Analytic geometry of semisimple coalescent Frobenius structures

ANALYTIC GEOMETRY OF SEMISIMPLE COALESCENT FROBENIUS STRUCTURES

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Abstract. We present some results of a joint paper with B. Dubrovin (see references), as exposed at the Workshop "Asymptotic and Computational Aspects of Complex Differential Equations" at the CRM in Pisa, in February 2017. The analytical description of semisimple Frobenius manifolds is extended at semisimple coalescence points, namely points with some coalescing canonical coordinates although the corresponding Frobenius algebra is semisimple. After summarizing and revisiting the theory of the monodromy local invariants of semisimple Frobenius manifolds, as introduced by B. Dubrovin, it is shown how the definition of monodromy data can be extended also at semisimple coalescence points. Furthermore, a local Isomonodromy Theorem at semisimple coalescence points is presented. Some examples of computation are taken from the quantum cohomologies of complex Grassmannians.

Contents

1. Introduction

In these proceedings, we present some of the results from our papers [CDG17a, CDG17b], as presented in occasion of the Workshop "Asymptotic and Computational Aspects of Complex Differential Equations" at the CRM in Pisa, in February 2017.

Frobenius manifolds, introduced by B. Dubrovin in [Dub92] in the context of topological field theories, are complex manifolds endowed with a very rich structure, a flat pseudo-metric and a compatible multiplication on the holomorphic tangent bundle. They play a prominent role in many areas of contemporary Geometry and Mathematical Physics, usually providing unexpected and sometimes still conjectural links between theories apparently of very different nature, e.g. singularity theory, integrable systems, symplectic geometry, derived geometry and others.

The local geometry of a Frobenius manifold at its generic points (the *semisimple* ones, see Section 2.2) has been shown to be equivalent to isomonodromic deformations of a certain system of differential equations on the complex domain (see [Dub96, Dub98, Dub99b], and also Section 2). In particular, a procedure of classification for germs of semisimple Frobenius manifolds has been developed by introducing some *monodromy local moduli* (see Sections 2.4, 2.5), from whose knowledge the whole structure can be reconstructed ([Dub96, Dub98, Dub99b], [Guz01]).

In this paper, we outline how such a local description of the Frobenius structure can be extended also at some *non-generic points*, lying in a stratum of codimension one in the semisimple part of the Frobenius manifold. The non-genericity of these points is due to the presence of some coalescences in a system of local coordinates $(u_i)_{i=1}^n$, called *canonical* (Section 2.2).

From the point of view of the theory of isomonodromic deformations, the problem which we addressed is *de facto* equivalent to the study of solutions and deformations of systems admitting an irregular singularity at $z=\infty$, of the type

$$\frac{dY}{dz}=\left(A_0+\frac{A_1}{z}\right)Y,$$

where the matrix A_0 is supposed to be semisimple (i.e. diagonalizable), but *not necessarily with simple spectrum*, and A_1 is an antisymmetric matrix. Thus, the assumption that A_0 has simple spectrum, a fundamental one in the classical analytical theory of isomonodromic deformations, does not hold (see [JMU81, JM81a, JM81b], [FIKN06] and references therein). A general analytical theory for such systems has been developed in [CDG17a]. In the general case, because of the assumptions of non-genericity on the spectrum of A_0 and without requiring further conditions on A_1, several divergence issues (w.r.t. the deformation parameters) of both asymptotical and genuine solutions of the system must be faced (see Section 2.5.2). Remarkably, the geometric case associated to Frobenius structure completely satisfy all sharp assumptions found in [CDG17a], and in such a case an isomonodromic theory can be extended also in cases of coalescences (see Theorems 2.6, 4.1).

We underline the fact that such an extension of the local description of Frobenius manifolds is necessary for at least two reasons:

- as examples coming from singularity theory show ([CDG17b]), it may happen that, though a Frobenius structure is globally and explicitely known, the computation of the local monodromy moduli (defined in Sections 2.4, 2.5) is unfeasible – or extremely difficult – at a generic semisimple point where $u_i \neq u_j$ for $i \neq j$. The system simplifies at semisimple points of coalescence, so that we can explicitely solve it and compute the monodromy data.
- It may happen that a Frobenius structure is explicitly known *only at semisimple points of coalescence*. Hence, monodromy data can be computed only at these points. Our results justify the extension of the data so computed to the whole manifold. This is the case of Grassmannians (see Section 5).

1.1. **Plan of the paper.** In Section 2 we summarize B. Dubrovin's general theory of local monodromy moduli for semisimple Frobenius manifolds as developed in [Dub96, Dub98, Dub99b]. Some minor mistakes of B. Dubrovin's description of monodromy data are discussed and corrected, namely for what concerns the ambiguities and freedom up to which they are defined (Theorems 2.3, 2.4). A decomposition of a semisimple Frobenius manifold in *chambers* is introduced (Section 2.5.1), on which the classical B. Dubrovin's Isomonodromy Theorem holds (Theorem 2.8). We extend the definition of the monodromy data also at semimsimple points of coalescence (see Theorem 2.6). In Section 3, we briefly summarize up to which freedom the monodromy data of a semisimple Frobenius manifold are defined, by describing the action of several groups on them (some of which are implicitly described in [Dub96, Dub99b] with some minor mistakes). The notions of *lexicographical* and more general *triangular* orders of canonical coordinates are discussed in details, as well as the action of the braid group on the monodromy data. In Section 4 the results of [CDG17a] are applied in order to deduce the main result of Isomonodromicity at coalescence points (Theorem 4.1). In Section 5 an application to the study of quantum cohomology of complex Grassmannians is discussed, namely for a conjectural link between the enumerative geometry of Fano manifolds with their derived category of coherent sheaves (*Dubrovin's conjecture*, [Dub98, Dub13, CDG17b, CDG]).

2. Analytic geometry of Frobenius manifolds

2.1. **Frobenius manifolds and main examples.** The following main notion was introduced and extensively developed by B. Dubrovin in [Dub92, Dub96, Dub98, Dub99b]. The original motivation for its study was a differential geometric approach of axiomatization of the work of R. Dijkgraaf, E. Verlinde and H. Verlinde in the context of topological strings and two dimensional quantum gravity (see [DVV91]).

Definition 2.1. A *Frobenius manifold* structure on a complex manifold M of dimension n is defined by giving

(FM1) a symmetric $\mathcal{O}(M)$-bilinear metric tensor $\eta \in \Gamma\left(\bigodot^2 T^*M\right)$, whose corresponding Levi-Civita connection ∇ is flat;

(FM2) a $(1,2)$-tensor $c \in \Gamma\left(TM \otimes \bigodot^2 T^*M\right)$ such that
- the induced multiplication of vector fields $X \circ Y := c(-, X, Y)$, for $X, Y \in \Gamma(TM)$, is *associative*,
- $c^\flat \in \Gamma\left(\bigodot^3 T^*M\right)$,
- $\nabla c^\flat \in \Gamma\left(\bigodot^4 T^*M\right)$;

(FM3) a vector field $e \in \Gamma(TM)$, called the *unity vector field*, such that
- the bundle morphism $c(-, e, -)\colon TM \to TM$ is the identity morphism,
- $\nabla e = 0$;

(FM4) a vector field $E \in \Gamma(TM)$, called the *Euler vector field*, such that
- $\mathfrak{L}_E c = c$,
- $\mathfrak{L}_E \eta = (2-d)\cdot\eta$, where $d \in \mathbb{C}$ is called the *charge* of the Frobenius manifold.

From these axioms it directly follows that for any point $p \in M$ the triple $(T_pM, \eta_p, \circ_p)$ is a Frobenius algebra, namely an associative commutative algebra with unity whose product is compatible with the metric, in the sense that

$$\eta_p(a \circ_p b, c) = \eta_p(a, b \circ_p c), \quad \text{for all } a, b, c \in T_pM. \tag{2.1}$$

Remark 2.1. Because of flatness and the conformal Killing condition, the Euler vector field is affine, i.e.

$$\nabla\nabla E = 0.$$

Hence, by introducing ∇-flat coordinates $(t^\alpha)_{\alpha=1}^n$ on M, w.r.t. which the metric η is constant and the connection ∇ coincides with partial derivatives, we have that

$$E = \sum_{\alpha=1}^{n} \left((1-q_\alpha)t^\alpha + r_\alpha\right)\frac{\partial}{\partial t^\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}.$$

Following [Dub96, Dub98, Dub99b], we choose flat coordinates so that $\frac{\partial}{\partial t^1} \equiv e$ and $r_\alpha \neq 0$ only if $q_\alpha = 1$ (this can always be done, up to an affine change of coordinates). Let $\eta_{\alpha\beta} = \eta(\partial_\alpha, \partial_\beta)$, and $c^\gamma_{\alpha\beta} = c(dt^\gamma, \partial_\alpha, \partial_\beta)$, so that $\partial_\alpha \circ \partial_\beta = c^\gamma_{\alpha\beta}\partial_\gamma$. Condition (FM2) means that $c_{\alpha\beta\gamma} := \eta_{\alpha\rho}c^\rho_{\beta\gamma}$ and $\partial_\alpha c_{\beta\gamma\delta}$ are symmetric in all indices. This implies the local existence of a function F such that

$$c_{\alpha\beta\gamma} = \partial_\alpha\partial_\beta\partial_\gamma F.$$

The associativity of the algebra is equivalent to the following conditions for F, called WDVV-equations:

$$\partial_\alpha\partial_\beta\partial_\gamma F\ \eta^{\gamma\delta}\partial_\delta\partial_\epsilon\partial_\nu F = \partial_\nu\partial_\beta\partial_\gamma F\ \eta^{\gamma\delta}\partial_\delta\partial_\epsilon\partial_\alpha F,$$

while axiom (FM4) is equivalent to

$$\eta_{\alpha\beta} = \partial_1\partial_\alpha\partial_\beta F, \quad \mathfrak{L}_E F = (3-d)F + Q(t),$$

with $Q(t)$ a quadratic expression in t_α's. Conversely, given a solution of the WDVV equations, satisfying the quasi-homogeneity conditions above, a structure of Frobenius manifold is naturally defined on an open subset of the space of parameters t^α's.

2.1.1. *Examples of Frobenius manifolds.* There are two main classes of non-trivial Frobenius manifolds: the first are Frobenius structures arising from Singularity Theory, while the second ones from the Gromov-Witten Theory of Kähler (or symplectic, more in general) manifolds. In physicists' terminology, these structures correspond to the study of Topological Landau-Ginzburg models, and Topological non-linear σ-models of A-type respectively (see [CK99] and references therein). The isomorphy of Frobenius manifolds of two different classes can be interpreted as one version of the *Mirror Symmetry* (see [BK98], [Bar99], [Dub99b], [Man99], [Giv96, Giv97, Giv98, Giv01], [Voi96]).

Remarkably, Frobenius manifolds structures defined on the base space of a semiuniversal unfolding of an isolated hypersurface singularity implicitly appeared in the work of K. Saito [Sai83b, Sai83a] in his theory of Primitive forms, and further investigated by M. Saito [Sai89]. For an exposition of the construction of these structures see [Sab08], [Her02], [Tak98]. For general and explicit Frobenius structures defined on the orbit space of a (finite) Coxeter group see [Dub96, Dub99a, Dub99b].

In the present paper we will consider only an example of the second class of Frobenius manifolds, namely the *(small) Quantum Cohomology* of a smooth projective variety X (see Section 5.1). In particular, we will show how to compute the *monodromy data* for this *coalescent* Frobenius manifold (see Sections 2.4, 2.5, 4): according to Theorem 4.1 these data provide local invariants. Furthermore, a conjectural meaning of these local invariants is discussed in terms of objects of the derived category of coherent sheaves of X (*Dubrovin's Conjecture*, [Dub98, Dub13, CDG]).

2.2. **Semisimple Frobenius manifolds.**

Definition 2.2. A point p of a Frobenius manifold M is *semisimple* if the corresponding Frobenius algebra $(T_pM, \eta_p, \circ_p)$ satisfies one of the following equivalent[1] conditions:

[1]The equivalence of (1),(2) and (4) is the content of the Wedderburn-Artin Theorem; the equivalence of (2) and (3) is a consequence of the Nakayama Lemma.

(1) it is semisimple,
(2) it has vanishing Jacobson radical,
(3) it is without nilpotents,
(4) it isomorphic to $\mathbb{C}^n$, where $n = \dim_{\mathbb{C}} M$.

If there is an open dense subset M_{ss} of M of semisimple points, then M is called a *semisimple Frobenius manifold.* If p is a semisimple point, there are tangent vectors $\pi_1, \dots, \pi_n \in T_pM$ such that

$$\pi_i \circ_p \pi_j = \delta_{ij}\pi_i, \quad i, j = 1, \dots, n.$$

They are called *idempotents vectors* at p. The family $(\pi_i)_{i=1}^n$ of idempotents vectors at p is unique up to reordering. Notice that by (2.1) idempotents vectors are mutually orthogonal.

