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Gregorio Falqui, and Cesare Reina

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$N=2$ super Riemann surfaces and algebraic geometry

Gregorio Falqui and Cesare Reina
S.I.S.S.A. (International School for Advanced Studies), Strada Costiera 11, 34014 Trieste, Italy

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The geometric framework for $N=2$ superconformal field theories are described by studying *susy*$_2$ *curves*—a nickname for $N=2$ super Riemann surfaces. It is proved that "single" susy$_2$ curves are actually split supermanifolds, and their local model is a Serre self-dual locally free sheaf of rank two over a smooth algebraic curve. Superconformal structures on these sheaves are then examined by setting up deformation theory as a first step in studying moduli problems.

I. INTRODUCTION

Supersymmetric extensions of algebraic curves have been recently studied in the physical and mathematical literature (see, e.g., Refs. 1 and 2). Among the physical motivations, a complete understanding of $N=1$ susy curves and their moduli spaces is needed in superstring theory in order to give meaning to computations in the Polyakov approach: Besides, they provide the natural arena for higher genus superconformal field theories.

From the mathematical point of view, superalgebraic curves are the simplest candidates for testing "new directions in geometry" in the spirit advocated by Manin.[3] Although studying "two supersymmetries" may seem the most obvious step beyond $N=1$, it is already a nontrivial matter, as noticed in some works.[4,5] In fact, for $N>1$ one is lead to consider locally free sheaves of rank greater than 1 on algebraic curves, a topic that is not completely under control as compared to the complete understanding of invertible sheaves. Luckily enough, for $N=2$, the superconformal structure to be imposed on such objects will bypass most of the subtleties related to moduli of vector bundles over variable curves: On the contrary, for $N\geqslant 3$, these enter the stage in a substantial way. Thus susy$_2$ curves are in a way the last "easy going" supersymmetric extension of algebraic curves, a fact that deserves special attention.

From the physical point of view, there is some "stringy" interest in the study of $N=2$ superconformal models; for instance, a recent work[6] has pointed out that space-time $N=1$ supersymmetry requires $N=2$ world-sheet supersymmetry. It is also widely believed that viewing $N=1$ supermoduli spaces as embedded in $N=2$ supermoduli spaces could be a keen standpoint for investigating the peculiarities of the former (provided that one has a good control of the latter).

The plan of this work is as follows. In Sec. II we investigate the geometry of susy$_2$ curves in connection with the theory of rank two locally free sheaves over an algebraic curve. Some nice features due to the existence of a superconformal structure, such as the splitness of "isolated" susy$_2$ curves, are proved. We also show that isolated susy$_2$ curves are the same thing as the datum of a Serre self-dual vector bundle on a curve and we classify such bundles completely. In Sec. III we set up a deformation theory and construct the local model for $N=2$ supermoduli spaces. Finally, Sec. IV is devoted to a detailed discussion of the global structure of the reduced moduli spaces of untwisted susy$_2$ curves.

II. SUSY$_2$ CURVES

This paper deals with susy$_2$ curves from the point of view of the theory of Kostant–Leites supermanifolds. In this framework, N supersymmetry is encoded in a $\mathbb{Z}_2$-graded extension $\mathscr{A}_X$ of the structure sheaf of a (complex) manifold such that $\mathscr{A}_X$ is locally isomorphic to the total wedge product of a rank N locally free analytic sheaf $\mathscr{E}$ [hereinafter called the *characteristic* sheaf of the supermanifold $(X, \mathscr{A}_X)$] over X (for a full definition, see, e.g., Ref. 2).

Recall that susy$_1$ curves are $1|1$-dimensional supercurves that come equipped with a distinguished distribution $\mathscr{D}$ in the tangent sheaf, spanned by the supersymmetry generator. In the same way the structure sheaf $\mathscr{A}_C$ of a susy$_2$ curve is quite special since it should embody the idea of the superconformal structure. In the physical literature this is realized in terms of coordinate transformations.[4,5] Here we give a definition that naturally extends that of susy$_1$ curves.[7,8]

Definition 1: A family of susy$_2$ curves $(C, \mathscr{A}_C)$ parametrized by the complex superspace $(S, \mathscr{A}_S)$—a susy$_2$ curve over S—is the datum of (i) a sheaf homomorphism $\pi^{\#}: \pi^{-1}\mathscr{A}_S \to \mathscr{A}_C$ of relative dimension $1|2$ over a proper surjective flat map $C \xrightarrow{\pi} S$ and (ii) a $0|2$-dimensional locally free distribution $\mathscr{D}_\pi$ in the relative tangent sheaf $\mathscr{T}_\pi$ such that the commutator mod $\mathscr{D}_\pi$,

$$\{,\}_{\mathscr{D}}: \mathscr{D}_\pi \otimes \mathscr{D}_\pi \to \mathscr{T}_\pi / \mathscr{D}_\pi,$$

is a symmetric nondegenerate bilinear map of sheaves of $\mathscr{A}_C$ modules.

