberg picture, inside the Hilbert space where the theory is defined at finite times.

From the results in the previous works it has become clear that:
- it is necessary to overcome the perturbative approach to complete and justify them since the yet open problems require an approach to the infrared problem not restricted to the scattering matrix or to the identification of the scattering space at most;
- the various aspects of the infrared problem present many relations, so that they require a more insight into the structure of the theory.
Therefore the previous results stimulated both a general study in the context of Wigthman's approach to quantum field theory (see [11], [12], [13], [14], [15], [16]) and a rigorous analysis, inside the hamiltonian scheme, of models describing quantum mechanical matter interacting with the ra-

diation field.

0.2 Analysis of models.

The inquired models are simplified versions of Q.E.D. with the following general features:
- the interaction between the current and the e.m. field is not the interaction of quantized fields because the charge is not described by a field, so that in this approximation pair production disappears;
- to avoid ultraviolet divergences an ultraviolet cut-off is (generally) performed in the interaction.
The underlying conjecture is that, in spite of the approximations above, these models retain the main infrared features of Q.E.D., in the asymptotic limit $|t| \to \infty$ and for low energy configurations. Another source of interest in studying these models is directly connected with the realm of phenomena related to atomic physics. In this context, many studies are recently devoted to a rigorous analysis of binding ([17], [18], [19], [20], {21]), scattering by light and the relaxation to the ground state for isolated atoms ([22], [23], [25], [26]), all phenomena where the zero photon mass implies not-trivial spectral problems.

In this thesis we aim to shed light on the infrared difficulties related to the translation invariant case, for only one charged particle; namely, we aim at a rigorous treatment of "Compton scattering" in a scalar infrared model: Nelson's massless model.
Our analysis will be focused on the relation between the absence of one-particle states and the asymptotic dynamics, both for the photon field and for the charged particle.
In order to define the conceptual and the technical problems we will discuss later, we briefly review and compare the answers given by various infrared models, many of them solvable. As we point out in the next lines, important indications come from these models but, at the same time ,the analysis is clearly affected by too strong approximations and the answers are not completely clear and satisfactory thereby.
In this direction we consider both the scalar and the vectorial interaction between non-relativistic particles and massless bosons, at different levels of approximation. Many of these models are solvable, they are approximate versions of (fully) interacting models like Nelson's massless model and non-relativistic Q.E.D.. Since in the rest of the thesis we discuss Nelson's massless model in details, we now provide its mathematical definition and then we turn to the approximate versions already considered in literature. A remark on notation: we will call both the spin 1 and spin 0 massless bosons as photons, the charged particle as electron.

Nelson's massless model.

The physical system consists of a non-relativistic spin-less quantum particle of mass m, linearly coupled to the massless scalar quantized boson field. The non-relativistic particle is described by position and momentum variables with usual *canonical commutation rules* (c.c.r.) $[x_i, p_j] = i\delta_{i,j}$ $(\hbar = 1)$; the (scalar) boson field at time $t = 0$ is:

$$A(0, \mathbf{y}) = \frac{1}{\sqrt{2\pi}^3} \cdot \int \left(a^\dagger(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{y}} + a(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{y}} \right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}},$$

$(c = \hbar = 1)$, where $a^{\dagger}(\mathbf{k}), a(\mathbf{k})$ are standard creation and annihilation operator valued tempered distributions obeying the c.c.r.

$$\left[a(\mathbf{k}), a^{\dagger}(\mathbf{q})\right] = \delta^{3}(\mathbf{k}-\mathbf{q}) \ , \ \left[a(\mathbf{k}), a(\mathbf{q})\right] = \left[a^{\dagger}(\mathbf{k}), a^{\dagger}(\mathbf{q})\right] = 0.$$

The spatial translations are implemented by the total momentum

$$\mathbf{P} = \mathbf{p} + \int \mathbf{k} a^{\dagger}(\mathbf{k}) a(\mathbf{k}) \, d^{3}k \, .$$

The dynamics of the system is generated by the covariant hamiltonian ($[H, \mathbf{P}] = 0$)

$$H = \frac{\mathbf{p}^{2}}{2m} + g \int_{0}^{\kappa} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a^{\dagger}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \frac{d^{3}k}{\sqrt{2}\,|\mathbf{k}|^{\frac{1}{2}}} + H^{ph}$$

where κ is the ultraviolet *cut-off*, g is the coupling constant and H^{ph} is the free hamiltonian of the (scalar) photon field

$$\int |\mathbf{k}| \, a^{\dagger}(\mathbf{k}) \, a(\mathbf{k}) \, d^{3}k \, .$$

The Hilbert space of the system is H$= L^{2}(R^{3}) \otimes F$ where F is the Fock space with respect to the creation and annihilation operator valued distributions $\{a^{\dagger}(\mathbf{k}), a(\mathbf{k})\}$. An element of H is a sequence $\{\psi_{n}\}$ of functions on R^{3n+1} with $||\psi|| < \infty$, where

$$||\psi||^{2} = \sum_{n=0}^{\infty} \int \overline{\psi^{n}(\mathbf{x}, \mathbf{k}_{1}, ..\mathbf{k}_{n})} \psi^{n}(\mathbf{x}, \mathbf{k}_{1}, ..\mathbf{k}_{n}) \, d^{3}k_{1}...d^{3}k_{n} d^{3}x$$

and each $\psi^{n}(\mathbf{x}, \mathbf{k}_{1}, ..\mathbf{k}_{n})$ is symmetric in $\mathbf{k}_{1}, ...\mathbf{k}_{n}$. The $n = 0$ component belongs to the tensor product of the vacuum subspace with the non-relativistic particle space $L^{2}(R^{3})$.

Standard results about H and P:

i) The operators $\mathbf{P} = \mathbf{p} \otimes 1 + 1 \otimes \int \mathbf{k} a^{\dagger}(\mathbf{k}) \, a(\mathbf{k}) \, d^{3}k$ are essentially self-adjoint (e.s.a.) in

$$D \equiv \bigvee_{n \in N} h \otimes \psi^{n}$$

which is the set of finite linear combinations of vectors of wave function $h(\mathbf{x}) \psi^{n}(\mathbf{k}_{1}, ...\mathbf{k}_{n})$, where $h(\mathbf{x}) \in S(R^{3})$, $\psi^{n}(\mathbf{k}_{1}, ...\mathbf{k}_{n}) \in S(R^{3n})$ is symmetric and $n \in N$. Since $\mathbf{p}$ and $\int \mathbf{k} a^{\dagger}(\mathbf{k}) \, a(\mathbf{k}) \, d^{3}k$ are e.s.a. in $S(R^{3})$ and $\bigvee_{n \in N} \psi^{n}$ respectively, the result easily follows for the $\mathbf{P}$ operators.

ii) The interaction term in the hamiltonian is a Kato's infinitesimal small perturbation with respect to

$$H_{0} \equiv \frac{\mathbf{p}^{2}}{2m} + H^{ph} \, .$$

Therefore H is e.s.a. in $D \equiv \bigvee_{n \in \aleph} h \otimes \psi^{n}$ and its self-adjointness domain, $D(H)$, coincides with $D(H_{0})$.

iii) The groups $e^{i\mathbf{a}\cdot\mathbf{P}}$ and $e^{i\tau H}$ ($\tau, a^i \in R$) commute.

iv) The joined spectral decomposition of the Hilbert space with respect to the $\mathbf{P}$ operators is H= $\int^{\oplus} \mathrm{H}_{\mathbf{P}} d^3P$ where $\mathrm{H}_{\mathbf{P}}$ is isomorphic to F.
Indeed, to the improper eigenvectors (of the $\mathbf{P}$ operators) $\psi^n_{\mathbf{P}}$

$$\psi^n_{\mathbf{P}}(\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n) = (2\pi)^{-\frac{3}{2}} e^{i(\mathbf{P}-\mathbf{k}_1-...-\mathbf{k}_n)\cdot\mathbf{x}} \psi^n_{\mathbf{P}}(\mathbf{k}_1, ..\mathbf{k}_n) \qquad \psi^n_{\mathbf{P}}(\mathbf{k}_1, ..\mathbf{k}_n) \in S\left(R^{3n}\right) \; and\, simm.$$

we can relate a natural scalar product: $(\phi^n_{\mathbf{P}}, \psi^m_{\mathbf{P}}) = \delta_{n,m} \int \overline{\phi^n_{\mathbf{P}}(\mathbf{k}_1, ..\mathbf{k}_n)} \psi^m_{\mathbf{P}}(\mathbf{k}_1, ..\mathbf{k}_n)\, d^3k_1...d^3k_n.$ (0)
The vector space $\overline{\bigvee_{n\in\aleph} \psi^n_{\mathbf{P}}}$ is obtained as the closure of the finite linear combinations of the $\psi^n_{\mathbf{P}}$, in the norm which arises from the scalar product (0). Starting from this space we uniquely define the linear application

$$I_{\mathbf{P}} : \overline{\bigvee_{n\in N} \psi^n_{\mathbf{P}}} \to F^b$$

by the prescription:

$$I_{\mathbf{P}}\left(\psi^n_{\mathbf{P}}(\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n)\right) = \frac{1}{\sqrt{n!}} \int b^\dagger(\mathbf{k}_1)....b^\dagger(\mathbf{k}_n)\, \psi^n_{\mathbf{P}}(\mathbf{k}_1, ..\mathbf{k}_n)\, d^3k_1...d^3k_n \psi_0\,,$$

where $b(\mathbf{k}), b^\dagger(\mathbf{k})$ formally correspond to $a(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, a^\dagger(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}$. They are annihilation and creation operator-valued tempered distributions in the Fock space $F^b \cong F$. The norm given by (0) for $\psi^n_{\mathbf{P}}$ is equal to $||I_{\mathbf{P}}(\psi^n_{\mathbf{P}})||_F$ ($||||_F$ is the Fock norm).

v) Since $[H, \mathbf{P}] = 0$, we have $H = \int H_{\mathbf{P}} d^3P$, where $H_{\mathbf{P}} : \mathrm{H}_{\mathbf{P}} \to \mathrm{H}_{\mathbf{P}}$ is e.s.a. in $D^b \equiv \bigvee_{n\in N} \psi^n_{\mathbf{P}}$; in terms of the variables $\mathbf{P}$, $b(\mathbf{k}), b^\dagger(\mathbf{k})$, the operator $H_{\mathbf{P}}$ is written as follows:

$$H_{\mathbf{P}} = \frac{\left(\mathbf{P}^{ph} - \mathbf{P}\right)^2}{2m} + g \int_0^\kappa \left(b(\mathbf{k}) + b^\dagger(\mathbf{k})\right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}} + H^{ph}\,.$$

Approximation of fixed charge.

As first approximation we consider the particle energy independent on momentum, i.e. $\frac{\mathbf{P}^2}{2m} \to m$:

$$H_m = m + g \int_0^\kappa \left(a(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \frac{d^3k}{\sqrt{2}\,|\mathbf{k}|^{\frac{1}{2}}} + H^{ph}$$

The hamiltonian H_m is e.s.a. in $L^2\left(R^3\right) \otimes \bigvee_{n\in N} \psi^n$. The approximation makes the model (explicitly) solvable: the electron dynamics is trivial, there is no scattering.

From the differential equation for distributions

$$\frac{da(\mathbf{k}, t)}{dt} = -i\,|\mathbf{k}|\, a(\mathbf{k}, t) - ig \frac{\chi^\kappa_0(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\,|\mathbf{k}|}}$$

we obtain

$$a(\mathbf{k},t) = a(\mathbf{k})\, e^{-i|\mathbf{k}|t} - ig\frac{\chi_0^\kappa(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2|\mathbf{k}|}} \cdot \frac{1-e^{-i|\mathbf{k}|t}}{i|\mathbf{k}|},$$

analogous result for $a^\dagger(\mathbf{k},t)$.
The asymptotic field is defined starting from the L.S.Z. (Lehmann, Symanzik and Zimmermann) smeared fields

$$e^{iHt}e^{-iH^{ph}t}\int a^\sharp(\mathbf{k})\,\widetilde{\varphi}(\mathbf{k})\, d^3k e^{iH^{ph}t}e^{-iHt}, \quad a^\sharp(\mathbf{k}) = a(\mathbf{k}) \; or \; a^\dagger(\mathbf{k}) \quad and \quad \widetilde{\varphi}(\mathbf{k}) \in S\left(R^3\right)$$

and taking into account the solution $a^\sharp(\mathbf{k},t)$.
The above expression converges for $t \to \infty$ on $L^2\left(R^3\right) \otimes \bigvee_{n\in N} \psi^n$. In the case of $a(\mathbf{k},t)$, the limit is

$$\int a(\mathbf{k})\,\widetilde{\varphi}(\mathbf{k})\, d^3k + g\int \widetilde{\varphi}(\mathbf{k})\,\frac{\chi_0^\kappa(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}} d^3k$$

from which

$$a^{out}(\mathbf{k}) = a^{in}(\mathbf{k}) = a(\mathbf{k}) + g\frac{\chi_0^\kappa(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}}.$$

Starting from the joined spectral decomposition of H with respect the $\mathbf{x}$ operators H$=\int^{\oplus}\mathrm{H}_{\mathbf{x}}d^3x$, it is simple to check that each $\mathbf{x}$-fiber space carries a non-Fock coherent representation of the asymptotic Weyl algebra. These representations are however equivalent each other.

The expression of H_m at fixed total momentum $\mathbf{P}$ is

$$\int |\mathbf{k}|\left(b^\dagger(\mathbf{k}) + \frac{g\chi_0^\kappa}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}}\right)\left(b(\mathbf{k}) + \frac{g\chi_0^\kappa}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}}\right)d^3k - g^2\int_0^\kappa \frac{1}{2|\mathbf{k}|^2}d^3k + m$$

from which derives that the hamiltonian H_m admits a one-particle state if we perform the following non Fock coherent transformation on $\left\{b(\mathbf{k}), b^\dagger(\mathbf{k})\right\}$:

$$b(\mathbf{k}) \longrightarrow b(\mathbf{k}) - \frac{g}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}} \quad for\, \mathbf{k}: \quad 0 \le |\mathbf{k}| \le \kappa$$

(it is sufficient in a neighbourhood of $\mathbf{k}=0$).

The claim is justified considering that:
- the hamiltonian has the following expression

$$\int |\mathbf{k}|\, a^{out^\dagger}(\mathbf{k})\, a^{out}(\mathbf{k})\, d^3k - g^2\int_0^\kappa \frac{1}{2|\mathbf{k}|^2}d^3k + m,$$

in terms of the asymptotic field operators;
- after the coherent transformation, each $\mathrm{H}_{\mathbf{x}}$ carries a Fock representation of the Weyl algebra associated to the asymptotic field.
The main interest has been generally concentrated on the described renormalization of the one-particle states. For our purpose, this transformation is however not very interesting to discuss since

in the fully interacting model the analogous transformation is physically meaningless (superselection of the total momentum). Because of that, we are interested in a physical description inside the Hilbert space H which is assumed in the definition of the model, with a Fock representation for the interacting field. Having in mind the final result of this thesis, we now present a construction for the generic asymptotic state, in Heisenberg picture, that can be considered as the zero order in the $\frac{1}{m}$-expansion of the "minimal asymptotic electron" constructed in Part III.
We want to describe an electron out of the scattering with the always present soft photons cloud linked to the non-Fock representation of the asymptotic field in the Hilbert space H.
Let us start from a one-particle state for the hamiltonian $H_{m,\sigma}$ (i.e with an infrared cut-off σ in the interaction)

$$\int W^{\dagger}\psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P \equiv e^{g\int_{\sigma}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P$$

where $h\left(\mathbf{P}\right)$ is the wave function in the total momentum $\mathbf{P}$ and $\psi_{0,\mathbf{P}}$ is the vacuum related to the fiber space $\mathrm{H}_{\mathbf{P}}$ (in other formulas, sometimes, we implicitly assume the isomorphism $I_{\mathbf{P}}$ and we use ψ_0 for $\psi_{0,\mathbf{P}}$). The above vector has the property

$$H_{m,\sigma}e^{g\int_{\sigma}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P = \left(m-g^2\int_{\sigma}^{\kappa}\frac{1}{2\left|\mathbf{k}\right|^2}d^3k\right)e^{g\int_{\sigma}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P\,.$$

Now we consider the time dependent vector ($0<\kappa_1<\kappa$, $\quad E^{\sigma}=m-g^2\int_{\sigma}^{\kappa}\frac{1}{2|\mathbf{k}|^2}d^3k$):

$$e^{iHt}e^{-iH^{ph}t}e^{-g\int_{\sigma}^{\kappa_1}\frac{a(\mathbf{k})-a^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}e^{-iH^{ph}t}e^{-iE^{\sigma}t}e^{g\int_{\sigma}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P =$$

$$= e^{iHt}e^{-g\int_{\sigma}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|t}-a^{\dagger}(\mathbf{k})e^{-i|\mathbf{k}|t}}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}e^{-iH_{\sigma}t}e^{g\int_{\sigma}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P ==$$

that for $\sigma\to 0$ has a well defined limit

$$e^{ig^2\int_0^{\kappa_1}\sin(\mathbf{k}\cdot\mathbf{x})\frac{d^3k}{|\mathbf{k}|^3}}e^{-ig^2\int_0^{\kappa_1}\sin(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)\frac{d^3k}{|\mathbf{k}|^3}}e^{-g\int_0^{\kappa_1}\frac{a(\mathbf{k})-a^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}e^{g\int_0^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P$$

which converges for $t\to+\infty$ to the following expression

$$= e^{i\frac{g^2}{2}\int_0^{\kappa_1}\sin(\mathbf{k}\cdot\mathbf{x})\frac{d^3k}{|\mathbf{k}|^3}}e^{4\pi i g^2\int_0^{+\infty}\sin(|\mathbf{q}|)\frac{d|\mathbf{q}|}{|\mathbf{q}|}}e^{-g\int_0^{\kappa_1}\frac{a(\mathbf{k})\left(1-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-c.c.}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}e^{g\int_{\kappa_1}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\int \psi_{0,\mathbf{P}}h\left(\mathbf{P}\right)d^3P\,.$$

We remark that the above construction is extremely simplified because the ground energy "$E\left(\mathbf{P}\right)$" is independent of $\mathbf{P}$ ($E=m-g^2\int_0^{\kappa}\frac{1}{2|\mathbf{k}|^2}d^3k$) so that $\nabla E\left(\mathbf{P}\right)=0$. It implies that the "static" coherent factor and the "dynamical" one trivially coincide: the first one is associated to the $\left\{b\left(\mathbf{k}\right),b^{\dagger}\left(\mathbf{k}\right)\right\}$ representation in which the one-particle states are defined, the second one is related to the representation of the asymptotic field in H. We will see that a less trivial relation holds in the fully interacting case, where the coherent factors are not c-numbers but depend on an operator.

Simplified spectral model.

Regarding the behavior at the infimum of the spectrum of the hamiltonians $H_{\mathbf{P}}$, a better qualitative approximation of Nelson's massless model is provided by the following solvable model (see also [3]): consider for each $\mathbf{P}$: $|\mathbf{P}| \leq m$ the operator

$$\widetilde{H}_{\mathbf{P},\sigma} \equiv H_{\mathbf{P},\sigma} - \frac{\mathbf{P}^{ph^2}}{2m}$$

that after a simple manipulation can be expressed as

$$\int |\mathbf{k}| \left(1 - \widehat{\mathbf{k}} \cdot \frac{\mathbf{P}}{m}\right) \left(b^\dagger + \frac{g\chi_\sigma^\kappa}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}} \left(1 - \widehat{\mathbf{k}} \cdot \frac{\mathbf{P}}{m}\right)}\right) \left(b + g\frac{g\chi_\sigma^\kappa}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}} \left(1 - \widehat{\mathbf{k}} \cdot \frac{\mathbf{P}}{m}\right)}\right) d^3k - g^2 \int_\sigma^\kappa \frac{1}{2\,|\mathbf{k}|^2 \left(1 - \widehat{\mathbf{k}} \cdot \frac{\mathbf{P}}{m}\right)^2} d^3k$$

where χ_σ^κ is the characteristic function of the set $\{\mathbf{k} : \sigma \leq |\mathbf{k}| \leq \kappa\}$.
The ground state of $\widetilde{H}_{\mathbf{P},\sigma}$, in the Fock space F ($F \simeq$H$_{\mathbf{P}}$), is the coherent state (ψ_0 is the vacuum state):

$$W_\sigma^\dagger(\mathbf{P})\,\psi_0 \equiv e^{g\int_\sigma^\kappa \frac{b(\mathbf{k}) - b^\dagger(\mathbf{k})}{|\mathbf{k}|\left(1 - \widehat{\mathbf{k}} \cdot \frac{\mathbf{P}}{m}\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}}\,\psi_0 \,.$$

In the limit $\sigma \to 0$, this vector goes weakly to zero. The ground state belongs to a $\mathbf{P}-$dependent representation of $\{b^\dagger, b\}$, not equivalent to the Fock one and each other for different $\mathbf{P}$.
The phenomenon can also be seen in terms of the transformed hamiltonians $\widetilde{H}^w_{\mathbf{P},\sigma}$ acting on F and obtained according to the Weyl transformation $W_\sigma(\mathbf{P})$

$$\widetilde{H}^w_{\mathbf{P},\sigma} \equiv W_\sigma(\mathbf{P})\, \widetilde{H}_{\mathbf{P},\sigma} W_\sigma^\dagger(\mathbf{P}) = H^{ph} - g^2 \int_\sigma^\kappa \frac{1}{2\,|\mathbf{k}|^2 \left(1 - \widehat{\mathbf{k}} \cdot \frac{\mathbf{P}}{m}\right)} d^3k \,.$$

If we call $\widetilde{\phi}^g_{\mathbf{P}}$ the corresponding ground states, we have $\widetilde{\phi}^g_{\mathbf{P}} \equiv \psi_0$ for each $0 \leq \sigma \leq \kappa$; in particular, the "limit" vector for $\sigma \to 0$ is still in F with an obvious property of regularity in $\mathbf{P}$.

As we will check in Part II, an analogous phenomenon happens in Nelson's massless model. We define a sequence of ground eigenvectors $\phi^g_{\mathbf{P}}$ of properly transformed hamiltonians $H^w_{\mathbf{P},\sigma}$ according to a $\mathbf{P}-$dependent Weyl operator which becomes an intertwiner in the limit $\sigma \to 0$, and we prove that $\phi^g_{\mathbf{P}}$ is strong convergent in H$_{\mathbf{P}} \simeq F$ for $\sigma \to 0$. As straightforward consequence, in the limit $\sigma \to 0$, the ground eigenvector $\psi^g_{\mathbf{P}}$ of $H_{\mathbf{P},\sigma}$ has a coherent non-Fock behavior for $\mathbf{k} \to 0$ and goes weakly to zero in H$_{\mathbf{P}} \simeq F$.
This simplified spectral model is not however interesting as far as asymptotic fields are concerned. Differently from the fixed charge approximation, the last one does not admit an asymptotic field.

Pauli-Fierz-Blanchard model.

The model was proposed by Pauli and Fierz [27] and later studied by Blanchard [28]. It is a model where an electron interacts with the radiation field according to the minimal coupling in the dipole approximation:

$$\frac{\left(\mathbf{p}-e\mathbf{A}^{(\rho)}(\mathbf{x})\right)^2}{2m} \to \frac{\mathbf{p}^2}{2m} - \frac{e\mathbf{p}\cdot\mathbf{A}^{(\rho)}(0)}{m}$$

where

$$\mathbf{A}(\mathbf{y}) = \sum_s \int \widehat{e_s}(\mathbf{k}) \left\{ a_s(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{y}} + a_s^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{y}} \right\} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}$$

is the vector potential in Coulomb gauge, $\widehat{e_s}(\mathbf{k})$ are the polarizations, $s = 1, 2$, *c.c.r.* are assumed for $\left\{a_s^\dagger(\mathbf{k}), a_s(\mathbf{k})\right\}$, and ρ is a form factor to avoid ultraviolet divergences.
At the same time a potential $V(\mathbf{x})$ is considered, so that the dynamics is not trivial. $V(\mathbf{x})$ should mimic the effect of the radiation reaction. Because of the dipole approximation and of the potential $V(\mathbf{x})$, the model is not translationally invariant.
We discuss the model in a formalism and with an interest slightly different from Blanchard's original paper; it is suggested by the general analysis of the infrared problem in the Wightman framework [14].
We assume working in $L^2(R^3) \otimes F$, where F is now the Fock space with respect to the creation and annihilation operator valued distributions $\left\{a_s^\dagger(\mathbf{k}), a_s(\mathbf{k})\right\}$, and we consider first the model in Heisenberg picture and without the potential $V(\mathbf{x})$. Given the hamiltonian

$$H_d = \frac{\mathbf{p}^2}{2m} - \frac{e\mathbf{p}\cdot\mathbf{A}^{(\rho)}(0)}{m} + H^{ph}$$

the corresponding evolution is provided by $e^{-iH_dt} = e^{-iH_0t}e^{-i\frac{\mathbf{p}^2}{2m(t)}}e^{-i\mathbf{p}\cdot\mathbf{A}(t)}$ where

$\mathbf{A}(t) = -\frac{e}{m}\sum_s \int \frac{\widetilde{\rho}(|\mathbf{k}|)}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}} \widehat{e_s}(\mathbf{k}) \left\{ a_s(\mathbf{k})\, u(|\mathbf{k}|, t) + a_s^\dagger(\mathbf{k})\, \overline{u}(|\mathbf{k}|, t) \right\} d^3k$

$\frac{1}{m(t)} = \frac{1}{m} - \frac{2e^2}{3m^2}\int \frac{\widetilde{\rho}^2(|\mathbf{k}|)}{|\mathbf{k}|^2} \cdot \left(1 - \frac{\sin(|\mathbf{k}|t)}{|\mathbf{k}|t}\right) d^3k$

$u(|\mathbf{k}|, t) = i\left(e^{-i|\mathbf{k}|t} - 1\right) \quad , \quad g_s(\mathbf{k}) = \frac{e}{m}\left(\mathbf{p}\cdot\widehat{e_s}(\mathbf{k})\right) \frac{\widetilde{\rho}(|\mathbf{k}|)}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}}$

At time t, the H_d- evolved dynamical variables are:

$\mathbf{x} \to \mathbf{x} + \frac{\mathbf{p}}{m(t)}t + \mathbf{A}(t)$

$\mathbf{p} \to \mathbf{p}$

$a_s(\mathbf{k}) \to a_s(\mathbf{k})\, e^{-i|\mathbf{k}|t} + ig_s(\mathbf{k})\, u(|\mathbf{k}|, t)$ ($a_s^\dagger(\mathbf{k}, t)$ by complex conjugation).

Blanchard observed that the hamiltonian H_d plays the role of the "free" hamiltonian with respect to $H = H_d + V(\mathbf{x})$ for proper potentials $V(\mathbf{x})$ (for details see [28]), namely the limits

$$s - \lim_{t\to\pm\infty} e^{iHt}e^{-iH_dt}$$

are well defined on the dense set of H, $\bigvee_{n\in\aleph} h \otimes \psi^n$ where $h \in C_0^\infty(R^3)$ and ψ^n defined as for Nelson's model apart from the polarizations (omitted in the notation) . Therefore it is important to specify which are the "free" or asymptotic configurations of this scattering problem. Such a characterization is the most important point for our purposes. The problem consists in expressing

$$e^{-iH_d t}\varphi \qquad \varphi \in \bigvee_{n\in\aleph} h\otimes\psi^n$$

in terms of the asymptotic variables related to the "free" evolution provided by H_d:

$$\mathbf{p} \quad and \quad \begin{cases} a_s^{out}(\mathbf{k}) = a_s^{in}(\mathbf{k}) = a_s(\mathbf{k}) - \frac{e}{m}\left(\mathbf{p}\cdot\widehat{e_s}(\mathbf{k})\right)\frac{\widetilde{\rho}(|\mathbf{k}|)}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}} \\ a_s^{\dagger out}(\mathbf{k}) = a_s^{\dagger in}(\mathbf{k}) = a_s^{\dagger}(\mathbf{k}) - \frac{e}{m}\left(\mathbf{p}\cdot\widehat{e_s}(\mathbf{k})\right)\frac{\widetilde{\rho}(|\mathbf{k}|)}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}} \end{cases}$$

where $\left\{a_s^{out(in)}(\mathbf{k}), a_s^{\dagger out(in)}(\mathbf{k})\right\}$ are obtained from the asymptotic limit of the L.S.Z. smeared fields. The expression of H_d in terms of the asymptotic fields is the following:

$$H_d = H_{ch}^{out(in)} + H_{ph}^{out(in)}$$

where

$$H_{ch}^{out(in)} = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{2m^2}\sum_s\int\left(\mathbf{p}\cdot\widehat{e_s}(\mathbf{k})\right)^2\frac{\widetilde{\rho}(|\mathbf{k}|)}{|\mathbf{k}|^2}d^3k = \frac{\mathbf{p}^2}{2m'} \quad , \quad H_{ph}^{out(in)} = \sum_s\int|\mathbf{k}|\,a_s^{\dagger out}(\mathbf{k})\,a_s^{out}(\mathbf{k})\,d^3k$$

and $m' > 0$ by constraints on $\widetilde{\rho}$. Now let us consider the state $e^{-iH_d t}\psi_0\otimes h$, for instance. It formally corresponds to

$$e^{-i\left(H_{ch}^{out(in)}+H_{ph}^{out(in)}\right)t}\int W_{\mathbf{p}}W_{\mathbf{p}}^{\dagger}\psi_{0,\mathbf{p}}h(\mathbf{p})\,d^3p$$

where $W_{\mathbf{p}} = e^{\sum_s\int_\sigma^\kappa \frac{e}{m}\left(\mathbf{p}\cdot\widehat{e_s}(\mathbf{k})\right)\widetilde{\rho}(|\mathbf{k}|)\frac{a_s^{out}(\mathbf{k})-a_s^{out\dagger}(\mathbf{k})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$ and $\int W_{\mathbf{p}}^{\dagger}\psi_{0,\mathbf{p}}h(\mathbf{p})\,d^3p$ is a vacuum state for the asymptotic operator valued distributions $a_s^{out}(\mathbf{k})$.
The emerging physical picture is that of an asymptotic electron $\int W_{\mathbf{p}}^{\dagger}\psi_{0,\mathbf{p}}h(\mathbf{p})\,d^3p$ surrounded by a cloud of soft photons represented by the not-unitary Weyl operators $W_{\mathbf{p}}$, which are freely moving according the energy dispersion laws $E_{ch}(\mathbf{p}) = \frac{\mathbf{p}^2}{2m'}$ and $\omega(\mathbf{k}) = |\mathbf{k}|$ respectively.
Since the asymptotic particle momentum $\mathbf{p}$ changes in the scattering because of the potential $V(\mathbf{x})$, the *in* and *out* states differ for an infinite number of photons, as pointed out by Chung in his proposal for the Q.E.D scattering amplitudes.

The models discussed so far contain the simplifying hypothesis of a trivial dynamics for the charged particle which is not realistic and makes completely unclear the interaction charge-radiation as far as the radiation reaction is concerned. This approximation is also at the origin of the coherent states' appearance.
An important step towards a more complete picture of the scattering configurations is contained in a paper by Hoegh-Krohn [29] where the existence of the asymptotic boson field for a class of fully interacting scalar models is proved by Cook's argument. His results can be easily exported to the vectorial interaction case.

Asymptotic fields in (fully) interacting models.

The general situation considered by Hoegh-Krohn [29] is that of a scalar boson field locally interacting with a fixed number N of non-relativistic particles.

The L.S.Z. smeared field converges by proving the norm integrability in t of

$$\frac{d}{dt}\left\{e^{iHt}e^{-iH^{ph}t}\int a(\mathbf{k})\sqrt{2|\mathbf{k}|}\cdot\widetilde{\varphi}(\mathbf{k})\,d^3ke^{iH^{ph}t}e^{-iHt}\right\}=\sum_{j=1}^{N}\varphi(\mathbf{x}_j(t),t)\qquad\widetilde{\varphi}(\mathbf{k})\in S(R^3)$$

where $\varphi(\mathbf{x}_j(t),t)=\int e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}_j(t)}\widetilde{\varphi}(\mathbf{k})\,d^3k$ (analogous quantities for the smeared creation operator). The Hilbert norm of the single multiplication operator $\varphi(\mathbf{x}_j(t),t)$ is

$$\sup_{\mathbf{x}_j\in R^3}|\varphi(\mathbf{x}_j,t)|$$

which is simple to estimate, by stationary phase methods.
At this point a remark about Hoegh-Krohn's results is in order:

i) Cook's argument, described so far, is completely sufficient for massive bosons, in this case the dispersion law

$$\omega(|\mathbf{k}|)=\sqrt{|\mathbf{k}|^2+m^2}$$

guarantees a decrease at an integrable rate for $\sup_{\mathbf{x}_j\in R^3}|\varphi(\mathbf{x}_j,t)|$ (see [30]);

ii) in the massless case, it is necessary "to avoid" the light cone directions $|\mathbf{x}_j|=t$ where the decrease of $|\varphi(\mathbf{x}_j,t)|$ is only of order $\frac{1}{|t|}$, not integrable thereby.
For a relativistic electron, due to the energy dispersion law, such a region is automatically avoided. Indeed, a propagation estimate for the evolved $\mathbf{x}_j(t)$ particle position operators yields a fast decrease outside any region contained in the light-cone for a dense set of states.
For a non-relativistic electron, a subspace in the Hilbert space has to be singled out corresponding to asymptotic electrons "inside" the light-cone except fast time-decreasing tails. In [24] it is proved for the subspace corresponding to hamiltonian spectral values below $\frac{M}{2}(c=1)$, in agreement with the classical situation. An analogous condition is exploited in the scattering states constructed in Part III for the one-particle sector of Nelson's massless model, by imposing an electron asymptotic mean velocity less than 1, already in the construction. The result relies on the propagation estimate

$$e^{iHt}f\left(\frac{\mathbf{x}}{t}\right)e^{-iHt}\rightarrow_{t\rightarrow\infty}f(\nabla E(\mathbf{P}))$$

for proper functions f, as it will be explained later.

Froehlich's analysis of Nelson's model.

The first systematic analysis of the fully interacting and translationally invariant Nelson's model was performed in two papers [3] and [31], which strongly inspired the present thesis. In such a

study the previous information from solvable models is mastered, several concepts for a collision theory are developed and many technical tools are provided. The following cases are investigated and compared:
- the massive and the massless cases as far as the boson field is concerned;
- both the non-relativistic and the relativistic dispersion law for the charged particle energy .

The scattering problem is studied in the "Haag-Ruelle" framework [32] adapted to the mixed character of the actual model which joins quantum mechanical matter and a quantum relativistic field. This approach is satisfactory as far as one particle states for the charge are available: it happens in the massive case and for an infrared-cut interaction in the massless case. By the one-particle states and the asymptotic limit of the L.S.Z smeared field, the asymptotic picture, in the one-particle sector, is simply given by a free electron and free bosons in Fock representation.
Such a physical picture fails in the true (without any infrared regularization) Nelson's massless model and two alternative scattering descriptions are considered in the paper.
The first one tries a generalized Haag-Ruelle theory, by a limiting construction analogous to the one already showed in the "fixed charge approximation" of the model. This approach is discussed and developed in the next sections of the present thesis, where it is proved to be consistent.
The second one assumes the existence of the asymptotic boson free algebra to define the generators of time-space translations of the asymptotic charge as a difference (for details see [3]). Similar concepts were later exploited for Q.E.D. (see [14]) in the Wightman framework of quantum field theory .

The first approach requires a careful analysis of the one-particle improper states or in other words of the one-particle states corresponding to smaller and smaller infrared-cutoff σ. The "spectrally simplified model" indeed hints at the absence of one-particle state and a coherent structure less trivial than in the "fixed charge approximation" model.
The study of the ground state of the hamiltonians $H_{\mathbf{P}}$ is performed by a method already used by Glimm and Jaffe [33]. Very roughly, the method consists in (for details see [31]):
- the discretization of the photon momentum space, which implies that the free hamiltonian ($H^o_{\mathbf{P}} = = \frac{(\mathbf{P}-\mathbf{P}^{ph})^2}{2m} + H^{ph}$) spectrum is discrete; since the (discretized) interaction is a small Kato-perturbation, the (discretized) hamiltonian $H_{\mathbf{P}}$ has discrete spectrum as well and, in particular, it has a non-degenerate ground state;
- the removal of the discretization, which is the difficult step.
For Nelson's massless model, the main achieved results are:
- the absence, in the not-infrared regularized case, of the ground state for $H_{\mathbf{P}}$ in the Hilbert space $\mathrm{H}_{\mathbf{P}} \simeq F$ and its existence in the $\mathbf{P}-$dependent coherent representation of $\{b(\mathbf{k}), b^\dagger(\mathbf{k})\}$ with coherent function:

$$-\frac{g}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E(\mathbf{P})\right)} \qquad for \quad \mathbf{k}\to 0$$

- the following properties for the ground energy $E(\mathbf{P})$:

- $E(\mathbf{P}) = E(|\mathbf{P}|)$ is absolutely continuous $\Rightarrow \frac{\partial E(\mathbf{P})}{\partial|\mathbf{P}|}$ exists almost everywhere;
- $\left|\frac{\partial E(\mathbf{P})}{\partial|\mathbf{P}|}\right| < 1$ for $|\mathbf{P}| < m$ in the non-relativistic case, for each $\mathbf{P}$ in the relativistic case

($\sqrt{\mathbf{p}^2+m^2}$ instead of $\frac{\mathbf{p}^2}{2m}$ in the hamiltonian).

Other important properties for $E(\mathbf{P})$ are mentioned and used in chapter 1.
A crucial technical ingredient enters in the above results: the pull-through formula. It is concerned with the action of $b(\mathbf{k})$ on the ground state $\psi^{\sigma}_{\mathbf{P}}$ of the hamiltonian $H_{\mathbf{P},\sigma}$(i.e. with a σ infrared-cut interaction):

$$b(\mathbf{k})\,\psi^{\sigma}_{\mathbf{P}}=\frac{g}{\sqrt{2\,|\mathbf{k}|}}\left(\frac{1}{E^{\sigma}(\mathbf{P})-|\mathbf{k}|-H_{\mathbf{P}-\mathbf{k},\sigma}}\right)\psi^{\sigma}_{\mathbf{P}}\qquad \sigma\le|\mathbf{k}|\le\kappa\,.$$

The formula above plays an important role (also in next chapters) because it contains a structural information about the logarithmic divergence, in the infrared limit, of the boson number evaluated on the ground state. It also accounts for the "asymptotic free" infrared character of the model, and of Q.E.D. by analogy.

As far as the scattering is concerned, the (first) approach in [3] consists in the removal of the infrared cut-off σ in a time-dependent expression, $\psi_h(t)$, explained below. The limiting vector $\lim_{t\to\infty}\psi_h(t)$, has to represent, in Heisenberg picture, a freely moving electron surrounded by a freely moving soft photon cloud. The wave function for the asymptotic photons is suggested by the spectral analysis, according to the same reasoning discussed in the approximated models. The comparison with perturbative results provides further confirmations for the considered construction.
To construct the wave function of such a generic vector that we call $\psi_h(t)$, let us start from the wave function of a one-particle state with respect to the charged particle position operator and the photon momentum variables. Let it be given by the sequence

$$\left\{\psi^{\sigma^{(n)}}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n)\right\}\qquad (\sum_n\int\left|\psi^{\sigma^{(n)}}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n)\right|^2 d^3x d^3k_1...d^3k_n<\infty)$$

where

$$\psi^{\sigma^{(n)}}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n)=\int e^{i\mathbf{p}\cdot\mathbf{x}}\widetilde{\psi}^{\sigma^{(n)}}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3p\quad \widetilde{\psi}^{\sigma^{(n)}}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)\ simm.$$

(in $\psi_h(t)$, the h is referred to the wave function in $\mathbf{P}$ of the one-particle state according to the notations later used in chapter 4; in the above expression the $\mathbf{P}$-dependence is hidden in $\widetilde{\psi}^{\sigma^{(n)}}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)$).
By the substitution $\mathbf{p}=\mathbf{P}-\mathbf{k}_1-...-\mathbf{k}_n$

$$\psi^{\sigma^{(n)}}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n)=\int e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i(\mathbf{k}_1+..+\mathbf{k}_n)\cdot\mathbf{x}}\widetilde{\psi}^{\sigma^{(n)}}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3p=\int e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}}h(\mathbf{P})\,\psi^{\sigma^{(n)}}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3\mathbf{P}$$

where

$$\widetilde{\psi}^{\sigma^{(n)}}_{\mathbf{p}=\mathbf{P}-\mathbf{k}_1-..-\mathbf{k}_n}(\mathbf{k}_1,...,\mathbf{k}_n)=\frac{1}{(2\pi)^{\frac{3}{2}}}h(\mathbf{P})\,\psi^{\sigma^{(n)}}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n)\qquad with\qquad \sum_n\int\left|\psi^{\sigma^{(n)}}_{\mathbf{P}}\right|^2 d^3k_1....d^3k_n=1$$

Since we know the action of $\{a(\mathbf{k}),a^{\dagger}(\mathbf{k})\}$ on $\widetilde{\psi}^{\sigma^{(n)}}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)$, we can consider an expression like

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int\int a(\mathbf{k})\,\frac{g\chi^{\kappa}_{0}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)}e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}}h(\mathbf{P})\,\psi^{\sigma^{(n)}}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3P d^3k=$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \int a(\mathbf{k}) \frac{g\chi_0^{\kappa}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} e^{i\mathbf{P}\cdot\mathbf{x}} e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}} \widetilde{\psi}^{\sigma(n)}_{\mathbf{P}-\mathbf{k}_1-..-\mathbf{k}_n}(\mathbf{k}_1,...,\mathbf{k}_n)\, d^3P d^3k =$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{i\mathbf{P}\cdot\mathbf{x}} e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}} \int a(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{g\chi_0^{\kappa}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} \psi^{\sigma(n)}_{\mathbf{P}-\mathbf{k}_1-..-\mathbf{k}_n}(\mathbf{k}_1,...,\mathbf{k}_n)\, d^3P d^3k =$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{n} \int e^{i\mathbf{P}^{ph}\cdot\mathbf{x}} e^{-i\mathbf{P}\cdot\mathbf{x}} \int \frac{g\chi_{\sigma}^{\kappa}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} e^{-i\mathbf{k}\cdot\mathbf{x}} \widetilde{\psi}^{\sigma(n)}_{\mathbf{P}-\mathbf{k}-..-\mathbf{k}_{j-1}-\mathbf{k}_{j+1}-.-\mathbf{k}_n}(\mathbf{k}_1,.\mathbf{k}_{j-1},\mathbf{k},\mathbf{k}_{j+1}.,\mathbf{k}_n)\, d^3k d^3P$$

analogous reasonings for the action of $\int a^{\dagger}(\mathbf{k}) \frac{g\chi_{\sigma}^{\kappa}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} d^3k$ and of H^{ph}.

