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A mathematical analysis of a one-dimensional model for dynamic debonding.

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CONTENTS

Abstract

In this thesis we develop a mathematical analysis for a dynamic model of peeling test in dimension one. In the first part we give existence and uniqueness results for dynamic evolutions. In the second part we study the quasistatic limit of such evolutions, i.e., the limit as inertia tends to zero.

In the model the wave equation $u_{tt} - u_{xx} = 0$ is coupled with a Griffith's criterion for the propagation of the debonding front. Our first results provide existence and uniqueness for the solution to this coupled problem under different assumptions on the data. This analysis is extended when we study the initiation of the debonding process. We also give an existence and uniqueness result for solutions to the damped wave equation $u_{tt} - u_{xx} + u_t = 0$ in a time-dependent domain whose evolution depends on the given debonding front.

We then analyse the quasistatic limit without damping. We find that the limit evolution satisfies a stability condition; however, the activation rule in Griffith's (quasistatic) criterion does not hold in general, thus the limit evolution is not rate-independent. This behaviour is due to the oscillations of the kinetic energy and of the presence of an acceleration term in the limit. The same phenomenon is observed even in the case of a singularly perturbed second order equation $\varepsilon^2 \ddot{u}_\varepsilon + V_x(t, u_\varepsilon(t)) = 0$, where $V(t, x)$ is a potential. We assume that $u_0(t)$ is one of its equilibrium points such that $V_x(t, u_0(t)) = 0$ and $V_{xx}(t, u_0(t)) > 0$. We find that, under suitable initial data, the solutions u_ε converge uniformly to u_0, by imposing mild hypotheses on V. However, a counterexample shows that such assumptions cannot be weakened. Thus, inertial effects can not, in general, be captured by a pure quasistatic analysis.

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A mathematical analysis of a one-dimensional model for dynamic debonding.

Lorenzo Nardini

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Introduction

In this thesis we analyse a one-dimensional model of dynamic peeling test. This model is one of the few cases where a complete mathematical analysis can be performed in the dynamical case, i.e., when the momentum equation includes an inertial term.

In the first part of the thesis we introduce and analyse the model of dynamic peeling test, giving existence and uniqueness results; the second part focuses on the quasistatic limit, i.e., the limit as inertia tends to zero and internal oscillations are neglected. We show that the kinetic energy plays a relevant role even if the inertia is very small and we therefore call into question the validity of the quasistatic assumption.

Existence and uniqueness of solutions for a dynamic peeling test. The study of crack growth based on Griffith's criterion has become of great interest in the mathematical community. The starting point was the seminal paper [29], where a precise variational scheme for the quasistatic evolution was proposed. This strategy has been exploited under different hypotheses in [26, 13, 30, 19, 22, 42]. The approximation of brittle crack growth by means of phase-field models in the quasistatic regime has been studied in [33]. A comprehensive presentation of the variational approach to quasistatic fracture mechanics can be found in [9]. For the relationships between this approach and the general theory of rate-independent systems we refer to the recent book [54].

In the dynamic case no general formulation has been yet proposed and only preliminary results are available (see [57, 20, 24, 21]). A reasonable model for dynamic fracture should combine the equations of elasto-dynamics for the displacement u out of the crack with an evolution law which connects the crack growth with u. The only result in this direction, without strong geometrical assumptions on the cracks, has been obtained for a phase-field model [43], but the convergence of these solutions to a brittle crack evolution has not been proved in the dynamic case. In the latter model the equation of elasto-dynamics for u is coupled with a suitable minimality condition for the phase-field ζ at each time. Other models in materials science, dealing with damage or delamination, couple a second order hyperbolic equation for a function u with a first order flow rule for an internal variable ζ (see, e.g., [32, 8, 7, 58, 59, 37, 36] for viscous flow rules on ζ and [62, 64, 63, 60, 65, 6, 66, 49] for rate-independent evolutions of ζ).

In this work we contribute to the study of dynamic fracture by analysing a simpler one-dimensional model already considered in [31, Section 5.5.1]. This model exhibits some of the

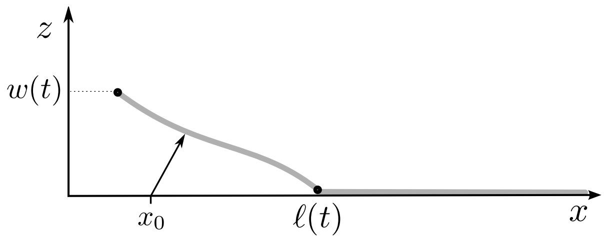

Figure 1: Peeling test.

relevant mathematical difficulties due to the time dependence of the domain of the wave equation. More precisely, following [28, 48] we study a model of a dynamic peeling test for a thin film, initially attached to a planar rigid substrate; the process is assumed to depend only on one variable. This hypothesis is crucial for our analysis, since we frequently use d'Alembert's formula for the wave equation.

The film is described by a curve, which represents its intersection with a vertical plane with horizontal coordinate x and vertical coordinate y. The positive x-axis represents the substrate as well as the reference configuration of the film. In its deformed configuration the film at time $t \geq 0$ is parametrised by $(v(t,x), u(t,x))$, where $v(t,x)$ (resp. $u(t,x)$) is the horizontal (resp. vertical) displacement of the point with reference coordinate x. The film is assumed to be perfectly flexible, inextensible, and glued to the rigid substrate on the half line $\{x \geq \ell(t)\,,\, y=0\}$, where $\ell(t)$ is a nondecreasing function which represents the debonding front, with $\ell_0 := \ell(0) > 0$. This implies $v(t,x) = u(t,x) = 0$ for $x \geq \ell(t)$. At the end point $x = 0$ we prescribe a vertical displacement $u(t,0) = w(t)$ depending on time $t \geq 0$, and a fixed tension so that the speed of sound in the film is constant. Using the linear approximation and the inextensibility it turns out that v can be expressed in terms of u as

$$v(t,x) = \frac{1}{2}\int_x^{+\infty} u_x(t,z)^2 \,\mathrm{d}z,$$

and u solves the problem

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad t > 0\,,\, 0 < x < \ell(t), \tag{0.1a}$$
$$u(t,0) = w(t), \quad t > 0, \tag{0.1b}$$
$$u(t,\ell(t)) = 0, \quad t > 0, \tag{0.1c}$$

where we normalised the speed of sound to one. The system is supplemented by the initial conditions

$$u(0,x) = u_0(x), \quad 0 < x < \ell_0, \tag{0.1d}$$
$$u_t(0,x) = u_1(x), \quad 0 < x < \ell_0, \tag{0.1e}$$

where u_0 and u_1 are prescribed functions.

Notice that the peeling test is closely related to fracture. The debonded part of the film, here parametrised on the interval $(0,\ell(t))$, corresponds to the uncracked part of a body subject to fracture; both domains are monotone in time, though in opposite directions, increasing in our case, decreasing in the fracture problem. The debonding propagation $t \mapsto \ell(t)$ corresponds to the evolution of a crack tip. The debonding front $\ell(t)$ has the role of a free boundary just as a crack. However, notice that cracks are discontinuity sets for the displacement, where a homogeneous Neumann condition is satisfied since they are traction free; in contrast, in the peeling test the displacement is continuous at $\ell(t)$ because of the Dirichlet constraint (0.1c): the debonding front is a discontinuity set for the displacement derivatives and represents a free boundary between $\{x : u(x, s) = 0 \text{ for every } s \leq t\}$ and $\{x : u(x, s) \neq 0 \text{ for some } s \leq t\}$.

The first result of Chapter 1 is that, under suitable assumptions on the functions u_0, u_1, ℓ, and w, problem (0.1) has a unique solution $u \in H^1$, with the boundary and initial conditions intended in the sense of traces (cf. Theorem 1.8). In particular, we always assume that $\dot{\ell} < 1$, which means that the debonding speed is less than the speed of sound.

In order to prove this theorem, we observe that, by d'Alembert's formula, u is a solution of (0.1a)&(0.1b) if and only if

$$u(t, x) = w(t+x) - f(t+x) + f(t-x), \tag{0.2}$$

for a suitable function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$. Moreover, the boundary condition (0.1c) is satisfied if and only if

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)). \tag{0.3}$$

Using this formula, together with the monotonicity and continuity of ℓ, we can determine the values of $f(s)$ for $s \in [-\ell_0, t + \ell(t)]$ from the values of $f(s)$ for $s \in [-\ell_0, t - \ell(t)]$.

It is easy to see (see Proposition 1.6) that (0.2) implies that f is uniquely determined on $[-\ell_0, \ell_0]$ by the initial conditions u_0 and u_1 through an explicit formula (see (1.19)). If s_1 is the unique time such that $s_1 - \ell(s_1) = \ell_0$, formula (0.3) allows us to extend f to the interval $[-\ell_0, s_1+\ell(s_1)]$. Then, we consider the unique time s_2 such that $s_2-\ell(s_2) = s_1+\ell(s_1)$ and, using again formula (0.3), we are able to extend f to $[-\ell_0, s_2 + \ell(s_2)]$. In this way we can construct recursively a sequence s_n such that f is extended to $[-\ell_0, s_n + \ell(s_n)]$ and (0.3) holds for every $0 \leq t \leq s_n$. Since it is easy to see that $s_n \to +\infty$, we are able to extend f to $[-\ell_0, +\infty)$ in such a way that (0.3) holds for every $t > 0$. This contruction allows us also to obtain the expected regularity for u from our hypotheses on u_0, u_1, ℓ, and w, see Figure 2.

In the second part of Chapter 1 only u_0, u_1, and w are given while the evolution of the debonding front ℓ has to be determined on the basis of an additional energy criterion.

To formulate this criterion we fix once and for all the initial conditions u_0 and u_1 and we consider the internal energy of u as a functional depending on ℓ and w. More precisely,

$$\mathcal{E}(t; \ell, w) := \frac{1}{2} \int_0^{\ell(t)} u_x(t, x)^2 \,\mathrm{d}x + \frac{1}{2} \int_0^{\ell(t)} u_t(t, x)^2 \,\mathrm{d}x,$$

where u is the unique solution corresponding to u_0, u_1, ℓ, and w; the first term is the potential energy and the second one is the kinetic energy.

A crucial role is played by the dynamic energy release rate, which is defined as a (sort of) partial derivative of $\mathcal{E}$ with respect to the elongation of the debonded region. More precisely, to define the dynamic energy release rate $G_\alpha(t_0)$ at time t_0 corresponding to a speed $0 < \alpha < 1$ of the debonding front, we modify the debonding front ℓ and the vertical displacement w using the functions

$$\lambda(t) = \begin{cases} \ell(t), & t \leq t_0, \\ (t - t_0)\alpha + \ell(t_0), & t > t_0, \end{cases} \qquad z(t) = \begin{cases} w(t), & t \leq t_0, \\ w(t_0), & t > t_0, \end{cases}$$

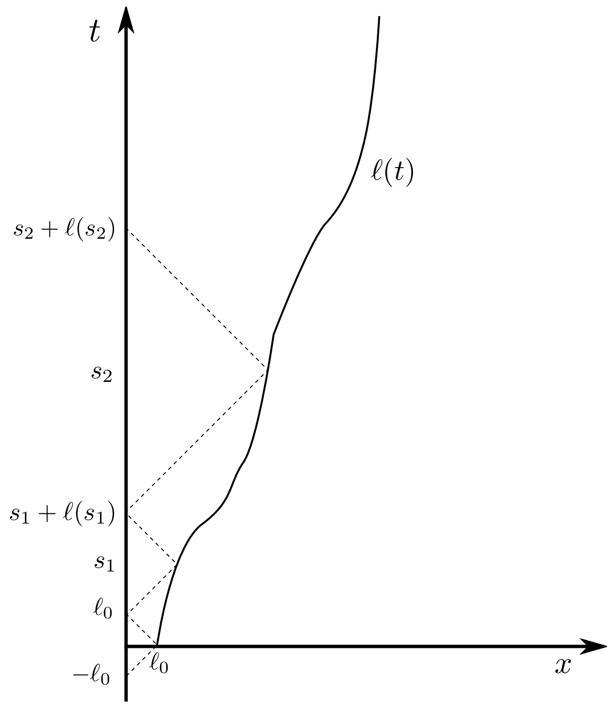

Figure 2: Extension of f to the interval $[-\ell_0, s_2+\ell(s_2)]$.

and we set

$$G_\alpha(t_0) := \lim_{t\to t_0^+} \frac{\mathcal{E}(t_0;\lambda,z)-\mathcal{E}(t;\lambda,z)}{(t-t_0)\alpha}.$$

We prove in Proposition 1.15 that, given ℓ and w, the limit above exists for a.e. $t_0>0$ and for every $\alpha\in(0,1)$. Moreover, we prove that

$$G_\alpha(t_0) = 2\frac{1-\alpha}{1+\alpha}\dot{f}(t_0-\ell(t_0))^2, \tag{0.4}$$

where f is the function which appears in (0.2). This formula shows, in particular, that $G_\alpha(t_0)$ depends only on α and on the values of $u(t,x)$ for $t\le t_0$ (see the discussion which leads to (1.35)).

We assume that the energy dissipated to debond a segment $[x_1,x_2]$, with $0\le x_1<x_2$ is given by

$$\int_{x_1}^{x_2}\kappa(x)\,\mathrm{d}x,$$

where $\kappa\colon[0,+\infty)\to(0,+\infty)$ represents the local toughness of the glue between the film and the substrate. At this stage, we assume that it depends only on the position, while in Sections 1.4 and 4.4 we consider the case in which κ depends also on the debonding speed $\dot{\ell}$.

In Section 1.2, starting from a maximum dissipation dissipation principle, we prove that the debonding front must satisfy the following energy criterion, called Griffith's criterion:

$$\dot\ell(t) \geq 0, \tag{0.5a}$$

$$G_{\dot\ell(t)}(t) \leq \kappa(\ell(t)), \tag{0.5b}$$

$$\dot\ell(t)\left[G_{\dot\ell(t)}(t) - \kappa(\ell(t))\right] = 0, \tag{0.5c}$$

for a.e. $t > 0$. The first condition asserts that the debonding can only grow (unidirectionality). The second condition states that the dynamic energy release rate is always bounded by the local toughness, while, accordingly to the third one, ℓ can increase with positive speed at t only when the dynamic energy release rate is critical at t, i.e., $G_{\dot\ell(t)}(t) = \kappa(\ell(t))$.

The main results of Chapter 1 are Theorems 1.18, 1.21, and 1.22, where we show existence and uniqueness of the solution (u, ℓ) to the coupled problem (0.1)&(0.5) under various assumptions on the data.

The strategy for the proof of these results is to write (0.5) as an ordinary differential equation for ℓ depending on the unknown function f. More precisely, starting from (0.4) we find that (0.5) is equivalent to

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, & \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{0.6}$$

As observed above, f is uniquely determined on the interval $[-\ell_0, \ell_0]$ by the initial conditions u_0 and u_1. Therefore, we can solve (0.6) in a maximal interval $[0, s_1]$, where s_1 is the unique point such that $s_1 - \ell(s_1) = \ell_0$. We can now apply formula (0.3) to extend f to the interval $[-\ell_0, s_1 + \ell(s_1)]$. Then we can extend the solution ℓ of (0.6) to a larger interval $[0, s_2]$, where s_2 is the only point such that $s_2 - \ell(s_2) = s_1 + \ell(s_1)$. Arguing recursively, we can find $f\colon [-\ell_0, +\infty) \to \mathbb{R}$ and $\ell\colon [0, +\infty) \to [0, +\infty)$ such that (0.6) is satisfied in $[0, +\infty)$, see Figure 3. The three Theorems 1.18, 1.21, and 1.22 consider different assumptions on u_0, u_1, w, and κ, which require different techniques to solve the differential equation (0.6).

These results are then used in the second part of this thesis to study the limit of (a rescaled version of) the solutions, as the speed of external loading tends to zero. In particular we examine the relationships between these limits and different notions of quasistatic evolution.

The case of a speed-dependent local toughness. A generalisation of the model analysed in this thesis is to consider a wider class of local toughnesses. In particular, we take into account a dependence of κ on the debonding speed $\dot\ell$, as mentioned in works by L. B. Freund (see e.g. [31]).

In Section 1.4, we will consider the problem of existence and uniqueness of a pair (u, ℓ), where u is a solution of (0.1) and ℓ satisfies

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t), \dot\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t), \dot\ell(t))} \vee 0, & \text{for a.e. } t > 0, \\ \ell(0) = \ell_0, \end{cases} \tag{0.7}$$

that is equivalent to Griffith's criterion

$$\begin{cases} \dot\ell(t) \geq 0, \\ G_{\dot\ell(t)}(t) \leq \kappa(\ell(t), \dot\ell(t)), \\ \dot\ell(t)\left[G_{\dot\ell(t)}(t) - \kappa(\ell(t), \dot\ell(t))\right] = 0. \end{cases}$$

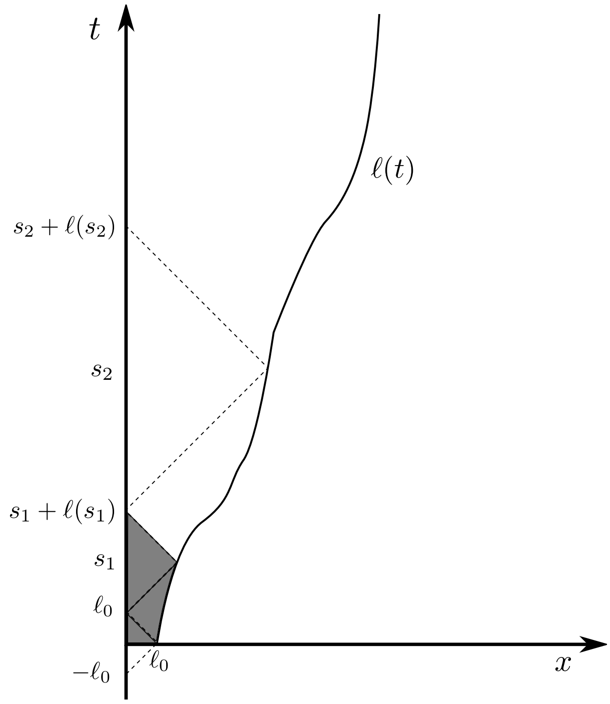

Figure 3: Construction of the pair (ℓ, f) with the iterative scheme. The knowledge of f in $[-\ell_0, s_1 + \ell(s_1)]$ gives the solution u to the wave equation in the grey area. Then, one solves (0.6) to extend ℓ to $[0, s_2]$.

Our motivation is two-fold: on the one hand we analyse a more realistic model for the peeling test, on the other we take into account the effects of a viscous term in Griffith's criterion. In fact, we will consider a local toughness κ of the form given by, e.g. $\kappa = c + \alpha\dot\ell$, with $\alpha > -2$, or $\kappa = \tilde\kappa\frac{1+\dot\ell}{1-\dot\ell}$, with $\tilde\kappa$ depending only on the position: in both cases we have the presence of a viscous term in the equation $G_{\dot\ell(t)} = \kappa(\ell(t), \dot\ell(t))$.

Notice that the Cauchy problem (0.7) is not expressed in normal form. Moreover, (0.7) is equivalent to $\Phi = 0$, where

$$\Phi(t, z, \mu) := \begin{cases} \mu - \dfrac{2\dot f(z)^2 - \kappa(t-z,\mu)}{2\dot f(z)^2 + \kappa(t-z,\mu)}, & \text{if } 2\dot f(z)^2 \geq \kappa(t-z,\mu), \\ \mu, & \text{if } 2\dot f(z)^2 < \kappa(t-z,\mu). \end{cases}$$

Our strategy then relies on inverting the function Φ assuming a growth condition from below on $\mu \mapsto \kappa(x, \mu)$. By also assuming Lipschitz regularity for κ, we find in Theorem 1.26 existence and uniqueness for a solution (u, ℓ) to the coupled problem (0.1) & (0.7).

The study of the quasistatic limit is then considered in the second part of this thesis.

The initiation problem. The problem of the initiation of the debonding is considered in Chapter 2. The analogous problem in fracture mechanics of crack initiation has been considered in [14, 9]. In this case we have $\ell_0 = 0$ so that the evolution of the debonding front is given by $\ell\colon [0, +\infty) \to [0, +\infty)$. Then, the vertical displacement u solves the problem (0.1a)–(0.1c) without initial conditions.

In the case where $t \mapsto \ell(t)$ is given, $u \in H^1$ is again represented by means of d'Alembert's formula (0.2) through a function $f\colon [0, +\infty) \to [0, +\infty)$. Moreover, by iterating several times (0.3), we find that f satisfies

$$f(s) = \lim_n \left[\sum_{k=0}^{n-1} w(\omega^k(s)) + f(\omega^n(s))\right], \tag{0.8}$$

where $\omega(s) := \varphi(\psi^{-1}(s)))$, $\varphi(s) := s - \ell(s)$, $\psi(s) := s + \ell(s)$, and the power k represents the composition of ω with itself k times, see Figure 4.

In order to find $u \in H^1$ solution to (0.1a)–(0.1c) with $\ell_0 = 0$, the sum in (0.8) needs to be finite. This happens for instance in the trivial case where the evolution of the debonding front is given by a straight line $\ell(t) = pt$ and w is affine. In this case we find

$$u(t, x) = w(t)(t - \tfrac{x}{p}) \quad \text{and} \quad f(t) = \frac{1+p}{2p} w(t).$$

However, we find that, if we only look for solutions in H^1_{loc}, then even in this case there is no uniqueness. See Section 2.1.1).

This example motivate us to look again for solution in H^1 and to assume a constraint for the evolution of the debonding:

$$0 < c_0 \leq \dot\ell(t) < 1, \tag{0.9}$$

in a neighbourhood of the origin. Under this condition, we find an upper bound for the sum in (0.8). Then, in Theorem 2.1 we find existence and uniqueness of solutions $u \in H^1$ to problem (0.1a)–(0.1c).

We next consider the problem where the evolution of the debonding front is unknown. In order to guarantee that the solution ℓ satisfies condition (0.9), we give restrictive assumptions

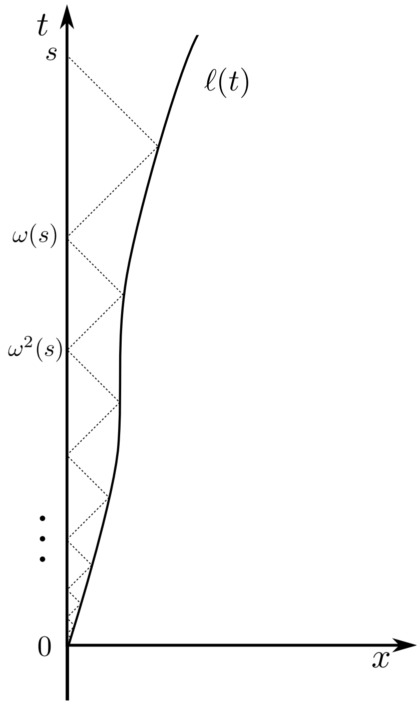

Figure 4: Iteration of the "bounce formula" (0.3).

on the regularity and on the growth of κ, w, and some of their derivatives. Then, the idea is to fix $\delta > 0$ and to consider a problem in which κ and $\dot{w}$ are constant in $[0, \delta]$. In this case, we find that there exists a unique pair (u^δ, ℓ^δ) solving the coupled problem, where ℓ is chosen among the functions that are linear in $[0, \delta]$. More precisely, there exists a unique $p_\delta \in (0, 1)$ such that, setting $\ell^\delta(t) = p_\delta t$ and $u_\delta = w(t)(t - x/p_\delta)$, then (u_δ, ℓ_δ) solves (0.1c) & (0.6) for $t, x \leq \delta$. Next, we extend (u^δ, ℓ^δ) to $[0, +\infty)$ using Thereom 1.22 and the values at time δ as initial data. The problem is then to let $\delta \to 0$. Using the additional conditions on κ and w, we find that the functions are equi-bounded in $W^{2,\infty}$. This allows us to pass to the limit in δ and to find a pair (u, ℓ) solution to problem (0.1a)–(0.1c) coupled with (0.6). Notice that we are unable to prove uniqueness.

Existence and uniqueness for the damped peeling test. In Chapter 3 we consider another model for the peeling test in which the wave equation for the vertical displacement u is damped. More precisely, we replace problem (0.1) with the following one:

$$u_{tt}(t, x) - u_{xx}(t, x) + u_t(t, x) = 0, \quad t > 0\,,\ 0 < x < \ell(t), \tag{0.10a}$$

$$u(t, 0) = w(t), \quad t > 0, \tag{0.10b}$$

$$u(t, \ell(t)) = 0, \quad t > 0, \tag{0.10c}$$

$$u(0, x) = u_0(x), \quad 0 < x < \ell_0, \tag{0.10d}$$

$$u_t(0, x) = u_1(x), \quad 0 < x < \ell_0, \tag{0.10e}$$

where $t \mapsto \ell(t)$ is given. The time derivative of u is a damping of the internal oscillations due to friction between the film and the surrounding air. This equation is sometimes referred to as the damped wave equation or telegraph equation.

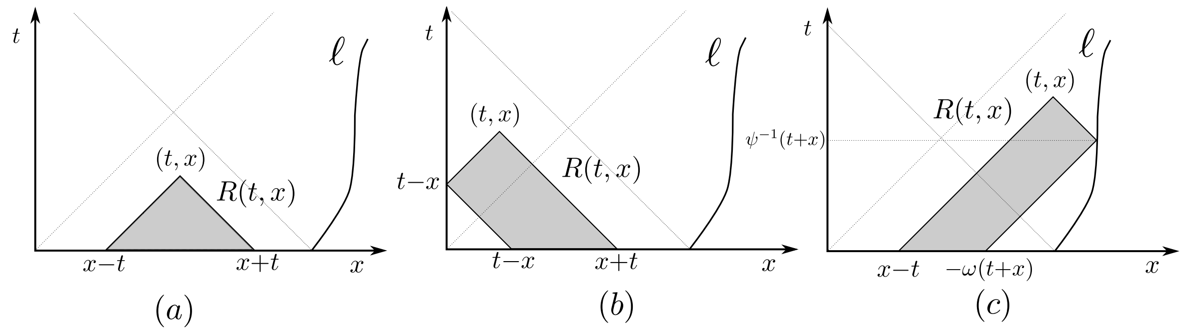

Figure 5: The set $R(t,x)$ in three typical cases. See Chapter 3.

Here, we only consider the problem of finding the vertical displacement u when $t \mapsto \ell(t)$ is a given Lipschitz function with $0 \le \dot\ell(t) < 1$ and $\ell(0) =: \ell_0 > 0$, to generalise Theorem 1.8. The main difficulty is that we are unable to find a d'Alembert's representative of u as in (0.2). Indeed, by the presence of the damping term, $u(t,x)$ is not just an explicit combination of the data w, u_0, u_1, and ℓ of the problem, but it also depends on the integration of u_t on the cone of dependence $R(t,x)$ (see Fig. 5). Specifically, we have

$$u(t,x) = A(t,x) - \frac{1}{2}\iint_{R(t,x)} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \tag{0.11}$$

where $A(t,x) = f(t-x) + g(t+x)$ depends explicitly on the data of the problem. Equation (0.11) is a necessary condition for a solution of the problem (if it exists). Our strategy then relies on considering the map $F\colon H^1 \to H^1$ given by

$$F(v) := A(t,x) - \frac{1}{2}\iint_{R(t,x)} v_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

and to prove that it is a contraction if the domain is sufficiently small, see Theorems 3.5 and 3.8. Then, one repeatedly applies these theorems to extend the solution u, see Remarks 3.7 and 3.9). This eventually gives existence and uniqueness of $u \in H^1$ as stated in Theorem 3.10.

The problem in which ℓ is unknown is more involved and still under investigation. Thus it is not part of the thesis. In our opinion, in order to state Griffith's criterion in this case, we would need more regularity since the dynamic energy release rate can not be expressed in terms of the one dimensional function f, as we did in (0.4). Indeed, if u is more regular, then the dynamic energy release rate reads as

$$G_\alpha(t) = \frac{1}{2}(1-\alpha^2)u_x(t,\ell(t))^2.$$

This formula makes sense if u_x has a trace well defined on the curve $t \mapsto (t,\ell(t))$, which also depends on the regularity of ℓ.

In the second part of this thesis we deal with the analysis of the quasistatic limit of this model, i.e., the limit as inertia tends to zero and the internal oscillations can be neglected. We prove that the kinetic energy plays a relevant role that can not be captured by a quasistatic analysis.

Quasistatic limit of dynamic evolutions. In Chapter 4 we pass to the analysis of the quasistatic limit for the peeling test that is studied in Chapter 1. We recall that in models that predict the growth of cracks in structures, it is often assumed that the process is quasistatic. The quasistatic hypothesis is that the inertial effects can be neglected since the time scale of the external loading is very slow, or equivalently the speed of the internal oscillations is very large if compared with the speed of loading. The resulting evolutions are rate-independent, i.e., the system is invariant under time reparametrisations.

Starting from the scheme proposed in [29], quasistatic crack growth has been extensively studied in the mathematical literature. The existence of quasistatic evolutions in fracture mechanics has been proved in several papers concerning globally minimising evolutions [26, 13, 30, 19, 27, 22, 12, 44, 16] and vanishing-viscosity solutions [56, 11, 38, 39, 47, 4, 2, 15]. We refer to [9] for a presentation of the variational approach to fracture and to [54] for the relations with the abstract theory of rate-independent systems. These results also show that quasistatic evolutions may present phases of brutal crack growth (appearing as time discontinuities in the quasistatic scale). In order to study fast propagations of cracks, a dynamical analysis is needed, since inertial effects have to be accounted for.

On the other hand, in the case of dynamic fracture, only preliminary existence results were given [57, 20, 24, 21]. The main difficulty is that the equations of elasto-dynamics for the displacement have to be satisfied in a time dependent domain (i.e., the body in its reference configuration, minus the growing crack), while the evolution of the domain is prescribed by a first-order flow rule. The resulting PDE system is strongly coupled, as in other models of damage or delamination (see, e.g., [32, 8, 7, 58, 59, 37, 36] for viscous flow rules and [62, 63, 43, 60, 6, 5, 64, 65, 49, 66, 50, 61] for rate-independent evolutions of internal variables).

In few cases, it has been shown that the quasistatic hypothesis is a good approximation, that is, the dynamic solutions converge to a rate-independent evolution as inertia tends to zero. This was proved in [64, 49] for damage models, including a damping term in the wave equation, and in [25] in the case of perfect plasticity. On the other hand, even in finite dimension there are examples of singularly perturbed second order potential-type equations (where the inertial term vanishes and the formal limit is an equilibrium equation), such that the dynamic solutions do not converge to equilibria as it is proved in Chapter 5. In finite dimension, if the equations include a friction term whose coefficient tends to zero as inertia vanishes, then the dynamic evolutions converge to a solution of the equilibrium equation [1].

In Chapter 4 we develop a "vanishing inertia" analysis for the model of dynamic debonding in dimension one that is introduced in Chapter 1. Here, we study the behaviour of this system when the speed of loading and the initial velocity of the displacement are very small. More precisely, the prescribed vertical displacement is given by $w_\varepsilon(t) := w(\varepsilon t)$ where w is a given function and $\varepsilon > 0$ is a small parameter. The initial vertical displacement and its initial velocity are respectively u_0 and εu_1, where u_0 and u_1 are two functions of x satisfying some suitable assumptions. We use the notation $(u_\varepsilon, \ell_\varepsilon)$ to underline the dependence of the solution on ε. Assuming that the speed of sound is constant and normalised to one, the problem satisfied by

u_ε is

$$
\begin{aligned}
&(u_\varepsilon)_{tt}(t,x) - (u_\varepsilon)_{xx}(t,x) = 0, && t>0,\ 0<x<\ell_\varepsilon(t), && \text{(0.12a)}\\
&u_\varepsilon(t,0) = w_\varepsilon(t), && t>0, && \text{(0.12b)}\\
&u_\varepsilon(t,\ell_\varepsilon(t)) = 0, && t>0, && \text{(0.12c)}\\
&u_\varepsilon(0,x) = u_0(x), && 0<x<\ell_0, && \text{(0.12d)}\\
&(u_\varepsilon)_t(0,x) = \varepsilon u_1(x), && 0<x<\ell_0. && \text{(0.12e)}
\end{aligned}
$$

In Section 4.2 we consider a more general dependence of w, u_0, and u_1 on ε, see (4.1).

The evolution of the debonding front ℓ_ε is determined by rephrasing Griffith's criterion (0.5) to this setting. The flow rule for the evolution of the debonding front now reads as

$$
\begin{aligned}
&\dot\ell_\varepsilon(t) \geq 0, && \text{(0.13a)}\\
&G_\varepsilon(t) \leq \kappa(\ell_\varepsilon(t)), && \text{(0.13b)}\\
&\dot\ell_\varepsilon(t)\,[G_\varepsilon(t) - \kappa(\ell_\varepsilon(t))] = 0, && \text{(0.13c)}
\end{aligned}
$$

for a.e. $t>0$, where $\ell_\varepsilon(0) = \ell_0$.