Lemma 2.1. *Let M be a Frobenius manifold. A point $p \in M$ is semisimple if and only if there exists a vector $v \in T_pM$ such that the operator $v\circ_p \colon T_pM \to T_pM$ has simple spectrum.*

Definition 2.3 (Caustic and Bifurcation Set, [Her02],[CDG17b])**.** Let M be a semisimple Frobenius manifold. We call *caustic* the set

$$\mathcal{K}_M := \{p \in M \colon T_pM \text{ is not a semisimple Frobenius algebra}\}.$$

We call *bifurcation set* of the Frobenius manifold the set

$$\mathcal{B}_M := \{p \in M \colon \operatorname{spec}(E\circ_p \colon T_pM \to T_pM) \text{ is not simple}\}.$$

By Lemma 2.1, we have $\mathcal{K}_M \subseteq \mathcal{B}_M$. Semisimple points in $\mathcal{B}_M \setminus \mathcal{K}_M$ are called *coalescence semisimple points.*

The bifurcation set $\mathcal{B}_M$ and the caustic $\mathcal{K}_M$ are either empty or an hypersurface, invariant w.r.t. the unit vector field e (see [Her02]). For Frobenius manifolds defined on the base space of semiuniversal unfoldings of a singularity, these sets coincide with the *bifurcation diagram* and the *caustic* as defined in the classical setting of singularity theory ([Arn93, Arn90]). In this context, the set $\mathcal{B}_M \setminus \mathcal{K}_M$ is called *Maxwell strata.* Remarkably, all these subsets typically admit a naturally induced Frobenius submanifold structure ([Str01, Str04]). In what follows we will assume that the semisimple Frobenius manifold M admits nonempty bifurcation set and caustic.

Definition 2.4 (D-untwisted open sets)**.** Let M a complex manifold, and $D \subseteq M$ a complex analytic hypersurface. A connected open subset $\Omega \subseteq M \setminus D$ will be said to be *D-untwisted* if for any $z \in \Omega$ the inclusions

$$\Omega \xhookrightarrow{\alpha} M \setminus D \xhookrightarrow{\beta} M$$

induce morphisms in homotopy

$$\pi_1(\Omega, z) \xrightarrow{\alpha_*} \pi_1(M \setminus D, z) \xrightarrow{\beta_*} \pi_1(M, z)$$

such that $\operatorname{im}(\alpha_*) \cap \ker(\beta_*) = \{0\}$. Roughly speaking, Ω does not "encircle" the hypersurface D.

Theorem 2.1 ([Dub92, Dub96, Dub98, Dub99b, CDG17b])**.** *Let M be a semisimple Frobenius manifold.*

(1) *If $\Omega \subseteq M_{ss}$ is a $\mathcal{K}_M$-untwisted connected open set, then the idempotents vector fields are holomorphic on Ω. More precisely, a coherent ordering of idempotents vectors can be chosen at any point of Ω, so that the resulting local vector fields are holomorphic.*
(2) *Furthermore, the idempotents vector fields $(\pi_i)_{i=1}^n$ commute,*

$$[\pi_i, \pi_j] = 0, \quad i, j = 1, \dots, n.$$

Hence, there exist local coordinates $(u_i)_{i=1}^n$, holomorphic on any $\mathcal{K}_M$-untwisted connected open $\Omega \subseteq M_{ss}$, such that

$$\frac{\partial}{\partial u_i} = \pi_i.$$

Any such a system of local coordinates will be called canonical. *They are uniquely defined up to re-ordering and shifts by constants ($u_i' := u_i + c_i$, $c_i \in \mathbb{C}$).*

(3) *If $(u_i)_{i=1}^n$ are canonical coordinates near a semisimple point of a Frobenius manifold M, then (up to shifts) the following relations hold*

$$\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_i} = \delta_{ij} \frac{\partial}{\partial u_i}, \qquad e = \sum_{i=1}^n \frac{\partial}{\partial u_i}, \qquad E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i}.$$

According to (3) of Theorem 2.1, in this paper we will fix the shifts of canonical coordinates so that they coincide with the eigenvalues of the (1,1)-tensor $E\circ$.

Note that the structure group of the tangent bundle TM_{ss} can be reduced to the symmetric group $\mathfrak{S}_n$, and that in virtue of Theorem 2.1 it holds a *local* isomorphism of sheaves of $\mathcal{O}_{M_{ss}}$-algebras

$$\mathscr{T}_{M_{ss}} \cong \mathcal{O}_{M_{ss}}^{\oplus n}.$$

Such an isomorphism is not global: the caustic $\mathcal{K}_M$, indeed, represents the branch locus for idempotents vector fields, and canonical coordinates. By prolonging the semisimple Frobenius structure to an unramified covering of M_{ss} of degree at most $n!$, the isomorphism of sheaves of algebras holds *globally* (see [Man99]).

2.3. **Deformed extended flat connection.** Let us consider

- the canonical projection $\pi\colon \mathbb{P}^1_{\mathbb{C}} \times M \to M$,
- the pull-back π^*TM of the tangent bundle TM:

$$\begin{tikzcd} \pi^*TM \arrow[r] \arrow[d] & TM \arrow[d] \\ \mathbb{P}^1_{\mathbb{C}} \times M \arrow[r, "\pi"] & M \end{tikzcd}$$

- two holomorphic (1,1)-tensors $\mathcal{U}, \mu \in \Gamma(TM \otimes T^*M)$ on M defined by

$$\mathcal{U}(X) := E \circ X, \quad \mu(X) := \frac{2-d}{2} X - \nabla_X E, \quad \text{for all } X \in \Gamma(TM).$$

  Note that while the operator $\mathcal{U}$ is η-symmetric, the operator μ is η-skew-symmetric. Moreover, let us *assume* that the nilpotent part of μ vanishes, i.e. the matrix μ is diagonalizable, and actually (after an affine change of flat coordinates) in diagonal form

$$\mu = \operatorname{diag}(\mu_1, ..., \mu_n).$$

In what follows, we will denote by

(1) $\mathscr{T}_M$ the sheaf of sections of TM,
(2) $\pi^*\mathscr{T}_M$ the sheaf of sections of π^*TM,
(3) $\pi^{-1}\mathscr{T}_M$ is the sheaf of sections of π^*TM constant on the fiber of π.

All the tensors $\eta, e, c, E, \mathcal{U}, \mu$ can be lifted to π^*TM, and their lift will be denoted with the same symbol. So, also the Levi-Civita connection ∇ is lifted on π^*TM, and it acts so that

$$\nabla_{\partial_z} Y = 0 \quad \text{for } Y \in (\pi^{-1}\mathscr{T}_M)(M).$$

Let us now twist this connection by using the multiplication of vectors and the operators $\mathcal{U}, \mu$.

Definition 2.5 ([Dub96, Dub98, Dub99b])**.** Let $\widehat{M} := \mathbb{C}^* \times M$. The *deformed extended connection* $\widehat{\nabla}$ on the vector bundle $\pi^*TM|_{\widehat{M}} \to \widehat{M}$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + z \cdot X \circ Y,$$
$$\widehat{\nabla}_{\partial_z} Y = \nabla_{\partial_z} Y + \mathcal{U}(Y) - \frac{1}{z}\mu(Y)$$

for $X, Y \in (\pi^* \mathscr{T}_M)(\widehat{M})$.

The crucial fact is that the deformed extended connection $\widehat{\nabla}$ is *flat.*

Theorem 2.2 ([Dub96],[Dub99b])**.** *The flatness of $\widehat{\nabla}$ is equivalent to the following conditions on M*

- $\nabla c^\flat$ *is completely symmetric,*
- *the product on each tangent space of M is associative,*
- $\nabla\nabla E = 0$*,*
- $\mathfrak{L}_E c = c$*.*

This integrabilty condition implies the existence of solutions of the equation

$$\widehat{\nabla} d\tilde{t} = 0, \quad \text{where} \quad \tilde{t}^\alpha = \tilde{t}^\alpha(t, z), \quad d := \sum_\alpha \frac{\partial}{\partial t^\alpha} dt^\alpha.$$

A set of independent solutions $(\tilde{t}^1, ..., \tilde{t}^n)$ will be called a system of *deformed flat coordinates*: together with the function z, they define a set of $\widehat{\nabla}$-flat coordinates on $\widehat{M}$. If ζ denotes the η-gradient of a deformed flat coordinate, then the previous equation can be written in the frame $(\frac{\partial}{\partial t^\alpha})_\alpha$ as the system

$$\partial_\alpha \zeta = z \mathcal{C}_\alpha \zeta, \quad \alpha = 1, \dots, n \tag{2.2}$$

$$\partial_z \zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta, \tag{2.3}$$

where $\mathcal{C}_\alpha$ is the matrix $(\mathcal{C}_\alpha)^\gamma_\beta = c^\gamma_{\alpha\beta}$. Studying the monodromy phenomenon of the system (2.2)-(2.3), more precisely of its last equation, we will define a set of local invariants of the Frobenius manifold.

2.4. Spectrum of a Frobenius manifold and its Monodromy Data at $z = 0$.

Let M be a Frobenius manifold, not necessarily semisimple. Let us fix a point t of the Frobenius manifold M, and let us focus on the equation

$$\partial_z \zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta, \quad \mathcal{U} = \mathcal{U}(t), \quad \mu = \mu(t), \tag{2.4}$$

for which the following relations hold

$$\mathcal{U}^T \eta = \eta \mathcal{U}, \tag{2.5}$$

$$\mu = \operatorname{diag}(\mu_1, \dots, \mu_n), \quad \mu\eta + \eta\mu = 0. \tag{2.6}$$

Note that for any $t \in M$ the system (2.4) has a Fuchsian singularity at $z = 0$, and an irregular singularity at $z = \infty$ of Poincaré rank 1.

In order to give a geometric and instrinsic description of normal forms of solutions near $z = 0$, we introduce the concept of the *spectrum* of a Frobenius manifold ([Dub99b, Dub04]). Let (V, η, μ) be the datum of

- an n-dimensional complex vector space V,
- a bilinear symmetric non-degenerate form η on V,
- a diagonalizable endomorphism $\mu \in \operatorname{End}(V)$ which is η-antisymmetric

$$\eta(\mu a, b) + \eta(a, \mu b) = 0 \quad \text{for any } a, b \in V.$$

Let $\mathrm{spec}(\mu)=(\mu_1,...,\mu_n)$ and let V_{μ_α} be the eigenspace of a μ_α.

Definition 2.6 ([Dub99b, Dub04])**.** Let (V,η,μ) as above. We say that an endomorphism $A\in\mathrm{End}(V)$ is *μ-nilpotent* if

$$AV_{\mu_\alpha}\subseteq\bigoplus_{m\geq 1}V_{\mu_\alpha+m}\quad\text{for any }\mu_\alpha\in\mathrm{spec}(\mu).$$

In particular such an operator is nilpotent in the usual sense. A μ-nilpotent operator A can be uniquely decomposed in components $A_k\in\mathrm{End}(V)$, $k\geq 1$, such that

$$A_kV_{\mu_\alpha}\subseteq V_{\mu_\alpha+k}\quad\text{for any }\mu_\alpha\in\mathrm{spec}(\mu),\qquad A=\sum_{k\geq 1}A_k,$$

so that the following identities hold:

$$z^\mu Az^{-\mu}=A_1z+A_2z^2+A_3z^3+\dots,\qquad [\mu,A_k]=kA_k\quad\text{for }k=1,2,3,\dots.$$

Definition 2.7 ([Dub99b, CDG17b])**.** Let (V,η,μ) as above. Let us define on V a new non-degenerate bilinear form $\{\cdot,\cdot\}$ by the equation

$$\{a,b\}:=\eta\left(e^{i\pi\mu}a,b\right),\qquad\text{for all }a,b\in V.$$

We define the (η,μ)*-parabolic orthogonal group*, denoted by $\mathcal{G}(\eta,\mu)$, as the complex Lie group of all $\{\cdot,\cdot\}$-isometries $G\in\mathrm{GL}(V)$ of the form

$$G=\mathbb{1}_V+\Delta$$

with Δ a μ-nilpotent operator. Its Lie algebra $\mathfrak{g}(\eta,\mu)$ coincides with the set of all μ-nilpotent operators R which are also *μ-skew-symmetric* in the sense that

$$\{Rx,y\}+\{x,Ry\}=0.$$

In particular, any such operator R commutes with the operator $e^{2\pi i\mu}$.

Lemma 2.2 ([Dub99b, CDG17b])**.** *Let (V,η,μ) as above, and let us fix a basis $(v_i)_{i=1}^n$ of eigenvectors of μ.*

(1) *The operator $A\in\mathrm{End}(V)$ is μ-nilpotent if and only if its associate matrix w.r.t. the basis $(v_i)_{i=1}^n$ satisfies the condition*

$$(A)^\alpha_\beta=0\ \textit{unless}\ \mu_\alpha-\mu_\beta\in\mathbb{N}^*.$$

(2) *If $A\in\mathrm{End}(V)$ is a μ-nilpotent operator, then the matrices associated to its components $(A_k)_{k\geq1}$ w.r.t. the basis $(v_i)_{i=1}^n$ satisfy the condition*

$$(A_k)^\alpha_\beta=0\ \textit{unless}\ \mu_\alpha-\mu_\beta=k,\quad k\in\mathbb{N}^*.$$

(3) *A μ-nilpotent operator $A\in\mathrm{End}(V)$ is an element of $\mathfrak{g}(\eta,\mu)$ if and only if the matrices of its components $(A_k)_{k\geq1}$ w.r.t. $(v_i)_{i=1}^n$ satisfy the further conditions*

$$A_k^T=(-1)^{k+1}\eta A_k\eta^{-1},\quad k\geq 1.$$

The parabolic orthogonal group $\mathcal{G}(\eta,\mu)$ acts canonically on its Lie algebra $\mathfrak{g}(\eta,\mu)$ by the adjoint representation $\mathrm{Ad}\colon\mathcal{G}(\eta,\mu)\to\mathrm{Aut}(\mathfrak{g}(\eta,\mu))$:

$$\mathrm{Ad}_G(R):=G\cdot R\cdot G^{-1},\quad\text{for all }G\in\mathcal{G}(\eta,\mu),\quad R\in\mathfrak{g}(\eta,\mu).$$

Such an action, in general is not free.