In the following, a susy$_2$ curve over the trivial superspace $\{*\}$ will be called a single susy$_2$ curve. The connection between Definition 1 and the usual coordinate approach, as given, e.g., in Ref. 9, is a simple generalization of the $N=1$ case (see Refs. 7–10). Indeed, one can easily prove that there exist generators D^i for $\mathscr{D}_\pi$ and $\partial/\partial z$ for $\mathscr{T}_\pi/\mathscr{D}_\pi$ such that

$$\{D^i, D^j\}_{\mathscr{D}} = \delta^{ij}\frac{\partial}{\partial z}.$$

A simple computation then shows that $D^i = \partial/\partial\theta^i + \theta^i(\partial/\partial z)$.

Besides matching with physical applications, Definition 1 allows an immediate characterization of single susy$_2$ curves.

Proposition 1: Let $(C, \mathscr{A}_C)$ be a single susy$_2$ curve with reduced canonical sheaf ω. Then there exists a rank two lo-

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cally free sheaf $\mathscr{E}$ such that (i) $\mathscr{A}_C \simeq \Lambda\mathscr{E}$, i.e., $\mathscr{A}_C$ is split; and (ii) $\mathscr{E} \simeq \mathscr{E}^* \otimes \omega$, i.e., $\mathscr{E}$ is Serre self-dual.

Proof: Let $(U_\alpha, z_\alpha, \theta^i{}_\alpha)$ be a canonical atlas with transition functions

$$z_\alpha = f_{\alpha\beta}(z_\beta) + g_{\alpha\beta}\epsilon_{ij}\theta^i\theta^j,$$
$$\theta_\alpha{}^i = [m_{\alpha\beta}]^i_j \, \theta_\beta{}^j.$$

The existence of the distribution $\mathscr{D}_\pi$ is then equivalent to the superconformal condition

$$D^i{}_\beta z_\alpha = \theta^k{}_\alpha D^i{}_\beta \theta^k{}_\alpha$$

(sum over repeated latin indices), which gives

$$\epsilon_{ij} g_{\alpha\beta} + \delta_{ij} f'_{\alpha\beta} = [{}^t m_{\alpha\beta} m_{\alpha\beta}]_{ij},$$

where $f'_{\alpha\beta} = \partial f_{\alpha\beta}/\partial z_\beta$. Looking at the symmetric and antisymmetric parts of this equation we have (i) $g_{\alpha\beta} = 0$, so that $\mathscr{A}_C$ splits to $\Lambda\mathscr{E}$, where $\mathscr{E}$ is locally generated by the $\theta^i{}_\alpha$'s; and (ii) ${}^t m_{\alpha\beta}\, m_{\alpha\beta} = \mathbf{1}\cdot f'_{\alpha\beta}$, where $m_{\alpha\beta}$ are the transition functions of $\mathscr{E}$. Thus $m_{\alpha\beta} = {}^t m^{-1}_{\alpha\beta} f'_{\alpha\beta}$, i.e., $\mathscr{E} \simeq \mathscr{E}^* \otimes \omega$. ■

We want to remark at this point on the power of superconformal structures. Indeed, a generic supercurve of dimension $1|2$ is by no means split, as opposed to the trivial $1|1$ case. Nevertheless, susy$_2$ curves are split, a peculiarity that does not survive to higher supersymmetric extensions.

According to the physical literature, a susy$_2$ curve is called *twisted* whenever the $O(2)$ symmetry of the (anti) commutation relations for the local supersymmetry generators D^i_α cannot be reduced to an SO(2) symmetry.[4] This is related to the vanishing of a class in $H^2(C, \mathbf{Z}_2)$ obtained by taking the determinant of the transition functions for the locally free sheaf $\mathscr{E}$. Namely, since any rank two locally free sheaf can be represented as the extension of an invertible sheaf $\mathscr{L}_1$ by another $\mathscr{L}_2$ fitting the exact sequence $0 \to \mathscr{L}_2 \to \mathscr{E} \to \mathscr{L}_1 \to 0$, we have $\det \mathscr{E} = \mathscr{L}_1 \otimes \mathscr{L}_2$. However, Serre self-duality implies $\det \mathscr{E} = \omega \otimes \mathscr{N}_2$, where $\mathscr{N}_2$ is a point of order two on the Jacobian of C. Then a susy$_2$ curve is untwisted whenever $\mathscr{N}_2$ is trivial.