For the existence of a unique solution $(u_\varepsilon, \ell_\varepsilon)$ in a weak sense, we use the results of Chapter 1. Next, we perform an asymptotic analysis of (0.12)&(0.13) as ε tends to zero, i.e., we study the limit of the system for quasistatic loading. Some results in this direction were given in [28, 48] in the specific case of a piecewise constant toughness.

It is convenient to consider the rescaled functions

$$
(u^\varepsilon(t,x), \ell^\varepsilon(t)) := (u_\varepsilon(\tfrac{t}{\varepsilon}, x), \ell_\varepsilon(\tfrac{t}{\varepsilon})). \tag{0.14}
$$

After this time rescaling, the problem solved by $(u^\varepsilon, \ell^\varepsilon)$ consists of the equation of elastodynamics complemented with initial and boundary conditions

$$
\begin{aligned}
&\varepsilon^2 u^\varepsilon_{tt}(t,x) - u^\varepsilon_{xx}(t,x) = 0, && t>0,\ 0<x<\ell^\varepsilon(t), && \text{(0.15a)}\\
&u^\varepsilon(t,0) = w(t), && t>0, && \text{(0.15b)}\\
&u^\varepsilon(t,\ell^\varepsilon(t)) = 0, && t>0, && \text{(0.15c)}\\
&u^\varepsilon(0,x) = u_0(x), && 0<x<\ell_0, && \text{(0.15d)}\\
&u^\varepsilon_t(0,x) = u_1(x), && 0<x<\ell_0, && \text{(0.15e)}
\end{aligned}
$$

and coupled with Griffith's criterion

$$
\begin{aligned}
&\dot\ell^\varepsilon(t) \geq 0, && \text{(0.16a)}\\
&G^\varepsilon(t) \leq \kappa(\ell^\varepsilon(t)), && \text{(0.16b)}\\
&\dot\ell^\varepsilon(t)\,[G^\varepsilon(t) - \kappa(\ell^\varepsilon(t))] = 0, && \text{(0.16c)}
\end{aligned}
$$

where $G^\varepsilon(t) = G_\varepsilon(\frac{t}{\varepsilon})$ and $\ell^\varepsilon(0) = \ell_0$. Notice that the speed of sound is now $\frac{1}{\varepsilon}$. Indeed, in the quasistatic limit the time scale of the internal oscillations is much faster than the time scale of the loading.

The existence of a unique solution $(u^\varepsilon, \ell^\varepsilon)$ to the coupled problem (0.15)&(0.16) for a fixed $\varepsilon > 0$ is guaranteed by Theorem 1.22, provided the data are Lipschitz and the local toughness is piecewise Lipschitz; moreover it turns out that u^ε is Lipschitz in time and space and ℓ^ε is

Lipschitz in time. (See also Theorem 4.3 below.) As above, we write u^ε in terms of a one-dimensional function f^ε; more precisely, $u^\varepsilon(t,x)$ depends on $f^\varepsilon(x \pm \varepsilon t)$ through the D'Alembert formula (4.4). On the other hand, the dynamic energy release rate G^ε can also be expressed as a function of f^ε, so Griffith's criterion (0.16) reduces to a Cauchy problem which has a unique solution.

In order to study the limit of the solutions $(u^\varepsilon, \ell^\varepsilon)$ as $\varepsilon \to 0$, we use again the one-dimensional structure of the model. First, we derive an a priori bound for the internal energy, uniform with respect to ε; to this end, it is convenient to write the internal energy in terms of f^ε, see Proposition 4.4. The uniform bound allows us to find a limit pair (u, ℓ). More precisely, since the functions ℓ^ε are non-decreasing and $\ell^\varepsilon(T) < L$, Helly's Theorem provides a subsequence ε_k such that ℓ^{ε_k} converges for every t to a (possibly discontinuous) non-decreasing function ℓ. On the other hand, the uniform bound on u^{ε_k} in $L^2(0,T;H^1(0,L))$ guarantees the existence of a weak limit u. We call (u,ℓ) the quasistatic limit of $(u^\varepsilon, \ell^\varepsilon)$.

The issue is now to pass to the limit in (0.15)&(0.16) and to understand the properties of the quasistatic limit. As for the vertical displacement, in the first main result of Chapter 4 (Theorem 4.8) we find that the equilibrium equations are satisfied, i.e., u solves the limit problem

$$u_{xx}(t,x) = 0, \quad t > 0,\ 0 < x < \ell(t), \tag{0.17a}$$
$$u(t,0) = w(t), \quad t > 0, \tag{0.17b}$$
$$u(t,\ell(t)) = 0, \quad t > 0. \tag{0.17c}$$

More precisely, for a.e. t, $u(t,\cdot)$ is affine in $(0,\ell(t))$ and $u(t,x) = -\frac{w(t)}{\ell(t)}x + w(t)$. To prove this, we exploit a technical lemma stating that the graphs of ℓ^{ε_k} converge to the graph of ℓ in the Hausdorff metric, see Lemma 4.7. We remark that in general the initial conditions (0.15d)&(0.15e) do not pass to the limit since there may be time discontinuities, even at $t = 0$.

Next we study the flow rule satisfied by the limit debonding evolution ℓ. We question whether it complies with the quasistatic formulation of Griffith's criterion,

$$\dot\ell(t) \geq 0, \tag{0.18a}$$
$$G_{qs}(t) \leq \kappa(\ell(t)), \tag{0.18b}$$
$$\dot\ell(t)\,[G_{qs}(t) - \kappa(\ell(t))] = 0, \tag{0.18c}$$

where G_{qs} is the quasistatic energy release rate, that is the partial derivative of the quasistatic internal energy with respect to the elongation of the debonded region. Notice that in the quasistatic setting the internal energy consists of the potential term only, so (0.18) is the formal limit of (0.16) as $\varepsilon \to 0$.

Condition (0.18a) is guaranteed by Helly's Theorem. By passing to the limit in (0.16b), we also prove that (0.18b) holds. For this result we use again the D'Alembert formula for u^ε and find the limit f of the one-dimensional functions f^ε. In fact, $\dot f$ turns out to be related to u_x through an explicit formula, as we see in Theorem 4.14, which is the second main result of Chapter 4.

In contrast, (0.18c) is in general not satisfied. This was already observed in the earlier paper [48], which presents an example of dynamic solutions whose limit violates (0.18c). The singular behaviour of these solutions is due to the choice of a toughness with discontinuities. Indeed, when the debonding front meets a discontinuity in the toughness, a shock wave is generated. The interaction of such singularities causes strong high-frequency oscillations of the kinetic energy, which affects the limit as the wave speed tends to infinity.

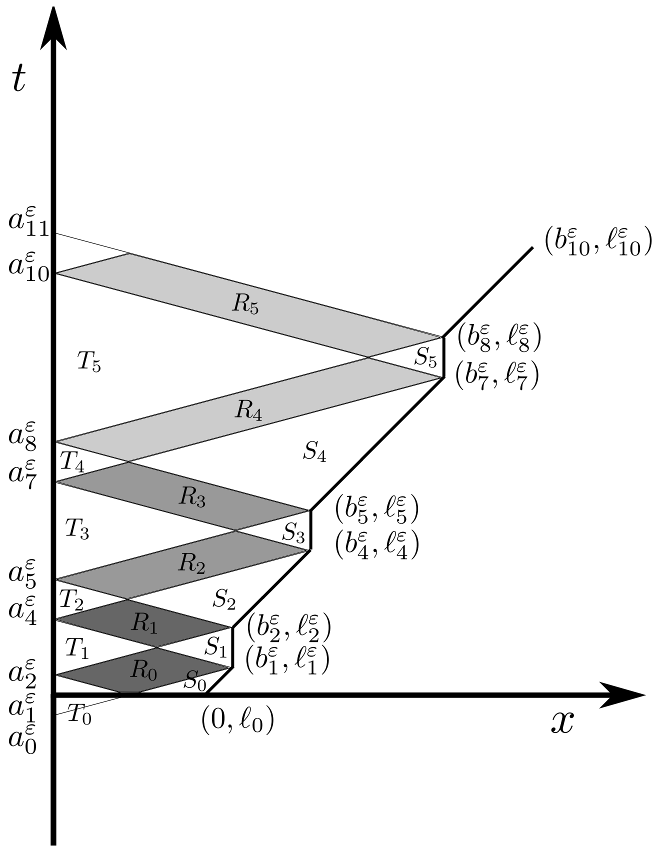

Figure 6: The sectors highlighted above give different contributions to the kinetic energy K^{ε}. The darker the shade of grey, the larger is $u_t^{\varepsilon}(t,x)^2$ in that region. White sectors give a negligible contribution. See Section 4.3

In Section 4.3, we continue the discussion of this kind of behaviour by providing an explicit case where (0.18c) does not hold in the limit even if the local toughness is constant and the other data are smooth. (See also Remark 4.18.) In our new example, the initial conditions are not at equilibrium, in particular the initial position u_0 is not affine in $(0, \ell_0)$. Therefore, due to the previous results, the quasistatic limit cannot satisfy the initial condition, i.e., it has a time discontinuity at $t = 0$. Moreover, our analysis of the limit evolution (u, ℓ) shows that the kinetic energy given through the initial conditions is not relaxed instantaneously; its effects persist in a time interval where the evolution does not satisfy (0.18c). See Figure 6. The surplus of initial energy, instantaneously converted into kinetic energy, cannot be quantified in a purely quasistatic analysis. For this reason the usual quasistatic formulation (0.18) is not suited to describe the quasistatic limit of our dynamic process.

Quasistatic limit in the case of the speed-depending local toughness. We then take into account the quasistatic limit in the case of a speed-dependent local toughness $\kappa = \kappa(\ell(t), \dot{\ell}(t))$ in order to understand the effect of internal oscillations when Griffith's activation criterion features viscous terms.

Our analysis shows that we get the same results of the case with no dependence of κ on the debonding speed. First, we consider the pair $(u^\varepsilon, \ell^\varepsilon)$, where u^ε is solution to (0.15) and ℓ^ε solves

$$\begin{cases} \dot\ell^\varepsilon(t) = \dfrac{1}{\varepsilon}\dfrac{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 - \kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t))}{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 + \kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t))} \vee 0, \quad \text{for a.e. } t > 0, \\ \ell^\varepsilon(0) = \ell_0, \end{cases}$$

that is equivalent to Griffith's criterion

$$\begin{aligned} &\dot\ell^\varepsilon(t) \geq 0, \\ &G^\varepsilon_{\dot\ell^\varepsilon(t)}(t) \leq \kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t)), \\ &\dot\ell^\varepsilon(t)\left[G^\varepsilon_{\dot\ell^\varepsilon(t)}(t) - \kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t))\right] = 0. \end{aligned}$$

Then, by proving equiboundedness in ε, we show that $u^\varepsilon \rightharpoonup u$ weakly in $L^2(H^1)$, with u satisfying the equilibrium equation (0.17). Moreover, the limit evolution of debonding front ℓ is such that $\dot\ell \geq 0$. In Remark 4.20 we prove that Griffith's inequality is satisfied if we assume that κ is upper semicontinuous in both variables.

Again, in Section 4.4.1 we show that the activation condition in Griffith's criterion does not hold by presenting an example modeled as the one of Section 4.3. We consider a local toughness of the form

$$\kappa(x, \mu) = a + b\mu,$$

where a and b are positive constants. We see that the limit as $\varepsilon \to 0$ features a non-quasistatic phase. This proves that the kinetic energy released as soon as the process starts is not absorbed by the system even if the local toughness depends on the speed of the debonding.

In Section 4.4.2 we also prove that this dependence on the debondng speed does not penalise fast propagations, as one could expect. We analyse the effect of a local toughness κ such that

$$\lim_{\mu \to 1^-} \kappa(x, \mu) = +\infty.$$

We show another example where we observe the presence of a limit jump when we consider as local toughness

$$\kappa(x, \mu) = \tilde\kappa(x)\frac{1+\mu}{1-\mu},$$

with $\kappa(x)$ depending only on the position.

In the last part of this thesis, we analyse a simpler problem where we consider the convergence of solution to singularly perturbed second order equations. Our study shows that in general is not possible to ignore inertial effects even in lower dimensional problems.

Convergence of singularly perturbed second order potential type equations. A problem of interest in various areas of applied mathematics is to find stable equilibrium points for time-dependent energies. In a simplified setting, the problem is to find an evolution $t \to u(t)$ such that

$$\begin{cases} V_x(t, u(t)) = 0, \\ V_{xx}(t, u(t)) > 0, \end{cases} \tag{0.19}$$

where $V(t, x)$ is a potential, V_x denotes the gradient with respect to x, and V_{xx} the corresponding Hessian. This problem can be locally solved by means of the Implicit Function Theorem, which provides a smooth solution defined in a neighborhood of $t = 0$.

Problem (0.19) has also been studied in finite dimension as the limit case of ε-gradient flows. A first general result was given by C. Zanini in [68], where the author studies the system

$$\varepsilon \dot{u}_\varepsilon(t) + V_x(t, u_\varepsilon(t)) = 0. \tag{0.20}$$

In [68] it is proved that the solutions $u_\varepsilon(t)$ to (0.20) converge to a solution $u(t)$ to (0.19), obtained by connecting smooth branches of solutions to the equilibrium equation (0.19) through suitable heteroclinic solutions of the ε-gradient flows (0.20).

In [1] V. Agostiniani analysed the second order approximation with a dissipative term:

$$\varepsilon^2 A\ddot{u}_\varepsilon(t) + \varepsilon B\dot{u}_\varepsilon(t) + V_x(t, u_\varepsilon(t)) = 0, \tag{0.21}$$

where A and B are positive definite and symmetric matrices. It turns out that $(u_\varepsilon, \varepsilon B\dot{u}_\varepsilon) \to (u, 0)$, where u is piecewise continuous and satisfies (0.19). Moreover the behaviour of the system at jump times is described by trajectories connecting the states before and after the jumps; such trajectories are given by a suitable autonomous second order system related to A, B, and V_x.

We remark that studying the asymptotic behaviour of solutions, as $\varepsilon \to 0$, in systems of the form (0.21) with $A \neq 0$ and $B = 0$ (vanishing inertia), or $A = 0$ and $B \neq 0$ (vanishing viscosity), or $A, B \neq 0$ (vanishing viscosity and inertia), may give a selection principle for rate-independent evolutions (namely those evolutions whose loading is assumed to be so slow that at every time the system is at equilibrium and internal oscillation can be neglected). This approach has been succesfully adopted in various situations in the case of vanishing viscosity (cf. e.g. [53, 52, 18, 38, 47, 40, 17]) and in the case of vanishing viscosity and inertia (cf. e.g. [62, 64, 49, 67, 25]). We remark that in [25] viscosity can be neglected under suitable assumptions.

The above mentioned results [1, 68] require strong smoothness assumptions on V (C^3-regularity is required). The aim of Chapter 5 is to weaken the assumptions under which second order perturbed problems converge to (0.19). More precisely, we consider a second order equation of the form (0.21) without the dissipative term $B\dot{u}_\varepsilon$. (Notice that in general, when $B > 0$, it is easier to prove the convergence of solutions.) We therefore study the asymptotic behaviour of the solutions $u_\varepsilon(t)$ of the problem

$$\varepsilon^2 \ddot{u}_\varepsilon(t) + V_x(t, u_\varepsilon(t)) = 0 \tag{0.22}$$

to a continuous stable equilibrium $u_0(t)$ of (0.19). The main result of Chapter 5 is that the convergence $u_\varepsilon \to u_0$ still holds under some regularity and growth conditions on V that are weaker than those required in [1, 68]. Furthermore we provide a counterexample to that convergence when such assumptions do not hold.

More precisely we require continuity for V in both variables and we assume that $V(t, \cdot) \in C^2$. We also suppose that there is a function $V_t(t, x)$ of class C^1-Carathéodory (i.e., $V_t(\cdot, x)$ is measurable and $V_t(t, \cdot)$ is of class C^1) such that

$$V(t_2, x) - V(t_1, x) = \int_{t_1}^{t_2} V_t(t, x)\, \mathrm{d}t,$$

for a.e. t_1, t_2. With some further boundedness conditions on V (listed in Section 5.1) we prove that $u_0(t)$ is absolutely continuous and we obtain the convergence result, see Theorem 5.6 in Section 5.2. Specifically, we find that solutions to (0.22) satisfy

$$u_\varepsilon \to u_0 \text{ uniformly and } \varepsilon \|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1} \to 0 \tag{0.23}$$

as $\varepsilon \to 0$.

In Section 5.3 we show that, if we weaken the assumptions on V, we are not able to get (0.23). More precisely we provide a counterexample for a model case where the time-dependent energy is given by

$$V(t,x) := \frac{1}{2}|x - u_0(t)|^2.$$

We remark that, when $u_0 \in W^{1,1}(0,T)$, then V in its turn satisfies the assumptions of Section 5.2. In this case, solutions u_ε of (0.22) converge uniformly to $u_0(t)$. On the other hand we show that, if u_0 is the Cantor-Vitali function, then (0.23) can not be satisfied (see Example 5.9). In fact, we prove that no subsequences of solutions to (0.22) could converge to u_0 and that the continuous functions u_0 with this property are infinitely many (see Proposition 5.7 and Remark 5.8).

Our non-convergence result in Section 5.3 can therefore be regarded as an example in which, in the absence of a damping viscous term, dynamic solutions do not converge to stable equilibria even in very simple situations. This is consistent with our examples in Sections 4.3 & 4.4.

The material contained in the present thesis has partly been published on journals. Specifically, Sections 1.1–1.3 are presented in [23], Sections 4.1–4.3 are part of [45], Sections 1.4 and 4.4 appear in the Preprint [46], Chapter 5 can be found in [55], while Chapters 2 and 3 are in preparation.

Notation

In this chapter we fix the notation that will be used throughout the thesis.

Basic notation.

$\alpha \wedge \beta, \min\{\alpha, \beta\}$	minimum between α and β
$\alpha \vee \beta, \max\{\alpha, \beta\}$	maximum between α and β
$a \cdot b$	scalar product between two vectors $a, b \in \mathbb{R}^n$
$\lvert \cdot \rvert$	modulus, euclidean norm of vectors in $\mathbb{R}^n$ or matrices in $\mathbb{R}^{n \times n}$
$\\| \cdot \\|_X$	norm of the normed space X
t	time variable
x	space variable
u_t, u_x	time and space (weak) first derivatives of the function u
u_{tt}, u_{tx}, u_{xx}	(mixed) time and space (weak) second derivatives of the function u
$\dot{f}$	(weak) first derivative of the function of only one variable f
$\ddot{f}$	(weak) second derivative of the function of only one variable f
$\overline{A}$	closure of the set A
$B_r(x)$	ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$

Functional spaces. Let X be a metric space, Ω an open set in $\mathbb{R}^n$, and $T > 0$.

$C^k(X;\mathbb{R}^m)$	space of $\mathbb{R}^m$-valued functions defined in X and with k continuous derivatives
$C^{k,1}(X;\mathbb{R}^m)$	space of $C^k(X;\mathbb{R}^m)$ functions whose k-th derivative is Lipschitz
$C^0_c(X;\mathbb{R}^m)$	space of $\mathbb{R}^m -$ valued continuous functions with compact support in X
$C^0_0(X;\mathbb{R}^m)$	closure of $C^0_c(X;\mathbb{R}^m)$ with respect to the supremum norm in X
$\mathcal{D}'(\Omega)$	space of distributions over Ω
$L^p(\Omega)$	Lebesgue space with $1 \leq p \leq +\infty$
$W^{k,p}(\Omega)$	Soboloev space with k derivatives and $\leq p \leq +\infty$
$H^1(\Omega), H^2(\Omega)$	the Sobolev spaces $W^{1,2}(\Omega)$ and $W^{2,2}(\Omega)$ respectively
$H^1_{\text{loc}}(\Omega)$	space of functions that are locally in $H^1(\Omega)$
$H^1_0(\Omega)$	space of $H^1(\Omega)$ functions with zero trace)
$H^{-1}(\Omega)$	dual of $H^1_0(\Omega)$
$L^2(0,T;L^2(\Omega))$	L^2 space of $L^2(\Omega)$-valued functions defined over the interval $(0,T)$
$L^2(0,T;H^1(\Omega))$	L^2 space of $H^1(\Omega)$-valued functions defined over the interval $(0,T)$
$H^1(0,T;L^2(\Omega))$	H^1 space of $L^2(\Omega)$-valued functions defined over the interval $(0,T)$

In the previous spaces $\mathbb{R}^n$ is omitted when $n = 1$.

Part I

Existence and uniqueness results

CHAPTER 1

Dynamic evolutions for a peeling test in dimension one

This chapter is devoted to the study of a model for a peeling test in dimension one.

In the first section, we establish existence and uniqueness of a solution u to (0.1), when the evolution of the debonding front $t \mapsto \ell(t)$ is already known. In the second section we introduce a precise notion for the dynamic energy release rate, which will be used to derive Griffith's criterion for the evolution of the debonding front. In the third section we analyse the coupled problem for u and ℓ and eventually find that there exists a unique solution with u satisfying (0.1) and ℓ such that Griffith's criterion is satisfied in the equivalent form given by (0.6). Finally, in the fourth section we consider the case of a speed-dependent local toughness, giving existence and uniqueness result for the pair (u, ℓ) satisfying the coupled problem (0.1) & (0.7).

The results of Sections 1.1–1.3 have been published in the paper [23], a joint work in collaboration with G. Dal Maso and G. Lazzaroni, while Section 1.4 is part of a forthcoming paper in collaboration with G. Lazzaroni.

1.1 The problem for prescribed debonding front

In this section we make precise the notion of solution of problem (0.1) when the evolution of the debonding front is prescribed. More precisely, we fix $\ell_0 > 0$ and $\ell\colon [0, +\infty) \to [\ell_0, +\infty)$, and we assume that for every $T > 0$ there exists $0 < L_T < 1$ such that

$$0 \le \ell(t_2) - \ell(t_1) \le L_T(t_2 - t_1), \text{ for every } 0 \le t_1 < t_2 \le T, \tag{1.1a}$$

$$\ell(0) = \ell_0. \tag{1.1b}$$

It will be convenient to introduce the following functions:

$$\varphi(t) := t - \ell(t) \text{ and } \psi(t) := t + \ell(t). \tag{1.2}$$

We observe that φ and ψ are strictly increasing, so we can define

$$\omega\colon [\ell_0, +\infty) \to [-\ell_0, +\infty), \quad \omega(t) := \varphi \circ \psi^{-1}(t). \tag{1.3}$$

Observe that

$$\frac{1-L_T}{1+L_T}(t_2-t_1) \le \omega(t_2)-\omega(t_1) \le t_2-t_1, \text{ for every } 0 \le t_1 < t_2 \le T. \tag{1.4}$$

For every $a \in \mathbb{R}$, we introduce the space

$$\widetilde{H}^1(a,+\infty) := \{u \in H^1_{\text{loc}}(a,+\infty) : u \in H^1(a,b), \text{ for every } b > a\}.$$

We assume that

$$w \in \widetilde{H}^1(0,+\infty). \tag{1.5}$$

As for the initial data we require

$$u_0 \in H^1(0,\ell_0), \quad u_1 \in L^2(0,\ell_0), \tag{1.6a}$$

and the compatibility conditions

$$u_0(0) = w(0), \quad u_0(\ell_0) = 0. \tag{1.6b}$$

We set

$$\Omega := \{(t,x) : t > 0\,,\, 0 < x < \ell(t)\},$$

and

$$\Omega_T := \{(t,x) : 0 < t < T\,,\, 0 < x < \ell(t)\}.$$

We will look for solutions in the space

$$\widetilde{H}^1(\Omega) := \{u \in H^1_{\text{loc}}(\Omega) : u \in H^1(\Omega_T), \text{ for every } T > 0\}.$$

Moreover, we set for $k \ge 0$

$$\widetilde{C}^{k,1}(0,+\infty) := \{f \in C^k([0,+\infty)) : f \in C^{k,1}([0,T]) \text{ for every } T > 0\}. \tag{1.7}$$

and

$$\widetilde{C}^{k,1}(\Omega) := \{u \in C^k(\overline{\Omega}) : u \in C^{k,1}(\overline{\Omega}_T), \text{ for every } T > 0\}.$$

Definition 1.1. *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. in $H^1(\Omega_T)$) is a solution of* (0.1a)–(0.1c) *if $u_{tt} - u_{xx} = 0$ holds in the sense of distributions in Ω (resp. in Ω_T) and the boundary conditions are intended in the sense of traces.*

Given a solution $u \in \widetilde{H}^1(\Omega)$ in the sense of Definition 1.1, we extend u to $(0,+\infty)^2$ (still denoting it by u), by setting $u = 0$ in $(0,+\infty)^2 \setminus \Omega$. Note that this agrees with the interpretation of u as vertical displacement of the film which is still glued to the substrate for $(t,x) \notin \Omega$. For a fixed $T > 0$, we define $Q_T := (0,T)\times(0,\ell(T))$ and we observe that $u \in H^1(Q_T)$ because of the boundary conditions (0.1b)&(0.1c). Further, we need to impose the initial position and velocity of u. While condition in (0.1d) can be formulated in the sense of traces, we have to give a precise meaning to the second condition. Since $H^1((0,T)\times(0,\ell_0)) = H^1(0,T;L^2(0,\ell_0)) \cap L^2(0,T;H^1(0,\ell_0))$, we have $u_t, u_x \in L^2(0,T;L^2(0,\ell_0))$. This implies that $u_t, u_{xx} \in L^2(0,T;H^{-1}(0,\ell_0))$ and, by the wave equation, $u_{tt} \in L^2(0,T;H^{-1}(0,\ell_0))$. Therefore $u_t \in H^1(0,T;H^{-1}(0,\ell_0)) \subset C^0([0,T];H^{-1}(0,\ell_0))$ and we can impose condition (0.1e) as an equality between elements of $H^{-1}(0,\ell_0)$. This discussion shows that the following definition makes sense.

Definition 1.2. *We say that* $u \in \widetilde{H}^1(\Omega)$ *(resp.* $H^1(\Omega_T)$*) is a solution of* (0.1) *if Definition 1.1 holds and the initial conditions* (0.1d)*&*(0.1e) *are satisfied in the sense of* $L^2(0,\ell_0)$ *and* $H^{-1}(0,\ell_0)$*, respectively.*

In the following discussion T and u are fixed as above. We consider the change coordinates

$$\begin{cases} \xi = t - x, \\ \eta = t + x, \end{cases} \tag{1.8}$$

which maps the set Ω_T into $\widetilde{\Omega}$. In terms of the new function

$$v(\xi,\eta) := u(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}), \tag{1.9}$$

the wave equation (0.1a) (weakly formulated) reads as

$$v_{\eta\xi} = 0 \quad \text{in } \mathcal{D}'(\widetilde{\Omega}). \tag{1.10}$$

This means that for every test function $\alpha \in \mathcal{C}_c^\infty(\widetilde{\Omega})$ we have

$$0 = \langle v_{\eta\xi}, \varphi \rangle = -\int_{\widetilde{\Omega}} v_\eta(\xi,\eta)\, \alpha_\xi(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{1.11}$$

For every $\xi \in \mathbb{R}$ let

$$\widetilde{\Omega}^\xi := \{\eta \in \mathbb{R} : (\xi,\eta) \in \widetilde{\Omega}\},$$

and, similarly, for every $\eta \in \mathbb{R}$ let

$$\widetilde{\Omega}_\eta := \{\xi \in \mathbb{R} : (\xi,\eta) \in \widetilde{\Omega}\}.$$

Notice that, thanks to (1.1a), $\widetilde{\Omega}^\xi$ and $\widetilde{\Omega}_\eta$ are intervals. Moreover $\widetilde{\Omega}^\xi \neq \emptyset$ if and only if $\xi \in (-\ell_0, T)$ and similarly $\widetilde{\Omega}_\eta \neq \emptyset$ if and only if $\eta \in (0, T{+}\ell(T))$.

Lemma 1.3. *A function* $v \in H^1(\widetilde{\Omega})$ *is a solution to* (1.10) *if and only if there exist functions* $f \in H^1_{loc}(-\ell_0, T)$ *and* $g \in H^1_{loc}(0, T{+}\ell(T))$ *such that*

$$\int_{-\ell_0}^{T} \dot{f}(\xi)^2 |\widetilde{\Omega}^\xi|\, \mathrm{d}\xi < +\infty, \tag{1.12a}$$

$$\int_0^{T+\ell(t)} \dot{g}(\eta)^2 |\widetilde{\Omega}_\eta|\, \mathrm{d}\eta < +\infty, \tag{1.12b}$$

and

$$v(\xi,\eta) = f(\xi) + g(\eta), \qquad \textit{for a.e. } (\xi,\eta) \in \widetilde{\Omega}. \tag{1.13}$$

Proof. Let $v \in H^1(\widetilde{\Omega})$ be a solution to (1.10). Using a standard argument for the slicing of H^1 functions, we deduce from (1.11) that for a.e. $\eta \in (0, T{+}\ell(T))$ we have $v_\eta(\cdot,\eta) \in L^2(\widetilde{\Omega}_\eta)$ and

$$\int_{\widetilde{\Omega}_\eta} v_\eta(\xi,\eta)\dot{\beta}(\xi)\, \mathrm{d}\xi = 0, \quad \text{for every } \beta \in C_c^\infty(\widetilde{\Omega}_\eta).$$

This implies that v_η is in $H^1(\widetilde{\Omega}_\eta)$ and its derivative in the sense of distributions vanishes in $\widetilde{\Omega}_\eta$. Therefore for a.e. $\eta \in (0, T{+}\ell(T))$ there exists $\Phi(\eta) \in \mathbb{R}$ such that

$$v_\eta(\xi,\eta) = \Phi(\eta), \quad \text{for a.e. } \xi \in \widetilde{\Omega}_\eta. \tag{1.14}$$

Let us prove that $\Phi \in L^2_{\text{loc}}(0,T+\ell(T))$. First, by applying the Fubini Theorem to v_η, we deduce that the function Φ belongs to $L^2(\widetilde{\Omega}^\xi)$ for a.e. $\xi \in (-\ell_0,T)$. On the other hand, for every $\eta_0 \in (0,T+\ell(T))$ there exists $\xi_0 \in (-\ell_0,T)$ such that $\eta_0 \in \widetilde{\Omega}^\xi$ for every ξ in a suitable neighbourhood of ξ_0. Together with the previous result this gives $\Phi \in L^2_{\text{loc}}(0,T+\ell(T))$.

Let now g be a primitive of Φ, which clearly belongs to $H^1_{\text{loc}}(0,T+\ell(T))$. By (1.14) and the Fubini Theorem, for a.e. $\xi \in (-\ell_0,T)$ we have $v_\eta(\xi,\eta) = \dot g(\eta)$ for a.e. $\eta \in \widetilde{\Omega}^\xi$; therefore for a.e. $\xi \in (-\ell_0,T)$ there exists $f(\xi) \in \mathbb{R}$ such that $v(\xi,\eta) = f(\xi) + g(\eta)$ for a.e. $\eta \in \widetilde{\Omega}^\xi$. Using again the Fubini Theorem, for a.e. $\eta \in (0,T+\ell(T))$ we obtain $v(\xi,\eta) = f(\xi) + g(\eta)$ for a.e. $\xi \in \widetilde{\Omega}_\eta$. This implies that for a.e. $\eta \in (0,T+\ell(T))$ the function f belongs to $H^1(\widetilde{\Omega}_\eta)$. Arguing as above we deduce that $f \in H^1_{\text{loc}}(-\ell_0,T)$. In conclusion, for every solution v to (1.10), with $v \in H^1(\widetilde{\Omega})$, there exist $f \in H^1_{\text{loc}}(-\ell_0,T)$ and $g \in H^1_{\text{loc}}(0,T+\ell(T))$ such that (1.13) is satisfied.

Moreover, taking the derivative with respect to ξ we find that for a.e. $\eta \in (0,T+\ell(T))$, $v_\xi(\xi,\eta) = \dot f(\xi)$ for a.e. $\xi \in \widetilde{\Omega}_\eta$. By the Fubini Theorem

$$\int_{-\ell_0}^{T} \dot f(\xi)^2 |\widetilde{\Omega}^\xi| \,\mathrm{d}\xi = \int_{\widetilde{\Omega}} v_\xi(\xi,\eta)^2 \,\mathrm{d}\xi\,\mathrm{d}\eta < +\infty.$$

Similarly we prove that

$$\int_{0}^{T+\ell(T)} \dot g(\xi)^2 |\widetilde{\Omega}_\eta| \,\mathrm{d}\xi = \int_{\widetilde{\Omega}} v_\eta(\xi,\eta)^2 \,\mathrm{d}\xi\,\mathrm{d}\eta < +\infty.$$

Conversely, assume that $f \in H^1_{\text{loc}}(-\ell_0,T)$ and $g \in H^1_{\text{loc}}(0,T+\ell(T))$ satisfy (1.12) and define v as in (1.13). Then, by the Fubini Theorem, f and g belong to $H^1(\widetilde{\Omega})$. Moreover, $v \in H^1(\widetilde{\Omega})$ and (1.10) is satisfied. □

In the next proposition we return to the variables (t,x) and use Lemma 1.3 to characterise the solutions of problem (0.1a)–(0.1c) according to Definition 1.1. Notice that the boundary conditions imply a relationship between the functions f and g of the previous lemma, so that the solution can be written using either of them.