Definition 2.8 ([CDG17b])**.** Let $R\in\mathfrak{g}(\eta,\mu)$. We define the group $\widetilde{\mathcal{C}}_0(\mu,R)$ as the isotropy group of R for the adjoint representation $\mathrm{Ad}\colon\mathcal{G}(\eta,\mu)\to\mathrm{Aut}(\mathfrak{g}(\eta,\mu))$.

If M is a Frobenius manifold (not necessarily semisimple), using the Levi-Civita connection ∇ we can identify all tangent spaces, so that we can canonically associate to M a triple (V, η, μ) as above, called *spectrum* of M.

Definition 2.9 ([Dub99b, Dub04, CDG17b])**.** A Frobenius manifold M is called *resonant* if, for some $\alpha \neq \beta$, $\mu_\alpha - \mu_\beta \in \mathbb{Z}^*$. If no eigenvalues of μ differ by a nonzero integer, M will be called *non-resonant.*

For resonant Frobenius manifolds the corresponding (η, μ)-parabolic orthogonal group $\mathcal{G}(\eta, \mu)$ together with all its subgroups $\widetilde{\mathcal{C}}_0(\mu, R)$ are trivial. Since these groups are the responsible of a certain *freedom* in the choice of a normal forms for solutions of (2.4) (see Theorem 2.3), it follows that for non-resonant Frobenius manifolds such a choice is unique.

Theorem 2.3 ([Dub96, Dub99b, CDG17b])**.** *Let M be a Frobenius manifold (not necessarily semisimple).*

(1) *The system* (2.4) *admits fundamental matrix solutions of the form*
$$Z(t,z) = \Phi(t,z) \cdot z^\mu z^{R(t)},$$
$$\Phi(t,z) = \sum_{k\in\mathbb{N}} \Phi_k(t) z^k, \quad \Phi_0(t) \equiv \mathbb{1}, \quad \Phi(t,-z)^T \cdot \eta \cdot \Phi(t,z) = \eta,$$
where $\Phi_k \in \mathcal{O}(M) \otimes \mathfrak{gl}_n(\mathbb{C})$, and $R \in \mathcal{O}(M) \otimes \mathfrak{g}(\eta, \mu)$. A solution of such a form will be said to be in Levelt normal form *at $z = 0$. Because of the Fuchsian character of the singularity $z = 0$, the power series Φ is convergent, and defines a genuine analytic solution.*

(2) *Solutions of* (2.4) *in normal form are not unique. Given two of them*
$$Z(t,z) = \Phi(t,z) \cdot z^\mu z^{R(t)}, \quad \widetilde{Z}(t,z) = \widetilde{\Phi}(t,z) \cdot z^\mu z^{\widetilde{R}(t)},$$
there exists a unique holomorphic $\mathcal{G}(\eta, \mu)$-valued function
$$G(t) = \mathbb{1} + \Delta(t)$$
on M such that
$$\widetilde{Z}(t,z) = Z(t,z) \cdot G(t),$$
$$\widetilde{R}(t) = G(t)^{-1} \cdot R(t) \cdot G(t), \quad \widetilde{\Phi}(t,z) = \Phi(t,z) \cdot P_G(t,z),$$
where
$$\begin{aligned} P_G(t,z) :&= z^\mu \cdot G(t) \cdot z^{-\mu} \\ &= \mathbb{1} + z\Delta_1(t) + z^2 \Delta_2(t) + \dots, \end{aligned}$$
$(\Delta_k)_{k\geq 1}$ being the components of Δ. In particular, if $\widetilde{R} = R$, then G is $\widetilde{\mathcal{C}}_0(\mu, R)$-valued.

If instead of considering only the equation (2.4), we focus on the whole system (2.2)-(2.3), then the previous results can be further refined.

Theorem 2.4 (Isomonodromy Theorem I, [Dub96, Dub99b, CDG17b])**.** *Let M be a Frobenius manifold (not necessarily semisimple).*

(1) *The system* (2.2)-(2.3) *admits fundamental matrix solutions of the form*
$$Z(t,z) = \Phi(t,z) \cdot z^\mu z^R,$$
$$\Phi(t,z) = \sum_{k\in\mathbb{N}} \Phi_k(t) z^k, \quad \Phi_0(t) \equiv \mathbb{1}, \quad \Phi(t,-z)^T \cdot \eta \cdot \Phi(t,z) = \eta,$$
where $\Phi_k \in \mathcal{O}(M) \otimes \mathfrak{gl}_n(\mathbb{C})$, and $R \in \mathfrak{g}(\eta, \mu)$ is independent *of t. In particular the monodromy $M_0 = \exp(2\pi i\mu)\exp(2\pi i R)$ at $z = 0$ does not depend on t.*

(2) *Solutions of the whole system* (2.2)-(2.3) *in normal form are not unique. Given two of them*

$$Z(t,z)=\Phi(t,z)\cdot z^{\mu}z^{R},\quad \widetilde{Z}(t,z)=\widetilde{\Phi}(t,z)\cdot z^{\mu}z^{\widetilde{R}},$$

there exists a unique matrix $G\in\mathcal{G}(\eta,\mu)$, *say* $G=\mathbb{1}+\Delta$, *such that*

$$\widetilde{Z}(t,z)=Z(t,z)\cdot G,$$

$$\widetilde{R}=G^{-1}\cdot R\cdot G,\quad \widetilde{\Phi}(t,z)=\Phi(t,z)\cdot P_G(t,z),$$

where

$$\begin{aligned}P_G(t,z):&=z^{\mu}\cdot G\cdot z^{-\mu}\\ &=\mathbb{1}+z\Delta_1+z^2\Delta_2+\dots,\end{aligned}$$

$(\Delta_k)_{k\geq 1}$ *being the components of* Δ. *In particular, if* $\widetilde{R}=R$, *then* $G\in\widetilde{\mathcal{C}}_0(\mu,R)$.

Proof. Let $Z(z,t)$ be a solution of (2.2)-(2.3), and let $M_0(t)$ be the monodromy of $Z(\cdot,t)$ at $z=0$:

$$Z(e^{2\pi i}z,t)=Z(z,t)\cdot M_0(t).$$

The coefficients of the equations

$$\partial_\alpha Z(z,t)=z\mathcal{C}_\alpha(t)\cdot Z(z,t),\quad \alpha=1,\dots,n$$

being holomorphic in z, we have that

$$\begin{aligned}\partial_\alpha Z(z,t)\cdot Z(z,t)^{-1}&=\partial_\alpha Z(e^{2\pi i}z,t)\cdot Z(e^{2\pi i}z,t)^{-1}\\ &=\partial_\alpha\left(Z(z,t)\cdot M_0(t)\right)\cdot\left(Z(z,t)\cdot M_0(t)\right)^{-1}\\ &=\partial_\alpha Z(z,t)\cdot Z(z,t)^{-1}+Z(z,t)\cdot\partial_\alpha M_0(t)\cdot M_0(t)^{-1}\cdot Z(z,t)^{-1},\end{aligned}$$

for any α. Hence

$$\partial_\alpha M_0(t)=0,\quad \alpha=1,\dots,n.$$

By Theorem 2.3, we necessarily conclude that R is t-independent. □

Definition 2.10 ([Dub96, Dub99b])**.** Given a Frobenius manifold M, we will call *monodromy data of* M *at* $z=0$ the data $(\mu,[R])$, where $[R]$ denotes the $\mathcal{G}(\eta,\mu)$-class of exponents of formal solutions in Levelt normal form of the system (2.2)-(2.3) as in Theorem 2.4. A representative R can be chosen independent of the point $t\in M$.

Remark 2.2. A first description of the freedom and ambiguities in the definition of the monodromy data was given in [Dub96, Dub99b]. In particular, a complex Lie group $\mathcal{C}_0(\mu,R)$ was introduced in order to describe the freedom of normal forms of solutions of (2.2)-(2.3). Such a group *is too big*, and in particular does not preserve the orthogonality conditions of point (1) of Theorems 2.3, 2.4. it must be replaced by $\widetilde{\mathcal{C}}_0(\mu,R)$ of Definition 2.8, which is the correct one.

2.5. **Monodromy Data for a semisimple Frobenius manifold.** Having completely described the monodromy phenomenon near the singularity $z=0$ of equation (2.4), let us now focus on the singularity at $z=\infty$. At generic points of the Frobenius manifold M, the singularity $z=\infty$ of the corresponding equation (2.4) is irregular of Poincaré rank 1, and hence a Stokes phenomenon for the solutions must be studied.

Following the existing literature on general theory differential equations, in order to give complete the description of this Stokes phenomenon, in [Dub98], [Dub96] and [Dub99b] the following *main assumptions* are made:

(A1) the manifold M is semisimple, so that the operator $\mathcal{U}$ is diagonalizable on a non-empty dense open subset M_{ss};

(A2) the point $t \in M_{ss}$ at which we consider the system (2.4) is not in the bifurcation set $\mathcal{B}_M$, so that the eigenvalues of $\mathcal{U}(t)$ are pairwise distinct.

In this Section, we show how it is possible to drop Assumption (A2), by enlarging the definition of monodromy data to *all semisimple points*, including the semisimple coalescence points of Definition 2.3. In Section 4, we also show up to which extent the *Isomonodromy Property* extends to these points.

Remark 2.3. At points $t \in M$ where $\mathcal{U}(t) \equiv 0$, a phenomenon of Poincaré rank reduction manifests, and $z = \infty$ becomes a Fuchsian singularity for (2.4). These points are not semisimple, the Euler vector being nilpotent.

First of all, let us introduce a (non-tensorial) transform matrix from a frame of ∇-flat coordinate vector fields $(\frac{\partial}{\partial t^\alpha})_{\alpha=1}^n$ to the idempotent vielbein at any semisimple point of M.

Definition 2.11 (Matrix Ψ)**.** Let M be a semisimple Frobenius manifold, let $(t^\alpha)_{\alpha=1}^n$ be local flat coordinates such that $\frac{\partial}{\partial t^1} = e$, and let $u_1, \dots, u_n$ be canonical coordinates. Introducing the orthonormal basis

$$f_i := \frac{1}{\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right)^{\frac{1}{2}}} \frac{\partial}{\partial u_i}$$

for arbitrary choices of signs in the square roots, we define a matrix Ψ (depending on the point of the Frobenius manifold) whose elements $\Psi_{i\alpha}$ (i-th row, α-th column) are defined by the relation

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \Psi_{i\alpha} f_i \quad \alpha = 1, \dots, n.$$

Lemma 2.3 ([Dub96, Dub99b])**.** *Let M be a semisimple Frobenius manifold.*

(1) *The matrix Ψ is a one-valued holomorphic function on any $\mathcal{K}_M$-untwisted connected open subset $\Omega \subseteq M_{ss}$. Moreover, it satisfies the following relations:*

$$\Psi^T\Psi = \eta, \quad \Psi_{i1} = \eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right)^{\frac{1}{2}}, \quad f_i = \sum_{\alpha,\beta=1}^n \Psi_{i1}\Psi_{i\beta}\eta^{\beta\alpha}\frac{\partial}{\partial t^\alpha}, \quad c_{\alpha\beta\gamma} = \sum_{i=1}^n \frac{\Psi_{i\alpha}\Psi_{i\beta}\Psi_{i\gamma}}{\Psi_{i1}}.$$

(2) *The matrix Ψ diagonalizes the operator $\mathcal{U}$, the operator of multiplication by the Euler vector field:*

$$\Psi\mathcal{U}\Psi^{-1} = U := \mathrm{diag}(u_1, \dots, u_n).$$

(3) *The matrix $V := \Psi\mu\Psi^{-1}$ is antisymmetric, i.e. $V^T + V = 0$.*

The system (2.2)-(2.3) corresponding to points $t \in M_{ss}$ can be rewritten in the idempotent vielbein

$$y = \Psi\zeta, \tag{2.7}$$

in the following way

$$\partial_i y = (zE_i + V_i)y, \tag{2.8}$$

$$\partial_z y = \left(U + \frac{1}{z}V\right)y, \tag{2.9}$$

where

$$(E_i)^\alpha_\beta = \delta^\alpha_i\delta^\beta_i, \quad V_i = \partial_i\Psi \cdot \Psi^{-1},$$

$$U = \mathrm{diag}(u_1, \dots, u_n), \quad \text{with not necessarily } u_i \neq u_j,\ i \neq j.$$

2.5.1. *ℓ-chamber decomposition of a semisimple Frobenius manifold.* For the sake of clarity, before studying solutions of (2.8)-(2.9), we introduce a decomposition of a semisimple Frobenius manifold into *pieces*. Such a decomposition is subordinate to the choice of oriented rays in the universal cover $\mathcal{R} := \widetilde{\mathbb{C} \setminus \{0\}}$, or equivalently of a line in the complex plane. In what follows, we denote with $\mathrm{pr} : \mathcal{R} \to \mathbb{C}\setminus\{0\}$ the covering map.