From the holomorphic point of view, Serre self-dual rank two locally free sheaves are quite simple objects.

Lemma 1: The characteristic sheaf $\mathscr{E}$ of untwisted susy$_2$ curves decomposes as the direct sum $\mathscr{E} = \mathscr{L}_1 \oplus \mathscr{L}_2$, with $\mathscr{L}_1 \otimes \mathscr{L}_2 \simeq \omega$.

Proof: In a superconformal gauge the transition functions $\mu_{\alpha\beta}(z_\beta)^i{}_j$ of $\mathscr{E}$ satisfy ${}^t m_{\alpha\beta} \cdot m_{\alpha\beta} = f'_{\alpha\beta} \cdot \mathbf{1}$ and hence can be given the form

$$m_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ -b_{\alpha\beta} & a_{\alpha\beta} \end{pmatrix},$$

with $a^2_{\alpha\beta} + b^2_{\alpha\beta} = f'_{\alpha\beta}$. A simple computation shows that there is a one-cochain λ_α with values in the sheaf of GL (2, $\mathbb{C}$)-valued holomorphic functions that diagonalizes $m_{\alpha\beta}$, showing that actually, $\mathscr{E} = \mathscr{L}_1 \oplus \mathscr{L}_2$. Imposing the Serre self-duality condition in this gauge gives

$$\mathscr{L}_1 \oplus \mathscr{L}_2 \simeq \mathscr{L}_1^{-1} \otimes \omega \oplus \mathscr{L}_2^{-1} \otimes \omega.$$

This completes the proof since $\mathscr{L}_1 \oplus \mathscr{L}_2 \simeq \mathscr{L}'_1 \oplus \mathscr{L}'_2$ if and only if either $\mathscr{L}_1 \simeq \mathscr{L}'_1$ or $\mathscr{L}_1 \simeq \mathscr{L}'_2$. ■

Proposition 2: Any twisted Serre self-dual locally free sheaf $\mathscr{E}$ of rank two on C is holomorphically isomorphic to the direct sum of two different θ characteristics, i.e., $\mathscr{E} = \mathscr{L}_1 \oplus \mathscr{L}_2$, with $\mathscr{L}_i^2 = \omega$.

Proof: Recall that a rank two locally free sheaf $\mathscr{E}$ is called semistable if for any invertible subsheaf $\mathscr{L} \subset \mathscr{E}$,

$$c_1(\mathscr{L}) \leqslant c_1(\det \mathscr{E})/2:$$

it is stable if the above inequality holds in the strict sense. A classical result of the theory of locally free sheaves over algebraic curves[11] states that a stable locally free sheaf cannot be decomposable (i.e., it cannot be isomorphic to the direct sum of two invertible subsheaves).

We first prove that a twisted semistable Serre self-dual locally free sheaf is strictly semistable, i.e., it admits only degree $g-1$ invertible subsheaves. Notice that if $\mathscr{E}$ is untwisted, by Lemma 1 it is not stable. If $\mathscr{E}$ is twisted, there is a point $\mathscr{M}$ of order 4 on the Jacobian of C such that $\mathscr{E} \otimes \mathscr{M}$ is untwisted in the sense that $\det(\mathscr{E} \otimes \mathscr{M}) = \omega$. Since $\mathscr{E} \otimes \mathscr{M}$ is stable if and only if $\mathscr{E}$ is stable, we are again in the above situation.

Second, an unstable Serre self-dual locally free sheaf is strictly semistable as well. In fact, suppose that $\mathscr{E}$ is given as $0 \to \mathscr{L}_1 \to \mathscr{E} \to \mathscr{L}_2 \to 0$, with $c_1(\mathscr{L}_1) \geqslant g-1$. Serre dualizing, we obtain $0 \to \mathscr{L}_2^\vee \to \mathscr{E} \to \mathscr{L}_1^\vee \to 0$. Then, supposing $c_1(\mathscr{L}_1) > g-1$, Lemma 15 of Ref. 12 shows that $\mathscr{L}_1 \simeq \mathscr{L}_2^\vee$ and hence as $\det \mathscr{E} = \mathscr{L}_1 \otimes \mathscr{L}_2 = \omega$ we obtain a contradiction with the assumption of the twisting of $\mathscr{E}$.