On the basis of these definitions, after having expanded the exponential, an expression like

$$\left(e^{-iHt}\psi_h(t)\right)^{(n)}(\mathbf{x},\mathbf{k}_1,..\mathbf{k}_n) = \int \left(e^{-iH^{ph}t} e^{-g\int_{\sigma}^{\kappa_1} \frac{a(\mathbf{k})-a^{\dagger}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} d^3k} e^{iH^{ph}t} e^{-iE^{\sigma}(\mathbf{P})t} e^{i\mathbf{P}\cdot\mathbf{x}} e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}} h(\mathbf{P})\,\psi_{\mathbf{P}} \right)^{(n)} d^3P\,(\mathbf{k}_1,..\mathbf{k}$$

$$= \int e^{i\mathbf{P}\cdot\mathbf{x}} e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}} \left(e^{-g\int_{\sigma}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}-a^{\dagger}(\mathbf{k})e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} d^3k} e^{-iE^{\sigma}(\mathbf{P})t} h(\mathbf{P})\,\psi_{\mathbf{P}} \right)^{(n)} d^3P\,(\mathbf{k}_1,..\mathbf{k}_n)$$

can be handled.

For a fixed infrared cut-off σ, the time convergence of $\psi_h(t)$, is obtained according to Heep's method [34]. The proof is possible thanks to the estimates for the action of polynomials of $\{b(\mathbf{k})\}$ on the ground state $\psi^g_{\mathbf{P}}$ (generalized pull-through formula, see [3]) and because of an implicit propagation estimate for the electron, contained in the constraint $|\nabla E^{\sigma}(\mathbf{P})| < 1$.
The difficulties in handling the analogous expression without the cut-off σ are discussed in Part III, where an alternative expression is constructed to avoid technical obstructions.

Part II

Spectral analysis.

In this section, we shall mainly concern ourselves with the fate of one-particle states in Nelson's massless model, when no infrared regularization is considered in the interaction. Such a study consists in the determination of the limit for $\sigma \to 0$ of the ground states of the hamiltonians at a fixed total momentum $\mathbf{P}$ and with an infrared cut-off σ (in the interaction term). For this purpose, we use an iterative procedure different from the operatorial renormalization group [35]. It provides the strong convergence of the ground state, with an error estimable in terms of the infrared cut-off that we have to remove, if the hamiltonians are properly transformed and for small coupling constant g. Our method is based on the analytic perturbation; it exploits the "smallness" of the variation of the interaction term when we slightly modify the infrared energy scale.
The strong convergence proved in this section is a fundamental ingredient for our discussion of scattering in chapters 4 and 5. In addition, the performed constructive method aims at shedding light on the physical content of the limit states, which have been already found non-constructively [3]. By this method, we intend to partially answer simple questions, in particular, how two ground states at different cut-off or at different $\mathbf{P}$ are related, in which sense and to what extent the expected regularity, of certain physical quantities, is conserved.

Survey of results

Taking into account the definition of Nelson's model (Part I, pag 8), our first concern (chapter 1) is to construct the ground eigenvectors $\psi_{\mathbf{P}}^{\sigma_j}$ of the hamiltonians $H_{\mathbf{P},\sigma_j}$ acting on $\mathrm{H}_{\mathbf{P}} \simeq F$ and with an infrared cut-off σ_j in the interaction term

$$H_{\mathbf{P},\sigma_j} = \frac{\left(\mathbf{P}^{ph} - \mathbf{P}\right)^2}{2m} + g\int_{\sigma_j}^{\kappa} \left(b\left(\mathbf{k}\right) + b^{\dagger}\left(\mathbf{k}\right)\right) \frac{d^3k}{\sqrt{2\left|\mathbf{k}\right|}} + H^{ph}$$

where $\mathbf{P}$ belongs to $\Sigma \equiv \left\{\mathbf{P} : |\mathbf{P}| \leq \frac{m}{20}\right\}$, $\sigma_j = \kappa\epsilon^{\frac{j}{2}}$, $j \in N$, $0 < \epsilon < \left(\frac{1}{5}\right)^8$ and for small values of g (uniform in j).
In this construction, the underlying idea is to break the interaction and to construct the vector $\psi_{\mathbf{P}}^{\sigma_{j+1}}$ in terms of $\psi_{\mathbf{P}}^{\sigma_j}$ by iteration of the analytic perturbation. The "small" and analytic perturbation for the hamiltonian $H_{\mathbf{P},\sigma_j}$ is represented by the difference of the interaction terms

$$\Delta H_{\mathbf{P}} \left|_{\sigma_{j+1}}^{\sigma_j}\right. \equiv H_{\mathbf{P},\sigma_{j+1}} - H_{\mathbf{P},\sigma_j}$$

at subsequent infrared cutoffs σ_j and at fixed coupling constant g.
In developing this technique, the tensorial structure of the Fock space is crucial: it means that if the one-particle Hilbert space h is a direct sum $h_1 \oplus h_2$, then the bosonic Hilbert space F over h, $F\left(h\right)$, is isomorphic to $F_1 \otimes F_2$, where F_1 is the Fock space over h_1 and F_2 is the Fock space over h_2.
The technique substantially relies on the comparison between the resolvents of the hamiltonians $H_{\mathbf{P},\sigma_j}$ and $H_{\mathbf{P},\sigma_{j+1}}$; it recursively uses the Kato-Rellich theorem on the analytic perturbation of isolated eigenvalues (of self adjoint operators) to relate the corresponding ground eigenvectors $\psi_{\mathbf{P}}^{\sigma_j}$

and $\psi_{\mathbf{P}}^{\sigma_{j+1}}$.
At each step two pieces of information are required:

1) a lower bound for the gap of the hamiltonian $H_{\mathbf{P},\sigma_j}$ restricted to the subspace

$$F^{+}_{\sigma_{j+1}} \equiv F(h) \quad , \qquad h \equiv L^2\left(R^3 \setminus O_{\sigma_{j+1}}, d^3k\right) \; , \; O_{\sigma_{j+1}} \equiv \{\mathbf{k} : |\mathbf{k}| < \sigma_{j+1}\} \quad ;$$

2) an estimate of the difference $\Delta H_{\mathbf{P}} \mid^{\sigma_j}_{\sigma_{j+1}} \equiv H_{\mathbf{P},\sigma_{j+1}} \mid_{F^{+}_{\sigma_{j+1}}} - H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_{j+1}}}$ between two subsequent infrared cutoff hamiltonians; this is small with respect to $H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_{j+1}}}$ in a generalized sense, which means that it is possible to expand the spectral projection of $H_{\mathbf{P},\sigma_{j+1}} \mid_{F^{+}_{\sigma_{j+1}}}$, on the ground eigenvalue, in a perturbative series in terms of the resolvent of $H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_{j+1}}}$ and of the difference $\Delta H_{\mathbf{P}} \mid^{\sigma_j}_{\sigma_{j+1}}$.

The requirement 1) is provided by lemma 1.1, concerning the study of the ground eigenvector of the operator $H_{\mathbf{P},\sigma_j}$ restricted to the subspace $F^{+}_{\sigma_{j+1}}$. The result is that, if $\psi_{\mathbf{P}}^{\sigma_j}$ is the unique (up to a phase) ground eigenvector of $H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_j}}$ of energy $E_{\mathbf{P}}^{\sigma_j}$ with gap bigger than $\frac{\sigma_j}{2}$, then $H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_{j+1}}}$ has unique ground eigenvector $\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0$ (ψ_0 vacuum state) with a gap bigger than $\frac{3}{5}\sigma_{j+1}$.
The meaning of this result is that, if the operator $H_{\mathbf{P},\sigma_j}$ is applied on the larger space $F^{+}_{\sigma_{j+1}}$ keeping fixed the interaction till the cutoff σ_j, the contribution of the new terms which appear is positive in such a way that the ground state is the same as for $H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_j}}$ and the gap has a substantially "free" behavior (i.e is of order σ_{j+1}).

The requirement 2) is provided by lemma 1.3. Except for an ordinary factorization (see [35]) in the series expansion of the resolvent and a crucial consideration based on the joined spectral decomposition of commuting observables, the result is only due to the relevant estimate

$$\left\| \int f(\mathbf{k})\, b(\mathbf{k})\, d^3k \psi \right\| \leq \left(\int \frac{|f(\mathbf{k})|^2}{|\mathbf{k}|} d^3k \right)^{\frac{1}{2}} \left\| H^{ph\frac{1}{2}} \psi \right\|$$

(where the expression on the right side is supposed to be well defined).
Thanks to lemma 1.3, we can establish in theorem 1.4 that $\frac{1}{H_{\mathbf{P},\sigma_{j+1}} \mid_{F^{+}_{\sigma_{j+1}}} - E}$ has an analytic expansion in terms of $\frac{1}{H_{\mathbf{P},\sigma_j} \mid_{F^{+}_{\sigma_{j+1}}} - E}$ and the difference $\Delta H_{\mathbf{P}} \mid^{\sigma_j}_{\sigma_{j+1}}$, at small, but fixed, coupling constant g and for a properly chosen E. Then we define $\psi_{\mathbf{P}}^{\sigma_{j+1}}$ as $P_{\sigma_{j+1}} \psi_{\mathbf{P}}^{\sigma_j}$, where $P_{\sigma_{j+1}}$ is the spectral projection associated to $H_{\mathbf{P},\sigma_{j+1}}$ and centered on the ground eigenvalue of $H_{\mathbf{P},\sigma_j}$

$$\psi_{\mathbf{P}}^{\sigma_{j+1}} \equiv -\frac{1}{2\pi i} \oint \frac{1}{H_{\mathbf{P},\sigma_{j+1}} - E} dE\, \psi_{\mathbf{P}}^{\sigma_j}$$

where E belongs to a proper circle of integration in the complex plane according to the estimate of the gap. We obtain $\psi_{\mathbf{P}}^{\sigma_{j+1}}$ as $\psi_{\mathbf{P}}^{\sigma_j}$ plus a finite g−dependent reminder of order 1. Moreover the gap associated to $\psi_{\mathbf{P}}^{\sigma_{j+1}}$ is bigger than $\frac{1}{2}\sigma_j$ and $\left\|\psi_{\mathbf{P}}^{\sigma_{j+1}}\right\| \geq c \left\|\psi_{\mathbf{P}}^{\sigma_j}\right\|$ where $0 < c < 1$, provided g is sufficiently small. Such results allow us to implement the iteration and to construct the sequence $\{\psi_{\mathbf{P}}^{\sigma_j}\}$ thereby.

Then (chapter 2) we deal with the problem of the convergence of $\{\psi_{\mathbf{P}}^{\sigma_j}\}$ and we are forced to discuss a related sequence $\{\phi_{\mathbf{P}}^{\sigma_j}\}$.
Mathematically, we are faced with a problem of perturbation of eigenvalues at the threshold of the continuum spectrum, more specifically of the ground eigenvalue of the hamiltonian

$$H_{\mathbf{P}}^{0}=\frac{\left(\mathbf{P}^{ph}-\mathbf{P}\right)^{2}}{2m}+H^{ph}.$$

If the exponent of $|\mathbf{k}|$ in the interaction term of the hamiltonian $H_{\mathbf{P}}$ were larger than $-\frac{1}{2}$, the norm estimates about the resolvents would be sufficient not only to construct the sequence $\{\psi_{\mathbf{P}}^{\sigma_j}\}$ but also to gain the convergence. The relativistic field case $-\frac{1}{2}$ is a limiting case which requires inequivalent representations of the variables $\{b(\mathbf{k}), b^{\dagger}(\mathbf{k})\}$ at different $\mathbf{P}$. After having performed the required coherent transformation, a strong estimate of the series expansion of the difference between two subsequent ground eigenvectors is sufficient to prove the convergence. The known coherent transformation (in this respect see [3]) is reobtained thanks to an heuristic proof based on a virial type argument. Starting from the assumption of a ground state "coherent in the infrared region", such an argument works out the representation given by the following intertwiner

$$W(\nabla E(\mathbf{P}))=e^{-g\int_0^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E(\mathbf{P})\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}},$$

where $\nabla E(\mathbf{P})$ is the gradient of the ground energy (as a function of the total momentum $\mathbf{P}$).
Taking care of the above expression and by analogy with the *simplified spectral model* (Part I, paragraph 0.2) we turn to consider the transformed hamiltonians

$$H_{\mathbf{P},\sigma}^{w}\equiv W_{\sigma}\left(\nabla E^{\sigma}(\mathbf{P})\right)H_{\mathbf{P},\sigma}W_{\sigma}^{\dagger}\left(\nabla E^{\sigma}(\mathbf{P})\right)$$

where $W_{\sigma}\left(\nabla E^{\sigma}(\mathbf{P})\right)=e^{-g\int_{\sigma}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$.
Then we realize that $H_{\mathbf{P},\sigma}^{w}$ can be rearranged in the following "canonical" form:

$$\frac{1}{2m}\left(\Pi_{\mathbf{P},\sigma}-\frac{1}{\|\phi_{\mathbf{P}}^{\sigma}\|^{2}}\cdot\left(\phi_{\mathbf{P}}^{\sigma},\Pi_{\mathbf{P},\sigma}\phi_{\mathbf{P}}^{\sigma}\right)\right)^{2}+\int_{\sigma}^{\infty}\left(|\mathbf{k}|-\mathbf{k}\cdot\nabla E^{\sigma}(\mathbf{P})\right)b^{\dagger}(\mathbf{k})\,b(\mathbf{k})\,d^{3}k+c_{\mathbf{P}}(\sigma)$$

where

- $\Pi_{\mathbf{P},\sigma}=\mathbf{P}^{ph}-g\int_{\sigma}^{\kappa}\frac{\mathbf{k}\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)}d^{3}k$
- $\phi_{\mathbf{P}}^{\sigma}$ is the (unique) ground eigenvector of $H_{\mathbf{P},\sigma}^{w}$ (note that no problem with the normalization and the phase of $\phi_{\mathbf{P}}^{\sigma}$ arises in the above expression)
- $c_{\mathbf{P}}(\sigma)$ is an additive constant.

An iteration procedure as in section 1 can be performed on the hamiltonians $H^{w}_{\mathbf{P},\sigma_j}$ to construct the sequence of the corresponding ground states $\phi^{\sigma_j}_{\mathbf{P}}$, using the spectral information known for the hamiltonians $H_{\mathbf{P},\sigma_j}$. We find that thanks to the property

$$\left\{ \Pi^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}} - \frac{1}{\left\|\phi^{\sigma_j}_{\mathbf{P}}\right\|^2}\cdot\left(\phi^{\sigma_j}_{\mathbf{P}},\Pi^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right)\phi^{\sigma_j}_{\mathbf{P}} \right\} \perp \phi^{\sigma_j}_{\mathbf{P}} \quad i=1,2,3$$

the remainder $\phi^{\sigma_{j+1}}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}}$ is of order of some positive power of σ_j, in contrast to the sequence $\{\psi^{\sigma_j}_{\mathbf{P}}\}$. The final result is the content of **theorem 2.3** and **corollary 2.4** in which we prove the strong convergence of the sequence $\{\phi^{\sigma_j}_{\mathbf{P}}\}$ to a vector $\phi_{\mathbf{P}}$ in $\mathrm{H}_{\mathbf{P}} \simeq \mathbf{F}$.

As already seen in the *simplified spectral model*, we obtain that the sequence $\{\psi^{\sigma_j}_{\mathbf{P}}\}$ goes weakly to zero in $\mathrm{H}_{\mathbf{P}}$ and it converges to a vector $\psi_{\mathbf{P}}$ in the representation of $\{b(\mathbf{k}), b^{\dagger}(\mathbf{k})\}$ given by the non-Fock coherent transformation $W(\nabla E(\mathbf{P}))$. Since the representations of $\{b(\mathbf{k}), b^{\dagger}(\mathbf{k})\}$, associated to the intertwiners $W(\nabla E(\mathbf{P}))$, are not equivalent for different $\mathbf{P}$, the construction of a state "$\int \psi_{\mathbf{P}} d^3P$" is physically meaningless (it requires the superselection of the total momentum) [3]. For the construction of the vector $\int \phi_{\mathbf{P}} d^3P$ in H, we define a strong convergent vector $\phi^{\sigma}_{\mathbf{P}}$ for $\sigma \to 0$, where σ lies in the continuum. It carries a (strong) Hoelder property with respect to $\mathbf{P}$, uniformly in σ. All this matter is discussed in chapter 3. Such points are crucial in the construction of the asymptotic states in the scattering theory developed in Part III, chapters 4,5 (see also [3]).

1 Construction of the sequence $\{\psi^{\sigma_j}_{\mathbf{P}}\}$.

In the present section we only construct the sequence of eigenvectors of the hamiltonians $H_{\mathbf{P},\sigma_j}$. In order to do it, we introduce some preliminary lemmas (1.1, 1.2, 1.3) which are necessary to perform the iterative procedure in theorem 1.4. Finally in corollary 1.5 the sequence $\{\psi^{\sigma_j}_{\mathbf{P}}, j \in N\}$ is constructed. Lemma 1.3 is crucial in the proof of theorem 1.4. Starting from the relation between the resolvents of the hamiltonians $H_{\mathbf{P},\sigma_{j+1}}$ and $H_{\mathbf{P},\sigma_j}$, it allows us to establish that the norm difference between the corresponding ground eigenvectors $\psi^{\sigma_{j+1}}_{\mathbf{P}}$ and $\psi^{\sigma_j}_{\mathbf{P}}$ is of order 1.

The initial hypotheses are:

I) the considered infrared cut-off are $\sigma_j = \kappa\epsilon^{\frac{j}{2}}$ *where* $0<\epsilon<\left(\frac{1}{5}\right)^8$, $j \in N$;
II) the momenta $\mathbf{P}$ *are restricted to the set* $\Sigma \equiv \{\mathbf{P} : |\mathbf{P}| \le \frac{m}{20}\}$;
III) the relation $2\pi g^2\kappa + \frac{3}{5}\kappa\epsilon^{\frac{1}{2}} \le \frac{m}{200}$ *has to be satisfied.*

We synthesize the content of the three lemmas:

- in lemma 1.1 we study the operator $H_{\mathbf{P},\sigma_j}$ restricted to the subspace $F^{+}_{\sigma_{j+1}}$; under the initial assumptions, we prove that if $\psi^{\sigma_j}_{\mathbf{P}}$ is the unique ground eigenvector of $H_{\mathbf{P},\sigma_j}\mid_{F^{+}_{\sigma_{j+1}}}$ with gap bigger than $\frac{\sigma_j}{2}$, then $\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0$ is the unique ground eigenvector of $H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}}}$ of energy $E_{\mathbf{P}}^{\sigma_j}$ and the corresponding gap is bigger than $\frac{3}{5}\sigma_{j+1}$;

- in lemma 1.2 the ground energy is checked to be not decreasing in the infrared cut-off: $E_{\mathbf{P}}^{\sigma_j} \geq E_{\mathbf{P}}^{\sigma_{j+1}}$;

- in lemma 1.3 the meaning of the "smallness" of

$$\Delta H_{\mathbf{P}} \mid^{\sigma_j}_{\sigma_{j+1}} \equiv H_{\mathbf{P},\sigma_{j+1}} \mid_{F^+_{\sigma_{j+1}}} - H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}}}$$

with respect to $H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}}}$ is explained.

Remark

The ultraviolet *cut-off* κ and the mass m, with the initial constraints, are fixed. The value of g ($g > 0$) will be constrained several times during the procedure; at each time we call g the maximum value such that the constraint under examination as well as the previous constraints are satisfied. In chapter 2. we have to assume a small ratio $\frac{\kappa}{m}$ to prove the convergence of the transformed sequence $\{\phi_{\mathbf{P}}^{\sigma_j}\}$.

Lemma 1.1

If $\psi_{\mathbf{P}}^{\sigma_j}$ is the unique ground eigenvector of $H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_j}}$ with corresponding gap bigger than $\frac{\sigma_j}{2}$, then $\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0$ is the unique ground eigenvector of $H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}}}$ with the same eigenvalue $E_{\mathbf{P}}^{\sigma_j}$ and its gap is bigger than $\frac{3}{5}\sigma_{j+1}$.

Proof.

Let us decompose $F^+_{\sigma_{j+1}}$ as $F^+_{\sigma_j} \otimes F^{\sigma_j}_{\sigma_{j+1}}$, where $F^{\sigma_j}_{\sigma_{j+1}}$ is the tensorial sub-product defined as follows

$$F^{\sigma_j}_{\sigma_{j+1}} \equiv F(h) \quad , \qquad h \equiv L^2\left(O^{\sigma_j}_{\sigma_{j+1}}, d^3k\right) \, , \, O^{\sigma_j}_{\sigma_{j+1}} \equiv \{\mathbf{k} : \sigma_{j+1} < |\mathbf{k}| < \sigma_j\} \, .$$

Clearly the vector $\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0$ is an eigenvector of $H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}}}$ with eigenvalue $E_{\mathbf{P}}^{\sigma_j}$ and also we have that

$$H_{\mathbf{P},\sigma_j} : F^+_{\sigma_{j+1}} \ominus \{\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0\} \to F^+_{\sigma_{j+1}} \ominus \{\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0\} \, .$$

where $\{\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0\}$ denotes the subspace generated by $\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0$.
For this reason the gap that we want to estimate can be analyzed starting from $inf\,spec\left\{H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}} \ominus \{\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0\}}\right\}$, if the latter quantity is larger than $E_{\mathbf{P}}^{\sigma_j}$. In this case, the gap corresponds to

$$inf\,spec\left\{H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}} \ominus \{\psi_{\mathbf{P}}^{\sigma_j} \otimes \psi_0\}} - E_{\mathbf{P}}^{\sigma_j}\right\} \, .$$

Since it is useful in lemma 1.3, we prove a stronger result:

$$inf spec\left\{ H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}} \ominus \{\psi^{\sigma_j}_{\mathbf{P}} \otimes \psi_0\}} - \frac{1}{5} H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}} - E^{\sigma_j}_{\mathbf{P}} \right\} \geq \frac{3}{5}\sigma_{j+1}\,.$$

For this purpose, note that $\left[H_{\mathbf{P},\sigma_j}, N \mid^{\sigma_j}_{\sigma_{j+1}}\right] = 0$ where $N \mid^{\sigma_j}_{\sigma_{j+1}} = \int^{\sigma_j}_{\sigma_{j+1}} b^\dagger(\mathbf{k})\, b(\mathbf{k})\, d^3k$; it implies that our search of the infimum can be restricted to the analysis of the mean value of

$$H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}} \ominus \{\psi^{\sigma_j}_{\mathbf{P}} \otimes \psi_0\}} - \frac{1}{5} H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}}$$

on normalized vectors like $\varphi \otimes \eta$ ($\varphi \otimes \eta \perp \psi^{\sigma_j}_{\mathbf{P}} \otimes \psi_0$), where $\varphi \in F^+_{\sigma_j}$ is in the domain of $H_{\mathbf{P},\sigma_j}$, $\eta \in F^{\sigma_j}_{\sigma_{j+1}}$ is in the domain of $H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}}$.
At this point two different proofs are available. The first proof that we present is selfcontained in this paper but it requires a constraint on the ratio $\frac{g^2\kappa}{m}$ (*initial hypothesis III*). The second one requires no constraint for κ and it relies on some properties of the ground energy $E(\mathbf{P})$ proved in [31][1].

First proof.

We distinguish two different regimes depending on $q \equiv \left(\eta\,, H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}} \eta\right)$:

1) $q \leq \frac{m}{20}$
it implies $|\nabla E^{\sigma_j}(\mathbf{P}')| \leq \frac{1}{5}$ where $|\mathbf{P}'| \leq |\mathbf{P}| + q$, so that we can estimate

$$\left(\varphi \otimes \eta, \left\{ H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}} \ominus \{\psi^{\sigma_j}_{\mathbf{P}} \otimes \psi_0\}} - \frac{1}{5} H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}} - E^{\sigma_j}_{\mathbf{P}} \right\} \varphi \otimes \eta \right)$$

from below in terms of

$$min\left\{ \frac{1}{2}\sigma_j\,,\; inf_{\mathbf{q}:\frac{m}{20} \geq |\mathbf{q}| \geq \sigma_{j+1}} \left\{ E^{\sigma_j}_{\mathbf{P}-\mathbf{q}} - E^{\sigma_j}_{\mathbf{P}} + \frac{4}{5}|\mathbf{q}| \right\} \right\}$$

due to the fact that the gap of $H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_j}}$ is bigger than $\frac{\sigma_j}{2}$, by hypothesis. Moreover from the constraint on the gradient and being $\frac{1}{2}\sigma_j \geq \frac{3}{5}\sigma_{j+1}$, the above quantity is always larger than $\frac{3}{5}\sigma_{j+1}$.

2) $q > \frac{m}{20}$
in this case, let us start observing that

$$H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}}} - H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}} + 2\pi g^2 \kappa \geq 0$$

to provide the bound

$$\left(\varphi \otimes \eta, \left\{ H_{\mathbf{P},\sigma_j} \mid_{F^+_{\sigma_{j+1}} \ominus \{\psi^{\sigma_j}_{\mathbf{P}} \otimes \psi_0\}} - \frac{1}{5} H^{ph} \mid^{\sigma_j}_{\sigma_{j+1}} - E^{\sigma_j}_{\mathbf{P}} \right\} \varphi \otimes \eta \right) \geq \frac{4}{5}q - 2\pi g^2\kappa - E^{\sigma_j}_{\mathbf{P}} \geq \frac{m}{25} - 2\pi g^2 \kappa - E^{\sigma_j}_{\mathbf{P}}\,.$$

Now, from the *initial hypotheses* and from next lemma 1.2 we have

$$\frac{m}{25} - 2\pi g^2\kappa - E^{\sigma_j}_{\mathbf{P}} \geq \frac{m}{25} - 2\pi g^2 \kappa - E^{\sigma_0}_{\mathbf{P}} \geq \frac{m}{25} - 2\pi g^2\kappa - \frac{m}{40} \geq \frac{3}{5}\sigma_1 \geq \frac{3}{5}\sigma_{j+1}\,.$$

[1] I am indebted with J.Frohlich for having suggested me such proof and for an helpful discussion of this lemma.

Conclusion

$$inf\,spec\left\{H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}\ominus\{\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0\}}-\frac{1}{5}H^{ph}\mid^{\sigma_j}_{\sigma_{j+1}}-E^{\sigma_j}_{\mathbf{P}}\right\}\geq\frac{3}{5}\sigma_{j+1}\,.$$

Second proof.

It starts with the observation that

$$inf\,spec\left\{H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}\ominus\{\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0\}}-\frac{1}{5}H^{ph}\mid^{\sigma_j}_{\sigma_{j+1}}-E^{\sigma_j}_{\mathbf{P}}\right\}$$

can be estimated in terms of

$$min\left\{\frac{1}{2}\sigma_j\,,\;inf_{\mathbf{q}:|\mathbf{q}|\geq\sigma_{j+1}}\left\{E^{\sigma_j}_{\mathbf{P}-\mathbf{q}}-E^{\sigma_j}_{\mathbf{P}}+\frac{4}{5}|\mathbf{q}|\right\}\right\}\,.$$

Then we exploit the property

$$\left\{inf_{\mathbf{q}:|\mathbf{q}|\geq\sigma_{j+1}}\left\{E^{\sigma_j}_{\mathbf{P}-\mathbf{q}}-E^{\sigma_j}_{\mathbf{P}}+\frac{4}{5}|\mathbf{q}|\right\}\right\}\geq\frac{3}{5}|\mathbf{q}|\geq\frac{3}{5}\sigma_{j+1}$$

which holds for $\mathbf{P}\in\Sigma$ and derives from the concavity of $t(\mathbf{P},\sigma_j)\equiv E^{\sigma_j}_{\mathbf{P}}-\frac{\mathbf{P}^2}{2m}$ and from the inequalities $E^{\sigma_j}_{\mathbf{P}}\geq E^{\sigma_j}_0$, $t(0,\sigma_j)\geq t(\mathbf{P},\sigma_j)$; both the two properties are discussed in [31].

Lemma 1.2

The following relation between $E^{\sigma_j}_{\mathbf{P}}$ and $E^{\sigma_{j+1}}_{\mathbf{P}}$ (ground energy of $H_{\mathbf{P},\sigma_{j+1}}\mid_{F^+_{\sigma_{j+1}}}$) holds:

$$E^{\sigma_j}_{\mathbf{P}}\geq E^{\sigma_{j+1}}_{\mathbf{P}}\geq E^{\sigma_j}_{\mathbf{P}}-10\pi g^2\sigma_j\,.$$

Proof.

Considering that $H_{\mathbf{P},\sigma_{j+1}}\mid_{F^+_{\sigma_{j+1}}}=H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}+1\otimes g\int^{\sigma_j}_{\sigma_{j+1}}\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)\frac{d^3\mathbf{k}}{\sqrt{2}|\mathbf{k}|^{\frac{1}{2}}}$, the expectation value of $H_{\mathbf{P},\sigma_{j+1}}\mid_{F^+_{\sigma_{j+1}}}$ on $\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0$

$$\frac{\left(\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0,\,H_{\mathbf{P},\sigma_{j+1}}\mid_{F^+_{\sigma_{j+1}}}\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0\right)}{\left\|\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0\right\|^2}$$

coincides with $E^{\sigma_j}_{\mathbf{P}}$. $E^{\sigma_{j+1}}_{\mathbf{P}}$ is the infimum of the mean value of $H_{\mathbf{P},\sigma_{j+1}}\mid_{F^+_{\sigma_{j+1}}}$ on the normalized vectors in $F^+_{\sigma_{j+1}}$ belonging to the operator domain, by definition. Therefore $E^{\sigma_{j+1}}_{\mathbf{P}}\leq E^{\sigma_j}_{\mathbf{P}}$ and in general $E^{\sigma_2}_{\mathbf{P}}\leq E^{\sigma_1}_{\mathbf{P}}$ for $\sigma_1>\sigma_2$. Moreover:

- as proved in the previous lemma

$$\inf spec\left(H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}-\frac{1}{5}H^{ph}\mid^{\sigma_j}_{\sigma_{j+1}}\right)>E^{\sigma_j}_{\mathbf{P}}$$

- being a square

$$\frac{1}{5}H^{ph}\mid_{\sigma_{j+1}}^{\sigma_j}+g\int_{\sigma_{j+1}}^{\sigma_j}\left(b\left(\mathbf{k}\right)+b^{\dagger}\left(\mathbf{k}\right)\right)\frac{d^3k}{\sqrt{2}\left|\mathbf{k}\right|^{\frac{1}{2}}}+10\pi g^2\left(\sigma_j-\sigma_{j+1}\right)\geq 0$$

so that

$$E_{\mathbf{P}}^{\sigma_{j+1}}=\inf spec\left(H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}+g\int_{\sigma_{j+1}}^{\sigma_j}\left(b\left(\mathbf{k}\right)+b^{\dagger}\left(\mathbf{k}\right)\right)\frac{d^3k}{\sqrt{2}|\mathbf{k}|^{\frac{1}{2}}}\right)\geq$$
$$\geq \inf spec\left(H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}-\tfrac{1}{5}H^{ph}\mid_{\sigma_{j+1}}^{\sigma_j}-10\pi g^2\sigma_j\right)\geq E_{\mathbf{P}}^{\sigma_j}-10\pi g^2\sigma_j.$$

Lemma 1.3

For fixed and properly small g, $(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\equiv g\int_{\sigma_{j+1}}^{\sigma_j}\left(b\left(\mathbf{k}\right)+b^{\dagger}\left(\mathbf{k}\right)\right)\frac{d^3\mathbf{k}}{\sqrt{2|\mathbf{k}|}}$ is small of order 1 with respect to $H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}$ in the following sense:

given $E\left(j+1\right)\in\mathcal{C}$ such that $\left|E\left(j+1\right)-\inf spec\left(H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}\right)\right|=\left|E\left(j+1\right)-E_{\mathbf{P}}^{\sigma_j}\right|=\frac{11}{20}\sigma_{j+1}$

$$\frac{1}{H_{\mathbf{P},\sigma_j}+(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}-E\left(j+1\right)}\mid_{F^+_{\sigma_{j+1}}}=\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\sum_{n=0}^{+\infty}\left(-(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\right)^n\mid_{F^+_{\sigma_{j+1}}}$$

and

$$\left\|\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\left(-(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\right)^n\right\|_{F^+_{\sigma_{j+1}}}\leq\frac{20\left(C\left(g,m\right)\right)^n}{\sigma_{j+1}}$$

where $0<C\left(g,m\right)<\frac{1}{12}$ is a constant independent of σ_j.

Proof.

Let us analyze the n^{th} term of the sum

$$\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\sum_{n=0}^{+\infty}\left(-(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\right)^n\mid_{F^+_{\sigma_{j+1}}}$$

$$(-1)^n\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\cdots\cdots\cdots(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}=$$
$$=(-1)^n\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\cdots\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}(\Delta H_{\mathbf{P}})_{\sigma_{j+1}}^{\sigma_j}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\cdots\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}$$

where $\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}$ is defined starting from the spectral representation of $H_{\mathbf{P},\sigma_j}$ by using the convention to take the branch of the square root with smaller argument in $(-\pi,\pi]$.

Study of the norm of $\left\{\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}(\Delta H_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\}|_{F^+_{\sigma_{j+1}}}$.

$$\left\|\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}(\Delta H_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}} =$$

$$= \left\|\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\left(g\int_{\sigma_{j+1}}^{\sigma_j}\frac{1}{\sqrt{2}|\mathbf{k}|^{\frac{1}{2}}}\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)d^3k\right)\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}} \leq$$

$$\leq 2g\left\|\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}}\cdot\left\|\left(\int_{\sigma_{j+1}}^{\sigma_j}\frac{1}{\sqrt{2}|\mathbf{k}|^{\frac{1}{2}}}b(\mathbf{k})\,d^3k\right)\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}}$$

if the quantities written above exist.

The following estimate holds:

$$\left\|\int_{\sigma_{j+1}}^{\sigma_j} b(\mathbf{k})\frac{d^3\mathbf{k}}{\sqrt{2\,|\mathbf{k}|}}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}} \leq \sqrt{10\pi}\cdot\sigma_j^{\frac{1}{2}}\,. \qquad (1)$$

Indeed for vectors $\varphi \in D^b \bigcap F^+_{\sigma_{j+1}}$ ($D^b = \bigvee^n \psi^n_{\mathbf{P}}$, see point v), pag. 10,):

$$\left\|\int_{\sigma_{j+1}}^{\sigma_j} b(\mathbf{k})\frac{d^3\mathbf{k}}{\sqrt{2|\mathbf{k}|}}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\varphi\right\|^2_{F^+_{\sigma_{j+1}}} \leq 2\pi\sigma_j\cdot\left(\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\varphi\,,\,H^{ph}|^{\sigma_j}_{\sigma_{j+1}}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\varphi\right) \leq$$

$$\leq 2\pi\sigma_j\cdot\|\varphi\|\cdot\left\|H^{ph}|^{\sigma_j}_{\sigma_{j+1}}\left[\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right]^{\dagger}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\varphi\right\|_{F^+_{\sigma_{j+1}}}$$

(note that $\left[H_{\mathbf{P},\sigma_j}, H^{ph}|^{\sigma_j}_{\sigma_{j+1}}\right] = 0$).

The operator norm of

$$H^{ph}|^{\sigma_j}_{\sigma_{j+1}}\cdot\left[\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right]^{\dagger}\cdot\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}|_{F^+_{\sigma_{j+1}}}$$

can be studied separately on $\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0$ and on $F^+_{\sigma_{j+1}}\ominus\{\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0\}$.
The operator vanishes on $\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0$ (put $H^{ph}|^{\sigma_j}_{\sigma_{j+1}}$on the right). The discussion is then restricted to the subspace $F^+_{\sigma_{j+1}}\ominus\{\psi^{\sigma_j}_{\mathbf{P}}\otimes\psi_0\}$.

As already seen in lemma 1.1, we have

$$\inf spec\left(H_{\mathbf{P},\sigma_j}|_{F^+_{\sigma_{j+1}}} - \frac{1}{5}H^{ph}|^{\sigma_j}_{\sigma_{j+1}}\right) \geq E^{\sigma_j}_{\mathbf{P}} + \frac{3}{5}\sigma_{j+1}$$

from which

$$\inf spec\left(H_{\mathbf{P},\sigma_j}|_{F^+_{\sigma_{j+1}}} - \frac{1}{5}H^{ph}|^{\sigma_j}_{\sigma_{j+1}} - Re E(j+1)\right) \geq \frac{3}{5}\sigma_{j+1} - \frac{11}{20}\sigma_{j+1} > 0\,.$$

Going to the **joined spectral representation** of $H_{\mathbf{P},\sigma_j}$ and $H^{ph}\mid_{\sigma_{j+1}}^{\sigma_j}$, we obtain

$$\left\| H^{ph}\mid_{\sigma_{j+1}}^{\sigma_j} \left[\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\right)^{\frac{1}{2}}\right]^{\dagger}\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E\left(j+1\right)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}} \leq 5\,. \tag{2}$$

Conclusion

For g sufficiently small the thesis is proved since $\left\|\left(\frac{1}{H_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}}$ is of order $\left(\frac{1}{\sigma_{j+1}}\right)^{\frac{1}{2}}$.

Theorem 1.4

The hamiltonian $H_{\mathbf{P},\sigma_{j+1}}\mid_{F^+_{\sigma_{j+1}}}$ has a unique ground eigenvector $\psi_{\mathbf{P}}^{\sigma_{j+1}}$ of energy $E_{\mathbf{P}}^{\sigma_{j+1}}$ and the corresponding gap is bigger than $\frac{\sigma_{j+1}}{2}$; the (unnormalized) vector $\psi_{\mathbf{P}}^{\sigma_{j+1}}$ is so defined

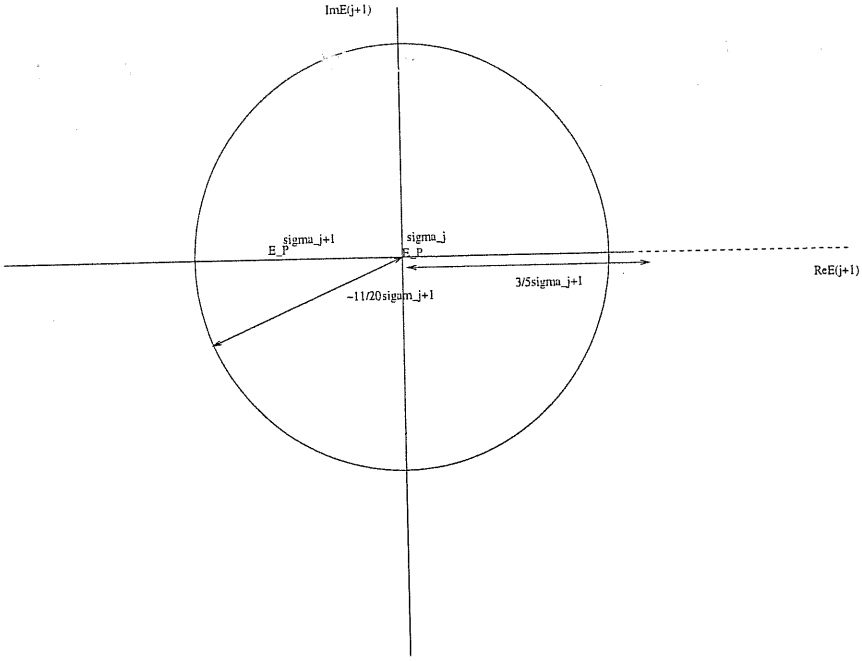
$$\psi_{\mathbf{P}}^{\sigma_{j+1}} \equiv -\frac{1}{2\pi i}\oint \frac{1}{H_{\mathbf{P},\sigma_{j+1}}-E\left(j+1\right)}dE\left(j+1\right)\,\psi_{\mathbf{P}}^{\sigma_j} \tag{3}$$

where $E\left(j+1\right)\in\mathcal{C}$ and $\left|E\left(j+1\right)-E_{\mathbf{P}}^{\sigma_j}\right|=\frac{11}{20}\sigma_{j+1}$.

Proof.

Continuity argument.

We distinguish the coupling constant g in $H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}$ from that one in $\left(\Delta H_{\mathbf{P}}\right)_{\sigma_{j+1}}^{\sigma_j}$, and we denote the latter by $g^{\sharp}$. Kato-Rellich theorem ensures that (3) is verified for sufficiently small $g^{\sharp}$, since the gap of $H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}$ is bigger or equal to $\frac{3}{5}\sigma_{j+1}$ and $\left(\Delta H_{\mathbf{P}}\left(g^{\sharp}\right)\right)_{\sigma_{j+1}}^{\sigma_j}$ is a small Kato perturbation with respect to $H_{\mathbf{P},\sigma_j}\mid_{F^+_{\sigma_{j+1}}}$. Now look at the figure



if $g^\sharp$ increases, the equation (3) is valid till the eigenvalue $E_{\mathbf{P}}^{\sigma_{j+1}}(g^\sharp)$ remains inside the circle of integration and the remaining spectrum of $H_{\mathbf{P},\sigma_{j+1}}|_{\mathcal{F}^+_{\sigma_{j+1}}}$ remains outside of the circle of integration.

There exists a limiting value $\overline{g^\sharp}$ for which the expression $-\frac{1}{2\pi i}\oint \left\| \frac{1}{H_{\mathbf{P},\sigma_{j+1}}|_{\mathcal{F}^+_{\sigma_{j+1}}} - E(j+1)} \right\| dE\,(j+1)$ diverges, because the spectrum intersects the circle of integration.

According to the estimates given in lemma 1.3, we can conclude that:

1. the integral $-\frac{1}{2\pi i}\oint \left\| \frac{1}{H_{\mathbf{P},\sigma_{j+1}}|_{\mathcal{F}^+_{\sigma_{j+1}}} - E(j+1)} \right\| dE\,(j+1)$ exists for $0 \le g^\sharp \le g$;

2. the ground state of $H_{\mathbf{P},\sigma_{j+1}}|_{\mathcal{F}^+_{\sigma_{j+1}}}$ is unique and it is not zero since

$$\psi_{\mathbf{P}}^{\sigma_{j+1}} = \psi_{\mathbf{P}}^{\sigma_j} - \frac{1}{2\pi i}\sum_{n=1}^{+\infty}\oint \frac{1}{H_{\mathbf{P},\sigma_j} - E\,(j+1)}\left(-\left(\Delta H_{\mathbf{P}}\right)^{\sigma_j}_{\sigma_{j+1}} \frac{1}{H_{\mathbf{P},\sigma_j} - E\,(j+1)}\right)^n dE\,(j+1)\,\psi_{\mathbf{P}}^{\sigma_j} \tag{4}$$

   where the norm of the remainder is less than $\frac{11C(g,m)}{1-C(g,m)}\left\|\psi_{\mathbf{P}}^{\sigma_j}\right\|$, therefore

$$\left\|\psi_{\mathbf{P}}^{\sigma_{j+1}}\right\| \ge \left\|\psi_{\mathbf{P}}^{\sigma_j}\right\| - \frac{11C\,(g,m)}{1-C\,(g,m)}\left\|\psi_{\mathbf{P}}^{\sigma_j}\right\| \ge \frac{1-12C\,(g,m)}{1-C\,(g,m)}\left\|\psi_{\mathbf{P}}^{\sigma_j}\right\| > 0\,;$$

3. since for lemma 1.3 $E_{\mathbf{P}}^{\sigma_{j+1}} \le E_{\mathbf{P}}^{\sigma_j}$, the new gap is bigger than $\frac{\sigma_{j+1}}{2}$.