In this characterisation we use the functions φ and ψ defined in (1.2). We extend ψ^{-1} to $[0,+\infty)$ by setting $\psi^{-1}(s) := 0$ for $s \in [0,\ell_0)$. Notice that all integrands in (1.15) are nonnegative and recall that $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

Proposition 1.4. *Let $T > 0$ and assume* (1.1) *and* (1.5). *There exists a weak solution $u \in H^1(\Omega_T)$ to problem* (0.1a)–(0.1c) *(in the sense of Definition 1.1) if and only if there exists a function $f \in H^1_{loc}(-\ell_0,T+\ell(T))$ with*

$$\int_{-\ell_0}^{T-\ell(T)} \dot f(s)^2(\varphi^{-1}(s) - (s \vee 0))\,\mathrm{d}s + \int_{T-\ell(T)}^{T} \dot f(s)^2(T - (s \vee 0))\,\mathrm{d}s < +\infty, \tag{1.15a}$$

$$\int_{0}^{T+\ell(T)} (\dot w(s) - \dot f(s))^2((s \wedge T) - \psi^{-1}(s))\,\mathrm{d}s < +\infty, \tag{1.15b}$$

whose continuous representative satisfies $f(0) = 0$ and

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)), \quad \textit{for every } t \in (0,T). \tag{1.16}$$

In this case u is given by

$$u(t,x) = w(t+x) - f(t+x) + f(t-x), \quad \textit{for a.e. } (t,x) \in \Omega_T. \tag{1.17}$$

Proof. Using (1.8), (1.9), and (1.13), we can assert that every weak solution $u \in H^1(\Omega_T)$ of problem (0.1) has the form

$$u(t,x) = f(t-x) + g(t+x), \quad \text{for a.e. } (t,x) \in \Omega_T, \tag{1.18}$$

for some functions $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$ and $g \in H^1_{\mathrm{loc}}(0, T+\ell(t))$ satisfying (1.12). Then, by the boundary condition (0.1b) and by the continuity of f, g, and w in $(0,T)$,

$$u(t,0) = w(t) = f(t) + g(t), \quad \text{for a.e. } t \in (0,T).$$

From now on we consider the consider the continuous representatives of f, g, and w. We observe that $g = w - f$ everywhere in $(0,T)$ and $w - f$ is continuous in $[0,T)$ (indeed, w is continuous in $[0,T]$ because $w \in H^1(0,T)$, while f is continuous in $(-\ell_0,T)$ because $f \in H^1_{\mathrm{loc}}(-\ell_0,T)$). Therefore we can extend g at zero by continuity. Analogously, f can be extended at T by continuity, so that $w(t) = f(t) + g(t)$ for every t in $[0,T]$. We can also extend f by setting $f = w - g$ in $(T, T+\ell(T))$, so that $f \in H^1_{\mathrm{loc}}(-\ell_0, T+\ell(T))$. In particular, (1.18) reduces to (1.17). Moreover, by the second boundary condition $u(t,\ell(t)) = 0$ we obtain (1.16). By a direct computation of $|\widetilde{\Omega}^\xi|$ and of $|\widetilde{\Omega}_\eta|$, one sees that conditions (1.12) are equivalent to (1.15). The condition $f(0) = 0$ can be obtained adding a suitable constant. □

Remark 1.5. The results obtained up to now hold also in the case $\ell_0 = 0$, provided that $\ell(t) > 0$ for every $t > 0$ and that $w(0) = 0$.

In the remaining part of the section we focus on the case $\ell_0 > 0$. We begin with a Proposition which gives the connection between f and the initial conditions (0.1d)&(0.1e).

Proposition 1.6. *Let $T > 0$ and assume* (1.1), (1.5), *and* (1.6a). *Let $f \in H^1_{loc}(-\ell_0, T+\ell(T))$ satisfy* (1.15), (1.16), *and $f(0) = 0$, and let u be defined by* (1.17). *Then, u is solution to problem* (0.1) *in $H^1(\Omega_T)$, according to Definition 1.2, if and only if*

$$f(s) = w(s) - \frac{u_0(s)}{2} - \frac{1}{2}\int_0^s u_1(x)\,\mathrm{d}x - w(0) + \frac{u_0(0)}{2}, \quad \textit{for every } s \in [0,\ell_0], \tag{1.19a}$$

$$f(s) = \frac{u_0(-s)}{2} - \frac{1}{2}\int_0^{-s} u_1(x)\,\mathrm{d}x - \frac{u_0(0)}{2}, \quad \textit{for every } s \in (-\ell_0, 0]. \tag{1.19b}$$

Proof. We already know, by Proposition 1.4, that u is a solution to problem (0.1a)–(0.1c). We compute the time derivative of u using (1.17) and we obtain

$$u_t(t,x) = \dot{w}(t+x) - \dot{f}(t+x) + \dot{f}(t-x), \quad \text{for a.e. } (t,x) \in \Omega_T.$$

Assume that (0.1d)&(0.1e) holds. By (1.17) and (1.25a), taking $(t,x) = (0,s)$, we deduce that

$$u_0(s) = w(s) - f(s) + f(-s), \quad \text{for every } s \in [0,\ell_0), \tag{1.20a}$$

$$u_1(s) = \dot{w}(s) - \dot{f}(s) + \dot{f}(-s), \quad \text{for a.e. } s \in (0,\ell_0), \tag{1.20b}$$

where we have used the continuity property of f and the initial conditions according to Definition 1.2. By adding (1.20b) to the derivative of (1.20a), we find that

$$\dot{f}(s) = \dot{w}(s) - \frac{\dot{u}_0(s) + u_1(s)}{2}, \quad \text{for a.e. } s \in (0,\ell_0). \tag{1.21}$$

Therefore, integrating (1.21), we obtain (1.19). Equality (1.20a) enables us to determine f in the interval $[-\ell_0, 0]$, leading to (1.19b).

Conversely, assume that (1.19) holds. Then (1.20a) follows easily and, taking the derivative of (1.19), we obtain also (1.20b). Finally, (1.20), together with (1.17) and (1.25a), gives (0.1d)&(0.1e) in the sense of Definition 1.2. □

Remark 1.7. Conditions (1.5) and (1.6a), together with (1.19), show that $f \in H^1(-\ell_0, \ell_0)$ and (1.19b) holds for every $s \in [-\ell_0, 0]$.

We are now in a position to give the main result of this section, which gives existence and uniqueness of a solution to problem (0.1), according to Definition 1.2.

Theorem 1.8. *Assume* (1.1), (1.5), *and* (1.6). *Then there is a unique solution* $u \in \widetilde{H}^1(\Omega)$ *to problem* (0.1), *according to Definition 1.2. Moreover, there is a unique function* $f\colon [-\ell_0, +\infty) \to \mathbb{R}$, *with* $f(0) = 0$ *and* $f \in \widetilde{H}^1(-\ell_0, +\infty)$, *such that* (1.17) *holds.*

Proof. By Propositions 1.4 and 1.6, it is enough to construct a function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$, with $f \in \widetilde{H}^1(-\ell_0, +\infty)$, such that (1.19) holds and

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)), \tag{1.22}$$

for every $t \in [0, +\infty)$. We use (1.19) and Remark 1.7 to define f in $[-\ell_0, \ell_0]$. To conclude the proof we now have to extend it to $(\ell_0, +\infty)$ in such a way that (1.22) is satisfied.

We set $t_0 := \ell_0$ and $t_1 := \omega^{-1}(t_0)$ and we define f in $(t_0, t_1]$ by

$$f(t) = w(t) + f(\omega(t)), \tag{1.23}$$

for every $t \in (t_0, t_1]$. Since w, $f \in H^1(-\ell_0, t_0)$ (see Remark 1.7) and ω is bi-Lipschitz between (t_0, t_1) and $(-\ell_0, \ell_0)$ by (1.4), we have $f \in H^1(t_0, t_1)$. Using the compatibility conditions (1.6b) we deduce from (1.19) and (1.23) that $f(t_0^-) = f(t_0^+)$, hence $f \in H^1(-\ell_0, t_1)$. Moreover, by (1.23), we obtain that (1.22) is satisfied in $[0, \psi^{-1}(t_1))$.

We now define inductively a sequence t_i by setting $t_{i+1} := \omega^{-1}(t_i)$. Let us prove that $t_i \to +\infty$. From the definition of φ and ψ and from the inequality $\ell(t) \geq \ell_0$ we deduce that $\varphi^{-1}(t) \geq t + \ell_0$ and $\psi(t) \geq t + \ell_0$. By the monotonicity of ψ we thus find that

$$\omega^{-1}(t) \geq \psi(t + \ell_0) \geq t + 2\ell_0,$$

which implies $t_{i+1} - t_i \geq 2\ell_0$ and therefore $t_i \to +\infty$.

Assume that for some i the function f has already been defined in $[-\ell_0, t_i]$ so that $f \in H^1(-\ell_0, t_i)$ and (1.22) holds for every $t \in [0, \psi^{-1}(t_i))$. We define f in $[t_i, t_{i+1}]$ by (1.23) for every $t \in [t_i, t_{i+1}]$. With this construction (1.23) holds for every $t \in [t_0, t_i)$, hence (1.22) holds for every $t \in [0, \psi^{-1}(t_{i+1}))$. Since f is continuous at $t_{i-1} \in (-\ell_0, t_i)$, we deduce from (1.23) that f is continuous at t_i, which implies $f \in H^1(-\ell_0, t_{i+1})$.

Since $t_i \to +\infty$, this construction leads to $f \in H^1_{\mathrm{loc}}(-\ell_0, +\infty)$ satisfying (1.22) for every $t \in [0, +\infty)$. Condition (1.19) is obviously satisfied.

This construction shows that the function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$ satisfying (1.19) in $[-\ell_0, \ell_0]$ and (1.22) for every $t \in [0, +\infty)$ is uniquely determined. Thanks to Propositions 1.4 and 1.6 this gives the uniqueness of the solution u. □

Remark 1.9. Theorem 1.8 implies that the solution of problem (0.1) according to Definition 1.2 has a continuous representative which satisfies

$$u(t,x) = w(t+x) - f(t+x) + f(t-x), \quad \text{for every } (t,x) \in \Omega, \tag{1.24}$$

for a suitable function $f \in \widetilde{H}^1(-\ell_0, +\infty)$ such that

$$f(t) = w(t) + f(\omega(t)) \quad \text{for every } t \geq \ell_0.$$

From now on we shall identify u with its continuous representative. Equality (1.24) implies that, for every $t > 0$, the function $u(t,\cdot)$ belongs to $H^1(0,\ell(t))$. Moreover, for every $t > 0$, the partial derivatives, defined as the limits of the corresponding difference quotients, exist for a.e. $(t,x) \in \Omega$ and satisfy the equalities

$$u_t(t,x) = \dot{w}(t+x) - \dot{f}(t+x) + \dot{f}(t-x), \tag{1.25a}$$

$$u_x(t,x) = \dot{w}(t+x) - \dot{f}(t+x) - \dot{f}(t-x). \tag{1.25b}$$

Therefore, if we set $u = 0$ on $(0,+\infty)^2 \setminus \Omega$, taking into account the boundary condition $u(t,\ell(t)) = 0$, we obtain that $u(t,\cdot) \in H^1(0,+\infty)$ for every $t > 0$. Moreover $t \mapsto u(t,\cdot)$ belongs to $C^0([0,+\infty); H^1(0,+\infty))$, while $t \mapsto u_t(t,\cdot)$ and $t \mapsto u_x(t,\cdot)$ belong to $C^0([0,+\infty); L^2(0,+\infty))$.

Remark 1.10. We denote by ω^k the composition of ω with itself k times. The construction of f in the proof of the previous theorem shows that for every $s \in [\ell_0,+\infty)$ there exists a nonnegative integer n, depending on s and with $n \leq \frac{s+\ell_0}{2\ell_0}$, such that $\omega^n(s) \in [-\ell_0, \ell_0)$ and

$$f(s) = \sum_{k=0}^{n-1} w(\omega^k(s)) + f(\omega^n(s)). \tag{1.26}$$

Since $f(\omega^n(s))$ can be computed using (1.19), this provides an alternative formula of f in $[-\ell_0, +\infty)$, whose geometrical meaning is described in Figure 1.1.

Remark 1.11 (Causality)**.** In order to prove Theorem 1.8, we needed formula (1.17), which expresses $u(t,x)$ using $w(t+x) - f(t+x)$. Hence, $u(t,x)$ seems to depend on the value of the prescribed vertical displacement at a time larger than t. However, one can see that $u(t,x)$ can be alternatively written using the data of the problem (the initial conditions, the boundary condition w, and the prescribed debonding front ℓ) evaluated only at times smaller than t.

Indeed, if $t+x \leq \ell_0$, formula (1.19) shows that $w(t+x) - f(t+x)$ only depends on the initial conditions. On the other hand, for every (t,x) such that $t+x > \ell_0$ there exists $s > 0$ such that $t+x = s+\ell(s) = \psi(s)$, because ψ is invertible. Therefore, using (1.16) we get

$$w(t+x) - f(t+x) = f(\omega(t+x)). \tag{1.27}$$

Notice that $\omega(t+x) \leq \omega(t+\ell(t)) = t-\ell(t) < t$.

If the vertical displacement w is prescribed only in a time interval $[0,T]$, we can extend it to any $\tilde{w} \in \widetilde{H}^1(0,+\infty)$ such that $\tilde{w} = w$ in $[0,T]$ in order to apply Theorem 1.8. Then, by (1.27), the solution u will not depend on the chosen extension.

Remark 1.12 (Regularity)**.** The regularity of the solution to problem (0.1) depends on the data. If we assume that the debonding front ℓ is of class $C^{1,1}(0,+\infty)$, the loading w belongs to $\widetilde{C}^{1,1}(0,+\infty)$, and the initial conditions satisfy $u_0 \in C^{1,1}([0,\ell_0])$, $u_1 \in C^{0,1}([0,\ell_0])$, and

$$u_1(0) = \dot{w}(0), \tag{1.28a}$$

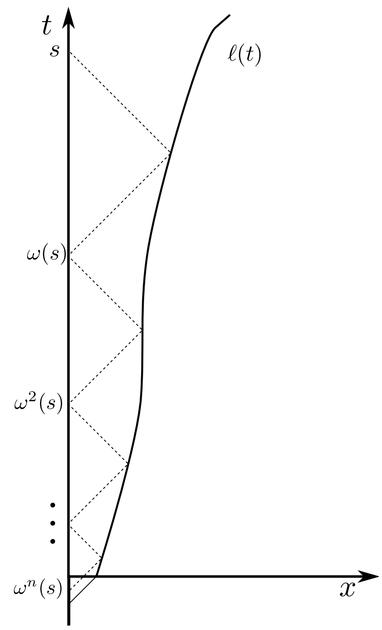

Figure 1.1: Construction of the sequence in Remark 1.10.

$$\dot{u}_0(\ell_0)\dot{\ell}(0) + u_1(\ell_0) = 0, \tag{1.28b}$$

then the solution u is of class $\widetilde{C}^{1,1}(\Omega)$, as one can see using the construction introduced in the proof of Theorem 1.8. Indeed, the function f constructed in Theorem 1.8 belongs to $C^{1,1}([-\ell_0, \ell_0])$ by (1.6b), (1.19), and (1.28a), while $f \in C^{1,1}([t_i, t_{i+1}])$ by (1.22). We already know that f is continuous at t_i by (1.6b); the continuity of $\dot{f}$ at t_i is a consequence of (1.19), (1.22), and (1.28b). This implies that $f \in \widetilde{C}^{1,1}(-\ell_0, +\infty)$ and guarantees the $C^{1,1}$-regularity of the solution u in the whole of Ω. If condition (1.28b) does not hold, we still have $f \in C^{1,1}([-\ell_0, \ell_0])$ and $f \in C^{1,1}([t_i, t_{i+1}])$ for every $i \geq 0$, but the function $\dot{f}$ may be discontinuous at the points t_i; in this case u is only piecewise regular in Ω. Similarly, if condition (1.28a) does not hold, we may have discontinuities of $\dot{f}$ at 0 and, by the "bounce formula" (1.16), at times $\omega^{-1}(0)$, $\omega^{-2}(0)$, ...

We conclude this section with some results on the energy balance for a solution to problem (0.1). For a solution $u \in \widetilde{H}^1(\Omega)$ to problem (0.1) the derivatives $u_x(t,x)$ and $u_t(t,x)$ are defined for every $t > 0$ and almost every $x > 0$ by Remark 1.9. The energy of u is defined for every $t \in [0, +\infty)$ by

$$\mathcal{E}(t) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2 \,\mathrm{d}x, \tag{1.29}$$

where the first term is the potential energy and the second one is the kinetic energy.

Proposition 1.13. *Let $u \in \widetilde{H}^1(\Omega)$ be a solution to problem* (0.1). *Then $\mathcal{E}\colon [0, +\infty) \to \mathbb{R}$ is absolutely continuous in $[0, T]$ for every $T > 0$. Moreover we have*

$$\mathcal{E}(t) = \int_{t-\ell(t)}^{t} \dot{f}(s)^2 \,\mathrm{d}s + \int_t^{t+\ell(t)} [\dot{w}(s) - \dot{f}(s)]^2 \,\mathrm{d}s \tag{1.30}$$

for every $t \in [0, +\infty)$, where f is as in Proposition 1.4.

Proof. Using (1.24) and (1.25), we can write

$$\begin{aligned}
&\frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\ell(t)} \Big[\big(\dot w(t{+}x) - \dot f(t{+}x) + \dot f(t{-}x)\big)^2 + \big(\dot w(t{+}x) - \dot f(t{+}x) - \dot f(t{-}x)\big)^2\Big]\,\mathrm{d}x \\
&= \int_{t-\ell(t)}^{t} \dot f(s)^2\,\mathrm{d}s + \int_t^{t+\ell(t)} [\dot w(s) - \dot f(s)]^2\,\mathrm{d}s,
\end{aligned}$$

where in the last equality we have used obvious changes of variables. Since the expression in last line of the last formula is absolutely continuous on $[0,T]$ for every $T>0$, the proof is complete. □

Proposition 1.14. *Let u and $\mathcal{E}$ be as in Proposition 1.13. Then $\mathcal{E}$ satisfies the energy balance*

$$\mathcal{E}(t) = \mathcal{E}(0) - 2\int_0^t \dot\ell(s)\frac{1{-}\dot\ell(s)}{1{+}\dot\ell(s)}\dot f(s{-}\ell(s))^2\,\mathrm{d}s - \int_0^t [\dot w(s) - 2\dot f(s)]\dot w(s)\,\mathrm{d}s, \tag{1.31}$$

for every $t\in[0,+\infty)$.

The second integral in (1.31) can be interpreted as the work corresponding to the prescribed displacement. The first integral is related to the notion of dynamic energy release rate as explained in Section 1.2.

Proof. Thanks to (1.30), for a.e. $t\in[0,+\infty)$ we have

$$\dot{\mathcal{E}}(t) = [\dot w(t{+}\ell(t)) - \dot f(t{+}\ell(t))]^2(1{+}\dot\ell(t)) - [\dot w(t) - \dot f(t)]^2 + \dot f(t)^2 - \dot f(t{-}\ell(t))^2(1{-}\dot\ell(t)). \tag{1.32}$$

The boundary condition $u(t,\ell(t)) = 0$ together with (1.24) gives $w(t{+}\ell(t)) - f(t{+}\ell(t)) + f(t{-}\ell(t)) = 0$ for every $t\geq 0$. By differentiating we obtain

$$\dot w(t{+}\ell(t))(1{+}\dot\ell(t)) - \dot f(t{+}\ell(t))(1{+}\dot\ell(t)) + \dot f(t{-}\ell(t))(1{-}\dot\ell(t)) = 0$$

for a.e. $t\in[0,+\infty)$. From this equality and from (1.32) we obtain, with easy algebraic manipulations,

$$\dot{\mathcal{E}}(t) = -2\dot\ell(t)\frac{1{-}\dot\ell(t)}{1{+}\dot\ell(t)}\dot f(t{-}\ell(t))^2 - [\dot w(t) - 2\dot f(t)]\dot w(t),$$

for a.e. $t\in[0,+\infty)$. This proves (1.31), since $\mathcal{E}$ is absolutely continuous on $[0,T]$ for every $T>0$. □

1.2 Dynamic energy release rate and Griffith's criterion

In this section we introduce in a rigorous way the dynamic energy release rate in our context; such a notion will be used to formulate Griffith's criterion throughout this thesis. To this end we assume that the debonding front $t\mapsto\ell(t)$ satisfies (1.1a). Let u be the solution to (0.1) in Ω, with $w\in\widetilde H^1(0,+\infty)$, $u_0\in H^1(0,\ell_0)$, and $u_1\in L^2(0,\ell_0)$, satisfying the compatibility conditions (1.6b). (See Remark 1.11.)

1.2.1 Dynamic energy release rate

To define the dynamic energy release rate we fix $\bar{t} > 0$ and consider virtual modifications z and λ of the functions w and ℓ after $\bar{t}$. We then consider the corresponding solution v to problem (0.1) and we study the dependence of its energy on z and λ. More precisely, we consider a function $z \in \widetilde{H}^1(0,+\infty)$ and a function $\lambda\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfying condition (1.1a), with

$$z(t) = w(t) \quad \text{and} \quad \lambda(t) = \ell(t) \quad \text{for every } t \le \bar{t}. \tag{1.33}$$

We consider the problem

$$\begin{cases} v_{tt}(t,x) - v_{xx}(t,x) = 0, & t > 0,\ 0 < x < \lambda(t), \\ v(t,0) = z(t), & t > 0, \\ v(t,\lambda(t)) = 0, & t > 0, \\ v(0,x) = u_0(x), & 0 \le x \le \ell_0, \\ v_t(0,x) = u_1(x), & 0 \le x \le \ell_0, \end{cases} \tag{1.34}$$

whose solution has to be interpreted in the sense of Definition 1.2 and of Remark 1.9. We recall that by Remark 1.11 $v(t,x) = u(t,x)$ for every $(t,x) \in \Omega_{\bar{t}}$. By the previous results, there exists a unique function $g \in \widetilde{H}^1(-\ell_0,+\infty)$ with $g(0) = 0$ such that

$$v(t,x) = z(t+x) - g(t+x) + g(t-x).$$

By Remark 1.11 we have

$$g = f \quad \text{in } [-\ell_0, \bar{t}]. \tag{1.35}$$

Recalling (1.29), we now define

$$\mathcal{E}(t;\lambda,z) := \frac{1}{2}\int_0^{\lambda(t)} \left[v_x(t,x)^2 + v_t(t,x)^2\right] \mathrm{d}x. \tag{1.36}$$

By Proposition 1.14 we have

$$\dot{\mathcal{E}}(t;\lambda,z) = -2\dot{\lambda}(t)\frac{1-\dot{\lambda}(t)}{1+\dot{\lambda}(t)}\dot{g}(t-\lambda(t))^2 - \dot{z}(t)[\dot{z}(t) - 2\dot{g}(t)] \quad \text{for a.e. } t > 0. \tag{1.37}$$

This is not enough for our purposes, since we want to compute the right derivative $\dot{\mathcal{E}}_r(\bar{t};\lambda,z)$ at $t = \bar{t}$. This will be done in the next proposition. We recall that, by definition, $\bar{t} \in [0,+\infty)$ is a right Lebesgue point of $\dot{\lambda}$ if there exists $\alpha \in \mathbb{R}$ such that

$$\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} \left|\dot{\lambda}(t) - \alpha\right| \mathrm{d}t \to 0, \quad \text{as } h \to 0^+. \tag{1.38}$$

We say that $\bar{t}$ is a right L^2-Lebesgue point for $\dot{z}$ and $\dot{g}$, respectively, if there exist β and γ in $\mathbb{R}$ such that

$$\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{z}(t) - \beta|^2 \,\mathrm{d}t \to 0 \quad \text{and} \quad \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{g}(t) - \gamma|^2 \,\mathrm{d}t \to 0, \quad \text{as } h \to 0^+. \tag{1.39}$$

It is easy to see that, in this case, we also have

$$\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{z}(t)(\dot{z}(t) - 2\dot{g}(t)) - \beta(\beta - 2\gamma)| \,\mathrm{d}t \to 0, \quad \text{as } h \to 0^+. \tag{1.40}$$

Proposition 1.15. *Assume* (1.1) *(with* $\ell_0 > 0$*),* (1.5)*, and* (1.6)*. Then there exists a set* $N \subset [0, +\infty)$*, with measure zero, depending only on* ℓ*,* w*,* u_0*, and* u_1*, such that the following property holds for every* $\bar{t} \in [0, +\infty) \setminus N$*: if* λ *and* z *are as above, if* v*,* g*, and* $\mathcal{E}(\cdot; \lambda, z)$ *are defined by* (1.34)*&*(1.36)*, if* $\dot{\lambda}$ *has a right Lebesgue point at* $\bar{t}$*, and if* $\dot{z}$ *has a right* L^2*-Lebesgue point at* $\bar{t}$*, then* $\bar{t}$ *is a right* L^2*-Lebesgue point for* $\dot{g}$ *and*

$$\dot{\mathcal{E}}_r(\bar{t}; \lambda, z) = -2\alpha \frac{1-\alpha}{1+\alpha} \dot{f}(\bar{t} - \ell(\bar{t}))^2 - \beta(\beta - 2\gamma), \tag{1.41}$$

where α*,* β*, and* γ *are as in* (1.38) *and* (1.39)*.*

Proof. We consider the points $\bar{t}$ with the following properties:

a) $\dot{f}$ exists at $\bar{t} - \ell(\bar{t})$ and $\displaystyle\lim_{h \to 0^+} \frac{1}{h} \int_{\bar{t}-\ell(\bar{t})}^{\bar{t}-\ell(\bar{t})+h} \left| \dot{f}(s)^2 - \dot{f}(\bar{t} - \ell(\bar{t}))^2 \right| \mathrm{d}s = 0$;

b$_1$) if $\bar{t} \le \ell_0$, $\bar{t}$ is an L^2-Lebesgue point for $\dot{u}_0$ and u_1;

b$_2$) if $\bar{t} \ge \ell_0$, $\bar{t}$ is a Lebesgue point for $\dot{\omega}$ and $\omega(\bar{t})$ is an L^2-Lebesgue point for $\dot{f}$.

We call E the set of the points satisfying all the properties above. It is well known that $N := [0, +\infty) \setminus E$ has measure zero. Let us fix $\bar{t} \in E$.

Let us prove that $\dot{g}$ has a right L^2-Lebesgue point at $\bar{t}$. This is clear if $\bar{t} < \ell_0$. Assume $\bar{t} \ge \ell_0$; using (1.23) and (1.35), we have

$$g(t) = z(t) + g(\omega(t)) = z(t) + f(\omega(t)), \text{ for every } t \in [\ell_0, \bar{t} + \ell(\bar{t})].$$

Then we have

$$\dot{g}(t) = \dot{z}(t) + \dot{f}(\omega(t))\dot{\omega}(t), \qquad \text{for a.e. } t \in (\ell_0, \bar{t} + \ell(\bar{t})].$$

Since $\dot{z}$ has a right L^2-Lebesgue point at $\bar{t}$, it is enough to prove that $\dot{f}(\omega(t))\dot{\omega}(t)$ has a right L^2-Lebesgue point at $t = \bar{t}$. Since $\bar{t} \in E$, there exist a and b in $\mathbb{R}$ such that

$$\frac{1}{h} \int_{\omega(\bar{t})}^{\omega(\bar{t})+h} \left| \dot{f}(s) - a \right|^2 \mathrm{d}s \to 0 \quad \text{ and } \quad \frac{1}{h} \int_{\bar{t}}^{\bar{t}+h} |\dot{\omega}(s) - b|^2 \,\mathrm{d}s \to 0, \tag{1.42}$$

where in the last formula we used the fact that $\dot{\omega}$ is bounded. We now have

$$\begin{aligned} &\frac{1}{h} \int_{\bar{t}}^{\bar{t}+h} \left| \dot{f}(\omega(t))\dot{\omega}(t) - ab \right|^2 \mathrm{d}t \\ \le &\frac{2}{h} \int_{\bar{t}}^{\bar{t}+h} \left| \dot{f}(\omega(t)) - a \right|^2 \dot{\omega}(t)^2 \,\mathrm{d}t + \frac{2}{h} \int_{\bar{t}}^{\bar{t}+h} a^2 \left| \dot{\omega}(t) - b \right|^2 \mathrm{d}t. \end{aligned} \tag{1.43}$$

Using the change of variables $s = \omega(t)$ and the inequalities $0 \le \dot{\omega} \le 1$, we deduce from (1.42) that the right hand side in (1.43) tends to zero. This proves that $\bar{t}$ is a right L^2-Lebesgue point for $\dot{g}$.

We now prove the formula for the right derivative of the energy at $\bar{t} \in E$. By (1.37), we have

$$\begin{aligned}
&\left| \frac{\mathcal{E}(\bar{t}+h;\lambda,z) - \mathcal{E}(\bar{t};\lambda,z)}{h} - \left(-2\alpha \frac{1-\alpha}{1+\alpha} \dot{f}(\bar{t}-\ell(\bar{t}))^2 - \beta(\beta-2\gamma) \right) \right| \\
\leq &\left| \frac{2}{h} \int_{\bar{t}}^{\bar{t}+h} \left(\dot{\lambda}(t) \frac{1-\dot{\lambda}(t)}{1+\dot{\lambda}(t)} \dot{g}(t-\lambda(t))^2 - \alpha \frac{1-\alpha}{1+\alpha} \dot{f}(\bar{t}-\ell(\bar{t}))^2 \right) \mathrm{d}t \right| \\
&+ \left| \frac{1}{h} \int_{\bar{t}}^{\bar{t}+h} (\dot{z}(t)(\dot{z}(t) - 2\dot{g}(t)) - \beta(\beta-2\gamma)) \, \mathrm{d}t \right| \\
\leq &\frac{2}{h} \int_{\bar{t}}^{\bar{t}+h} \dot{\lambda}(t) \frac{1-\dot{\lambda}(t)}{1+\dot{\lambda}(t)} \left| \dot{g}(t-\lambda(t))^2 - \dot{f}(\bar{t}-\ell(\bar{t}))^2 \right| \mathrm{d}t \\
&+ \frac{2}{h} \dot{f}(\bar{t}-\ell(\bar{t}))^2 \int_{\bar{t}}^{\bar{t}+h} \left| \dot{\lambda}(t) \frac{1-\dot{\lambda}(t)}{1+\dot{\lambda}(t)} - \alpha \frac{1-\alpha}{1+\alpha} \right| \mathrm{d}t \\
&+ \frac{1}{h} \int_{\bar{t}}^{\bar{t}+h} |\dot{z}(t)(\dot{z}(t) - 2\dot{g}(t)) - \beta(\beta-2\gamma)| \, \mathrm{d}t =: I_h^1 + I_h^2 + I_h^3. \qquad (1.44)
\end{aligned}$$

By (1.35) we can replace $\dot{g}(\cdot)$ by $\dot{f}(\cdot)$ in I_h^1. Hence

$$\begin{aligned}
I_h^1 \leq &\frac{2}{h} \int_{\bar{t}}^{\bar{t}+h} (1-\dot{\lambda}(t)) \left| \dot{f}(t-\lambda(t))^2 - \dot{f}(\bar{t}-\ell(\bar{t}))^2 \right| \mathrm{d}t \\
\leq &\frac{2}{h} \int_{\bar{t}-\ell(\bar{t})}^{\bar{t}-\ell(\bar{t})+h} \left| \dot{f}(s)^2 - \dot{f}(\bar{t}-\ell(\bar{t}))^2 \right| \mathrm{d}s \to 0, \quad \text{as } h \to 0^+,
\end{aligned}$$

where we have used the change of variables $s = t - \lambda(t)$ and the fact that $\ell(\bar{t}) = \lambda(\bar{t}) \leq \lambda(\bar{t}+h)$. Moreover, since the function $x \mapsto x\frac{1-x}{1+x}$ is Lipschitz and since $\bar{t}$ is a right Lebesgue point for $\dot{\lambda}$, we conclude that

$$I_h^2 \to 0, \quad \text{as } h \to 0^+. \qquad (1.45)$$

Equations (1.44)–(1.45), together with (1.40), prove (1.41). □

Remark 1.16. The set N introduced in Proposition 1.15 can be chosen in such a way that $N \cap [0,t]$ depends only on the restriction of ℓ and w to $[0,t]$, cf. also (1.3). Moreover, (1.26) shows that (1.39) does not depend on the choice of λ but only on z.