Definition 2.12 (Stokes rays, [CDG17b])**.** Let $t \in M_{ss}$, and let us consider the spectrum $\mathrm{spec}\,(\mathcal{U}(t)) = \{u_i(t)\}_{i=1}^n$. For all pairs $(u_i(t), u_j(t))$, such that $u_i(t) \neq u_j(t)$, we fix a determination $\alpha_{ij}(t)$ of the argument $\arg(u_i(t) - u_j(t))$ so that it belongs to the interval $[0; 2\pi[$, and let

$$\tau_{ij}(t) := \frac{3\pi}{2} - \alpha_{ij}(t).$$

We call *Stokes rays* at $t \in M_{ss}$ the rays in the universal covering $\mathcal{R}$ defined by

$$R_{ij,k}(t) := \{z \in \mathcal{R} \colon \arg z = \tau_{ij}(t) + 2k\pi\}$$

for any $k \in \mathbb{Z}$. The projections onto the $\mathbb{C}$-plane

$$R_{ij}(t) := \mathrm{pr}\,(R_{ij,k}(t))$$

will also be called *Stokes rays* at $t \in M_{ss}$.

Note that the projected Stokes rays coincide with the ones defined in [Dub99b], namely

$$R_{ij} := \{z \in \mathbb{C} \colon z = -i\rho(\overline{u_i} - \overline{u_j}),\ \rho > 0\}\,. \tag{2.10}$$

Note that Stokes rays have a natural orientation (from 0 to ∞).

Remark 2.4. The characterisation of $R_{ij,k}$ is that $z \in R_{ij,k}$ if and only if

$$\mathrm{Re}((u_i - u_j)z) = 0.$$

For $z \in \mathbb{C}$ we have

$$\begin{aligned} |e^{zu_i}| &= |e^{zu_j}| && \text{if } z \in R_{ij},\\ |e^{zu_i}| &> |e^{zu_j}| && \text{if } z \text{ is on the left of } R_{ij},\\ |e^{zu_i}| &< |e^{zu_j}| && \text{if } z \text{ is on the right of } R_{ij}. \end{aligned}$$

Definition 2.13 (Admissible Rays and Line, [CDG17b])**.** Let $\phi \in \mathbb{R}$ and let us define the rays

$$\ell_+(\phi) := \{z \in \mathcal{R} \colon \arg z = \phi\}\,,$$
$$\ell_-(\phi) := \{z \in \mathcal{R} \colon \arg\ z = \phi - \pi\}\,.$$

We will say that these rays are *admissible at $t \in M_{ss}$* if they do not coincide with any Stokes rays $R_{ij,k}(t)$ for any i, j s.t. $u_i(t) \neq u_j(t)$ and any $k \in \mathbb{Z}$. Moreover, we call *admissible at $t \in M_{ss}$* a line ℓ of the complex plane such that

$$\mathrm{Re}\ z(u_i(t) - u_j(t))|_{z \in \ell \setminus 0} \neq 0$$

for any i, j s.t. $u_i(t) \neq u_j(t)$. In other words, a line is admissible if it does not contain any Stokes rays $R_{ij}(t)$. Thus the projections $\mathrm{pr}\,(\ell_\pm(\phi))$ are contained in an admissible line, on which we can define an orientation by choosing as positive part $\mathrm{pr}\,(\ell_+(\phi))$.

Definition 2.14 (ℓ-Chambers, [CDG17b])**.** Given a semisimple Frobenius manifold M, and fixed a line ℓ in the complex plane, consider the open dense subset of points $p \in M_{ss}$ such that

- the eigenvalues of U at p are pairwise distinct,
- the line ℓ is admissible at p.

We call any connected component Ω_ℓ of this set an *ℓ-chamber.*

Remark 2.5. The topology of an ℓ-chamber can be highly non-trivial and quite mysterious so far. In particular, it should not be confused with the simple topology of an ℓ-cell (see Definition 4.1). For example, in [Guz05] the analytic continuation of the Frobenius structure of the Quantum Cohomology of $\mathbb{P}^2$ is studied (see Section 5.1): it is shown that there exist points $(u_1, u_2, u_3) \in \mathbb{C}^3$ with $u_i \neq u_j$ with $i \neq j$ which do not correspond to any true geometric point of the Frobenius manifold. This is due to singularities of the change of coordinates $u \mapsto t$.

Note that points of the bifurcation set $\mathcal{B}_M$ are (by definition) in the complement of the union of ℓ-chambers for any choice of the line ℓ.

2.5.2. *Fundamental solutions at semisimple points.* Let us consider a $\mathcal{B}_M$-untwisted connected open subset $\Omega \subseteq M \setminus \mathcal{B}_M$. On such an open set, a single-valued holomorphic brach of the idempotent vector fields, of canonical coordinates and of the matrix Ψ will be fixed (see Theorem 2.1, and Lemma 2.3), so that the corresponding system (2.8)-(2.9) can be studied.

Theorem 2.5 ([Dub96, Dub99b, CDG17b]). *Let M be a semisimple Frobenius manifold, $\Omega \subseteq M \setminus \mathcal{B}_M$ a $\mathcal{B}_M$-untwisted connected open subset, and let us consider*

- *a real number $\phi \in [0; 2\pi[$,*
- *the line ℓ in the complex plane containing the projections $\operatorname{pr}(\ell_\pm(\phi))$,*
- *an ℓ-chamber Ω_ℓ,*
- *the sectors*

$$\Pi_{\text{right}}(\phi) := \{z \in \mathcal{R} \colon \phi - \pi < \arg\ z < \phi\},$$
$$\Pi_{\text{left}}(\phi) := \{z \in \mathcal{R} \colon \phi < \arg\ z < \phi + \pi\}.$$

Then the following results hold.

(1) *For any point $t \in \Omega$ there exists a unique* formal *solution of the system* (2.8)-(2.9) *of the form*

$$Y_{\text{formal}}(z,t) = F(z,t) \cdot \exp(zU(t)), \quad F \in (\mathcal{O}(\Omega) \otimes \mathfrak{gl}_n(\mathbb{C}))\left[\left[\frac{1}{z}\right]\right],$$
$$F(z,t) = \sum_{m \in \mathbb{N}} \frac{F_m(t)}{z^m}, \quad F_0 \equiv \mathbb{1}, \quad F(-z,t)^T \cdot F(z,t) = \mathbb{1}.$$

(2) *For any point $t \in \Omega_\ell$, and any $k \in \mathbb{Z}$, there exist two solutions $Y^{(k)}_{\text{right}}(z,t)$, and $Y^{(k)}_{\text{left}}(z,t)$ of the system* (2.8)-(2.9) *uniquely determined by the asymptotic expansion*

$$Y^{(k)}_{\text{right/left}}(z,t) \sim Y_{\text{formal}}(z,t), \quad |z| \to +\infty, \quad z \in e^{2\pi k i}\Pi_{\text{right/left}}(\phi),$$

uniformly in $t \in \Omega_\ell$.

(3) *For any $t \in \Omega_\ell$, for any $k \in \mathbb{Z}$ we have that*

$$Y^{(k)}_{\text{right/left}}(e^{2\pi k i} z, t) = Y^{(0)}_{\text{right/left}}(z,t), \quad z \in \mathcal{R}.$$

Remark 2.6. The precise meaning of the asymptotic relation in (2) of Theorem 2.5 is the following:

$$\forall K \Subset \Omega_\ell,\ \forall h \in \mathbb{N},\ \forall \overline{\mathcal{S}} \subsetneq e^{2\pi k i}\Pi_{\text{right/left}}(\phi),\ \exists C_{K,h,\overline{\mathcal{S}}} > 0 \colon \text{ if } z \in \overline{\mathcal{S}} \setminus \{0\} \text{ then}$$
$$\sup_{t \in K} \left\| Y^{(k)}_{\text{right/left}}(z,t) \cdot \exp(-zU(t)) - \sum_{m=0}^{h-1} \frac{F_m(t)}{z^m} \right\| < \frac{C_{K,h,\overline{\mathcal{S}}}}{|z|^h}.$$

Here $\overline{\mathcal{S}}$ denotes any unbounded closed sector of $\mathcal{R}$ with vertex at 0.

Let us now consider the case of a semisimple coalescence point $t_0 \in \mathcal{B}_M \setminus \mathcal{K}_M$. *A priori* one should expect the following:

- the system (2.8)-(2.9) at $t_0 \in \mathcal{B}_M \setminus \mathcal{K}_M$ does not admit a formal solution of the form as in (1) of Theorem 2.5. In general, at irregular singularities with coalescing eigenvalues formal solutions have a more complicated form (see [BJL79], [CDG17a], Section 4);
- if such a formal solution exists, it is not unique;
- if $t_0 \in \overline{\Omega_\ell}$, then the coefficients $F_m \in \mathcal{O}(\Omega_\ell)$, and the solutions $\left(Y^{(k)}_{\text{right/left}}\right)_{k\in\mathbb{Z}}$ of Theorem 2.5 typically diverge as $t \to t_0$.

Nevertheless, according to the analysis extensively developed in [CDG17a], the system (2.8)-(2.9) has certain properties ensuring that the following result holds:

Theorem 2.6 ([CDG17b]). *Let M be a semisimple Frobenius manifold, let $t_0 \in \mathcal{B}_M \setminus \mathcal{K}_M$ be a semisimple coalescence point, let us fix a choice of ordering for canonical coordintes $(u_i(t_0))_{i=1}^n$ and a determination of $\Psi(t_0)$, and let us consider the corresponding system*

$$\partial_i y = (zE_i + V_i(t_0))\, y, \quad \partial_z y = \left(U(t_0) + \frac{1}{z}V(t_0)\right) y. \tag{2.11}$$

(1) *The matrix $V(t_0)$ satisfies the vanishing condition*

$$V_{ij}(t_0) = 0 \quad \text{if } u_i(t_0) = u_j(t_0).$$

(2) *This vanishing condition implies the existence of a* unique *formal solution of the form*

$$\mathring{Y}_{\text{formal}}(z) = \mathring{F}(z) \exp\left(zU(t_0)\right), \quad \mathring{F} \in \mathfrak{gl}_n(\mathbb{C})\left[\left[\frac{1}{z}\right]\right]$$

$$\mathring{F}(z) = \sum_{m\in\mathbb{N}} \frac{\mathring{F}_m}{z^m}, \quad \mathring{F}_0 \equiv \mathbb{1}, \quad \mathring{F}(-z)^T \cdot \mathring{F}(z) = \mathbb{1}.$$

(3) *If Ω is a $\mathcal{B}_M$-untwisted connected open subset of M such that $t_0 \in \overline{\Omega}$, on which branches of u_i's and of the matrix Ψ are fixed so that their continuous extensions on $\overline{\Omega} \cap (\mathcal{B}_M \setminus \mathcal{K}_M)$ induces the same determinations chosen at t_0, i.e.*

$$\lim_{t\to t_0} u_i(t) = u_i(t_0), \quad \lim_{t\to t_0} \Psi(t) = \Psi(t_0),$$

and $F_m \in \mathcal{O}(\Omega)$ are the coefficients of the formal solution as in Theorem 2.5, then

$$\lim_{t\to t_0} F_m(t) = \mathring{F}_m.$$

Hence, the functions F_m's extend to holomorphic functions on any $\mathcal{K}_M$-untwisted connected open subset of M.

(4) *Let us now fix an admissible oriented line ℓ at t_0 of slope $\phi \in [0, 2\pi[$, and let $\Pi_{right/left}(\phi)$ be the sectors defined as in Theorem 2.5. Then for any $k \in \mathbb{Z}$, there exists two fundamental solutions $\mathring{Y}^{(k)}_{\text{right}}, \mathring{Y}^{(k)}_{\text{left}}$ of (2.11) uniquely determined by the asymptotic expansion*

$$\mathring{Y}^{(k)}_{\text{right/left}}(z) \sim \mathring{Y}_{\text{formal}}(z), \quad |z| \to +\infty, \quad z \in e^{2\pi k i}\Pi_{\text{right/left}}(\phi).$$

In the remaining part of this Section, we will denote by $Y^{(k)}_{\text{left/right}}(z,t)$ the solutions of (2.8)-(2.9) of Theorem 2.5 if $t \in M \setminus \mathcal{B}_M$, and of Theorem 2.6 if $t \in \mathcal{B}_M \setminus \mathcal{K}_M$ respectively. Note, in particular, that *a priori* one should expect that the function so defined is *discontinuous* w.r.t. t: Theorem 4.1 will show that this is not the case.

Definition 2.15 (Stokes and Central Connection matrices, [Dub99b, CDG17b])**.** Let M be a semisimple Frobenius manifold, $\Omega \subseteq M$ a $\mathcal{K}_M$-untwisted connected open subset, and let ℓ be an oriented line of slope $\phi \in [0; 2\pi[$ in the complex plane. For any point $t \in \Omega$ at which ℓ is admissible, we define two matrices $\mathbb{S}(t), \mathbb{S}_-(t), C(t) \in GL_n(\mathbb{C})$ such that for all $z \in \mathcal{R}$ one has

$$\begin{aligned} Y^{(0)}_{\text{left}}(z,t) &= Y^{(0)}_{\text{right}}(z,t) \cdot \mathbb{S}(t),\\ Y^{(0)}_{\text{left}}(e^{2\pi i}z,t) &= Y^{(0)}_{\text{right}}(z,t) \cdot \mathbb{S}_-(t),\\ Y^{(0)}_{\text{right}}(z,t) &= Y_0(z,t) \cdot C(t), \end{aligned}$$

where $Y_0(z,t) = \Psi(t) \cdot \Phi(z,t) \cdot z^{\mu} z^{R}$ is a solution of (2.8)-(2.9) in Levelt normal form at $z = 0$ as in Theorem 2.4.