Corollary 1.5

The sequence $\left\{\psi_{\mathbf{P}}^{\sigma_j},\, j\in N\right\}$ is well defined.

Proof.

Thanks to the results of lemmas 1.1, 1.2, 1.3 and theorem 1.4, it is possible to iterate the projection at fixed g, starting from the vacuum state ψ_0 at the level $j=0$. The iteration is consistent and does not stop since the vector obtained at the step $j+1$ has norm bigger than a fixed fraction of the norm of the vector at the j step. At each step the infrared cut-off is reduced by a factor $\epsilon^{\frac{1}{2}}$.

2 Convergence of the ground states of the transformed hamiltonians $H^w_{\mathbf{P},\sigma_j}$.

By analogy with the *simplified spectral model*, we conjecture that the hamiltonians $H_{\mathbf{P}}$ have a ground state for representations of $\left\{b(\mathbf{k}),b^{\dagger}(\mathbf{k})\right\}$ which are coherent in the infrared region ($\mathbf{k}\to 0$). Then an argument is developed which explicitly identifies the expected coherent factor in the case $\mathbf{P}=0$ and implicitly in the case $\mathbf{P}\neq 0$, $\mathbf{P}\in\Sigma$. Such an heuristic information is then used in a rigorous proof which is based on the iterative procedure of construction of the ground state.

Derivation of the coherent factor[2].

Let us assume that $\psi_{\mathbf{P}}$ is an eigenvector of $H_{\mathbf{P}}$ and that it is "coherent in the infrared region", which means

$$b(\mathbf{k})\,\psi_{\mathbf{P}}\approx f_{\mathbf{P}}(\mathbf{k})\,\psi_{\mathbf{P}}\qquad \textit{for }\mathbf{k}\to 0$$

where the meaning of the limit is given only "a posteriori". Then the coherent function $f_{\mathbf{P}}(\mathbf{k})$ has to be such that the following relation is satisfied:

$$(\psi_{\mathbf{P}},[H_{\mathbf{P}},b(\mathbf{k})]\,\psi_{\mathbf{P}})=0\quad for\,\mathbf{k}\to 0\,.$$

From computations the expected behavior for $f_{\mathbf{P}}(\mathbf{k})$ is

$$f_{\mathbf{P}}(\mathbf{k})\approx_{\mathbf{k}\to 0}-\frac{g}{\sqrt{2\,|\mathbf{k}|}}\cdot\frac{1}{\left(|\mathbf{k}|+\frac{|\mathbf{k}|^2}{2m}-\frac{\mathbf{P}\cdot\mathbf{k}}{m}+\frac{\mathbf{k}\cdot\left(\psi_{\mathbf{P}},\mathbf{P}^{ph}\psi_{\mathbf{P}}\right)}{m\cdot\|\psi_{\mathbf{P}}\|^2}\right)}=-\frac{g}{\sqrt{2\,|\mathbf{k}|}}\cdot\frac{1}{\left(|\mathbf{k}|+\frac{|\mathbf{k}|^2}{2m}-\frac{\mathbf{k}\cdot\left(\mathbf{P}-\left\langle\mathbf{P}^{ph}\right\rangle_{\psi_{\mathbf{P}}}\right)}{m}\right)},$$

the coherent factor is therefore labelled by

$$\mathbf{P}-\left\langle\mathbf{P}^{ph}\right\rangle_{\psi_{\mathbf{P}}}\equiv\mathbf{P}-\frac{\left(\psi_{\mathbf{P}},\mathbf{P}^{ph}\psi_{\mathbf{P}}\right)}{\|\psi_{\mathbf{P}}\|^2}.$$

[2] I am indebted with G.Morchio for having suggested me this nice argument and for many discussions and advice.

The argument proves that if the ground state is "coherent in the infrared region", it does not belong to the Fock space. Starting from this result we act with a proper coherent transformation on the variables $\{b(\mathbf{k}), b^\dagger(\mathbf{k})\}$ of the hamiltonian $H_\mathbf{P}$ and we look for a ground state of the so transformed hamiltonian in $\mathrm{H}_\mathbf{P}$.

2.1 Transformed hamiltonians $H^w_{\mathbf{P},\sigma_j}$.

Let us consider the coherent transformation

$$b(\mathbf{k}) \longrightarrow b(\mathbf{k}) - \frac{g}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\frac{\mathbf{P}-\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}}{m}\right)} \qquad for\,\mathbf{k}: \quad \sigma \le |\mathbf{k}| \le \kappa$$

obtained by the unitary operator

$$W_\sigma\left(\frac{\mathbf{P}-\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}}{m}\right) = e^{-g\int_\sigma^\kappa \frac{b(\mathbf{k})-b^\dagger(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\frac{\mathbf{P}-\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}}{m}\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{5}$$

which becomes an intertwiner between inequivalent representations (of $\{b(\mathbf{k}), b^\dagger(\mathbf{k})\}$) in the limit $\sigma = 0$. From the perturbation of the isolated eigenvalue $E^\sigma_\mathbf{P}$ of $H_{\mathbf{P},\sigma}\,|_{F^+_\sigma}$ (see [3]) , one can easily check that $\mathbf{P}-\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}$ corresponds to $m\nabla E^\sigma(\mathbf{P})$.

Note that $\nabla E^\sigma(\mathbf{P})$ satisfies the equation:

$$\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}} = \mathbf{P} - m\nabla E^\sigma(\mathbf{P}) = \frac{1}{\|\phi^\sigma_\mathbf{P}\|^2}(\phi^\sigma_\mathbf{P}, \Pi_{\mathbf{P},\sigma}\phi^\sigma_\mathbf{P}) + g^2\int_\sigma^\kappa \frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)^2}d^3k \tag{6}$$

where $\phi^\sigma_\mathbf{P}$ is ground eigenvector of the transformed hamiltonian

$$W_\sigma(\nabla E^\sigma(\mathbf{P}))\,H_{\mathbf{P},\sigma}W^\dagger_\sigma(\nabla E^\sigma(\mathbf{P}))$$

and $\Pi_{\mathbf{P},\sigma} = \mathbf{P}^{ph} - g\int_\sigma^\kappa \frac{\mathbf{k}\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}d^3k.$

Transformed hamiltonian $W_\sigma(\nabla E^\sigma(\mathbf{P}))\,H_{\mathbf{P},\sigma}W^\dagger_\sigma(\nabla E^\sigma(\mathbf{P}))$.

Let us rewrite $H_{\mathbf{P},\sigma}$, $\mathbf{P} = m\nabla E^\sigma(\mathbf{P}) + \langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}$, as

$$H_\mathbf{P} = \frac{\mathbf{P}^2}{2m} - \frac{\left(m\nabla E^\sigma(\mathbf{P})+\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}\right)\cdot\mathbf{P}^{ph}}{m} + \frac{\mathbf{P}^{ph2}}{2m} + g\int_\sigma^\kappa\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)\frac{d^3\mathbf{k}}{\sqrt{2|\mathbf{k}|}} + H^{ph} =$$

$$= \frac{\mathbf{P}^2}{2m} + \frac{\mathbf{P}^{ph2}}{2m} - \frac{\langle\mathbf{P}^{ph}\rangle_{\psi^\sigma_\mathbf{P}}\cdot\mathbf{P}^{ph}}{m} + \int_\kappa^\infty\left(|\mathbf{k}| - \mathbf{k}\cdot\nabla E^\sigma(\mathbf{P})\right)b^\dagger(\mathbf{k})\,b(\mathbf{k})\,d^3k +$$

$$+\int_\sigma^\kappa (|\mathbf{k}|-\mathbf{k}\cdot\nabla E^\sigma(\mathbf{P}))\left(b^\dagger(\mathbf{k})+\frac{g}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}\right)\left(b(\mathbf{k})+\frac{g}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}\right)d^3k+$$

$$+\int_0^\sigma (|\mathbf{k}|-\mathbf{k}\cdot\nabla E^\sigma(\mathbf{P}))\,b^\dagger(\mathbf{k})\,b(\mathbf{k})\,d^3k-g^2\int_\sigma^\kappa \frac{1}{2|\mathbf{k}|^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}d^3k\,.$$

Performing the coherent transformation (5):

$$W_\sigma(\nabla E^\sigma(\mathbf{P}))\,H_{\mathbf{P},\sigma}W_\sigma^\dagger(\nabla E^\sigma(\mathbf{P}))=$$

$$=\frac{1}{2m}\left(\mathbf{P}^{ph}-g\int_\sigma^\kappa \frac{\mathbf{k}}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)d^3k+g^2\int_\sigma^\kappa \frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)^2}d^3k\right)^2+$$

$$-\frac{\left(\mathbf{P}^{ph}\right)_{\psi_{\mathbf{P},\sigma}}}{m}\left(\mathbf{P}^{ph}-g\int_\sigma^\kappa \frac{\mathbf{k}}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)d^3k+g^2\int_\sigma^\kappa \frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)^2}d^3k\right)+$$

$$+\int_0^\infty (|\mathbf{k}|-\mathbf{k}\cdot\nabla E^\sigma(\mathbf{P}))\,b^\dagger(\mathbf{k})\,b(\mathbf{k})\,d^3k+\frac{\mathbf{P}^2}{2m}-g^2\int_\sigma^\kappa \frac{1}{2|\mathbf{k}|^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}d^3k\,.$$

By substitution we obtain

$$H^w_{\mathbf{P},\sigma}\equiv W_\sigma(\nabla E^\sigma(\mathbf{P}))\,H_{\mathbf{P},\sigma}W_\sigma^\dagger(\nabla E^\sigma(\mathbf{P}))=$$

$$=\frac{1}{2m}\left(\Pi_{\mathbf{P},\sigma}-\frac{1}{\|\phi^\sigma_{\mathbf{P}}\|^2}\cdot(\phi^\sigma_{\mathbf{P}},\Pi_{\mathbf{P},\sigma}\phi^\sigma_{\mathbf{P}})\right)^2+\int_0^\infty (|\mathbf{k}|-\mathbf{k}\cdot\nabla E^\sigma(\mathbf{P}))\,b^\dagger(\mathbf{k})\,b(\mathbf{k})\,d^3k+c_{\mathbf{P}}(\sigma)\qquad(7)$$

where $c_{\mathbf{P}}(\sigma)=\frac{\mathbf{P}^2}{2m}-\frac{1}{2m}\left[\mathbf{P}-m\nabla E^\sigma(\mathbf{P})\right]^2-g^2\int_\sigma^\kappa \frac{1}{2|\mathbf{k}|^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma\right)}d^3k\,.$

From lemma A2, point 1, (appendix A), $|c_{\mathbf{P}}(\sigma)|$ is bounded uniformly in σ for $\mathbf{P}\in\Sigma$. The selfadjointness (s.a.) domain of the transformed hamiltonian $H^w_{\mathbf{P},\sigma}$ coincides with $D^b(H_{\mathbf{P},\sigma})$ (see an analogous proof in [2]).

Definitions

i) To compress formulas in next steps we will use the notation

$$\Gamma^i_{\mathbf{P},\sigma_j}\equiv\Pi^i_{\mathbf{P},\sigma_j}-\frac{\left(\phi^{\sigma_j}_{\mathbf{P}},\Pi^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right)}{\|\phi^{\sigma_j}_{\mathbf{P}}\|^2}\,.$$

ii) In proving the convergence of the sequence $\{\phi^{\sigma_j}_{\mathbf{P}}\}$ we take advantage of the intermediate hamiltonians:

$$\widehat{H}^w_{\mathbf{P},\sigma_{j+1}}\equiv W_{\sigma_{j+1}}(\nabla E^{\sigma_j}(\mathbf{P}))\,H_{\mathbf{P},\sigma_{j+1}}W^\dagger_{\sigma_{j+1}}(\nabla E^{\sigma_j}(\mathbf{P}))=$$

$$= W_{\sigma_{j+1}}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) W^{\dagger}_{\sigma_{j+1}}\left(\nabla E^{\sigma_{j+1}}(\mathbf{P})\right) H^{w}_{\mathbf{P},\sigma_{j+1}} W_{\sigma_{j+1}}\left(\nabla E^{\sigma_{j+1}}(\mathbf{P})\right) W^{\dagger}_{\sigma_{j+1}}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) =$$

$$= \frac{1}{2m}\left(\Gamma_{\mathbf{P},\sigma_j} - g\int_{\sigma_{j+1}}^{\sigma_j} \frac{\mathbf{k}\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)} d^3k + g^2\int_{\sigma_{j+1}}^{\sigma_j} \frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)^2} d^3k\right)^2 +$$

$$+\int_0^{\infty}\left(|\mathbf{k}| - \mathbf{k}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right) b^{\dagger}(\mathbf{k})\, b(\mathbf{k})\, d^3k + \widehat{c}_{\mathbf{P}}(j+1)$$

where $\widehat{c}_{\mathbf{P}}(j+1) = \frac{\mathbf{P}^2}{2m} - \frac{1}{2m}\left[\mathbf{P} - m\nabla E^{\sigma_j}(\mathbf{P})\right]^2 - g^2\int_{\sigma_{j+1}}^{\kappa} \frac{1}{2|\mathbf{k}|^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)} d^3k$.

iii) Analogously we define

$$\widehat{\Pi}^{i}_{\mathbf{P},\sigma_{j+1}} \equiv W_{\sigma_{j+1}}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) W^{\dagger}_{\sigma_{j+1}}\left(\nabla E^{\sigma_{j+1}}(\mathbf{P})\right) \Pi^{i}_{\mathbf{P},\sigma_{j+1}} W_{\sigma_{j+1}}\left(\nabla E^{\sigma_{j+1}}(\mathbf{P})\right) W^{\dagger}_{\sigma_{j+1}}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) =$$

$$= \mathbf{P}^{ph^i} - g\int_{\sigma_{j+1}}^{\kappa} \frac{k^i\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}_{\mathbf{P}}\right)} d^3k + \frac{g^2}{2}\int_{\sigma_{j+1}}^{\kappa} \frac{k^i}{|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}_{\mathbf{P}}\right)^2} d^3k - \frac{g^2}{2}\int_{\sigma_{j+1}}^{\kappa} \frac{k^i}{|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{j+1}}_{\mathbf{P}}\right)^2} d^3k;$$

$\widehat{\Gamma}^{i}_{\mathbf{P},\sigma_{j+1}} \equiv \widehat{\Pi}^{i}_{\mathbf{P},\sigma_{j+1}} - \frac{\left(\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}, \widehat{\Pi}^{i}_{\mathbf{P},\sigma_{j+1}}\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}\right)}{\left\|\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}\right\|^2}$, where $\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}$ is ground eigenvector of $\widehat{H}^{w}_{\mathbf{P},\sigma_{j+1}}$ and is defined in the iterative procedure explained in the next paragraph.

Remarks

1) Note that the two transformations

$$H_{\mathbf{P},\sigma_j} \rightarrow \quad H^{w}_{\mathbf{P},\sigma_j} \equiv W_{\sigma_j}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) H_{\mathbf{P},\sigma_j} W^{\dagger}_{\sigma_j}\left(\nabla E^{\sigma_j}(\mathbf{P})\right)$$

$$H_{\mathbf{P},\sigma_{j+1}} \rightarrow \quad \widehat{H}^{w}_{\mathbf{P},\sigma_{j+1}} \equiv W_{\sigma_{j+1}}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) H_{\mathbf{P},\sigma_{j+1}} W^{\dagger}_{\sigma_{j+1}}\left(\nabla E^{\sigma_j}(\mathbf{P})\right)$$

are different in the infrared cutoff but not in the coherent factor.

2) The hamiltonians $H_{\mathbf{P},\sigma_j}$,$H^{w}_{\mathbf{P},\sigma_j}$ and $\widehat{H}^{w}_{\mathbf{P},\sigma_j}$ are s.a. on the same domain and the formal relations are well defined from an operatorial point of view.

2.2 Convergent sequence.

In order to arrive at a strongly convergent sequence $\{\phi^{\sigma_j}_{\mathbf{P}}\}$ of ground eigenvectors, we start from the vector ψ_0, ψ_0 vacuum state. From the results of the previous chapter and by unitarity the following properties hold $\forall j$ (these properties are exploited in lemma A1, appendix A):

i) $H^{w}_{\mathbf{P},\sigma_j}\mid_{F^{+}_{\sigma_j}}$ has ground eigenvalue $E^{\sigma_j}_{\mathbf{P}}$ with the corresponding gap bigger than $\frac{\sigma_j}{2}$;

ii) $H^{w}_{\mathbf{P},\sigma_j}\mid_{F^{+}_{\sigma_{j+1}}}$ has ground eigenvalue $E^{\sigma_j}_{\mathbf{P}}$ with the corresponding gap bigger than $\frac{3}{5}\sigma_{j+1}$.

Comparing the resolvents of the hamiltonians $H^{w}_{\mathbf{P},\sigma_j}$ and $\widehat{H}^{w}_{\mathbf{P},\sigma_{j+1}}$ we can build $\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}$in terms of $\phi^{\sigma_j}_{\mathbf{P}}$ by projection, thanks to the estimates cointained in Lemma A1, in Appendix, which is the analogue of lemma 1.3 for the hamiltonians $H^w_{\mathbf{P},\sigma_j}$

$$\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}} = -\frac{1}{2\pi i}\oint \sum_{n=1}^{\infty} \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\left(-(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right)^n \phi^{\sigma_j}_{\mathbf{P}}\, dE(j+1)\,. \quad (8)$$

where $(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} = \widehat{H}^w_{\mathbf{P},\sigma_{j+1}} + c_{\mathbf{P}}(j) - \widehat{c_{\mathbf{P}}}(j+1) - H^w_{\mathbf{P},\sigma_j}$.
Then we define

$$\phi^{\sigma_{j+1}}_{\mathbf{P}} \equiv W_{\sigma_{j+1}}(\nabla E^{\sigma_{j+1}}(\mathbf{P}))\, W^{\dagger}_{\sigma_{j+1}}(\nabla E^{\sigma_j}(\mathbf{P}))\, \widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}\,.$$

The construction of $\{\phi^{\sigma_j}_{\mathbf{P}}\}$ is therefore performed by the same method described in the previous section (in some sense it is already given in terms of $\{\psi^{\sigma_j}_{\mathbf{P}}\}$).

Outline of the proof of the convergence.

In inquiring the (strong) convergence of the vectors $\phi^{\sigma_j}_{\mathbf{P}}$ for $j \to \infty$, we have to compare the following vectors one after the other:

$$\phi^{\sigma_j}_{\mathbf{P}} \to \widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}} \to \phi^{\sigma_{j+1}}_{\mathbf{P}} \to \widehat{\phi}^{\sigma_{j+2}}_{\mathbf{P}} \to \phi^{\sigma_{j+2}}_{\mathbf{P}}$$

(in the special case $\mathbf{P} = 0$, there is a simplification because $\phi^{\sigma_j}_{\mathbf{P}} \equiv \widehat{\phi}^{\sigma_j}_{\mathbf{P}}$, being $\nabla E^{\sigma}(0) = 0$).
First note that one needs a more refined estimate of the difference between the generic vectors $\phi^{\sigma_j}_{\mathbf{P}}$ and $\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}$. At this point a crucial difference with respect the previous sequence $\{\psi^{\sigma_j}_{\mathbf{P}}\}$ emerges. In particular, break the interaction

$$(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} = \widehat{H}^w_{\mathbf{P},\sigma_{j+1}} + c_{\mathbf{P}}(j) - \widehat{c_{\mathbf{P}}}(j+1) - H^w_{\mathbf{P},\sigma_j}$$

in

$$\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix} + \left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{quad.}$$

where

$$\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix} \equiv \Gamma_{\mathbf{P},\sigma_j}\cdot\left(-\frac{g}{2m}\int_{\sigma_{j+1}}^{\sigma_j} \frac{\mathbf{k}\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)} d^3k + \frac{g^2}{2m}\int_{\sigma_{j+1}}^{\sigma_j} \frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)^2} d^3k\right) + c.c.$$

and consider again the expression (8).
Due to the mixed terms the norm estimate provided in lemma A1 for

$$\left\| \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right)^{\frac{1}{2}} \left(-(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right) \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right)^{\frac{1}{2}} \right\|$$

is only of order 1, so that it is not sufficient to evaluate the norm vector $\left\|\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}}\right\|$ with a quantity of order a positive power of the cutoff σ_{j+1}.
We are able to give a more refined estimate of the norm of $\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}}$ by a careful analysis of the first factor in each term of the sum in (8)

$$\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right)^{\frac{1}{2}} \left(-\left((\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right)\right) \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right) \phi^{\sigma_j}_{\mathbf{P}}\,.$$

In particular note that if for the vector

$$\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)^{\frac{1}{2}}\left[(\Delta H^{w}_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix}\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)^{\frac{1}{2}}\phi^{\sigma_j}_{\mathbf{P}}$$

the following inequalities were true

$$\left\|\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)^{\frac{1}{2}}\left[(\Delta H^{w}_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix}\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)^{\frac{1}{2}}\phi^{\sigma_j}_{\mathbf{P}}\right\|\leq\frac{\epsilon^{\frac{j+1}{8}}}{4}\qquad(9)$$

the estimate for $\left\|\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}-\phi^{\sigma_j}_{\mathbf{P}}\right\|$ would be less than or equal to $\epsilon^{\frac{j+1}{8}}$.
Indeed, from lemma A1, we know that the norm of the contribution due to the quadratic terms can be bounded by $\frac{\epsilon^{\frac{j+1}{8}}}{40}$ (for a proper g) :

$$\left\|\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)^{\frac{1}{2}}\left[(\Delta H^{w}_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{quad}\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)^{\frac{1}{2}}\phi^{\sigma_j}_{\mathbf{P}}\right\|\leq\frac{\epsilon^{\frac{j+1}{8}}}{40}$$

We know also from Lemma A1 how to bound the norm of the other factors of the product corresponding to each term of the sum: such a norm is of order 1, in particular less than $\frac{1}{12}$. Therefore we would have

$$\left\|\sum_{n=1}^{\infty}\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E(j+1)}\left(-(\Delta H^{w}_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{n}\phi^{\sigma_j}_{\mathbf{P}}\right\|\leq$$

$$\leq\left|\left(\frac{1}{E^{\sigma_j}_{\mathbf{P}}-E(j+1)}\right)\right|^{\frac{1}{2}}\cdot\left\|\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E(j+1)}\right)\right\|^{\frac{1}{2}}_{F^{+}_{\sigma_{j+1}}}\cdot\epsilon^{\frac{j+1}{8}}\cdot\sum_{n=1}^{\infty}\left(\frac{11}{40}\right)\left(\frac{1}{12}\right)^{n-1}\leq\left|\left(\frac{1}{E^{\sigma_j}_{\mathbf{P}}-E(j+1)}\right)\right|\cdot\epsilon^{\frac{j+1}{8}}\ .$$

Finally, as it is shown in corollary 2.4, an estimate like $\left\|\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}-\phi^{\sigma_j}_{\mathbf{P}}\right\|\leq\epsilon^{\frac{j+1}{8}}$ implies the convergence of the sequence $\{\phi^{\sigma_j}_{\mathbf{P}}\}$.

The conclusion of the previous reasoning is that, turning to strong estimates for the first factor in all the terms of the series expansion in (8), we are able to prove the convergence of the sequence if the inequality (9) holds. Taking care of the expression of $\left[(\Delta H^{w}_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix}$, after a few steps one can check that the inequality (9) is implied by the following

$$g^{2}\left|\left(\Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}},\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j}-E\,(j+1)}\right)\Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right)\right|<const\cdot\left(\frac{1}{\sigma_j}\right)^{\frac{1}{4}}\qquad(10)$$

(where $E\,(j+1)$ is s.t. $\left|E\,(j+1)-E^{\sigma_j}_{\mathbf{P}}\right|=\frac{11}{20}\sigma_{j+1}$) where *const* means uniform in j and it is supposed sufficiently small.
We arrive at (10) by two technical lemmas: lemma 2.1 and lemma 2.2. In lemma 2.1 we start from

of the spectral representation of the operator $H^{w}_{\mathbf{P},\sigma_j} - ReE\,(j+1)$ and we provide a form bound of the type

$$\left(\varphi, \left|\frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E\,(j+1)}\right| \varphi\right) \leq const \cdot \left|\left(\varphi, \frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E\,(j+1)}\varphi\right)\right|$$

for φ belonging to the subspace of $F^{+}_{\sigma_j}$ orthogonal to the ground state $\phi^{\sigma_j}_{\mathbf{P}}$ and such that the two quantities are well defined; at this point we take advantage of it and of the canonical form (7) of the hamiltonian $H^{w}_{\mathbf{P},\sigma}$, which was rearranged to exploit the property

$$\Gamma^{i}_{\mathbf{P},\sigma}\phi^{\sigma}_{\mathbf{P}} \equiv \left\{\Pi^{i}_{\mathbf{P},\sigma}\phi^{\sigma}_{\mathbf{P}} - \frac{1}{\|\phi^{\sigma}_{\mathbf{P}}\|^2} \cdot \left(\phi^{\sigma}_{\mathbf{P}}, \Pi^{i}_{\mathbf{P},\sigma}\phi^{\sigma}_{\mathbf{P}}\right)\phi^{\sigma}_{\mathbf{P}}\right\} \perp \phi^{\sigma}_{\mathbf{P}}$$

In lemma 2.2 we deal with the relevant term in (9); by calling a term "relevant" we mean to say that the other ones have a better infrared behavior or can be reduced to the "relevant term" plus smaller terms. We estimate the norm

$$\left\|\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E\,(j+1)}\right)^{\frac{1}{2}} \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^{\dagger}(\mathbf{k})}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}}\left(1 - \widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)} d^3k \cdot \Gamma^{i}_{\mathbf{P},\sigma_j}\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E\,(j+1)}\right)^{\frac{1}{2}} \phi^{\sigma_j}_{\mathbf{P}}\right\|$$

It requires the study of the series expansion linked to the commutation of $\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)} b^{\dagger}(\mathbf{k})\, d^3k$ with $\left(\frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E(j+1)}\right)$ from the right side. At this point we assume $\frac{\kappa}{m}$ sufficiently small. Being standard computations, they are done in appendix A.

The last step is theorem 2.3, in which, by induction, we provide the estimate (10).

Lemma 2.1

The following inequalities hold:

$$\text{I)} \left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^{\dagger}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha^{\sigma_j}_{\mathbf{P}}(\widehat{\mathbf{k}})} d^3k\, \Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}, \left|\frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E(j+1)}\right| \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^{\dagger}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha^{\sigma_j}_{\mathbf{P}}(\widehat{\mathbf{k}})} d^3k\, \Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right) \leq$$

$$\leq \sqrt{122}\left|\left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^{\dagger}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha^{\sigma_j}_{\mathbf{P}}(\widehat{\mathbf{k}})} d^3k\, \Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}, \frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E(j+1)} \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^{\dagger}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha^{\sigma_j}_{\mathbf{P}}(\widehat{\mathbf{k}})} d^3k\, \Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right)\right|$$

$$\text{II)} \left(\Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}, \left|\frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E(j+1)}\right| \Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right) \leq Q(\epsilon) \cdot \left|\left(\Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}, \frac{1}{H^{w}_{\mathbf{P},\sigma_j} - E(j+1)}\Gamma^{i}_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right)\right|$$

where $\alpha^{\sigma_j}_{\mathbf{P}}(\widehat{\mathbf{k}}) \equiv \left(1 - \widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)$, $Q(\epsilon) \equiv \sqrt{1 + \left(\frac{11\sqrt{\epsilon}}{10 - 11\sqrt{\epsilon}}\right)^2}$.

Proof

Let us define the wave functions $\zeta_I(z), \zeta_{II}(z)$, respectively of $\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}} \alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\, \Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}}$ and $\Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}}$, in the spectral variable of the operator $H^w_{\mathbf{P},\sigma_j} - ReE(j+1)$ (we do not explicit the other degrees of freedom).

Note that:

- the operator $H^w_{\mathbf{P},\sigma_j} - ReE(j+1)$, applied to the vector $\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}} \alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\, \Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}}$, takes spectral values bigger or equal to $\frac{1}{20}\sigma_{j+1}$ $\left(= \frac{3}{5}\sigma_{j+1} - \frac{11}{20}\sigma_{j+1}\right)$ because of lemma 1.1;

- the operator $H^w_{\mathbf{P},\sigma_j} - ReE(j+1)$ takes spectral values bigger or equal to $\frac{10-11\sqrt{\epsilon}}{20}\sigma_j$ $\left(= \frac{1}{2}\sigma_j - \frac{11}{20}\sigma_{j+1}\right)$ if applied to the vector $\Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}}$ because of theorem 1.4.

Let us write the scalar products I) and II) by using the spectral representation of the operator $H^w_{\mathbf{P},\sigma_j} - ReE(j+1)$ with spectral measure $d\mu(z)$ and ignoring the remaining degrees of freedom. In the chosen spectral representation, the following inequalities are clear:

$$\left|\int \frac{|\zeta_{I,II}(z)|^2}{z - iIm(E(j+1))} d\mu(z)\right| = \left\{\left|\int \frac{z|\zeta_{I,II}(z)|^2}{z^2 + [Im(E(j+1))]^2} d\mu(z)\right|^2 + [Im(E(j+1))]^2 \left|\int \frac{|\zeta_{I,II}(z)|^2}{z^2+[Im(E(j+1))]^2} d\mu(z)\right|^2\right\}^{\frac{1}{2}} \geq$$

$$\geq \left|\int \frac{|\zeta_{I,II}(z)|^2}{\sqrt{z^2+[Im(E(j+1))]^2}} \cdot \frac{z}{\sqrt{z^2+[Im(E(j+1))]^2}} d\mu(z)\right| \geq \frac{z_{min}}{\sqrt{z^2_{min} + |E(j+1) - E^{\sigma_j}_{\mathbf{P}}|^2}} \cdot \left|\int \frac{|\zeta_{I,II}(z)|^2}{\sqrt{z^2+[Im(E(j+1))]^2}} d\mu(z)\right| \geq$$

$$\geq \frac{z_{min}}{\sqrt{z^2_{min} + |E(j+1) - E^{\sigma_j}_{\mathbf{P}}|^2}} \cdot \left|\int \frac{|\zeta_{I,II}(z)|^2}{\sqrt{z^2+|\mathbf{k}|^2+2|\mathbf{k}|z+[Im(E(j+1))]^2}} d\mu(z)\right|$$

It follows that:

- in the case I), being $z_{min} \geq \frac{1}{20}\sigma_{j+1}$, $\frac{z_{min}}{\sqrt{z^2_{min} + |E(j+1) - E^{\sigma_j}_{\mathbf{P}}|^2}} \geq \frac{1}{\sqrt{122}}$

$$\left|\int \frac{|\zeta_I(z)|^2}{\sqrt{z^2+[Im(E(j+1))]^2}} d\mu(z)\right| \leq \sqrt{122} \cdot \left|\int \frac{|\zeta_I(z)|^2}{z - iIm(E(j+1))} d\mu(z)\right|$$

- in the case II), being $z_{min} \geq \frac{10-11\sqrt{\epsilon}}{20} \cdot \sigma_j$, $\frac{z_{min}}{\sqrt{z^2_{min} + |E(j+1) - E^{\sigma_j}_{\mathbf{P}}|^2}} \geq \frac{1}{\sqrt{1+\left(\frac{11\sqrt{\epsilon}}{10-11\sqrt{\epsilon}}\right)^2}} = \frac{1}{Q(\epsilon)}$

$$\left|\int \frac{|\zeta_{II}(z)|^2}{\sqrt{z^2+|\mathbf{k}|^2+2|\mathbf{k}|z+[Im(E(j+1))]^2}} d\mu(z)\right| \leq \left|\int \frac{|\zeta_{II}(z)|^2}{\sqrt{z^2+[Im(E(j+1))]^2}} d\mu(z)\right| \leq Q(\epsilon) \cdot \left|\int \frac{|\zeta_{II}(z)|^2}{z - iIm(E(j+1))} d\mu(z)\right|.$$

Lemma 2.2

The following inequality holds for a sufficiently small ratio $\frac{\kappa}{m}$:

$$\left\| \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right)^{\frac{1}{2}} \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}} \alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k \cdot \Gamma^i_{\mathbf{P},\sigma_j} \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right)^{\frac{1}{2}} \phi^{\sigma_j}_{\mathbf{P}} \right\|^2 \leq$$

$$\leq 2 \cdot Q(\epsilon) \cdot \sqrt{122} \cdot \left\{ \int_{\sigma_{j+1}}^{\sigma_j} \frac{{k^i}^2}{2|\mathbf{k}|^3 \alpha_{\mathbf{P}}^{\sigma_{j+1}}(\widehat{\mathbf{k}})^2} d^3k \right\} \left| \frac{1}{E^{\sigma_j}_{\mathbf{P}} - E(j+1)} \right| \left| \left(\Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}}, \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right) \Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}} \right) \right|.$$

Proof

See lemma A3 in appendix A.

Theorem 2.3

For g and $\frac{\kappa}{m}$ sufficiently small the inequality

$$\left\| \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right)^{\frac{1}{2}} \left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} \right]^{mix} \left(\frac{1}{E^{\sigma_j}_{\mathbf{P}} - E(j+1)} \right)^{\frac{1}{2}} \phi^{\sigma_j}_{\mathbf{P}} \right\| \leq \frac{\epsilon^{\frac{j+1}{8}}}{4} \tag{11}$$

holds uniformly in j.

Proof

Due to the result of lemma 2.2 and the previous discussion about the convergence, the inequality (11) is true if the following estimate holds:

$$g^2 \left| \left(\Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}}, \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right) \Gamma^i_{\mathbf{P},\sigma_j} \phi^{\sigma_j}_{\mathbf{P}} \right) \right| < \frac{M}{\epsilon^{\frac{j}{4}}} \qquad (i = 1, 2, 3) \tag{12}$$

where M is a proper constant uniform in j.
The inequality (12) implies the thesis of the theorem and also the bound $\left\| \widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}} \right\| \leq \epsilon^{\frac{j+1}{8}}$, as discussed so far in *Outline of the proof of the convergence.*

In order to prove the inequality (12) we start applying to both the factors of the scalar product the unitary operator

$$W_{\sigma_j}(\nabla E^{\sigma_{j-1}}(\mathbf{P})) W^\dagger_{\sigma_j}(\nabla E^{\sigma_j}(\mathbf{P}))$$

(such an operation is not required if $\mathbf{P} = 0$) to obtain

$$\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j} \widehat{\phi}^{\sigma_j}_{\mathbf{P}}, \left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j} - E(j+1)} \right) \widehat{\Gamma}^i_{\mathbf{P},\sigma_j} \widehat{\phi}^{\sigma_j}_{\mathbf{P}} \right). \tag{13}$$

Adding and subtracting $\Gamma^i_{\mathbf{P},\sigma_{j-1}} \phi^{\sigma_{j-1}}_{\mathbf{P}}$ on the left and on the right of the scalar product (13), we bound the new terms that we get, using elementary properties of the scalar product:

$$g^2\left|\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}},\left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right)\right|\leq$$

$$\leq 2g^2\left|\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right)\right|+ \tag{14}$$

$$+2g^2\left|\left(\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right| \tag{15}$$

At this point, what we want to do is to reduce the quantity (15) to the (12) at the level $j-1$, times a constant less than 1, and to estimate the remainder (14) by a quantity of order $\frac{1}{\epsilon^{\frac{j}{4}}}$.

Treatment of the remainder (14)

$$2g^2\left|\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j}-H^w_{\mathbf{P},\sigma_{j-1}}+H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right)\right|\leq$$

(for g sufficiently small, taking into account lemma 1.2 and using a procedure as in lemma A1)

$$\leq 4g^2\left|\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left|\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right|\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right)\right|\leq$$

$$\leq 8g^2\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right)\right\|^2+8g^2\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\Gamma^i_{\mathbf{P},\sigma_{j-1}}\left(\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right\|^2\leq$$

$$\leq\frac{R_1(g)}{\epsilon^{\frac{j}{4}}}+\frac{R_2(g)}{\epsilon^{\frac{j}{4}}}\cdot\left(\frac{\left\|\frac{\widehat{\phi}^{\sigma_j}_{\mathbf{P}}}{\left\|\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right\|}-\frac{\phi^{\sigma_{j-1}}_{\mathbf{P}}}{\left\|\phi^{\sigma_{j-1}}_{\mathbf{P}}\right\|}\right\|+\epsilon^{\frac{j}{8}}}{4\epsilon^{\frac{j}{8}}}\right)^2$$

where $R_1(g)$ and $R_2(g)$ are independent of j and vanish for $g\to 0$. They are obtained considering the following facts:

i) from the relation (6) pag. 32 and the definition of $\widehat{\Pi}^i_{\mathbf{P},\sigma_{j+1}}$ at pag. 34:

$$\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}=\Gamma^i_{\mathbf{P},\sigma_{j-1}}-g\int_{\sigma_j}^{\sigma_{j-1}}\frac{k^i\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{j-1}}_{\mathbf{P}}\right)}d^3k+m\left(\nabla E^{\sigma_j}(\mathbf{P})-\nabla E^{\sigma_{j-1}}(\mathbf{P})\right)-g^2\int_{\sigma_j}^{\sigma_{j-1}}\frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)^2}d^3k$$

$$\textbf{ii)}\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\Gamma^i_{\mathbf{P},\sigma_{j-1}}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right)\right\|\leq$$

$$\leq\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\left(g\int_{\sigma_j}^{\sigma_{j-1}}\frac{k^i\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{j-1}}_{\mathbf{P}}\right)}d^3k\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right)\right\|+$$

$$+\left|m\nabla E^{\sigma_j}(\mathbf{P})-m\nabla E^{\sigma_{j-1}}(\mathbf{P})\right|\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right\|+$$

$$+\left|g^2\int_{\sigma_j}^{\sigma_{j-1}}\frac{k^i}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)}d^3k\right|\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\right\|.$$

iii) the following estimates hold with constants $C^{\nabla E}, C', C''$ uniform in j for g sufficiently small

- $|\nabla E^{\sigma_j}(\mathbf{P})-\nabla E^{\sigma_{j-1}}(\mathbf{P})|\leq C^{\nabla E}\left(\left\|\frac{\widehat{\phi}^{\sigma_j}_{\mathbf{P}}}{\|\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\|}-\frac{\phi^{\sigma_{j-1}}_{\mathbf{P}}}{\|\phi^{\sigma_{j-1}}_{\mathbf{P}}\|}\right\|+\epsilon^{\frac{j-1}{8}}\right)$ (see lemma A2)
- $\left\|\int_{\sigma_j}^{\sigma_{j-1}}\frac{k^i\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{j-1}}_{\mathbf{P}}\right)}d^3k\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_j}}\leq C'\cdot\epsilon^{\frac{j}{4}}$ (by steps as in lemma A1 and for the result in lemma 1.2)
- $\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\Gamma^i_{\mathbf{P},\sigma_{j-1}}\right\|_{F^+_{\sigma_j}}=\left\|\Gamma^i_{\mathbf{P},\sigma_{j-1}}\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_j}}\leq\frac{C''}{\epsilon^{\frac{j}{4}}}$ (by steps as in lemma A1 and for the result in lemma 1.2)
- $\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)^{\frac{1}{2}}\Gamma^i_{\mathbf{P},\sigma_{j-1}}\left(\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right\|\leq\frac{C''}{\epsilon^{\frac{j}{4}}}\left\|\widehat{\phi}^{\sigma_j}_{\mathbf{P}}-\phi^{\sigma_{j-1}}_{\mathbf{P}}\right\|.$

Treatment of (15)

$$2g^2\left|\left(\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j}-H^w_{\mathbf{P},\sigma_{j-1}}+H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right|\leq$$

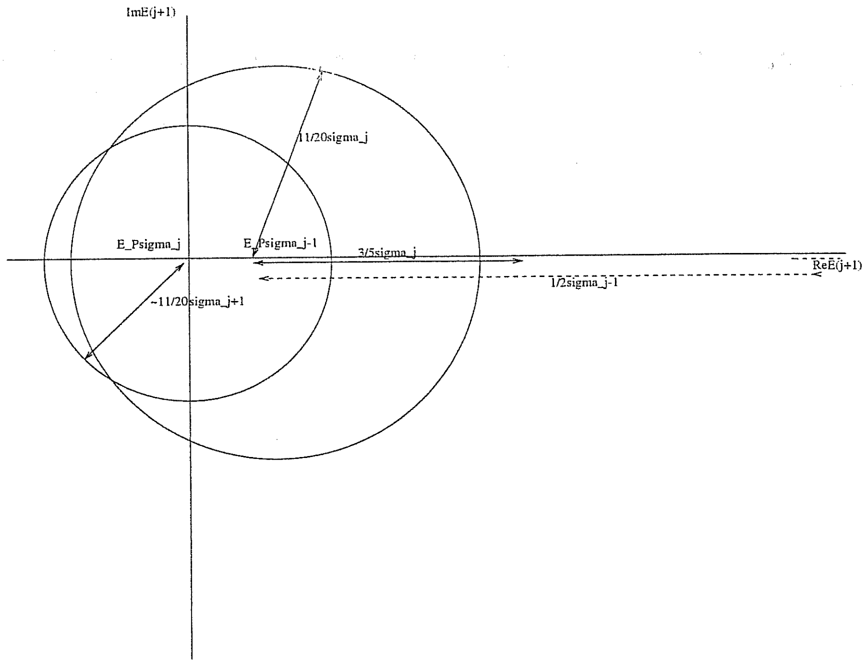
(for g sufficiently small, taking into account lemma 1.2 and using a procedure as in lemma A1)

$$\leq 4g^2\left|\left(\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left|\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)\right|\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right|\leq$$

$$\leq 4g^2\cdot Q(\epsilon)\left|\left(\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j+1)}\right)\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right|\leq$$

$$\leq 4g^2\cdot b\cdot Q^2(\epsilon)\left|\left(\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}},\left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}}-E(j)}\right)\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)\right|$$

where $b\preceq 2$; the last step is due to the fact that $\left(\phi^{\sigma_{j-1}}_{\mathbf{P}},\Gamma^i_{\mathbf{P},\sigma_{j-1}}\phi^{\sigma_{j-1}}_{\mathbf{P}}\right)=0$ and g is sufficiently small, on this respect see the figure below



Now let us analyze the norm $\left\| \phi_{\mathbf{P}}^{\sigma_j} - \widehat{\phi}_{\mathbf{P}}^{\sigma_j} \right\|$; note that

$$\phi_{\mathbf{P}}^{\sigma_j} \equiv W_{\sigma_j}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) W_{\sigma_j}^{\dagger}\left(\nabla E^{\sigma_{j-1}}(\mathbf{P})\right) \widehat{\phi}_{\mathbf{P}}^{\sigma_j}$$

by definition, from which

$$\left\| \phi_{\mathbf{P}}^{\sigma_j} - \widehat{\phi}_{\mathbf{P}}^{\sigma_j} \right\| = \left\| W_{\sigma_j}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) W_{\sigma_j}^{\dagger}\left(\nabla E^{\sigma_{j-1}}(\mathbf{P})\right) \widehat{\phi}_{\mathbf{P}}^{\sigma_j} - \widehat{\phi}_{\mathbf{P}}^{\sigma_j} \right\| =$$
$$= \left\| W_{\sigma_j}^{\dagger}\left(\nabla E^{\sigma_{j-1}}(\mathbf{P})\right) W_{\sigma_j}\left(\nabla E^{\sigma_j}(\mathbf{P})\right) \psi_{\mathbf{P}}^{\sigma_j} - \psi_{\mathbf{P}}^{\sigma_j} \right\| .$$

Our upper estimate of the norm above is given by

$$g \cdot Z \cdot \left| \nabla E_{\mathbf{P}}^{\sigma_j} - \nabla E_{\mathbf{P}}^{\sigma_{j-1}} \right| \left| \ln\left(\epsilon^{\frac{j}{2}}\right) \right|$$

where

- Z is a constant dependent on m, κ and uniform in j;

- the logarithmically divergent quantity arises from

$$\left(\int_{\sigma_j}^{\kappa} \left\| b(\mathbf{k})\, \psi_{\mathbf{P},\sigma_j} \right\|^2 d^3k \right)^{\frac{1}{2}}$$

taking into account that $b(\mathbf{k})\,\psi_{\mathbf{P},\sigma_j} = \frac{g}{\sqrt{2|\mathbf{k}|}}\left(\frac{1}{E^{\sigma_j}(\mathbf{P})-|\mathbf{k}|-H_{\mathbf{P}-\mathbf{k},\sigma_j}}\right)\psi_{\mathbf{P},\sigma_j}$ for $\{\mathbf{k}:\sigma_j \leq |\mathbf{k}| \leq \kappa\}$ (see [3]) and $\|\psi_{\mathbf{P},\sigma_j}\| < 1$ and from

$$\left(\int_{\sigma_j}^{\kappa}\left(\frac{1}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{j+1}}(\mathbf{P})\right)}\right)^2 d^3k\right)^{\frac{1}{2}} \quad ;$$

- the (infinitesimal) quantity $\left|\nabla E^{\sigma_j}_{\mathbf{P}} - \nabla E^{\sigma_{j-1}}_{\mathbf{P}}\right|$ is connected to the difference between the coherent factors in the Weyl operators.