We have only to discuss the case $0 \to \mathscr{L}_1 \to \mathscr{E} \to \mathscr{L}_2 \to 0$, with $c_1(\mathscr{L}_i) = g-1$, $\mathscr{L}_1 \otimes \mathscr{L}_2 \neq \omega$. If $\mathscr{E}$ is not decomposable, again Lemma 15 of Ref. 12 tells us that there would be a unique invertible subsheaf $\mathscr{L} \subset \mathscr{E}$ of degree $g-1$, contradicting the assumption that $\mathscr{L}_1 \not\simeq \mathscr{L}_2^\vee$. Finally, given $\mathscr{E} \simeq \mathscr{L}_1 \oplus \mathscr{L}_2$ the Serre self-duality condition implies that $\mathscr{L}_i^2 \simeq \omega$. ■

In summary, we have that superconformal structures force the characteristic sheaf $\mathscr{E}$ to be, in the twisted case, the direct sum of two nonisomorphic square roots of the canonical bundle. The untwisted case has a richer structure since here $\mathscr{E}$ decomposes as $\mathscr{L} \oplus \omega \otimes \mathscr{L}^{-1}$, $\mathscr{L} \in \operatorname{Pic} C$. As pointed out in Ref. 13, this fact has interesting consequences both from the mathematical and physical standpoints. We simply notice that to ensure semistability of the sheaf $\mathscr{E}$ also in the untwisted sector, one has to be restricted to the case $\deg \mathscr{L} = g-1$. Here a convenient and physically reasonable parametrization of $\mathscr{E}$ is $\mathscr{E} = \mathscr{L} \otimes \mathscr{N} \oplus \omega \otimes (\mathscr{L} \otimes \mathscr{N})^{-1}$, with $\mathscr{N} \in \operatorname{Pic}_0(C)$ and $\mathscr{L}$ a θ characteristic on C.

Actually, this is not the whole story since for susy$_2$ curves the above analysis is somewhat blind. Indeed, we have to work in a finer category than the holomorphic one because two susy$_2$ curves may very well be holomorphically equivalent, but by no means superconformally equivalent. This finer classification is entirely an outspring of physics and we wish to uncover it in full detail by studying deformation theory of susy$_2$ curves.

III. DEFORMATIONS OF SUSY$_2$ CURVES

The first step in studying moduli space of algebraic objects is to find their local structure, as given by the base spaces of versal deformations.

Definition 2: A deformation of a susy$_2$ curve C over a pointed superspace $(B, \{*\})$ is a family $\mathscr{C} \xrightarrow{\pi} B$ of susy$_2$ curves together with an isomorphism ψ of C with the "central fiber" $\pi^{-1}(\{*\})$ fitting the commutative diagram

$$\begin{array}{ccc} C & \hookrightarrow & \mathscr{C} \\ \downarrow & & \downarrow \pi \\ \{*\} & \hookrightarrow & B \end{array}.$$

As usual, the starting point for setting up a deformation theory is to identify the sheaf of infinitesimal automorphisms of the object to be deformed. In our case this is the subsheaf $T_\pi^{\mathscr{D}}$ of the relative tangent sheaf whose elements are germs of vector fields along the fibers which preserve $\mathscr{D}$:

$$T_\pi^{\mathscr{D}} := \{X \in T_\pi \mid [D, X] \in \mathscr{D}\ \forall D \in \mathscr{D}\}.$$

In perfect analogy with the case of $N = 1$ susy curves we find the following lemma.

Lemma 2: There is an isomorphism $T_\pi^{\mathscr{D}} \simeq (T_\pi)_{\rm red} \otimes \mathscr{A}_C$ as sheaves of $\pi^{-1}(\mathscr{A}_B)$ modules.

Proof: The condition for X to belong to $T_\pi^{\mathscr{D}}$ reads as $[D^i, X] \in \mathscr{D}$, where D^i are generators of $\mathscr{D}$. Introducing the canonical coordinates (z, θ^i), so that $D^i = \partial/\partial\theta^i + \theta^i(\partial/\partial z)$, and setting $X = a \cdot \partial/\partial z + b_i \cdot D^i$, one has

$$[D^i, X] = D^i a \frac{\partial}{\partial z} - (-1)^{|X|} b_k \delta^{ki} \frac{\partial}{\partial z} + D^i b_k \cdot D^k.$$

Therefore, $X \in T_\pi^{\mathscr{D}}$ if and only if $b_i = (-1)^{|a|} D^i a$ and the isomorphism is given by $a \cdot \partial/\partial z \rightsquigarrow a(\partial/\partial z) + (-1)^{|a|} D^i a \cdot D^i$. ■

Thanks to this lemma we have, for $\mathscr{E}$ semistable, the following proposition.

Proposition 3: Versal deformations of susy$_2$ curves exist. The dimension of the base of such deformations is $3g - 3 + g - a|4g - 4$, with $a = 0$, 1 in the untwisted and twisted cases, respectively.