We are now in a position to introduce the notion of dynamic energy release rate, which measures the amount of energy spent during the debonding evolution. It is defined as a sort of partial derivative of $\mathcal{E}$ with respect to the elongation of the debonded region. More precisely, we fix $\bar{t} > 0$, we consider an arbitrary virtual extension λ of $\ell|_{[0,\bar{t}]}$ with right speed α at $\bar{t}$ in the sense of (1.38), and we freeze the loading after time $\bar{t}$ at the level $w(\bar{t})$. The derivative of the energy $\mathcal{E}$ with respect to the elongation is obtained by taking the time derivative and dividing it by the velocity α.

Definition 1.17. *For a.e. $\bar{t} > 0$ and every $\alpha \in (0,1)$ the dynamic energy release rate corresponding to the velocity α of the debonding front is defined as*

$$G_\alpha(\bar{t}) := -\frac{1}{\alpha} \dot{\mathcal{E}}_r(\bar{t};\lambda,\bar{z}),$$

where $\lambda\colon [0,+\infty) \to [\ell_0,+\infty)$ is an arbitrary extension of $\ell|_{[0,\bar{t}]}$ satisfying conditions (1.1a), (1.33), *and* (1.38), *while $\bar{z}(t) = w(t)$ for every $t \leq \bar{t}$ and $\bar{z}(t) = w(\bar{t})$ for every $t > \bar{t}$.*

Proposition 1.15 implies that

$$G_\alpha(\bar t) = 2\frac{1-\alpha}{1+\alpha}\dot f(\bar t - \ell(\bar t))^2 \quad \text{for a.e. } \bar t > 0. \tag{1.46}$$

In particular, $G_\alpha(\bar t)$ depends on λ only through α, so the definition is well posed.

Straightforward computations based on (1.25b) show that, when the solution is regular enough so that $u_x(\bar t, \ell(\bar t))$ is well defined for a.e. $\bar t > 0$, the dynamic energy release rate can also be expressed as

$$G_\alpha(\bar t) = \frac{1}{2}(1-\alpha^2)u_x(\bar t, \ell(\bar t))^2. \tag{1.47}$$

This is consistent with the formulas given in [28].

The dynamic energy release rate can be extended to the case $\alpha = 0$, by continuity, as

$$G_0(\bar t) := 2\dot f(\bar t - \ell(\bar t))^2. \tag{1.48}$$

We observe that, by (1.46), $G_\alpha(\bar t)$ is continuous and strictly monotone with respect to α and

$$G_\alpha(\bar t) < G_0(\bar t), \text{ for every } \alpha \in (0,1), \quad G_\alpha(\bar t) \to 0 \text{ for } \alpha \to 1^-, \tag{1.49}$$

for a.e. $\bar t > 0$.

1.2.2 Griffith's criterion

To introduce Griffith's criterion for the debonding model we consider the notion of local toughness of the glue between the substrate and the film. This is a measurable function $\kappa\colon [0,+\infty) \to [c_1, c_2]$, with $0 < c_1 < c_2$, with the following mechanical interpretation: the energy dissipated to debond a segment $[x_1, x_2]$, with $0 \le x_1 < x_2$ is given by

$$\int_{x_1}^{x_2} \kappa(x)\,\mathrm{d}x.$$

This implies that, for every $t > 0$, the energy dissipated in the debonding process in the time interval $[0,t]$ is

$$\int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x.$$

In our model we postulate the following energy-dissipation balance: for every $t > 0$ we have

$$\mathcal{E}(t;\ell,w) + \int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x = \mathcal{E}(0;\ell,w) - \int_0^t \dot w(s)[\dot w(s) - 2\dot f(s)]\,\mathrm{d}s, \tag{1.50}$$

where the last term is the work of the external loading. By (1.31), (1.46), and (1.48) we obtain that (1.50) is equivalent to

$$\int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x = \int_0^t G_{\dot\ell(s)}(s)\dot\ell(s)\,\mathrm{d}s,$$

which, in turn, is equivalent to

$$\kappa(\ell(t))\dot\ell(t) = G_{\dot\ell(t)}(t)\dot\ell(t), \quad \text{for a.e. } t > 0. \tag{1.51}$$

In addition to the energy-dissipation balance we postulate the following maximum dissipation principle, as proposed in [41]: for a.e. $t > 0$

$$\dot{\ell}(t) = \max\{\alpha \in [0,1) : \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}. \tag{1.52}$$

This means that the debonding front must move as fast as possible, consistent with the energy-dissipation balance (1.50). We observe that the set $\{\alpha \in [0,1) : \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}$ has at most one element different from zero, by the strict monotonicity of $\alpha \mapsto G_\alpha(t)$. Therefore the maximum dissipation principle (1.52) simply states that the debonding front must move when this is possible.

Our postulates imply the following properties.

- For a.e. $t > 0$, if $\dot{\ell}(t) > 0$, then $\kappa(\ell(t)) = G_{\dot{\ell}(t)}(t)$.
- For a.e. $t > 0$, if $\dot{\ell}(t) = 0$ then $\kappa(\ell(t)) \geq G_{\dot{\ell}(t)}(t) = G_0(t)$. Indeed, if the opposite inequality holds, by continuity and by (1.49) then there exists $\alpha > 0$ such that $\kappa(\ell(t)) = G_\alpha(t)$, which contradicts (1.49).

This amounts to the following system, which will be called Griffith's criterion in analogy to the corresponding criterion in Fracture Mechanics: for a.e. $t > 0$

$$\dot{\ell}(t) \geq 0, \tag{1.53a}$$
$$G_{\dot{\ell}(t)}(t) \leq \kappa(\ell(t)), \tag{1.53b}$$
$$\left[G_{\dot{\ell}(t)}(t) - \kappa(\ell(t))\right] \dot{\ell}(t) = 0. \tag{1.53c}$$

Conversely, we now show that Griffith's criterion implies both the energy-dissipation balance and the maximum dissipation. Indeed, the third condition in Griffith's criterion implies (1.51) which is equivalent to the energy-dissipation balance. As for the maximum dissipation, (1.51) implies that $\dot{\ell}(t) \in \{\alpha \in [0,1) : \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}$. Recalling that this set has at most one positive element, we only need to prove that if $\dot{\ell}(t) = 0$, then there is no positive $\alpha > 0$ such that $G_\alpha(t) = \kappa(\ell(t))$. This is a consequence of the inequality in (1.49) and of (0.13b).

We conclude this section by proving that Griffith's criterion is equivalent to the following ordinary differential equation:

$$\dot{\ell}(t) = \frac{2\dot{f}(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot{f}(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0 \quad \text{for a.e. } t \in [0, +\infty). \tag{1.54}$$

We recall that $G_0(t) = 2\dot{f}(t - \ell(t))^2$, by (1.48). If $G_0(t) \leq \kappa(\ell(t))$, then the right hand side of (1.54) is zero. Moreover, by the strict monotonicity of $\alpha \mapsto G_\alpha(t)$ we have $G_\alpha(t) < \kappa(\ell(t))$ for every $\alpha > 0$, hence (0.13c) gives $\dot{\ell}(t) = 0$. Therefore (1.54) is satisfied in this case. Conversely, if $G_0(t) > \kappa(\ell(t))$, then the right hand side of (1.54) is strictly positive and $\dot{\ell}(t)$ is the unique $\alpha \in (0,1)$ such that $G_\alpha(t) = \kappa(\ell(t))$. Using (1.46), one sees that (1.54) holds.

1.3 Evolution of the debonding front

In this section we prove existence and uniqueness of a pair $(u(t,x), \ell(t))$ where u solves problem (0.1) (in the sense of Definitions 1.1 and 1.2) and ℓ satisfies Griffith's criterion (1.53) as formulated in the discussion above. By (1.54) we look for functions $t \mapsto f(t)$, $t \mapsto \ell(t)$ satisfying

$$\begin{cases} \dot{\ell}(t) = \dfrac{2\dot{f}(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot{f}(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, \quad \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{1.55}$$

We recall that, in order to solve system (0.1) in Ω_T for some $T > 0$, it is sufficient to apply Proposition 1.4 and find the related function f defined in $[-\ell_0, T+\ell(T)]$; the solution u is then given by (1.17). The pair (f, ℓ) is found by recursively applying an alternate scheme where the two systems (0.1) and (1.55) are solved separately and iteratively. More precisely, one starts from the definition of f in $[-\ell_0, \ell_0]$, given by Proposition 1.6. Thus (1.55) can be solved in a time interval $[0, s_1]$ such that the right-hand side of the differential equation is defined; this is illustrated in the proof of the theorem below. The debonding front $\ell\colon [0, s_1] \to [\ell_0, +\infty)$ turns out to be as in the assumptions of Section 1.1, hence f can be defined in a subsequent interval $[\ell_0, t_1]$ thanks to the "bounce formula" (1.16). This alternate scheme is then iterated in order to find the solution in the whole domain.

We are now in a position to state the first existence result under regularity assumptions on the data. The main point is to solve (1.55) in the first time interval $[0, s_1]$.

Theorem 1.18. *Let $u_0 \in C^{1,1}([0, \ell_0])$, $u_1 \in C^{0,1}([0, \ell_0])$, and $w \in \widetilde{C}^{1,1}(0, +\infty)$ be such that* (1.6b) *and* (1.28a) *hold. Assume that the local toughness $\kappa\colon [0, +\infty) \to [c_1, c_2]$ belongs to $\widetilde{C}^{0,1}(0, +\infty)$. Assume in addition that*

$$u_1(\ell_0) + \dot{u}_0(\ell_0) \left\{ \frac{2\left[-\frac{\dot{u}_0(\ell_0)}{2} + \frac{u_1(\ell_0)}{2}\right]^2 - \kappa(\ell_0)}{2\left[-\frac{\dot{u}_0(\ell_0)}{2} + \frac{u_1(\ell_0)}{2}\right]^2 + \kappa(\ell_0)} \vee 0 \right\} = 0. \tag{1.56}$$

Then, there exists a unique pair $(u, \ell) \in \widetilde{H}^1(\Omega) \times \widetilde{C}^{0,1}(0, +\infty)$ satisfying (0.1)*&*(1.55)*. Moreover, one has $(u, \ell) \in \widetilde{C}^{1,1}(\Omega) \times \widetilde{C}^{1,1}(0, +\infty)$ and $0 \le \dot{\ell}(t) < 1$ for every $t \in [0, +\infty)$.*

Proof. We define f in the interval $[-\ell_0, \ell_0]$ by (1.19). Our regularity assumptions and the condition (1.28a) guarantee that $f \in C^{1,1}([-\ell_0, \ell_0])$. Therefore the right hand side of the differential equation in (1.55) is Lipschitz and bounded by a constant strictly smaller than one. We now set $t_0 := \ell_0$. We can thus find a unique solution to (1.55) defined up to the unique time s_1 with $s_1 - \ell(s_1) = t_0$. Notice that $\ell \in C^{1,1}([0, s_1])$. Moreover, by (1.19), (1.28a), and (1.55), $\dot{\ell}(0)$ coincides with the term in curly brackets in (1.56), hence condition (1.28b) is satisfied. With the aid of the "bounce formula" (1.16), we can now find the value of f in the interval $[t_0, t_1]$ where $t_1 = s_1 + \ell(s_1)$. By Remark 1.12, f and $\dot{f}$ are continuous at t_0. By now, the problem is uniquely solved with a pair (u, ℓ), with ℓ defined in $[0, s_1]$ and u defined (through formula (1.24)) in $\overline{\Omega}_{s_1} \cup \{(t, x) : t \in [s_1, t_1],\ 0 \le x \le t_1 - t\}$, that is the grey part in Figure 1.2. We also notice that $f \in C^{1,1}([t_0, t_1])$, so that we can repeat the previous argument in order to find a unique solution to the differential equation in (1.55), with initial conditions given by $\ell(s_1)$, in the time interval $[s_1, s_2]$, where $s_2 - \ell(s_2) = t_1$. Applying again (1.16) we can define f on the interval $[t_1, t_2]$, where $t_2 = s_2 + \ell(s_2)$. Arguing as in Remark 1.12, we can deduce that $f \in C^{1,1}([t_1, t_2])$ and f, $\dot{f}$ are continuous at t_1. Formula (1.24) leads to a unique solution u of problem (0.1) defined in $\overline{\Omega}_{s_2} \cup \{(t, x) : t \in [s_2, t_2],\ 0 \le x \le t_2 - t\}$. By iterating this argument we construct two sequences $\{s_i\}$ and $\{t_i\}$, with $t_i < s_{i+1} < t_{i+1}$ and $t_{i+1} = s_{i+1} + \ell(s_{i+1}) \ge t_i + \ell_0$ and we extend progressively the definitions of ℓ and f to the intervals $[0, s_i]$ and $[-\ell_0, t_i]$ respectively.

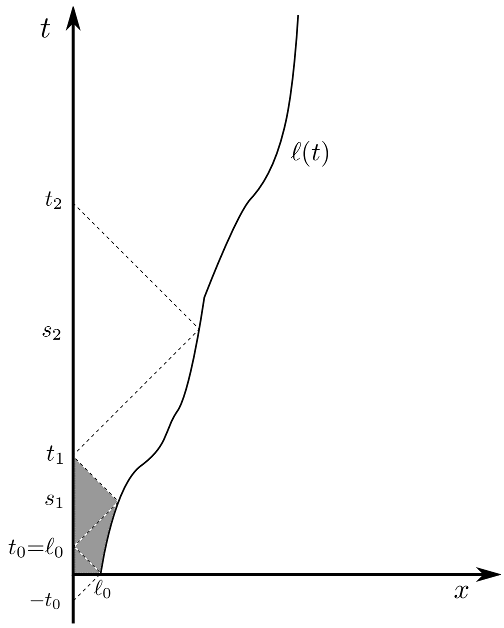

Figure 1.2: Construction of the solution $(\ell(t), u(t,x))$.

Since $t_i \to +\infty$, we are able to find a unique solution (u, ℓ) to the coupled problem defined in $\Omega\times[0,+\infty)$. The inequality $0 \le \dot{\ell}(t) < 1$ follows easily from the equation (1.55). □

Remark 1.19. We make some remarks on the role of conditions (1.28) in Theorem 1.18. (Recall that (1.28b) follows from (1.56).) When they are not satisfied, arguing as in the previous proof we see that $f \in \widetilde{C}^{0,1}(-\ell_0, +\infty)$ and $\ell \in C^{0,1}(0, +\infty)$, and they are only piecewise $C^{1,1}$. Indeed, $\dot{f}$ may have discontinuities at times 0 and ℓ_0 (and their subsequent times $\omega^{-1}(0)$, $\omega^{-1}(\ell_0)$, etc., according to the previous construction). Such discontinuities generate forward and backward shock waves travelling with speed 1 and -1, respectively, and represented by lines $R_1^+ := \{(t,t) : t \in [0, \varphi^{-1}(0)]\}$ and $S_1^- := \{(t, \ell_0 - t) : t \in [0, \ell_0]\}$. At time $t = \varphi^{-1}(0)$, R_1^+ intersects the front of debonding, causing a discontinuity for $\dot{\ell}$; the forward shock wave is then reflected into a backward shock wave $R_2^- := \{(t, \omega^{-1}(0) - t) : t \in [\varphi^{-1}(0), \omega^{-1}(0)]\}$. Analogously, the backward shock wave S_1^- intersects the axis $x = 0$ and it is transformed into a forward shock wave $S_2^+ := \{(t, t - \ell_0) : t \in [\ell_0, \varphi^{-1}(\ell_0)]\}$. By iterating this argument we construct lines where the following Rankine-Hugoniot conditions for the derivatives of u hold:

$$[\![u_x]\!] + [\![u_t]\!] = 0 \text{ on } \bigcup_{i=1}^{\infty} \left(R_{2i-1}^+ \cup S_{2i}^+\right) \text{ and } [\![u_x]\!] - [\![u_t]\!] = 0 \text{ on } \bigcup_{i=0}^{\infty} \left(R_{2i}^- \cup S_{2i-1}^-\right),$$

where $[\![\cdot]\!]$ denotes the difference between the values of the functions across the discontinuity line.

Remark 1.20. Under the assumptions of Theorem 1.18 we have the equality

$$\kappa(\ell_0) = G_{\dot{\ell}(0)}(0). \tag{1.57}$$

Indeed, the formula for $\dot\ell(0)$ in the proof implies that, if $\dot\ell(0)>0$, we have

$$u_1(\ell_0)+\dot u_0(\ell_0)\,\frac{2\left[-\frac{\dot u_0(\ell_0)}{2}+\frac{u_1(\ell_0)}{2}\right]^2-\kappa(\ell_0)}{2\left[-\frac{\dot u_0(\ell_0)}{2}+\frac{u_1(\ell_0)}{2}\right]^2+\kappa(\ell_0)}=0,$$

which implies

$$\kappa(\ell_0)=\frac{1}{2}\left[\dot u_0(\ell_0)^2-u_1(\ell_0)^2\right]=G_{\dot\ell(0)}(0),$$

where the last equality follows from (1.47) and (1.28b). If instead $\dot\ell(0)=0$, by analogous computations we find that

$$\kappa(\ell_0)=\frac{1}{2}\dot u_0(\ell_0)^2=G_0(0),$$

which concludes the proof of (1.57).

We now prove existence and uniqueness for the coupled system (0.1)&(1.55) under weaker regularity assumptions on the data. More precisely, we assume

$$u_0\in C^{0,1}([0,\ell_0]),\quad u_1\in L^\infty(0,\ell_0),\ \text{ and } w\in\widetilde C^{0,1}(0,+\infty). \tag{1.58}$$

In Theorem 1.21 we assume that the local toughness κ is constant, while in Theorem 1.22 we consider a nonconstant toughness. Since the arguments in the proof are different, we prefer to present both cases separately.

Theorem 1.21. *Let u_0, u_1, and w satisfy* (1.6b) *and* (1.58) *and let the local toughness κ be a positive constant. Then, there exists a unique pair $(u,\ell)\in\widetilde H^1(\Omega)\times\widetilde C^{0,1}(0,+\infty)$ satisfying* (0.1)*&*(1.55)*. Moreover, one has $u\in\widetilde C^{0,1}(\Omega)$ and for every $T>0$ there exists $L_T<1$ such that*

$$0\le\dot\ell(t)\le L_T\quad\textit{for a.e. } t\in(0,T). \tag{1.59}$$

Proof. We define f in $[-\ell_0,\ell_0]$ by (1.19). Since our regularity assumptions imply only that $f\in C^{0,1}([-\ell_0,\ell_0])$, we now have to justify existence and uniqueness of a local solution to (1.55). This is done by reducing the problem to an autonomous equation, using the fact that κ is constant. Set $z(t):=t-\ell(t)$. Then the Cauchy problem (1.55) reduces to

$$\begin{cases}\dot z(t)=F(z),\\ z(0)=-\ell_0,\end{cases} \tag{1.60}$$

where

$$F(z):=1-\frac{\left(2\dot f(z)^2-\kappa\right)\vee 0}{2\dot f(z)^2+\kappa}.$$

Since $\dot f$ is bounded on $[-\ell_0,\ell_0]$, there exists a constant $c_0\in(0,1)$ such that $F(z)\ge c_0$ for a.e. $z\in[-\ell_0,\ell_0]$. The standard formula for the solution of autonomous Cauchy problems implies that, setting

$$s_1=\int_{-\ell_0}^{\ell_0}\frac{\mathrm{d}z}{F(z)},$$

problem (1.60) has a unique solution $z \in C^{0,1}([0, s_1])$ and that this solution satisfies

$$\int_{-\ell_0}^{z(t)} \frac{\mathrm{d}z}{F(z)} = t, \qquad \text{for every } t \in [0, s_1].$$

Notice that s_1 is the unique point such that $s_1 - \ell(s_1) = \ell_0$. Since $\dot{\ell}(t) = 1 - \dot{z}(t) < 1 - c_0$, we have that $\omega(t)$ (see (1.3)) is bi-Lipschitz and thus, by the bounce formula (1.23), $f \in C^{0,1}([t_0, t_1])$, where $t_0 = \ell_0$ and $t_1 = \omega^{-1}(t_0) = s_1 + \ell(s_1)$. Then one can argue iteratively imitating the proof of Theorem 1.18, without the part concerning the continuity of $\dot{f}$. We thus find a unique solution (u, ℓ) on $\Omega \times [0, +\infty)$ which now belongs to $\widetilde{C}^{0,1}(\Omega) \times \widetilde{C}^{0,1}(0, +\infty)$. □

We extend this result to a wider class of local toughnesses.

Theorem 1.22. *Let u_0, u_1, and w satisfy* (1.6b) *and* (1.58) *and let $\kappa \in \widetilde{C}^{0,1}(\ell_0, +\infty)$ with $c_1 \le \kappa \le c_2$. Then, there exists a unique pair $(u, \ell) \in \widetilde{H}^1(\Omega) \times \widetilde{C}^{0,1}(0, +\infty)$ satisfying* (0.1)*&*(1.55)*. Moreover, $u \in \widetilde{C}^{0,1}(\Omega)$ and for every $T > 0$ there exists $L_T < 1$ such that* (1.59) *is satisfied.*

Proof. As in the proof of Theorem 1.21, we only have to study (1.55) in a first time interval $[0, s_1]$. Set $z(t) = t - \ell(t)$. We look for solutions to the system

$$\begin{cases} \dot{z}(t) = \dfrac{2\kappa(t-z)}{2\dot{f}(z)^2 + \kappa(t-z)} \wedge 1, \\ z(0) = -\ell_0. \end{cases}$$

Any solution must satisfy $\dot{z} > 0$ a.e. and therefore $t \mapsto z(t)$ is invertible. The equation solved by $t(z)$ is

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \left(\frac{1}{2} + \frac{\dot{f}(z)^2}{\kappa(t-z)} \right) \vee 1 =: \Phi(z, t), \tag{1.61}$$

with initial condition $t(-\ell_0) = 0$. Recalling that $\dot{f}$ is bounded in $[-\ell_0, \ell_0]$, it is easy to prove that Φ is locally Lipschitz in t, uniformly with respect to z.

We can thus apply classical results of ordinary differential equations (see, e.g., [35, Theorem 5.3]) and get a unique solution $z \mapsto t(z)$ to (1.61). Then z is found by inverting the function $t(z)$ and finally $\ell(t) = t - z(t)$ is the unique solution to (1.55) up to time $s_1 = t(\ell_0)$, which is the unique point such that $s_1 - \ell(s_1) = \ell_0$. Property (1.59) follows from the differential equation. The proof is concluded by an iterative argument based on the "bounce formula" (1.16) as for the previous theorems. □

Remark 1.23. The previous result can be adapted to the case where κ is piecewise Lipschitz. More precisely, we assume that there exist a finite or infinite sequence $\ell_0 = x_0 < x_1 < x_2 < \dots$, without accumulation points, and a sequence κ_n of Lipschitz functions on $[x_{n-1}, x_n]$ such that $\kappa(x) = \kappa_n(x)$ for $x \in [x_{n-1}, x_n)$. Using the arguments of Theorem 1.22, we can solve the coupled system for (u, ℓ) with κ replaced by κ_1. It may happen that $\ell(t) < x_1$ for every t. In this case the problem is solved and the discontinuities play no role. Assume, in contrast, that there exists τ_1 such that $\ell(\tau_1) = x_1$. To extend ℓ after this time, we solve the equation in (1.55) with κ replaced by κ_2 and initial condition $\ell(\tau_1) = x_1$ and then we apply the iterative procedure of Theorem 1.22 with κ replaced by κ_2 as long as $\ell(t) < x_2$. If there exists τ_2 such that $\ell(\tau_2) = x_2$, then we iterate this argument using as local toughness κ_3.

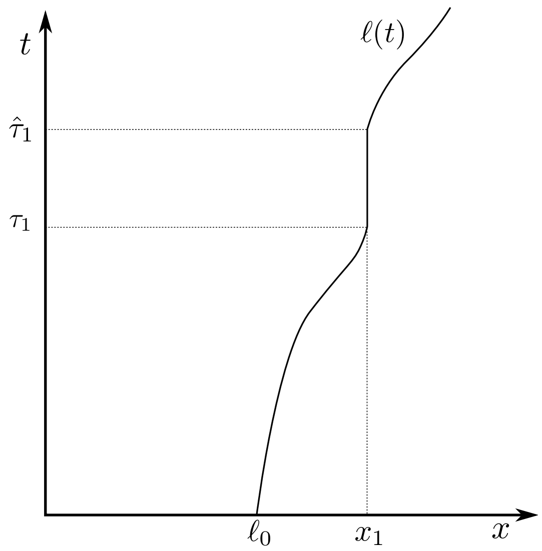

Figure 1.3: A jump of the local toughness at x_1 may lead to a solution lingering at x_1 in a time interval $[\tau_1, \hat\tau_1]$.

Note that the equation may lead to a solution satisfying $\ell(t) = x_1$ for every $t \in [\tau_1, \hat\tau_1]$, for some $\hat\tau_1 > \tau_1$. This happens if and only if $2\dot f(t-\ell(t))^2 - \kappa_2(\ell(t)) \le 0$ for a.e. $t \in [\tau_1, \hat\tau_1]$, that is $G_0(t) \le \kappa_2(x_1)$ for a.e. $t \in [\tau_1, \hat\tau_1]$.

Particular cases of piecewise constant local toughnesses κ have been studied in detail in [28, 48]. Our analysis proves the uniqueness of the solution obtained in those papers.

1.4 The case of a speed-dependent local toughness

In this section we consider a generalisation of our problem to the case of a local toughness κ depending also on the debonding speed. We assume that $\kappa(x, \mu)$ is a function of the position x in the reference configuration and of the debonding speed μ,

$$\kappa\colon [0,+\infty)\times[0,+\infty) \to [c_1, +\infty), \tag{1.62a}$$

where $c_1 > 0$. We require that κ is piecewise Lipschitz in the first variable with a finite number of discontinuities $x_1 < \dots < x_N$ and with $x_0 = \ell_0$.

$$|\kappa(x_1,\mu) - \kappa(x_2,\mu)| \le L|x_1 - x_2|(\kappa(x_1,\mu) + \kappa(x_2,\mu)) \quad \text{for } x_1, x_2 \in (x_j, x_{j+1}),\ j \ge 0. \tag{1.62b}$$

Moreover, for every $x \ge \ell_0$ and $\mu_1, \mu_2 \ge 0$ we assume

$$\frac{\kappa(x,\mu_2) - \kappa(x,\mu_1)}{\mu_2 - \mu_1} > -c_3 \frac{(\sqrt{\kappa(x,\mu_2)} + \sqrt{\kappa(x,\mu_1)})^2}{4}, \tag{1.62c}$$

where $c_3 < 2$. Notice that this condition is automatically satisfied when κ is non-decreasing with respect to μ; in general, it requires a bound on its slope. It will be used in Lemma (1.24).

Existence and uniqueness of a solution u to (0.1), when the evolution of the debonding front $t \mapsto \ell(t)$ is prescribed, is again guaranteed by Theorem 1.8 since κ plays no role at this stage of the discussion. Then, the arguments of Section 1.2 are repeated in order to state Griffith's criterion that now reads as follows:

$$\begin{cases} \dot\ell(t) \geq 0, \\ G_{\dot\ell(t)}(t) \leq \kappa(\ell(t), \dot\ell(t)), \\ \dot\ell(t)\left[G_{\dot\ell(t)}(t) - \kappa(\ell(t), \dot\ell(t))\right] = 0. \end{cases}$$

Using (1.46), it is rephrased in terms of a Cauchy problem for the evolution of the debonding front $t \mapsto \ell(t)$. We obtain indeed the equivalent formulation

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t), \dot\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t), \dot\ell(t))} \vee 0, \quad \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{1.63}$$

Since the local toughness depends also on $\dot\ell(t)$, our main difficulty is that the ordinary differential equation in (1.63) is not expressed in normal form. To overcome this difficulty, we introduce the variable $z(t) := t - \ell(t)$ and we consider the function

$$\Phi\colon [0, +\infty)\times[-\ell_0, \ell_0]\times[0, +\infty) \to \mathbb{R}$$

defined (for every t, μ and a.e. z) by

$$\Phi(t, z, \mu) := \begin{cases} \mu - \dfrac{2\dot f(z)^2 - \kappa(t-z, \mu)}{2\dot f(z)^2 + \kappa(t-z, \mu)}, & \text{if } 2\dot f(z)^2 \geq \kappa(t-z, \mu), \\ \mu, & \text{if } 2\dot f(z)^2 < \kappa(t-z, \mu). \end{cases} \tag{1.64}$$

Our strategy is then to prove that $\mu \mapsto \Phi(t, z, \mu)$ is invertible for fixed t, z. This will ensure that the Cauchy problem can be recast in normal form.

By Proposition 1.6, we obtain the one-dimensional function f in the interval $[-\ell_0, \ell_0]$. We then want to solve (1.63) as long as $z(t) = t - \ell(t) \in [-\ell_0, \ell_0]$, knowing that $f \in C^{0,1}([-\ell_0, \ell_0])$.

Lemma 1.24. *Let κ be as in* (1.62), *$f \in C^{0,1}([-\ell_0, \ell_0])$, and Φ as in* (1.64). *Then , the*

$$\frac{\Phi(t, z, \mu_2) - \Phi(t, z, \mu_1)}{\mu_2 - \mu_1} > 1 - \frac{c_3}{2},$$

for every $t > 0$, a.e. $z > -\ell_0$, and every $0 < \mu_1 \leq \mu_2$, where c_3 is given in (1.62c).

Proof. Let $t > 0$ and $z > -\ell_0$. We first observe that if $\Phi(t, z, \mu) = \mu$ as in the second line of (1.64), then the thesis trivially holds. Next we prove it when Φ is given by the first line. This leads to the conclusion, since Φ is the minimum of two functions whose difference quotients are controlled from below.

We can conclude by showing that Φ is increasing also when it is equal to $\mu - \frac{2\dot f(z)^2 - \kappa(t-z,\mu)}{2\dot f(z)^2 + \kappa(t-z,\mu)}$.

Let $0 < \mu_1 \leq \mu_2$ and assume that $\Phi(t, z, \mu_i) = \mu_i - \frac{2\dot f(z)^2 - \kappa(t-z,\mu_i)}{2\dot f(z)^2 + \kappa(t-z,\mu_i)}$ for $i = 1, 2$. Then,

$$\begin{aligned} \frac{\Phi(t, z, \mu_2) - \Phi(t, z, \mu_1)}{\mu_2 - \mu_1} &= 1 - \frac{1}{\mu_2 - \mu_1}\frac{2\dot f(z)^2 - \kappa(t-z, \mu_2)}{2\dot f(z)^2 + \kappa(t-z, \mu_2)} + \frac{1}{\mu_2 - \mu_1}\frac{2\dot f(z)^2 - \kappa(t-z, \mu_1)}{2\dot f(z)^2 + \kappa(t-z, \mu_1)} \\ &= 1 + \frac{4\dot f(z)^2[\kappa(t-z, \mu_2) - \kappa(t-z, \mu_1)]}{(2\dot f(z)^2 + \kappa(t-z, \mu_1))(2\dot f(z)^2 + \kappa(t-z, \mu_2))(\mu_2 - \mu_1)} \\ &> 1 - \frac{c_3}{2}\frac{2\dot f(z)^2(\sqrt{\kappa(t-z, \mu_1)} + \sqrt{\kappa(t-z, \mu_2)})^2}{(2\dot f(z)^2 + \kappa(t-z, \mu_1))(2\dot f(z)^2 + \kappa(t-z, \mu_2))}. \end{aligned} \tag{1.65}$$

This holds for every $t > 0$ and a.e. $z > -\ell_0$. Notice that in the last line we used (1.62c). Moreover, it is easy to see that

$$\frac{\alpha}{(\alpha+\kappa(t-z,\mu_1))(\alpha+\kappa(t-z,\mu_2))} \leq \frac{1}{(\sqrt{\kappa(t-z,\mu_1)}+\sqrt{\kappa(t-z,\mu_2)})^2}$$

for every $\alpha \geq 0$. Therefore, we can continue (1.65) and deduce that

$$\frac{\Phi(t,z,\mu_2)-\Phi(t,z,\mu_1)}{\mu_2-\mu_1} > 1-\frac{c_3}{2},$$

where $c_3 < 2$ as stated in (1.62c). □

By Lemma 1.24, the function $\mu \mapsto \Phi(t,z,\mu)$, that maps $[0,+\infty)$ into itself, is globally invertible for every $t \geq 0$ and a.e. $z \in [-\ell_0,\ell_0]$. Let then $\Psi\colon [0,+\infty)\times[-\ell_0,\ell_0]\times[0,+\infty) \to [0,+\infty)$ be the function such that, given $\sigma \in [0,+\infty)$,

$$\Phi(t,z,\Psi(t,z,\sigma)) = \sigma \quad \text{and} \quad \Psi(t,z,\Phi(t,z,\mu)) = \mu,$$

for every $t \geq 0$ and a.e. $z \in [-\ell_0,\ell_0]$. We can thus rephrase problem (1.63) as

$$\begin{cases} \dot\ell(t) = \Psi(t,t-\ell,0) & \text{for a.e. } t>0,\\ \ell(0)=\ell_0, \end{cases}$$

now expressed in normal form. In analogy to Theorem 1.22, it is convenient to use the equivalent form

$$\begin{cases} 1-\dot z(t) = \Psi(t,z,0) & \text{for a.e. } t>0,\\ z(0)=-\ell_0. \end{cases} \tag{1.66}$$

We now prove that $t \mapsto \Psi(t,z,0)$ is Lipschitz for fixed z.