Now, let be g sufficiently small such that

- $G_1^{\infty} \equiv \sum_{k=1}^{\infty} g\cdot Z\cdot 4C^{\nabla E}\cdot \epsilon^{\frac{k}{8}}\left|\ln\left(\epsilon^{\frac{k}{2}}\right)\right| \leq \frac{1}{12}$

- the previous constraints hold, in particular the bound (12) is valid for $j=1$ and

$$0 < R_1(g) + R_2(g) = R(g) \leq \left(1 - 4bQ^2(\epsilon)\,\epsilon^{\frac{1}{4}}\right)\cdot M$$

where $4b\cdot Q^2(\epsilon)\,\epsilon^{\frac{1}{4}} \ll 1$ for $b \preceq 2$ and $0 < \epsilon < \left(\frac{1}{5}\right)^8$.

inductive hypothesis

Let us assume that for the chosen value of g the property (12) holds for $j-1$ and that

$$\left\|\phi^{\sigma_{j-1}}_{\mathbf{P}} - \phi^{\sigma_0}_{\mathbf{P}}\right\| \leq g\cdot Z\cdot 4C^{\nabla E}\cdot\epsilon^{\frac{j-1}{8}}\left|\ln\left(\epsilon^{\frac{j-1}{2}}\right)\right| + \epsilon^{\frac{j-1}{8}} + \ldots + g\cdot Z\cdot 4C^{\nabla E}\cdot\epsilon^{\frac{1}{8}}\left|\ln\left(\epsilon^{\frac{1}{2}}\right)\right| + \epsilon^{\frac{1}{8}} = \sum_{k=1}^{j-1}\epsilon^{\frac{j}{8}} + G_1^{j-1}$$

where $G_1^{j-1} = g\cdot\sum_{k=1}^{j-1} Z\cdot 4C^{\nabla E}\cdot\epsilon^{\frac{k}{8}}\left|\ln\left(\epsilon^{\frac{k}{2}}\right)\right|$.

thesis

As a consequence of the inductive hypothesis we have

- $\left\|\widehat{\phi}^{\sigma_j}_{\mathbf{P}} - \phi^{\sigma_{j-1}}_{\mathbf{P}}\right\| \leq \epsilon^{\frac{j}{8}}$
- $\left\|\frac{\widehat{\phi}^{\sigma_j}_{\mathbf{P}}}{\|\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\|} - \frac{\phi^{\sigma_{j-1}}_{\mathbf{P}}}{\|\phi^{\sigma_{j-1}}_{\mathbf{P}}\|}\right\| \leq 2\left\|\frac{\widehat{\phi}^{\sigma_j}_{\mathbf{P}} - \phi^{\sigma_{j-1}}_{\mathbf{P}}}{\|\widehat{\phi}^{\sigma_j}_{\mathbf{P}}\|}\right\| \leq 3\epsilon^{\frac{j}{8}}$ since $\left\|\phi^{\sigma_{j-1}}_{\mathbf{P}}\right\| \geq 1 - \left\|\phi^{\sigma_{j-1}}_{\mathbf{P}} - \phi^{\sigma_0}_{\mathbf{P}}\right\| > 1 - \sum_{k=1}^{\infty}\epsilon^{\frac{1}{8}} - G_1^{\infty} > \frac{2}{3}$
- $\left|\nabla E^{\sigma_j}_{\mathbf{P}} - \nabla E^{\sigma_{j-1}}_{\mathbf{P}}\right| \leq 4C^{\nabla E}\cdot\epsilon^{\frac{j}{8}}$ (see lemma A2).

Then

$$g^2 \left| \left(\widehat{\Gamma}^i_{\mathbf{P},\sigma_j} \widehat{\phi}^{\sigma_j}_{\mathbf{P}}, \left(\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_j} - E(j+1)} \right) \widehat{\Gamma}^i_{\mathbf{P},\sigma_j} \widehat{\phi}^{\sigma_j}_{\mathbf{P}} \right) \right| \leq$$

$$\leq \frac{R(g)}{\epsilon^{\frac{j}{4}}} + 4g^2 \cdot b \cdot Q^2(\epsilon) \left| \left(\Gamma^i_{\mathbf{P},\sigma_{j-1}} \phi^{\sigma_{j-1}}_{\mathbf{P}}, \left(\frac{1}{H^w_{\mathbf{P},\sigma_{j-1}} - E(j)} \right) \Gamma^i_{\mathbf{P},\sigma_{j-1}} \phi^{\sigma_{j-1}}_{\mathbf{P}} \right) \right| \leq$$

$$\leq \frac{R(g)}{\epsilon^{\frac{j}{4}}} + 4b \cdot Q^2(\epsilon) \frac{M}{\epsilon^{\frac{j-1}{4}}} \leq \frac{M}{\epsilon^{\frac{j}{4}}}.$$

Moreover

$$\left\| \phi^{\sigma_j}_{\mathbf{P}} - \phi^{\sigma_0}_{\mathbf{P}} \right\| \leq \left\| \widehat{\phi}^{\sigma_j}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}} \right\| + \left\| \widehat{\phi}^{\sigma_j}_{\mathbf{P}} - \phi^{\sigma_{j-1}}_{\mathbf{P}} \right\| + \left\| \phi^{\sigma_{j-1}}_{\mathbf{P}} - \phi^{\sigma_0}_{\mathbf{P}} \right\| \leq$$

$$\leq g \cdot Z \cdot 4C^{\nabla E} \cdot \epsilon^{\frac{j}{8}} \left| \ln \left(\epsilon^{\frac{j}{2}} \right) \right| + \epsilon^{\frac{j}{8}} + \sum_{k=1}^{j-1} \epsilon^{\frac{k}{8}} + G_1^{j-1} \leq \sum_{k=1}^{j} \epsilon^{\frac{k}{8}} + G_1^j.$$

Corollary 2.4

Given the result of theorem 2.3, the sequence $\left\{ \phi^{\sigma_j}_{\mathbf{P}} \right\}$ ($\phi^{\sigma_0}_{\mathbf{P}} \equiv \psi_0$, ψ_0 vacuum state) converges strongly to a non-vanishing vector for a sufficiently small coupling constant g.

Proof

By the estimates of theorem 2.3 we easily conclude that $\left\{ \phi^{\sigma_j}_{\mathbf{P}} \right\}$ is a Cauchy sequence: $\forall l, j \;\; l \geq j$

$$\left\| \phi^{\sigma_l}_{\mathbf{P}} - \phi^{\sigma_j}_{\mathbf{P}} \right\| \leq \sum_{k=j+1}^{l} \epsilon^{\frac{k}{8}} + G^l_{j+1} \leq \epsilon^{\frac{j+1}{8}} \cdot \left(\frac{1}{1 - \epsilon^{\frac{1}{8}}} \right) + G^l_{j+1}$$

The limit does not vanish, since

$$\left\| \phi^{\sigma_j}_{\mathbf{P}} \right\| \geq 1 - \left(\frac{\epsilon^{\frac{1}{8}}}{1 - \epsilon^{\frac{1}{8}}} + G_1^{\infty} \right) \geq \frac{2}{3} \, .$$

3 Regularity.

In this section we define a normalized vector $\phi^\sigma_{\mathbf{P}}$, that is ground state of $H^w_{\mathbf{P},\sigma}\,|_{F^+_\sigma}$ $(\sigma \leq \kappa\epsilon)$. It has to possess a regularity property in $\mathbf{P}$ to be exploited in the construction of the scattering states in chapter 4. We arrive at the vector $\phi^\sigma_{\mathbf{P}}$ through an intermediate (not normalized) vector $\overline{\phi}^\sigma_{\mathbf{P}}$. $\phi^\sigma_{\mathbf{P}}$ and $\overline{\phi}^\sigma_{\mathbf{P}}$ may differ only by a phase term, apart from the normalization. The phase is fixed with an eye to the following:

- the norm convergence of the vector $\phi^\sigma_{\mathbf{P}}$ to a vector $\phi_{\mathbf{P}}$, for $\sigma \to 0$;

- the Hoelder property, with respect to $\mathbf{P}$:

$$\left\|\phi^\sigma_{\mathbf{P}+\Delta\mathbf{P}} - \phi^\sigma_{\mathbf{P}}\right\| \leq C\,|\Delta\mathbf{P}|^{\frac{1}{16}}$$

for $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$ and $\Delta\mathbf{P} \in I$, $I \subset \Sigma$ is a set defined in the next lines and C is uniform in $\mathbf{P}, \Delta\mathbf{P}$ and σ.

For these purposes we consider infrared sequences starting from $\{\kappa\epsilon' : \kappa\epsilon \geq \kappa\epsilon' \geq \kappa\epsilon\sqrt{\epsilon}\}$ and a coupling constant g such that it is possible to perform the iterative procedure uniformly in ϵ', $\epsilon \geq \epsilon' \geq \epsilon\sqrt{\epsilon}$, and in $\mathbf{P} \in \Sigma$, with the properties already shown in the case of the factor ϵ. Therefore, we assume the results of theorem 2.3 and corollary 2.4. We also require that for the chosen value g:

$$\left|\left(\phi^{\kappa\epsilon'}_{\mathbf{P}}, \psi_0\right)\right| > \frac{1}{3} \qquad \forall \epsilon'\,,\ \epsilon \geq \epsilon' \geq \epsilon\sqrt{\epsilon}\,,\ \forall \mathbf{P} \in \Sigma.$$

Definition of $\overline{\phi}^\sigma_{\mathbf{P}}$.

Given a σ ranging between σ_j and σ_{j+1}, $j \geq 2$, we can always write it as $\kappa\epsilon'^{\frac{j}{2}}$ where $\epsilon' \equiv \epsilon'(\sigma) = = \left(\frac{\sigma}{\kappa}\right)^{\frac{2}{j}}$. By performing the iteration shown in the previous section, we define

$$\overline{\phi}^\sigma_{\mathbf{P}} \equiv \phi^{\kappa\epsilon'(\sigma)^{\frac{j}{2}}}_{\mathbf{P}}\,.$$

Lemma 3.1

$\left(\overline{\phi}^\sigma_{\mathbf{P}}, \psi_0\right) \neq 0 \quad \forall \sigma \leq \kappa\epsilon,\ \forall \mathbf{P} \in \Sigma.$

Proof

Knowing that $\left\|\phi^{\kappa\epsilon'}_{\mathbf{P}} - \phi^{\kappa\epsilon'^{\frac{j}{2}}}_{\mathbf{P}}\right\| \leq \frac{1}{3}$ from corollary 2.4, we have:

$$\left|\left(\overline{\phi}^\sigma_{\mathbf{P}}, \psi_0\right)\right| \geq \left|\left|\left(\phi^{\kappa\epsilon'(\sigma)}_{\mathbf{P}}, \psi_0\right)\right| - \left|\left(\overline{\phi}^\sigma_{\mathbf{P}} - \phi^{\kappa\epsilon'(\sigma)}_{\mathbf{P}}, \psi_0\right)\right|\right| > 0\,.$$

Definition of $\phi_{\mathbf{P}}^{\sigma}$.

Since $\overline{\phi}_{\mathbf{P}}^{o}$ is ground eigenvector of $H_{\mathbf{P},\sigma}^{w}\mid_{F_{\sigma}^{+}}$ with a gap bigger than $\frac{\sigma}{2}$ by construction, thanks to lemma 3.1 the normalized vector

$$\phi_{\mathbf{P}}^{\sigma} \equiv \frac{-\frac{1}{2\pi i}\oint \frac{1}{H_{\mathbf{P},\sigma}^{w}-E} dE\, \psi_0}{\left\| -\frac{1}{2\pi i}\oint \frac{1}{H_{\mathbf{P},\sigma}^{w}-E} dE\, \psi_0 \right\|}$$

(where $E \in \mathcal{C}$ and s.t. $|E - E_{\mathbf{P}}^{\sigma}| = \frac{\sigma}{4}$) is ground state of $H_{\mathbf{P},\sigma}^{w}\mid_{F_{\sigma}^{+}}$.

Theorem 3.2

For $\mathbf{P} \in \Sigma$ the limits $s - \lim_{\sigma\to 0} \phi_{\mathbf{P}}^{\sigma} (\equiv \phi_{\mathbf{P}})$ and $\lim_{\sigma\to 0} E_{\mathbf{P}}^{\sigma}$ exist.

Proof

Again write $\overline{\phi}_{\mathbf{P}}^{\sigma_2} - \overline{\phi}_{\mathbf{P}}^{\sigma_1}$ in the following way

$$\overline{\phi}_{\mathbf{P}}^{\sigma_2} - \overline{\phi}_{\mathbf{P}}^{\sigma_1} = \overline{\phi}_{\mathbf{P}}^{\sigma_2} - \phi_{\mathbf{P}}^{\kappa\epsilon_2(\sigma_2)^{\frac{l}{2}}} + \phi_{\mathbf{P}}^{\kappa\epsilon_2(\sigma_2)^{\frac{l}{2}}} - \phi_{\mathbf{P}}^{\kappa\epsilon_1(\sigma_1)^{\frac{m}{2}}} + \phi_{\mathbf{P}}^{\kappa\epsilon_1(\sigma_1)^{\frac{m}{2}}} - \overline{\phi}_{\mathbf{P}}^{\sigma_1} .$$

Now, given an arbitrarily small δ, there exist $l(\delta), m(\delta)$ sufficiently large and a phase $e^{i\eta(\delta)}$ for which

$$\left\| \phi_{\mathbf{P}}^{\kappa\epsilon_1(\sigma_1)^{\frac{m}{2}}} - e^{i\eta(\delta)} \phi_{\mathbf{P}}^{\kappa\epsilon_2(\sigma_2)^{\frac{l}{2}}} \right\| \leq \delta$$

This is essentially due to the convergence established in corollary 2.4 and because the ground state is unique until there is a cut-off, by construction.

Therefore the norm $\left\| \overline{\phi}_{\mathbf{P}}^{\sigma_2} - e^{-i\eta(\delta)} \overline{\phi}_{\mathbf{P}}^{\sigma_1} \right\|$ can be bounded with a quantity of order $\sigma_2^{\frac{1}{4}} + \sigma_1^{\frac{1}{4}} + \delta$. Moreover we have that

$$\begin{aligned}
&\left\| P_{\overline{\phi}_{\mathbf{P}}^{\sigma_1}} \psi_0 - P_{\overline{\phi}_{\mathbf{P}}^{\sigma_1}} \psi_0 \right\| \equiv \left\| \overline{\phi}_{\mathbf{P}}^{\sigma_1} \left(\overline{\phi}_{\mathbf{P}}^{\sigma_1}, \psi_0 \right) - \overline{\phi}_{\mathbf{P}}^{\sigma_2} \left(\overline{\phi}_{\mathbf{P}}^{\sigma_2}, \psi_0 \right) \right\| \leq \\
&\leq \left\| \overline{\phi}_{\mathbf{P}}^{\sigma_1} \left(\overline{\phi}_{\mathbf{P}}^{\sigma_1}, \psi_0 \right) - e^{i\eta(\delta)} \overline{\phi}_{\mathbf{P}}^{\sigma_2} \left(\overline{\phi}_{\mathbf{P}}^{\sigma_1}, \psi_0 \right) \right\| + \left\| \overline{\phi}_{\mathbf{P}}^{\sigma_2} \left(e^{-i\eta(\delta)} \overline{\phi}_{\mathbf{P}}^{\sigma_1}, \psi_0 \right) - \overline{\phi}_{\mathbf{P}}^{\sigma_2} \left(\overline{\phi}_{\mathbf{P}}^{\sigma_2}, \psi_0 \right) \right\| \leq \\
&\leq \left\| \overline{\phi}_{\mathbf{P}}^{\sigma_1} - e^{i\eta(\delta)} \overline{\phi}_{\mathbf{P}}^{\sigma_2} \right\| \cdot \left| \left(\overline{\phi}_{\mathbf{P}}^{\sigma_1}, \psi_0 \right) \right| + \left\| \overline{\phi}_{\mathbf{P}}^{\sigma_2} \right\| \cdot \|\psi_0\| \cdot \left\| e^{-i\eta(\delta)} \overline{\phi}_{\mathbf{P}}^{\sigma_1} - \overline{\phi}_{\mathbf{P}}^{\sigma_2} \right\|
\end{aligned}$$

It follows that $\phi_{\mathbf{P}}^{\sigma} \equiv \frac{P_{\overline{\phi}_{\mathbf{P}}^{\sigma}} \psi_0}{\left\| P_{\overline{\phi}_{\mathbf{P}}^{\sigma}} \psi_0 \right\|}$ converges strongly to a vector $\phi_{\mathbf{P}}$ in $\mathrm{H}_{\mathbf{P}}$, with an error of order $\sigma^{\frac{1}{4}}$. The convergence of $E_{\mathbf{P}}^{\sigma}$ easily follows from the iterative construction.

Lemma 3.3

The following Hoelder estimate on the ground energy gradient holds:

$$|\nabla E^{\sigma}(\mathbf{P}) - \nabla E^{\sigma}(\mathbf{P}+\Delta\mathbf{P})| \leq C\,|\Delta\mathbf{P}|^{\frac{1}{16}}$$

where the constant C is uniform in $0 \leq \sigma < \kappa\epsilon$, in $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$ and $\Delta\mathbf{P} \in \widehat{I}$, where $\widehat{I} \equiv \left\{ \Delta\mathbf{P} : \frac{|\Delta\mathbf{P}|}{m} \leq \left(\frac{1}{3C_{\widehat{I}}}\right)^{\frac{8}{3}} \right\}$ and $C_{\widehat{I}}$ is a constant sufficiently larger than 1.

Proof

The idea is to perturb, in $\mathbf{P}$, the gradient

$$\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) \equiv \left(\psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \frac{\mathbf{P}-\mathbf{P}^{ph}}{m}\psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right)$$

where $\psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}$ is the (normalized) ground eigenvector of $H_{\mathbf{P},m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}$, which we simply denote as $H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}$(analogous notation for $F^{+}_{|\Delta\mathbf{P}|^{\frac{1}{4}}}$).
For this purpose we use the series expansion of the resolvent

$$\frac{1}{H_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}|_{F^{+}_{|\Delta\mathbf{P}|^{\frac{1}{4}}}} - E}$$

(where $E \in C$ and s.t. $\left|E - E_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right| = \frac{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}{4}$) on the basis of the following information:

- $H_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} - H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} = -\frac{\Delta\mathbf{P}}{m}\cdot\mathbf{P}^{ph} + \frac{\Delta\mathbf{P}}{m}\cdot\mathbf{P} + \frac{|\Delta\mathbf{P}|^2}{2m}$;
- $H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}|_{F^{+}_{|\Delta\mathbf{P}|^{\frac{1}{4}}}}$ has unique ground state $\psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}$ of energy $E_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}$ and its gap is bounded from below by $\frac{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}{2}$ (theorem 1.4 in the continuum case);
- the norm $m^{\frac{3}{8}}\left|\Delta P^{i}\right|^{\frac{1}{8}}\left\|\frac{\left(\mathbf{P}^{i}-\mathbf{P}^{ph^{i}}\right)}{\sqrt{m}}\left(\frac{1}{H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}|_{F^{+}_{|\Delta\mathbf{P}|^{\frac{1}{4}}}} - E}\right)^{\frac{1}{2}}\right\|$ is uniformly bounded in $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$; therefore, we have that

$$\sum_i \left|\Delta P^{i}\right|^{\frac{1}{4}}\left\|\left(\frac{1}{H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}|_{F^{+}_{|\Delta\mathbf{P}|^{\frac{1}{4}}}} - E}\right)^{\frac{1}{2}}\cdot\left(\frac{\left(\mathbf{P}^{i}-\mathbf{P}^{ph^{i}}\right)}{m} + \frac{|\Delta\mathbf{P}|}{2m}\right)\left(\frac{1}{H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}|_{F^{+}_{|\Delta\mathbf{P}|^{\frac{1}{4}}}} - E}\right)^{\frac{1}{2}}\right\| < C_{\widehat{I}}m^{-\frac{3}{4}}$$

- $\left\| -\frac{1}{2\pi i}\oint \frac{1}{H_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}-E} dE\, \psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \right\| \geq 1 - \sum_{n=1}^{\infty} \left(\frac{|\Delta\mathbf{P}|^{\frac{3}{4}}}{m^{\frac{3}{4}}} \cdot C_{\widehat{I}} \right)^n \geq \frac{1}{2}$ for $\Delta\mathbf{P}$ belonging to $\widehat{I} \equiv \left\{ \Delta\mathbf{P} : \frac{|\Delta\mathbf{P}|}{m} \leq \left(\frac{1}{3C_{\widehat{I}}} \right)^{\frac{8}{3}} \right\}$

From the above considerations it follows that:

- $\psi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \equiv \frac{-\frac{1}{2\pi i}\oint \frac{1}{H_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}-E} dE\, \psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}}{\left\| -\frac{1}{2\pi i}\oint \frac{1}{H_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}-E} dE\, \psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \right\|}$ is ground state of $H_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}$
- $\left\| \psi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} - \psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \right\| \leq C' |\Delta\mathbf{P}|^{\frac{3}{8}}$ where C' is a constant uniform in $\mathbf{P}$, $\mathbf{P}+\Delta\mathbf{P} \in \Sigma$ and $\Delta\mathbf{P} \in \widehat{I}$.

Since:

1) $\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P}) - \nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) = \left(\psi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \frac{\mathbf{P}+\Delta\mathbf{P}-\mathbf{P}^{ph}}{m} \psi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \right) - \left(\psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \frac{\mathbf{P}-\mathbf{P}^{ph}}{m} \psi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \right)$

2) $H_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} + 2\pi g^2 \kappa - \frac{(\mathbf{P}^{ph}-\mathbf{P})^2}{2m} \geq 0,$

we can conclude that

$$\left| \nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P}) \right| \leq C'' |\Delta\mathbf{P}|^{\frac{3}{8}} \tag{16}$$

where C'' is a constant uniform in $\mathbf{P}$, $\mathbf{P}+\Delta\mathbf{P} \in \Sigma$ and $\Delta\mathbf{P} \in \widehat{I}$.
If $\sigma < m^{\frac{3}{4}} |\Delta\mathbf{P}|^{\frac{1}{4}}$, in order to prove the claim of the lemma, we take advantage of the result of lemma A2 (in appendix A) together with theorem 2.3 :

$$\left| \nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \nabla E^{\sigma}(\mathbf{P}) \right| \leq C |\Delta\mathbf{P}|^{\frac{1}{16}} \qquad for \quad \mathbf{P} \in \Sigma$$

If $\sigma \geq m^{\frac{3}{4}} |\Delta\mathbf{P}|^{\frac{1}{4}}$, an estimate analogous to (16) holds.

Theorem 3.4

Under the hypotheses of lemma 3.3, the norm difference between $\phi_{\mathbf{P}}^{\sigma}$ and $\phi_{\mathbf{P}+\Delta\mathbf{P}}^{\sigma}$ is Hoelder in $|\Delta\mathbf{P}|$ with coefficient $\frac{1}{16}$, the multiplicative constant is uniform in $0 \leq \sigma < \kappa\epsilon$, in $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$ and $\Delta\mathbf{P} \in I \subset \widehat{I}$.

Proof

Preliminary definitions:

$$H^w_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} - H^w_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} =$$

$$= c_{\mathbf{P}+\Delta\mathbf{P}}\left(m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}\right) - c_{\mathbf{P}}\left(m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}\right) + \int\left(\mathbf{k}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \mathbf{k}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right) b^{\dagger}(\mathbf{k})\, b(\mathbf{k})\, d^3k +$$

$$+\frac{1}{2m}\left(g\int_{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}^{\kappa} \frac{\mathbf{k}\left(\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P})\right)}\left(b(\mathbf{k})+b(\mathbf{k})^{\dagger}\right) d^3k + \Delta\Pi_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\right)^2 +$$

$$+\frac{1}{2m}\Gamma_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\cdot\left(g\int_{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}^{\kappa} \frac{\mathbf{k}\left(\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P})\right)}\left(b(\mathbf{k})+b(\mathbf{k})^{\dagger}\right) d^3k + \Delta\Pi_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\right) +$$

$$+\left(g\int_{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}^{\kappa} \frac{\mathbf{k}\left(\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P})\right)}\left(b(\mathbf{k})+b(\mathbf{k})^{\dagger}\right) d^3k + \Delta\Pi_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\right)\cdot$$

$$\cdot\frac{1}{2m}\Gamma_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}$$

where $\Delta\Pi_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} = \left(\phi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \Pi_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\phi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right) - \left(\phi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \Pi_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\phi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right)$

Considering that for $|\Delta\mathbf{P}| \in \widehat{I}$

- the estimate (16) in lemma 3.3 holds: $\left|\nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}) - \nabla E^{|\Delta\mathbf{P}|^{\frac{1}{4}}}(\mathbf{P}+\Delta\mathbf{P})\right| \le C''|\Delta\mathbf{P}|^{\frac{3}{8}}$;
- $\left|\left(\phi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \Pi^i_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\phi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right) - \left(\phi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}, \Pi^i_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}\phi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right)\right|$ is bounded by a quantity of order $|\Delta\mathbf{P}|^{\frac{3}{8}}$(see equation (6) pag.32)
- the operator $\frac{H^w_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}} - H^w_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}}{|\Delta\mathbf{P}|^{\frac{3}{8}}}$ is form-bounded with respect to $H^w_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}$ uniformly in $|\Delta\mathbf{P}|$;
- the gap of $E_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}$(as ground energy of $H^w_{\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}|_{F^+_{|\Delta\mathbf{P}|^{\frac{1}{4}}}}$) is bounded from below by $\frac{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}{2}$

the vector $\phi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} \equiv \frac{-\frac{1}{2\pi i}\oint \frac{1}{H^w_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}-E} dE\, \psi_0}{\left\|-\frac{1}{2\pi i}\oint \frac{1}{H^w_{\mathbf{P}+\Delta\mathbf{P},|\Delta\mathbf{P}|^{\frac{1}{4}}}-E} dE\, \psi_0\right\|}$ (where $E \in \mathcal{C}$ and s.t. $\left|E - E_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right| = \frac{m^{\frac{3}{4}}|\Delta\mathbf{P}|^{\frac{1}{4}}}{4}$)
can be obtained perturbing $\phi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}$ for $|\Delta\mathbf{P}| \in I \subset \widehat{I}$, I sufficiently small.

From the perturbation we have the estimate:

$$\left\|\phi_{\mathbf{P}+\Delta\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}} - \phi_{\mathbf{P}}^{|\Delta\mathbf{P}|^{\frac{1}{4}}}\right\| \le C'''|\Delta\mathbf{P}|^{\frac{1}{16}} \qquad (17)$$

where the constant C''' is uniform in $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$ and $|\Delta\mathbf{P}| \in I$.

For $\sigma < m^{\frac{3}{4}} |\Delta\mathbf{P}|^{\frac{1}{4}}$, the thesis is proved using theorem 3.2 and the inequality:

$$\left\|\phi^{\sigma}_{\mathbf{P}+\Delta\mathbf{P}} - \phi^{\sigma}_{\mathbf{P}}\right\| = \left\|\phi^{\sigma}_{\mathbf{P}+\Delta\mathbf{P}} - \phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}+\Delta\mathbf{P}} + \phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}+\Delta\mathbf{P}} - \phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}} + \phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}} - \phi^{\sigma}_{\mathbf{P}}\right\| \leq$$

$$\leq \left\|\phi^{\sigma}_{\mathbf{P}+\Delta\mathbf{P}} - \phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}+\Delta\mathbf{P}}\right\| + \left\|\phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}+\Delta\mathbf{P}} - \phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}}\right\| + \left\|\phi^{|\Delta\mathbf{P}|^{\frac{1}{4}}}_{\mathbf{P}} - \phi^{\sigma}_{\mathbf{P}}\right\|$$

If $\sigma \geq m^{\frac{3}{4}} |\Delta\mathbf{P}|^{\frac{1}{4}}$ an estimate analogous to (17) holds.

Part III
Scattering theory.

The infrared features of the model produce some difficulties in understanding the scattering:
- due to the arbitrarily large number of photons involved in the scattering, a problem of consistency might apparently arise between the total emission in time of an infinite number of massless particles and a "free" asymptotic dynamics for the electron;
- because of the massless dispersion, it is not clear if the L.S.Z. Weyl photon operators

$$e^{iHt}e^{-iH^{ph}t}e^{i\left(a(\varphi)+a^{\dagger}(\varphi)\right)}e^{iH^{ph}t}e^{-iHt} \qquad \varphi(\mathbf{y})=\int e^{-i\mathbf{k}\mathbf{y}}\widetilde{\varphi}(\mathbf{k})\,d^3k\in S\left(R^3\right)$$

converge for $t\to\infty$ on generic vectors of the Hilbert space, as pointed out in Part I.

In Part 1 we outlined the physical situation emerging from perturbative computations and from rigorous results in solvable models. We saw how all this information give an intuitive picture for asymptotic states consisting of a "free" electron surrounded by a cloud of asymptotic bosons, whose distribution in the photon momenta space is linked to the electron asymptotic velocity according to "Bloch and Nordsieck" type factors (depending on the considered interactions). The spectral counterpart is the absence of one-particle states, which avoids any asymptotic description inspired by the Haag-Ruelle collision theory for quantum fields. For Nelson's massless model, in the paper by Froehlich, starting from this intuitive picture, a tentative recipe is provided for the generic vector $\psi_h(t)$, whose limit, for $t\to\infty$, has to correspond to an asymptotic electron (Heisenberg picture), as described in Part 1. The convergence of the vector $\psi_h(t)$, without any infrared cut-off, would provide a definition of "free" dynamics for the infraparticle (the electron), which is the main open problem.

In our opinion, the way drawn in [3] is the correct one to understand the scattering behavior. In some sense, it is already the physical answer. Therefore we reconsider this work with the aim to overcome the technical difficulties in the proof of the convergence, by clarifying some aspects in the definition of the approximating vector $\psi_h(t)$ and developing some new ideas, namely:
- a stronger use of the "non-relativistic locality", since the decoupling mechanism in the Haag-Ruelle theory can be reproduced in terms of fixed time locality properties of the photon field and of the "current density field" of the electron;
- a convergence by a diagonal limiting procedure, that means that the infrared cut-off in the approximating vectors is removed only asymptotically.

The purposes of such a construction is twofold:
- the first is to provide a minimal (with respect to the photon cloud) description of the electron out of the scattering;
- the second one consists in picking out a subspace of states which will be used to prove the asymptotic convergence of the massless field. This subspace can be seen as an one-particle subspace, up to an energy threshold for the photons.

The final result is that, once we are given the strong convergence in time of the approximating vectors $\psi_h(t)$, it is simple to define *in* and *out* subspaces where the asymptotic boson field and the asymptotic mean velocity of the electron are well defined.

Analysis of the minimal asymptotic electron.

Let us consider again the proposal by Froehlich (Part I). No problem arises in the norm control and in the convergence of $\psi_h(t)$ till $\sigma \neq 0$, since we can control the series expansion of the Weyl operators and the regularity properties in $\mathbf{P}$ of $\psi^{\sigma}_{\mathbf{P}}$ (notations as in Part I). The situation changes for $\sigma = 0$. There is already a problem at the level of definition of the vector: namely the existence of the integral when the infrared cutoff σ is removed. The previous control of the norm is not available because of divergences appearing in the terms obtained from the expansion of the Weyl operators. The need of the series expansion is technically forced by the fact that the Weyl operators

$$e^{-g\int_{\sigma}^{\kappa_1} \frac{a(\mathbf{k})-a^{\dagger}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}(\mathbf{P})\right)} d^3k}$$

do not preserve the $\mathbf{P}-$fibers. The definition is well founded by assuming the regularity properties of certain functions (see for details [3]) which can probably be reconciled with the existence of the second derivative of the ground energy $E(\mathbf{P})$ and which are confirmed by perturbative computations. In this respect, the existence of the second derivative of the ground energy has recently been proved by Chen [36] for non-relativistic Q.E.D..

Our proposal to avoid such technical obstructions is to consider a time dependent cut-off σ_t, which is removed only asymptotically at a rate faster than $\frac{1}{t}$ in accordance with the uncertainty principle, and to transform the integral (in d^3P) in a Riemann sum by a (time dependent) cell-partition of the $\mathbf{P}$-momentum space. The time dependent cut-off σ_t and the Riemann sum can be considered as regularization tools to define an integral in the limit, which might exist anyway but that we are not able to control otherwise. Another step towards the convergence consists in using a phase factor already present in the tentative construction by Froehlich for the case $\sigma = 0$.

Let us see in details the reasons and the expected advantages of these constructive modifications.

1) By introducing a time-dependent cut-off σ_t, we aim at having a better control on quantities like the norm of $\psi_h(t)$. Our conjecture relies on two facts:
- the unitarity property of the Weyl operators with a cut-off σ_t can provide an a-priori estimate on sums of contributions which are divergent in the infrared limit;
- the infrared divergences in electrodynamics are poor ones because of their logarithmic growth so that our task should become easier since we do not need optimal estimates of the vanishing quantities.

2) The transformation of the integral to a Riemann sum is a tool towards the same purpose, because:
- we get a well defined expression in terms of bounded operators in Hilbert space for all finite times, that we can control without considering any particular wave function representation (except for

certain estimates);
- it allows us to replace the series expansion of the Weyl operators by a "cell-expansion" which can be more easily controlled; in this sense it is a "regularization" of the fiber-changing.

3) The phase factor is employed in the applications of Cook's argument. From this point of view there is an analogy with Dollard's treatment of Coulomb scattering even if the present phase factor is only a technical tool. It is convergent in the limit $(t \to \infty)$ differently from the Coulomb phase. It is probably avoidable even if it is helpful in our framework because provides some useful subtractions.

What we have described represent the building blocks of a strategy employed to control the logarithmic divergences. More technically, this strategy requires the use of different time scales. In particular, we trigger the rate of the partition (governed by an exponent ϵ) and another regularization parameter (δ) according to the time scale arising from Cook's argument, since the latter is substantially independent of the regularization devices. Moreover by the asymptotic removal of the cut-off σ_t we can extend to the limit some properties which hold for the model with a fixed infrared cut-off. More specifically:
- the propagation estimate for proper functions f

$$e^{iHt}f\left(\frac{\mathrm{x}}{t}\right)e^{-iHt} \to_{t\to\infty} f\left(\nabla E\right)$$

starting from the analogous property on one-particle states corresponding to a fixed $\sigma-$cutoff dynamics (already used in semiclassical analysis [37]);
- the fact that for a fixed $\sigma-$cutoff dynamics the one particle states are vacua for the annilihation operator of the asymptotic bosonic field; by using it, we can treat the off-diagonal terms, with respect to the partition, of the squared norm $\|\psi_h(t)\|^2$ and $\|\psi_h(t_2)-\psi_h(t_1)\|^2$.

The main difference with the analogous construction by Froehlich [3] is that, in the new recipe for the vector $\psi_h(t)$, the infrared cut-off removal is an "a posteriori" result and a byproduct of the decoupling. Therefore, in our opinion, it is simpler to use locality by the support properties of functions of the electron position $\mathbf{x}(t)$ (in such a way we exploit Huyghens' principle for our observables, at fixed time t). Our construction should be hopefully made simpler in order to consider generalizations, for instance more than one electron. Some regularization device is not probably necessary in a modified framework of construction. However the present construction may be a starting point for simpler descriptions of the asymptotic decoupling and for a more precise analysis of the involved time scales. We have to stress that, in our construction, we assume an hypothesis (see pag. 55) on the second derivative of the ground energy, which is not proved in the spectral analysis contained in chapters 2 and 3.

Mathematical scheme.

The generic vector $\psi_h(t)$ is constructed starting from a one-particle state for the hamiltonian H_{σ_t}, of wave function h in $\mathbf{P}-$variables.
A $\mathbf{P}-$dependent L.S.Z. Weyl operator, in properly evolved photon variables is applied on the considered one-particle state. The smearing function in Weyl operator has frequency support from σ_t $(\sigma_t \to 0$ for $|t| \to +\infty)$ to an arbitrarily small $\kappa_1 \neq 0$. Its spectral distribution, near $\mathbf{k}=0$, is

labelled by the asymptotic electron mean velocity (constructed "a posteriori"). The relation with the coherent "static" factor

$$-g\frac{1}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}\left(1-\widehat{\mathbf{k}}\cdot\nabla E\right)}$$

is clarified if we consider a $\sigma-$cut-off dynamics and the corresponding asymptotic electron mean velocity first. This operator coincides with ∇E^{σ} if it is applied on the one -particle states. Through a non-rigorous removal of the cut-off, the coherent factor can be thought of as a function of the asymptotic electron mean velocity to be constructed.
All this is converted in a Riemann sum by a time dependent cell-partition of the $\mathbf{P}-$space, which implies the discretization of the "velocity" ∇E^{σ_t}.

Having constructed the generic vector $\psi_h(t)$, we prove the existence of the limit

$$s-\lim_{t\to\pm\infty}\psi_h(t)\equiv\psi_h^{out(in)}.$$

By analogy with the regularized case, we define the invariant (under space-time translation) subspaces

$$H^{1out(in)}\equiv\overline{\left\{\bigvee\psi_h^{out(in)}\ h(\mathbf{P})\in C_0^1\left(R^3\setminus 0\right),\mathbf{P}\in\Sigma\right\}}.$$

On these states the functions f, continuous and of compact support, of the electron mean velocity have limit :

$$s-\lim_{t\to\pm\infty}e^{iHt}f\left(\frac{\mathbf{x}}{t}\right)e^{-iHt}\psi_h^{out(in)}=s-\lim_{t\to\pm\infty}e^{iHt}f\left(\frac{\mathbf{x}}{t}\right)e^{-iHt}\psi_h(t).$$

The next step consists in adding "hard" asymptotic photons as result of the limits

$$s-\lim_{t\to\pm\infty}e^{iHt}e^{-iH^{ph}t}e^{i\left(a(\varphi)+a^{\dagger}(\varphi)\right)}e^{iH^{ph}t}e^{-iHt}\psi_h(t)\equiv\psi_{h,\varphi}^{out(in)}\qquad\varphi(\mathbf{y})=\int e^{-i\mathbf{ky}}\widetilde{\varphi}(\mathbf{k})\,d^3k,\ \widetilde{\varphi}(\mathbf{k})\in C_0^{\infty}\left(R^3\setminus 0\right.$$

The proposed scattering subspaces are then

$$H^{out(in)}\equiv\overline{\left\{\bigvee\psi_{h,\varphi}^{out(in)}\ h\in C_0^1\left(R^3\setminus 0\right),\widetilde{\varphi}\in C_0^{\infty}\left(R^3\setminus 0\right)\right\}}.$$

A remark is necessary at this point. These definitions are indeed arbitrary in some sense, especially the coherent function in the definition of the minimal asymptotic electron states, which is arbitrary except at the infrared limit. Nevertheless, through the (artificial) separation between $\mathrm{H}^{1out(in)}$ and $\mathrm{H}^{out(in)}$ we want to point out that:

- from a technical point of view, our construction of the scattering subspaces is based on some $\mathrm{H}^{1out(in)}$ (arbitrary to some extent);

- from a physical point of view, even if the photon cloud described by the smearing functions φ is totally removable, the photon cloud linked to the vectors in $\mathrm{H}^{1out(in)}$ is not completely removable; all the scattering states always contain asymptotic photons, precisely those ones of the spaces $\mathrm{H}^{1out(in)}$ involved in the construction of the spaces $\mathrm{H}^{out(in)}$.

Content of the chapters.

In chapter 4, the approximating vector $\psi_h(t)$ for the generic state of minimal asymptotic electron is defined. Then we study its norm in time, and finally we prove the strong convergence for $t \to \infty$ (paragraphs 4.1 and 4.2).
Chapter 5 contains the construction of the scattering subspaces $\mathrm{H}^{out(in)}$. On these subspace the asymptotic limits of the functions, continuous and of compact support, of the electron mean velocity and of the asymptotic limits of L.S.Z. Weyl operators are proved and their commutation properties are discussed.

The construction will be explicitly performed in the case "out". The case "in" is completely analogous.

4 Approximating vector $\psi_h(t)$.

Let us consider a cubic region of volume $V = L^3$ in the $\mathbf{P}$−space, inside $\Sigma = \left\{\mathbf{P} : |\mathbf{P}| < \frac{m}{20}\right\}$. For $\mathbf{P}$ in this region, we have:

- the existence of the ground state of $H_{\mathbf{P},\sigma}$ with the properties which follow from the results in chapters 2 and 3;
- small electron velocities $|\nabla E^\sigma(\mathbf{P})| < 1 \quad \forall\sigma$ (see lemma A2);
- $|\nabla E^\sigma(\mathbf{P}+\mathbf{k})| < 1 \quad \forall\sigma$, for $\mathbf{k} : 0 \le |\mathbf{k}| \le \kappa_1 \ll 1$ (κ_1 is supposed sufficiently small).