Theorem 2.7 ([Dub99b, CDG17b])**.** *The Stokes matrices $\mathbb{S}(t), \mathbb{S}_-(t)$ and the central connection matrix $C(t)$, computed w.r.t. all the choices done as in Definition 2.15, satisfy the following properties:*

(1) *for all $k \in \mathbb{Z}$, $t \in \Omega$ and all $z \in \mathcal{R}$ one has*

$$\begin{aligned} Y^{(k)}_{\text{left}}(z,t) &= Y^{(k)}_{\text{right}}(z,t) \cdot \mathbb{S}(t),\\ Y^{(k)}_{\text{left}}(z,t) &= Y^{(k+1)}_{\text{right}}(z,t) \cdot \mathbb{S}_-(t),\\ Y^{(k)}_{\text{right}}(z,t) &= Y_0(z,t) \cdot M_0^{-k} \cdot C(t), \end{aligned}$$

where $M_0 = \exp(2\pi i \mu)\exp(2\pi i R)$;

(2) *for all $k \in \mathbb{Z}$, $t \in \Omega$ and $z \in \mathcal{R}$ one has*

$$\begin{aligned} Y^{(k)}_{\text{right}}(e^{2\pi i}z,t) &= Y^{(k)}_{\text{right}}(z,t) \cdot \left(\mathbb{S}_-(t) \cdot \mathbb{S}(t)^{-1}\right),\\ Y^{(k)}_{\text{left}}(e^{2\pi i}z,t) &= Y^{(k)}_{\text{right}}(z,t) \cdot \left(\mathbb{S}(t)^{-1} \cdot \mathbb{S}_-(t)\right); \end{aligned}$$

(3) *for all $t \in \Omega$ one has that*

$$\begin{gathered} \mathbb{S}_-(t) = \mathbb{S}(t)^T,\\ \mathbb{S}(t)_{ii} = 1, \quad i = 1, \dots, n,\\ \mathbb{S}(t)_{ij} \neq 0 \text{ with } i \neq j \text{ only if } u_i \neq u_j \text{ and } R_{ij}(t) \subset \operatorname{pr}(\Pi_{\text{left}}(\phi)). \end{gathered}$$

Furthermore, the Stokes and Central Connection matrices must satisfy the following constraints at any point $t \in \Omega$:

(4.a) $C(t) \cdot \mathbb{S}(t)^T \cdot \mathbb{S}(t)^{-1} \cdot C(t)^{-1} = M_0 = \exp(2\pi i \mu)\exp(2\pi i R)$,
(4.b) $\mathbb{S}(t) = C(t)^{-1} \cdot e^{-i\pi R} \cdot e^{-i\pi\mu} \cdot \eta^{-1} \cdot (C(t)^T)^{-1}$,
(4.c) $\mathbb{S}(t)^T = C(t)^{-1} \cdot e^{i\pi R} \cdot e^{i\pi\mu} \cdot \eta^{-1} \cdot (C(t)^T)^{-1}$.

2.5.3. *Isomonodromy Property on ℓ-chambers.* The following result is of fundamental importance for the theory of semisimple Frobenius manifold.

Theorem 2.8 (Isomonodromy Theorem II, [Dub96, Dub99b, CDG17b])**.** *The Stokes matrix $\mathbb{S}$ and the central connection matrix C, computed w.r.t. an oriented line ℓ, are constant on any ℓ-chamber Ω_ℓ. The values of $\mathbb{S}, C$ in two different ℓ-chambers are related by an action of the braid group of Section 3.*

The full set of data $(\mu, R, \mathbb{S}, C)$ can be used to locally classify semisimple Frobenius manifolds. In particular, given a point $u^0 = (u^0_1, \dots, u^0_n) \in \mathbb{C}^n$ with $u^0_i \neq u^0_j$ for $i \neq j$ and given data $(\mu, R, \mathbb{S}, C)$, a Riemann-Hilbert problem can be formulated. If a unique solution of the RH-problem exists, then it can be shown that the solution exists for u sufficiently close to u^0, that it is an analytic function of u, and that actually can be *meromorphically* extended to the whole universal cover of

$$\mathbb{C}^n \setminus \bigcup_{i<j} \{u_i = u_j\},$$

as shown in [Mal83a, Mal83b], [Miw81]. Using the solution of this RH-problem the Frobenius structure can be explicitly reconstructed through the formulæ in [Dub96, Dub98, Dub99b], and also [Guz01] where explicit examples of reconstruction are given.

3. Freedom of Monodromy Data and Braid Group action

In defining the full set of monodromy data $(\mu, R, \mathbb{S}, C)$ several non-unique choices must be done, and consequently some freedom of these invariants is allowed. Clearly, these admissible transformations of these local invariants preserve the constraints (4.a),(4.b),(4.c) of Theorem 2.7.

While the operator μ is completely determined by the choice of flat coordinates as in Remark 2.1, we have seen in Theorem 2.4 that the invariant R is defined only up to conjugacy class of the (η, μ)-parabolic orthogonal group $\mathcal{G}(\eta, \mu)$.

For the remaining local invariants $\mathbb{S}$ and C notice that their definition is subordinate to the following choices:

(1) an oriented line in the complex plane;
(2) an integer number $k \in \mathbb{Z}$, which correspond to fixing a determination for the slope of ℓ, and corresponding solutions $Y^{(k)}_{\text{left/right}}$;
(3) the choice of an ordering of the canonical coordinates on each ℓ-chamber Ω_ℓ;
(4) the choice of the branch of the square roots defining the matrix Ψ on each ℓ-chamber Ω_ℓ;
(5) the choice of different solutions Y_0 in Levelt normal form with the same exponent R.

We first describe (2),(3),(4) and (5). The transformations of the data depending on the choice of the oriented line ℓ will be studied in the next Section. We have the following actions on the data $(\mathbb{S}, C)$:

- *Action of the additive group* $\mathbb{Z}$: according to Theorem 2.7 this action is non trivial only on the central connection matrix C,
$$C(t) \mapsto M_0^{-k} \cdot C(t), \quad k \in \mathbb{Z}, \quad M_0 = e^{2\pi i \mu} e^{2\pi i R}, \quad t \in \Omega_\ell.$$
The Stokes matrix $\mathbb{S}$ remains invariant.
- *Action of the group of permutations* $\mathfrak{S}_n$: corresponding to any reordering of the canonical coordinates at a point $t \in M \setminus \mathcal{K}_M$
$$(u_1, \dots, u_n) \mapsto (u_{\tau(1)}, \dots, u_{\tau(n)}), \quad u_i := u_i(t), \quad \tau \in \mathfrak{S}_n,$$
there is the following natural transformation of solutions of the system (2.8)-(2.9)
$$Y^{(k)}_{\text{left/right}} \longmapsto P Y^{(k)}_{\text{left/right}} P^{-1}, \qquad Y_0 \longmapsto P Y_0, \quad P \in GL_n(\mathbb{C}), \quad P_{ij} := \delta_{j\tau(i)}.$$
With such a choice of reordering, the monodromy data transform as follows:
$$\mathbb{S} \mapsto P \cdot \mathbb{S} \cdot P^{-1}, \quad C \mapsto C P^{-1}. \tag{3.1}$$
- *Action of the group* $(\mathbb{Z}/2\mathbb{Z})^{\times n}$: by choosing opposite signs for the normalized idempotents (matrix Ψ), we can change the sign of the entries of the matrices $\mathbb{S}$ and C. If $\mathcal{I}$ is a matrix with 1's or (-1)'s on the diagonal, corresponding to the transformations $Y_{\text{left/right}} \mapsto \mathcal{I} Y_{\text{left/right}} \mathcal{I}$, $Y_0 \mapsto \mathcal{I} Y_0$ the monodromy data transform as
$$\mathbb{S} \mapsto \mathcal{I} \cdot \mathbb{S} \cdot \mathcal{I}^{-1}, \quad C \mapsto C \mathcal{I}^{-1}.$$

- *Action of the group* $\widetilde{\mathcal{C}}_0(\mu, R)$: according to Theorem 2.4, different solutions Y_0 in Levelt form at $z = 0$ with the same exponent R are related by multiplication to the right by elements of the isotropy group $\widetilde{\mathcal{C}}_0(\mu, R)$. Correspondingly, the central connection matrix transforms as $C \mapsto G \cdot C$, where $G \in \widetilde{\mathcal{C}}_0(\mu, R)$.

Among all possible ordering of the canonical coordinates, we will consider the following particularly useful class.

Definition 3.1 (Triangular order, [CDG17b])**.** Let ℓ be an oriented line in the complex plane, and let $t \in M \setminus \mathcal{K}_M$. We will say that $u_1(t), ..., u_n(t)$ are in *triangular order at* t w.r.t. the line ℓ whenever $\mathbb{S}(t)$ is upper triangular.

An example of triangular order is given the *lexicographical order* w.r.t an admissible line ℓ. If ℓ is an admissible line at $t \in M \setminus \mathcal{K}_M$ of slope $\phi \in \mathbb{R}$, let us consider the rays starting from the points $u_1(t), \dots, u_n(t)$ in the complex plane

$$L_j := \left\{ u_j + \rho e^{i\left(\frac{\pi}{2} - \phi\right)} \colon \rho \in \mathbb{R}_+ \right\}, \quad u_i \equiv u_i(t), \quad j = 1, \dots, n,$$

and for any complex number z_0 let us define the oriented line

$$L_{z_0,\phi} := \left\{ z_0 + \rho e^{-i\phi} \colon \rho \in \mathbb{R} \right\}$$

where the orientation is induced by $\mathbb{R}$. In this way we have a natural total order $\preceq$ on the points of $L_{z_0,\phi}$. We can choose z_0, with $|z_0|$ sufficiently large, so that the intersections

$$L_j \cap L_{z_0,\phi} =: \{p_j\}$$

are non-empty.

Definition 3.2 (Lexicographical order, [Dub96, Dub99b, CDG17b])**.** We will say that the canonical coordinates u_j's are in *ℓ-lexicographical order at $t \in M \setminus \mathcal{K}_M$* if

$$p_1 \preceq p_2 \preceq p_3 \preceq \cdots \preceq p_n.$$

It is clear that the definition does not depend on the choice of $z_0 \in \mathbb{C}$, with $|z_0|$ sufficiently large.

Lemma 3.1 ([CDG17b])**.** *Let M be a semisimple Frobenius manifold, $t \in M \setminus \mathcal{K}_M$, and let ℓ be an oriented line in the complex plane, admissible at t.*

(1) *Any ℓ-lexicographical order at t of the canonical coordinates $(u_i(t))_{i=1}^n$ is triangular.*
(2) *If $t \in M \setminus \mathcal{B}_M$, then the ℓ-lexicographical order at t of the canonical coordinates $(u_i(t))_{i=1}^n$ is* unique*.*
(3) *If $t \in \mathcal{B}_M \setminus \mathcal{K}_M$, then* any *triangular order at t of the canonical coordinates $(u_i(t))_{i=1}^n$ is ℓ-lexicographical.*

Note that it is not true that in any ℓ-chamber Ω_ℓ there is a unique triangular order (for further comments, see Remark 4.1).

3.1. **Action of the braid group $\mathcal{B}_n$.** We consider the problem of studying

(1) either the exact relations between the data computed in two different ℓ-chambers $\Omega_\ell^1, \Omega_\ell^2$ of a semisimple Frobenius manifold M w.r.t. a fixed oriented line ℓ,
(2) or the dependence of the monodromy data $(\mathbb{S}(t), C(t))$ at a point $t \in M \setminus \mathcal{B}_M$ on the choice of an oriented line ℓ admissible at t.

Both sides of the problem can be described through an action of the braid group $\mathcal{B}_n$. In the first case the ℓ-chambers are fixed, in the second case they change: indeed, the ℓ-chamber decomposition of M depends on the choice of ℓ, so that the given point of $M \setminus \mathcal{B}_M$ may fall in two different ℓ-chambers if ℓ is changed. Below we consider the change of ℓ as given by a continuous rotation. Let us always label the canonical coordinates $(u_1, \dots, u_n)$ in lexicographical order w.r.t. ℓ

- in both the fixed ℓ-chambers $\Omega_\ell^1, \Omega_\ell^2$, in case (1)
- both before and after the rotation of ℓ, in case (2).

so that, in particular, any Stokes matrix is always in upper triangular form. Thus, if we introduce the configuration space $X := (\mathbb{C}^n \setminus \Delta)/\mathfrak{S}_n$, where Δ stands for the union of all diagonals in $\mathbb{C}^n$, i.e. the sets $\{u_i = u_j\}$ with $i \neq j$, and if we identify a point $t \in M \setminus \mathcal{B}_M$ with its ℓ-lexicographical n-tuple representative of canonical coordinates set $\{u_i(t)\}_{i=1}^n \in X$, we can analyze properties of the analytic continuation of the whole Frobenius structure by varying the configuration point $\{u_1, \dots, u_n\}$ in the universal cover $\widetilde{X}$. The fundamental group $\mathcal{B}_n = \pi_1(X)$, called *braid group* with n strands, is generated by $n-1$ elementary braids $\beta_{12}, \beta_{23}, \dots, \beta_{n-1,n}$ with the relations

$$\beta_{i,i+1}\beta_{j,j+1} = \beta_{j,j+1}\beta_{i,i+1} \quad \text{for } i+1 \neq j, j+1 \neq i,$$

$$\beta_{i,i+1}\beta_{i+1,i+2}\beta_{i,i+1} = \beta_{i+1,i+2}\beta_{i,i+1}\beta_{i+1,i+2}.$$

Equivalently, the braid group $\mathcal{B}_n$ can be defined as the mapping class group of a disk with n punctures, i.e. $\mathcal{B}_n \cong \mathfrak{M}(\mathbb{D}_n)$. Hence, any continuous deformation of the n-tuple $(u_1, \dots, u_n) \in \mathbb{C}^n \setminus \Delta$, represented as a deformation of n points in $\mathbb{C}$ never colliding, can be decomposed into *elementary* ones. In particular, an elementary deformation $\beta_{i,i+1}$ consists of a counter-clockwise rotation of u_i w.r.t. u_{i+1}, so that the two exchange. All other points u_j's are subjected to a sufficiently small perturbation, so that the corresponding Stokes' rays almost do not move. The braid $\beta_{i,i+1}$ corresponds to

- a clockwise rotation of the Stokes' ray $R_{i,i+1}$ crossing the line ℓ,
- or, dually, a counter-clockwise rotation of the line ℓ crossing the Stokes' ray $R_{i,i+1}$.