Proposition 1.25. *Consider Φ and Ψ as above. Let κ be as in* (1.62)*. Then, there exists $C > 0$ such that*

$$|\Psi(t_2,z,\sigma)-\Psi(t_1,z,\sigma)| \leq C|t_2-t_1|,$$

for every $t_1, t_2 > 0$ and a.e. $z > -\ell_0$ such that $t_1, t_2 \in (x_j+z, x_{j+1}+z)$ for some $j \geq 0$, and for every $\sigma > 0$.

Proof. We start from showing that Φ is Lipschitz in t. Let $t_1, t_2 > 0$, $z \in [-\ell_0,\ell_0]$, and $\mu > 0$ as in the statement, such that $\Phi(t_1,z,\mu)$ and $\Phi(t_2,z,\mu)$ are defined. Then, we consider $x_1, x_2 > \ell_0$ such that $x_i = t_i - z$ for $i = 1,2$. We have that

$$\begin{aligned}
\Phi(t_1,z,\mu)-\Phi(t_2,z,\mu) &= \frac{2\dot f(z)^2-\kappa(\ell_1,\mu)}{2\dot f(z)^2+\kappa(\ell_1,\mu)} - \frac{2\dot f(z)^2-\kappa(\ell_2,\mu)}{2\dot f(z)^2+\kappa(\ell_2,\mu)}\\
&= \frac{2\dot f(z)^2-\kappa(\ell_1,\mu)}{2\dot f(z)^2+\kappa(\ell_1,\mu)} - \frac{2\dot f(z)^2-\kappa(\ell_2,\mu)}{2\dot f(z)^2+\kappa(\ell_1,\mu)} + \frac{2\dot f(z)^2-\kappa(\ell_2,\mu)}{2\dot f(z)^2+\kappa(\ell_1,\mu)} - \frac{2\dot f(z)^2-\kappa(\ell_2,\mu)}{2\dot f(z)^2+\kappa(\ell_2,\mu)}\\
&= \frac{\kappa(\ell_2,\mu)-\kappa(\ell_1,\mu)}{2\dot f(z)^2+\kappa(\ell_1,\mu)} + (\kappa(\ell_2,\mu)-\kappa(\ell_1,\mu))\frac{2\dot f(z)^2-\kappa(\ell_2,\mu)}{(2\dot f(z)^2+\kappa(\ell_1,\mu))(2\dot f(z)^2+\kappa(\ell_2,\mu))}\\
&= (\kappa(\ell_2,\mu)-\kappa(\ell_1,\mu))\frac{4\dot f(z)^2}{(2\dot f(z)^2+\kappa(\ell_1,\mu))(2\dot f(z)^2+\kappa(\ell_2,\mu))}.
\end{aligned}$$

This means that, by (1.62b),

$$\begin{aligned}
&|\Phi(t_1,z,\mu)-\Phi(t_2,z,\mu)| \\
&\le 4L|\ell_1-\ell_2|\dot f(z)^2 \frac{\kappa(\ell_1,\mu)+\kappa(\ell_2,\mu)}{(2\dot f(z)^2+\kappa(\ell_1,\mu))(2\dot f(z)^2+\kappa(\ell_2,\mu))} \\
&\le 4L|\ell_1-\ell_2| = 4L|t_1-t_2|, \qquad (1.67)
\end{aligned}$$

where in the last line we used the fact that

$$(\kappa(\ell_1,\mu)+\kappa(\ell_2,\mu))\dot f(z)^2 \le (2\dot f(z)^2+\kappa(\ell_1,\mu))(2\dot f(z)^2+\kappa(\ell_2,\mu)).$$

We now notice that, for every $\sigma_1,\sigma_2>0$, we have

$$\begin{aligned}
1 &= \frac{\Phi(t,z,\Psi(t,z,\sigma_2))-\Phi(t,z,\Psi(t,z,\sigma_1))}{\sigma_2-\sigma_1} \\
&= \frac{\Phi(t,z,\Psi(t,z,\sigma_2))-\Phi(t,z,\Psi(t,z,\sigma_1))}{\Psi(t,z,\sigma_2)-\Psi(t,z,\sigma_1)} \frac{\Psi(t,z,\sigma_2)-\Psi(t,z,\sigma_1)}{\sigma_2-\sigma_1}.
\end{aligned}$$

Therefore, by Lemma 1.24,

$$\frac{|\Psi(t,z,\sigma_2)-\Psi(t,z,\sigma_1)|}{\sigma_2-\sigma_1} < \frac{1}{1-\frac{c_3}{2}}. \qquad (1.68)$$

Moreover, for every $\mu>0$, we have

$$\begin{aligned}
0 &= \frac{\Psi(t_2,z,\Phi(t_2,z,\mu))-\Psi(t_1,z,\Phi(t_1,z,\mu))}{t_2-t_1} \\
&= \frac{\Psi(t_2,z,\Phi(t_2,z,\mu))-\Psi(t_1,z,\Phi(t_2,z,\mu))}{t_2-t_1} \\
&\quad + \frac{\Psi(t_1,z,\Phi(t_2,z,\mu))-\Psi(t_1,z,\Phi(t_1,z,\mu))}{\Phi(t_2,z,\mu)-\Phi(t_1,z,\mu)} \frac{\Phi(t_2,z,\mu)-\Phi(t_1,z,\mu)}{t_2-t_1}.
\end{aligned}$$

Finally, for every $\sigma>0$ there exists $\mu>0$ such that $\sigma=\Phi(t_2,z,\mu)$ (by invertibility of $\mu\mapsto\Phi(t_2,z,\mu)$) and, by (1.67) and (1.68),

$$|\Psi(t_2,z,\sigma)-\Psi(t_1,z,\sigma)| \le \frac{4L}{1-\frac{c_3}{2}}|t_2-t_1|.$$

This concludes the proof. □

The following result shows existence and uniqueness of a pair (u,ℓ) solving the coupled problem (0.1) & (1.63). This generalises Theorem 1.22 to the case of a speed-dependent toughness.

Theorem 1.26. *Assume that the local toughness satisfies* (1.62) *and let* u_0, u_1, *and* w *be as in* (1.58) *such that* (1.6b) *is satisfied. Then, there exists a unique pair* $(u,\ell)\in\widetilde H^1(\Omega)\times\widetilde C^{0,1}(0,+\infty)$ *solving* (0.1)*&*(1.63). *Moreover,* $u\in\widetilde C^{0,1}(\Omega)$ *and for every* $T>0$ *there exists* $L_T<1$ *such that* $\dot\ell\le L_T$.

Proof. We first consider the case where κ is continuous. We have to construct a function f satisfying (1.22) and a function ℓ satisfying (1.63). By Proposition 1.8 we are provided f in the interval $[-\ell_0, \ell_0]$ and we know that f is Lipschitz. Next we solve the Cauchy problem (1.66) as long as $t - \ell(t) \in [-\ell_0, \ell_0]$. We have that Ψ is measurable in z because

$$\{z > -\ell_0 : \Psi(t, z, \sigma) < \mu\} = \{\sigma > 0 : \sigma < \Phi(t, z, \mu)\},$$

for every $t, \mu > 0$, and Φ is measurable because $\dot{f} \in L^\infty(-\ell_0, \ell_0)$ and $\kappa > c_1$ is piecewise Lipschitz. Moreover, by Proposition 1.25, $t \mapsto \Psi(t, z, 0)$ is locally Lipschitz for a.e. $z \in (-\ell_0, \ell_0)$. We now notice that there exists $0 < c < 1$ such that

$$\Psi(t, z, 0) \in [0, 1 - c].$$

Indeed, starting from $\Phi(t, z, \Psi(t, z, 0)) = 0$, we find that

$$\Psi(t, z, 0) = \frac{2\dot{f}(z)^2 - \kappa(t - z, \Psi(t, z, 0))}{2\dot{f}(z)^2 + \kappa(t - z, \Psi(t, z, 0))} \vee 0.$$

Therefore, every solution to (1.66) must satisfy $\dot{z}(t) > 0$ for a.e. $t > 0$ and it is thus invertible. The function $z \mapsto t(z)$ solves the problem

$$\begin{cases} \dot{t}(z) = \dfrac{1}{1 - \Psi(t, z, 0)} & \text{for a.e. } z > -\ell_0, \\ t(-\ell_0) = 0. \end{cases} \tag{1.69}$$

Since $0 \leq \dot{t}(z) \leq \frac{1}{c}$, $\Psi(t, z, 0)$ is Lipschitz in t uniformly in z, and it is measurable in z, we can apply classical results on ordinary differential equations (see, e.g., [35, Theorem 5.3]) and get a unique solution $z \mapsto t(z)$ to (1.69). Then z is found by inverting the function $t(z)$ and finally $\ell(t) = t - z(t)$ is the unique solution to (1.63) up to time $t(\ell_0)$, satisfying $\dot{\ell} \leq L_T$. Next we employ (1.22) to extend f to $(\ell_0, t(\ell_0) + \ell(t(\ell_0))]$, so the ordinary differential equation can be solved in this interval, hence ℓ and f are further extended. The proof is concluded by the iterative argument based on the "bounce formula" (1.22) that we explained in Theorem 1.18.

In the case that κ has a finite number of discontinuities $x_1, \dots, x_N$, we may apply the previous argument to solve (1.69) as long as $t(z) - z < x_1$. If there is z_1 such that $t(z_1) = x_1 + z_1$, we extend the solution for $z \geq z_1$ by solving the Cauchy problem with initial datum $t(z_1) = x_1 + z_1$ as long as $t(z) - z < x_2$, recalling the monotonicity of $z \mapsto t(z) - z$. Iterating this argument allows us to conclude. □

CHAPTER 2

The problem of debond initiation

In this Chapter we consider the case of debond initiation, i.e., the case $\ell_0 = 0$ for our model of dynamic peeling test. In the first section we analyse the problem of finding u solution to (0.1a)–(0.1c) when the evolution of the debonding front is given. In the second section we couple this problem with Griffith's criterion and find existence of a pair (u, ℓ) solution to the coupled problem. The results of this chapter are part of a forthcoming paper in collaboration with G. Lazzaroni.

2.1 The problem for prescribed debonding front

In the case of debond initiation the problem for u is

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad t > 0\,,\, 0 < x < \ell(t), \tag{2.1a}$$
$$u(t,0) = w(t), \quad t > 0, \tag{2.1b}$$
$$u(t,\ell(t)) = 0, \quad t > 0, \tag{2.1c}$$

where $\ell\colon [0,+\infty) \to [0,+\infty)$ is a given Lipschitz function and it satisfies

$$0 \le \dot\ell(t) < 1, \text{ for a.e. } t > 0, \tag{2.2a}$$
$$\ell(0) = 0, \tag{2.2b}$$
$$\ell(t) > 0, \text{ for every } t > 0. \tag{2.2c}$$

Defining $\varphi, \psi\colon [0,\infty) \to [0,\infty)$ as in (1.2) and ω as in (1.3), we now observe that $\omega(t) = 0$ if and only if $t = 0$. Moreover,

$$0 < \omega(t) < t, \qquad \text{for every } t > 0. \tag{2.3}$$

Indeed, by (2.2c), for every $t > 0$ we have

$$\psi(t) > t \text{ and } \varphi(t) < t.$$

Therefore $\psi^{-1}(t) < t$ and thus $\omega(t) < t$.

We assume

$$w \in \widetilde{H}^1(0,+\infty), \tag{2.4}$$

with the compatibility condition

$$w(0) = 0. \tag{2.5}$$

By Proposition 1.4, given w of the form (2.4), $u \in H^1(\Omega_T)$ is a solution of (2.1) in the sense of Definition 1.1 if and only if there exists a function of one variable $f \in H^1_{\mathrm{loc}}(0, T + \ell(T))$ such that

$$\int_0^{T-\ell(T)} \dot{f}(s)^2(\varphi^{-1}(s) - s)\,\mathrm{d}s + \int_{T-\ell(T)}^{T} \dot{f}(s)^2(T - s)\,\mathrm{d}s < +\infty, \tag{2.6a}$$

$$\int_0^{T+\ell(T)} (\dot{w}(s) - \dot{f}(s))^2((s \wedge T) - \psi^{-1}(s))\,\mathrm{d}s < +\infty, \tag{2.6b}$$

whose continuous representative satisfies $f(0) = 0$ and the "bounce formula" (1.16). Using the change of variables $s = \psi(t)$ we re-write (1.16) as

$$f(s) = w(s) + f(\omega(s)), \qquad \text{for every } s \in (0, T + \ell(T)). \tag{2.7}$$

By Remark 1.10 we write, for every $n \geq 1$,

$$f(s) = \sum_{k=0}^{n-1} w(\omega^k(s)) + f(\omega^n(s)). \tag{2.8}$$

Notice that for every $s \in (0, T + \ell(T))$ we have

$$\omega^k(s) \to 0, \text{ as } k \to \infty.$$

Indeed, let $s_k := \omega^k(s)$. By (2.3), the sequence $\{s_k\}$ is decreasing and thus $s_k \to \bar{s}$ for some $\bar{s} \in [0, T + \ell(T)]$. By the continuity of ω,

$$\omega(\bar{s}) = \omega(\lim_k s_k) = \lim_k \omega(s_k) = \lim_k s_{k+1} = \bar{s}.$$

This means that $\bar{s} = 0$.

Since (2.8) is constant in n, then

$$f(s) = \lim_{n\to\infty} \left[\sum_{k=0}^{n-1} w(\omega^k(s)) + f(\omega^n(s)) \right], \quad \text{for every } s \in (0, T + \ell(T)).$$

Moreover, if

$$\sum_{k=0}^{\infty} w(\omega^k(s)) < \infty,$$

then $\lim_n f(\omega^n(s))$ exists. Notice that in the case $f \in H^1(0, T+\ell(T))$ we have $\lim_n f(\omega^n(s)) = 0$ for every $s \in [0, T + \ell(T)]$ which leads to

$$f(s) = \sum_{k=0}^{\infty} w(\omega^k(s)). \tag{2.9}$$

Our aim is then to find conditions such that the series above is finite in order to use formula (2.9) to find the solution u to problem (2.1) via the function f. See Figure 2.1.

The following example gives a motivation to move in this direction.

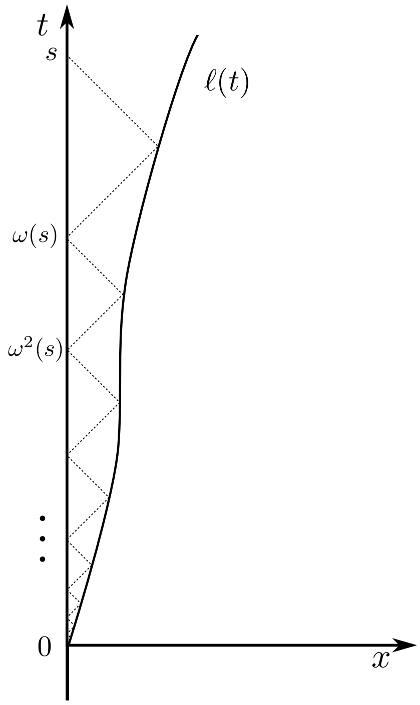

Figure 2.1: Infinite iteration of the "bounce formula" (2.7).

2.1.1 The straight line case

Before proceeding with a general analysis, we first discuss the sample case of $\ell(t) = pt$, where $0 < p \leq 1$, and $w(t) = \alpha t$, where $\alpha > 0$. This will give us hints to solve more general cases.

By solving the wave equation in the domain Ω, we find a solution $u(t,x) = \alpha t - \alpha \frac{x}{p}$. By (1.17), we have

$$\alpha t - \alpha \frac{x}{p} = \alpha t + \alpha x - f(t+x) + f(t-x) \quad \text{for every } (t,x) \in \Omega.$$

Then, for $t = x = \frac{s}{2}$, we find that

$$f(s) = \alpha \frac{1+p}{2p} s. \tag{2.10}$$

This is consistent with (2.9) because, since $\omega(t) = \frac{1-p}{1+p} t$ and since

$$\sum_{k=0}^{\infty} \left(\frac{1-p}{1+p} \right)^k = \frac{1+p}{2p},$$

then

$$f(s) = \sum_{k=0}^{\infty} w(\omega^k(s)) = \sum_{k=0}^{\infty} \alpha \omega^k(s) = \sum_{k=0}^{\infty} \alpha \left(\frac{1-p}{1+p} \right)^k s = \alpha \frac{1+p}{2p} s.$$

Therefore, in the case of a given debonding evolution of the form $\ell(t) = pt$, we can construct explicitly the function f corresponding to the solution of the wave equation on the time-dependent domain Ω.

We now motivate our choice to look for solutions in H^1 as required in Definition 1.1. To this end, we show that even in the case of the straight line we are able to find more than one solution (in fact, infinitely many) to (2.1), when u is only in $H^1_{\text{loc}}(\Omega)$. Indeed, problem (2.1) admits a unique solution if and only if

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \tag{2.11a}$$
$$u(t,0) = 0, \tag{2.11b}$$
$$u(t,pt) = 0, \tag{2.11c}$$

has $u(t,x) = 0$ as its unique solution. By D'Alembert's formula, and using the condition (2.11b), we find

$$u(t,x) = f(t-x) - f(t+x). \tag{2.12}$$

Moreover, the boundary condition on (t,pt) implies that $f((1-p)t) = f((1+p)t)$. We define $\mu := \frac{1-p}{1+p}$ and obtain

$$f(\mu t) = f(t). \tag{2.13}$$

We now look for a solution f of the form $f(t) = F(\log t)$, so that condition (2.13) can be written as

$$F(\log t) = F(\log t + \log \mu).$$

This implies that any $(\log \mu)$-periodic function F gives a solution to problem (2.11).

We can however prove that there exists a unique $u \in H^1(\Omega)$ solution to (2.11). Indeed, by (2.6), we have

$$+\infty > \int_0^1 s\dot{f}(s)^2 \,\mathrm{d}s = \int_0^1 \frac{\dot{F}(\log s)^2}{s} \,\mathrm{d}s = \int_{-\infty}^0 \dot{F}(\sigma)^2 \,\mathrm{d}\sigma$$

Since F is periodic, also $\dot{F}$ is periodic. Therefore

$$\int_{-\infty}^0 \dot{F}(\sigma)^2 \,\mathrm{d}\sigma < \infty,$$

if and only if $\dot{F} = 0$ almost everywhere. This implies that F is constant and thus the same holds for f. Finally, by (2.12) we obtain $u \equiv 0$ and therefore there is a unique solution $u \in H^1(\Omega)$ to (2.11). On the other hand, if we seek solutions $u \in H^1_{\text{loc}}(\Omega)$, then F need not be constant and infinitely many solution to (2.11) are thus found because of the periodicity of F.

2.1.2 Case with controlled debonding front speed

To rule out the previous examples, we now present existence and uniqueness for solutions $u \in H^1(\Omega)$ according to Definition 1.1 when in addition to (1.1) we assume

$$0 < c_0 \leq \dot{\ell}(t) < 1 \quad \text{for a.e. } t > 0. \tag{2.14}$$

Notice that this condition implies that ω is a contraction with

$$\dot{\omega}(t) \leq \frac{1-c_0}{1+c_0}. \tag{2.15}$$

Theorem 2.1. *Let $1 < p \leq \infty$ and $T > 0$. Assume (2.2) and (2.14), and let $w \in H^1(0,T)$ such that (2.5) is satisfied. Then, there exists a unique solution $u \in H^1(\Omega_T)$ to problem (2.1) such that $f \in H^1(0,T)$.*

Proof. We consider the space

$$X_T := \{f \in H^1(0,T), f(0) = 0\}.$$

We want to prove that there exists a unique function $f \in H^1(0,T)$ such that (2.7) is satisfied. We thus consider the map $S\colon X_T \to X_T$ defined by

$$S(g) := g \circ \omega + w,$$

for every $g \in X_T$. We notice that S maps X_T into itself because $\omega \in C^{0,1}([0,T])$ and $w \in H^1(0,T)$, with $w(0) = 0$ and $\omega(0) = 0$. We observe that

$$\begin{aligned}
\|\tfrac{\mathrm{d}}{\mathrm{d}t}(Sf_1 - Sf_2)\|^p_{L^2(0,T)} &= \|\left(\dot{f}_1 \circ \omega - \dot{f}_2 \circ \omega\right)\dot{\omega}\|^2_{L^2(0,T)} \\
&= \int_0^T \left(\dot{f}_1(\omega(t)) - \dot{f}_2(\omega(t))\right)^2 \,\dot{\omega}(t)\dot{\omega}(t)\,\mathrm{d}t \\
&\le \frac{1-c_0}{1+c_0} \int_{\omega(0)}^{\omega(T)} |f_1(s) - f_2(s)|^2\,\mathrm{d}s \\
&\le \frac{1-c_0}{1+c_0}\|f_1 - f_2\|^2_{H^1(0,T)},
\end{aligned}$$

because ω is a contraction. Moreover,

$$\begin{aligned}
\|Sf_1 - Sf_2\|^2_{L^2} &\le \int_0^T |f_1(\omega(t)) - f_2(\omega(t))|^2\,\mathrm{d}t \\
&= \int_0^T |(f_1(\omega(t)) - f_2(\omega(t))) - (f_1(\omega(0)) - f_2(\omega(0)))|^2\,\mathrm{d}t \\
&= \int_0^T \left|\int_{\omega(0)}^{\omega(t)} \frac{\mathrm{d}}{\mathrm{d}s}(f_1(s) - f_2(s))\,\mathrm{d}s\right|^2 \mathrm{d}t \\
&\le \int_0^T \|f_1 - f_2\|^p_{H^1(0,T)}(\omega(t) - \omega(0))\,\mathrm{d}t \\
&\le \|f_1 - f_2\|^2_{H^1(0,T)}\frac{1-c_0}{1+c_0}\int_0^T t\,\mathrm{d}t \\
&= \frac{1-c_0}{1+c_0}\frac{T^2}{2}\|f_1 - f_2\|^2_{H^1(0,T)}.
\end{aligned}$$

Notice that we used Hölder's inequality and (2.15). If T is sufficiently small, then S is a contraction in X_T. By the contractions lemma, there exists a unique function $f \in X_T$ such that $Sf(t) = f(\omega(t)) + w(t) = f(t)$, that is (1.23). Through formula (1.17), we have a unique solution $u \in H^1(\Omega_T)$ to problem (2.1). We finally notice that, in the case $\frac{1-c_0}{1+c_0}\frac{T}{2} \ge 1$, then S is not a contraction in X_T. However, we can find $\tau \in (0,T)$ such that S is a contraction in X_τ and then we extend f to $[0,T]$ using an iterative argument based on (1.23). Therefore, there exists a unique solution $u \in H^1(\Omega_T)$ to problem (2.1). □

2.2 Evolution of the debonding front via Griffith's criterion

After having established a general case where there is existence of a solution u when the evolution of the debonding front is already known, we question whether it is possible to select those evolutions satisfying Griffith's criterion (1.53). Our aim is then now to construct a pair (u, ℓ) solution to the coupled problem (2.1)&(1.55). To this end, we first consider again the case in which the evolution of the debonding front is given by a straight line (see Section 2.1.1).

Lemma 2.2. *Assume that $w(t) = \alpha t$ with $\alpha \in (0,1)$ and let the local toughness κ be constant. Then the pair (u, ℓ) with $\ell(t) = pt$ and $u(t,x) = \alpha(t - \frac{x}{p})$ (with related function f defined in* (2.10)*) is a solution to the coupled problem* (2.1) *&* (1.55).

Proof. We prove that there exists only one possible value for $p \in (0,1)$ such that (1.53) is satisfied and $\ell(t) = pt$. From $u(t,x) = \alpha t - \alpha \frac{x}{p}$, we find $u_x(t,x) = -\frac{\alpha}{p}$ for almost every t and x. Thus, from (1.47), we obtain that the dynamic energy release rate is

$$G_{\dot{\ell}(t)}(t) = \frac{1}{2}(1-p^2)\frac{\alpha^2}{p^2}.$$

Using (1.53c), we have that evolution occurs when

$$\frac{1}{2}(1-p^2)\frac{\alpha^2}{p^2} = \kappa.$$

Therefore,

$$p = \frac{\alpha}{\sqrt{\alpha^2 + 2\kappa}} \tag{2.16}$$

is the only possible slope between 0 and 1 such that Griffith's criterion is satisfied. □

Fix now $T > 0$. Let the local toughness $\kappa\colon [0, +\infty) \to [c_1, c_2]$ be a Lipschitz function and let the external loading $w \in C^{0,1}([0,T])$, with $w(0) = 0$ and $\dot{w}(0) > 0$. Then, for $0 < \delta < T$ sufficiently small there exists a pair $(u^\delta, \ell^\delta) \in C^{0,1}(\overline{\Omega}_T) \times C^{0,1}([0,T])$ solution to the coupled problem (2.1) & (1.55) with w and κ substituted by

$$w^\delta(t) = \begin{cases} \dfrac{w(\delta)}{\delta} t, & \text{if } 0 \le t \le \delta, \\ w(t), & \text{if } t \ge \delta, \end{cases} \tag{2.17}$$

and

$$\kappa^\delta(x) = \begin{cases} \kappa(p_\delta \delta), & \text{if } 0 \le x \le p_\delta \delta, \\ \kappa(x), & \text{if } x \ge p_\delta \delta, \end{cases} \tag{2.18}$$

with $\ell^\delta(t) = p_\delta t$ when $0 \le t \le \delta$ and

$$p_\delta = \frac{\alpha_\delta}{\sqrt{2\kappa_\delta + \alpha_\delta^2}}. \tag{2.19}$$

Indeed, let $\alpha_\delta := \frac{w(\delta)}{\delta}$ and $\kappa_\delta := \kappa(p_\delta \delta)$. By (2.16) and since $\dot{w}(0) > 0$, if δ is sufficiently small, there exists a unique $p_\delta \in (0,1)$ of the form (2.19) such that $\ell^\delta(t) = p_\delta t$ is an evolution of the debonding front in $[0, \delta]$ with corresponding u^δ given through (1.17) by

$$f^\delta(t) = \frac{\sqrt{2\kappa_\delta + \alpha_\delta^2} + \alpha_\delta}{2} t, \quad \text{with } 0 \le t \le \delta \tag{2.20}$$

that solves (2.1) with $t \in [0, \delta]$ and satisfies Griffith's criterion. At time $t = \delta$ we can explicitly compute $u(\delta, x)$ and $u_t(\delta, x)$ by (1.17) obtaining

$$u_0^\delta := u(\delta, x) = \alpha_\delta \delta - \sqrt{2\kappa_\delta + \alpha_\delta^2} x$$

and

$$u_1^\delta := u_t(\delta, x) = \alpha_\delta.$$

Moreover,

$$\ell_0^\delta := \ell^\delta(\delta) = p_\delta \delta.$$

Notice that $u_0^\delta(0) = w(\delta)$ and $u_0^\delta(p_\delta \delta) = 0$. Therefore, u_0^δ satisfies (1.6b) and thus, using Theorem 1.8 with data given by w^δ, κ^δ, u_0^δ, u_1^δ, and ℓ_0^δ, we extend (u^δ, ℓ^δ) to a solution of (2.1) & (1.55) for every $0 \le t \le T$. Moreover, $(u^\delta, \ell^\delta) \in C^{0,1}(\overline{\Omega}_T) \times C^{0,1}([0, T])$.

Therefore, there exists a solution to (2.1) & (1.55) that satisfies Griffith's criterion (1.53) with w^δ and κ^δ as in (2.17) and (2.18). Notice that we are unable to prove uniqueness at this stage. Our aim is then to study the limit as $\delta \to 0$ in order to obtain a limit pair (u, ℓ) solution to the coupled problem (2.1) & (1.55). Nevertheless, we shall make some technical assumption on the regularity of w in a small neighborhood of 0.

We now set $\kappa_0 := \kappa(0)$ and prove a technical lemma which will be used to find assumptions allowing a bound for f^δ uniform in δ.

Lemma 2.3. *Assume that $\dot{w}(0) > 0$ and $\kappa\colon [c_1, +\infty) \to [c_1, c_2]$ with $0 < c_1 \le c_2$ be a Lipschitz function. Then, there exist λ, μ such that*

$$\lambda > \sqrt{\frac{\kappa_0}{2}}, \tag{2.21a}$$

$$\dot{w}(0) + \frac{\kappa_0}{2\mu} > \lambda, \tag{2.21b}$$

$$\dot{w}(0) + \frac{\kappa_0}{2\lambda} < \mu. \tag{2.21c}$$

In particular, we have $\lambda < \nu < \mu$, where $\nu := \frac{\dot{w}(0) + \sqrt{2\kappa_0 + \dot{w}(0)^2}}{2}$ is the fixed point of $x \mapsto \dot{w}(0) + \frac{\kappa_0}{2x}$.

Proof. We first notice that, since $x \mapsto \dot{w}(0) + \frac{\kappa_0}{2x}$ is decreasing (recall that $\kappa_0 \ge c_1$) and since ν is its fixed point, (2.21b) & (2.21c) imply $\lambda < \nu < \mu$. Moreover, we can re-write (2.21b) & (2.21c) in the equivalent form:

$$\frac{\kappa_0}{2} \frac{1}{\mu - \dot{w}(0)} < \lambda < \dot{w}(0) + \frac{\kappa_0}{2\mu}.$$

Notice that, if $\mu > \nu > \dot{w}(0)$, then the left hand side of the previous inequality is positive. We choose μ such that

$$\dot{w}(0) + \frac{\kappa_0}{2\mu} > \frac{\kappa_0}{2} \frac{1}{\mu - \dot{w}(0)}.$$

This is equivalent to

$$2\mu^2 - 2\dot{w}(0)\mu - \kappa_0 > 0,$$

that is satisfied because for every $\mu > \nu$.

We may now take λ satisfying also (2.21a), i.e.,

$$\left(\frac{\kappa_0}{2}\frac{1}{\mu-\dot w(0)}\right)\vee\sqrt{\frac{\kappa_0}{2}}<\lambda<\dot w(0)+\frac{\kappa_0}{2\mu},$$

which is possible by the choice of μ. Notice that if $\dot w(0)\neq\frac{2}{k_0}$ then we have a further restriction on μ since

$$\dot w(0)+\frac{\kappa_0}{2\mu}>\sqrt{\frac{\kappa_0}{2}}$$

implies that

$$\mu<\sqrt{\frac{\kappa_0}{2}}\frac{1}{1-\dot w(0)\sqrt{\frac{\kappa_0}{2}}}.$$

□

We now set

$$\beta>-\frac{\log 3}{\log a_0},\qquad\text{with}\qquad a_0=\frac{\kappa_0(1+\sigma)}{\kappa_0+\dot w(0)^2+\dot w(0)\sqrt{2\kappa_0+\dot w(0)^2}}.\tag{2.22}$$

We fix λ and μ satisfying (2.21) and choose $\eta,\sigma>0$ sufficiently small so that

$$\eta\dot w(0)+\sigma\frac{\kappa_0}{2\mu}<\left(\dot w(0)+\frac{\kappa_0}{2\mu}-\lambda\right)\wedge\left(\mu-\dot w(0)-\frac{\kappa_0}{2\lambda}\right),\tag{2.23a}$$

$$\frac{\kappa_0}{2}(1+\sigma)<\lambda^2,\tag{2.23b}$$

and $a_0\in(0,1)$ so that $\beta>0$. Notice that the right hand side of (2.23a) is positive by choice of λ and μ.