We also assume the following hypothesis , that is physically reasonable and strongly supported by Chen's result [Ch]:

hypothesis B1 $\quad \exists m_r > 0 \quad$ s.t. $\quad \forall\sigma, \forall\mathbf{P} \in \Sigma \quad \frac{\partial E^\sigma(\mathbf{P})}{\partial|\mathbf{P}|} \ge \frac{|\mathbf{P}|}{m_r} \quad$ and $\quad \frac{\partial^2 E^\sigma(\mathbf{P})}{\partial^2|\mathbf{P}|} \ge \frac{1}{m_r}$

Starting from this hypothesis, we obtain that the application $\mathbf{J}_\sigma : \mathbf{P} \to \nabla E^\sigma(\mathbf{P})$ is one to one and that the determinant of the jacobian satisfies the inequality:

$$det\,\mathbf{dJ}_\sigma = \frac{1}{|\mathbf{P}|^2}\left(\frac{\partial E^\sigma(\mathbf{P})}{\partial|\mathbf{P}|}\right)^2 \cdot \frac{\partial^2 E^\sigma(\mathbf{P})}{\partial^2|\mathbf{P}|} \ge \frac{1}{m_r^3}$$

($E^\sigma(\mathbf{P})$, as function of $\mathbf{P}$, is invariant under rotations and belongs to $C^\infty(R^3)$ (for $\sigma > 0$), see [3]).

Now we consider a time-dependent, $t \gg 1$, cell-partition of the volume $V = L^3$, defined as follows: at time t, the linear dimension of each cell is $\frac{L}{2^n}$ where $n \in N,\ n \ge 1$, is such that

$$(2^n)^{\frac{1}{\epsilon}} \le t < \left(2^{n+1}\right)^{\frac{1}{\epsilon}} \qquad \epsilon > 0$$

and the exponent ϵ is fixed only "a posteriori".
Such definition implies that the total number of cells is $N(t) = (2^n)^3$ at time t, where $n = [\log_2 t^\epsilon]$.
We call Γ_j the j^{th} cell, centered in $\overline{\mathbf{P}_j}$.

Constructive prescription of $\psi_h(t)$.

1) We start from the vector $\psi^{(t)}_{j,\sigma_t} = \int_{\Gamma_j} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_t} d^3P$, where:

- $h(\mathbf{P}) \in C^1_0\left(R^3 \setminus 0\right)$ has support inside the volume V ;

- $\psi^{\sigma_t}_{\mathbf{P}} \equiv W^{b\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}(\mathbf{P})\right) \phi^{\sigma_t}_{\mathbf{P}}$ is the ground state of $H_{\mathbf{P},\sigma_t}$;
warning: we use the index b in $W^{b\dagger}_{\sigma_t}$ in order to distinguish it from the dressing Weyl operators in the $\{a, a^\dagger\}$ variables. $W^b_\sigma\left(\nabla E^\sigma(\mathbf{P})\right)$ corresponds to the operator $W_\sigma\left(\nabla E^\sigma(\mathbf{P})\right)$ of chapter 2.

- (t) is referred to the partition at time t; note that $\left\|\psi^{(t)}_{j,\sigma_t}\right\| = \left(\int_{\Gamma_j} |h(\mathbf{P})|^2 d^3P\right)^{\frac{1}{2}}$ is of order $(N(t))^{-\frac{1}{2}}$.

2) We dress each $\psi^{(t)}_{j,\sigma_t}$ by the proper "$e^{iHt}e^{-iH^{ph}t}W_{\sigma_t}(\mathbf{v}_j)\, e^{iH^{ph}t}e^{-iH_{\sigma_t}t}$"; by this operation the so obtained vector remains inside the Hilbert space H under the removal of the infrared cut-off σ_t.
In particular we define:

$$\begin{aligned}\psi_h(t) &\equiv e^{iHt}e^{-iH^{ph}t}\sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j)\, e^{iH^{ph}t}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)}e^{-iE^{\sigma_t}(\mathbf{P})t}\psi^{(t)}_{j,\sigma_t} \\ &= e^{iHt}\sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j, t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)}e^{-iE^{\sigma_t}t}\psi^{(t)}_{j,\sigma_t}\end{aligned}$$

being

- $W_{\sigma_t}(\mathbf{v}_j) = e^{-g\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})-a^\dagger(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$, where $\mathbf{v}_j \equiv \nabla E^{\sigma_t}\left(\overline{\mathbf{P}}_j\right)$ is the velocity corresponding to the center $\overline{\mathbf{P}_j}$ of the cell Γ_j (belonging to the cell-partition at time t), κ_1 $(0 < \kappa_1 \ll 1)$ is the integration upper bound for the frequency;

- $W_{\sigma_t}(\mathbf{v}_j, t) = e^{-iH^{ph}t}W_{\sigma_t}(\mathbf{v}_j)\, e^{iH^{ph}t} = e^{-g\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|t}-a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|t}}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$;

- $\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), t) = -\int_1^t \left\{g^2\int_{\sigma_t}^{\sigma^S_\tau}\frac{\cos(\mathbf{k}\cdot\nabla E^{\sigma_t}(\mathbf{P})\tau-|\mathbf{k}|\tau)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d^3k\right\}d\tau =$
$= -\int_1^t \left\{g^2\int_{\tau\sigma_t}^{\tau\sigma^S_\tau}\frac{\cos(\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P})-|\mathbf{q}|)}{\left(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_j\right)}\frac{d^3q}{\tau}\right\}d\tau$
$e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)}$ is a phase factor whose origin and definition comes from Cook's argument.

At this point we only comment the two integration bounds for $|\mathbf{k}|$. The integration bound σ_τ^S is a "slow" cut-off, $\sigma_\tau^S = \tau^{-\alpha}$ where α is a positive number sufficiently less than 1. The "fast" cut-off σ_τ is of order $\tau^{-\beta}$, where β is sufficiently bigger than 1. To get more easily uniform estimates, the derivative of the phase factor has the role of killing a term arising from Cook's argument. In the proofs, we consider $\alpha = \frac{39}{40}$ from the beginning. On the basis of partial estimates, eventually β is chosen equal to 128 in order to gain the strong convergence of the vector $\psi_h(t)$.

4.1 Control of the norm of $\psi_h(t)$.

The squared norm $(\psi_h(t), \psi_h(t))$ corresponds to

$$\sum_{l,j=1}^{N(t)} \left(W_{\sigma_t}(\mathbf{v}_l, t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}(\mathbf{P}), t)} e^{-iH_{\sigma_t} t} \psi_{l,\sigma_t}^{(t)}, W_{\sigma_t}(\mathbf{v}_j, t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), t)} e^{-iH_{\sigma_t} t} \psi_{j,\sigma_t}^{(t)} \right).$$

In proving that the $\psi_h(t)$ squared norm has a not vanishing limit for $t \to +\infty$, our strategy starts from the diagonal terms in the above sum. The diagonal terms are easily under control because their sum is constant in time, indeed

$$\sum_{j=1}^{N(t)} \left(\psi_{j,\sigma_t}^{(t)}, \psi_{j,\sigma_t}^{(t)} \right) = \sum_{j=1}^{N(t)} \int_{\Gamma_j} |h(\mathbf{P})|^2 d^3P = \int_V |h(\mathbf{P})|^2 d^3P$$

The next and more difficult step consists in proving that each mixed term in the sum $\sum_{l,j=1}^{N(t)}$ (we mean off-diagonal term) asymptotically vanishes with an order in t substantially independent on the dimension of the cell. For this purpose, we take advantage of the time dependent cut-off σ_t. At the end, we obtain that the sum of the mixed terms $\sum_{l,j=1,l\neq j}^{N(t)}$ vanishes for $t \to +\infty$, by properly choosing the exponent ϵ that determines the growth rate of the total number of cells, $N(t) \leq t^{3\epsilon}$.

Remarks on the notations.

1)In the estimates that we produce later, we generically call C all the multiplicative constants which are uniform in the infrared cut-off and in the cells partition of the volume V. The provided bounds are intended from above, up to a different explicit warning. The time t is intended much greater than 1.

2) The operators, $\nabla E^\sigma(\mathbf{P})$, $W_\sigma^b(\nabla E^\sigma(\mathbf{P}))$, are functions of the total momentum $\mathbf{P}$. For reasons of space the dependence on $\mathbf{P}$ is not always explicit.

Control of the mixed terms

The generic mixed term is $(l \neq j)$

$$M_{l,j}(t) = \left(e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}(\mathbf{P}), t)} \psi_{l,\sigma_t}^{(t)}, e^{iH_{\sigma_t} t} W_{\sigma_t, l, j}(t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), t)} e^{-iH_{\sigma_t} t} \psi_{j,\sigma_t}^{(t)} \right)$$

where $W_{\sigma_t,l,j}(t) \equiv e^{-\int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|t} - a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|t}}{|\mathbf{k}|} \cdot h_{l,j}\left(\widehat{\mathbf{k}}\right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$ and $h_{l,j}\left(\widehat{\mathbf{k}}\right) \equiv \frac{g\widehat{\mathbf{k}}\cdot(\mathbf{v}_j - \mathbf{v}_l)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}$.

Now, let us consider $M_{l,j}(t)$ as a two-variable function, by distinguishing the time variable t, which parameterizes the partition and infrared cut-off σ_t, from the time variable s of the dynamical evolution. For this purpose, we define for $s \geq t$:

$$\widehat{M}_{l,j}(t,s) \equiv \left(e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}(\mathbf{P}), s)}\psi^{(t)}_{l,\sigma_t}\,,\, e^{iH_{\sigma_t}s}W_{\sigma_t,l,j}(s)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\right)$$

where

$$\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}(\mathbf{P}), s) \equiv \begin{cases} -\int_1^s \left\{ g^2 \int_{\tau\cdot\sigma_t}^{\tau\cdot\tau^{-\frac{39}{40}}} \frac{\cos(\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P}) - |\mathbf{q}|)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)} \frac{d^3q}{\tau} \right\} d\tau & for\ s: \quad s^{-\frac{39}{40}} \geq \sigma_t \\ -\int_1^{\sigma_t^{-\frac{40}{39}}} \left\{ g^2 \int_{\tau\cdot\sigma_t}^{\tau\cdot t^{-\frac{39}{40}}} \frac{\cos(\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P}) - |\mathbf{q}|)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_j)} \frac{d^3q}{\tau} \right\} d\tau & for\ s: \quad s^{-\frac{39}{40}} < \sigma_t \end{cases}$$

the property $\widehat{M}_{l,j}(t,t) = M_{l,j}(t)$ follows by definition.

We verify that:

I) $\widehat{M}_{l,j}(t,+\infty) = \lim_{s\to+\infty} \widehat{M}_{l,j}(t,s) = 0$

II) $|M_{l,j}(t)| = \left|\widehat{M}_{l,j}(t,t) - \widehat{M}_{l,j}(t,+\infty)\right| \leq$

$\leq \left| \int_t^{+\infty} \frac{d}{ds}\left(e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}, s)}\psi^{(t)}_{l,\sigma_t}, e^{iH_{\sigma_t}s}W_{\sigma_t,l,j}(s)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t},s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\right) ds \right| \leq C\cdot t^{-7\epsilon}$

where some constraints on ϵ and on another regularization parameter (δ) are assumed.

Proof of I)

For $s \geq t$, let us consider

$$\widehat{M}_{l,j}(\lambda,t,s) \equiv \left(e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}(\mathbf{P}), s)}\psi^{(t)}_{l,\sigma_t}\,,\, e^{iH_{\sigma_t}s}W^{\lambda}_{\sigma_t,l,j}(s)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\right)$$

where $W^{\lambda}_{\sigma_t,l,j}(s) \equiv e^{-\lambda\int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|s} - a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|s}}{|\mathbf{k}|} \cdot h_{l,j}\left(\widehat{\mathbf{k}}\right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$ and λ is a real parameter.
From the derivative with respect to the real parameter λ, we arrive at the differential equation:

$$\frac{d\widehat{M}_{l,j}(\lambda,t,s)}{d\lambda} = -\lambda C_{l,j}\cdot\widehat{M}_{l,j}(\lambda,t,s) + r_\sigma(\lambda,t,s)$$

(note that $\psi^{(t)}_{j,\sigma_t} \in D(H_{\sigma_t})$, then it belongs to $D(a(f))$ and $D\left(a^\dagger(f)\right)$, $f \in L^2(R^3)$, and the derivative with respect to λ is therefore well defined) where

$C_{l,j}=\int_{\sigma_t}^{\kappa_1}\left|h_{l,j}\left(\widehat{\mathbf{k}}\right)\right|^2\frac{d^3k}{2|\mathbf{k}|^3}$;

$r_{\sigma_t}\left(\lambda,t,s\right)=$
$=-\left(W^{\lambda\dagger}_{\sigma_t,l,j}\left(s\right)e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{l,\sigma_t},\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}}{|\mathbf{k}|}\cdot h_{l,j}\left(\widehat{\mathbf{k}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\right)+$
$+\left(\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}}{|\mathbf{k}|}\cdot h_{l,j}\left(\widehat{\mathbf{k}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{l,\sigma_t},W^{\lambda}_{\sigma_t,l,j}\left(s\right)e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\right)$

The solution of the differential equation at $\lambda=1$ is

$$\widehat{M}_{l,j}\left(1,t,s\right)=e^{-\frac{C_{l,j}}{2}}\widehat{M}_{l,j}\left(0,t,s\right)+\int_0^1 r_{\sigma_t}\left(\lambda',t,s\right)e^{-\frac{C_{l,j}}{2}(1-\lambda'^2)}d\lambda'\,. \tag{18}$$

Now, note that:

- $\widehat{M}_{l,j}\left(0,t,s\right)=0\ \forall t,s$, since the $\mathbf{P}-$supports of $\psi^{(t)}_{j,\sigma_t}$ and $\psi^{(t)}_{l,\sigma_t}$ $(l\neq j)$ are disjoint;
- thanks to theorem B6 (appendix B) one can verify the existence of

  $s-\lim_{s\to+\infty}e^{iH_{\sigma_t}s}\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}}{|\mathbf{k}|}\cdot h_{l,j}\left(\widehat{\mathbf{k}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{-iH_{\sigma_t}s}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\equiv$
  $\equiv a^{out}_{\sigma_t}\left(h_{l,j}\right)e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),\sigma_t^{-\frac{40}{39}}\right)}\psi^{(t)}_{j,\sigma_t}$

- $\lim_{s\to+\infty}r_{\sigma_t}\left(\lambda,t,s\right)=0$ since the vector $\psi^{(t)}_{j,\sigma_t}$ is a vacuum vector for $\left\{a^{out}_{\sigma_t}\left(\mathbf{k}\right)\right\}$ (see theorem B7, appendix B).

Therefore, starting from the equation (18), we have

$$\widehat{M}_{l,j}\left(t,+\infty\right)=\widehat{M}_{l,j}\left(\lambda=1,t,+\infty\right)=\int_0^1 r_{\sigma_t}\left(\lambda',t,+\infty\right)e^{-\frac{C_{l,j}}{2}(1-\lambda'^2)}d\lambda'=0\,.$$

Proof of II)

Let us consider:

$$\frac{d}{ds}\left(e^{iH_{\sigma_t}s}W_{\sigma_t}\left(\mathbf{v}_l,s\right)e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iH_{\sigma_t}s}\right)\psi^{(t)}_{l,\sigma_t}=$$

$$=ie^{iH_{\sigma_t}s}W_{\sigma_t}\left(\mathbf{v}_l,s\right)\left(\varphi_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)+\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}\left(\mathbf{P}\right),s\right)}{ds}\right)e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{l,\sigma_t} \tag{19}$$

where $\varphi_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)\equiv g^2\int_{\sigma_t}^{\kappa_1}\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}d^3k$.
Now we discuss some preliminary quantities to estimate the norm of the expression (19):

i) from the definition $\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}, s) \equiv -\int_1^s \left\{ g^2 \int_{\sigma_t \cdot \tau}^{\tau^{\frac{1}{40}}} \frac{\cos(\mathbf{q}\cdot\nabla E^{\sigma_t} - |\mathbf{q}|)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)} \frac{d^3 q}{\tau} \right\} d\tau$, we have

$$\frac{d\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}, s)}{ds} = -g^2 \int_{\sigma_t \cdot s}^{s^{\frac{1}{40}}} \frac{\cos(\mathbf{q}\cdot\nabla E^{\sigma_t} - |\mathbf{q}|)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)} \frac{d^3 q}{s} \qquad for\ s < \sigma_t^{-\frac{40}{39}}$$

$$= 0 \qquad for\ s \geq \sigma_t^{-\frac{40}{39}}$$

by analogy we define

$$\frac{d\gamma_{\sigma_t}(\mathbf{v}_l, \frac{\mathbf{x}}{s}, s)}{ds} = -g^2 \int_{\sigma_t \cdot s}^{s^{\frac{1}{40}}} \frac{\cos(\mathbf{q}\cdot\frac{\mathbf{x}}{s} - |\mathbf{q}|)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)} \frac{d^3 q}{s} \qquad for\ s < \sigma_t^{-\frac{40}{39}}$$

$$= 0 \qquad for\ s \geq \sigma_t^{-\frac{40}{39}} \quad .$$

ii) the function $\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s)$ can be decomposed in $\varphi^1_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) + \varphi^2_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s)$, which are so defined

$$\varphi^1_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) \equiv \begin{cases} g^2 \int_{s\cdot\sigma_t}^{s^{\frac{1}{40}}} \frac{\cos(\mathbf{k}\cdot\frac{\mathbf{x}}{s} - |\mathbf{k}|)}{(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l)} \frac{d^3 k}{s} & for \quad s \leq \sigma_t^{-\frac{40}{39}} \\ 0 & for \quad s > \sigma_t^{-\frac{40}{39}} \end{cases}$$

$$\varphi^2_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) \equiv \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) - \varphi^1_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) = \begin{cases} g^2 \int_{s^{-\frac{39}{40}}}^{\kappa_1} \int \frac{\cos(\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|s)}{(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l)} d^3 k & for\ s \leq \sigma_t^{-\frac{40}{39}} \\ \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) & for\ s > \sigma_t^{-\frac{40}{39}} \end{cases}$$

iii) for technical estimates we have to consider a "regularized characteristic function" of each cell: for this purpose, in appendix B, the generic function $\chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}, s)$ is defined with the following property:

$$\chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}), s) \rightarrow_{s\to+\infty} 1_{\Gamma_l}(\mathbf{P})$$

where $1_{\Gamma_l}(\mathbf{P})$ is the characteristic function of the Γ_l cell. In particular an exponent δ, parameter of the regularization, is introduced, to which a scale length $t^{-\frac{\delta}{6}}$ corresponds . This scale length has to be less than the ($\epsilon-$dependent) linear dimension of the cell Γ_l in order to be consistent (see definition B1, appendix B).

What we want to check is that the norm of (19) goes to zero independently on the partition in a sense made clear by the next estimates. For this purpose we break the expression (19) in many contributions which exploit the decomposition of $\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s)$ in $\varphi^1_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s) + \varphi^2_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s)$ and the "regularized characteristic function" $\chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}, s)$. We can disregard the unitary operator $e^{iH_{\sigma_t}s} W_{\sigma_t}(\mathbf{v}_l, s)$ in order to study the norm of the expression (19).

Analysis of the expression (19) :

$$\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\, e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} + \frac{d\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}{ds} e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} =$$

$$= \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\, e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \left(1_{\Gamma_l}(\mathbf{P}) - \chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}),s)\right) \psi^{(t)}_{l,\sigma_t} +$$

$$+ \frac{d\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}{ds} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} e^{-iE^{\sigma_t}(\mathbf{P})s} \left(1_{\Gamma_l}(\mathbf{P}) - \chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}),s)\right) \psi^{(t)}_{l,\sigma_t} +$$

$$+ \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s) \left(\chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}),s) - \chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)\right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} +$$

$$+ \varphi^2_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\, \chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} +$$

$$+ \left(\varphi^1_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\, \chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right) + \tfrac{d\gamma_{\sigma_t}}{ds}\left(\mathbf{v}_l,\tfrac{\mathbf{x}}{s},s\right) \chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)\right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} +$$

$$- \tfrac{d\gamma_{\sigma_t}}{ds}\left(\mathbf{v}_l,\tfrac{\mathbf{x}}{s},s\right) \left(\chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right) - \chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}),s)\right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} +$$

$$+ \left(-\tfrac{d\gamma_{\sigma_t}}{ds}\left(\mathbf{v}_l,\tfrac{\mathbf{x}}{s},s\right) + \tfrac{d\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}{ds}\right) e^{-iE^{\sigma_t}(\mathbf{P})s} \chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t},s)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t}$$

In lemma B8 each term is discussed. The asymptotic behavior is governed by:

- the "physical rate" (we term it physical because of the connection with the asymptotic decoupling) which is estimated from above by

$$\left| sup_{\mathbf{x}} \left(\varphi^2_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\, \chi^{(t)}_{\mathbf{v}_l}\left(\frac{\mathbf{x}}{s},s\right)\right)\right| \leq s^{-2} \cdot s^{\frac{39}{40}} ;$$

- the rate connected with the propagation estimate

$$\left\| \left(\chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}),s)\, \psi^{(t)}_{l,\sigma_t} - \chi^{(t)}_{\mathbf{v}_l}\left(\frac{\mathbf{x}}{s},s\right)\right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t},s)} \psi^{(t)}_{l,\sigma_t} \right\| \leq C \cdot s^{-\frac{1}{112}} \cdot s^{2\delta} \cdot |\ln \sigma_t| \cdot t^{-\frac{3\epsilon}{2}} ;$$

- the rate linked to the regularization of $1_{\Gamma_l}(\mathbf{P})$

$$\left\| \left(1_{\Gamma_l}(\mathbf{P}) - \chi^{(t)}_{\mathbf{v}_l}(\nabla E^{\sigma_t}(\mathbf{P}),s)\right) \psi^{(t)}_{l,\sigma_t} \right\| \leq C \cdot \frac{1}{s^{\frac{\delta}{12}}} \cdot t^{-\epsilon}.$$

The important fact is that the physical rate, the propagation estimate rate and the $\delta-$regularization scale are substantially independent on the partition, so that by assuming the constraints

- $\delta > 72\epsilon$
- $2\delta + 4\epsilon < \frac{1}{112}$

we obtain:

$$|M_{l,j}(t)| = \left|\widehat{M}_{l,j}(t,t) - \widehat{M}_{i,j}(t,+\infty)\right| \leq$$

$$\leq \left| \int_t^{+\infty} \tfrac{d}{ds} \left(e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)} \psi^{(t)}_{l,\sigma_t} \,,\, e^{iH_{\sigma_t}s} W_{\sigma,l,j}(s)\, e^{-iH_{\sigma_t}s} e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)} \psi^{(t)}_{j,\sigma_t} \right) ds \right| \leq$$

$$\leq \int_t^{+\infty} 2 \left\| \frac{d}{ds} \left\{ e^{iH_{\sigma_t} s} W_{\sigma_t}(\mathbf{v}_l, s)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}(\mathbf{P}), s)} e^{-iH_{\sigma_t} s} \psi_{l,\sigma_t}^{(t)} \right\} \right\| \left\| \psi_{j,\sigma_t}^{(t)} \right\| ds \leq C \cdot t^{-7\epsilon}$$

Provided the constraints for ϵ, δ are satisfied, as it is assumed in the following paragraphs, the sum of the mixed terms is bounded by $C \cdot t^{-\epsilon}$ thereby.

4.2 Strong convergence of $\psi_h(t)$ for $t \to +\infty$.

We display the Cauchy property of $\psi_h(t)$, by studying the norm of the vector:

$$\psi_h(t_2) - \psi_h(t_1)$$

for arbitrary times $t_2 > t_1 \gg 1$.
For $t_2 - t_1$ sufficiently large we have that $N(t_2) \neq N(t_1)$, i.e. different partitions correspond to t_2 and t_1 respectively. The t_2-partition sum, $\sum_{j=1}^{N(t_2)}$, is therefore generally written as $\sum_{j=1}^{N(t_1)} \sum_{l(j)}$, where $l(j)$ is the index which counts the sub-cells, relative to the t_2-partition, which are contained in the j^{th} cell of the t_1-partition, $1 \leq l(j) \leq \frac{N(t_2)}{N(t_1)}$.

With the notations above, the vector $\psi_h(t_2) - \psi_h(t_1)$ corresponds to

$$e^{iHt_2} \sum_{j=1}^{N(t_1)} \sum_{l(j)} W_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, t_2)\, e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}(\mathbf{P}), t_2)} e^{-iH_{\sigma_{t_2}} t_2} \psi_{l(j),\sigma_{t_2}}^{(t_2)} -$$

$$-e^{iHt_1} \sum_{j=1}^{N(t_1)} W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}(\mathbf{P}), t_1)} e^{-iH_{\sigma_{t_1}} t_1} \psi_{j,\sigma_{t_1}}^{(t_1)}$$

Our final goal is to prove that $\|\psi_h(t_2) - \psi_h(t_1)\| \leq C \cdot \frac{|\ln t_2|^2}{t_1^\rho}$, $\rho > 0$, so that we can easily prove the strong Cauchy property of $\psi_h(t)$ by a telescopic argument.

To estimate the norm difference we perform some intermediate steps from $\psi_h(t_2)$ to $\psi_h(t_1)$ and we study the norm of each contribution. In the next lines, for the considered intermediate variations, D1),D2),D3), D4.1), D4.2) and D4.3, we point out which quantities are involved and which physical properties and technical tools are exploited to estimate their norms (from above) by a quantity of order less or equal to $\frac{|\ln t_2|^2}{t_1^\rho}$.

As first step we consider the variation of the "dressing" term $W_{\sigma_{t_2}}(\mathbf{v}_j, t_2)$ and of the phase factor $e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}(\mathbf{P}), t_2)}$ when the cell-partition changes from t_2 to t_1, remaining fixed all other variables:

D1)

$$e^{iHt_2}\sum_{j=1}^{N(t_1)}\sum_{l(j)} W_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E^{\sigma_{t_2}}(\mathbf{P}),t_2\right)} e^{-iH_{\sigma_{t_2}}t_2}\psi^{(t_2)}_{l(j),\sigma_{t_2}} \rightarrow$$

$$\rightarrow e^{iHt_2}\sum_{j=1}^{N(t_1)} W_{\sigma_{t_2}}\left(\mathbf{v}_j,t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),t_2\right)} e^{-iH_{\sigma_{t_2}}t_2}\psi^{(t_1)}_{j,\sigma_{t_2}}$$

After the change of the partition, we separately consider the time evolution and the infrared cut-off variation.

The backwards time evolution, at fixed cut-off σ_{t_2}, corresponds to

D2)

$$\sum_j e^{iHt_2} W_{\sigma_{t_2}}\left(\mathbf{v}_j,t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_2\right)} e^{-iE^{\sigma_{t_2}}t_2}\psi^{(t_1)}_{j,\sigma_{t_2}} \rightarrow \sum_j e^{iHt_1} W_{\sigma_{t_2}}\left(\mathbf{v}_j,t_1\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_1\right)} e^{-iE^{\sigma_{t_2}}t_1}\psi^{(t_1)}_{j,\sigma_{t_2}}$$

the study of the above quantity requires Cook's argument.

In the analysis of the variation of the infrared cut-off, $\sigma_{t_2} \rightarrow \sigma_{t_1}$

$$\sum_j e^{iHt_1} W_{\sigma_{t_2}}\left(\mathbf{v}_j,t_1\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_1\right)} e^{-iE^{\sigma_{t_2}}t_1}\psi^{(t_1)}_{j,\sigma_{t_2}} \rightarrow \sum_j e^{iHt_1} W_{\sigma_{t_1}}\left(\mathbf{v}_j,t_1\right) e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1\right)} e^{-iE^{\sigma_{t_1}}t_1}\psi^{(t_1)}_{j,\sigma_{t_1}},$$

for convenience, we consider each vector in the (first) sum as the product of two blocks

$$\left[e^{iHt_1} W_{\sigma_{t_2}}\left(\mathbf{v}_j,t_1\right) W^{b\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}\right)\right]\cdot\left[e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_1\right)} e^{-iE^{\sigma_{t_2}}t_1} W^b_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}\right)\psi^{(t_1)}_{j,\sigma_{t_2}}\right],$$

where $W^b_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}(\mathbf{P})\right) = e^{-g\int_{\sigma_t}^{\kappa}\frac{b(\mathbf{k})-b^\dagger(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{t_2}}(\mathbf{P})\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$. Let us term "dressing" block the first one and "regular" block the second one.

The contribution due to the infrared cut-off variation $\sigma_{t_1} \rightarrow \sigma_{t_2}$ consists of :

- a very simple part concerned with the convergence of the "regular" block in the limit $\sigma \rightarrow 0$

D3)

$$e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_1\right)} e^{-iE^{\sigma_{t_2}}t_1} W^b_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}\right)\psi^{(t_1)}_{j,\sigma_{t_2}} \rightarrow e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1\right)} e^{-iE^{\sigma_{t_1}}t_1} W^b_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}\right)\psi^{(t_1)}_{j,\sigma_{t_1}}$$

- a more delicate one which comes from the variation of the "dressing" block

D4)

$$W_{\sigma_{t_2}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) \to W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}).$$

Let us analyze the variation **D4)** in details. It can be written as

$$W_{\sigma_{t_2}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) \to W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})$$

(where $W^{b}_{\sigma_{t_2}}(\mathbf{v}_j) = e^{-g\int_{\sigma_t}^{\kappa_1} \frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$) so that we can perform three smaller steps:

D4.1)

$$W_{\sigma_{t_2}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) \to W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})$$

this step is controlled thanks to the strong Hoelder property in $\mathbf{P}$ of the "regular" block

$$e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)}\, e^{-iE^{\sigma_{t_1}} t_1}\, W^{b}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\, \psi^{(t_1)}_{j,\sigma_{t_1}}$$

that neutralizes the logarithmic divergence arising from the variation of the lower bound of the Weyl operator $W_{\sigma_{t_2}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_2}}(\mathbf{v}_j)$;

D4.2)

$$W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) \to W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}})$$

D4.3)

$$W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}}) \to W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})$$

which account for the difference between the gradient $\nabla E^{\sigma_{t_2}}(\mathbf{P})$ and the mean velocity $\mathbf{v}_j$, corresponding to the cell center $\overline{\mathbf{P}_j}$.

analysis of D1)

Let us examine the square norm:

$$\left\| e^{iH_{\sigma_{t_2}} t_2} \sum_{j=1}^{N(t_1)} \sum_{l(j)} \left(W_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, t_2)\, e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}(\mathbf{P}), t_2)} - W_{\sigma_{t_2}}(\mathbf{v}_j, t_2)\, e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}(\mathbf{P}), t_2)} \right) e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l(j),\sigma_{t_2}} \right\|^2.$$

To compress the formulas, let us define $\widehat{W}_{\sigma_{t_2}(\mathbf{v}_j, t_2)} \equiv W_{\sigma_{t_2}}(\mathbf{v}_j, t_2)\, e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_2)}$; the square norm above corresponds to

$$\sum_{j,j'=1}^{N(t_1)} \sum_{l(j),l'(j')} \left(e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l(j),\sigma_{t_2}}, \left(\widehat{W}^{\dagger}_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, t_2) - \widehat{W}^{\dagger}_{\sigma_{t_2}}(\mathbf{v}_j, t_2) \right) \left(\widehat{W}_{\sigma_{t_2}}(\mathbf{v}_{l'(j')}, t_2) - \widehat{W}_{\sigma_{t_2}}(\mathbf{v}_{j'}, t_2) \right) e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l'(j'),\sigma_{t_2}} \right)$$

(note that $\psi_{j,\sigma_{t_2}}^{(t_1)} = \int_{\Gamma_j} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_{t_2}} d^3P = \sum_{l(j)} \int_{\Gamma_{l(j)}} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_{t_2}} d^3P$ by definition).

The sum of the terms where $j' \neq j$ and $l'(j) \neq l(j)$ vanishes, for $t_2 \to +\infty$. Its rate is surely bounded (from above) by of a quantity of order $t_2^{-\epsilon}$, as we can estimate by the same procedure used in the norm control of $\psi_h(t)$. The remainder is given by terms of this type

$$\left(e^{-iH_{\sigma_{t_2}} t_2} \psi_{l(j),\sigma_{t_2}}^{(t_2)}, \left(2 - \widehat{W}_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j, t_2)\, \widehat{W}_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, t_2) - \widehat{W}_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_{l(j)}, t_2)\, \widehat{W}_{\sigma_{t_2}}(\mathbf{v}_j, t_2)\right) e^{-iH_{\sigma_{t_2}} t_2} \psi_{l(j),\sigma_{t_2}}^{(t_2)}\right).$$

Let us examine:

$$\left(e^{-iH_{\sigma_{t_2}} t_2} \psi_{l(j),\sigma_{t_2}}^{(t_2)}, e^{-i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_2)} W_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j, t_2)\, W_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, t_2)\, e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}, t_2)} e^{-iH_{\sigma_{t_2}} t_2} \psi_{l(j),\sigma_{t_2}}^{(t_2)}\right).$$

As in the norm control, for $s \geq t_2$ we consider

$$\left(e^{-iH_{\sigma_{t_2}} s} \psi_{l(j),\sigma_{t_2}}^{(t_2)}, e^{-i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, s)} W_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j, s)\, W_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, s)\, e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}, s)} e^{-iH_{\sigma_{t_2}} s} \psi_{l(j),\sigma_{t_2}}^{(t_2)}\right).$$

The limit for $s \to +\infty$ of the above expression is:

$$e^{-\frac{C_{j,l(j)}}{2}} \left(e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}(\mathbf{P}), \sigma_{t_2}^{-\frac{40}{39}}\right)} \psi_{l(j),\sigma_{t_2}}^{(t_2)}, e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}(\mathbf{P}), \sigma_{t_2}^{-\frac{40}{39}}\right)} \psi_{l(j),\sigma_{t_2}}^{(t_2)} \right)$$

where $C_{j,l(j)} \equiv \int_{\sigma_{t_2}}^{\kappa_1} \left| h_{j,l(j)}\left(\widehat{\mathbf{k}}\right)\right|^2 \frac{d^3k}{2|\mathbf{k}|^3}$.
Summing over the cells, the total error is surely bounded by a quantity of order $t_2^{-\epsilon}$. Then the discussion is restricted to

$$\sum_{j=1}^{N(t_1)} \sum_{l(j)} \left(\psi_{l(j),\sigma_{t_2}}^{(t_2)}, \left(2 - e^{i\Delta\gamma_{\sigma_{t_2}}(\mathbf{v}_{l(j)} - \mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_2)} e^{-\frac{C_{l(j),j}}{2}} - e^{i\Delta\gamma_{\sigma_{t_2}}(\mathbf{v}_j - \mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}, t_2)} e^{-\frac{C_{j,l(j)}}{2}}\right) \psi_{l(j),\sigma_{t_2}}^{(t_2)}\right)$$

(where $e^{i\Delta\gamma_{\sigma_{t_2}}(\mathbf{v}_j - \mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}, t_2)} \equiv e^{-i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, \sigma_{t_2}^{-\frac{40}{39}}\right)} e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, \nabla E^{\sigma_{t_2}}, \sigma_{t_2}^{-\frac{40}{39}}\right)}$) that we can control by

$$\sum_{j=1}^{N(t_1)} \sum_{l(j)} \left(\left\| \psi_{l(j),\sigma_{t_2}}^{(t_2)} \right\|^2 \cdot C \cdot t_1^{-\frac{\epsilon}{16}} \cdot |\ln(\sigma_{t_2})| \right) \leq C \cdot t_1^{-\frac{\epsilon}{16}} \cdot |\ln(\sigma_{t_2})|$$

since

- $C_{l(j),j} \equiv \int_{\sigma_{t_2}}^{\kappa_1} \left| h_{l(j),j}\left(\widehat{\mathbf{k}}\right)\right|^2 \frac{d^3k}{2|\mathbf{k}|^3}$ is bounded by $C \cdot t_1^{-\frac{\epsilon}{16}} \cdot |\ln(\sigma_{t_2})|$ because of the difference $\mathbf{v}_j - \mathbf{v}_{l(j)}$ and because of lemma 3.3 ;
- $\left| e^{i\Delta\gamma_{\sigma_{t_2}}(\mathbf{v}_{l(j)} - \mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_2)} \right|$ is bounded by $C \cdot t_1^{-\frac{\epsilon}{16}} \cdot |\ln(\sigma_{t_2})|$ uniformly in $\mathbf{P} \in \Sigma$ (see lemma B2).

analysis of D2)

The term **D2)** corresponds to

$$\sum_{j=1}^{N(t_1)}\left\{e^{iHt_2}W_{\sigma_{t_2}}(\mathbf{v}_j,t_2)\,e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_2)}e^{-iE^{\sigma_{t_2}}t_2}\psi^{(t_1)}_{j,\sigma_{t_2}}-e^{iHt_1}W_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_1)}e^{-iE^{\sigma_{t_2}}t_1}\psi^{(t_1)}_{j,\sigma_{t_2}}\right\}$$

By Cook's argument we estimate the contribution for each cell by expressing the difference of the two vectors as the following integral from t_1 to t_2:

$$\int_{t_1}^{t_2}\frac{d}{ds}\left\{e^{iHs}W_{\sigma_{t_2}}(\mathbf{v}_j,s)\,e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},s)}e^{-iE^{\sigma_{t_2}}s}\psi^{(t_1)}_{j,\sigma_{t_2}}\right\}ds$$

The integral is bounded by the integral of the norm of the following derivative:

$$\frac{d}{ds}\left\{e^{iHs}W_{\sigma_{t_2}}(\mathbf{v}_j,s)\,e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},s)}e^{-iE^{\sigma_{t_2}}s}\psi^{(t_1)}_{j,\sigma_{t_2}}\right\}=$$

$$=ie^{iHs}W_{\sigma_{t_2}}(\mathbf{v}_j,s)\left(\varphi_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)+\frac{d\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),s)}{ds}\right)e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_2)}e^{-iE^{\sigma_{t_2}}s}\psi^{(t_1)}_{j,\sigma_{t_2}}+\quad(20)$$

$$+ie^{iHs}W_{\sigma_{t_2}}(\mathbf{v}_j,s)\left(H-H_{\sigma_{t_2}}\right)e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},s)}e^{-iE^{\sigma_{t_2}}s}\psi^{(t_1)}_{j,\sigma_{t_2}}\quad(21)$$

The formal steps are operatorially well defined because $\psi^{(t_1)}_{j,\sigma_{t_2}}\in D\left(H_{\sigma_{t_2}}\right)\equiv D(H)$(see also lemma B8).

estimate of (20)

Analogously to the proof of **II)** in paragraph 4.1, we decompose $\varphi_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)$ in

$$\varphi^1_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)+\varphi^2_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)$$

having defined:

- $\varphi^1_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)\equiv g^2\int_{\sigma_{t_2}}^{s^{-\frac{39}{40}}}\int\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d\Omega d|\mathbf{k}|$
- $\varphi^2_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)\equiv g^2\int_{s^{-\frac{39}{40}}}^{\kappa_1}\int\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d\Omega d|\mathbf{k}|$.

By a procedure analogous to the one used in proof **II)** of paragraph 4.1, we obtain the following estimate for the norm of the expression (20):

$$C\cdot t_1^{-7\epsilon}\cdot(\ln\sigma_{t_2})^2$$

estimate of (21)

The norm of the vector

$$\left(H-H_{\sigma_{t_2}}\right)e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_2)}\psi^{(t_1)}_{j,\sigma_{t_2}}=e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j,\nabla E^{\sigma_{t_2}},t_2)}g\int_0^{\sigma_{t_2}}\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\psi^{(t_1)}_{j,\sigma_{t_2}}$$

can be estimated in the following way:

$$\left\| g \int_0^{\sigma_{t_2}} \left(b(\mathbf{k}) + b^\dagger(\mathbf{k}) \right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}} \psi^{(t_1)}_{j,\sigma_{t_2}} \right\| \leq \left\| g \int_0^{\sigma_{t_2}} b(\mathbf{k}) \frac{d^3k}{\sqrt{2|\mathbf{k}|}} \psi^{(t_1)}_{j,\sigma_{t_2}} \right\| + \left\| g \int_0^{\sigma_{t_2}} b^\dagger(\mathbf{k}) \frac{d^3k}{\sqrt{2|\mathbf{k}|}} \psi^{(t_1)}_{j,\sigma_{t_2}} \right\| \leq$$

$$\leq \left(g^2 \int_0^{\sigma_{t_2}} \frac{d^3k}{2|\mathbf{k}|} \right)^{\frac{1}{2}} \cdot \left\| \psi^{(t_1)}_{j,\sigma_{t_2}} \right\| \leq C \cdot (\sigma_{t_2}) \cdot t_1^{-\frac{3\epsilon}{2}}$$

(note that $b(\mathbf{k})\, \psi^{(t_1)}_{j,\sigma_{t_2}} = 0$ for $\mathbf{k}: \ |\mathbf{k}| \leq \sigma_{t_2}$) .

Therefore the norm of the term *D2)* is bounded by a quantity of order

$$t_1^{-\epsilon} \cdot (\ln \sigma_{t_2})^2 + t_2 \cdot (\sigma_{t_2}) \cdot t_1^{\frac{3\epsilon}{2}}$$

(remember that the constraint $2\delta + 4\epsilon < \frac{1}{112}$ is assumed).

analysis of D3)

$$\sum_{j=1}^{N(t_1)} \left[e^{iHt_1} W_{\sigma_{t_2}}(\mathbf{v}_j, t_1)\, W^{b^\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) \right] \left[e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_1)} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, e^{-iE^{\sigma_{t_2}} t_1} - e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\, e^{-iE^{\sigma_{t_1}}}\right.$$

Let us analyze the difference for a single cell. Its norm vanishes with a rate related to the decrease of the cut-off. In fact

$$\left\| e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_1)} e^{-iE^{\sigma_{t_2}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, \psi^{(t_1)}_{j,\sigma_{t_2}} - e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\, \psi^{(t_1)}_{j,\sigma_{t_1}} \right\| \leq$$

$$\leq \left\| e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_1)} e^{-iE^{\sigma_{t_2}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, \psi^{(t_1)}_{j,\sigma_{t_2}} - e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_2}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, \psi^{(t_1)}_{j,\sigma_{t_2}} \right\| +$$

$$+ \left\| e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_2}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, \psi^{(t_1)}_{j,\sigma_{t_2}} - e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, \psi^{(t_1)}_{j,\sigma_{t_2}} \right\| +$$

$$+ \left\| e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, \psi^{(t_1)}_{j,\sigma_{t_2}} - e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\, \psi^{(t_1)}_{j,\sigma_{t_1}} \right\|$$

first term of the sum

The absolute value of $e^{i\gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}, t_1)} - e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)}$ can be estimated in terms of the absolute value of the difference between the arguments of the exponentials (lemma B2):

$$\left| \gamma_{\sigma_{t_2}}(\mathbf{v}_j, \nabla E^{\sigma_{t_2}}(\mathbf{P}), t_1) - \gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}(\mathbf{P}), t_1) \right| \leq C \cdot (\sigma_{t_1})^{\frac{1}{4}} \int_1^{t_1} \tau^{-\frac{19}{20}} d\tau + C \cdot t_1 \cdot \sigma_{t_1}$$

where $\mathbf{P} \in \Sigma$.
Then we surely have a bound by a quantity of order $t_1 \cdot (\sigma_{t_1})^{\frac{1}{4}}$.

second term of the sum

Taking in account the iterative procedure ($\mathbf{P} \in \Sigma$)

$$\left| e^{-iE^{\sigma_{t_2}}(\mathbf{P}) t_1} - e^{-iE^{\sigma_{t_1}}(\mathbf{P}) t_1} \right| \leq C \cdot (\sigma_{t_1}) \cdot t_1 \,.$$

third term of the sum

$$\left\| e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\,\psi^{(t_1)}_{j,\sigma_{t_2}} - e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}} \right\| =$$

$$= \left\| \int_{\Gamma_j} h(\mathbf{P})\,(\,W^b_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_2}} - W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,)\,\psi_{\mathbf{P},\sigma_{t_1}} d^3P \right\| \le C\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot t_1^{-\frac{3\epsilon}{2}}$$

as follows from theorem 3.2.