This determines the following mutations of the monodromy data, as shown in [Dub96] and [Dub99b]:

$$\mathbb{S}^{\beta_{i,i+1}} := A^{\beta_{i,i+1}}(\mathbb{S}) \cdot \mathbb{S} \cdot A^{\beta_{i,i+1}}(\mathbb{S})^T \tag{3.2}$$

where

$$\left(A^{\beta_{i,i+1}}(\mathbb{S})\right)_{hh} = 1 \quad h = 1, \dots, n \quad h \neq i, i+1$$

$$\left(A^{\beta_{i,i+1}}(\mathbb{S})\right)_{i+1,i+1} = -s_{i,i+1}$$

$$\left(A^{\beta_{i,i+1}}(\mathbb{S})\right)_{i,i+1} = \left(A^{\beta_{i,i+1}}(\mathbb{S})\right)_{i+1,i} = 1.$$

For a generic braid β, which is a product of N elementary braids $\beta = \beta_{i_1,i_1+1} \dots \beta_{i_N,i_N+1}$, the action is

$$\mathbb{S} \mapsto \mathbb{S}^\beta := A^\beta(\mathbb{S}) \cdot \mathbb{S} \cdot A^\beta(\mathbb{S})^T \tag{3.3}$$

where

$$A^\beta(\mathbb{S}) = A^{\beta_{i_N,i_N+1}}\left(\mathbb{S}^{\beta_{i_{N-1},i_{N-1}+1}}\right) \cdot \dots \cdot A^{\beta_{i_2,i_2+1}}\left(\mathbb{S}^{\beta_{i_1,i_1+1}}\right) \cdot A^{\beta_{i_1,i_1+1}}(\mathbb{S}).$$

The action on the central connection matrix (in lexicographical order) is

$$C \mapsto C^\beta := C \cdot (A^\beta)^{-1}. \tag{3.4}$$

Since a complete 2π-rotation of ℓ can be described by the action of a generator of $\mathbb{Z}$ on the monodromy data (as discussed in the previous Section), we have the following result.

Lemma 3.2 ([CDG17b]). *The braid corresponding to a complete counter-clockwise 2π-rotation of ℓ is the braid*

$$(\beta_{12}\beta_{23}\dots\beta_{n-1,n})^n,$$

and its acts on the monodromy data as follows:

- *trivially on Stokes matrices,*
- *the central connection matrix is transformed as $C \mapsto M_0^{-1}C$.*

4. Isomonodromy Theorem at coalescence points

Let $t_0 \in \mathcal{B}_M \setminus \mathcal{K}_M$ be a semisimple coalescence point of a Frobenius manifold M, and let Ω be a $\mathcal{K}_M$-untwisted open connected neighborhood of t_0, on which we fix an ordering of canonical coordinates $u\colon \Omega \to \mathbb{C}^n$ and a holomorphic branch of the function $\Psi\colon \Omega \to GL_n(\mathbb{C})$. We define

$$\Delta_\Omega := u(\Omega \cap \mathcal{B}_M), \qquad \text{so that} \quad \emptyset \neq \Delta_\Omega \subseteq \Delta \subseteq \mathbb{C}^n.$$

In particular $u^{(0)} := u(t_0) \in \Delta_\Omega$. Without loss of generality, up to a permutation of the canonical coordinates, we can assume that $u^{(0)}$ is

$$u_1^{(0)} = \dots = u_{p_1}^{(0)} =: \lambda_1 \tag{4.1}$$

$$u_{p_1+1}^{(0)} = \dots = u_{p_1+p_2}^{(0)} =: \lambda_2 \tag{4.2}$$

$$\vdots \tag{4.3}$$

$$u_{p_1+\dots+p_{s-1}+1}^{(0)} = \dots = u_{p_1+\dots+p_{s-1}+p_s}^{(0)} =: \lambda_s, \tag{4.4}$$

where p_1, ..., p_s are the multiplicities of the eigenvalues of $U(u^{(0)}) = \operatorname{diag}(u_1^{(0)}, ..., u_n^{(0)})$, with $s \leq n$, $p_1 + \dots + p_s = n$. Let

$$\delta_i := \frac{1}{2}\min\left\{\left|\lambda_i - \lambda_j + \rho e^{i\left(\frac{\pi}{2}-\phi\right)}\right|,\ j \neq i,\ \rho \in \mathbb{R}\right\},$$

and let ϵ_0 be a small positive number such that

$$\epsilon_0 < \min_{1\leq i\leq s} \delta_i. \tag{4.5}$$

We will assume that ϵ_0 is sufficiently small so that the neighborhood of $u^{(0)}$ defined by[2]

$$\mathcal{U}_{\epsilon_0}(u^{(0)}) := \bigtimes_{i=1}^{s} \overline{B}(\lambda_i; \epsilon_0)^{\times p_i}$$

is completely contained in the image of the chart $u(\Omega)$. Note that, for the choice of ϵ_0, if u varies in $\mathcal{U}_{\epsilon_0}(u^{(0)})$, the sets

$$I_1 := \{u_1,\ \dots, u_{p_1}\},\ I_2 := \{u_{p_1+1},\ \dots, u_{p_1+p_2}\},\ \dots,\ I_s := \{u_{p_1+\dots+p_{s-1}+1},\ \dots, u_{p_1+\dots+p_{s-1}+p_s}\} \tag{4.6}$$

do never intersect. Thus, $u^{(0)}$ is a point of *maximal coalescence* in $\mathcal{U}_{\epsilon_0}(u^{(0)})$. We will say that a coordinate u_a *is close to a* λ_j if it belongs to I_j. Equivalently, if $u_a \in \overline{B}(\lambda_j; \epsilon_0)$.

Let us fix an oriented line ℓ admissible for the point t_0, of slope $\phi \in [0; 2\pi[$. For $u = (u_1, ..., u_n) \in \mathcal{U}_{\epsilon_0}(u^{(0)})$, consider the subset $\mathfrak{R}(u)$ of Stokes rays associated to couples of eigenvalues u_a and u_b, such that u_a is close to a λ_i and u_b is closed to a λ_j, with $i \neq j$. The choice of ϵ_0 as in (4.5) is motivated

[2] Here $\overline{B}(\lambda_i; \epsilon_0)$ is the closed ball in $\mathbb{C}$ with center λ_i and radius ϵ_0. Note that if the uniform norm $|u| = \max_i |u_i|$ is used, as in [CDG17a], then

$$\mathcal{U}_{\epsilon_0}(u^{(0)}) = \left\{u \in \mathbb{C}^n \;\middle|\; |u - u^{(0)}| \leq \epsilon_0\right\}.$$

by the following fact: consider the pair u_a, u_b, and the associated Stokes ray R_{ab} (projection of $R_{ab,k} \in \mathfrak{R}(u)$). As u varies, the ray R_{ab} continuously move, but it never crosses the line ℓ as long as $u_a \in \overline{B}(\lambda_i;\epsilon_0)$ and $u_b \in \overline{B}(\lambda_j;\epsilon_0)$. See Figure 1.

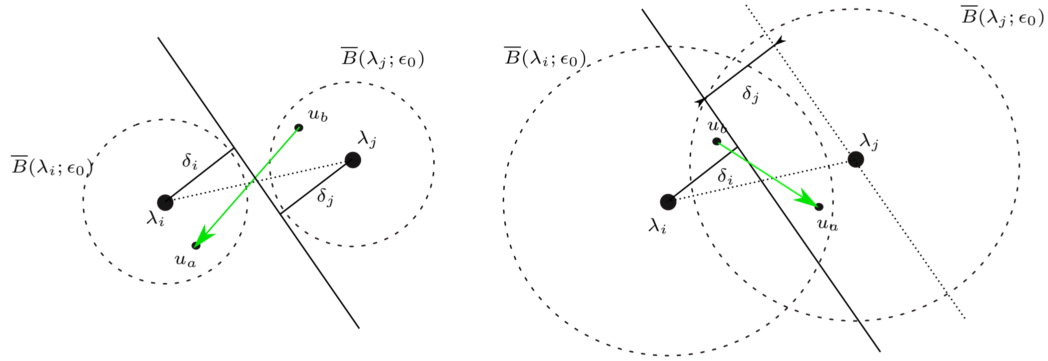


FIGURE 1. Points λ_i's and u_a's are represented in the same complex plane. The thick line has slope $\pi/2-\phi$. As u varies, for values of ϵ_0 sufficiently small (left figure) the Stokes rays R_{ab} associated to u_a in the disk $\overline{B}(\lambda_i;\epsilon_0)$ and u_b in the disk of $\overline{B}(\lambda_j;\epsilon_0)$ do not cross the line ℓ in the z-plane. If the disks have radius exceeding $\min_{1\le i\le s}\delta_i$ as in (4.5) (see right figure), then the Stokes rays R_{ab} cross the line ℓ.

The choice of the line ℓ induces a cell decomposition of $\mathcal{U}_{\epsilon_0}(u^{(0)})$.

Definition 4.1. An ℓ-cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$ is any connected component of the open dense subset of points $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$ such that $u_1,...,u_n$ are pairwise distinct and ℓ is admissible for the n-tuple $(u_1,\dots,u_n)$.

Proposition 4.1 ([CDG17a]). *An ℓ-cell is a topological cell, namely it is homeomorphic to a ball.*

For each point $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$, the monodromy and Stokes phenomenon at $z=\infty$ of the system

$$\frac{dY}{dz}=\left(U+\frac{V(u)}{z}\right)Y,\quad u\in\mathcal{U}_{\epsilon_0}(u^{(0)}),\tag{4.7}$$

is completely described by Theorems 2.5, 2.6, 2.7. In particular, we have well-defined Stokes and central connection matrices $\mathbb{S}(u), C(u)$ at each point, computed w.r.t.

- solutions $Y^{(0)}_{\text{right/left}}(z,u)$ as in Theorem 2.5 for $u \notin \Delta_\Omega \cap \mathcal{U}_{\epsilon_0}(u^{(0)})$,
- solutions $\mathring{Y}^{(0)}_{\text{right/left}}(z,u)$ as in Theorem 2.6 for $u \in \Delta_\Omega \cap \mathcal{U}_{\epsilon_0}(u^{(0)})$,

which are in both cases uniquely determined by their asymptotic expansion in sectors $\Pi_{\text{right/left}}(\phi)$ given by the *unique* formal solution

$$Y_{\text{formal}}(z,u)=\left(\mathbb{1}+\sum_{h=1}^{\infty}\frac{F_h(u)}{z^h}\right)\exp(zU),\quad F_m\in\mathcal{O}(\mathcal{U}_{\epsilon_0}(u^{(0)})),$$

$$Y_{\text{formal}}(z,u)\equiv\mathring{Y}_{\text{formal}}(z,u)\quad\text{if } u\in\Delta_\Omega\cap\mathcal{U}_{\epsilon_0}(u^{(0)}).$$

Note that actually, if for any $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$ we consider the sectors $\mathcal{S}_{\text{right}}(u)$ and $\mathcal{S}_{\text{left}}(u)$ which contain the sectors $\Pi_{\text{right}}(\phi)$ and $\Pi_{\text{left}}(\phi)$ respectively and extend up to the nearest Stokes rays, then for each $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$ the asymptotic expansion above holds on each $\mathcal{S}_{\text{right/left}}(u)$ sector. The central connection matrix $C(u)$ is somputed w.r.t. a fixed solution $Y_0(z,u)$ in Levelt-normal form at $z=0$ holomorphic in $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$.