Let us introduce further assumptions on w and κ. We consider an external loading $w\in\widetilde C^{0,1}(0,+\infty)$ with $w(0)=0$ and a local toughness κ as above such that the following conditions are satisfied in an interval $(0,\delta_0)$:

$$\kappa\in C^{0,1}([0,\delta_0]),\tag{2.24a}$$

$$(1-\sigma)\kappa_0\le\kappa(x)\le(1+\sigma)\kappa_0,\quad\text{for every } x\in[0,\delta_0],\tag{2.24b}$$

$$\dot\kappa(x)\le C_1x^\beta,\quad\text{for a.e. } s\in(0,\delta_0),\tag{2.24c}$$

$$w\in W^{2,\infty}(0,\delta_0),\tag{2.25a}$$

$$\dot w(0)>0,\tag{2.25b}$$

$$(1-\eta)\dot w(0)<\dot w(s)<(1+\eta)\dot w(0),\quad\text{for every } s\in[0,\delta_0],\tag{2.25c}$$

$$\ddot w(s)\le C_2s^\beta,\quad\text{for a.e. } s\in(0,\delta_0),\tag{2.25d}$$

where $C_1,C_2>0$.

We now consider a pair (f,ℓ) such that (1.55) holds and f satisfies the "bounce formula" (2.7). We compute its derivative that exists for a.e. $s>0$:

$$\dot f(s)=\dot w(s)+\dot f(\omega(s))\dot\omega(s)=\dot w(s)+\dot f(\omega(s))\frac{1-\dot\ell(\psi^{-1}(s))}{1+\dot\ell(\psi^{-1}(s))}.\tag{2.26}$$

The following result provides conditions to deduce boundedness of $\dot f$ if it is controlled in a first time interval.

Proposition 2.4. *Let η, σ as in (2.23) and assume (2.24) and (2.25). Then, the following implication holds for a.e. $s \in (0, \delta_0)$:*

$$\lambda \leq \dot{f}(\omega(s)) \leq \mu \quad \Rightarrow \quad \lambda \leq \dot{f}(s) \leq \mu. \tag{2.27}$$

Proof. We first notice that, by (1.46) and our assumption we have

$$G_{\dot{\ell}(\psi^{-1}(s))}(\psi^{-1}(s)) = 2\frac{1-\dot{\ell}(\psi^{-1}(s))}{1+\dot{\ell}(\psi^{-1}(s))}\dot{f}(\omega(s))^2 > 2\frac{1-\dot{\ell}(\psi^{-1}(s))}{1+\dot{\ell}(\psi^{-1}(s))}\lambda^2.$$

Moreover, by (1.53b) and (2.24b)

$$G_{\dot{\ell}(\psi^{-1}(s))}(\psi^{-1}(s)) \leq \kappa(\psi^{-1}(s)) \leq \kappa_0(1+\sigma).$$

It then follows that

$$\frac{1-\dot{\ell}(\psi^{-1}(s))}{1+\dot{\ell}(\psi^{-1}(s))} < \frac{(1+\sigma)\kappa_0}{2\lambda^2} < 1,$$

by (2.23b). Therefore, $\dot{\ell}(\psi^{-1}(s)) > 0$, so that, by (1.55), we write (2.26) as

$$\dot{f}(s) = \dot{w}(s) + \frac{\kappa(\psi^{-1}(s))}{2\dot{f}(\omega(s))},$$

for a.e. $s \in (0, \delta_0)$. Using now (2.25c), we have that (2.27) is satisfied if

$$(1-\eta)\dot{w}(0) + \frac{\kappa_0(1-\sigma)}{2\mu} \geq \lambda \quad \text{and} \quad (1+\eta)\dot{w}(0) + \frac{\kappa_0(1+\sigma)}{2\lambda} \leq \mu.$$

These two conditions hold by (2.23a). □

We now have that $\dot{f}^\delta$ is equibounded in $(0, \delta)$ since, by (2.20) and (2.24), we have

$$\dot{f}^\delta(s) = \frac{\sqrt{2\kappa_\delta + \alpha_\delta^2} + \alpha_\delta}{2}, \qquad \text{for a.e. } s \in (0, \delta). \tag{2.28}$$

Notice also that, by (2.25a) we have that $\alpha_\delta \to \dot{w}(0)$ and thus $\dot{f}^\delta(s) \to \nu$ in $(0, \delta)$. Then, there exists $\delta_1 < \delta_0$ such that for every $0 < \delta < \delta_1$ we have

$$\lambda < \frac{\sqrt{2\kappa_\delta + \alpha_\delta^2} + \alpha_\delta}{2} < \mu.$$

By Proposition 2.4, we obtain that condition (2.27) is satisfied for a.e. $s \in (0, \delta_0)$ so that $\dot{f}^\delta(s)$ is equibounded in $(0, \delta_0)$. Using (1.55) and since $x \mapsto \frac{2x^2-\kappa_0}{2x^2+\kappa_0}$ is increasing and $\kappa_0 \mapsto \frac{2x^2-\kappa_0}{2x^2+\kappa_0}$ is decreasing, we get the following bounds on the debonding speed for a.e. $t \in (0, \delta_0)$:

$$0 < c_0 := \frac{2\lambda^2 - \kappa_0(1+\sigma)}{2\lambda^2 + \kappa_0(1+\sigma)} \leq \dot{\ell}^\delta(t) \leq \frac{2\mu^2 - \kappa_0(1-\sigma)}{2\mu^2 + \kappa_0(1-\sigma)}, \qquad \text{for every } \delta < \delta_1. \tag{2.29}$$

We now consider $\omega_\delta(s) := \varphi_\delta(\psi_\delta^{-1}(s))$, where $\varphi_\delta(s) := s - \ell^\delta(s)$ and $\psi_\delta(s) := s + \ell^\delta(s)$. Since $\dot{\omega}_\delta = \frac{1-\dot{\ell}}{1+\dot{\ell}}$ and $x \mapsto \frac{1-x}{1+x}$ is decreasing, we obtain, by (2.29),

$$(1-\sigma)\frac{\kappa_0}{2\mu^2} \leq \dot{\omega}_\delta(s) \leq \frac{\kappa_0}{2\lambda^2}(1+\sigma)$$

for a.e. $s \in (0, \delta_0)$ and every $0 < \delta < \delta_1$. Notice in particular that, since $\lambda < \nu$, then $\dot{\omega}(s) \leq \frac{\kappa_0(1+\sigma)}{2\nu^2} = a_0$, where a_0 is the constant appearing in (2.22). In particular, since $\omega_\delta(0) = 0$ for every $\delta > 0$, we have

$$\omega_\delta(s) \leq a_0 s \tag{2.30}$$

for every $s \in [0, \delta_0]$ and every $0 < \delta < \delta_1$.

Proposition 2.5. *Assume* (2.24) *and* (2.25)*. Then, there exists* $M > 0$ *such that*

$$\|f^\delta\|_{W^{2,\infty}(0,\delta_0)} \leq M$$

for every $0 < \delta < \delta_1$.

Proof. Since f^δ is already bounded in $W^{1,\infty}$ by Proposition 2.4, we seek a uniform bound for the second derivative of f^δ. Starting from (2.26) and using (2.25a), we obtain for a.e. $s \in (0, \delta_0)$

$$\ddot{f}^\delta(s) = \ddot{w}^\delta(s) - \frac{2\ddot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1 + \dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\dot{f}^\delta(\omega_\delta(s)) + \dot{\omega}_\delta^2(s)\ddot{f}^\delta(\omega_\delta(s)).$$

We now compute the second derivative of $\ell^\delta(t)$ for a.e. $t \in (0, \delta_0)$, recalling (2.24) and (2.25). Starting from (1.55) and since $\dot{\ell}^\delta \geq c_0 > 0$ by (2.29), we have

$$\ddot{\ell}^\delta(t) = \frac{8\kappa(\ell^\delta(t))(1 - \dot{\ell}^\delta(t))\dot{f}^\delta(t - \ell^\delta(t))}{[2\dot{f}^\delta(t - \ell^\delta(t))^2 + \kappa(\ell^\delta(t))]^2}\ddot{f}^\delta(t - \ell^\delta(t)) - \frac{2\dot{\kappa}(\ell^\delta(t))\dot{\ell}^\delta(t)}{[2\dot{f}^\delta(t - \ell^\delta(t))^2 + \kappa(\ell^\delta(t))]^2}\dot{f}^\delta(t - \ell^\delta(t))^2.$$

For $t = \psi_\delta^{-1}(s)$, the last two equations give

$$\begin{aligned}
\ddot{f}^\delta(s) = &\ddot{w}^\delta(s) + \left(\dot{\omega}_\delta^2(s) - \frac{16\kappa(\psi_\delta^{-1}(s))\dot{f}^\delta(\omega_\delta(s))^2}{[2\dot{f}^\delta(\omega_\delta(s))^2 + \kappa(\psi_\delta^{-1}(s))]^2}\frac{1 - \dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1 + \dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\right)\ddot{f}^\delta(\omega_\delta(s)) \\
&+ \frac{4\dot{\kappa}(\ell^\delta(\psi_\delta^{-1}(s)))}{[2\dot{f}^\delta(\omega_\delta(s))^2 + \kappa(\psi_\delta^{-1}(s))]^2}\frac{\dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1 + \dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\dot{f}^\delta(\omega_\delta(s))^3 \\
=: \; &\ddot{w}^\delta(s) + A\ddot{f}^\delta(\omega_\delta(s)) + B\dot{f}^\delta(\omega_\delta(s))^3
\end{aligned}$$

Using $\dot{\omega}_\delta \leq 1$ and the inequality $\frac{cd}{(c+d)^2} \leq \frac{1}{4}$ with $c = 2\dot{f}^\delta(\omega_\delta(s))^2$ and $\kappa(\psi_\delta^{-1}(s))$, we find that

$$|A| \leq \left|\dot{\omega}_\delta^2(s)\right| + \left|\frac{16\kappa(\psi_\delta^{-1}(s))\dot{f}^\delta(\omega_\delta(s))^2}{[2\dot{f}^\delta(\omega_\delta(s))^2 + \kappa(\psi_\delta^{-1}(s))]^2}\frac{1 - \dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1 + \dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\right| \leq 3. \tag{2.31}$$

Moreover, since $\dot{f}^\delta$ is bounded in $(0, \delta_0)$ and using (4.12) and (2.24c), there exists $\tilde{M} > 0$ such that

$$B\dot{f}^\delta(\omega_\delta(s))^3 = \frac{4\dot{\kappa}(\ell^\delta(\psi_\delta^{-1}(s)))}{[2\dot{f}^\delta(\omega_\delta(s))^2 + \kappa(\psi_\delta^{-1}(s))]^2}\frac{\dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1 + \dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\dot{f}^\delta(\omega_\delta(s))^3 \leq \tilde{M}\ell^\delta(\psi_\delta^{-1}(s))^\beta. \tag{2.32}$$

Since $\ell^\delta(\psi_\delta^{-1}(s)) \leq s$, we use (2.31) and (2.32) together to get

$$|\ddot{f}^\delta(s)| \leq |\ddot{w}^\delta(s)| + \tilde{M}s^\beta + 3|\ddot{f}^\delta(\omega_\delta(s))|.$$

Then, we can iterate the previous formula and obtain

$$|\ddot{f}^\delta(s)| \leq |\ddot{w}^\delta(s)| + 3|\ddot{w}^\delta(\omega_\delta(s))| + \tilde{M}s^\beta + \tilde{M}\omega_\delta(s)^\beta + 9|\ddot{f}^\delta(\omega_\delta^2(s))|.$$

Notice that, for a.e. $s \in (0, \delta_0)$ there exists $n_\delta(s) \geq 0$ such that $\omega_\delta^k \in (0, \delta)$ for every $k \geq n_\delta(s)$. Notice that, for every such k we have that $\dot{f}^\delta$ is constant and its value is given by (2.28). Moreover, by considering $n_{\delta_0}^\delta := n_\delta(\delta_0)$. By (2.25c), for a.e. $s \in (0, \delta_0)$ we have $\ddot{f}^\delta(\omega_\delta^k(s)) = 0$ for every $k \geq n_{\delta_0}^\delta$ because $\dot{f}^\delta$ is constant in $(0, \delta)$. Therefore, for a.e. $s \in (0, \delta_0)$ we have

$$\begin{aligned}
|\ddot{f}^\delta(s)| &\leq \sum_{k=0}^{n_{\delta_0}^\delta} 3^k |\ddot{w}^\delta(\omega_\delta^k(s))| + \tilde{M} \sum_{k=0}^{n_{\delta_0}^\delta} 3^k |\omega_\delta^k(s)|^\beta \\
&\leq (C_2 + \tilde{M}) \sum_{k=0}^{n_{\delta_0}^\delta} 3^k |\omega_\delta^k(s)|^\beta \\
&\leq (C_2 + \tilde{M}) \sum_{k=0}^{n_{\delta_0}^\delta} 3^k |a_0^k s|^\beta \\
&\leq (C_2 + \tilde{M}) \delta_0^\beta \sum_{k=0}^{n_{\delta_0}^\delta} (3a_0^\beta)^k \\
&\leq (C_2 + \tilde{M}) \delta_0^\beta \sum_{k=0}^{\infty} (3a_0^\beta)^k \leq M.
\end{aligned}$$

Notice that $3a_0^\beta < 1$ by (2.22) and that we used (2.25d) and (2.30) in the previous inequalities. □

We are now in a position to state the main result.

Theorem 2.6. *Let (f^δ, ℓ^δ) be as above and $\kappa\colon [c_1, +\infty) \to [c_1, c_2]$ with $0 < c_1 \leq c_2$ be a Lipschitz function. Assume* (2.23), (2.24), *and* (2.25). *Then, there exists a subsequence $\delta_n \to 0$ and a pair (f, ℓ) such that $f^{\delta_n} \to f$ strongly in $W^{1,\infty}$, $\ell^{\delta_n} \to \ell$ uniformly, and (u, ℓ) is a solution to the coupled problem* (2.1) *&* (1.55), *with $u(t, x) = w(t+x) - f(t+x) + f(t-x)$.*

Proof. By Proposition 2.5, we find that f^δ is bounded in $W^{2,\infty}$ uniformly in $0 < \delta < \delta_1$. The internal energy of the system $\mathcal{E}^\delta(t; \ell^\delta, w^\delta)$ satisfies the following energy balance:

$$\mathcal{E}^\delta(t; \ell^\delta, w^\delta) - \mathcal{E}^\delta(0; \ell^\delta, w^\delta) + \int_0^{\ell^\delta(t)} \kappa^\delta(x) \,\mathrm{d}x + \int_0^t [\dot{w}^\delta(s) - 2\dot{f}^\delta(s)]^2 \dot{w}^\delta(s) \,\mathrm{d}s = 0,$$

see Proposition 1.13. Notice that, since $\ell_0 = 0$, then $\mathcal{E}^\delta(0; \ell^\delta, w^\delta) = 0$. Moreover, by (2.5) and the bounds on κ, we obtain

$$\mathcal{E}^\delta(t; \ell^\delta, w^\delta) \leq C,$$

for some $C > 0$ and for every $t \in (0, \delta_0)$ and $0 < \delta < \delta_1$. It follows that ℓ^δ is uniformly bounded in δ. Furthermore, $\dot{\ell}^\delta \leq L_{\delta_0} < 1$ by Theorem 1.8 for every $0 < \delta < \delta_1$. By using the Ascoli-Arzelà theorem, there exists a subsequence δ_n such that ℓ^{δ_n} uniformly converges to a limit evolution ℓ. Since ℓ^δ is monotone non-decreasing for every δ and we have uniform convergence, then ℓ is monotone non-decreasing.

By the uniform $W^{2,\infty}$-bound for f^{δ_n}, up to extracting a further subsequence, we have that there exists $f \in W^{2,\infty}(0,\delta_0)$ such that

$$f^{\delta_n} \to f \text{ strongly in } W^{1,\infty}. \tag{2.33}$$

Moreover, by the uniform convergence of ℓ^{δ_n} to ℓ, we can pass in the limit in the "bounce formula" for f^{δ_n}

$$f^{\delta_n}(t+\ell^{\delta_n}(t)) = w^{\delta_n}(t+\ell^{\delta_n}(t)) + f^{\delta_n}(t-\ell^{\delta_n}(t)),$$

so that, in the limit as $n \to \infty$, we obtain that f satisfies (1.16). Indeed, $w^{\delta_n} \to w$ strongly in $W^{1,\infty}$ as $n \to \infty$. Besides, starting from

$$u^{\delta_n}(t,x) = w^{\delta_n}(t+x) - f^{\delta_n}(t+x) + f^{\delta_n}(t-x)$$

and using (2.33), one gets strong convergence in $W^{1,\infty}$ to a function u which satisfies (1.17). Since f satisfies (1.16), then u solves problem (2.1) (see Proposition 1.4). Finally, using again the strong convergence of f^{δ_n} in $W^{1,\infty}$ and the uniform convergence of ℓ^{δ_n} to the limit debonding front ℓ, one passes to the limit in

$$\begin{cases} \dot{\ell}^{\delta_n}(t) = & \dfrac{2\dot{f}^{\delta_n}(t-\ell^{\delta_n}(t))^2 - \kappa^{\delta_n}}{2\dot{f}^{\delta_n}(t-\ell^{\delta_n}(t))^2 + \kappa^{\delta_n}} \vee 0 \\ \ell(0) = & 0. \end{cases}$$

This implies that the pair (u,ℓ) is solution to (2.1) & (1.55), with $u \in W^{2,\infty}(\Omega_{\delta_0})$ and $\ell \in C^{1,1}([0,\delta_0])$. □

The previous discussion shows how to determine existence of a solution u satisfying Griffith's criterion for the evolution of the debonding front ℓ when we consider the problem of initiation of the debonding. We remark that the higher regularity required in (2.24) and (2.25) was assumed only in a small neighborhood of zero. Out of this interval we can continue our solution using Theorem 1.8, assuming only the natural assumptions on the data as in Theorem 1.22.

CHAPTER 3

Evolutions with the damped wave equation and prescribed debonding front

In this Chapter we consider the case of the damped wave equation for our model of one-dimensional peeling test. We will establish existence and uniqueness of a vertical displacement u, solution to problem 0.10 in the case in which the evolution of the debonding front $t \mapsto \ell(t)$ is a given function of the form (1.1). We consider a given external loading w as in (1.5), initial conditions u_0 and u_1 as in (1.6) such that the compatibility conditions (1.6b) are satisfied.

We start from a generic definition of solution for the problem without initial conditions, as we did in Definition 1.1.

Definition 3.1. *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. in $H^1(\Omega_T)$) is a solution of* (0.10a)–(0.10c) *if $u_{tt} - u_{xx} + u_t = 0$ holds in the sense of distributions in Ω (resp. in Ω_T) and the boundary conditions are intended in the sense of traces.*

Given a solution $u \in \widetilde{H}^1(\Omega)$ in the sense of Definition 1.1, we extend u to $(0,+\infty)^2$ (still denoting it by u), by setting $u = 0$ in $(0,+\infty)^2 \setminus \Omega$. Note that this agrees with the interpretation of u as vertical displacement of the film which is still glued to the substrate for $(t,x) \notin \Omega$. For a fixed $T > 0$, we define $Q_T := (0,T)\times(0,\ell(T))$ and we observe that $u \in H^1(Q_T)$ because of the boundary conditions (0.10b)&(0.10c). Further, we need to impose the initial position and velocity of u. While condition in (0.10d) can be formulated in the sense of traces, we have to give a precise meaning to the second condition. Since $H^1((0,T)\times(0,\ell_0)) = H^1(0,T;L^2(0,\ell_0)) \cap L^2(0,T;H^1(0,\ell_0))$, we have $u_t, u_x \in L^2(0,T;L^2(0,\ell_0))$. This implies that $u_t, u_{xx} \in L^2(0,T;H^{-1}(0,\ell_0))$ and, by the wave equation, $u_{tt} \in L^2(0,T;H^{-1}(0,\ell_0))$. Therefore $u_t \in H^1(0,T;H^{-1}(0,\ell_0)) \subset C^0([0,T];H^{-1}(0,\ell_0))$ and we can impose condition (0.10e) as an equality between elements of $H^{-1}(0,\ell_0)$. This discussion shows that the following definition makes sense (cf. Definition 1.2).

Definition 3.2. *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega_T)$) is a solution of* (0.10) *if Definition 3.1 holds and the initial conditions* (0.10d)*&*(0.10e) *are satisfied in the sense of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.*

In the following discussion $T > 0$ is fixed, $L := \ell(T)$, and $u \in H^1(\Omega)$. We consider the

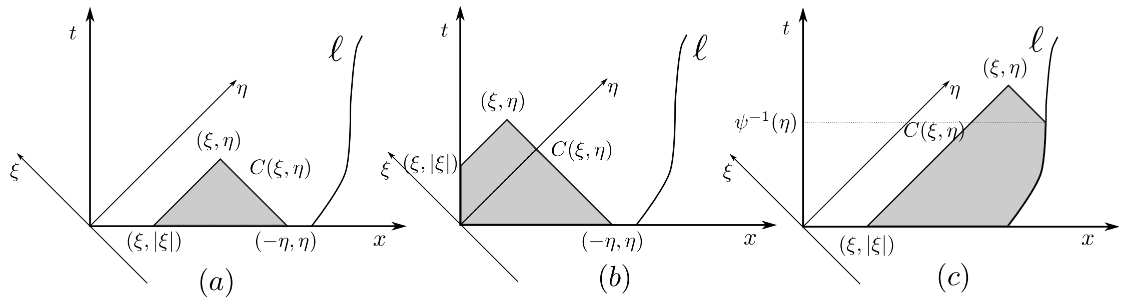

Figure 3.1: The set $C(\xi,\eta)$ in three typical cases.

change coordinates (1.8) and the new function

$$v(\xi,\eta) := u\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}\right).$$

Then, u is a solution to the damped wave equation (0.10a) if it satisfies

$$v_{\eta\xi} = -\tilde{u}_t \tag{3.1}$$

where $\tilde{u}_t(\xi,\eta) := u_t(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2})$. Since $\tilde{u}_t$ is obtained from u_t just via a smooth change of variables, $\tilde{u}_t \in L^2(0,T;L^2(0,L))$ as u_t. By integrating (3.1) between $|\xi|$ and η in the direction η, we obtain

$$v_\xi(\xi,\eta) = v_\xi(\xi,|\xi|) - \int_{|\xi|}^{\eta} \tilde{u}_t(\xi,y)\,\mathrm{d}y.$$

Then, we integrate in the direction η between $(-\eta)\vee\psi^{-1}(\eta)$ and η obtaining

$$v(\xi,\eta) = \begin{cases} v(-\eta,\eta) + \displaystyle\int_{-\eta}^{\xi} v_\xi(z,|z|)\,\mathrm{d}z - \iint_{C(\xi,\eta)} \tilde{u}_t(z,y)\,\mathrm{d}z\,\mathrm{d}y, & \text{if } \eta < \ell_0, \\ v(\psi^{-1}(\eta),\eta) + \displaystyle\int_{\omega(\eta)}^{\xi} v_\xi(z,|z|)\,\mathrm{d}z - \iint_{C(\xi,\eta)} \tilde{u}_t(z,y)\,\mathrm{d}z\,\mathrm{d}y, & \text{if } \eta \geq \ell_0, \end{cases} \tag{3.2}$$

see Figure 3.1. The set $C(\xi,\eta)$ in the double integral above is the cone of dependence of (ξ,η) and it is defined as

$$C(\xi,\eta) := \{(z,y)\in\Omega : z\leq\xi \text{ and } y\leq\eta\}.$$

Changing the variables back to (t,x), we get the following representation formula for a solution u of problem (0.10) for a.e. $(t,x)\in\Omega$:

$$u(t,x) = \tilde{f}(t-x) + \tilde{g}(t+x) - \frac{1}{2}\iint_{C(t,x)} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \tag{3.3}$$

where $\tilde{f}(\xi)$ and $\tilde{g}(\eta)$ are one-variable functions, defined by

$$\tilde{f}(\xi) := \int_0^{\xi} v_\xi(z,|z|)\,\mathrm{d}z, \quad \text{and} \quad \tilde{g}(\eta) := \begin{cases} v(-\eta,\eta) + \displaystyle\int_{-\eta}^{0} v_\xi(z,|z|)\,\mathrm{d}z, & \text{if } \eta < \ell_0, \\ v(\psi^{-1}(\eta),\eta) + \displaystyle\int_{\omega(\eta)}^{0} v_\xi(z,|z|)\,\mathrm{d}z, & \text{if } \eta < \ell_0. \end{cases}$$

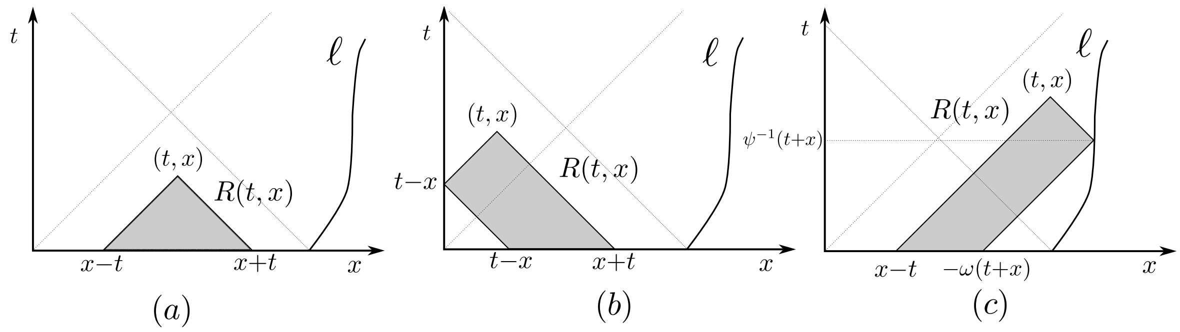

Figure 3.2: The set $R(t,x)$ in three typical cases.

Notice that in the change of variables $C(\xi,\eta)$ is written as $C(t,x)$, where

$$C(t,x) := \{(\tau,\sigma) \in \Omega : 0 \le \tau \le t,\ x-t+\tau \le \sigma \le x+t-\tau\}.$$

We start by finding an explicit representation formula for u in

$$\Omega' := (\{t < x\} \cup \{t{+}x < \ell_0\}) \cap \Omega.$$

It is then useful to introduce the set $R(t,x)$ defined for $(t,x) \in \Omega'$ as

$$R(t,x) := \begin{cases} C(t,x), & \text{if } t < x \text{ and } t+x < \ell_0, \quad (a) \\ C(t,x) \setminus C(t{-}x, 0), & \text{if } t > x \text{ and } t+x < \ell_0, \quad (b) \\ C(t,x) \setminus C(\psi^{-1}(t{+}x), \ell(\psi^{-1}(t{+}x))), & \text{if } t < x \text{ and } t+x > \ell_0, \quad (c) \end{cases}$$

(see Fig. 3.2). Notice that the three cases *a, b,* and *c* are highlighted in Figure 3.2 and represent a partition of Ω'. Notice that

$$R(t,x) = \{(\tau,\sigma) \in \Omega : 0 \le \tau \le t, \gamma_1(\tau;t,x) \le \sigma \le \gamma_2(\tau;t,x)\},$$

where

$$\gamma_1(\tau;t,x) = \begin{cases} x{-}t{+}\tau, & \text{if } t < x \text{ and } t+x < \ell_0, \quad (a) \\ |x{-}t{+}\tau|, & \text{if } t > x \text{ and } t+x < \ell_0, \quad (b) \\ x{-}t{+}\tau, & \text{if } t < x \text{ and } t+x > \ell_0, \quad (c) \end{cases} \tag{3.4}$$

and

$$\gamma_2(\tau;t,x) = \begin{cases} x{+}t{-}\tau, & \text{if } t < x \text{ and } t+x < \ell_0, \quad (a) \\ x{+}t{-}\tau, & \text{if } t > x \text{ and } t+x < \ell_0, \quad (b) \\ \tau{-}\omega(t{+}x), & \text{if } t < x,\ \ t+x > \ell_0, \text{ and } \tau \le \psi^{-1}(t+x), \quad (c) \\ x{+}t{-}\tau, & \text{if } t < x,\ \ t+x > \ell_0, \text{ and } \tau > \psi^{-1}(t+x), \quad (c). \end{cases} \tag{3.5}$$

Equation 3.3 gives us a general representation for a solution u to our problem. We now present an explicit formula for u in the three cases *a, b,* and *c*. First, we notice that in the cases b and c the set $R(t,x)$ is obtained by subtracting from $C(t,x)$ a part depending only on $t-x$ or $t+x$. This means that we can rewrite (3.3) by introducing two functions $f(t-x)$ and $g(t+x)$

obtained by adding this part to $\tilde{f}(t-x)$ and $\tilde{g}(t+x)$, respectively. We obtain for a.e. $(t,x) \in \Omega$ such that $t+x \le \ell_0$ or $t+x > \ell_0$ and $t < x$

$$u(t,x) = f(t-x) + g(t+x) - \frac{1}{2} \iint_{R(t,x)} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \tag{3.6}$$
$$= f(t-x) + g(t+x) - \frac{1}{2} \int_0^t \int_{\gamma_1(\tau;t,x)}^{\gamma_2(\tau;t,x)} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Notice that u_t is now integrated over $R(t,x)$ instead of $C(t,x)$. We now denote by $A(t,x) := f(t-x) + g(t+x)$. We can explicitly write $A(t,x)$ using only the initial data u_0 and u_1 and the external loading w. To this aim, we consider (3.2) and use the fact that

$$v_\xi(z,|z|) = \begin{cases} \frac{1}{2}u_t(0,-z) - \frac{1}{2}u_x(0,-z), & \text{if } z < 0, \\ \frac{1}{2}u_t(z,0) - \frac{1}{2}u_x(z,0), & \text{if } z \ge 0, \end{cases}$$

as one easily gets from (1.9). We thus obtain

$$A(t,x) = \begin{cases} \frac{1}{2}u_0(x-t) + \frac{1}{2}u_0(x+t) + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\,\mathrm{d}s, & \text{if } t \le x \text{ and } t+x \le \ell_0, \quad (a) \\ w(t-x) - \frac{1}{2}u_0(t-x) + \frac{1}{2}u_0(t+x) + \frac{1}{2}\int_{t-x}^{x+t} u_1(s)\,\mathrm{d}s, & \text{if } t > x \text{ and } t+x \le \ell_0, \quad (b) \\ \frac{1}{2}u_0(x-t) - \frac{1}{2}u_0(-\omega(t+x)) + \frac{1}{2}\int_{x-t}^{-\omega(t+x)} u_1(s)\,\mathrm{d}s, & \text{if } t \le x \text{ and } t+x > \ell_0, \quad (c). \end{cases} \tag{3.7}$$

For any $F \in L^2(\Omega')$ we now consider $\iint_{R(t,x)} F$ and we study its derivatives in t and x.

Proposition 3.3. *Let $F \in L^2(\Omega')$ and for a.e. $(t,x) \in \Omega'$ let*

$$H(t,x) = \int_0^t \int_{\gamma_1(\tau;t,x)}^{\gamma_2(\tau;t,x)} F(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Then, $H \in H^1(\Omega')$ and

$$H_t(t,x) = \int_0^t [F(\tau,\gamma_2(\tau;t,x))(\gamma_2)_t(\tau;t,x) - F(\tau,\gamma_1(\tau;t,x))(\gamma_1)_t(\tau;t,x)]\,\mathrm{d}\tau, \tag{3.8}$$

$$H_x(t,x) = \int_0^t [F(\tau,\gamma_2(\tau;t,x))(\gamma_2)_x(\tau;t,x) - F(\tau,\gamma_1(\tau;t,x))(\gamma_1)_x(\tau;t,x)]\,\mathrm{d}\tau. \tag{3.9}$$

We first need a preliminary lemma.