Therefore, the norm of ***D3)*** is surely bounded by:

$$C\cdot t_1^{\frac{3\epsilon}{2}+1}\cdot(\sigma_{t_1})^{\frac{1}{4}}\,.$$

analysis of D4.1)

$$\sum_{j=1}^{N(t_1)} e^{iHt_1} W_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_2}}(\mathbf{v}_j)\,W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}} +$$

$$-\sum_{j=1}^{N(t_1)} e^{iHt_1} W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\,W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}}$$

Having defined

$$\varphi_j = W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}} t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}}$$

the expression above can be written as:

$$\sum_{j=1}^{N(t_1)} e^{iHt_1}\left(W_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_2}}(\mathbf{v}_j) - W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\right)\varphi_j\,.$$

We can restrict the analysis to a single cell. For this purpose we examine

$$W_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_2}}(\mathbf{v}_j) - W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j) =$$

$$= W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\left(e^{ig^2\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\sin(|\mathbf{k}|t_1-\mathbf{k}\cdot\mathbf{x})}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d^3k}\; e^{-g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-a^\dagger(\mathbf{k})\left(e^{-i|\mathbf{k}|t_1}-e^{-i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} - I \right) =$$

$$= W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\left(e^{ig^2\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\sin(|\mathbf{k}|t_1-\mathbf{k}\cdot\mathbf{x})}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d^3k}\right)\left(e^{-g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-a^\dagger(\mathbf{k})\left(e^{-i|\mathbf{k}|t_1}-e^{-i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} - I \right) +$$

$$+ W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\left(e^{ig^2\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\sin(|\mathbf{k}|t_1-\mathbf{k}\cdot\mathbf{x})}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d^3k} - I \right)$$

Since the vector φ_j belongs to the domain of the generator, we have

$$\left(e^{-g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-a^\dagger(\mathbf{k})\left(e^{-i|\mathbf{k}|t_1}-e^{-i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}-I\right)\varphi_j=$$

$$=-\int_0^1 e^{-g\lambda\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-a^\dagger(\mathbf{k})\left(e^{-i|\mathbf{k}|t_1}-e^{-i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}d\lambda\cdot g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-c.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\varphi_j$$

and

$$\left\|g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-c.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\varphi_j\right\|\le\left\|g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\varphi_j\right\|+$$

$$+\left\|g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{a^\dagger(\mathbf{k})\left(e^{-i|\mathbf{k}|t_1}-e^{-i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\varphi_j\right\|\le$$

$$\le 2\left\|g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{b(\mathbf{k})\left(e^{i|\mathbf{k}|t_1-i\mathbf{k}\cdot\mathbf{x}}-1\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\varphi_j\right\|+\left(\varphi_j,g^2\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{\left|e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2}d^3k\varphi_j\right)^{\frac{1}{2}}$$

In lemma B9 the vanishing of the two expressions above is studied. Moreover the norm

$$\left\|\left(e^{2i\int\frac{\sin(|\mathbf{k}|t_1-\mathbf{k}\cdot\mathbf{x})}{|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d^3k}-I\right)\varphi_j\right\|$$

can be easily reconciled with the second term in the expression above.

At the end we obtain that the norm of the term **D4.1)** is surely bounded by a quantity of order:

$$t_1\cdot t_1^{2\epsilon}\cdot|\ln\sigma_{t_2}|\cdot(\sigma_{t_1})^{\frac{1}{16}}\ .$$

analysis of D4.2)

$$\sum_{j=1}^{N(t_1)}e^{iHt_1}W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\,W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)}e^{-iE^{\sigma_{t_1}}t_1}W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}}\to$$

$$\to\sum_{j=1}^{N(t_1)}e^{iHt_1}W_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\,W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}})\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)}e^{-iE^{\sigma_{t_1}}t_1}W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}}$$

For a single cell, by a reasoning used in lemma B9, we have

$$\left\|\left(W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})-W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}})\right)W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}}\right\|\le C\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot|\ln\sigma_{t_2}|\cdot t_1^{-\frac{3\epsilon}{2}}$$

so that the norm of D4.2) is bounded by

$$C\cdot t_1^{\frac{3\epsilon}{2}}\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot|\ln\sigma_{t_2}|\,.$$

analysis of D4.3)

$\sum_{j=1}^{N(t_1)} e^{iHt_1} W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}})\, e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1\right)} e^{-iE^{\sigma_{t_1}} t_1} W^{b}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\, \psi^{(t_1)}_{j,\sigma_{t_1}} \rightarrow$

$\rightarrow \sum_{j=1}^{N(t_1)} e^{iHt_1} W_{\sigma_{t_1}}(\mathbf{v}_j, t_1)\, e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1\right)} e^{-iE^{\sigma_{t_1}} t_1} \psi^{(t_1)}_{j,\sigma_{t_1}}$

This variation can be written as

$$\sum_{j=1}^{N(t_1)} e^{iHt_1} W_{\sigma_{t_1}}(\mathbf{v}_j, t_1) \left(W^{b}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}}) - I \right) e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j, \nabla E_{\sigma_{t_1}}, t_1\right)} e^{-iE_{\sigma_{t_1}} t_1} \psi^{(t_1)}_{j,\sigma_{t_1}}$$

with the definitions

- $W^{b}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\mathbf{v}_j) \equiv W^{b\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\, W^{b}_{\sigma_{t_2}}(\mathbf{v}_j) = e^{-g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{b(\mathbf{k}) - b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1 - \widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$
- $W^{b\dagger}\,|^{\sigma_{t_2}}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}) \equiv W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}})\, W^{b}_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}) = e^{g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{b(\mathbf{k}) - b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1 - \widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{t_1}}\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$

The discussion of this contribution requires the study of the squared norm and the control of the mixed terms, with respect to the cell-partition, in the corresponding scalar product. We verify that the sum of the mixed terms vanishes and we estimate the rate. Then we turn to the diagonals terms.

mixed terms

Let us consider the generic l-j term. We observe that it is possible to reply the same procedure as in par.4.1. Indeed, for the given expression, the (annihilation) operators, that we obtain from the derivative with respect to the real parameter λ, substantially pass through $\left(W^{b}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}})\right)$, in the limit $s \to \infty$. By similar steps an analogous estimate can be obtained: the sum of the absolute values of the mixed terms is bounded by $C \cdot t_1^{-\epsilon}$.

diagonal terms

Considering that:

- the norm
$$\left\| \left(W^{b}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_1}}(\mathbf{P})) - I \right) \psi^{(t_1)}_{j,\sigma_{t_1}} \right\|$$
can be estimated by a quantity of order
$$\sup_{\mathbf{P}\in\Gamma_j} |\mathbf{v}_j - \nabla E^{\sigma_{t_1}}(\mathbf{P})| \cdot |\ln \sigma_{t_2}| \cdot t_1^{-\frac{3\epsilon}{2}}$$

- $\sup_{\mathbf{P}\in\Gamma_j} \left|\nabla E^{\sigma_{t_1}}(\mathbf{P}) - \nabla E^{\sigma_{t_1}}(\overline{\mathbf{P}_j})\right| \leq C \cdot \sup_{\mathbf{P}\in\Gamma_j} \left|\mathbf{P} - \overline{\mathbf{P}_j}\right|^{\frac{1}{16}} \leq C \cdot t_1^{-\frac{\epsilon}{16}}$

  (for the last step, see lemma 3.3)

the sum of the diagonal terms provides a contribution bounded by

$$C \cdot t_1^{-\frac{\epsilon}{16}} \cdot |\ln \sigma_{t_2}| \ .$$

Therefore it follows that the norm of the term **D4.3)** is bounded by

$$C \cdot t_1^{-\frac{\epsilon}{16}} \cdot |\ln \sigma_{t_2}| \ .$$

Theorem 4.1

The vector $\psi_h(t)$ converges strongly for $t \to +\infty$, with an error of order $\frac{1}{t^\rho}$ where $\rho > 0$ is a properly small coefficient.

Proof

Let us consider the bounds obtained for the norms of the quantities D1,...D4.3. Having tuned the time scales related to $\epsilon$and δ in accordance with the constraints

- $72\epsilon < \delta$.
- $2\delta + 4\epsilon < \frac{1}{112}$

and after having chosen $\sigma_t = t^{-128}$, for instance, we can observe that each term D1,...D4.3 is bounded by a quantity of order

$$\left(\frac{\ln(t_2)}{t_1^{3\rho}}\right)^2$$

where $\rho > 0$. Then we can estimate

$$\|\psi_h(t_2) - \psi_h(t_1)\| < C \cdot \left(\frac{\ln(t_2)}{t_1^{3\rho}}\right)^2$$

where $\rho > 0$ e $C > 0$ are independent of t_1 and t_2 (for $t_2 \geq t_1 > \bar{t} \gg 1$).

Now, let us consider the sequence $t_1, t_1^2, t_1^3, ...t_1^n, ...$ and put $t_1^n \leq t_2 < t_1^{n+1}$. Due to the norm properties, it follows that:

$$\|\psi_h(t_2) - \psi_h(t_1)\| \leq \left\|\psi_h(t_1^2) - \psi_h(t_1)\right\| + \left\|\psi_h(t_1^3) - \psi_h(t_1^2)\right\| + ... + \|\psi_h(t_2) - \psi_h(t_1^n)\| \leq$$

$$\leq C \cdot \left\{\left(\frac{2\ln(t_1)}{t_1^{3\rho}}\right)^2 + \left(\frac{3\ln(t_1)}{t_1^{2\cdot 3\rho}}\right)^2 + \left(\frac{4\ln(t_1)}{t_1^{3\cdot 3\rho}}\right)^2 + ... + \left(\frac{(n+1)\ln(t_1)}{t_1^{n\cdot 3\rho}}\right)^2\right\} \leq$$

$$\leq \frac{C}{t_1^\rho} \cdot \left\{\left(\frac{2}{t_1^\rho} \cdot \frac{\ln(t_1)}{t_1^\rho}\right)^2 + \left(\frac{3}{t_1^{2\rho}} \cdot \frac{\ln(t_1)}{t_1^\rho}\right)^2 + \left(\frac{4}{t_1^{3\rho}} \cdot \frac{\ln(t_1)}{t_1^\rho}\right)^2 + ... + \left(\frac{n+1}{t_1^{n\rho}} \cdot \frac{\ln(t_1)}{t_1^\rho}\right)^2\right\}$$

For t_1 sufficiently large, $t_1 \geq \widehat{t_1} > \bar{t} \gg 1$, the series inside the brackets is convergent, and it is bounded by a constant less than 1. We can conclude that $\forall t_1, t_2$, where $t_2 \geq t_1 \geq \widehat{t_1}$,

$$\|\psi_h(t_2) - \psi_h(t_1)\| \leq \frac{C}{t_1^\rho} \ .$$

5 Scattering subspaces and asymptotic observables.

In this chapter, at first we construct the subspace $\mathrm{H}^{1out(in)}$ as the norm closure of the finite linear combinations of the vectors $\left\{\psi_h^{out(in)}\right\}$. The invariance requirement, under space-time translations, for the space $\mathrm{H}^{1out(in)}$ implies a more general definition of the vector $\psi_h^{out(in)}$, in the sense that for each given $\psi_h^{out(in)}$ we consider the family of vectors $\psi_{h,\tau,\mathbf{a}}^{out(in)}$ corresponding to the evolution in the time τ and to the translation of a quantity $\mathbf{a}$ of the previous vector.
Then, we verify that on the subspace $\mathrm{H}^{1out(in)} \equiv \overline{\left\{\bigvee \psi_{h,\tau,\mathbf{a}}^{out(in)}\ h \in C_0^1\left(R^3 \setminus 0\right), \, supph \subset \Sigma \, , \tau \in R \, , \mathbf{a} \in R^3\right\}}$

the strong limits of the functions, continuous and of compact support, of the electron mean velocity exist and the strong limits of the L.S.Z. Weyl operator associated to the photon field as well. Due to these results, we can define the vectors $\psi_{h,\varphi}^{out(in)}$ ($\tau, \mathbf{a}$ are omitted) obtained by applying the L.S.Z. Weyl operators, with smearing functions $\left\{\varphi : \widetilde{\varphi}\left(\mathbf{k}\right) \in C_0^\infty\left(R^3 \setminus 0\right)\right\}$, to the total set of $\mathrm{H}^{1out(in)}$.
The norm closure of the finite linear combinations of the $\left\{\psi_{h,\varphi}^{out(in)}\right\}$ is a reasonable candidate for the scattering subspaces $\mathrm{H}^{out(in)}$. The physical meaning of this definition is in the characterization of the $\mathrm{H}^{out(in)}$ states in terms of quantum numbers of the asymptotic variables which are well defined on them: the asymptotic photon Weyl operators and the asymptotic electron mean velocity.
For sure, the spectral restriction on the electron velocity (strictly less than 1) implies a restriction of $\mathrm{H}^{out(in)}$ as subspaces of H, explained with the partial non-relativistic character of the model.
In theorem 5.1, the vectors $\left\{\psi_{h,\varphi}^{out(in)}\right\}$ are constructed. In theorem 5.2, we prove the convergence of continuous and of compact support functions of the electron mean velocity on the vectors of $\mathrm{H}^{out(in)}$. The corollary 5.3 is a check of the fact that on $\mathrm{H}^{out(in)}$ the strong limits of the L.S.Z. Weyl operators generate a canonic Weyl algebra $\mathcal{A}^{out(in)}$, to which a free massless scalar field is associated .

Definition of the vector $\psi_{h,\tau,\mathbf{a}}^{out}$.

Applying the operator $e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}$ to the generic vector ψ_h^{out} , we obtain:

$$e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}\psi_h^{out} =$$

$$= s - \lim_{t\to+\infty} e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}e^{iHt}\sum_{j=1}^{N(t)} W_{\sigma_t}\left(\mathbf{v}_i, t\right) e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t\right)}e^{-iE^{\sigma_t}(t-\tau)}e^{-iE^{\sigma_t}\tau}\psi_{j,\sigma_t}^{(t)} =$$

$$= s\text{−}\lim_{t\to+\infty} e^{iHt}\sum_{j=1}^{N(t+\tau)} e^{-g\int_{\sigma_{t+\tau}}^{\kappa_1}\frac{a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{a}}e^{i|\mathbf{k}|(t+\tau)}-c.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}e^{i\gamma_{\sigma_{t+\tau}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t+\tau}}(\mathbf{P}),t+\tau\right)}e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iE^{\sigma_{t+\tau}}t}e^{-iE^{\sigma_{t+\tau}}\tau}\psi_{j,\sigma_{t+\tau}}^{(t+\tau)}$$

$$= s - \lim_{t\to+\infty}\psi_{h,\tau,\mathbf{a}}\left(t\right)$$

where $\psi_{h,\tau,\mathbf{a}}\left(t\right)$ corresponds to the approximating vector $\psi_h\left(t\right)$ with translated wave function.
The last equality follows by the same proof provided for ψ_h^{out}, apart from some little marginal

differences. The definition

$$\psi^{out}_{h,\tau,\mathbf{a}} \equiv e^{-i\mathbf{a}\cdot\mathbf{P}} e^{-iH\tau} \psi^{out}_h$$

is meaningful, thereby.

The subspace of the minimal asymptotic electron states is defined as $\mathrm{H}^{1out} \equiv \overline{\bigvee \psi^{out}_{h,\tau,\mathbf{a}}}$ where we simply denote $\{\psi^{out}_h\}$ the total set that generates H^{1out}.

Theorem 5.1

The strong limits

$$s - lim_{t\to+\infty} e^{iHt} e^{i\left(a(\varphi_t)+a^\dagger(\varphi_t)\right)} e^{-iHt} \psi^{out}_h$$

exist, where $\widetilde{\varphi}(\mathbf{k}) \in C^\infty_0\left(R^3 \setminus 0\right)$, $\varphi_t(\mathbf{y}) = \int e^{-i\mathbf{k}\mathbf{y}+i|\mathbf{k}|t} \widetilde{\varphi}(\mathbf{k})\, d^3k$ and $a^\dagger(\varphi) \equiv (a(\varphi))^\dagger = \left(\int a(\mathbf{k})\, \widetilde{\varphi}(\mathbf{k})\, d^3k\right)^\dagger$.

Proof

The convergence follows from the integrability of the norm

$$\left\| \frac{d\left(e^{iHt} e^{i\left(a(\varphi_t)+a^\dagger(\varphi_t)\right)} e^{-iHt}\right)}{dt} \psi^{out}_h \right\| =$$

$$= \left\| e^{i\left(a(\varphi_t)+a^\dagger(\varphi_t)\right)} \varphi(\mathbf{x},t)\, e^{-iHt} \psi^{out}_h \right\| \leq \left\| \varphi(\mathbf{x},t)\, e^{-iHt} \left(\psi^{out}_h - \psi_h(t)\right) \right\| + \left\| \varphi(\mathbf{x},t)\, e^{-iHt} \psi_h(t) \right\|$$

where $\varphi(\mathbf{x},t) \equiv g \int \left(\widetilde{\varphi}(\mathbf{k})\, e^{i|\mathbf{k}|t - i\mathbf{k}\cdot\mathbf{x}} + \overline{\widetilde{\varphi}}(\mathbf{k})\, e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} \right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}}$.

Both the two norms on the right hand side are bounded by quantities of order $\frac{1}{t^{1+\rho'}}$, $\rho' > 0$:
- the first one because $\sup_{\mathbf{x}} |\varphi(\mathbf{x},t)| \leq \frac{M}{t}$ and $\|(\psi^{out}_h - \psi_h(t))\| \leq \frac{C}{t^\rho}$ (M and C are constants);

- as regards the second one, starting from the identity

$$\varphi(\mathbf{x},t)\, e^{-iHt} \psi_h(t) = \varphi(\mathbf{x},t) \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j,t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)} e^{-iE^{\sigma_t}t} \psi^{(t)}_{j,\sigma_t} =$$

$$= \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j,t)\, \varphi(\mathbf{x},t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)} e^{-iE^{\sigma_t}t} \int_{\Gamma_j} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_t} d^3P =$$

$$= \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j,t)\, \varphi(\mathbf{x},t) \left(1_{\Gamma_j}(\mathbf{P}) - \chi^{(t)}_{\mathbf{v}_j}(\nabla E^{\sigma_t},t)\right) e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)} e^{-iE^{\sigma_t}t} \int_{\Gamma_j} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_t} d^3P + \quad \text{(I)}$$

$$+ \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j,t)\, \varphi(\mathbf{x},t) \left(\chi^{(t)}_{\mathbf{v}_j}(\nabla E^{\sigma_t},t) - \chi^{(t)}_{\mathbf{v}_j}\left(\tfrac{\mathbf{x}}{t},t\right)\right) e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)} e^{-iE^{\sigma_t}t} \int_{\Gamma_j} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_t} d^3P + \quad \text{(II)}$$

$$+ \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j,t)\, \varphi(\mathbf{x},t)\, \chi^{(t)}_{\mathbf{v}_j}\left(\tfrac{\mathbf{x}}{t},t\right) e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),t)} e^{-iE^{\sigma_t}t} \int_{\Gamma_j} h(\mathbf{P})\, \psi_{\mathbf{P},\sigma_t} d^3P \quad \text{(III)}$$

we exploit lemma B1 for (I), lemma B3 for (II), and Huygens' principle in order to estimate

$sup_{\mathrm{x}} \left| \varphi(\mathrm{x}, t)\, \chi^{(t)}_{\mathrm{v}_j}\left(\frac{\mathrm{x}}{t}, t\right) \right|$ in the expression (III) .

The scattering subspaces $\mathrm{H}^{out(in)} \equiv \overline{\bigvee \psi^{out(in)}_{h,\varphi}}$ are invariant under space-time translations because the subspaces $\mathrm{H}^{1out(in)}$ are invariant, by construction.

Theorem 5.2

The continuous and of compact support functions f of the electron mean velocity have strong limits in H^{out} for $t \to +\infty$, in particular

$$s - lim_{t\to+\infty} e^{iHt} f\left(\frac{\mathrm{X}}{t}\right) e^{-iHt} \psi^{out}_{h,\varphi} = \psi^{out}_{h\cdot\widehat{f},\varphi}$$

where $\widehat{f}(\mathbf{P}) \equiv lim_{\sigma\to 0} f(\nabla E^{\sigma}(\mathbf{P}))$.

Proof

It is sufficient to prove it on the vectors $\psi^{out}_{h,\varphi}$ for functions $f \in C_0^{\infty}(R^3)$. Exploiting theorem 5.1 and the uniform boundness in t of the operators $f\left(\frac{\mathrm{x}}{t}\right)$ and $e^{iHt} e^{i\left(a(\varphi_t)+a^{\dagger}(\varphi_t)\right)} e^{-iHt}$ we obtain

$$s - lim_{t\to+\infty} e^{iHt} f\left(\tfrac{\mathrm{x}}{t}\right) e^{-iHt} \psi^{out}_{h,\varphi} = s - lim_{t\to+\infty} e^{iHt} f\left(\tfrac{\mathrm{x}}{t}\right) e^{i\left(a(\varphi_t)+a^{\dagger}(\varphi_t)\right)} e^{-iHt} \psi^{out}_{h} =$$
$$= s - lim_{t\to+\infty} e^{iHt} e^{i\left(a(\varphi_t)+a^{\dagger}(\varphi_t)\right)} f\left(\tfrac{\mathrm{x}}{t}\right) e^{-iHt} \psi^{out}_{h} =$$

$$= s - lim_{t\to+\infty} e^{iHt} e^{i\left(a(\varphi_t)+a^{\dagger}(\varphi_t)\right)} e^{-iHt} \psi^{out}_{h\cdot\widehat{f}}$$

(the last step is proved by the technique used in lemma B3).

The extension to all of H^{out} is automatic since $f\left(\frac{\mathrm{x}}{t}\right)$ is uniformly bounded in t and the set $\bigvee \psi^{out}_{h,\varphi}$ is dense in H^{out}, by construction of H^{out} .

Corollary 5.3

In the space H^{out}, the asymptotic photon algebra $\mathcal{A}^{out}$ is defined as the norm closure of the *algebra generated by the set of unitary operators $\left\{W^{out}(\mu) : \widetilde{\mu} \in C_0^{\infty}\left(R^3 \setminus 0\right)\right\}$ constructed in H^{out} as follows:

$$W^{out}(\mu) = s - \lim_{t\to+\infty} e^{iHt} e^{i\left(a(\mu_t)+a^{\dagger}(\mu_t)\right)} e^{-iHt} \tag{22}$$

The following properties hold:

1) the generators $\left\{W^{out}(\mu) : \widetilde{\mu} \in C_0^{\infty}\left(R^3 \setminus 0\right)\right\}$ satisfy the Weyl commutation rules

$$W^{out}(\mu)\, W^{out}(\eta) = W^{out}(\eta)\, W^{out}(\mu)\, e^{-h(\mu,\eta)}$$

where $h(\mu,\eta) = 2iIm \int \widetilde{\mu}(\mathbf{k})\, \overline{\widetilde{\eta}}(\mathbf{k})\, d^3k$;

2) for each fixed region $R^3 \setminus O_r$ where O_r is a ball of radius $r \neq 0$, centered in the origin of R^3, the group generated by the operators $W^{out}(\mu)$, $\widetilde{\mu} \in C_0^\infty\left(R^3 \setminus O_r\right)$, is strongly continuous with respect to $\widetilde{\mu}$ in the $L^2\left(R^3 \setminus O_r, d^3k\right)$-norm,

3) given the τ-evolved generators

$$W_\tau^{out}(\mu) = s - \lim_{t\to+\infty} e^{iH(t+\tau)} e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)} e^{-iH(t+\tau)} = W^{out}(\mu_{-\tau}) \tag{23}$$

(where $\mu_{-\tau}$ is the freely evolved test function μ, in the time τ) an automorphism α_τ of $\mathcal{A}^{out}$ is uniquely defined starting from

$$\alpha_\tau\left(W^{out}(\mu)\right) = W^{out}(\mu_{-\tau}).$$

Therefore, we can conclude that $\mathcal{A}^{out}$ is the Weyl algebra associated to the scalar massless field.

4) the algebra $\mathcal{A}^{out}$ commutes with the asymptotic electron mean velocity defined through theorem 5.2.

$\Leftrightarrow$

Proof

The existence of $s - \lim_{t\to+\infty} e^{iHt} e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)} e^{-iHt} \psi_{h,\varphi}^{out}$ is substantially the content of theorem 5.1. The bounded operators $W^{out}(\mu)$ can be linearly extended from the dense set $\bigvee \psi_{h,\varphi}^{out}$ to all of H^{out}, by continuity. They leave the space H^{out} invariant and they are unitary in H^{out}.

1) The operators in $W^{out}(\mu)\, W^{out}(\eta) : \mathrm{H}^{out} \to \mathrm{H}^{out}$ is the time limit, at the same time t, of the product of the corresponding approximating operators (22). The last ones satisfy the Weyl rules by construction. Therefore, the property is satisfied in the limit.

2) Let us start proving that $W^{out}(\mu)$ is strongly continuous with respect to $\widetilde{\mu}$ if it is applied to the total set $\left\{\psi_{h,\varphi}^{out}\right\}$.

We know that:

- $W^{out}(\mu)\, \psi_{h,\varphi}^{out} = s - \lim_{t\to+\infty} e^{iHt} e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)} e^{-iHt} \psi_{h,\varphi}(t)$;
- at fixed t, the vector $e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)} e^{-iHt} \psi_{h,\varphi}(t)$ is strongly continuous with respect to $\widetilde{\mu} \in C_0^\infty\left(R^3 \setminus O_r\right)$, as follows from the identity

$$e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)} \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j, t)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), t)} e^{-iE^{\sigma_t} t} \psi_{j,\sigma_t}^{(t)} =$$

$$= \sum_{j=1}^{N(t)} W_{\sigma_t}(\mathbf{v}_j, t)\, e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)} e^{-h(\mu_t, \xi_t(\mathbf{v}_j))} e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}), t)} e^{-iE^{\sigma_t} t} \psi_{j,\sigma_t}^{(t)}$$

(where $\widetilde{\xi}(\mathbf{v}_j,\mathbf{k}) \equiv -ig\frac{\chi^{\kappa_1}_{\sigma_t}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}$) and from the fact that $\psi^{(t)}_{j,\sigma_t}$ is in the domain of the generator of $e^{i\left(a(\mu_t)+a^{\dagger}(\mu_t)\right)}$.

Being $\widetilde{\mu}_t \in C^{\infty}_0\left(R^3 \setminus O_r\right)$ and $\|\widetilde{\mu}_t\|_{L^2(R^3\setminus O_r,d^3k)} = \|\widetilde{\mu}\|_{L^2(R^3\setminus O_r,d^3k)}$, the vector $W^{out}(\mu)\,\psi^{out}_{h,\varphi}$ is strongly continuous with respect to $\widetilde{\mu}$.

Since $\|W^{out}(\mu)\| = 1$, the property holds for each vector in H^{out}.

3) The τ-evolved generators $e^{iH\tau}W^{out}(\mu)\,e^{-iH\tau}$ are well defined on H^{out} because $e^{-iH\tau}$:$\mathrm{H}^{out}\rightarrow\mathrm{H}^{out}$. By inserting the expression (22) for $W^{out}(\mu)$, we easily arrive at (23). The Weyl commutation rules are conserved by α_{τ} since

$$h\left(\mu_{-\tau},\eta_{-\tau}\right) = 2iIm\int\widetilde{\mu}(\mathbf{k})\,\overline{\widetilde{\eta}}(\mathbf{k})\,d^3k = h(\mu,\eta)$$

4) Such a property is implicit in the construction of the asymptotic electron mean velocity (theorem 5.2).

APPENDIX A

Preliminary remark on lemma A1

As in lemma 1.3, we want to prove that the operator

$$(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} \equiv \widehat{H}^w_{\mathbf{P},\sigma_{j+1}} + c_{\mathbf{P}}(j) - \widehat{c}_{\mathbf{P}}(j+1) - H^w_{\mathbf{P},\sigma_j}$$

is small of order 1 with respect to $H^w_{\mathbf{P},\sigma_j}\,|_{F^+_{\sigma_{j+1}}}$ in a generalized sense, for g sufficiently small. We aim at expanding the resolvent

$$\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_{j+1}} - (E(j+1) + \widehat{c}_{\mathbf{P}}(j+1) - c_{\mathbf{P}}(j))}\,|_{F^+_{\sigma_{j+1}}}$$

(where $E(j+1) \in \mathcal{C}$ s.t. $\left|E(j+1) - E^{\sigma_j}_{\mathbf{P}}\right| = \frac{11\sigma_{j+1}}{20}$, and $\widehat{c}_{\mathbf{P}}(j+1) - c_{\mathbf{P}}(j) = -g^2 \int^{\sigma_j}_{\sigma_{j+1}} \frac{1}{2|\mathbf{k}|^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)} d^3k$) in terms of $\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\,|_{F^+_{\sigma_{j+1}}}$ and $(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}$.

Lemma A1

Given the spectral properties pointed out in paragraph 2.1 pag. 35, $(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}$ is small with respect to $H^w_{\mathbf{P},\sigma_j}$ for sufficiently small g and $\mathbf{P} \in \Sigma$, in the following sense:

given $E(j+1) \in \mathcal{C}$ s.t. $\left|E(j+1) - E^{\sigma_j}_{\mathbf{P}}\right| = \frac{11}{20}\sigma_{j+1}$,

$$\frac{1}{\widehat{H}^w_{\mathbf{P},\sigma_{j+1}} - (E(j+1) + \widehat{c}_{\mathbf{P}}(j+1) - c_{\mathbf{P}}(j))}\,|_{F^+_{\sigma_{j+1}}} =$$

$$= \frac{1}{H^w_{\mathbf{P},\sigma_j} + (\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} - c_{\mathbf{P}}(j) + \widehat{c}_{\mathbf{P}}(j+1) - (E(j+1) - c_{\mathbf{P}}(j) + \widehat{c}_{\mathbf{P}}(j+1))}\,|_{F^+_{\sigma_{j+1}}} =$$

$$= \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\,|_{F^+_{\sigma_{j+1}}} + \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \sum_{n=1}^{+\infty} \left(-(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right)^n |_{F^+_{\sigma_{j+1}}}$$

where

- $\left\| \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \left(-(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}} \frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)} \right)^n \right\|_{F^+_{\sigma_{j+1}}} \le \frac{20(C(g,m))^n}{\sigma_{j+1}}$
- $0 < C(g,m) < \frac{1}{12}$.

Proof

As in lemma 1.3, we discuss the norm of

$$\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}|_{F^+_{\sigma_{j+1}}}$$

where

$$(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}=\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix}+\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{quad.}$$

$$\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{quad.}=\frac{1}{2m}\left(g\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}\big(b(\mathbf{k})+b^\dagger(\mathbf{k})\big)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\big(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\big)}d^3k-g^2\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}}{2|\mathbf{k}|^3\big(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\big)^2}d^3k\right)^2+ \quad \text{(a1)}$$

$$\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix}= \quad \text{(a2)}$$

$$=\frac{1}{2m}\left\{\Gamma_{\mathbf{P},\sigma_j}\cdot\left(-g\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}\big(b(\mathbf{k})+b^\dagger(\mathbf{k})\big)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\big(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\big)}d^3k+g^2\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}}{2|\mathbf{k}|^3\big(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\big)^2}d^3k\right)+c.c.\right\}$$

In order to control the above quantities, we use the following estimate again and again

$$\left\|\int_{\sigma_{j+1}}^{\sigma_j}k^ib(\mathbf{k})\frac{d^3\mathbf{k}}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)}\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}}\leq\frac{5}{4}\sqrt{10\pi}\cdot\sigma_j^{\frac{1}{2}},$$

which substantially comes from the estimate (1) of lemma 1.3, by performing an unitary transformation. So we can provide a bound of order σ_j for the norm of the "quadratic terms":

$$\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{quad.}\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}}\leq\sigma_j\cdot C_1(g,m)$$

For the mixed terms (a2) containing the $\Gamma_{\mathbf{P},\sigma_j}$, we exploit the fact that of the norm

$$\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\Gamma^i_{\mathbf{P},\sigma_j}\right\|_{F^+_{\sigma_{j+1}}}=\left\|\Gamma^i_{\mathbf{P},\sigma_j}\left[\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right]^\dagger\right\|_{F^+_{\sigma_{j+1}}}$$

are of order $\sigma_j^{-\frac{j+1}{2}}$, which follows from the form inequality

$$\left(\Gamma^i_{\mathbf{P},\sigma_j}\right)^2\leq 2m\left(H^w_{\mathbf{P},\sigma_j}-c_{\mathbf{P}}(j)\right).$$

Therefore a bound of order 1 is worked out for the mixed part:

$$\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\left[(\Delta H^w_{\mathbf{P}})^{\sigma_j}_{\sigma_{j+1}}\right]^{mix}\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)^{\frac{1}{2}}\right\|_{F^+_{\sigma_{j+1}}}\leq C_2(g,m).$$

Conclusion

If g is less than a maximum value, the σ_j-independent constants $C_{1,2}(g,m)$ can be tuned in order to arrive at the thesis.

Lemma A2

Results about $\nabla E^{\sigma}(\mathbf{P})$, where $\mathbf{P} \in \Sigma$:

1) $|\nabla E^{\sigma}(\mathbf{P})| < v^{max} < \frac{1}{\sqrt{80}}$ $\forall\sigma$ (better estimates independent of the ultraviolet cut-off κ are provided in [28]);

2) $|\nabla E^{\sigma_{j+1}}(\mathbf{P}) - \nabla E^{\sigma_j}(\mathbf{P})| < C^{\nabla E} \cdot \left(\left\| \frac{\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}}{\|\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}\|} - \frac{\phi^{\sigma_j}_{\mathbf{P}}}{\|\phi^{\sigma_j}_{\mathbf{P}}\|} \right\| + \epsilon^{\frac{j}{8}} \right)$ for g sufficiently small and $C^{\nabla E}$ uniform in j;

$\Longleftarrow\!\!\Longrightarrow$

1)

$$|\nabla E^{\sigma}(\mathbf{P})| = \frac{\left[\sum_i \left(\psi^{\sigma}_{\mathbf{P}}, \mathbf{P}^i - \mathbf{P}^{ph^i}\psi^{\sigma}_{\mathbf{P}}\right)^2\right]^{\frac{1}{2}}}{m\|\psi^{\sigma}_{\mathbf{P}}\|^2} \le \frac{\|\psi^{\sigma}_{\mathbf{P}}\|[2m(\psi^{\sigma}_{\mathbf{P}}, H_{\mathbf{P},\sigma}+2\pi g^2\kappa\,\psi^{\sigma}_{\mathbf{P}})]^{\frac{1}{2}}}{m\|\psi^{\sigma}_{\mathbf{P}}\|^2} \le \sqrt{\frac{2}{m}}\frac{[(\psi^{\sigma}_{\mathbf{P}}, H_{\mathbf{P},\sigma}+2\pi g^2\kappa\,\psi^{\sigma}_{\mathbf{P}})]^{\frac{1}{2}}}{\|\psi^{\sigma}_{\mathbf{P}}\|}$$

According to the *initial hypotheses* at pag.23, we have

$$\frac{|(\psi^{\sigma}_{\mathbf{P}}, H_{\mathbf{P},\sigma}\,\psi^{\sigma}_{\mathbf{P}})|}{\|\psi^{\sigma}_{\mathbf{P}}\|^2} \le |(\psi_0, H_{\mathbf{P},\sigma}\,\psi_0)| = \frac{\mathbf{P}^2}{2m} \le \frac{m}{2\cdot 400}$$

$$\frac{|(\psi^{\sigma}_{\mathbf{P}}, 2\pi g^2\kappa\,\psi^{\sigma}_{\mathbf{P}})|}{\|\psi^{\sigma}_{\mathbf{P}}\|^2} \le \frac{m}{2\cdot 100}$$

so that the thesis is proved.

2)
Let us analyze the difference between the gradients of the ground energy at subsequent infrared cutoffs. Starting from the relation (at pag. 32)

$$m\nabla E^{\sigma_j}_{\mathbf{P}} = \mathbf{P} - \frac{(\phi^{\sigma_j}_{\mathbf{P}}, \Pi_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}})}{\|\phi^{\sigma_j}_{\mathbf{P}}\|^2} - g^2\int_{\sigma_j}^{\kappa} \frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}_{\mathbf{P}}\right)^2} d^3k\,,$$

we obtain

$$\begin{aligned} &m\nabla E^{\sigma_j}_{\mathbf{P}} - m\nabla E^{\sigma_{j+1}}_{\mathbf{P}} - g^2\int_{\sigma_{j+1}}^{\kappa}\frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{j+1}}_{\mathbf{P}}\right)^2}d^3k + g^2\int_{\sigma_j}^{\kappa}\frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}\right)^2}d^3k = \\ &= \frac{1}{\|\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}\|^2}\left(\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}, \widehat{\Pi}_{\mathbf{P},\sigma_{j+1}}\widehat{\phi}^{\sigma_{j+1}}_{\mathbf{P}}\right) - \frac{1}{\|\phi^{\sigma_j}_{\mathbf{P}}\|^2}\left(\phi^{\sigma_j}_{\mathbf{P}}, \Pi_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right). \end{aligned} \tag{a3}$$

By simple steps

$$m\nabla E_{\mathbf{P}}^{\sigma_j} - m\nabla E_{\mathbf{P}}^{\sigma_{j+1}} + g^2\int_{\sigma_j}^{\kappa} \frac{\mathbf{k}\left(-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}+\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)\left(2-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^2} d^3k =$$

$$= \frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \widehat{\Pi}_{\mathbf{P},\sigma_{j+1}}\left(\frac{\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|} - \frac{\phi_{\mathbf{P}}^{\sigma_j}}{\|\phi_{\mathbf{P}}^{\sigma_j}\|}\right)\right) + g^2\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)^2}d^3k +$$

$$+\frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|\|\phi_{\mathbf{P}}^{\sigma_j}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \widehat{\Pi}_{\mathbf{P},\sigma_{j+1}}\phi_{\mathbf{P}}^{\sigma_j}\right) - \frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|\|\phi_{\mathbf{P}}^{\sigma_j}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \Pi_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right) +$$

$$+\frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|\|\phi_{\mathbf{P}}^{\sigma_j}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \Pi_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right) - \frac{1}{\|\phi_{\mathbf{P}}^{\sigma_j}\|^2}\left(\phi_{\mathbf{P}}^{\sigma_j}, \Pi_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right).$$

Considering that

$$\widehat{\Pi}^i_{\mathbf{P},\sigma_{j+1}} - \Pi^i_{\mathbf{P},\sigma_j} =$$

$$= -g\int_{\sigma_{j+1}}^{\sigma_j}\frac{k^i\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\nabla E_{\mathbf{P}}^{\sigma_j}\right)}d^3k + \frac{g^2}{2}\int_{\sigma_{j+1}}^{\kappa}\frac{k^i\left(-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}+\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)\left(2-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)^2}d^3k$$

the equation (a3) can be written in the following way

$$m\nabla E_{\mathbf{P}}^{\sigma_j} - m\nabla E_{\mathbf{P}}^{\sigma_{j+1}} + g^2\int_{\sigma_j}^{\kappa} \frac{\mathbf{k}\left(-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}+\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)\left(2-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^2} d^3k +$$

$$-\frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|\|\phi_{\mathbf{P}}^{\sigma_j}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \phi_{\mathbf{P}}^{\sigma_j}\right) g^2\int_{\sigma_j}^{\kappa} \frac{\mathbf{k}\left(-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}+\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)\left(2-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)}{4|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^2} d^3k =$$

$$= \frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \widehat{\Pi}_{\mathbf{P},\sigma_{j+1}}\left(\frac{\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|} - \frac{\phi_{\mathbf{P}}^{\sigma_j}}{\|\phi_{\mathbf{P}}^{\sigma_j}\|}\right)\right) + \frac{1}{\|\phi_{\mathbf{P}}^{\sigma_j}\|}\left(\left(\frac{\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|} - \frac{\phi_{\mathbf{P}}^{\sigma_j}}{\|\phi_{\mathbf{P}}^{\sigma_j}\|}\right), \Pi_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right) +$$

$$+g^2\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right)^2}d^3k - \frac{1}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\|\|\phi_{\mathbf{P}}^{\sigma_j}\|}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, g\int_{\sigma_{j+1}}^{\sigma_j}\frac{\mathbf{k}\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)}d^3k\phi_{\mathbf{P}}^{\sigma_j}\right).$$

On the left hand side of the equation, there is a quantity whose absolute value is bigger than $C\cdot\left|\nabla E_{\mathbf{P}}^{\sigma_j} - \nabla E_{\mathbf{P}}^{\sigma_{j+1}}\right|$ for $g\to 0$, where C is a positive constant that is uniform in j and converges to m for $g\to 0$. It is due to the result in point **1)**.
On the right hand side, there is a quantity whose module is bounded by a $g-$dependent constant times

$$\left(\left\|\frac{\widehat{\phi}_{\mathbf{P}}^{\sigma_j}}{\|\widehat{\phi}_{\mathbf{P}}^{\sigma_j}\|} - \frac{\phi_{\mathbf{P}}^{\sigma_j}}{\|\phi_{\mathbf{P}}^{\sigma_j}\|}\right\| + \epsilon^{\frac{j}{8}}\right).$$

It follows taking into account the bounds below:

- $$\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, g\int_{\sigma_{j+1}}^{\sigma_j}\frac{k^i\left(b(\mathbf{k})+b^\dagger(\mathbf{k})\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)}d^3k\,\phi_{\mathbf{P}}^{\sigma_j}\right) = \left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, g\int_{\sigma_{j+1}}^{\sigma_j}\frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)}d^3k\,\phi_{\mathbf{P}}^{\sigma_j}\right) =$$

$$= g\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i}{\sqrt{2}|\mathbf{k}|^2\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^{\frac{3}{2}}}\left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, |\mathbf{k}|^{\frac{1}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^{\frac{1}{2}} b^\dagger(\mathbf{k})\,\phi_{\mathbf{P}}^{\sigma_j}\right) d^3k \le$$

$$\le g\left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{(k^i)^2}{2|\mathbf{k}|^4\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^3} d^3k\right)^{\frac{1}{2}} \left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \int (|\mathbf{k}| - \mathbf{k}\cdot\nabla E^{\sigma_j}(\mathbf{P}))\, b^\dagger(\mathbf{k})\, b(\mathbf{k})\, d^3k\, \widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\right)^{\frac{1}{2}} \|\phi_{\mathbf{P}}^{\sigma_j}\| \le$$

$$\le g\left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{(k^i)^2}{2|\mathbf{k}|^4\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_j}\right)^3} d^3k\right)^{\frac{1}{2}} \left(E_{\mathbf{P}}^{\sigma_{j+1}} - \widehat{c}_{\mathbf{P}}(j+1)\right)^{\frac{1}{2}} \left(\widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}, \widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\right)^{\frac{1}{2}} \|\phi_{\mathbf{P}}^{\sigma_j}\|$$

- $\left\|\widehat{\Pi}_{\mathbf{P},\sigma_{j+1}}^i \widehat{\phi}_{\mathbf{P}}^{\sigma_{j+1}}\right\| = \left\|\Pi_{\mathbf{P},\sigma_{j+1}}^i \phi_{\mathbf{P}}^{\sigma_{j+1}}\right\| \le \left\|P^{ph^i}\psi_{\mathbf{P}}^{\sigma_{j+1}}\right\| + const\cdot\|\psi_{\mathbf{P}}^{\sigma_{j+1}}\| \le$
  $\le \sqrt{2m}\cdot\left(2\pi g^2\kappa + E_{\mathbf{P}}^{\sigma_{j+1}}\right)^{\frac{1}{2}}\cdot\|\psi_{\mathbf{P}}^{\sigma_{j+1}}\| + const\cdot\|\psi_{\mathbf{P}}^{\sigma_{j+1}}\|.$

Lemma A3

The following inequality holds for a sufficiently small ratio $\frac{\kappa}{m}$:

$$\left\|\left(\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right)^{\frac{1}{2}} \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\cdot\Gamma_{\mathbf{P},\sigma_j}^i \left(\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right)^{\frac{1}{2}} \phi_{\mathbf{P}}^{\sigma_j}\right\|^2 \le$$

$$\le 2\cdot Q(\epsilon)\cdot\sqrt{122}\cdot\left\{\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^{i2}}{2|\mathbf{k}|^3\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})^2} d^3k\right\} \left|\frac{1}{E_{\mathbf{P}}^{\sigma_j} - E(j+1)}\right| \left|\left(\Gamma_{\mathbf{P},\sigma_j}^i\phi_{\mathbf{P}}^{\sigma_j}, \left(\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right)\Gamma_{\mathbf{P},\sigma_j}^i\phi_{\mathbf{P}}^{\sigma_j}\right)\right|$$

(having defined $\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}}) \equiv \left(1 - \widehat{\mathbf{k}}\cdot\nabla E^{\sigma_j}(\mathbf{P})\right)$).