Remarkably, by Theorem 2.8, on each ℓ-cell in $\mathcal{U}_{\epsilon_0}(u^{(0)})$ the system (4.7) is isomonodromic, so that $\mathbb{S}(u), C(u)$ do not depend on u varying in a ℓ-cell. The Main Theorem of [CDG17a], adapted and particularised to the case of Frobenius manifolds, becomes the following:

Theorem 4.1 ([CDG17a, CDG17b])**.** *Let $u^{(0)}$ be a colaescence point. Let $\mathcal{U}_{\epsilon_0}(u^{(0)})$ be as above. Let the manifold be semisimple in $\mathcal{U}_{\epsilon_0}(u^{(0)})$. Then*

- *The solutions $Y^{(0)}_{\rm left}(z,u)$, $Y^{(0)}_{\rm right}(z,u)$, can be u-analytically continued as single-valued holomorphic functions on $\mathcal{U}_{\epsilon_0}(u^{(0)})$. Moreover*
$$Y^{(0)}_{\rm left/right}(z,u^{(0)}) = \mathring{Y}^{(0)}_{\rm left/right}(z).$$
- *For any $\epsilon_1 < \epsilon_0$, the asymptotic relations*
$$Y^{(0)}_{\rm left/right}(z,u) \sim Y_{\rm formal}(z,u),\ \ z\to\infty \ \text{in}\ \Pi_{\rm left/right}(\phi), \tag{4.8}$$
hold uniformly in $u\in\mathcal{U}_{\epsilon_1}(u^{(0)})$. In particular they hold also at points of $\Delta_\Omega\cap\mathcal{U}_{\epsilon_1}(u^{(0)})$ and at $u^{(0)}$.
- *For any $u\in\mathcal{U}_{\epsilon_0}(u^{(0)})$ consider the sectors $\widehat{\mathcal{S}}_{\rm right}(u)$ and $\widehat{\mathcal{S}}_{\rm left}(u)$ which contain the sectors $\Pi_{\rm right}(\phi)$ and $\Pi_{\rm left}(\phi)$ respectively, and extend up to the nearest Stokes rays in the set $\mathfrak{R}(u)$ defined above. Let*
$$\widehat{\mathcal{S}}_{\rm left/right} = \bigcap_{u\,\in\,\mathcal{U}_{\epsilon_0}(u^{(0)})} \widehat{\mathcal{S}}_{\rm left/right}(u).$$
Observe that for sufficiently small $\varepsilon>0$ the sectors
$$\Pi^{\varepsilon}_{\rm right}(\phi) := \{z\in\mathcal{R}\colon \phi-\pi-\varepsilon<\arg\ z<\phi+\varepsilon\},$$
$$\Pi^{\varepsilon}_{\rm left}(\phi) := \{z\in\mathcal{R}\colon \phi-\varepsilon<\arg\ z<\phi+\pi+\varepsilon\},$$
are strictly contained in $\widehat{\mathcal{S}}_{\rm right}$ and $\widehat{\mathcal{S}}_{\rm left}$ respectively. The asymptotic relations (4.8) actually hold in the sectors $\widehat{\mathcal{S}}_{\rm left/right}$.
- *The system (4.7) is isomonodromic in $\mathcal{U}_{\epsilon_1}(u^{(0)})$: the monodromy data $(\mu,R,\mathbb{S},C)$ of system (4.7), defined and constant in any ℓ-cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$, are actually defined and constant at any $u\in\mathcal{U}_{\epsilon_1}(u^{(0)})$. The entries of $\mathbb{S}$ satisfy the vanishing condition*
$$\mathbb{S}_{ij}=\mathbb{S}_{ji}=0 \quad \text{for all } i\neq j \text{ such that} \quad u^{(0)}_i=u^{(0)}_j. \tag{4.9}$$

This Theorem allows us to obtain the monodromy data in a neighbourhood of a coalescence point just by computing them *at* the coalescence point.

Remark 4.1. Suppose that S is upper triangular. By formula (4.9), it follows that in any ℓ-cell of $\mathcal{U}_{\epsilon_1}(u^{(0)})$ the order of the canonical coordinates in triangular, according to Definition 3.1, and only in one cell the order is lexicographical (Definition 3.2). Starting from this cell we can apply the action of the braid group to S and C, as dictated by formulae (3.2), (3.4). In this way, the monodromy data for any other chamber of the manifold are obtained, as explained in Section 3.1.

5. Quantum cohomologies of complex Grassmannians.

5.1. **Gromov-Witten Theory and Quantum Cohomology.** Let X be a smooth projective complex variety with vanishing odd cohomology, i.e. $H^{2k+1}(X;\mathbb{C})\cong 0$ for $0\leq k$. Let us fix a homogeneous basis $(T_0,T_1,\dots,T_N)$ of $H^\bullet(X;\mathbb{C})=\bigoplus_k H^{2k}(X;\mathbb{C})$ such that

- $T_0=1$ is the unity of the cohomology ring;
- $T_1,\dots,T_r$ span $H^2(X;\mathbb{C})$.

We will denote by $\eta\colon H^\bullet(X;\mathbb{C})\times H^\bullet(X;\mathbb{C})\to\mathbb{C}$ the Poincaré metric

$$\eta(\xi,\zeta):=\int_X \xi\cup\zeta,\quad \text{with Gram matrix } \eta:=(\eta_{\alpha\beta})_{\alpha,\beta},\quad \eta_{\alpha\beta}:=\int_X T_\alpha\cup T_\beta.$$

If $\beta\in H_2(X;\mathbb{Z})/\text{torsion}$, we denote by $\overline{\mathcal{M}}_{g,n}(X,\beta)$ the Kontsevich-Manin moduli stack of n-pointed, genus g stable maps to X of degree β, which parametrizes equivalence classes of pairs $((C_g,\mathbf{x});f)$, where $(C_g,\mathbf{x})$ is an n-pointed algebraic curve of genus g, with at most nodal singularities and with n marked points $\mathbf{x}=(x_1,\dots,x_n)$, and $f\colon C_g\to X$ is a morphism such that $f_*[C_g]\equiv\beta$. Two pairs $((C_g,\mathbf{x});f)$ and $((C_g',\mathbf{x}');f')$ are defined to be equivalent if there exists a bianalytic map $\varphi\colon C_g\to C_g'$ such that $\varphi(x_i)=x_i'$, for all $i=1,\dots,n$, and $f'=\varphi\circ f$. The morphisms f are required to be *stable*: if f is constant on any irreducible component of C_g, then that component should have only a finite number of automorphisms as pointed curves (in other words, it must have at least 3 distinguished points, i.e. points that are either nodes or marked ones).

We will denote by $\mathrm{ev_i}\colon\overline{\mathcal{M}}_{g,n}(X,\beta)\to X\colon((C_g,\mathbf{x});f)\mapsto f(x_i)$ the naturally defined evaluations maps, and by $\psi_i\in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$ the Chern classes of tautological cotangent line bundles

$$\mathcal{L}_i\to\overline{\mathcal{M}}_{g,n}(X,\beta),\quad \mathcal{L}_i|_{((C_g,\mathbf{x});f)}=T^*_{x_i}C_g,\quad \psi_i:=c_1(\mathcal{L}_i).$$

Using the construction of [BF97] of a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\rm virt}$ in the Chow ring $A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$, and of degree equal to the expected dimension

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\rm virt}\in A_D(\overline{\mathcal{M}}_{g,n}(X,\beta)),\quad D=(1-g)(\dim_{\mathbb{C}}X-3)+n+\int_\beta c_1(X),$$

a good theory of intersection is allowed on the Kontsevich-Manin moduli stack.

We can thus define the *Gromov-Witten invariants (with descendants) of genus g, with n marked points and of degree β* of X as the integrals (whose values are rational numbers)

$$\langle\tau_{d_1}\gamma_1,\dots,\tau_{d_n}\gamma_n\rangle^X_{g,n,\beta}:=\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\rm virt}}\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\cup\psi_i^{d_i},\tag{5.1}$$

$$\gamma_i\in H^\bullet(X;\mathbb{C}),\quad d_i\in\mathbb{N},\quad i=1,\dots,n.$$

Since by *effectiveness* (see [Man99], [KM94]) the integral is non-vanishing only for effective classes $\beta\in\mathrm{Eff}(X)\subseteq H_2(X;\mathbb{Z})$, the generating function of rational numbers (5.1), called *total descendent potential* (or also *gravitational Gromov-Witten potential*, or even *Free Energy*) of *genus g* is defined as the formal series

$$\mathcal{F}_g^X(\gamma,\mathbf{Q}):=\sum_{n=0}^\infty\sum_{\beta\in\mathrm{Eff}(X)}\frac{\mathbf{Q}^\beta}{n!}\langle\underbrace{\gamma,\dots,\gamma}_{n\text{ times}}\rangle^X_{g,n,\beta},$$

where we have introduced (infinitely many) coordinates $\mathbf{t}:=(t^{\alpha,p})_{\alpha,p}$

$$\gamma=\sum_{\alpha,p}t^{\alpha,p}\tau_pT_\alpha,\quad \alpha=0,\dots,N,\ p\in\mathbb{N},$$

and formal parameters

$$\mathbf{Q}^\beta:=Q_1^{\int_\beta T_1}\cdot\dots\cdot Q_r^{\int_\beta T_r},\quad Q_i\text{'s elements of the Novikov ring }\Lambda:=\mathbb{C}[\![Q_1,\dots,Q_r]\!].$$

The free energy $\mathcal{F}_g^X\in\Lambda[\![\mathbf{t}]\!]$ can be seen as a function on the *large phase-space*, and restricting the free energy to the *small phase space* (naturally identified with $H^\bullet(X;\mathbb{C})$),

$$F_g^X(t^{1,0},\dots,t^{N,0}):=\mathcal{F}_g^X(\mathbf{t})|_{t^{\alpha,p}=0,\ p>0},$$

one obtains the generating function of the Gromov-Witten invariants of genus g.

By the Divisor axiom, the genus 0 Gromov-Witten potential $F_0^X(t)$, can be seen as an element of the ring $\mathbb{C}[\![t^0, Q_1e^{t^1}, \dots, Q_re^{t^r}, t^{r+1}, \dots, t^N]\!]$: in what follows we will be interested in cases in which F_0^X is the analytic expansion of an analytic function, i.e.

$$F_0^X \in \mathbb{C}\left\{t^0, Q_1e^{t^1}, \dots, Q_re^{t^r}, t^{r+1}, \dots, t^N\right\}.$$

Without loss of generality, we can put $Q_1 = Q_2 = \dots = Q_r = 1$, and $F_0^X(t)$ defines an analytic function in an open neighborhood $\mathcal{D} \subseteq H^\bullet(X;\mathbb{C})$ of the point

$$t^i = 0, \quad i = 0, r+1, \dots, N, \tag{5.2}$$

$$\operatorname{Re} t^i \to -\infty, \quad i = 1, 2, \dots, r. \tag{5.3}$$

The function F_0^X is a solution of WDVV equations (for a proof see [KM94], [Man99], [CK99]), and thus it defines an analytic Frobenius manifold structure on $\mathcal{D}$.

Definition 5.1 (Small and Big Quantum Cohomology, [Vaf91], [KM94], [Man99], [Dub96, Dub98, Dub99b])**.** The Frobenius manifold structure defined on the domain of convergence $\mathcal{D}$ of the Gromov-Witten potential F_0^X, solution of the WDVV problem, is called *Quantum Cohomology* of X, and denoted by $QH^\bullet(X)$. Note that

- the flat metric is given by the Poincaré metric η;
- the unity vector field is $T_0 = 1$, using the canonical identifications of tangent spaces

$$T_p\mathcal{D} \cong H^\bullet(X;\mathbb{C})\colon \partial_{t^\alpha} \mapsto T_\alpha;$$

- the Euler vector field is

$$E := c_1(X) + \sum_{\alpha=0}^N \left(1 - \frac{1}{2}\deg T_\alpha\right) t^\alpha T_\alpha.$$

By *small quantum cohomology* (or *small quantum locus*) we mean the Frobenius structure attached to points in $\mathcal{D} \cap H^2(X;\mathbb{C})$. In case of convergence, the domain $\mathcal{D}$ is non-empty and the potential (and hence the whole Frobenius structure) can be maximally analytically continued to an unramified covering of $\mathcal{D}$. We refer to this global Frobenius structure as the *Big Quantum Cohomology*.

There is an intriguing conjecture ([Dub98, Dub13]) which relates the enumerative geometry of a projective variety X with semisimple quantum cohomology to its derived category of coherent sheaves $\mathcal{D}^b(X)$. Conjecturally, the semisimplicity of the quantum cohomology $QH^\bullet(X)$ should be equivalent to the existence of a full exceptional collection in the derived category $\mathcal{D}^b(X)$, i.e. a collection of objects $(E_1, \dots, E_n)$ with the semi-orthogonal property

$$\operatorname{Hom}^\bullet(E_j, E_i) := \bigoplus_k \operatorname{Hom}_{\mathcal{D}^b(X)}(E_j, E_i[k]) = 0, \quad \text{for } j > i.$$

Furthermore, the monodromy data $(\mathbb{S}, C)$ should be expressed in terms of characteristic classes and Euler-Poincaré-Grothendieck products of the objects E_i's, i.e.

$$\chi(E_i, E_j) := \sum_k (-1)^k \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}^b(X)}(E_j, E_i[k]).$$

In particular, the Stokes matrix $\mathbb{S}$ computed in each ℓ-chamber of $QH^\bullet(X)$, and w.r.t. a triangular order, should correspond to the Gram matrix of χ w.r.t. an exceptional collection. Further details and a refined and precise version of this conjecture will be the content of a forthcoming paper ([CDG]).

5.2. **The case of complex Grassmannians.** Although there is no general result guaranteeing the convergence of the Gromov-Witten potential, in the case X is a Fano manifold (i.e. with ample anti-canonical bundle $-K_X \equiv \det TX$) it is known that the sum defining F_0^X at points with coordinates $t^0 = t^{r+1} = \dots = t^N = 0$, i.e. of the small quantum locus, is a polynomial (for a proof see [CK99]). Furthermore, for some homogeneous spaces such as Grassmannians $\mathbb{G}(k,n)$ of k-subspaces in $\mathbb{C}^n$ it is known that the points of the small quantum locus are all semisimple (see [CMP10]). Depending on the pair (k,n), the small quantum cohomology of $\mathbb{G}(k,n)$ may be contained (or not) in the semisimple coalescent stratum $\mathcal{B}_M \setminus \mathcal{K}_M$, $M := QH^\bullet(\mathbb{G}(k,n))$. In this case we will say that the Grassmannian $\mathbb{G}(k,n)$ is *coalescing*. Remarkably, this coalescence phenomenon of small quantum cohomologies of Grassmannians is strictly related to the prime factorization of n, and actually *almost all* Grassmannians are coalescing.