Proof: From the Kodaira–Spencer deformation theory, we know that possible obstructions lie in the second cohomology group of the sheaf of infinitesimal automorphisms $T_\pi^{\mathscr{D}}$. By Lemma 2 one obtains

$$T_\pi^{\mathscr{D}} = \omega^{-1} \oplus \Pi(\omega^{-1} \otimes \mathscr{E}) \oplus \det \mathscr{E} \otimes \omega^{-1},$$

where the above sum is the direct sum of sheaves of $\mathscr{O}_C$ modules. (The parity change operator Π has, strictly speaking, no effective meaning in this context; we just use it as a parity bookkeeper.) Then $H^2(T_\pi^{\mathscr{D}}) = \{0\}$, showing the existence of versal deformations. The second part of Proposition 3 follows from Serre self-duality of $\mathscr{E}$ and Proposition 2. Indeed, $\dim H^1(\omega^{-1} \oplus \det \mathscr{E} \otimes \omega^{-1}) = \dim H^1(\omega^{-1} \oplus \mathscr{N})$, where $\mathscr{N} = \det \mathscr{E} \otimes \omega^{-1} = \mathscr{O}$ for untwisted susy$_2$ curves, while it is a point of order 2 in the Jacobian of C in the twisted case. As for the odd dimension, notice that $\dim H^1(\omega^{-1} \otimes (\mathscr{L}_1 \oplus \mathscr{L}_2)) = \dim H^1(\mathscr{L}_1^{-1}) + \dim H^1(\mathscr{L}_2^{-1})$. ■

Remark: As for the computation of the odd dimension q of the would-be moduli space of susy$_2$ curves in the general untwisted case, one can argue as follows. Since $\mathscr{E} \simeq \mathscr{L} \oplus \omega \otimes \mathscr{L}^{-1}$, $H^1(C, \mathscr{E})$ is invariant under the Kummer map $\mathscr{L} \rightsquigarrow \omega \otimes \mathscr{L}^{-1}$. Hence one can be restricted to discussing the case $\deg \mathscr{L} \equiv d \geqslant g - 1$ only. By the Riemann–Roch theorem one has (i) if $g - 1 \leqslant d < 2g - 2$, then $q = 4g - 4$; (ii) if $2g - 2 \leqslant d \leqslant 3g - 3$ *and $\mathscr{L}$ is generic*, then $q = 4g - 4$; (iii) if $3g - 3 \leqslant d \leqslant 4g - 4$ *and $\mathscr{L}$ is generic*, then $q = d + g - 1$; and (iv) if $4g - 4 < d$, then $q = d + g - 1$. Notice that in cases (ii) and (iii) the odd dimension of "moduli space" jumps on analytic submanifolds of the reduced space, a fact that renders its structure quite subtle in the framework of Kostant–Leites supermanifold theory.

From a more computative point of view, one can consider infinitesimal deformations, i.e., deformations over the superspace $\hat{S} = (\{*\}; \mathbb{C}(t, \eta)/(t^2, t\zeta))$, as being given by deforming the clutching functions of the central fiber. From Sec. II, we learn that these are of the form

$$z_\alpha = f_{\alpha\beta}(z_\beta),$$
$$\theta_\beta^i = [m_{\alpha\beta}(z_\beta)]_j^i \theta_\beta^j,$$

where the matrix $\mu_{\alpha\beta}(z_\beta)^i{}_j$ is of the form

$$\mu_{\alpha\beta}(z_\beta)^i_j = \begin{pmatrix} g_{1\alpha\beta} & 0 \\ 0 & g_{2\alpha\beta} \end{pmatrix},$$

with either $g_{i_{\alpha\beta}}^2 = f'_{\alpha\beta}$ or $g_{1_{\alpha\beta}} \cdot g_{2_{\alpha\beta}} = f'_{\alpha\beta}$.

The most general deformation of such clutching functions, i.e., those generated by a vector field in the whole $\mathscr{T}_\pi$, over $\hat{S}$ is given by

$$z_\alpha = f_{\alpha\beta}(z_\beta) + t(b_{\alpha\beta}(z_\beta) + \tfrac{1}{2} g_{\alpha\beta}(z_\beta) \epsilon_{ij} \theta_\beta^i \theta_\beta^j) + \zeta \eta_{i_{\alpha\beta}}(z_\beta) \theta_\beta^i,$$
$$\theta_\beta^i = [m_{\alpha\beta}(z_\beta)]_j^i \theta_\beta^j + t[l_{\alpha\beta}(z_\beta)]_j^i \theta_\beta^j + \zeta(\psi_{\alpha\beta}^i(z_\beta) + \tfrac{1}{2} \nu_{\alpha\beta}^i \epsilon_{jk} \theta_\beta^j \theta_\beta^k).$$