Lemma 3.4. *Let $F \in L^2(0,T;L^2(0,L))$ and for a.e. $(t,x) \in [0,T]\times[0,L]$ consider the curves γ_1 and γ_2 introduced in* (3.4) *and* (3.5)*. Then,*

$$G^i(t,x) := \int_0^t F(\tau,\gamma_i(\tau;t,x))\,\mathrm{d}\tau$$

is in $L^2(0,T;L^2(0,L))$. *Moreover, if* $F \in H^1(\Omega_T)$ *then* $G \in H^1(\Omega_T)$ *and we have for a.e.* $(t,x) \in [0,T]\times[0,L]$

$$
\begin{aligned}
G^i_t(t,x) &= F(t,\gamma_i(t;t,x)) + \int_0^t F_x(\tau,\gamma_i(\tau;t,x))(\gamma_i)_t(\tau;t,x)\,\mathrm{d}\tau,\\
G^i_x(t,x) &= \int_0^t F_x(\tau,\gamma_i(\tau;t,x))(\gamma_i)_x(\tau;t,x)\,\mathrm{d}\tau.
\end{aligned}
$$

Proof. We consider the case of $\gamma_1(\tau;t,x)$. The case of $\gamma_2(\tau;t,x)$ is analogous. We look for an upper bound on the L^2-norm of G. We first use Jensen's inequality:

$$
\begin{aligned}
\int_0^T \int_0^L G(t,x)^2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_0^L t^2 \left(\frac{1}{t}\int_0^t F(\tau,\gamma_1(\tau;t,x))\,\mathrm{d}\tau\right)^2 \mathrm{d}x\,\mathrm{d}t\\
\le \int_0^T t \int_0^L \int_0^t F(\tau,\gamma_1(\tau;t,x))^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

We now use the explicit formula for γ_1 in (3.4) and obtain

$$
\begin{aligned}
&\int_0^T t \int_0^t \int_0^t F(\tau,|x-t+\tau|)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t + \int_0^T t \int_t^L \int_0^t F(\tau,x-t+\tau)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\\
&\le 2T^2\|F\|^2_{L^2([0,T]\times[0,L])}.
\end{aligned}
$$

Notice that to get the last passage we integrate in the variables (τ,x) recalling that for fixed t we have

$$
\{\gamma_1(\tau;t,x) : \tau \in [0,t],\ x \in [0,L]\} \subset [0,T]\times[0,L].
$$

Hence, the double integral in (τ,x) is controlled by $\|F\|^2_{L^2([0,T]\times[0,L])}$, which is independent of t. The second part is a direct consequence of differentiation under the integral. □

We now prove Proposition 3.3

Proof. Consider

$$
G(t,x) := \int_0^t [F(\tau,\gamma_2(\tau;t,x))(\gamma_2)_t(\tau;t,x) - F(\tau,\gamma_1(\tau;t,x))(\gamma_1)_t(\tau;t,x)]\,\mathrm{d}\tau.
$$

By (3.4) and (3.5), we have $|(\gamma_1)_t(\tau;t,x)| = 1$ and $|(\gamma_2)_t(\tau;t,x)| = 1$ for every $(t,x) \in \Omega'$ and $\tau \in [0,t]$, thus

$$
|G(t,x)| \le \int_0^t |F(\tau,\gamma_2(\tau;t,x))|\,\mathrm{d}\tau + \int_0^t |F(\tau,\gamma_1(\tau;t,x))|\,\mathrm{d}\tau
$$

is in $L^2(\Omega')$; by Lemma 3.4, $G \in L^2(\Omega')$. We prove that, as $h \to 0$,

$$
\frac{H(t+h,x) - H(t,x)}{h} \to G(t,x) \quad \text{in} \quad L^2(\Omega'), \tag{3.10}
$$

which implies $G = H_t$. The analogous argument is done to prove (3.9). We shall consider, for simplicity, the case a and therefore $\gamma_1(\tau;t,x) = x - t + \tau$, $\gamma_2(\tau;t,x) = x + t - \tau$. The other two cases are analogous, see Figure 3.2. We write the left hand side of (3.10) changing the order of

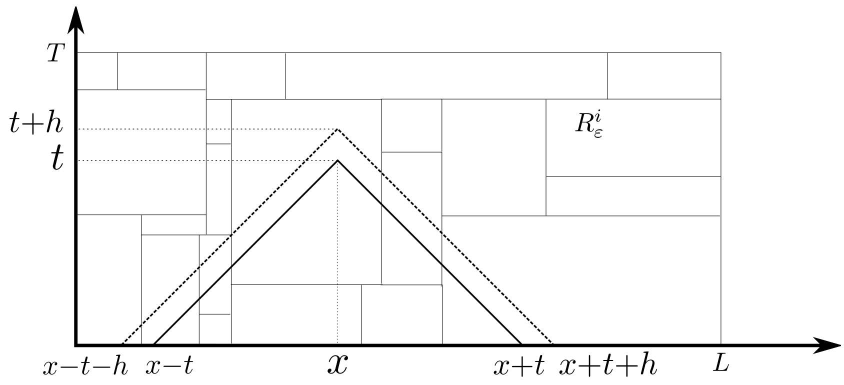

Figure 3.3: In the case $t < x$ and $t + x < \ell_0$ the region of integration in (3.11) is the difference of the two triangles. It is possible to find h so small that for every $i = 1, \dots, N_\varepsilon$ the points $(t - x + \sigma, \sigma)$ and $(\tau + t - x + \sigma, \sigma)$ belong to the same rectangle R^i_ε.

integration. We obtain

$$\frac{H(t+h,x) - H(t,x)}{h} = \frac{1}{h}\left[\int_{x-t-h}^{x}\int_{0\vee(t-x+\sigma)}^{t+h-x+\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma + \int_{x}^{x+t+h}\int_{0\vee(t+x-\sigma)}^{t+h+x-\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma\right]. \tag{3.11}$$

We will separately study the convergence of the two summands in the right hand side of (3.11) as follows. We claim that

$$\frac{1}{h}\int_{x-t-h}^{x}\int_{0\vee(t-x+\sigma)}^{t+h-x+\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma \to \int_{x-t}^{x} F(t-x+\sigma,\sigma)\,\mathrm{d}\sigma = \int_0^t F(\tau, x-t+\tau)\,\mathrm{d}\tau, \tag{3.12}$$

$$\frac{1}{h}\int_{x}^{x+t+h}\int_{0\vee(t+x-\sigma)}^{t+h+x-\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma \to \int_{x}^{x+t} F(t+x-\sigma,\sigma)\,\mathrm{d}\sigma = \int_0^t F(\tau, x+t-\tau)\,\mathrm{d}\tau. \tag{3.13}$$

Let us consider (3.12), the other being analogous. By Hölder's inequality,

$$\begin{aligned}&\frac{1}{h^2}\int_{\Omega'}\left(\int_{x-t-h}^{x-t}\int_0^{t+h-x+\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma\right)^2\,\mathrm{d}x\,\mathrm{d}t\\ \leq&\frac{1}{h^2}\int_{\Omega'}\frac{h^2}{2}\int_{x-t-h}^{x-t}\int_0^{t+h-x+\sigma} F(\tau,\sigma)^2\,\mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which tends to zero as $h \to 0$ by the absolute continuity of integrals. Therefore, we shall only study the convergence of the remaining part $\frac{1}{h}\int_{x-t}^{x}\int_{t-x+\sigma}^{t+h-x+\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma$ to the right hand side of (3.12). We have

$$\begin{aligned}&\frac{1}{h}\int_{x-t}^{x}\int_{t-x+\sigma}^{t+h-x+\sigma} F(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma - \int_{x-t}^{x} F(t-x+\sigma,\sigma)\,\mathrm{d}\sigma\\ =&\frac{1}{h}\int_{x-t}^{x}\int_0^h \left[F(\tau+t-x+\sigma,\sigma) - F(t-x+\sigma,\sigma)\right]\,\mathrm{d}\tau\,\mathrm{d}\sigma\end{aligned} \tag{3.14}$$

Since $F \in L^2(\Omega')$, we extend it to $L^2(0,T;L^2(0,L))$ by setting $F = 0$ in $[0,T]\times[0,L]\setminus\Omega'$. Then, for fixed $\varepsilon > 0$ there exists $N_\varepsilon > 0$ and a simple function

$$S_\varepsilon(t,x) = \sum_{i=1}^{N_\varepsilon} s_\varepsilon^i \mathbb{1}_{R_\varepsilon^i}(t,x),$$

where $s_\varepsilon^i \in \mathbb{R}$ and the sets R_ε^i are open rectangles of the form $(\sigma_\varepsilon^i, \tau_\varepsilon^i)\times(y_\varepsilon^i, z_\varepsilon^i)$, such that

$$\|F - S_\varepsilon\|_{L^2(0,T;L^2(0,L))} < \varepsilon.$$

Notice that, since $\mathbb{1}_{R_\varepsilon^i}(t,x) = \mathbb{1}_{(\sigma_\varepsilon^i,\tau_\varepsilon^i)}(t)\mathbb{1}_{(y_\varepsilon^i,z_\varepsilon^i)}(x)$ for every $i = 1,\dots,N_\varepsilon$,

$$\frac{1}{h}\int_{x-t}^{x}\int_0^h [S_\varepsilon(\tau+t-x+\sigma,\sigma) - S_\varepsilon(t-x+\sigma,\sigma)]\,\mathrm{d}\tau\,\mathrm{d}\sigma \tag{3.15}$$

$$= \sum_{i=1}^{N_\varepsilon}\frac{1}{h}\int_{x-t}^{x} s_\varepsilon^i \mathbb{1}_{(y_\varepsilon^i,z_\varepsilon^i)}(\sigma)\int_0^h \left[\mathbb{1}_{(\sigma_\varepsilon^i,\tau_\varepsilon^i)}(\tau+t-x+\sigma) - \mathbb{1}_{(\sigma_\varepsilon^i,\tau_\varepsilon^i)}(t-x+\sigma)\right]\,\mathrm{d}\tau\,\mathrm{d}\sigma. \tag{3.16}$$

We consider all the points (t,x) such that the lines $\sigma \mapsto (t-x+\sigma,\sigma)$ do not intersect any of the vertices of the rectangles R_ε. This is possible for a.e. (t,x). If we consider h so small that for every $i = 1,\dots,N_\varepsilon$ the points $(t-x+\sigma,\sigma)$ and $(\tau+t-x+\sigma,\sigma)$ for every $\tau \in [0,h]$ belong to the same rectangle R_ε^i, then the previous integral is zero, see Fig. 3.3. Then, by the Dominated Convergence Theorem, (3.10) holds for $F = S_\varepsilon$.

Hence, we can argue by approximation with simple functions. We define $F_\varepsilon := F - S_\varepsilon$. Adding and subtracting S_ε, we obtain that (3.14) is equal to

$$\frac{1}{h}\int_{x-t}^{x}\int_0^h [F_\varepsilon(\tau+t-x+\sigma,\sigma) - F_\varepsilon(t-x+\sigma,\sigma)]\,\mathrm{d}\tau\,\mathrm{d}\sigma$$
$$+\frac{1}{h}\int_{x-t}^{x}\int_0^h [S_\varepsilon(\tau+t-x+\sigma,\sigma) - S_\varepsilon(t-x+\sigma,\sigma)]\,\mathrm{d}\tau\,\mathrm{d}\sigma,$$

and the second summand tends to zero, by the previous argument. To conclude, we find an upper bound for the L^2-norm of the first one:

$$\int_0^T\int_0^L\left|\frac{1}{h}\int_0^h\int_{x-t}^{x}[F_\varepsilon(\tau+t-x+\sigma,\sigma) - F_\varepsilon(t-x+\sigma,\sigma)]\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq\frac{2}{h}\int_0^T\int_0^L\int_0^h\int_{x-t}^{x} F_\varepsilon(\tau+t-x+\sigma,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t + 2L\|F_\varepsilon\|^2_{L^2(0,T;L^2(0,L))},$$

where in the last inequality we used the same argument of Lemma 3.4. Moreover, by changing the order of integration,

$$\frac{2}{h}\int_0^h\int_0^T\int_0^L\int_{x-t}^{x} F_\varepsilon(\tau+t-x+\sigma,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau$$
$$\leq\frac{2}{h}\int_0^h\int_0^T\int_0^L\int_{\tau}^{L+\tau} F_\varepsilon(t-x+\sigma,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau$$
$$\leq 2L\|F_\varepsilon\|^2_{L^2(0,T;L^2(0,L+\delta))},$$

for $\delta > 0$. It is then enough to choose S_ε such that $\|F_\varepsilon\| \to 0$ as $\varepsilon \to 0$. To this end, it is enough to extend F in a slightly larger set than $(0,T)\times(0,L+\delta)$ Finally, (3.13) follows with the same argument and therefore we obtain (3.10). To prove (3.9), we argue in the same way. □

Our aim is to determine the existence and uniqueness of a solution u to problem (0.10) in the sense of Definition 3.2. We will proceed by applying the Contraction Lemma several times starting from the triangle $C(\ell_0, 0)$.

Theorem 3.5. *Assume* (1.6) *and consider the space of functions* $u \in H^1(C(\ell_0, 0))$ *such that* u *verifies conditions* (0.10b), (0.10d), *and* (0.10e). *Let* $\Phi \colon H^1(C(\ell_0, 0)) \to H^1(C(\ell_0, 0))$ *be defined as*

$$\Phi(u)(t, x) := A(t, x) - \frac{1}{2} \iint_{R(t,x)} u_t(\tau, \sigma) \, \mathrm{d}\tau \, \mathrm{d}\sigma,$$

where $A(t, x)$ *is as in* (3.7). *Then, if* $\ell_0 < \frac{1}{\sqrt{2}}$, *the map* Φ *is a contraction of* $H^1(C(\ell_0, 0))$.

Proof. First, we prove that $\Phi(u)$ belongs to the same class of functions of u. Since $u_t \in L^2(C(\ell_0, 0))$ then, by Lemma 3.4 and Proposition 3.3, $\Phi(u) \in H^1(C(\ell_0, 0))$. Moreover, $\Phi(u)$ satisfies (0.10b), (0.10d), and (0.10e) by the definition of $A(t, x)$. Hence, we are left to prove that Φ is a contraction.

We start from the case $t < x$. Let u^1 and u^2 be two functions in the above class. We want to find a constant $C < 1$ such that

$$\|\Phi(u^1) - \Phi(u^2)\|_{H^1(C(\frac{\ell_0}{2}, \frac{\ell_0}{2}))} \leq C \|u^1 - u^2\|_{H^1(C(\ell_0, 0))}.$$

Since u^1 and u^2 satisfy the same initial conditions, it follows that they share the same term $A(t, x)$. Therefore,

$$\begin{aligned}
\left|\Phi(u^1) - \Phi(u^2)\right| &= \frac{1}{2} \left| \iint_{R(t,x)} (u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)) \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| \\
&\leq \frac{1}{2} \iint_{R(t,x)} \left|u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)\right| \, \mathrm{d}\sigma \, \mathrm{d}\tau.
\end{aligned}$$

We now integrate with respect to x and t the square of the above quantity. By Jensen's inequality and the parametrisation of $R(t, x)$ introduced in (3.4) & (3.5), we find that

$$\begin{aligned}
\int_0^{\frac{\ell_0}{2}} \int_t^{\ell_0 - t} \left|\Phi(u^1) - \Phi(u^2)\right|^2 \mathrm{d}x \, \mathrm{d}t &\leq \int_0^{\frac{\ell_0}{2}} \int_t^{\ell_0 - t} \frac{|R(t, x)|}{4} \int_0^t \int_{x - t + \tau}^{x + t - \tau} \left|u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)\right|^2 \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \frac{\ell_0^2}{8} \int_0^{\frac{\ell_0}{2}} \int_t^{\ell_0 - t} \int_0^t \int_{x - t + \tau}^{x + t - \tau} \left|u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)\right|^2 \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

because $|R(t, x)| \leq \frac{\ell_0^2}{2}$ in $C(\ell_0, 0)$. We continue as follows:

$$\begin{aligned}
&\frac{\ell_0^2}{8} \int_0^{\frac{\ell_0}{2}} \int_t^{\ell_0 - t} \int_0^t \int_{x - t + \tau}^{x + t - \tau} \left|u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)\right|^2 \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \frac{\ell_0^2}{8} \int_0^{\frac{\ell_0}{2}} \int_t^{\ell_0 - t} \int_0^t \int_\tau^{\ell_0 - \tau} \left|u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)\right|^2 \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \frac{\ell_0^2}{8} \int_0^{\frac{\ell_0}{2}} \int_t^{\ell_0 - t} \int_0^{\frac{\ell_0}{2}} \int_\tau^{\ell_0 - \tau} \left|u^1_t(\tau, \sigma) - u^2_t(\tau, \sigma)\right|^2 \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \frac{\ell_0^4}{8} \|u^1 - u^2\|^2_{H^1(C(\ell_0, 0))}.
\end{aligned}$$

A similar argument is carried out for the derivative of Φ with respect to t. By Proposition 3.3, we have

$$\begin{aligned}
&\left|\partial_t\Phi(u^1)-\partial_t\Phi(u^2)\right| \\
=&\frac{1}{2}\left|\int_0^t (u_t^1(\tau,x+t-\tau)-u_t^2(\tau,x+t-\tau)+u_t^1(\tau,x-t+\tau)-u_t^2(\tau,x-t+\tau))\,\mathrm{d}\tau\right| \\
\leq&\frac{1}{2}\int_0^t \left|u_t^1(\tau,x+t-\tau)-u_t^2(\tau,x+t-\tau)\right|\,\mathrm{d}\tau+\frac{1}{2}\int_0^t \left|u_t^1(\tau,x-t+\tau)-u_t^2(\tau,x-t+\tau)\right|\,\mathrm{d}\tau.
\end{aligned}$$

We again integrate with respect to x and t the square of the previous quantity. Since $(a+b)^2 \leq 2a^2+2b^2$, we have that

$$\begin{aligned}
&\int_0^{\frac{\ell_0}{2}}\int_t^{\ell_0-t}\left|\partial_t\Phi(u^1)-\partial_t\Phi(u^2)\right|^2\,\mathrm{d}x\,\mathrm{d}t \\
\leq\quad& 2\int_0^{\frac{\ell_0}{2}}\int_t^{\ell_0-t}\frac{t}{4}\int_0^t\left|u_t^1(\tau,x+t-\tau)-u_t^2(\tau,x+t-\tau)\right|^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
&+2\int_0^{\frac{\ell_0}{2}}\int_t^{\ell_0-t}\frac{t}{4}\int_0^t\left|u_t^1(\tau,x-t+\tau)-u_t^2(\tau,x-t+\tau)\right|^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
\leq&\int_0^{\frac{\ell_0}{2}}t\int_0^t\int_t^{\ell_0-t}\left|u_t^1(\tau,x)-u_t^2(\tau,x)\right|^2\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \\
\leq&\int_0^{\frac{\ell_0}{2}}t\int_0^{\frac{\ell_0}{2}}\int_t^{\ell_0-t}\left|u_t^1(\tau,x)-u_t^2(\tau,x)\right|^2\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \\
\leq&\ell_0^2\|u^1-u^2\|^2_{H^1(C(\ell_0,0))}.
\end{aligned}$$

Notice that we used again the same argument of Lemma 3.4. Analogously, we find the same result for the derivative of Φ with respect to x, using (3.9) and therefore

$$\|\nabla\Phi(u^1)-\nabla\Phi(u^2)\|_{L^2(C(\ell_0,0))}\leq 2\ell_0^2\|u^1-u^2\|^2_{H^1(C(\ell_0,0))}.$$

The case $t > x$ follows with the same argument, using analogous estimates. In conlusion, we have that Φ is a contraction of $C(\ell_0, 0)$ if $\max\{\frac{\ell_0^4}{8}, 2\ell_0^2\} < 1$, which is verified if $\ell_0 < \frac{1}{\sqrt{2}}$. □

Remark 3.6. In the sub-triangle $C(\frac{\ell_0}{2},\frac{\ell_0}{2})$ we have that Φ is still a contraction provided that $\max\{\ell_0, \sqrt{2\ell_0}\} < 1$. Exploiting the previous proof, we only need to consider the case $t < x$ and therefore condition (0.10b) does not play any role here.

Remark 3.7. By Theorem 3.5, if $\ell_0 < \frac{1}{\sqrt{2}}$, then there exists a unique function $u \in H^1(C(\ell_0, 0))$ satisfying (0.10a), (0.10b), (0.10d), and (0.10e) and therefore it is the solution of our problem in $C(\ell_0, 0)$.

If $\ell_0 > \frac{1}{\sqrt{2}}$, we can argue as follows. By Theorem 3.5, there exists a unique solution u in $C(\frac{1}{2}, 0)$ of class H^1. Then we can find $0 < x_1 < \frac{1}{2}$ such that $x_1 + \frac{1}{2} \leq \ell_0$. By Remark 3.6, there exists a unique solution in $C(\frac{1}{4},\frac{1}{4}) + (0, x_1)$. Moreover, the two solutions we have found so far coincide in the intersection between $C(\frac{1}{2}, 0)$ and $C(\frac{1}{4},\frac{1}{4}) + (0, x_1)$ and therefore we can extend u to their union. We repeat the previous argument for every $0 < x_1 \leq \ell_0 - \frac{1}{2}$, extending u up to the set $C(\ell_0, 0) \cap \{t < \frac{1}{4}\}$ and preserving the H^1 regularity (see Figure 3.4). In particular,

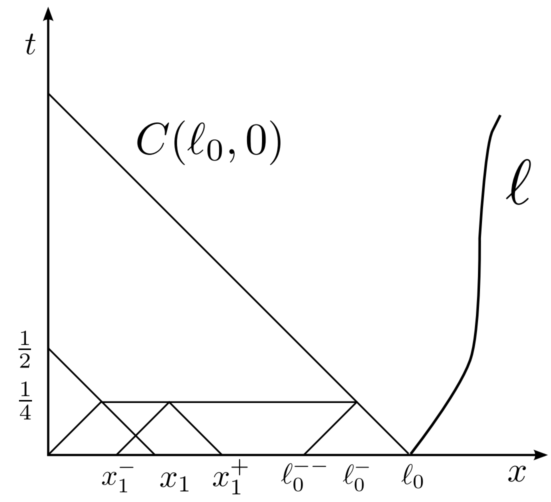

Figure 3.4: The construction of u in Remark 3.7. We used the following notation: $x_1^- := x_1 - \frac{1}{4}$, $x_1^+ := x_1 + \frac{1}{4}$, $\ell_0^{--} := \ell_0 - \frac{1}{2}$, and $\ell_0^- := \ell_0 - \frac{1}{4}$.

there exists $\bar{t} \le \frac{1}{4}$ such that $u(\bar{t}, x)$ is well defined for a.e. $x \in [0, \ell_0 - \frac{1}{4}]$. This allows us to start again the previous argument from time $\bar{t}$. Eventually, after a finite number of steps, we have the solution $u \in H^1(C(\ell_0, 0))$ that is the unique solution of our problem.

We now extend u out of $C(\ell_0, 0)$, using the evolution of the debonding front $t \mapsto \ell(t)$. We thus consider $C(\varphi^{-1}(0), \ell(\varphi^{-1}(0)))$ and recall that, by (3.6) and (3.7), in this region u has the following representation:

$$\begin{aligned} u(t,x) = \ & \frac{1}{2}u_0(x-t) - \frac{1}{2}u_0(-\omega(t+x)) + \frac{1}{2}\int_{x-t}^{-\omega(t+x)} u_1(s)\,\mathrm{d}s \\ & - \frac{1}{2}\int_0^{\psi^{-1}(t+x)} \int_{x-t+\tau}^{\tau-\omega(t+x)} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau - \frac{1}{2}\int_{\psi^{-1}(t+x)}^{t} \int_{x-t+\tau}^{x+t-\tau} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \end{aligned}$$

for a.e. $(t, x) \in C(\varphi^{-1}(0), \ell(\varphi^{-1}(0))) =: C_*$.

Theorem 3.8. *Assume* (1.1), (1.6), *and consider the space X_0 of functions $v \in H^1(\Omega')$ such that $v = u$ in $C(\ell_0, 0)$, where u is the function obtained in Remark 3.7 and v satisfies* (0.10c). *Let $\Psi\colon X_0 \to X_0$ be defined as*

$$\Psi(v)(t,x) := A(t,x) - \frac{1}{2}\iint_{R(t,x)} u_t(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma,$$

for a.e. $(t, x) \in C_*$. Then, the map Ψ is a contraction in X_0 provided that ℓ_0 is sufficiently small.

Proof. We first notice that $\Psi(v)$ is in the same class of functions of v for every v that satisfies the assumptions of the Theorem. Indeed, for a.e. $(t, x) \in C_*$ such that $t + x \le \ell_0$, we have that $\Psi(v)(t, x) = \Phi(v)(t, x) = u(t, x)$, where Φ is the contraction of Theorem 3.5. Moreover, $\Psi(v) \in H^1(C_*)$ by Lemma 3.4 and Proposition 3.3 and $\Psi(v)(t, \ell(t)) = 0$ for a.e. $t \in [0, \varphi^{-1}(0)]$ by definition.

We now take u^1 and u^2 in the above class of functions and we notice that

$$\left|\Psi(u^1)(t,x) - \Psi(u^2)(t,x)\right| = \frac{1}{2}\left|\iint_{R(t,x)} (u^1_t - u^2_t)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| \leq \frac{1}{2}\iint_{R(t,x)} \left|u^1_t - u^2_t\right|\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Therefore, for every $t \in [0, \varphi^{-1}(0)]$, we have by Jensen's inequality

$$\begin{aligned}
&\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} \left|\Psi(u^1)(t,x) - \Psi(u^2)(t,x)\right|^2\,\mathrm{d}x\,\mathrm{d}t \\
\leq &\frac{1}{4}\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} |R(t,x)| \iint_{R(t,x)} \left|u^1_t - u^2_t\right|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
\leq &\frac{1}{4}\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} |R(t,x)| \int_0^{\varphi^{-1}(0)} \int_\tau^{\ell(\tau)} \left|u^1_t - u^2_t\right|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
\leq &\frac{1}{8}\varphi^{-1}(0)^3 \ell_0 \|u^1 - u^2\|^2_{H^1(C_*)}.
\end{aligned}$$

Notice that $\ell(t) - t \leq \ell_0$ by (1.1) and that $|R(t,x)| \leq \frac{\varphi^{-1}(0)^2}{2}$ in C_*. We now consider the time derivative of Ψ. We have that

$$\begin{aligned}
\partial_t\Psi(u^1)(t,x) - \partial_t\Psi(u^2)(t,x) = &-\frac{1}{2}\int_0^t (u^1_t - u^2_t)(\tau, x - t + \tau)\,\mathrm{d}\tau \\
&+ \frac{1}{2}\dot{\omega}(t{+}x)\int_0^{\psi^{-1}(t+x)} (u^1_t - u^2_t)(\tau, \tau - \omega(t{+}x))\,\mathrm{d}\tau \\
&- \frac{1}{2}\int_{\psi^{-1}(t+x)}^t (u^1_t - u^2_t)(\tau, x + t - \tau)\,\mathrm{d}\tau.
\end{aligned}$$

Hence, by integrating over C_*, we obtain

$$\begin{aligned}
&\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} \left|\partial_t\Psi(u^1)(t,x) - \partial_t\Psi(u^2)(t,x)\right|^2\,\mathrm{d}x \\
\leq\ & 3\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} \frac{t}{4}\int_0^t |u^1_t - u^2_t|(\tau, x - t + \tau)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
&+ 3\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} \frac{t}{4}\dot{\omega}(t{+}x)^2 \int_0^{\psi^{-1}(t+x)} |u^1_t - u^2_t|(\tau, \tau - \omega(t{+}x))^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
&+ 3\int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} \frac{t}{4}\int_{\psi^{-1}(t+x)}^t |u^1_t - u^2_t|(\tau, x + t - \tau)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
\leq\ & \frac{3}{4}\varphi^{-1}(0)\int_0^{\varphi^{-1}(0)} \int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} |u^1_t - u^2_t|(\tau, \gamma_1(\tau; t, x))^2\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \\
&+ \frac{6}{4}\varphi^{-1}(0)\int_0^{\varphi^{-1}(0)} \int_0^{\varphi^{-1}(0)} \int_t^{\ell(t)} |u^1_t - u^2_t|(\tau, \gamma_2(\tau; t, x))^2\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \\
\leq\ & \frac{9}{4}\varphi^{-1}(0)^2 \|u^1 - u^2\|^2_{H^1(C_*)}.
\end{aligned}$$

Notice that the factor 3 is consequence of the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and that we used that $\dot{\omega} \leq 1$ by (1.4). The analogous argument is carried out for the derivative of Ψ with

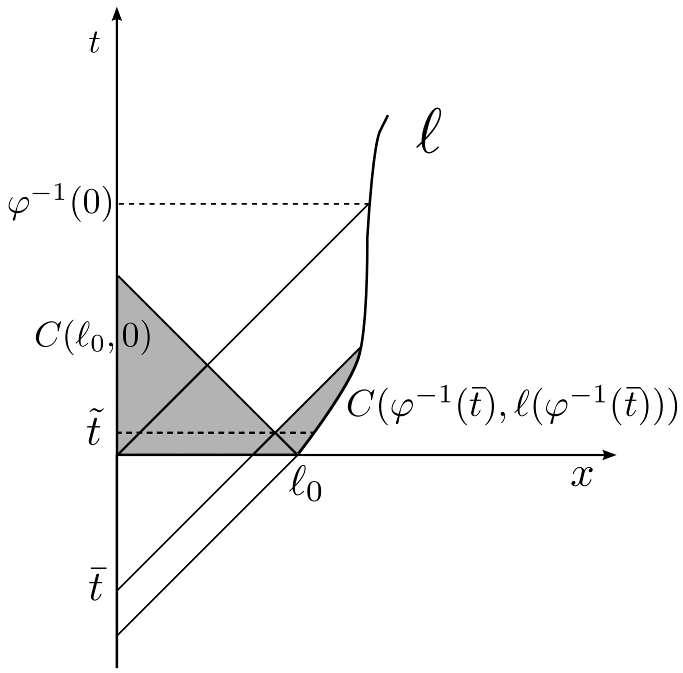

Figure 3.5: The construction of the solution u in $\Omega_{\tilde{t}}$.

respect to x. We finally obtain that Ψ is a contraction in $H^1(C_*)$ if

$$\max\left\{\frac{1}{8}\varphi^{-1}(0)^3\ell_0, \frac{9}{2}\varphi^{-1}(0)^2\right\} < 1. \tag{3.17}$$

By (1.1a), for every fixed $T > 0$ we have $\varphi^{-1}(0) \leq \frac{\ell_0}{1-L_T}$. Therefore, it is enough to require that

$$\max\left\{\frac{1}{8(1-L_T)^3}\ell_0^4, \frac{9}{2(1-L_T)^2}\ell_0^2\right\} < 1,$$

which is verified for sufficiently small ℓ_0. □

Remark 3.9. By Theorem 3.8, if ℓ_0 is sufficiently small, then we can extend our solution u that was constructed in Remark 3.7 to all $C(\ell_0, 0) \cup C_*$.

If this were not the case, we can argue as follows. There exists $-\ell_0 < \bar{t} < 0$ such that in $C(\varphi^{-1}(\bar{t}), \ell(\varphi^{-1}(\bar{t})))$ the map Ψ is a contraction. Indeed, since φ^{-1} is an increasing function that maps $[-\ell_0, 0]$ into $[0, \varphi^{-1}(0)]$, then there exists $-\ell_0 < \bar{t} < 0$ such that the corresponding $\varphi^{-1}(\bar{t})$ is so small that (3.17) is satisfied. Therefore, we now have a solution $u \in H^1(C(\ell_0, 0) \cup C(\varphi^{-1}(\bar{t}), \ell(\varphi^{-1}(\bar{t}))))$. This means that there exists $\tilde{t} < \min\{\ell_0, \varphi^{-1}(\bar{t})\}$ such that $u \in H^1(\Omega_{\tilde{t}})$ is the solution of our problem and $u(\tilde{t}, x)$ is well defined for a.e. $x \in [0, \ell(\tilde{t})]$. This allow us to start again the argument for the construction of u starting from time $\tilde{t}$ (see Fig. 3.5).

In conclusion, for every fixed $T > 0$, we can construct a unique function $u \in H^1(\Omega_T)$ solution to (0.10), by iteratively applying a finite number of times Remark 3.7 & Remark 3.9. We have thus obtained the following result:

Theorem 3.10. *Fix $T > 0$. Assume* (1.1), (1.5), *and* (1.6). *Then there exists a unique function $u \in H^1(\Omega_T)$ solution to Problem 0.10 according to Definition 3.2.*

Part II

Quasistatic Limits

CHAPTER 4

Quasistatic limit of dynamic evolutions for the peeling test in dimension one

In this chapter is presented the asymptotic analysis for the model introduced in Chapter 1 when the speed of the external loading becomes slower and slower. This is done by replacing the function w appearing in (4.1b) with $w(\varepsilon t)$, where $\varepsilon > 0$ is a small parameter. While in the first section we adapt the main results of Chapter 1 to find a unique pair $(u^\varepsilon, \ell^\varepsilon)$ which solves the coupled problem (0.15)&(0.16), in the second section we analyse the quasistatic limit as $\varepsilon \to 0$ and find that a limit solution (u, ℓ) does not, in general, satisfy the Griffith's criterion in its quasistatic formulation (0.18). Indeed, in Section 3 we give an explicit example in which this convergence fails. In Section

The results of Sections 4.1–4.3 are part of [45], while Section 4.4 is part of a forthcoming paper in collaboration with G. Lazzaroni.

4.1 Existence and uniqueness results

In this section we provide an outline of the results of existence and uniqueness for the coupled problem (0.15)&(0.16) for fixed $\varepsilon > 0$, proved in Chapter 1. The only difference is that the speed of sound is $\frac{1}{\varepsilon}$ instead of 1.