Proof

Let us start from

$$\left\|\left(\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right)^{\frac{1}{2}} \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\cdot\Gamma_{\mathbf{P},\sigma_j}^i \left(\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right)^{\frac{1}{2}} \phi_{\mathbf{P}}^{\sigma_j}\right\|^2 =$$

$$= \left|\frac{1}{E_{\mathbf{P}}^{\sigma_j} - E(j+1)}\right| \left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\Gamma_{\mathbf{P},\sigma_j}^i\phi_{\mathbf{P}}^{\sigma_j}, \left|\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right| \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} b^\dagger(\mathbf{k})\, d^3k\Gamma_{\mathbf{P},\sigma_j}^i\phi^{\sigma_j}\right).$$

Now, from lemma 2.1

$$\left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\Gamma_{\mathbf{P},\sigma_j}^i\phi_{\mathbf{P}}^{\sigma_j}, \left|\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right| \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} b^\dagger(\mathbf{k})\, d^3k\Gamma_{\mathbf{P},\sigma_j}^i\phi^{\sigma_j}\right) \le$$

$$\le \sqrt{122}\cdot\left|\left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\Gamma_{\mathbf{P},\sigma_j}^i\phi_{\mathbf{P}}^{\sigma_j}, \left(\frac{1}{H_{\mathbf{P},\sigma_j}^w - E(j+1)}\right) \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})} d^3k\Gamma_{\mathbf{P},\sigma_j}^i\phi_{\mathbf{P}}^{\sigma_j}\right)\right|$$

Starting from the expression $H_{\mathbf{P},\sigma_j}^w = \frac{1}{2m}\Gamma_{\mathbf{P},\sigma_j}^2 + \int |\mathbf{k}|\,\alpha_{\mathbf{P}}^{\sigma_j}(\widehat{\mathbf{k}})\, b^\dagger(\mathbf{k})\, b(\mathbf{k})\, d^3k + c_{\mathbf{P}}(j)$, the following identity holds in a distributional sense, for $\{\mathbf{k} : |\mathbf{k}| \leq \sigma_j\}$:

$$\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right) b^\dagger(\mathbf{k}) = b^\dagger(\mathbf{k}) \left(\frac{1}{\frac{1}{2m}\left(\Gamma_{\mathbf{P},\sigma_j} + \mathbf{k}\right)^2 + \int |\mathbf{k}|\, \alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) b^\dagger(\mathbf{k})\, b(\mathbf{k}) + |\mathbf{k}|\, \alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) + c_{\mathbf{P}}(j+1) - E(j+1)}\right.$$

Moreover, for $\frac{\kappa}{m}$ sufficiently small, for the assumptions made on $\left|E^{\sigma_j}_{\mathbf{P}} - E(j+1)\right|$ and being $\sigma_{j+1} \leq |\mathbf{k}| \leq \sigma_j$, the following bound holds in the subspace $F^+_{\sigma_j}$:

$$\left\| \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)^{\frac{1}{2}} \left(\frac{1}{m}\mathbf{k}\cdot\Gamma_{\mathbf{P},\sigma_j} + \frac{\mathbf{k}^2}{2m}\right) \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)^{\frac{1}{2}} \right\|_{F^+_{\sigma_j}} \leq \frac{1}{2}\,. \tag{a4}$$

Therefore the series expansion

$$\left(\frac{1}{\frac{1}{2m}\left(\Gamma_{\mathbf{P},\sigma_j} + \mathbf{k}\right)^2 + \int |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) b^\dagger(\mathbf{k}) b(\mathbf{k}) + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) + c_{\mathbf{P}}(j+1) - E(j+1)}\right) =$$
$$= \left(\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right) \sum_{n=0}^{+\infty} \left(-\left(\frac{1}{m}\mathbf{k}\cdot\Gamma_{\mathbf{P},\sigma_j} + \frac{\mathbf{k}^2}{2m}\right) \frac{1}{H^w_{\mathbf{P},} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)^n\right).$$

is well defined in $F^+_{\sigma_j}$.

Then we can write (note that $b(\mathbf{k})\,\phi^{\sigma_j}_{\mathbf{P}} = 0$ for $|\mathbf{k}| \leq \sigma_j$):

$$\left(\int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right)} d^3k\Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\,,\ \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} - E(j+1)}\right) \int_{\sigma_{j+1}}^{\sigma_j} \frac{k^i b^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right)} d^3k\Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right) = \tag{a5}$$
$$= \sum_{n=0}^{\infty} \int_{\sigma_{j+1}}^{\sigma_j} \frac{{k^i}^2}{2|\mathbf{k}|^3 \alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right)^2}\cdot$$
$$\cdot\left(\Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\,,\ \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)\left(-\left(\frac{1}{m}\mathbf{k}\cdot\Gamma_{\mathbf{P},\sigma_j} + \frac{\mathbf{k}^2}{2m}\right)\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)\right)^n \Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right) d^3k.$$

Now we prove that the module of the n^{th} term of the series can be reduced to the one of the first term, so that the whole sum is of the same order of the term at $n = 0$.

Exploiting the Schwartz inequality, we have that

$$\left|\left(\Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\,,\ \left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)\left(-\left(\frac{1}{m}\mathbf{k}\cdot\Gamma_{\mathbf{P},\sigma_j} + \frac{\mathbf{k}^2}{2m}\right)\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)\right)^n \Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right)\right| \leq$$
$$\leq \left\|\left[\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)^{\frac{1}{2}}\right]^\dagger \Gamma^i_{\mathbf{P},\sigma_j}\phi^{\sigma_j}_{\mathbf{P}}\right\|^2 \left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_j} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)^{\frac{1}{2}}\left(\frac{1}{m}\mathbf{k}\cdot\Gamma_{\mathbf{P},\sigma_j} + \frac{\mathbf{k}^2}{2m}\right)\left(\frac{1}{H^w_{\mathbf{P},} + |\mathbf{k}|\alpha^{\sigma_j}_{\mathbf{P}}\left(\widehat{\mathbf{k}}\right) - E(j+1)}\right)^{\frac{1}{2}}\right\|^n_F$$

Therefore, due to the estimate (a4) and to lemma 2.1, the absolute value of the scalar product (a5) is bounded by

$$\sum_{n=0}^{\infty}\left\{\int_{\sigma_{j+1}}^{\sigma_j}\frac{{k^i}^2}{2|\mathbf{k}|^3\alpha_{\mathbf{P}}^{\sigma_j}\left(\widehat{\mathbf{k}}\right)^2}\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}+|\mathbf{k}|\alpha_{\mathbf{P}}^{\sigma_j}\left(\widehat{\mathbf{k}}\right)-E(j+1)}\right)^{\frac{1}{2}}\Gamma^i_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right\|^2 d^3k\right\}\left(\frac{1}{2}\right)^n \leq$$

$$\leq 2\int_{\sigma_{j+1}}^{\sigma_j}\frac{{k^i}^2}{2|\mathbf{k}|^3\alpha_{\mathbf{P}}^{\sigma_j}\left(\widehat{\mathbf{k}}\right)^2}\cdot\left\|\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}+|\mathbf{k}|\alpha_{\mathbf{P}}^{\sigma_j}\left(\widehat{\mathbf{k}}\right)-E(j+1)}\right)^{\frac{1}{2}}\Gamma^i_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right\|^2 d^3k \leq$$

$$\leq 2\cdot Q(\epsilon)\cdot\int_{\sigma_{j+1}}^{\sigma_j}\frac{{k^i}^2}{2|\mathbf{k}|^3\alpha_{\mathbf{P}}^{\sigma_j}\left(\widehat{\mathbf{k}}\right)^2}d^3k\cdot\left|\left(\Gamma^i_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j},\left(\frac{1}{H^w_{\mathbf{P},\sigma_j}-E(j+1)}\right)\Gamma^i_{\mathbf{P},\sigma_j}\phi_{\mathbf{P}}^{\sigma_j}\right)\right|.$$

APPENDIX B

Preliminary remark

In the next lemmas we assume an hypothesis which is not proved in the spectral analysis but that is physically reasonable and strongly supported by Chen's result [CH]:
for $\mathbf{P} \in \Sigma$, there exists a constant $\frac{1}{m_r}$ such that the following inequalities hold, uniformly in $\sigma > 0$:

hypothesis B1 $\qquad \frac{\partial E^\sigma(\mathbf{P})}{\partial|\mathbf{P}|} \geq \frac{|\mathbf{P}|}{m_r}$ and $\frac{\partial^2 E^\sigma(\mathbf{P})}{\partial^2|\mathbf{P}|} \geq \frac{1}{m_r}$

Starting from this hypothesis, we obtain that the application $\mathbf{J}_\sigma : \mathbf{P} \to \nabla E^\sigma(\mathbf{P})$ is one to one and that the determinant of the jacobian satisfies the inequality:

$$det\, \mathbf{dJ}_\sigma = \frac{1}{|\mathbf{P}|^2}\left(\frac{\partial E^\sigma(\mathbf{P})}{\partial|\mathbf{P}|}\right)^2 \cdot \frac{\partial^2 E^\sigma(\mathbf{P})}{\partial^2|\mathbf{P}|} \geq \frac{1}{m_r^3}$$

(the function $E^\sigma(\mathbf{P})$ is invariant under rotations and belongs to $C^\infty\left(R^3\right)$, see [3]) . Then, for $O_\mathbf{P} \subset \Sigma$ and the corresponding region $O_{\nabla E^\sigma}$ in the $\nabla E^\sigma-$space, $O_\mathbf{P} = \mathbf{J}_\sigma^{-1}(O_{\nabla E^\sigma})$, we have the following relation between their volumes:

$$V_{O_\mathbf{P}} \leq m_r^3 V_{O_{\nabla E^\sigma}} .$$

Remark on the notations

As in the previous chapters, we use the convention to generically call C the constants which are uniform in the variables we are treating. The bounds are intended from above, up to a different explicit warning.

Definition of $\chi^{(t)}_{\mathbf{v}_j}(\nabla E^{\sigma_t}, s)$.

The function $\chi^{(t)}_{\mathbf{v}_j}(\nabla E^{\sigma_t}, s)$ has to approximate the characteristic function of the set $\mathbf{J}_{\sigma_t}(\Gamma_j)$ for $s \to +\infty$ where s is bigger or equal to the t ($\gg 1$) of the partition (the most general expression is $\chi^{(t_1)}_{\mathbf{v}_j}(\nabla E^{\sigma_{t_2}}, s)$ where the constraint is $s \geq t_1$).
In particular, in order to approximate the region $\mathbf{J}_{\sigma_t}(\Gamma_j)$ from inside, we define

$$\chi^{(t)}_{\mathbf{v}_j}(\nabla E^{\sigma_t}, s) = \sum_{m(j)} \Upsilon^{(t)}_{\mathbf{v}_{m(j)}}(\nabla E^{\sigma_t}, s)$$

where $supp_{\nabla E^{\sigma_t}} \chi^{(t)}_{\mathbf{v}_j}(\nabla E^{\sigma_t}, s) \subset supp \mathbf{J}_{\sigma_t}(\Gamma_j)$ and the functions $\Upsilon^{(t)}_{\mathbf{v}_{m(j)}}(\nabla E^{\sigma_t}, s)$ are constructed to fill the region $\mathbf{J}_{\sigma_t}(\Gamma_j)$, starting from the "model" function

$$\Upsilon^{(t)}(\mathbf{z}, s) = \Upsilon^{(t)}_1(z_1, s) \cdot \Upsilon^{(t)}_2(z_2, s) \cdot \Upsilon^{(t)}_3(z_3, s)$$

$$\Upsilon_k^{(t)}(z_k,s) \equiv \{ \begin{array}{ll} -s^{\frac{\delta}{2}} z_k + s^{\frac{\delta}{2}} \cdot \frac{1}{2s^{\frac{\delta}{6}}} & for \; -\frac{1}{s^{\frac{\delta}{2}}} + \frac{1}{2s^{\frac{\delta}{6}}} < z_k \leq \frac{1}{2s^{\frac{\delta}{6}}} \\ 1 & for \; -\frac{1}{2s^{\frac{\delta}{6}}} + \frac{1}{s^{\frac{\delta}{2}}} < z_k \leq \frac{1}{2s^{\frac{\delta}{6}}} - \frac{1}{s^{\frac{\delta}{2}}} \\ s^{\frac{\delta}{2}} z_k + s^{\frac{\delta}{2}} \cdot \frac{1}{2s^{\frac{\delta}{6}}} & for \; -\frac{1}{2s^{\frac{\delta}{6}}} < z_k \leq -\frac{1}{2s^{\frac{\delta}{6}}} + \frac{1}{s^{\frac{\delta}{2}}} \end{array}$$

through a translation $\mathbf{z} \to \mathbf{z} - \mathbf{v}_{m(j)}$. The sum $\sum_{m(j)}$ is clearly s and t dependent, since the number of sub-functions $\Upsilon_{\mathbf{v}_{m(j)}}^{(t)}(\nabla E^{\sigma_t}, s)$ is order $\frac{s^{\frac{\delta}{2}}}{t^{3\epsilon}}$ (the support of $\mathbf{J}_{\sigma_t}(\Gamma_i)$ has a volume of order $\frac{1}{t^{3\epsilon}}$).
In order to have a well defined $\chi_{\mathbf{v}_i}^{(t)}(\nabla E^{\sigma_t}, s)$ two requirements are necessary:
- the inequality $\delta > 6\epsilon$;
- a finite scale factor (related to m_r) for the variable z_k in the functions $\Upsilon_k^{(t)}(z_k, s)$; to simplify the notations we assume this factor equal to 1.

The $\widetilde{\Upsilon}_{\mathbf{v}_{m(j)}}^{(t)}(\mathbf{q}, s)$ behavior is therefore analogous to $\widetilde{\Upsilon}^{(t)}(\mathbf{q}, s) = \widetilde{\Upsilon}_1^{(t)}(q_1, s) \cdot \widetilde{\Upsilon}_2^{(t)}(q_2, s) \cdot \widetilde{\Upsilon}_3^{(t)}(q_3, s)$ where

$$\widetilde{\Upsilon}_k^{m(t)}(q_k, s) \propto s^{\frac{\delta}{2}} \cdot \frac{\cos\left(q_k \left(\frac{1}{2s^{\frac{\delta}{6}}} - \frac{1}{s^{\frac{\delta}{2}}}\right)\right) - \cos\left(q_k \cdot \frac{1}{2s^{\frac{\delta}{6}}}\right)}{q_k^2}$$

so that we can easily consider the bounds

$$\int \left|\widetilde{\Upsilon}^{(t)}(\mathbf{q}, s)\right| d^3q < C \cdot s^{3\frac{\delta}{2}} \quad , \quad \int_a^{\infty} \left|\widetilde{\Upsilon}^{(t)}(\mathbf{q}, s)\right| d^3q < C \cdot \frac{1}{a} \cdot s^{\frac{\delta}{2}} \cdot s^{\delta} \, .$$

In lemma B3 we exploit the inequalities

$$\int \left|\widetilde{\chi}_{\mathbf{v}_j}^{(t)}(\mathbf{q}, s)\right| d^3q \leq \sum_{m(j)} \int \left|\widetilde{\Upsilon}_{\mathbf{v}_{m(j)}}^{(t)}(\mathbf{q}, s)\right| d^3q \leq C \cdot \frac{L^3}{t^{3\epsilon}} \cdot s^{3\cdot\frac{\delta}{6}} \cdot s^{3\frac{\delta}{2}} \leq C \cdot s^{2\delta}$$

$$\int_a^{+\infty} \left|\widetilde{\chi}_{\mathbf{v}_j}^{(t)}(\mathbf{q}, s)\right| d^3q \leq \sum_{m(j)} \int \left|\widetilde{\Upsilon}_{\mathbf{v}_{m(j)}}^{(t)}(\mathbf{q}, s)\right| d^3q \leq C \cdot \frac{1}{t^{3\epsilon}} \cdot s^{3\frac{\delta}{6}} \frac{1}{a} \cdot s^{\frac{3\delta}{2}} < C \cdot \frac{1}{a} \cdot s^{2\delta} \, .$$

Lemma B1

The norm $\left\| \left(1_{\Gamma_j}(\mathbf{P}) - \chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t}(\mathbf{P}), s)\right) \psi_{j,\sigma_t}^{(t)} \right\|$ is bounded by a quantity of order $\frac{1}{s^{\frac{\delta}{12}}} \cdot t^{-\epsilon}$.

Proof

We define $\mathbf{J}_{\sigma_t}^{-1}\left(\widehat{O}_{\nabla E^{\sigma_t}}^j\right) \equiv \left\{\mathbf{P} \in \Sigma : \nabla E^{\sigma_t}(\mathbf{P}) \in supp\chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t}, s) \; and \, \chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t}, s) \neq 1\right\}$.

Taking into account:

- the definition of the application $\mathbf{J}_{\sigma_t}$ ($\mathbf{J}_{\sigma_t}(\mathbf{P}) \propto \mathbf{P}$) and the hypothesis B1;
- the definition of $\chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t}, s)$;

the volume $\mathbf{J}_{\sigma_t}^{-1}\left(\widehat{O}_{\nabla E^{\sigma_t}}^j\right)$ is bounded by a quantity of order $\frac{1}{s^{\frac{\delta}{3}}} \cdot t^{-3\epsilon}$. On the other hand, the volume of the region $supp\mathbf{J}_{\sigma_t}(\Gamma_j) \setminus supp_{\nabla E^{\sigma_t}} \chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t}, s)$ is bounded by a quantity of order $\frac{1}{s^{\frac{\delta}{6}}} \cdot t^{-2\epsilon}$.

Therefore

$$\left\|\left(1_{\Gamma_j}(\mathbf{P})-\chi^{(t)}_{\mathbf{v}_j}\left(\nabla E^{\sigma_t}(\mathbf{P}),s\right)\right)\psi^{(t)}_{i,\sigma_t}\right\|^2=\int_{\Gamma_j}\left|1_{\Gamma_j}(\mathbf{P})-\chi^{(t)}_{\mathbf{v}_j}\left(\nabla E^{\sigma_t},s\right)\right|^2|h(\mathbf{P})|^2\left\|\psi_{\mathbf{P},\sigma_t}\right\|^2 d^3P\leq$$
$$\leq C\cdot\frac{1}{s^{\frac{\delta}{6}}}\cdot t^{-2\epsilon}$$

from which the thesis follows .

Lemma B2

Results about the phase factor "$e^{i\gamma}$" (being $\mathbf{P}\in\Gamma_j\subset\Sigma$):

i) $\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),(\sigma_{t_2})^{-\frac{40}{39}}\right)-\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E^{\sigma_{t_2}}(\mathbf{P}),(\sigma_{t_2})^{-\frac{40}{39}}\right)\right|\leq C\cdot\left|\mathbf{v}_j-\mathbf{v}_{l(j)}\right|\ln\sigma_{t_2}$

where $\mathbf{v}_j$ and $\mathbf{v}_{l(j)}$ are referred to the partitions at time t_1 and t_2 respectively;

ii) $\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),t_1\right)-\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1\right)\right|\leq C\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot t_1;$

iii) $\left|e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right),s\right)}\right|\leq C\cdot s^{-\frac{1}{112}}$ for $\mathbf{q}:\ |\mathbf{q}|<s^{\frac{1}{20}}$

(notation warning: in the expressions below, $\sigma^S_t=t^{-\frac{39}{40}}$ is the slow cut-off).

Proof

i)
$$\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),(\sigma_{t_2})^{-\frac{40}{39}}\right)-\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E^{\sigma_{t_2}}(\mathbf{P}),(\sigma_{t_2})^{-\frac{40}{39}}\right)\right|=$$
$$=\left|\int_1^{(\sigma_{t_2})^{-\frac{40}{39}}}\left\{g^2\int_{\sigma_{t_2}}^{\sigma^S_\tau}\int\int\left(\frac{\cos\left(\mathbf{k}\cdot\nabla E^{\sigma_{t_2}}\tau(\mathbf{P})-|\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}-\frac{\cos\left(\mathbf{k}\cdot\nabla E^{\sigma_{t_2}}(\mathbf{P})-|\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_{l(j)}\right)}\right)d\Omega d|\mathbf{k}|\right\}d\tau\right|=$$
$$=\left|\int_1^{(\sigma_{t_2})^{-\frac{40}{39}}}\left\{g^2\int_{\sigma_{t_2}}^{\sigma^S_\tau}\int\int\left(\frac{\left(\widehat{\mathbf{k}}\cdot\mathbf{v}_j-\widehat{\mathbf{k}}\cdot\mathbf{v}_{l(j)}\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_{l(j)}\right)}\right)\cdot\cos\left(\mathbf{k}\cdot\nabla E^{\sigma_{t_2}}(\mathbf{P})\tau-|\mathbf{k}|\tau\right)d\Omega d|\mathbf{k}|\right\}d\tau\right|=$$
$$=\left|\int_1^{(\sigma_{t_2})^{-\frac{40}{39}}}\left\{g^2\int_{\tau\sigma_{t_2}}^{\tau\sigma^S_\tau}\int\int\left(\frac{\left(\widehat{\mathbf{q}}\cdot\mathbf{v}_j-\widehat{\mathbf{q}}\cdot\mathbf{v}_{l(j)}\right)}{\left(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_j\right)\left(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_{l(j)}\right)}\right)\cdot\cos\left(|\mathbf{q}|\,\widehat{\mathbf{q}}\cdot\nabla E^{\sigma_{t_2}}(\mathbf{P})-|\mathbf{q}|\right)d\Omega d|\mathbf{q}|\right\}\frac{d\tau}{\tau}\right|\leq$$
$$\leq C\cdot\left|\mathbf{v}_j-\mathbf{v}_{l(j)}\right|\cdot\left|\ln\sigma_{t_2}\right|$$

ii)
$$\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),t_1\right)-\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1\right)=$$
$$=-\int_1^{t_1}\left\{g^2\int_{\sigma_{t_1}}^{\sigma^S_\tau}\int\frac{\cos\left(\mathbf{k}\cdot\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\tau-|\mathbf{k}|\tau\right)-\cos\left(\mathbf{k}\cdot\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\tau-|\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d\Omega d|\mathbf{k}|\right\}d\tau+$$
$$-\int_1^{t_1}\left\{g^2\int_{\sigma_{t_2}}^{\sigma_{t_1}}\int\frac{\cos\left(\mathbf{k}\cdot\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\tau-|\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d\Omega d|\mathbf{k}|\right\}d\tau=$$

$$= 2\int_1^{t_1}\left\{ g^2 \int_{\sigma_{t_1}}^{\sigma_\tau^S}\int \frac{\sin\left(\frac{\mathbf{k}\cdot\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}-\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\tau}{2}\right)\sin\left(\frac{\mathbf{k}\cdot\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}+\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\tau-2|\mathbf{k}|\tau}{2}\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d\Omega d|\mathbf{k}| \right\} d\tau +$$

$$-\int_1^{t_1}\left\{ g^2 \int_{\sigma_{t_2}}^{\sigma_{t_1}}\int \frac{\cos\left(\mathbf{k}\cdot\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\tau-|\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d\Omega d|\mathbf{k}| \right\} d\tau$$

from which:

$$\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}(\mathbf{P}),t_1\right)-\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1\right)\right| \le C\cdot\left|\nabla E^{\sigma_{t_2}}(\mathbf{P})-\nabla E^{\sigma_{t_1}}(\mathbf{P})\right|\cdot\int_1^{t_1}\left(\sigma_\tau^S\right)^2\cdot\tau d\tau +$$
$$+C\cdot t_1\cdot\sigma_{t_1} \le C\cdot\left(\sigma_{t_1}\right)^{\frac{1}{4}}\cdot t_1$$

(the bound for $\left|\nabla E_{\mathbf{P}}^{\sigma_{t_2}}-\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right|$ comes from lemma A2, $\sigma_{t_1}=t_1^{-\beta}$ where $\beta \gg 1$, by hypothesis).

iii)

For $s \le \sigma_t^{-\frac{40}{39}}$, the absolute value of $\left|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}+\frac{\mathbf{q}}{s}}^{\sigma_t},s\right)}\right|$ can be estimated by the absolute value of the difference of the exponents:

$$\left|\int_1^s\left\{ g^2 \int_{\sigma_t}^{\sigma_\tau^S} \frac{\cos\left(\mathbf{k}\cdot\nabla E_{\mathbf{P}}^{\sigma_t}\tau-|\mathbf{k}|\tau\right)-\cos\left(\mathbf{k}\cdot\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\cdot\tau-|\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d\Omega d|\mathbf{k}| \right\} d\tau\right| =$$

$$= \left|2\int_1^s\left\{ g^2 \int_{\sigma_t}^{\sigma_\tau^S} \frac{\sin\left(\frac{\mathbf{k}\cdot\left(\nabla E_{\mathbf{P}}^{\sigma_t}-\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\tau}{2}\right)\sin\left(\frac{\mathbf{k}\cdot\left(\nabla E_{\mathbf{P}}^{\sigma_t}+\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\tau-2|\mathbf{k}|\tau}{2}\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d\Omega d|\mathbf{k}| \right\} d\tau\right| \le$$

$$\le C\cdot\left|\nabla E_{\mathbf{P}}^{\sigma_t}-\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right|\cdot\int_1^s\left(\sigma_\tau^S\right)^2\cdot\tau d\tau \le C\cdot\left|\frac{\mathbf{q}}{s}\right|^{\frac{1}{16}}\cdot\int_1^s\tau^{-\frac{19}{20}}d\tau \le C\cdot\left|\frac{s^{\frac{1}{20}}}{s}\right|^{\frac{1}{16}}\cdot s^{\frac{1}{20}} < C\cdot s^{-\frac{1}{112}}$$

The same estimate clearly holds for $s > \sigma_t^{-\frac{40}{39}}$.

Lemma B3

In the constructive hypotheses fixed at the beginning of chapter 4, we have

$$\left\|\int \widetilde{\chi}_{\mathbf{v}_j}^{(t)}(\mathbf{q},s)\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P})}-e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\right)d^3q e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi_{j,\sigma_t}^{(t)}\right\| \le$$

$$\le C\cdot s^{-\frac{1}{112}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}\cdot\left|\ln\sigma_t\right|$$

where $s \ge t$ and $\mathbf{v}_j$ is referred to the partition at time t.

Proof

Let us start from the following Hilbert inequality:

$$\left\|\int \widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P})}-e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\right)d^3q e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\|=$$

$$=\left\|e^{iE^{\sigma_t}(\mathbf{P})s}\int \widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P})}-e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\right)d^3q e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\|\leq$$

$$\leq\left\|\int \widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P})}-e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}\right)d^3q e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\|+ \qquad \mathbf{i)}$$

$$+\left\|\int \widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\| \qquad \mathbf{ii)}$$

To estimate the above integrals we distinguish "large" and "small" $\mathbf{q}$:
- for large $\mathbf{q}$, i.e for $|\mathbf{q}|>s^{\frac{1}{20}}$, we exploit the estimate of $\int_a^{+\infty}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_i}(\mathbf{q},s)\right|d^3q$;
- for small $\mathbf{q}$, $|\mathbf{q}|<s^{\frac{1}{20}}$, we exploit the Holder property in $\mathbf{P}$ of $\nabla E^{\sigma_t}(\mathbf{P})$ and of $\psi^{(t)}_{j,\sigma_t}$ in the expressions **i)** and **ii)** respectively.

$$\mathbf{i)}\ \left\|\int \widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P})}-e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}\right)d^3q e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t},s)}\psi^{(t)}_{j,\sigma_t}\right\|\leq$$

$$\leq C\cdot\left\|\psi^{(t)}_{j,\sigma}\right\|\cdot\int_{s^{\frac{1}{20}}}^{+\infty}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|d^3q+C\cdot\left\|\psi^{(t)}_{j,\sigma}\right\|\cdot\frac{\int_0^{s^{\frac{1}{20}}}|\mathbf{q}|\cdot\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|d^3q}{s^{\frac{19}{20}\cdot\frac{1}{16}}}$$

as follows from:

- $sE^{\sigma_t}(\mathbf{P})-sE^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)=-\mathbf{q}\cdot\nabla E^{\sigma_t}(\mathbf{P}')$ where $\mathbf{P}'$ satisfies $|\mathbf{P}-\mathbf{P}'|\leq\left|\frac{\mathbf{q}}{s}\right|$;
- for lemma 3.3: $|\nabla E^{\sigma_t}(\mathbf{P})-\nabla E^{\sigma_t}(\mathbf{P}')|\leq C\cdot|\mathbf{P}-\mathbf{P}'|^{\frac{1}{16}}\leq C\cdot\left|\frac{\mathbf{q}}{s}\right|^{\frac{1}{16}}$;
- $\int\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|d^3q\leq C\cdot s^{2\delta}$ and $\int_a^{+\infty}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|d^3q<C\cdot\frac{1}{a}\cdot s^{2\delta}$, by construction.

Then term **i)** is surely bounded by a quantity of order $s^{-\frac{1}{112}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}$.

$$\mathbf{ii)}\ \left\|\int \widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}\cdot\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\|\leq$$

$$\leq\left\|\int_{s^{\frac{1}{20}}}^{+\infty}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}\cdot\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\|+ \qquad \text{(b1)}$$

$$+\left\|\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}\cdot\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{j,\sigma_t}\right\| \qquad \text{(b2)}$$

term (b1)

It is bounded by $\left\|\psi^{(t)}_{i,\sigma_t}\right\|\cdot\int_{s^{\frac{1}{20}}}^{+\infty}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_i}(\mathbf{q},s)\right|d^3q$ then by

$$C\cdot s^{-\frac{1}{20}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}.$$

term (b2)

This term is controlled by adding and subtracting quantities to eventually obtain three expressions which can be estimated by:
- the convergence rate of the vector $\phi^{\sigma}_{\mathbf{P}}$;
- the regularity properties in $\mathbf{P}$ of $h(\mathbf{P})$ and $E^{\sigma_t}(\mathbf{P})$;
- the (vanishing) volume $O_{\frac{\mathbf{q}}{s}}$ which is the difference between the cell Γ_j and the same cell translated of a quantity $\frac{\mathbf{q}}{s}$.
In the expression below, take care of the following facts:

- both the vectors$\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}$, $e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\psi_{\mathbf{P},\sigma_t}$belong to the same fiber space $\mathrm{H}_{\mathbf{P}-\frac{\mathbf{q}}{s}}$;
- $I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\psi_{\mathbf{P},\sigma_t}\right)=I_{\mathbf{P}}\left(\psi_{\mathbf{P},\sigma_t}\right)$ (the isomorphism $I_{\mathbf{P}}$ is defined at pag.10);
- in an expression like $\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}h(\mathbf{P})\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}d^3P$, $\mathbf{P}$ is a variable of integration.

$$\left\|\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\,e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}\cdot\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right\|=$$

$$=\left\|\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}\cdot\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)h(\mathbf{P})\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi_{\mathbf{P},\sigma_t}d^3Pd^3q\right\|=$$

$$=\left\|\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}h(\mathbf{P})\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi_{\mathbf{P},\sigma_t}d^3Pd^3q+\right.$$

$$-\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}h(\mathbf{P})\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}d^3Pd^3q+$$

$$+\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}h(\mathbf{P})\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}d^3Pd^3q+$$

$$-\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P}-\frac{\mathbf{q}}{s})-E^{\sigma_t}(\mathbf{P})\right)s}h\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}d^3Pd^3q+$$

$$+\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P}-\frac{\mathbf{q}}{s})-E^{\sigma_t}(\mathbf{P})\right)s}h\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}d^3Pd^3q+$$

$$\left.-\int_0^{s^{\frac{1}{20}}}\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\int_{\Gamma_j}e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}h(\mathbf{P})\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi_{\mathbf{P},\sigma_t}d^3Pd^3q\right\|\leq$$

$$\leq\int_0^{s^{\frac{1}{20}}}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|\left\{\int_{\Gamma_j}|h(\mathbf{P})|^2\left\|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}I_{\mathbf{P}}\left(\psi_{\mathbf{P},\sigma_t}\right)-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|^2_F d^3P\right\}^{\frac{1}{2}}d^3q+ \quad \text{(b3.1)}$$

$$+\int_0^{s^{\frac{1}{20}}}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|\left\{\int_{\Gamma_j}\left|h\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)e^{i\left(E^{\sigma_t}(\mathbf{P}-\frac{\mathbf{q}}{s})-E^{\sigma_t}(\mathbf{P})\right)s}-G(\mathbf{P})\,e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}(\mathbf{P}+\frac{\mathbf{q}}{s})\right)s}\right|^2\left\|I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|^2_F d^3P\right\}^{\frac{1}{2}}d^3q- \quad \text{(b3.2)}$$

$$+\int_0^{s^{\frac{1}{20}}}\left|\widetilde{\chi}^{(t)}_{\mathbf{v}_j}(\mathbf{q},s)\right|\left\{\int_{O_{\frac{\mathbf{q}}{s}}}|h(\mathbf{P})|^2\left\|I_{\mathbf{P}}(\psi_{\mathbf{P},\sigma_t})\right\|_F^2 d^3P\right\}^{\frac{1}{2}} d^3q \tag{b3.3}$$

term (b3.1)

Knowing that $\mathbf{P}\in\Gamma_j\subset\Sigma$, let us estimate:

$$\left\|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}I_{\mathbf{P}}\left(\psi_{\mathbf{P},\sigma_t}\right)-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F\le$$

$$\le\left\|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}I_{\mathbf{P}}\left(\psi_{\mathbf{P},\sigma_t}\right)-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F+$$

$$+\left\|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F\le$$

$$\le\left\|I_{\mathbf{P}}\left(W^{b^\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)\psi_{\mathbf{P},\sigma_t}\right)-I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W^{b^\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F+$$

$$+\left|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\right|\left\|I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F\le$$

$$\le\left\|I_{\mathbf{P}}\left(W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)\psi_{\mathbf{P},\sigma_t}\right)-I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F+$$

$$+\left\|I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\left(W^{b^\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)-W^{b^\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\right)W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F+$$

$$+\left|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}-e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)}\right|\left\|I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F.$$

Note that:

- the norm $\left\|I_{\mathbf{P}}\left(W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)\psi_{\mathbf{P},\sigma_t}\right)-I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F$ is bounded by a quantity of order $\left(\frac{|\mathbf{q}|}{s}\right)^{\frac{1}{16}}$ (see theorem 3.4, consider the difference of notation for the Weyl operator).

- the norm of $\left\|I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\left(W^{b^\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)-W^{b^\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\right)W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F$ (b4)

can be estimated by the norm of

$$g\int_{\sigma_t}^{\kappa}\frac{\widehat{\mathbf{k}}\cdot\left(\nabla E^{\sigma_t}(\mathbf{P})-\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_t}(\mathbf{P})\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)}\left(b(\mathbf{k})-b^\dagger(\mathbf{k})\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W^{b}_{\sigma_t}\left(\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)$$

then it is substantially the product of the following quantities:

- $\left(\int_{\sigma_t}^{\kappa}\left(\frac{g\widehat{\mathbf{k}}\cdot\left(\nabla E^{\sigma_t}(\mathbf{P})-\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_t}(\mathbf{P})\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)}\right)^2 d^3k\right)^{\frac{1}{2}}$

  it is bounded by $C\cdot\left|\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)-\nabla E^{\sigma_t}(\mathbf{P})\right|\cdot|\ln\sigma_t|^{\frac{1}{2}}\le C\cdot\left|s^{-\frac{19}{20}}\right|^{\frac{1}{16}}\cdot|\ln\sigma_t|^{\frac{1}{2}}$ (see lemma

3.3);

- $\left(\int_{\sigma_t}^{\kappa}\left\|b(\mathbf{k})\, I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W^b_{\sigma_t}\left(\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F^2 d^3k\right)^{\frac{1}{2}} =$

$$= \left(\int_{\sigma_t}^{\kappa}\left\|I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W^b_{\sigma_t}\left(\nabla E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)\right)\left(b(\mathbf{k})+\frac{g\chi^{\kappa}_{\sigma_t}(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right)}\right)\psi_{\mathbf{P}+\frac{\mathbf{q}}{s},\sigma_t}\right)\right\|_F^2 d^3k\right)^{\frac{1}{2}};$$

by using techniques like in [3], it is possible to provide a bound uniformly in t and in s. For our purposes, a bound of order $|\ln\sigma_t|^{\frac{1}{2}}$ (uniform in s) is sufficient. For $\mathbf{P}\in\Sigma$, it comes from the formula obtained in [3]

$$b(\mathbf{k})\,\psi_{\mathbf{P},\sigma_t} = \frac{g}{\sqrt{2\,|\mathbf{k}|}}\left(\frac{1}{E^{\sigma_t}(\mathbf{P})-|\mathbf{k}|-H_{\mathbf{P}-\mathbf{k},\sigma_t}}\right)\psi_{\mathbf{P},\sigma_t} \qquad \sigma_t\leq|\mathbf{k}|\leq\kappa\,;$$

- finally $\left|e^{i\gamma_{\sigma_t}\left(\mathbf{v}_i,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} - e^{i\gamma_{\sigma_t}\left(\mathbf{v}_i,\nabla E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}},s\right)}\right| \leq C\cdot s^{-\frac{1}{112}}$ as proved in lemma B2, ii).

Term (b3.1) is therefore bounded by

$$C\cdot\left(s^{-\frac{19}{20}}\right)^{\frac{1}{32}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}} + C\cdot\left(s^{-\frac{19}{20}}\right)^{\frac{1}{16}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}\cdot|\ln\sigma_t| + C\cdot s^{-\frac{1}{112}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}$$

so that a leading order is

$$s^{-\frac{1}{112}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}\cdot|\ln\sigma_t|\,.$$

term (b3.2)

Being $h\in C^1_0\left(R^3\setminus 0\right)$ and using lemma 3.3 to estimate

$$e^{i\left(E^{\sigma_t}\left(\mathbf{P}-\frac{\mathbf{q}}{s}\right)-E^{\sigma_t}(\mathbf{P})\right)s} - e^{i\left(E^{\sigma_t}(\mathbf{P})-E^{\sigma_t}\left(\mathbf{P}+\frac{\mathbf{q}}{s}\right)\right)s}$$

we can provide a bound with the quantity

$$C\cdot|\mathbf{q}|\cdot\left(\frac{|\mathbf{q}|}{s}\right)^{\frac{1}{16}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}} \leq C\cdot s^{-\frac{1}{20}}\cdot\left(s^{-\frac{19}{20}}\right)^{\frac{1}{16}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}} \leq C\cdot s^{2\delta}\cdot s^{-\frac{1}{112}}\cdot t^{-\frac{3\epsilon}{2}}$$

term (b3.3)

Starting from a difference between volumes, the expression (b3.3) is bounded by a quantity of order $\left(\frac{|\mathbf{q}|}{s}\right)^{\frac{1}{2}}\cdot s^{2\delta}\cdot t^{-\epsilon}\leq s^{-\frac{19}{40}}\cdot s^{2\delta}\cdot t^{-\epsilon}$.

Conclusion

The sum of the terms **i)** and **ii)** is surely bounded by

$$C \cdot s^{-\frac{1}{112}} \cdot s^{2\delta} \cdot |\ln \sigma_t| \cdot t^{-\frac{3\epsilon}{2}}$$

(the constraint $2\delta + 4\epsilon < \frac{1}{112}$ is assumed).

Corollary B4

For $s \gg 1$ and such that $s^{-\frac{39}{40}} \geq \sigma_t$, the norm of

$$\left(g^2 \int_{\sigma_t \cdot s}^{s \cdot s^{-\frac{39}{40}}} \int\int \frac{\cos\left(\mathbf{q} \cdot \frac{\mathbf{x}}{s} - |\mathbf{q}|\right)}{\left(1 - \widehat{\mathbf{q}} \cdot \mathbf{v}_j\right)} d\Omega \frac{d\,|\mathbf{q}|}{s} - g^2 \int_{\sigma_t \cdot s}^{s \cdot s^{-\frac{39}{40}}} \int\int \frac{\cos\left(\mathbf{q} \cdot \nabla E^{\sigma_t} - |\mathbf{q}|\right)}{\left(1 - \widehat{\mathbf{q}} \cdot \mathbf{v}_j\right)} d\Omega \frac{d\,|\mathbf{q}|}{s} \right) e^{-iE^{\sigma_t} s} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)} \psi^{(t)}_{j,\sigma_t}$$

is surely bounded by a quantity of order $s^{-1} \cdot s^{-\frac{1}{112}} \cdot s^{2\delta} \cdot |\ln \sigma_t| \cdot t^{-\frac{3\epsilon}{2}}$.

Lemma B5

In this lemma we provide some upper estimates for the absolute value of the function $\varphi_{t^{-\alpha}, \mathbf{v}_j}(\mathbf{x}, t)$ ($t \gg 1$and $s \gg 1$)

$$\varphi_{t^{-\alpha}, \mathbf{v}_j}(\mathbf{x}, s) = g^2 \int_{t^{-\alpha}}^{\kappa_1} \int\int \frac{\cos\left(\mathbf{k} \cdot \mathbf{x} - |\mathbf{k}|\, s\right)}{\left(1 - \widehat{\mathbf{k}} \cdot \mathbf{v}_j\right)} d\Omega d\,|\mathbf{k}|$$

where $\alpha > 0$.

Proof

We provide the estimates by performing elementary integrations.