Imposing the superconformal condition shows that $g_{\alpha\beta}(z_\beta) = 0$ (this fact can be also grasped by writing explicitly the superconformal vector fields which generate the deformations) and the only independent data are $b_{\alpha\beta}(z_\beta)$, $\psi_{\alpha\beta}^i(z_\beta)$, and $[l_{\alpha\beta}(z_{\alpha\beta})]_j^i$. The cocycle condition leads easily to the identification of $\{b_{\alpha\beta}(z_\beta) \cdot \partial/\partial z_\alpha\}$ as a one-cocycle with values in the relative tangent sheaf ω_π^{-1} and $\{\psi_{\alpha\beta}^i(z_\beta)\}$ as a one-cycle with values in $\mathscr{E}^*$.

As for the role of the matrix $[l_{\alpha\beta}(z_\beta)]_j^i$, one can argue as follows. Since the even and odd infinitesimal deformations give decoupled equations, one can be limited to discussing a deformation of the form

$$z_\alpha = f_{\alpha\beta}(z_\beta) + t b_{\alpha\beta}(z_\beta),$$
$$\theta_\beta^i = [m_{\alpha\beta}(z_\beta)]_j^i \cdot \{\delta_k^j + t[m^{-1} \cdot l_{\alpha\beta}(z_\beta)]_k^j\} \cdot \theta_\beta^k.$$

The superconformal condition translates into $O_{\alpha\beta} + {}^t O_{\alpha\beta} = (1/f'_{\alpha\beta})(\partial b_{\alpha\beta}/\partial z_\beta) \cdot \mathbf{1}$ for the matrix $O_{\alpha\beta} \equiv m_{\alpha\beta}^{-1} \cdot l_{\alpha\beta}$. Hence

$$O_{\alpha\beta} = \begin{pmatrix} \tau_{\alpha\beta} & \alpha_{\alpha\beta} \\ -\alpha_{\alpha\beta} & \tau_{\alpha\beta} \end{pmatrix}$$

and its only free part is the off diagonal

$$\tilde{O}_{\alpha\beta} = \begin{pmatrix} 0 & \alpha_{\alpha\beta} \\ -\alpha_{\alpha\beta} & 0 \end{pmatrix}.$$

This decomposition is obviously due to the fact that when deforming the underlying curve C according to $b_{\alpha\beta}$, line bundles on C are deformed as well. The cocyle condition for $O_{\alpha\beta}$ gives

$$m_{\alpha\beta}O_{\alpha\beta}m_{\beta\gamma} + m_{\alpha\beta}m_{\beta\gamma}O_{\beta\gamma} = m_{\alpha\gamma}O_{\alpha\gamma} - b_{\alpha\beta}\, m'_{\alpha\beta}\, m_{\beta\gamma}.$$

Looking once again at the off-diagonal part of the above equation one has (multiplying on the left by $m_{\alpha\gamma}^{-1}$)

$$m_{\beta\gamma}^{-1}\tilde{O}_{\alpha\beta}m_{\beta\gamma} + \tilde{O}_{\beta\gamma} = \tilde{O}_{\alpha\gamma}.$$

A simple algebra shows that $\tilde{O}_{\alpha\beta}m_{\beta\gamma} = f'_{\beta\gamma}/\det m_{\beta\gamma}\cdot\tilde{O}_{\alpha\beta}$, yielding $(f'_{\beta\gamma}/\det m_{\beta\gamma})\tilde{O}_{\alpha\beta} + O_{\beta\gamma} = \tilde{O}_{\alpha\gamma}$. Then considering local generators $\{\varphi_\alpha\}$ of $\omega^{-1}\otimes\det\mathscr{E}$ one readily identifies the collection $\{\alpha_{\alpha\beta}\cdot\varphi_\beta\}$ as a one-cocycle with values in $\omega^{-1}\otimes\det\mathscr{E}$.

In summary, a complete infinitesimal deformation of a susy$_2$ curve consists of a deformation of the underlying algebraic curve and the couple of line bundles that define the "single" object plus the deformation specified by $l_{\alpha\beta}$. Since the latter is completely qualified by an element in $H^1(C,\omega^{-1}\otimes\det\mathscr{E})$ we find complete agreement with the results of Proposition 3. As a final remark, we notice that this latter space, which can also be thought of as the space of superconformally nonequivalent susy$_2$ structures on a fixed curve, coincides with the space of holomorphically nonequivalent extensions of a θ characteristic $\mathscr{L}_1$ by another one $\mathscr{L}_2$.