Theorem 5.1 ([Cot16])**.**

(1) *The complex Grassmannian $\mathbb{G}(k,n)$ is coalescing if and only if $P_1(n) \le k \le n - P_1(n)$, where $P_1(n)$ is the smallest prime divisor of n.*
(2) *In particular, if n is prime, the Grassmannians $\mathbb{G}(k,n)$ are not coalescing for any $0 < k < n$.*
(3) *Let us denote by $\tilde{\text{л}}_n$, for $n \ge 2$, the number of non-coalescing Grassmannians of proper subspaces of $\mathbb{C}^n$, i.e.*

$$\tilde{\text{л}}_n := \operatorname{card}\{k\colon 0<k<n, \quad \mathbb{G}(k,n) \text{ is not coalescing}\}.$$

We have that

$$\sum_{k=2}^{n} \tilde{\text{л}}_k \sim \frac{1}{2}\frac{n^2}{\log n}, \tag{5.4}$$

which means that non-coalescing Grassmannians are rare, *i.e. of density zero among all Grassmannians.*

Remark 5.1. In [Cot16], some properties of the analytic continuation of the generating Dirichlet series

$$\tilde{\text{Л}}(s) := \sum_{n=2}^{\infty} \frac{\tilde{\text{л}}_n}{n^s},$$

are partially studied: the global picture is still mysterious. $\tilde{\text{Л}}(s)$ is absolutely convergent in the half-plane $\operatorname{Re}(s) > 2$, where it can be represented by the infinite series

$$\tilde{\text{Л}}(s) = \sum_{p \text{ prime}} \frac{p-1}{p^s}\left(\frac{2\zeta(s)}{\zeta(s,p-1)} - 1\right),$$

involving the Riemann zeta function and the truncated Euler products defined for $h \in \mathbb{R}_{>0},\ s \in \mathbb{C}\setminus\{0\}$ as the functions

$$\zeta(s,h) := \prod_{\substack{p \text{ prime}\\ p \le h}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

By analytic continuation, $\tilde{\text{Л}}(s)$ can be extended to (the universal cover of) the punctured half-plane

$$\{s \in \mathbb{C}\colon\ \operatorname{Re}(s) > \overline{\sigma}\} \setminus \left\{ s = \frac{\rho}{k} + 1\colon \begin{array}{c} \rho \text{ pole or zero of } \zeta(s), \\ k \text{ squarefree positive integer} \end{array} \right\},$$

$$\overline{\sigma} := \limsup_{n\to\infty} \frac{1}{\log n} \cdot \log\left(\sum_{\substack{k \le n \\ k \text{ composite}}} \tilde{\text{л}}_k \right), \quad 1 \le \overline{\sigma} \le \frac{3}{2},$$

having logarithmic singularities at the punctures. In particular, we have the equivalence of the following statements:

- (RH) all non-trivial zeros of the Riemann zeta function $\zeta(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$;
- the derivative $\widetilde{\Pi}'(s)$ extends, by analytic continuation, to a meromorphic function in the half-plane $\frac{3}{2} < \mathrm{Re}(s)$ with a single pole of oder one at $s = 2$.

At the point $s = 2$ the following asymptotic estimate holds

$$\widetilde{\Pi}(s) = \log\left(\frac{1}{s-2}\right) + O(1), \quad s \to 2, \quad \mathrm{Re}(s) > 2,$$

from which it follows the asymptotic relation (5.4).

Thanks to Theorem 4.1, we know that the monodromy data for quantum cohomologies of Grassmannians can be computed at points of the small quantum locus, despite of coalescences of canonical coordinates. Furthermore, these data define local invariants and are constant in the ℓ-chambers encircling the small quantum cohomology. It is remarkable that the vanishing condition of Theorem 4.1 implies a constraint on the nature of the full exceptional collections that conjecturally should arise from the monodromy data according to [Dub98, Dub13]: in particular, if the canonical coordinates along the locus of small quantum cohomology of $\mathbb{G}(k,n)$ are as in equations (4.1)-(4.4) (and in triangular order), then the collections associated to the surrounding ℓ-chambers should have the structure

$$\mathcal{E} := (\underbrace{E_1,\dots,E_{p_1}}_{\mathcal{B}_1}, \underbrace{E_{p_1+1},\dots,E_{p_1+p_2}}_{\mathcal{B}_2},\dots,\underbrace{E_{p_1+\dots+p_{s-1}+1},\dots,E_{p_1+\dots+p_s}}_{\mathcal{B}_s}), \quad E_j \in \mathrm{Obj}\left(\mathcal{D}^b(X)\right),$$

where for each pair (E_i, E_j) in a same block $\mathcal{B}_k$ the following orthogonality conditions hold

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(E_i, E_j[\ell]) = 0, \quad \text{for any } \ell.$$

In particular, any reordering of the objects inside a single block $\mathcal{B}_j$ preserves the exceptionality of $\mathcal{E}$. Such a kind of exceptional collections, called *full s-block exceptional collections*, was firstly introduced and studied in [KN98]. More results about the nature of exceptional collections arising in this context and about their dispositions in the locus of small quantum cohomology for the class of complex Grassmannians will appear in a forthcoming paper [CDG].

Example 5.1. The first and simplest case where some coalescences of canonical coordinates manifest is the case of the complex Grassmannian $X := \mathbb{G}(2,4)$. Such a manifold is a complex 4-fold with Betti numbers

$$\beta_0(X) = \beta_2(X) = \beta_6(X) = \beta_8(X) = 1, \quad \beta_4(X) = 2.$$

Let us fix the Schubert basis $(\sigma_0, \sigma_1, \sigma_2, \sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2})$ of $H^\bullet(X,\mathbb{C})$ (see [GH78], [Man98]), and let us denote by $(t^1, \dots t^6)$ the corresponding flat coordinates of the six dimensional Frobenius manifold $QH^\bullet(X)$. The Frobenius structure is explicitly known only along the small quantum locus (see [Wit95], [Ber96, Ber97], [Buc03], [Cot16]): if $p = t^2\sigma_1 \in H^2(X,\mathbb{C})$, the corresponding Frobenius algebra defined on the tangent space is given by

$$QH^\bullet_p(X) \cong \frac{\mathbb{C}[x_1,x_2]^{\mathfrak{S}_2}[q]}{\langle h_3, h_4 + q\rangle}, \quad \sigma_\lambda = \frac{\begin{vmatrix} x_1^{\lambda_1+1} & x_1^{\lambda_2} \\ x_2^{\lambda_1+1} & x_2^{\lambda_2}\end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1\end{vmatrix}}, \quad q := \exp(t^2),$$

where x_1, x_2 are the Chern roots of the dual-tautological bundle $\mathcal{S}^*$ on X, and h_i is the i-th complete symmetric polynomial in x_1, x_2. The differential system defining *deformed flat 1-form* $\widehat{\nabla}\xi = 0$, with $\xi := \xi_j(z,t)dt^j$, and which defines the monodromy data of $QH^\bullet(X)$, is

$$\partial_z\xi_1 = 4\xi_2 + \frac{2}{z}\xi_1, \tag{5.5}$$
$$\partial_z\xi_2 = 4(\xi_3+\xi_4) + \frac{1}{z}\xi_2, \tag{5.6}$$
$$\partial_z\xi_3 = 4\xi_5, \tag{5.7}$$
$$\partial_z\xi_4 = 4\xi_5, \tag{5.8}$$
$$\partial_z\xi_5 = 4q\xi_1 + 4\xi_6 - \frac{1}{z}\xi_5, \tag{5.9}$$
$$\partial_z\xi_6 = 4q\xi_2 - \frac{2}{z}\xi_6. \tag{5.10}$$

As shown in [CDG17b], the study of the whole system can be reduced to the study of the quantum differential equation

$$\vartheta^5\Phi(w) - 1024w^4\vartheta\Phi(w) - 2048w^4\Phi(w) = 0, \quad \vartheta := w\frac{d}{dw}.$$

By using the Mellin transform, two solutions of the equation above can be found

$$\Phi_1(w) := \frac{1}{2\pi i}\int_\Lambda \frac{\Gamma(s)^5}{\Gamma(s+\frac{1}{2})}4^{-s}w^{-4s}ds, \quad \Phi_2(w) = \frac{1}{2\pi i}\int_\Lambda \Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}4^{-s}w^{-4s}ds,$$

where $\Lambda := c + i\mathbb{R}$, with $0 < c < \frac{1}{2}$, and it is possible to reconstruct both $\Xi_{\text{left/right}}$ for the system (5.5)-(5.10), w.r.t. a line ℓ of slope $0 < \phi < \frac{\pi}{6}$. See [CDG17b] for explicit formulæ of reconstruction of the solutions, and detailed asymptotic analysis. The Stokes matrix computed w.r.t. chosen oriented line ℓ at the point $t^2 = 0$ is given by

$$\Xi_{\text{left}}(z,0) = \Xi_{\text{right}}(z,0)\cdot\mathbb{S}, \qquad \mathbb{S} := \begin{pmatrix} 1 & 6 & -20 & 20 & -70 & 20 \\ 0 & 1 & -4 & 4 & -16 & 6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.11}$$

The central connection matrix computed at $t^2 = 0$ w.r.t. the *topological solution* of the system (5.5)-(5.10) (see [Dub92, Dub96, Dub99b, CDG17b])

$$\Xi_0(z,t) := \eta\cdot\Theta_{top}(z,t)\cdot z^{\mu}z^{c_1(X)\cup},$$

$$\Theta_{top}(z,t)^\gamma_\lambda := \delta^\gamma_\lambda + \sum_{k,n=0}^{\infty}\sum_{\beta\in\text{Eff}(X)}\sum_{\alpha_1,\dots,\alpha_k}\frac{h^\gamma_{\lambda,k,n,\beta,\underline{\alpha}}}{k!}\cdot t^{\alpha_1}\dots t^{\alpha_k}\cdot z^{n+1},$$

$$h^\gamma_{\lambda,k,n,\beta,\underline{\alpha}} := \sum_\nu \eta^{\nu\gamma}\int_{[\overline{\mathcal{M}}_{0,k+2}(X,\beta)]^{\text{virt}}} c_1(\mathcal{L}_1)^n\cup \text{ev}_1^*\sigma_\lambda\cup\text{ev}_2^*\sigma_\nu\cup\prod_{j=1}^k \text{ev}_{j+2}^*\sigma_{\alpha_j},$$

coincides with the matrix associated to the morphism

$$\mathit{\Pi}^-_X\colon K_0(X)\otimes_{\mathbb{Z}}\mathbb{C}\to H^\bullet(X,\mathbb{C})\colon E\mapsto \frac{1}{(2\pi)^2}\widehat{\Gamma}^-_X\cup e^{-c_1(X)\pi i}\cup \text{Ch}(E),$$

where

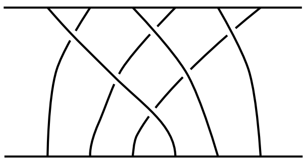
$$\widehat{\Gamma}^-_X := \prod_{j=1}^4\Gamma(1-\delta_j), \quad \Gamma(1-z) = \exp\left\{\gamma z + \sum_{n=2}^\infty\frac{\zeta(n)}{n}z^n\right\}, \quad \delta_j\text{'s are the Chern roots of } TX.$$

Such a matrix is computed w.r.t. the Schubert basis and the basis in the Grothendieck group $([E_1],\dots,[E_6])$, obtained by projection in K-theory of a full 5-block exceptional collection $(E_1,\dots,E_6)$.

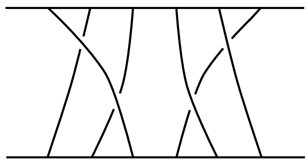
Such a collection is obtained from the Kapranov exceptional 5-block collection

$$\left(\mathbb{S}^0\mathcal{S}^*, \quad \mathbb{S}^1\mathcal{S}^*, \quad \begin{matrix}\mathbb{S}^2\mathcal{S}^*\\ \mathbb{S}^{1,1}\mathcal{S}^*\end{matrix}, \quad \mathbb{S}^{2,1}\mathcal{S}^*, \quad \mathbb{S}^{2,2}\mathcal{S}^*\right),$$

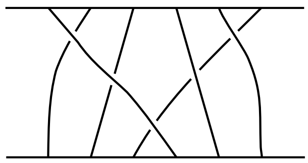
(here $\mathbb{S}^k$ denotes the k-th Schur functor, and $\mathcal{S}$ the tautological bundle on X, see [Kap88]) by mutation[3] under the inverse of the following braids in $\mathcal{B}_6$:



$\beta_{34}\beta_{12}\beta_{56}\beta_{23}\beta_{45}\beta_{34}$



$\beta_{12}\beta_{56}\beta_{23}\beta_{45}$



$\beta_{12}\beta_{56}\beta_{23}\beta_{45}\beta_{34}$

Note that the action of the braid β_{34} only consists in a permutation of the central objects of the 5-block collection. The inverse of the Stokes matrix $\mathbb{S}$ in (5.11) coincides with the Gram matrix of the Euler-Poincaré-Grothendieck product on $K_0(X)\otimes\mathbb{C}$ w.r.t. the basis $([E_1],\dots,[E_6])$. Theorem 4.1 guarantees that the monodromy data computed along the small quantum locus of $\mathbb{G}(2,4)$ coincide with the data of the surrounding chambers in the big quantum cohomology. The data in all other chambers of the Frobenius manifolds can be reconstructed through the action of the braid group, as exposed in Remark 4.1.

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