Estimate uniform in $\mathbf{x} \in R^3$.

For $\mathbf{x}$ in the set $\{\mathbf{x} : \; |\mathbf{x}| < (1 - \eta)\, s, \; 0 < \eta < 1\}$, we easily obtain

$$\left|\varphi_{t^{-\alpha}, \mathbf{v}_j}(\mathbf{x}, t)\right| = \left| g^2 \int\int \frac{\sin\left(\kappa_1 \widehat{\mathbf{k}} \cdot \mathbf{x} - \kappa_1 s\right) - \sin\left(t^{-\alpha} \widehat{\mathbf{k}} \cdot \mathbf{x} - t^{-\alpha} \cdot s\right)}{\left(1 - \widehat{\mathbf{k}} \cdot \mathbf{v}_j\right) \cdot \left(\widehat{\mathbf{k}} \cdot \mathbf{x} - s\right)} \cdot d\Omega \right| \leq \frac{1}{\eta s} \int\int \left| g^2 \frac{2}{\left(1 - \widehat{\mathbf{k}} \cdot \mathbf{v}_j\right)} \right| d\Omega$$

For $\mathbf{x}$ in the set $\{\mathbf{x} : \; |\mathbf{x}| \geq (1 - \eta)\, s, \; 0 < \eta < 1\}$ we first integrate by parts with respect to $d \cos\theta$, having set $\xi\left(\widehat{\mathbf{k}}, \mathbf{v}_i\right) = \xi\left(\theta, \varphi, \mathbf{v}_i\right) \equiv \frac{1}{\left(1 - \widehat{\mathbf{k}} \cdot \mathbf{v}_j\right)}$. Note that since $|\mathbf{v}_j| \leq v^{max} < 1$

$$\exists \;\; M \geq 0, M' \geq 0 \quad \Longrightarrow \quad \left|\xi\left(\widehat{\mathbf{k}}, \mathbf{v}_i\right)\right| < M \quad and \quad \left|\frac{d}{d\cos\theta}\left[\xi\left(\theta, \varphi, \mathbf{v}_i\right)\right]\right| < M' .$$

We obtain

$$
\begin{aligned}
& g^2 \int_{t^{-\alpha}}^{\kappa_1} \int\int \cos\left(\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|\, s\right) \cdot \xi\left(\widehat{\mathbf{k}}, \mathbf{v}_j\right) d\Omega d\,|\mathbf{k}| = \\
& = - \int_{t^{-\alpha}}^{\kappa_1} \int \frac{\sin(|\mathbf{k}|\cdot|\mathbf{x}|+|\mathbf{k}|s)}{|\mathbf{x}|} \cdot \frac{\xi(\pi,\varphi,\mathbf{v}_j)}{|\mathbf{k}|} d\,|\mathbf{k}|\, d\varphi + \int_{t^{-\alpha}}^{\kappa_1} \int \frac{\sin(|\mathbf{k}|\cdot|\mathbf{x}|-|\mathbf{k}|s)}{|\mathbf{x}|} \cdot \frac{\xi(0,\varphi,\mathbf{v}_j)}{|\mathbf{k}|} d\,|\mathbf{k}|\, d\varphi + \\
& + \int_{t^{-\alpha}}^{\kappa_1} \int\int \frac{\sin(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{|\mathbf{x}|} \cdot \frac{1}{|\mathbf{k}|} \frac{d}{d\cos\theta} \left[\xi\left(\theta,\varphi,\mathbf{v}_j\right)\right] d\,|\mathbf{k}|\, d\Omega\,.
\end{aligned}
\qquad \text{(b5)}
$$

each term on the right hand side can be easily bounded by a quantity of order $\frac{\ln t}{s-\eta s}$ thereby.
In conclusion there exists a cost ant C, that is uniform in $\mathbf{v}_j$, $|\mathbf{v}_j| < v^{max} < 1$, such that

$$\left|\varphi_{t^{-\alpha},\mathbf{v}_j}\left(\mathbf{x},s\right)\right| \leq C \cdot \frac{\ln t}{s-\eta s} \qquad \forall \mathbf{x} \in R^3.$$

Estimate restricted to the set $\{\mathbf{x} : \ (1-\eta')\, s < |\mathbf{x}| < (1-\eta)\, s\,,\ 0 < \eta < \eta' < 1\}$

To investigate the behavior of $\varphi_{t^{-\alpha},\mathbf{v}_i}(\mathbf{x},s)$ for $(1-\eta')\, s < |\mathbf{x}| < (1-\eta)\, s$, $0 < \eta < \eta' < 1$, let us start from the expression (b5).
We now provide an estimate for the first term

$$-\int_{t^{-\alpha}}^{\kappa_1} \int \frac{\sin\left(|\mathbf{k}|\cdot|\mathbf{x}| + |\mathbf{k}|\, s\right)}{|\mathbf{x}|} \cdot \frac{\xi\left(\pi,\varphi,\mathbf{v}_j\right)}{|\mathbf{k}|} d\,|\mathbf{k}|\, d\varphi$$

which however holds for the others by analogous steps, taking into account

$$|\mathbf{x}|\cos\theta - s \leq |\mathbf{x}| - s < (1-\eta)\, s - s \quad \Rightarrow \quad |\mathbf{x}|\cos\theta - s < -\eta s \quad \Rightarrow \quad \left||\mathbf{x}|\cos\theta - s\right| > \eta s\,.$$

Our estimate derives from a simple integration by parts with respect to $d\,|\mathbf{k}|$

$$
\begin{aligned}
& - \int_{t^{-\alpha}}^{\kappa_1} \int \frac{\sin(|\mathbf{k}|\cdot|\mathbf{x}|+|\mathbf{k}|s)}{|\mathbf{x}|} \cdot \frac{\xi(\pi,\varphi,\mathbf{v}_j)}{|\mathbf{k}|} d\,|\mathbf{k}|\, d\varphi = \int_{t^{-\alpha}}^{\kappa_1} \int \frac{d}{d|\mathbf{k}|} \left(\frac{\cos(|\mathbf{k}|\cdot|\mathbf{x}|+|\mathbf{k}|s)}{|\mathbf{x}|}\right) \cdot \frac{\xi(\pi,\varphi,\mathbf{v}_j)}{(|\mathbf{x}|+s)|\mathbf{k}|} d\,|\mathbf{k}|\, d\varphi = \\
& = \int \frac{\cos(\kappa_1\cdot|\mathbf{x}|+\kappa_1 s)}{|\mathbf{x}|} \cdot \frac{\xi(\pi,\varphi,\mathbf{v}_j)}{(|\mathbf{x}|+s)|\kappa_1|} d\varphi - \int \frac{\cos\left(t^{-\alpha}\cdot|\mathbf{x}|+t^{-\alpha}\cdot s\right)}{|\mathbf{x}|} \cdot \frac{\xi(\pi,\varphi,\mathbf{v}_j)}{(|\mathbf{x}|+s)\cdot t^{-\alpha}} d\varphi + \\
& + \int_{t^{-\alpha}}^{\kappa_1} \int \frac{\cos(|\mathbf{k}|\cdot|\mathbf{x}|+|\mathbf{k}|s)}{|\mathbf{x}|} \cdot \frac{\xi(\pi,\varphi,\mathbf{v}_i)}{(|\mathbf{x}|+s)|\mathbf{k}|^2} d\,|\mathbf{k}|\, d\varphi
\end{aligned}
$$

where each term on the right hand side is bounded by a quantity of order $\frac{t^\alpha}{s^2}$.

Conclusion: in the region $(1-\eta')\, s < |\mathbf{x}| < (1-\eta)\, s$ we have that

$$\left|\varphi_{t^{-\alpha},\mathbf{v}_j}\left(\mathbf{x},s\right)\right| \leq C_{\eta,\eta'} \cdot \frac{t^\alpha}{s^2}\,.$$

Theorem B6

Taking into account lemma B3, one can prove the existence of

$$s - \lim_{s\to+\infty} e^{iH_{\sigma_t} s} \int_{\sigma_t}^{\kappa_1} \frac{a\left(\mathbf{k}\right) e^{i|\mathbf{k}|s}}{|\mathbf{k}|} \cdot h_{l,j}\left(\widehat{\mathbf{k}}\right) \frac{d^3 k}{\sqrt{2\,|\mathbf{k}|}} e^{-iH_{\sigma_t} s} \psi^{(t)}_{j,\sigma_t} \equiv a^{out}_{\sigma_t}\left(h_{l,j}\right) \psi^{(t)}_{j,\sigma_t}$$

The vectors $a_{\sigma_t}^{out(in)}(h_{l,j})\,\psi_{j,\sigma_t}^{(t)}$ belong to $D(H_{\sigma_t})$.

Proof

We check that the following quantity is integrable with respect to s:

$$\left\| \frac{d\left\{ e^{iH_{\sigma_t}s} \int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|s}}{|\mathbf{k}|} \cdot h_{l,j}\left(\widehat{\mathbf{k}}\right) \frac{d^3k}{\sqrt{2|\mathbf{k}|}} e^{-iH_{\sigma_t}s} e^{i\gamma_{\sigma_t}(\mathbf{v}_j, \nabla E^{\sigma_t}(\mathbf{P}),s)} \psi_{j,\sigma_t}^{(t)} \right\}}{ds} \right\|$$

Formally:

$$\frac{d\left(e^{iH_{\sigma_t}s}e^{-iH^{ph}s}a(h_{l,j})e^{iH^{ph}s}e^{-iH_{\sigma_t}s}\right)}{dt} = \frac{d\left(e^{iH_{\sigma_t}s}a(h_{l,j\,s})e^{-iH_{\sigma_t}s}\right)}{dt} =$$

$$= e^{iH_{\sigma_t}s}\left\{ i\int_{\sigma_t}^{\kappa_1} \widetilde{h}_{l,j}(\mathbf{k})\, e^{i(|\mathbf{k}|s - \mathbf{k}\cdot\mathbf{x})} \frac{1}{2|\mathbf{k}|^2} d^3k \right\} e^{-iH_{\sigma_t}s}$$

The formal expression is well defined from an operatorial point of view in $D(H_{\sigma_t})$ (see the *Note* in lemma B8).

Having defined $\widehat{h_{l,j}}(\mathbf{x},s) \equiv \left\{ i\int_{\sigma_t}^{\kappa_1} \widetilde{h}_{l,j}(\mathbf{k})\, e^{i(|\mathbf{k}|s-\mathbf{k}\cdot\mathbf{x})}\frac{1}{2|\mathbf{k}|^2}d^3k \right\}$, we turn to consider the Hilbert inequality:

$$\left\| \frac{d\left(e^{iH_{\sigma_t}s}a(h_{l,j\,s})e^{-iH_{\sigma_t}s}\right)}{dt} \psi_{i,\sigma_t}^{(t)} \right\| \leq$$

$$\leq \left\| \widehat{h_{l,j}}(\mathbf{x},s)\left(1_{\Gamma_j}(\mathbf{P}) - \chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t},s)\right) e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} \right\| + \left\| \widehat{h_{l,j}}(\mathbf{x},s)\left(\chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t},s) - \chi_{\mathbf{v}_j}^{(t)}\left(\tfrac{\mathbf{x}}{s},s\right)\right) e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} \right\| +$$

$$+ \left\| \widehat{h_{l,j}}(\mathbf{x},s)\,\chi_{\mathbf{v}_j}^{(t)}\left(\tfrac{\mathbf{x}}{s},s\right) e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} \right\| \leq$$

$$\leq \sup_{\mathbf{x}} \left| \widehat{h_{l,j}}(\mathbf{x},s) \right| \cdot \left\| \left(1_{\Gamma_j}(\mathbf{P}) - \chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t},s)\right) e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} \right\| +$$

$$+ \sup_{\mathbf{x}} \left| \widehat{h_{l,j}}(\mathbf{x},s) \right| \cdot \left\| \left(\chi_{\mathbf{v}_j}^{(t)}(\nabla E^{\sigma_t},s) - \chi_{\mathbf{v}_j}^{(t)}\left(\tfrac{\mathbf{x}}{s},s\right)\right) e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} \right\| +$$

$$+ \sup_{\frac{\mathbf{x}}{s}\in \mathbf{J}_{\sigma_t}(\Gamma_j)} \left| \widehat{h_{l,j}}(\mathbf{x},s)\cdot \chi_{\mathbf{v}_j}\left(\tfrac{\mathbf{x}}{s},s\right) \right| \cdot \left\| \psi_{j,\sigma_t}^{(t)} \right\|$$

Because of lemmas B1, B2, B3 and B5, the first two terms on the right hand side are respectively bounded by $C\cdot s^{-\frac{\delta}{12}}\cdot \frac{|\ln \sigma_t|}{s}\cdot t^{-\epsilon}$ $(\delta > 72\epsilon)$ and by $C\cdot s^{-1}\cdot s^{-\frac{1}{112}}\cdot s^{2\delta}\cdot t^{-\frac{3\epsilon}{2}}\cdot (\ln\sigma_t)^2$ (the constraint $2\delta + 4\epsilon < \frac{1}{112}$ is assumed). As regards the third term, the hypotheses on $h(\mathbf{P})$and $\mathbf{v}_j$, the hypothesis B1 and Huyghens' principle ensure an integrable rate, for $s \to +\infty$.

The vectors $a_{\sigma_t}^{out(in)}(h_{l,j})\,\psi_{j,\sigma_t}^{(t)}$ *belong to* $D(H)$.

For each s, the vector $H_{\sigma_t}e^{iH_{\sigma_t}s}a(h_{l,j\,s})\,e^{-iH_{\sigma_t}s}\psi_{i,\sigma_t}^{(t)}$ is well defined because $a(h_{l,j\,s})\,e^{-iH_{\sigma_t}s}\psi_{i,\sigma_t}^{(t)} \subset D(H_{\sigma_t})$. Therefore, since H_{σ_t} is a closed operator, it is sufficient to prove the convergence, for

$s \to +\infty$, of:

$$H_{\sigma_t} e^{iH_{\sigma_t}s} a\left(h_{l,j\,s}\right) e^{-iH_{\sigma_t}s} \psi_{i,\sigma_t}^{(t)} = e^{iH_{\sigma_t}s} H_{\sigma_t} a\left(h_{l,j\,s}\right) e^{-iH_{\sigma_t}s} \psi_{i,\sigma_t}^{(t)} =$$

$$= \left\{ e^{iH_{\sigma_t}s} a\left(h_{l,j\,s}\right) e^{-iH_{\sigma_t}s} E^{\sigma_t}(\mathbf{P}) \psi_{i,\sigma_t}^{(t)} + e^{iH_{\sigma_t}s} \left[H_{\sigma_t} - H^{ph}, a\left(h_{l,j\,s}\right)\right] e^{-iH_{\sigma_t}s} \psi_{i,\sigma_t}^{(t)} + e^{iH_{\sigma_t}s} \left[H^{ph}, a\left(h_{l,j\,s}\right)\right] e^{-iH_{\sigma_t}s} \psi_{i,\sigma_t}^{(t)} \right\}$$

Looking at the first part of the theorem and being $\left[H^{ph}, a\left(h_{l,j\,s}\right)\right] = -\int_{\sigma_t}^{\kappa_1} a(\mathbf{k})\, e^{i|\mathbf{k}|s} \frac{\widetilde{h_{l,j}}(\mathbf{k})}{\sqrt{2|\mathbf{k}|}} d^3k$, each term in the above expression has limit.

Theorem B7

If for each $\mathbf{P}$ in Γ_j and for each $\mathbf{k}$ in $supp\widetilde{h_{l,j}}(\mathbf{k})$, $\widetilde{h_{l,j}}(\mathbf{k}) = \frac{h_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}} \chi_{\sigma_t}^{\kappa_1}(\mathbf{k})$, it happens that $\mathbf{P} + \mathbf{k} \in \Sigma$, then

$$a_{\sigma_t}^{out(in)}\left(h_{l,j}\right) \psi_{l,\sigma_t}^{(t)} = 0 \; .$$

Proof

Starting from the spectral decomposition with respect to $\mathbf{P}$ operators, we obtain that

$$\int a_{\sigma_t}^{out}(\mathbf{k})\, \widetilde{h_{l,j}}(\mathbf{k}) \left(\psi_{j,\sigma_t}^{(t)}\right)_{\mathbf{P}+\mathbf{k}} d^3k$$

is a vector in $\mathrm{H}_{\mathbf{P}}$ and that it belongs to the domain of $H_{\mathbf{P},\sigma_t}$. Then the procedure consists in studying the mean value of the positive operator $H_{\mathbf{P},\sigma_t} - E^{\sigma_t}(\mathbf{P}) + \delta$ (δ arbitrarily small positive number) on it. Taking into account the condition $|\nabla E^{\sigma_t}|_{\mathbf{P}+\mathbf{q}}| < 1 \;\forall \mathbf{q} \in supp\widetilde{h_{l,j}}$ (if $\mathbf{P} \in \Sigma$) which implies the estimate $E^{\sigma_t}(\mathbf{P}+\mathbf{k}) - |\mathbf{k}| - E^{\sigma_t}(\mathbf{P}) < 0$, we can conclude that the vector is zero.

Lemma B8

In this lemma we justify the derivation of the expression (19) in paragraph 4.1 and the upper bound for its norm estimate in time.

Proof

The term $ie^{iH_{\sigma_t}s} W_{\sigma_t}(\mathbf{v}_l, s)\, \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x}, s)\, e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}, s)} e^{-iH_{\sigma_t}s}$ is formally obtained from:

$$ie^{iH_{\sigma_t}s} \left[H_{\sigma_t} - H^{ph}, e^{-g\int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|s} - a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|s}}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \right] e^{-iH_{\sigma_t}s} =$$

$$= ie^{iH_{\sigma_t}s} \left[g\int_{\sigma_t}^{\kappa_1} \left(a(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \frac{d^3\mathbf{k}}{\sqrt{2|\mathbf{k}|}}, e^{-g\int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|s} - a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|s}}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \right] e^{-iH_{\sigma_t}s} =$$

$$= ie^{iH_{\sigma_t}s}e^{-g\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}-a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|s}}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}g^2\int_{\sigma_t}^{\kappa_1}\int\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}d\Omega d\left|\mathbf{k}\right|e^{-iH_{\sigma_t}s}$$

the last step follows from the commutator:

$$\left[g\int_{\sigma_t}^{\kappa_1}\left(a\left(\mathbf{k}\right)e^{i\mathbf{k}\cdot\mathbf{x}}+a^\dagger\left(\mathbf{k}\right)e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}},-g\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}-a^\dagger(\mathbf{k})e^{-i|\mathbf{k}|s}}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\right]=$$
$$= g^2\int_{\sigma_t}^{\kappa_1}\int\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}d\Omega d\left|\mathbf{k}\right|$$

Note

The formal steps are well defined in $D\left(H_{\sigma_t}\right)$ from the operatorial point of view, because:

- H_{σ_t}, H^{ph}, $H_0=\frac{\mathbf{p}^2}{2m}+H^{ph}$ and $g\int_{\sigma_t}^{\kappa_1}\left(a\left(\mathbf{k}\right)e^{i\mathbf{k}\cdot\mathbf{x}}+a^\dagger\left(\mathbf{k}\right)e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}$ have a common domain D of essential selfadjointness;
- $\dfrac{d\left(e^{iH^{ph}s}e^{-iH_{\sigma_t}s}\right)}{ds}$ is closable;
- the sequences, which are obtained from the formal calculus by approximating (in the norm $\|H_0\psi\|+\|\psi\|$) the vectors in $D\left(H_{\sigma_t}\right)$ with vectors in D, are convergent.

Norm estimate of the expression (19) :

We exploit the decomposition of $\varphi_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)$ in $\varphi^1_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)+\varphi^2_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)$ (paragraph 4.1) and the approximated characteristic function $\chi^{(t)}_{\mathbf{v}_l}\left(\nabla E^{\sigma_t},s\right)$. Indeed:

$$\varphi_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t}+\frac{d\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}{ds}e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t}=$$

$$=\varphi_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\left(1_{\Gamma_l}\left(\mathbf{P}\right)-\chi^{(t)}_{\mathbf{v}_l}\left(\nabla E^{\sigma_t}\left(\mathbf{P}\right),s\right)\right)\psi^{(t)}_{l,\sigma_t}+ \quad \text{i)}$$

$$+\frac{d\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}{ds}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}e^{-iE^{\sigma_t}(\mathbf{P})s}\left(1_{\Gamma_l}\left(\mathbf{P}\right)-\chi^{(t)}_{\mathbf{v}_l}\left(\nabla E^{\sigma_t}\left(\mathbf{P}\right),s\right)\right)\psi^{(t)}_{l,\sigma_t}+ \quad \text{ii)}$$

$$+\varphi_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)\left(\chi^{(t)}_{\mathbf{v}_l}\left(\nabla E^{\sigma_t}\left(\mathbf{P}\right),s\right)-\chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t}+ \quad \text{iii)}$$

$$+\varphi^2_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)\chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t}+ \quad \text{iv)}$$

$$+\left(\varphi^1_{\sigma_t,\mathbf{v}_l}\left(\mathbf{x},s\right)\chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)+\tfrac{d\gamma_{\sigma_t}}{ds}\left(\mathbf{v}_l,\tfrac{\mathbf{x}}{s},s\right)\chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t}+ \quad \text{v)}$$

$$-\tfrac{d\gamma_{\sigma_t}}{ds}\left(\mathbf{v}_l,\tfrac{\mathbf{x}}{s},s\right)\left(\chi^{(t)}_{\mathbf{v}_l}\left(\tfrac{\mathbf{x}}{s},s\right)-\chi^{(t)}_{\mathbf{v}_l}\left(\nabla E^{\sigma_t}\left(\mathbf{P}\right),s\right)\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t}+ \quad \text{vi)}$$

$$+\left(-\tfrac{d\gamma_{\sigma_t}}{ds}\left(\mathbf{v}_l,\tfrac{\mathbf{x}}{s},s\right)+\tfrac{d\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}{ds}\right)\chi^{(t)}_{\mathbf{v}_l}\left(\nabla E^{\sigma_t}\left(\mathbf{P}\right),s\right)e^{-iE^{\sigma_t}(\mathbf{P})s}e^{i\gamma_{\sigma_t}(\mathbf{v}_l,\nabla E^{\sigma_t}(\mathbf{P}),s)}\psi^{(t)}_{l,\sigma_t} \quad \text{vii)}$$

Now we explain how to control each term:

i) and ii) because of lemma B1 and of lemma B5 these terms are bounded by

$$C \cdot \left| \frac{\ln(\sigma_t)}{s} \right| \cdot \frac{1}{s^{\frac{\delta}{12}}} \cdot t^{-\epsilon} \qquad (\delta > 72\epsilon);$$

iii) we exploit the inequalities

$$|\varphi_{\sigma_t, \mathbf{v}_l}(\mathbf{x}, s)| \leq C \cdot \left| \frac{\ln(\sigma_t)}{s} \right|$$

$$\left\| \left(\chi^{(t)}_{\mathbf{v}_l} (\nabla E^{\sigma_t}(\mathbf{P}), s)\, \psi^{(t)}_{l,\sigma_t} - \chi^{(t)}_{\mathbf{v}_l} \left(\frac{\mathbf{x}}{s}, s \right) \right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}, s)} \psi^{(t)}_{l,\sigma_t} \right\| \leq C \cdot s^{-\frac{1}{112}} \cdot s^{2\delta} \cdot |\ln \sigma_t| \cdot t^{-\frac{3\epsilon}{2}}$$

discussed in lemma B5 and in lemma B3 respectively;

iv) the support restriction due to $\chi^{(t)}_{\mathbf{v}_l}\left(\frac{\mathbf{x}}{s}, s\right)$ leads to the estimate contained in lemma B5

$$\left| \varphi^2_{\sigma_t, \mathbf{v}_l}(\mathbf{x}, s)\, \chi^{(t)}_{\mathbf{v}_l} \left(\frac{\mathbf{x}}{s}, s \right) \right| \leq C \cdot s^{-2} \cdot s^{\frac{39}{40}};$$

v) this term is identically zero since we defined

$$\frac{d\gamma_{\sigma_t}}{ds} \left(\mathbf{v}_l, \frac{\mathbf{x}}{s}, s \right) \equiv -\varphi^1_{\sigma_t, \mathbf{v}_l}(\mathbf{x}, s);$$

vi) we exploit the inequalities

$$\left| \frac{d\gamma_{\sigma_t}}{ds} \left(\mathbf{v}_l, \frac{\mathbf{x}}{s}, s \right) \right| = |\varphi^1_{\sigma_t, \mathbf{v}_l}(\mathbf{x}, s)| \leq C \cdot \left| \frac{\ln(\sigma_t)}{s} \right|,$$

$$\left\| \left(\chi^{(t)}_{\mathbf{v}_l} (\nabla E^{\sigma_t}(\mathbf{P}), s)\, \psi^{(t)}_{i,\sigma_t} - \chi^{(t)}_{\mathbf{v}_l} \left(\frac{\mathbf{x}}{s}, s \right) \right) e^{-iE^{\sigma_t}(\mathbf{P})s} e^{i\gamma_{\sigma_t}(\mathbf{v}_l, \nabla E^{\sigma_t}, s)} \psi^{(t)}_{l,\sigma_t} \right\| \leq C \cdot s^{-\frac{1}{112}} \cdot s^{2\delta} \cdot |\ln \sigma_t| \cdot t^{-\frac{3\epsilon}{2}};$$

for which we use lemma B5 and lemma B3 respectively.

vi) from Corollary B4 a bound of order $s^{-1} \cdot s^{-\frac{1}{112}} \cdot s^{2\delta} \cdot |\ln \sigma_t| \cdot t^{-\frac{3\epsilon}{2}}$ follows.

Lemma B9

In this lemma we provide the proofs of the estimates used in the control of *D4.1*.

Analysis of

$$\left\| g \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{b(\mathbf{k}) \left(e^{i|\mathbf{k}|t_1 - i\mathbf{k}\cdot\mathbf{x}} - 1 \right)}{|\mathbf{k}| \left(1 - \widehat{\mathbf{k}} \cdot \mathbf{v}_j \right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}} \varphi_j \right\|.$$

1) We first check that the expression

$$b(\mathbf{k})\, \varphi_j \equiv b(\mathbf{k})\, W^b_{\sigma_{t_2}}(\mathbf{v}_j)\, W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\, e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j, \nabla E^{\sigma_{t_1}}, t_1)} e^{-iE^{\sigma_{t_1}}(\mathbf{P})\, t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\, \psi^{(t_1)}_{j,\sigma_{t_1}}$$

is a well defined vector in H and that it is strongly continuous in $\mathbf{k}$.
Then we easily get the following inequality:

$$\left\| \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{b(\mathbf{k})(e^{i|\mathbf{k}|t_1-i\mathbf{k}\cdot\mathbf{x}}-1)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_i\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}} W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}}t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}) \psi^{(t_1)}_{j,\sigma_{t_1}} \right\| \le$$

$$\le \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\left|1-\widehat{k}\cdot\mathbf{v}_j\right|^{-1}}{|\mathbf{k}|^{\frac{3}{2}}\sqrt{2}} \left\| b(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}e^{-i\mathbf{k}\cdot\mathbf{x}}-1\right) W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}}t_1} W^b_{\sigma_{t_1}}(\nabla E_{\sigma_{t_1}}) \psi^{(t_1)}_{j,\sigma_{t_1}} \right\| d^3k$$

$$\Longleftrightarrow$$

2) The second step is the estimate of

$$\left\| b(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}e^{-i\mathbf{k}\cdot\mathbf{x}}-1\right) W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)} e^{-iE^{\sigma_{t_1}}t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}) \psi^{(t_1)}_{j,\sigma_{t_1}} \right\|.$$

$$\Longleftrightarrow$$

1) In distributional sense, the following equality holds:

$$b(\mathbf{k}) W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P})) = W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P}))\left(b(\mathbf{k})+f(\mathbf{k},\mathbf{v}_j,\mathbf{P})\right)$$

where $f(\mathbf{k},\mathbf{v}_j,\mathbf{P}) = \frac{g\chi^{\kappa_1}_{\sigma_{t_2}}(\mathbf{k})}{|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} - \frac{g\chi^{\kappa}_{\sigma_{t_2}}(\mathbf{k})}{|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{t_2}}(\mathbf{P})\right)}$ ($\chi^{\kappa}_{\sigma_{t_2}}$ characteristic function of the set $\{\mathbf{k}: \ \sigma_{t_2} \le |\mathbf{k}| \le \kappa\}$);

Being $|\mathbf{k}| \le \sigma_{t_1}$, we have $b(\mathbf{k}) W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P})) \psi^{(t_1)}_{j,\sigma_{t_1}} = 0$ and then

$$b(\mathbf{k}) W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)} e^{-iE^{\sigma_{t_1}}t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}) \psi^{(t_1)}_{j,\sigma_{t_1}} =$$

$$= f(\mathbf{k},\mathbf{v}_j,\mathbf{P}) W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)} e^{-iE^{\sigma_{t_1}}t_1} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}) \psi^{(t_1)}_{j,\sigma_{t_1}}.$$

The strong continuity in $\mathbf{k}$, $\sigma_{t_2} \le |\mathbf{k}| \le \sigma_{t_1}$, comes from the continuity of the function $f(\mathbf{k},\mathbf{v}_j,\mathbf{P})$.

2) Starting from the identity:

$$\begin{aligned}
& e^{i|\mathbf{k}|t_1} W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}+\mathbf{k}}\right) e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}_{\mathbf{P}+\mathbf{k}},t_1\right)} e^{-iE^{\sigma_{t_1}}_{\mathbf{P}+\mathbf{k}}t_1} f(\mathbf{k},\mathbf{v}_j,\mathbf{P}+\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P})) \psi^{(t_1)}_{j,\sigma_{t_1}} + \\
& - W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P})) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)} e^{-iE^{\sigma_{t_1}}_{\mathbf{P}}t_1} f(\mathbf{k},\mathbf{v}_j,\mathbf{P}) W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P})) \psi^{(t_1)}_{j,\sigma_{t_1}} = \\
& = e^{i|\mathbf{k}|t_1} W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}+\mathbf{k}}\right) e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}_{\mathbf{P}+\mathbf{k}},t_1\right)} e^{-iE^{\sigma_{t_1}}_{\mathbf{P}+\mathbf{k}}t_1} f(\mathbf{k},\mathbf{v}_j,\mathbf{P}+\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P})) \psi^{(t_1)}_{j,\sigma_{t_1}} + \\
& - W^b_{\sigma_{t_2}}(\mathbf{v}_j) W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P})) e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)} e^{-iE^{\sigma_{t_1}}(\mathbf{P})t_1} f(\mathbf{k},\mathbf{v}_j,\mathbf{P}) e^{-i\mathbf{k}\cdot\mathbf{x}} W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P})) \psi^{(t_1)}_{j,\sigma_{t_1}} +
\end{aligned} \tag{b6}$$

$$+W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P}))\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)}e^{-iE^{\sigma_{t_1}}(\mathbf{P})t_1}f(\mathbf{k},\mathbf{v}_j,\mathbf{P})\,e^{-i\mathbf{k}\cdot\mathbf{x}}W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi^{(t_1)}_{j,\sigma_{t_1}}+$$
$$-W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P}))\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)}e^{-iE^{\sigma_{t_1}}(\mathbf{P})t_1}f(\mathbf{k},\mathbf{v}_j,\mathbf{P})\,W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi^{(t_1)}_{j,\sigma_{t_1}} \tag{b7}$$

we exploit the known continuity properties with respect to $\mathbf{k}$ in order to control the differences (b6) and (b7).

Estimate of (b6)

Considering that:

1. $\left\|\left(W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P}+\mathbf{k}))-W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}}(\mathbf{P}))\right)f(\mathbf{k},\mathbf{v}_j,\mathbf{P}+\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}}W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi^{(t_1)}_{j,\sigma_{t_1}}\right\|\leq$ $\leq C\cdot(\ln(\sigma_{t_2})\cdot\ln(\sigma_{t_1}))^{\frac{1}{2}}\cdot t_1^{-\frac{3\epsilon}{2}}\cdot|\mathbf{k}|^{\frac{1}{16}}\cdot|\mathbf{k}|^{-\frac{3}{2}}$ (it is proved starting from an estimate analogous to (b4) in lemma B3 and from the fact that $|\mathbf{k}|\leq\sigma_{t_1}\ll 1$);
2. $\left|e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}+\mathbf{k}),t_1)}-e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1)}\right|\leq C\cdot|\mathbf{k}|^{\frac{1}{16}}\cdot t_1^{\frac{1}{20}}$, see lemma B2 ($\mathbf{P}\in\Gamma_j\subset\Sigma$);
3. $\left|e^{i|\mathbf{k}|t_1}-1\right|<2t_1\cdot|\mathbf{k}|$;
4. $|f(\mathbf{k},\mathbf{v}_j,\mathbf{P})-f(\mathbf{k},\mathbf{v}_j,\mathbf{P}+\mathbf{k})|\leq C\cdot\frac{|\mathbf{k}|^{\frac{1}{16}}}{|\mathbf{k}|^{\frac{3}{2}}}$ ($\mathbf{P}\in\Gamma_j\subset\Sigma$);
5. $\left|e^{-iE^{\sigma_{t_1}}(\mathbf{P}+\mathbf{k})t_1}-e^{-iE^{\sigma_{t_1}}(\mathbf{P})t_1}\right|\leq C\cdot|\mathbf{k}|^{\frac{1}{16}}\cdot t_1$, see lemma 3.3 ($\mathbf{P}\in\Gamma_j\subset\Sigma$);

for $|\mathbf{k}|\leq\sigma_{t_1}\ll 1$, one can conclude that (b6) is bounded by

$$C\cdot|\mathbf{k}|^{\frac{1}{16}}\cdot|\mathbf{k}|^{-\frac{3}{2}}\cdot t_1\cdot t_1^{-\frac{3\epsilon}{2}}\cdot|\ln\sigma_{t_2}|\,.$$

Estimate of (b7)

The norm of the expression (b7) is equal to

$$\left\|e^{-i\mathbf{k}\cdot\mathbf{x}}\int_{\Gamma_j}h(\mathbf{P})\,W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}}d^3P-\int_{\Gamma_j}h(\mathbf{P})\,W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}}d^3P\right\|.$$

As in lemma B3 (discussion of *term (b2)*) we observe that

$$e^{-i\mathbf{k}\cdot\mathbf{x}}W^b_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)\psi_{\mathbf{P},\sigma_{t_1}}\in H_{\mathbf{P}-\mathbf{k}}$$

and that the following equality holds in F:

$$I_{\mathbf{P}-\mathbf{k}}\left(e^{-i\mathbf{k}\cdot\mathbf{x}}W^b_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)\psi_{\mathbf{P},\sigma_{t_1}}\right)=I_{\mathbf{P}}\left(W^b_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)\psi_{\mathbf{P},\sigma_{t_1}}\right).$$

At this point we can easily handle the difference below

$$e^{-i\mathbf{k}\cdot\mathbf{x}}\int_{\Gamma_j} h(\mathbf{P})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}} d^3P - \int_{\Gamma_j} h(\mathbf{P})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}} d^3P =$$

$$= \int_{\Gamma_j} h(\mathbf{P})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\left\{W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}}\right\} d^3P - \int_{\Gamma_j} h(\mathbf{P})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}-\mathbf{k}))\,\psi_{\mathbf{P}-\mathbf{k},\sigma_{t_1}} d^3P + \quad \text{(I)}$$

$$+ \int_{\Gamma_j} h(\mathbf{P})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}-\mathbf{k}))\,\psi_{\mathbf{P}-\mathbf{k},\sigma_{t_1}} d^3P - \int_{\Gamma_j} h(\mathbf{P}-\mathbf{k})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}-\mathbf{k}))\,\psi_{\mathbf{P}-\mathbf{k},\sigma_{t_1}} d^3P + \quad \text{(II)}$$

$$+ \int_{\Gamma_j} h(\mathbf{P}-\mathbf{k})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}-\mathbf{k}))\,\psi_{\mathbf{P}-\mathbf{k},\sigma_{t_1}} d^3P - \int_{\Gamma_j} h(\mathbf{P})\, W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}} d^3P\,. \quad \text{(III)}$$

We deduce that

- term (I) can be estimated starting from

$$\left\| I_{\mathbf{P}-\mathbf{k}}\left(e^{-i\mathbf{k}\cdot\mathbf{x}}\left\{W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}))\,\psi_{\mathbf{P},\sigma_{t_1}}\right\}\right) - I_{\mathbf{P}-\mathbf{k}}\left(W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}}(\mathbf{P}-\mathbf{k}))\,\psi_{\mathbf{P}-\mathbf{k},\sigma_{t_1}}\right)\right\|_F$$

and then it is norm bounded by $C\cdot|\mathbf{k}|^{\frac{1}{16}}\cdot t_1^{-\frac{3\epsilon}{2}}$, for theorem 3.4 ($\mathbf{P}\in\Gamma_j\subset\Sigma$);

- being $G\in C^1_0\left(R^3\setminus 0\right)$, the norm of the term (II) is bounded by $C\cdot|\mathbf{k}|\cdot t_1^{-\frac{3\epsilon}{2}}$;

- estimating a volume difference, the norm of the term (III) is bounded by a quantity of order $|\mathbf{k}|^{\frac{1}{2}}\cdot t_1^{-\epsilon}$.

In conclusion, the norm of the expression (b7) is bounded by $C\cdot|\mathbf{k}|^{\frac{1}{16}}\cdot|\mathbf{k}|^{\frac{3}{2}}\cdot t_1^{-\epsilon}$ ($|\mathbf{k}|\ll 1$).

Then $\left\| g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{b(\mathbf{k})\left(e^{i|\mathbf{k}|t_1-i\mathbf{k}\cdot\mathbf{x}}-1\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_i\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\varphi_j\right\| \le C\cdot t_1\cdot t_1^{-\epsilon}\cdot|\ln\sigma_{t_2}|\cdot(\sigma_{t_1})^{\frac{1}{16}}$.

Analysis of

$$\left(\varphi_j, g^2\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{\left|e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2}d^3k\varphi_j\right)^{\frac{1}{2}}$$

By analogous steps as in the previous discussion, one obtains that

$$\left\|\left(e^{i|\mathbf{k}|t_1}e^{-i\mathbf{k}\cdot\mathbf{x}}-1\right)W^b_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{b\dagger}_{\sigma_{t_2}}(\nabla E^{\sigma_{t_2}})\,e^{i\gamma_{\sigma_{t_1}}(\mathbf{v}_j,\nabla E^{\sigma_{t_1}},t_1)}e^{-iE^{\sigma_{t_1}}t_1}W^b_{\sigma_{t_1}}(\nabla E^{\sigma_{t_1}})\,\psi^{(t_1)}_{j,\sigma_{t_1}}\right\|$$

is bounded by a quantity of order $|\mathbf{k}|^{\frac{1}{16}}\cdot t_1\cdot t_1^{-\epsilon}\cdot|\ln\sigma_{t_2}|$.
Now, note that $\left|e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2 = 2-e^{-i|\mathbf{k}|t_1}e^{i\mathbf{k}\cdot\mathbf{x}}-e^{i|\mathbf{k}|t_1}e^{-i\mathbf{k}\cdot\mathbf{x}}$. Since $|\mathbf{k}|\le\sigma_{t_1}$, the following inequality follows :

$$\left(\varphi_j, g^2\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{\left|e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2}d^3k\varphi_j\right)^{\frac{1}{2}} \le C\cdot t_1^{\frac{1}{2}}\cdot t_1^{-\frac{5\epsilon}{4}}\cdot(|\ln\sigma_{t_2}|)^{\frac{1}{2}}\cdot(\sigma_{t_1})^{\frac{1}{16}}\ .$$

References

[1] F.Bloch and A.Nordsieck. *Helv. Phys. Acta.* 45: 303 (1972)

[2] E. Nelson. *J. Math. Phys.* 5:1190-1197 (1964)

[3] J. Froehlich. *Ann. Inst. H. Poincaré*, Sect. A, XIX (1): 1-103 (1973).

[4] B.Schroer. *Fortschr. Physik* 11:1-31 (1963).

[5] Jauch, Rohrlich. Theory of photons and electrons. Addison-Wesley.

[6] D.R. Yennie, S.C. Frautschi and H.Suura. *Ann.Phys.* (N.Y.) 13:375 (1961).

[7] V. Chung. *Phys.Rev.* 140B:1110 (1965).

[8] P.P Kulish and L.D. Fadeev, *Theor. Math.Phys.* 4:153 (1970).

[9] T.W. Kibble. *Phys. Rev* 173:1527 (1968); 174:1882 (1968); 175 :1624 (1968).

[10] J.D. Dollard. *Jour. Math.Phys.* 5:729 (1964).

[11] D. Buchholz. *Comm. Math. Phys.* 52: 147 (1977).

[12] D. Buchholz. *Comm. Math. Phys.* 85: 49 (1982).

[13] F.Strocchi and A.S. Wightman. *J.Math.Phys.* 15: 2198 (1974).

[14] J.Froehlich, G.Morchio and F.Strocchi. *Ann.Phys* (N.Y) 119:241 (1979).

[15] J.Froehlich, G.Morchio and F.Strocchi. *Phys.Lett.* 89B:61 (1979).

[16] G.Morchio and F.Strocchi. *Ann. Phys.* 211: 471 (1983).

[17] V.Bach, J. Froehlich and I.M. Sigal. *Adv. Math.*, 137(2): 299-395 (1998).

[18] C.Gerard. preprint mp_arc 99-158 (1999).

[19] M.Griesemer, E.H. Lieb and M.Loss. preprint. Los Alamos Preprint Archive, math-ph/0007014 (2000).

[20] J.Lorinczi, A.Minlos and H.Spohn. Los Alamos Preprint Archive, math-ph/0011043 (2000).

[21] Arai. mp_arc 00-478 (2000).

[22] H. Spohn. *J.Math.Phys.* 38(5): 2281-2296 (1997).

[23] J. Derezinski and C. Gerard. *Rev. Math.Phys.* 11: 383-450 (1999)

[24] J.Froehlich, M.Griesemer and B.Sclein. Los Alamos Preprint Archive, math-ph/0009033 (2000).

[25] Gerard. mp_arc 01-103 (2001).

[26] J.Froehlich, M.Griesemer and B.Sclein. math-ph/0103048 (2001).

[27] M.Fierz and W.Pauli. *Nuovo Cimento* 15: 167 (1938).

[28] Ph. Blanchard. *Comm. Math.Phys.* 15: 156 (1969).

[29] R. Hoegh-Krohn. *J. Math.Phys.,* 11: 185-188 (1970)

[30] M.Reed, B.Simon. *Methods of modern mathematical physics: Volumes 1,2,3,4.* Academic Press, 1975.

[31] J.Froehlich. Fort. der Phys. 22: 158-198 (1974).

[32] Haag. Local Quantum Physics, Fields Particles, Algebras. Berlin: Springer Verlag (1992).

[33] J.Glimm and A.Jaffe. *Ann. Math.* 91: 362 (1970).

[34] K.Heep. *Comm. Math.Phys.* 1: 95 (1965-1966).

[35] V.Bach, J. Froehlich and I.M. Sigal. *Adv. Math.*, 137: 205-298 (1998).

[36] T.Chen. preprint mp_arc 01-310 (2001).

[37] S. Teufel and H.Spohn. mp_arc 00-396 (2000)