IV. THE REDUCED MODULI SPACE OF UNTWISTED SUSY$_2$ CURVES

We can now give a detailed description of the reduced moduli spaces of susy$_2$ curves, which turns out to be complete in the untwisted case. According to Proposition 2, holomorphic isomorphism classes of twisted susy$_2$ curves are in one-to-one correspondence with isomorphism classes of couples $(C,\mathscr{L}_{12})$, where $\mathscr{L}_{12}$ is an unordered couple of nonequivalent θ characteristics. This sits inside the second symmetric power $\Sigma^{(2)}$ of the spin covering $\Sigma\to\mathscr{M}$ of the moduli space (at some fixed genus), a space that has a nice mathematical status.[14] Besides, according to Lemma 1, the reduced moduli space of untwisted susy$_2$ curves with a semistable characteristic sheaf $\mathscr{E}$ can be identified with the universal degree $g-1$ Picard variety $\mathrm{Pic}_{g-1}\to\mathscr{M}_g$ over the moduli variety of genus g algebraic curves, modulo the Kummer map $\mathscr{L}\rightsquigarrow\omega\otimes\mathscr{L}^{-1}$.

We next want to parametrize superconformal structures on a fixed curve C and a couple $\mathscr{L}_1\oplus\mathscr{L}_2$ of invertible sheaves fitting a Serre self-dual rank two locally free sheaf. The basic observation here is that the sheaf $\mathscr{E}$ should be regarded not merely as a holomorphic sheaf because the superconformal structure amounts to saying that it is the sheaf of sections of a vector bundle E with, as its structure group, the conformal group

$$G=\{m\in\mathrm{GL}(2,\mathbb{C})\,|\,{}^t m\cdot m=\lambda\,\mathbf{1}\}\equiv G_0\cup\eta\cdot G_0,$$

where G_0 is the identity component and

$$\eta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The map $\varphi:G\to\mathbb{C}^*$ given by $\varphi(m)={}^t m\cdot m$ gives rise to the exact diagram of complex groups:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & SO(2) & \to & G_0 & \xrightarrow{\varphi} & \mathbb{C}^* & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & O(2) & \to & G & \xrightarrow{\varphi} & \mathbb{C}^* & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2 & \to & 1 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \quad (1)$$

Notice that the first row is an exact sequence of central subgroups of the groups in the second row. This is vital at the level of exact sequences of sheaves of germs of group-valued functions associated to the above diagram. Indeed, pushing the induced cohomology sequences as far as possible, we obtain an exact diagram:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H^1(C,\mathscr{S}\mathscr{O}_2) & \to & H^1(C,\mathscr{G}_0) & \to & H^1(C,\mathscr{O}^*) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H^1(C,\mathscr{O}_2) & \to & H^1(C,\mathscr{G}) & \to & H^1(C,\mathscr{O}^*) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H^1(C,\mathbb{Z}_2) & \to & H^1(C,\mathbb{Z}_2) & \to & 1 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \quad . \quad (2)$$

Here we used some results of non-Abelian sheaf cohomology and the following lemma.

Lemma 3: The cohomology groups $H^*(C,\mathscr{O}^*)$ and $H^*(C,\mathscr{S}\mathscr{O}_2)$ coincide.

Proof: The exact sheaf sequence

$$0\to\mathbb{Z}\to\mathscr{O}\xrightarrow{m}\mathscr{S}\mathscr{O}_2\to 1,$$

where the map m is defined by

$$m(f)=\begin{pmatrix}\cos(2\pi i f) & \sin(2\pi i f)\\ -\sin(2\pi i f) & \cos(2\pi i f)\end{pmatrix},\quad \forall f\in\Gamma(U,\mathscr{O}),$$

fits together with the standard exponential sequence into the commutative diagram of sheaves (of Abelian groups)

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathcal{O} & \xrightarrow{\exp} & \mathcal{O}^* & \to & 1 \\ & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow m^* & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O} & \xrightarrow{m} & \mathcal{S}\mathcal{O}_2 & \to & 1 \end{array}, \quad (3)$$

where

$$m^*(\psi) = \begin{pmatrix} (\psi+\psi^{-1})/2 & (\psi-\psi^{-1})/2i \\ -(\psi-\psi^{-1})/2i & (\psi+\psi^{-1})/2 \end{pmatrix},$$
$$\forall \psi \in \Gamma(U,\mathcal{O}^*).$$

This gives rise to a long commutative sequence of cohomology groups, proving Lemma 3. ■

Remark: Notice that the above sequence shows that the cohomology group $H^1(C,\mathcal{S}\mathcal{O}_2)$ is isomorphic to the group Pic C of invertible sheaves on C.

The basic fact for our concern is the following lemma.

Lemma 4: The action of $H_1(C,\mathcal{S}\mathcal{O}_2)$ is transitive and free on the fiber of the bundle $H^1(C,\mathcal{G}_0) \to H^1(C,\mathcal{O}^*)$ over each class $\tau \in H^1(C,\mathcal{O}^*)$. The same is true for the action of $H^1(C,\mathcal{G}_0)$ on the fiber of $H^1(C,\mathcal{G}) \to H^1(C,\mathbb{Z}_2)$ over $\tau' \in H^1(C,\mathbb{Z}_2)$.

Proof: Since $\mathcal{S}\mathcal{O}_2 \hookrightarrow \mathcal{G}_0$ and $\mathcal{G}_0 \hookrightarrow \mathcal{G}$ are central and Abelian, we can apply a (simplified) argument of non-Abelian sheaf cohomology (see, e.g., Lemma 2.4 of Ref. 10) to obtain the proof. This runs as follows. Given an exact sequence of sheaves of groups $0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{R} \to 0$ in which $\mathcal{P}$ is central and Abelian and $\mathcal{R}$ is Abelian, one has the following results.

(i) There is a connecting map $H^1(\mathcal{R}) \xrightarrow{\delta_1} H^2(\mathcal{P})$, so that the sequence

$$H^1(\mathcal{Q}) \to H^1(\mathcal{R}) \xrightarrow{\delta_1} H^2(\mathcal{P})$$

is exact.

(ii) Whenever $\tau \in \mathrm{Ker}\,\delta_1$, $H^1(\mathcal{P})$ acts transitively on the fiber of $H^1(\mathcal{Q})$ over τ, with the kernel given by the image of $H^0(\mathcal{R}) \xrightarrow{\delta_0} H^1(\mathcal{Q})$. In our case Lemma 4 follows from the fact that $H^2(C,\mathcal{S}\mathcal{O}_2) \simeq H^2(C,\mathcal{O}^*) = \{0\}$ and the observation that elements in $H^0(C,\mathbb{Z}_2)$ $((H^0(C,\mathcal{O}^*))$ are mapped into locally constant matrices by the connecting homomorphisms δ_0 and thus are clearly trivial cocycles. ■

Using Lemma 4, we can give the following description of the (reduced) moduli space of untwisted susy_2 structures over a fixed curve C.

Proposition 4: Nonequivalent untwisted susy_2 structures on a fixed (smooth) algebraic curve C are parametrized by the fiber of $H^1(C,\mathcal{S}\mathcal{O}_2)$ in $H^1(C,\mathcal{G}_0)$ over $[\omega] \in H^1(C,\mathcal{O}^*)$.

Proof: This proof follows at once by noticing that the map $H^1(C,\mathcal{G}) \to H^1(C,\mathcal{O}^*)$ in diagram (2) is surjective and the map $H^1(C,\mathcal{O}_2) \to H^1(C,\mathcal{G})$ is injective. The last assertion follows by applying Lemma 4. ■

Remark: While in the general theory of supermanifolds the first infinitesimal neighborhood of M_{red} is an "ordinary" vector bundle and thus its vertical automorphisms are automorphisms of the supermanifold structure, when considering supermanifolds with contact structure, which are the most relevant to physics (see, e.g., Ref. 2), extra structures must be taken into account. Thus the classification of $N=2$ superconformal structures over an algebraic curve C is quite different from the classification of rank two vector bundles over C, which corresponds, as is well known, to the classification of all *split* supermanifolds of odd dimension 2 over C.

V. CONCLUSIONS AND OUTLOOK

In this paper we have reconsidered some features of the geometry of $N=2$ superconformal field theories in a proper geometric framework. We showed that most of the definitions and properties of $N=1$ super Riemann surfaces carry over, with obvious modifications, to the $N=2$ case. In particular, we pointed out the relations of the theory of susy_2 curves with the theory of Serre self-dual rank two locally free sheaves over algebraic curves.

This approach gives a full proof of the results that are usually obtained in the physical literature by studying degrees of freedom and "gauge invariance" of the $N=2$ supersymmetric action in two dimensions (see, e.g., Ref. 15). Namely, some pecularities of $N>1$ supersymmetry, such as the existence of modular parameters for the $U(1)$ current mixing the gravitinos, have been given a sound geometrical picture. In addition, a complete description of the reduced moduli space of susy_2 curves in the untwisted sector was given.

A detailed study of the global aspects of $N=2$ supermoduli spaces, together with a setup of the theory of superconformal fields on susy_2 curves along the lines of Refs. 16 and 17, will be the subject of future work.

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