We consider the following generalisation of problem (0.15),

$$\begin{aligned}
&\varepsilon^2 u^\varepsilon_{tt}(t,x) - u^\varepsilon_{xx}(t,x) = 0, && t > 0,\ 0 < x < \ell^\varepsilon(t), && (4.1a)\\
&u^\varepsilon(t,0) = w^\varepsilon(t), && t > 0, && (4.1b)\\
&u^\varepsilon(t,\ell^\varepsilon(t)) = 0, && t > 0 && (4.1c)\\
&u^\varepsilon(0,x) = u^\varepsilon_0(x), && 0 < x < \ell_0, && (4.1d)\\
&u^\varepsilon_t(0,x) = u^\varepsilon_1(x), && 0 < x < \ell_0. && (4.1e)
\end{aligned}$$

In analogy with (1.58), we require that

$$w^\varepsilon \in \widetilde{C}^{0,1}(0,+\infty), \quad u^\varepsilon_0 \in C^{0,1}([0,\ell_0]), \quad u^\varepsilon_1 \in L^\infty(0,\ell_0), \tag{4.2a}$$

where $\widetilde{C}^{0,1}(0,+\infty)$ is defined as in (1.7), and the compatibility conditions

$$u_0^\varepsilon(0) = w^\varepsilon(0), \quad u_0^\varepsilon(\ell_0) = 0. \tag{4.2b}$$

To give the notion of solution, for the moment we assume that the evolution of the debonding front $t \mapsto \ell^\varepsilon(t)$ is known. More precisely, we fix $\ell_0 > 0$ and $\ell^\varepsilon \colon [0,+\infty) \to [\ell_0,+\infty)$ Lipschitz and such that

$$0 \le \dot\ell^\varepsilon(t) < \frac{1}{\varepsilon}, \text{ for a.e. } t > 0, \tag{4.3a}$$

$$\ell^\varepsilon(0) = \ell_0. \tag{4.3b}$$

We introduce the sets

$$\begin{aligned} \Omega^\varepsilon &:= \{(t,x) : t > 0,\ 0 < x < \ell^\varepsilon(t)\}, \\ \Omega_T^\varepsilon &:= \{(t,x) : 0 < t < T,\ 0 < x < \ell^\varepsilon(t)\} \end{aligned}$$

and the spaces

$$\begin{aligned} \widetilde{H}^1(\Omega^\varepsilon) &:= \{u \in H^1_{\mathrm{loc}}(\Omega^\varepsilon) : u \in H^1(\Omega_T^\varepsilon), \text{ for every } T > 0\}, \\ \widetilde{C}^{0,1}(\Omega^\varepsilon) &:= \{u \in C^0(\Omega^\varepsilon) : u \in C^{0,1}(\Omega_T^\varepsilon) \text{ for every } T > 0\}, \end{aligned}$$

The notion of solution is given in the following sense.

Definition 4.1. *We say that $u^\varepsilon \in \widetilde{H}^1(\Omega^\varepsilon)$ (resp. in $u^\varepsilon \in H^1(\Omega_T^\varepsilon)$) is a solution to* (4.1) *if $\varepsilon^2 u_{tt}^\varepsilon - u_{xx}^\varepsilon = 0$ holds in the sense of distributions in Ω^ε (resp. Ω_T^ε), the boundary conditions* (4.1b)*&*(4.1c) *are intended in the sense of traces and the initial conditions* (4.1d)*&*(4.1e) *are satisfied in the sense of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.*

Condition (4.1e) makes sense since $u_x^\varepsilon \in L^2(0,T;L^2(0,\ell_0))$ and, by the wave equation, $u_{xx}^\varepsilon, u_{tt}^\varepsilon \in L^2(0,T;H^{-1}(0,\ell_0))$, therefore $u_t^\varepsilon \in H^1(0,T;H^{-1}(0,\ell_0)) \subset C^0([0,T];H^{-1}(0,\ell_0))$. Arguing as in Proposition 1.6 and Theorem 1.8, it is possible to uniquely solve (4.1) by means of the D'Alembert formula, as it is stated in the next proposition.

Proposition 4.2. *Assume* (4.2) *and* (4.3)*. Then, there exists a unique solution $u^\varepsilon \in H^1(\Omega^\varepsilon)$ to problem* (4.1)*, according to Definition 4.1. Moreover, $u^\varepsilon \in \widetilde{C}^{0,1}(\Omega^\varepsilon)$ and is expressed through the formula*

$$u^\varepsilon(t,x) = w^\varepsilon(t+\varepsilon x) - \frac{1}{\varepsilon} f^\varepsilon(t+\varepsilon x) + \frac{1}{\varepsilon} f^\varepsilon(t-\varepsilon x), \tag{4.4}$$

where $f^\varepsilon \in \widetilde{C}^{0,1}(-\varepsilon\ell_0,+\infty)$ is determined by

$$w^\varepsilon(t+\varepsilon\ell^\varepsilon(t)) - \frac{1}{\varepsilon} f^\varepsilon(t+\varepsilon\ell^\varepsilon(t)) + \frac{1}{\varepsilon} f^\varepsilon(t-\varepsilon\ell^\varepsilon(t)) = 0, \quad \textit{for every } t > 0, \tag{4.5}$$

and

$$f^\varepsilon(s) = \varepsilon w^\varepsilon(s) - \frac{\varepsilon}{2} u_0^\varepsilon(\tfrac{s}{\varepsilon}) - \frac{\varepsilon^2}{2} \int_0^{\frac{s}{\varepsilon}} u_1^\varepsilon(x)\,\mathrm{d}x - \varepsilon w^\varepsilon(0) + \frac{\varepsilon}{2} u_0^\varepsilon(0), \quad \textit{for every } s \in [0,\varepsilon\ell_0], \tag{4.6a}$$

$$f^\varepsilon(s) = \frac{\varepsilon}{2} u_0^\varepsilon(-\tfrac{s}{\varepsilon}) - \frac{\varepsilon^2}{2} \int_0^{-\frac{s}{\varepsilon}} u_1^\varepsilon(x)\,\mathrm{d}x - \frac{\varepsilon}{2} u_0^\varepsilon(0), \qquad \textit{for every } s \in (-\varepsilon\ell_0, 0]. \tag{4.6b}$$

By derivation of (4.4) we obtain

$$u_t^\varepsilon(t,x) = \dot w^\varepsilon(t+\varepsilon x) - \frac{1}{\varepsilon}\dot f^\varepsilon(t+\varepsilon x) + \frac{1}{\varepsilon}\dot f^\varepsilon(t-\varepsilon x), \tag{4.7a}$$

$$u_x^\varepsilon(t,x) = \varepsilon\dot w^\varepsilon(t+\varepsilon x) - \dot f^\varepsilon(t+\varepsilon x) - \dot f^\varepsilon(t-\varepsilon x). \tag{4.7b}$$

Formula (4.7a) guarantees that, for every t, $u_t^\varepsilon(t,\cdot)$ is defined a.e. in $(0,\ell^\varepsilon(t))$.

The last observation and the existence of a unique solution to (4.1) stated in Proposition 4.2 allow us to define the internal energy

$$\mathcal{E}^\varepsilon(t;\ell^\varepsilon,w^\varepsilon) := \int_0^{\ell^\varepsilon(t)} \left[\frac{\varepsilon^2}{2}u_t^\varepsilon(t,x)^2 + \frac{1}{2}u_x^\varepsilon(t,x)^2\right] \mathrm{d}x. \tag{4.8}$$

In the previous expression the internal energy is a functional of ℓ^ε and w^ε, while u^ε is the unique solution of (4.1) corresponding to the prescribed debonding evolution ℓ^ε and to the data of the problem. Using (4.7), then (4.8) reads as

$$\mathcal{E}^\varepsilon(t;\ell^\varepsilon,w^\varepsilon) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon\ell^\varepsilon(t)} [\varepsilon\dot w^\varepsilon(s) - \dot f^\varepsilon(s)]^2 \,\mathrm{d}s + \frac{1}{\varepsilon}\int_{t-\varepsilon\ell^\varepsilon(t)}^t \dot f^\varepsilon(s)^2 \,\mathrm{d}s. \tag{4.9}$$

We now give the notion of dynamic energy release rate which is used to give the criterion for the (henceforth unknown) evolution of the debonding front ℓ^ε. This is done as in Chapter 1, Section 2. Specifically, the dynamic energy release rate $G_\alpha^\varepsilon(t_0)$ at time t_0 corresponding to a speed $0<\alpha<\frac{1}{\varepsilon}$ of the debonding front, is defined as

$$G_\alpha^\varepsilon(t_0) := \lim_{t\to t_0^+} \frac{\mathcal{E}^\varepsilon(t_0;\lambda^\varepsilon,z^\varepsilon) - \mathcal{E}^\varepsilon(t;\lambda^\varepsilon,z^\varepsilon)}{(t-t_0)\alpha},$$

where $\lambda^\varepsilon \in C^{0,1}([0,+\infty))$ is such that $\lambda^\varepsilon(t) = \ell^\varepsilon(t)$ for every $0 \le t \le t_0$, $\dot\lambda^\varepsilon < \frac{1}{\varepsilon}$ for a.e. $t>0$, and

$$\frac{1}{h}\int_{t_0}^{t_0+h} \left|\dot\lambda^\varepsilon(t) - \alpha\right| \mathrm{d}t \to 0, \qquad \text{as } h \to 0^+,$$

while

$$z^\varepsilon(t) = \begin{cases} w^\varepsilon(t), & t \le t_0,\\ w^\varepsilon(t_0), & t > t_0.\end{cases}$$

As it is proved in Proposition 1.15, given ℓ^ε and w^ε, the limit above exists for a.e. $t_0>0$ and for every $\alpha \in (0,\frac{1}{\varepsilon})$. Moreover, it is expressed in terms of f^ε through the following formula:

$$G_\alpha^\varepsilon(t) = 2\frac{1-\varepsilon\alpha}{1+\varepsilon\alpha}\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2. \tag{4.10}$$

This also shows that G_α^ε depends on the choice of λ^ε only through α and therefore the definition is well posed. We also extend by continuity this definition to the case $\alpha = \dot\ell^\varepsilon(t) = 0$, by setting

$$G_0^\varepsilon(t) := 2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2.$$

Thus, by (4.10), we have monotonicity with respect to α:

$$G_\alpha^\varepsilon(t_0) < G_0^\varepsilon(t_0), \text{ for every } \alpha \in (0,\frac{1}{\varepsilon}), \quad G_\alpha^\varepsilon(t_0) \to 0 \text{ for } \alpha \to 1^-,$$

for a.e. $t_0 > 0$.

We require that the evolution of the debonding front ℓ^ε follows Griffith's criterion

$$\dot{\ell}^\varepsilon(t) \geq 0, \tag{4.11a}$$

$$G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t) \leq \kappa(\ell^\varepsilon(t)), \tag{4.11b}$$

$$\left[G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t) - \kappa(\ell^\varepsilon(t))\right] \dot{\ell}^\varepsilon(t) = 0, \tag{4.11c}$$

where the local toughness is assumed to be a piecewise Lipschitz function with a finite number of discontinuities

$$\kappa\colon [0, +\infty) \to [c_1, c_2], \quad 0 < c_1 < c_2. \tag{4.12}$$

Notice that $\dot{\ell}^\varepsilon(t)$ and $G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t)$ are well defined for a.e. t and (4.10) gives

$$G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t) = 2\frac{1 - \varepsilon\dot{\ell}^\varepsilon(t)}{1 + \varepsilon\dot{\ell}^\varepsilon(t)} \dot{f}^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2. \tag{4.13}$$

The criterion is derived by using the following maximum dissipation principle that is analogous to (1.52): for a.e. $t > 0$

$$\dot{\ell}^\varepsilon(t) = \max\{\alpha \in [0, \tfrac{1}{\varepsilon}) : \kappa(\ell^\varepsilon(t))\alpha = G^\varepsilon_\alpha(t)\alpha\}.$$

This implies that for a.e. $t > 0$, if $\dot{\ell}^\varepsilon(t) > 0$, then $\kappa(\ell^\varepsilon(t)) = G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t)$, while if $\ell^\varepsilon(t) = 0$, then $\kappa(\ell^\varepsilon(t)) \geq G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t) = G^\varepsilon_0(t)$, thus (4.11) follows. Combining (4.11) with (4.13), we have an equivalent formulation of this evolution criterion. Indeed, ℓ^ε satisfies Griffith's criterion if and only if it is solution of the following Cauchy problem:

$$\begin{cases} \dot{\ell}^\varepsilon(t) = \dfrac{1}{\varepsilon}\dfrac{2\dot{f}^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 - \kappa(\ell^\varepsilon(t))}{2\dot{f}^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 + \kappa(\ell^\varepsilon(t))} \vee 0, \\ \ell^\varepsilon(0) = \ell_0, \end{cases} \tag{4.14}$$

for a.e. $t > 0$.

The following existence and uniqueness result for the coupled problem (4.1)&(4.14) for fixed $\varepsilon > 0$ is the analogous of Theorem 1.22: we thus refer to it for its proof.

Theorem 4.3. *Let $T > 0$, assume (4.2), and let the local toughness κ be as in (4.12). Then, there is a unique solution $(u^\varepsilon, \ell^\varepsilon) \in C^{0,1}(\Omega^\varepsilon_T) \times C^{0,1}([0, T])$ to the coupled problem (4.1)&(4.14). Moreover, there exists a constant L^ε_T satisfying $\dot{\ell}^\varepsilon \leq L^\varepsilon_T < \frac{1}{\varepsilon}$.*

4.2 A priori estimate and convergence

In this section we study the limit as $\varepsilon \to 0$ of the solutions $(u^\varepsilon, \ell^\varepsilon)$ to the coupled problem (4.1)&(4.14). We fix $T > 0$ and make the following assumptions on the data: there exists $w \in C^{0,1}([0, T])$ such that

$$w^\varepsilon \overset{*}{\rightharpoonup} w \text{ weakly* in } W^{1,\infty}(0, T), \tag{4.15a}$$

$$u^\varepsilon_0 \text{ is bounded in } W^{1,\infty}(0, \ell_0), \tag{4.15b}$$

$$\varepsilon u^\varepsilon_1 \text{ is bounded in } L^\infty(0, \ell_0). \tag{4.15c}$$

Notice that (4.15b)&(4.15c) imply that the initial internal energy associated to $u^\varepsilon(0, \cdot)$ is uniformly bounded with respect to ε.

4.2.1 A priori bounds

We start from a uniform bound on the internal energy $\mathcal{E}^\varepsilon$. To this end, it is convenient to express it as in (4.9). As in (1.31), we find the energy balance for fixed $\varepsilon > 0$:

$$\mathcal{E}^\varepsilon(t;\ell^\varepsilon,w^\varepsilon) - \mathcal{E}^\varepsilon(0;\ell^\varepsilon,w^\varepsilon) + \int_{\ell_0}^{\ell^\varepsilon(t)} \kappa(x)\,\mathrm{d}x + \int_0^t [\varepsilon\dot{w}^\varepsilon(s) - 2\dot{f}^\varepsilon(s)]\dot{w}^\varepsilon(s)\,\mathrm{d}s = 0. \tag{4.16}$$

In the next proposition we derive an a priori bound for $\mathcal{E}^\varepsilon$, uniformly with respect to ε. First, we introduce the functions

$$\varphi^\varepsilon(t) := t - \varepsilon\ell^\varepsilon(t) \quad \text{and} \quad \psi^\varepsilon(t) := t + \varepsilon\ell^\varepsilon(t). \tag{4.17}$$

In view of Theorem 4.3, $\dot{\ell}^\varepsilon \leq L^\varepsilon_T < \frac{1}{\varepsilon}$ and therefore these functions are equi-Lipschitz. Then, we define

$$\omega^\varepsilon(t) := \varphi^\varepsilon((\psi^\varepsilon)^{-1}(t)),$$

which is also equi-Lipschitz, since

$$\dot{\omega} \leq \frac{1-\varepsilon L^\varepsilon_T}{1+\varepsilon L^\varepsilon_T} < 1$$

for a.e. $0 \leq t \leq T$.

Proposition 4.4. *Assume* (4.2), (4.15), *and let κ be as in* (4.12). *Then, there exists $C > 0$ such that $\mathcal{E}^\varepsilon(t) \leq C$ for every $\varepsilon > 0$ and for every $t \in [0,T]$. Moreover, we have*

$$\|\dot{f}^\varepsilon\|_{L^\infty(-\varepsilon\ell_0,T)} \leq C, \tag{4.18}$$

uniformly in ε.

Proof. We need to estimate the last term in (4.16). To this end, we notice that it is sufficient to get a uniform bound for $\dot{f}^\varepsilon$ in L^∞ as in (4.18). Then the conclusion readily follows from the bounds on the initial conditions and on the toughness.

In order to obtain (4.18), we first estimate $\dot{f}^\varepsilon$ in $[-\varepsilon\ell_0,\varepsilon\ell_0]$. By differentiating (4.6) and using the assumptions (4.15), we see that

$$\operatorname*{ess\,sup}_{t\in[-\varepsilon\ell_0,\varepsilon\ell_0]} |\dot{f}^\varepsilon(t)| \leq \varepsilon\|\dot{w}^\varepsilon\|_{L^\infty(0,T)} + \frac{1}{2}\|\dot{u}^\varepsilon_0\|_{L^\infty(0,\ell_0)} + \frac{\varepsilon}{2}\|u^\varepsilon_1\|_{L^\infty(0,\ell_0)} \leq C, \tag{4.19}$$

for some positive constant $C > 0$.

Then, we need to extend the estimate to $[0,T]$. To this end, we mimick the construction for the existence of a solution (see Theorem 1.8). More precisely, we define $t^\varepsilon_0 := \varepsilon\ell_0$ and, iteratively, $t^\varepsilon_i := (\omega^\varepsilon)^{-1}(t^\varepsilon_{i-1}) = \psi^\varepsilon((\varphi^\varepsilon)^{-1}(t^\varepsilon_{i-1}))$. Let also $s^\varepsilon_{i+1} := (\varphi^\varepsilon)^{-1}(t^\varepsilon_i)$ for $i \geq 0$. See Figure 4.1.

By differentiating the "bounce formula" (4.5), we find that

$$\dot{f}^\varepsilon(t+\varepsilon\ell^\varepsilon(t)) = \varepsilon\dot{w}^\varepsilon(t+\varepsilon\ell^\varepsilon(t)) + \frac{1-\varepsilon\dot{\ell}^\varepsilon(t)}{1+\varepsilon\dot{\ell}^\varepsilon(t)}\dot{f}^\varepsilon(t-\varepsilon\ell^\varepsilon(t)). \tag{4.20}$$

Then we have

$$\operatorname*{ess\,sup}_{t\in[t^\varepsilon_0,t^\varepsilon_1]} |\dot{f}(t)| = \operatorname*{ess\,sup}_{s\in[0,s^\varepsilon_1]} |\dot{f}^\varepsilon(s+\varepsilon\ell^\varepsilon(s))| \leq \varepsilon\|\dot{w}^\varepsilon\|_{L^\infty(0,T)} + \|\dot{f}^\varepsilon\|_{L^\infty(-\varepsilon\ell_0,\varepsilon\ell_0))} \leq C,$$

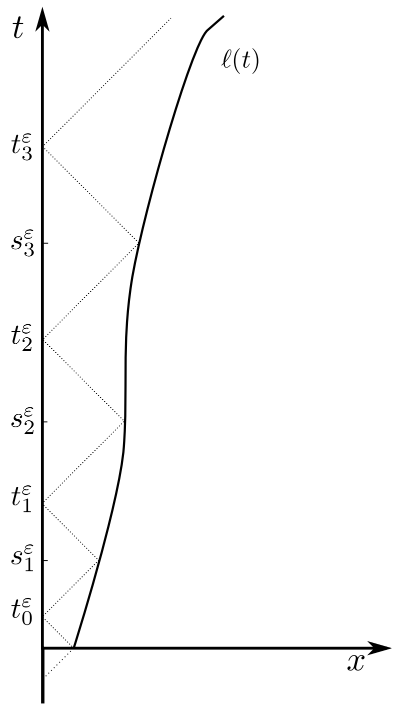

Figure 4.1: Construction of the sequences $\{s_i^\varepsilon\}$ and $\{t_i^\varepsilon\}$ employed in the proof of Proposition 4.4.

where the uniform bound follows from (4.19) up to changing the value of C. This implies that

$$\operatorname*{ess\,sup}_{t\in[-\varepsilon\ell_0,t_1^\varepsilon]} |\dot f(t)| \le C.$$

We iterate this argument and use the fact that the maximum number of "bounces", i.e., the number of times we apply formulas (4.5) and (4.20), is bounded. More precisely, there exists n_ε such that $T \in (t_{n_\varepsilon}^\varepsilon, t_{n_\varepsilon+1}^\varepsilon]$ and, since $\ell_0 > 0$, we have that $n_\varepsilon \le \frac{T}{2\varepsilon\ell_0}$. Therefore,

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in[t_{n_\varepsilon}^\varepsilon,T]} |\dot f^\varepsilon(t)| &\le \operatorname*{ess\,sup}_{t\in[t_{n_\varepsilon}^\varepsilon,t_{n_\varepsilon+1}^\varepsilon]} |\dot f^\varepsilon(t)| \le \varepsilon\|\dot w^\varepsilon\|_{L^\infty} + \operatorname*{ess\,sup}_{t\in[t_{n_\varepsilon-1}^\varepsilon,t_{n_\varepsilon}^\varepsilon]} |\dot f^\varepsilon(t)| \\ &\le 2\varepsilon\|\dot w^\varepsilon\|_{L^\infty} + \operatorname*{ess\,sup}_{t\in[t_{n_\varepsilon-2}^\varepsilon,t_{n_\varepsilon-1}^\varepsilon]} |\dot f^\varepsilon(t)| \le \dots \le n_\varepsilon\varepsilon\|\dot w^\varepsilon\|_{L^\infty} + \operatorname*{ess\,sup}_{t\in[t_0^\varepsilon,t_1^\varepsilon]} |\dot f^\varepsilon(t)| \\ &\le \frac{T}{2\ell_0}\|\dot w^\varepsilon\|_{L^\infty} + \operatorname*{ess\,sup}_{t\in[t_0^\varepsilon,t_1^\varepsilon]} |\dot f^\varepsilon(t)| \le C.\end{aligned}$$

Then, the uniform bound on $\dot f^\varepsilon$ holds in $[-\varepsilon\ell_0, T]$, thus (4.18) is proved. □

Remark 4.5. Formula (4.7a) guarantees that, for every $t \in [0,T]$, $u_t^\varepsilon(t,\cdot)$ is defined a.e. in $(0, \ell^\varepsilon(T))$. Moreover, the uniform bound on the internal energy implies that

$$\|\varepsilon u_t^\varepsilon(t,\cdot)\|_{L^2(0,\ell^\varepsilon(t))} \le C \text{ for every } t \in [0,T], \tag{4.21}$$

where $C > 0$ is independent of ε and t.

4.2.2 Convergence of the solutions

The a priori bound on the energy allows the passage to the limit in ℓ^ε.

Proposition 4.6. *Assume* (4.2), (4.15), *and* (4.12). *Let* $(u^\varepsilon, \ell^\varepsilon)$ *be the solution to the coupled problem* (4.1)*&*(4.14). *Then, there exists* $L > 0$ *such that* $\ell^\varepsilon(T) \leq L$. *Moreover, there exists a sequence* $\varepsilon_k \to 0$ *and an increasing function* $\ell\colon [0,T] \to [0,L]$ *such that*

$$\ell^{\varepsilon_k}(t) \to \ell(t)$$

for every $t \in [0,T]$.

Proof. Since the local toughness κ is bounded from below, a direct consequence of Proposition 4.4 is that the sequence of functions $\ell^\varepsilon(t)$ is bounded uniformly in ε. Indeed, the term $-\int_0^t \dot w^\varepsilon(s)[\varepsilon \dot w(s) - 2\dot f^\varepsilon(s)]\,\mathrm{d}s$ in the energy balance (4.16) is bounded, as one can see applying the Cauchy-Schwartz inequality and using (4.18). Therefore

$$\int_{\ell_0}^{\ell^\varepsilon(t)} \kappa(x)\,\mathrm{d}x = -\mathcal{E}^\varepsilon(t;\ell^\varepsilon, w^\varepsilon) + \mathcal{E}^\varepsilon(0;\ell^\varepsilon, w^\varepsilon) - \int_0^t \dot w^\varepsilon(s)[\varepsilon \dot w^\varepsilon(s) - 2\dot f^\varepsilon(s)]\,\mathrm{d}s$$

is uniformly bounded. Since $\kappa \geq c_1$, it follows that there exists $C > 0$ such that

$$c_1(\ell^\varepsilon(t) - \ell_0) \leq C, \tag{4.22}$$

uniformly in ε and for every $t \in [0,T]$. Then, using Helly's selection principle on the sequence of uniformly bounded and increasing functions ℓ^ε, it is possible to extract a sequence $\ell^{\varepsilon_k}(t)$ pointwise converging to an increasing function $\ell(t)$ for every $t \in [0,T]$. □

We now prove a technical lemma stating that the graphs of ℓ^{ε_k} converge to the graph of ℓ in the Hausdorff metric. We employ the following notation for the graph of a function:

$$\text{Graph}\,\ell := \{(t, \ell(t)) : 0 \leq t \leq T\}.$$

The same notation will be used for the graph of ℓ^ε. Since $t \mapsto \ell(t)$ may be discontinuous, we consider its extended graph

$$\text{Graph}^*\ell := \{(t,x) \in [0,T]\times[0,L] : \ell(t^-) \leq x \leq \ell(t^+)\},$$

where $\ell(t^-)$ (resp. $\ell(t^+)$) is the left-sided (resp. right-sided) limit of ℓ at t. Given $A \subset [0,T]\times[0,L]$ and $\eta > 0$ we set

$$(A)_\eta := \{(t,x) \in [0,T]\times[0,L] : d((t,x),A) < \eta\},$$

where d is the Euclidean distance, and we call $(A)_\eta$ the open η-neighbourhood of A. We also recall that, given two nonempty sets $A, B \subset [0,T]\times[0,L]$, their Hausdorff distance is defined by

$$d_{\mathcal{H}}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

Notice that

$$\text{if } d_{\mathcal{H}}(A,B) \leq \eta, \text{ then } A \subset (B)_\eta \text{ and } B \subset (A)_\eta. \tag{4.23}$$

We say that a sequence A_k converges to A in the sense of Hausdorff if $d_{\mathcal{H}}(A_k, A) \to 0$.

The Hausdorff convergence of Graph ℓ^{ε_k} to Graph$^*\ell$ will be used in the proof of Theorem 4.8. To prove that Graph ℓ^{ε_k} converges to Graph$^*\ell$ in the sense of Hausdorff, in the following lemma we employ the equivalent notion of Kuratowski convergence, whose definition is recalled below.

Lemma 4.7. *The sets* Graph ℓ^{ε_k} *converge to* Graph$^*\ell$ *in the sense of Hausdorff.*

Proof. In order to prove this result we show that Graph ℓ^{ε_k} converges to Graph$^*\ell$ in the sense of Kuratowski in the compact set $[0,T]\times[0,L]$. Since these sets are closed, the Kuratowski convergence implies that Graph ℓ^{ε_k} converges to Graph$^*\ell$ in the sense of Hausdorff, cf. [3, Proposition 4.4.14]. We recall that Graph ℓ^{ε_k} converges to Graph$^*\ell$ in the sense of Kuratowski if the following two conditions are both satisfied:

(i) Let $(t,x)\in[0,T]\times[0,L]$ and let $(t_k,x_k)\in$ Graph ℓ^{ε_k} be a sequence such that $(t_{k_n},x_{k_n})\to(t,x)$ for some subsequence. Then, $(t,x)\in$ Graph$^*\ell$.

(ii) For every $(t,x)\in$ Graph$^*\ell$ there exists a whole sequence such that $(t_k,x_k)\in$ Graph ℓ^{ε_k} and $(t_k,x_k)\to(t,x)$.

We prove condition (i) arguing by contradiction. Let thus $(t,x)\in[0,T]\times[0,L]$ and $(t_k,x_k)\in$ Graph ℓ^{ε_k} be such that $(t_k,x_k)\to(t,x)$ up to a subsequence (not relabelled) and assume that $(t,x)\notin$ Graph$^*\ell$, i.e., $x\notin[\ell(t^-),\ell(t^+)]$. We consider the case where $x<\ell(t^-)$, the case $x>\ell(t^+)$ being analogous. By assumption, there exists $k_0\in\mathbb{N}$ such that for every $k\geq k_0$ we have $\ell^{\varepsilon_k}(t_k)<\ell(t^-)$. By the definition of $\ell(t^-)$ and the monotonicity of ℓ, there exists $\eta>0$ such that $\ell^{\varepsilon_k}(t_k)<\ell(t-\eta)$ for every $k\geq k_0$. For k large, we have $t_k>t-\eta$. Therefore, by the monotonocity of ℓ^{ε_k}, we get

$$\ell^{\varepsilon_k}(t-\eta)\leq\ell^{\varepsilon_k}(t_k)<\ell(t-\eta),$$

which leads to contradiction, by the pointwise convergence of $\ell^{\varepsilon_k}(t-\eta)$.

We now prove condition (ii). Let $(t,x)\in$ Graph$^*\ell$. Then, for every $\eta>0$ we have $\ell(t-\eta)\leq x\leq\ell(t+\eta)$. We claim that there is a sequence $x_k\to x$ such that $x_k\in[\ell^{\varepsilon_k}(t-\eta),\ell^{\varepsilon_k}(t+\eta)]$. Specifically, if $\ell(t-\eta)<x<\ell(t+\eta)$ we take $x_k:=x$; if $x=\ell(t-\eta)$ we take $x_k:=\ell^{\varepsilon_k}(t-\eta)$; if $x=\ell(t+\eta)$ we take $x_k:=\ell^{\varepsilon_k}(t+\eta)$; in each case by pointwise convergence we conclude that $x_k\to x$. Then, by continuity and monotonicity of ℓ^{ε_k}, there exists $t_k\in[t-\eta,t+\eta]$ such that $\ell^{\varepsilon_k}(t_k)=x_k$. We conclude by the arbitrariness of η. □

We now investigate on the limit behaviour of u^ε. The next theorem shows that the limit displacement solves problem (0.17).

Theorem 4.8. *Assume* (4.2), (4.15), *and* (4.12). *Let* $(u^\varepsilon,\ell^\varepsilon)$ *be the solution to the coupled problem* (4.1)&(4.14). *Let* L *and* ε_k *be as in Proposition 4.6. Then,*

$$u^{\varepsilon_k}\rightharpoonup u \text{ weakly in } L^2(0,T;H^1(0,L)),\tag{4.24}$$

where

$$u(t,x)=\begin{cases}-\frac{w(t)}{\ell(t)}x+w(t) & \text{for a.e. } (t,x):x<\ell(t),\\ 0 & \text{for a.e. } (t,x):x\geq\ell(t).\end{cases}\tag{4.25}$$

Proof. We recall that $u^{\varepsilon_k}(t,x)=0$ whenever $x>\ell^{\varepsilon_k}(T)$. By Proposition 4.4 and by (4.8), $u_x^{\varepsilon_k}$ is bounded in $L^\infty(0,T;L^2(0,L))$ and therefore in $L^2(0,T;L^2(0,L))$ as well. We can thus extract a subsequence (not relabelled), and find a function $q\in L^2(0,T;L^2(0,L))$ such that

$$u_x^{\varepsilon_k}\rightharpoonup q \text{ in } L^2(0,T;L^2(0,L)).\tag{4.26}$$

We have

$$u^{\varepsilon_k}(t,x)=w^\varepsilon(t)+\int_0^x u_x^{\varepsilon_k}(t,\xi)\,\mathrm{d}\xi,\tag{4.27}$$

for every $(t,x) \in \Omega_T^{\varepsilon_k}$. In particular, u^{ε_k} is bounded in $L^2(0,T;L^2(0,L))$ and (up to extracting a further subsequence, not relabelled) there exists $u \in L^2(0,T;L^2(0,L))$ such that

$$u^{\varepsilon_k} \rightharpoonup u \text{ in } L^2(0,T;L^2(0,L)). \tag{4.28}$$

We remark that at this stage of the proof the limit displacement u may depend on the subsequence extracted in (4.28); however at the end of the proof we shall show the explicit formula (4.25), which implies that the limit is the same on the whole sequence ε_k extracted in Proposition 4.6.

We now pick a function $p(t,x) \in L^2(0,T;L^2(0,L))$ and integrate (4.27) over $(0,T)\times(0,L)$. By the Fubini Theorem we obtain

$$\begin{aligned}
\int_0^T\!\!\int_0^L u^{\varepsilon_k}(t,x)p(t,x)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\!\!\int_0^L w^{\varepsilon}(t)p(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_0^L p(t,x)\left(\int_0^x u_x^{\varepsilon_k}(t,\xi)\,\mathrm{d}\xi\right)\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\!\!\int_0^L w^{\varepsilon}(t)p(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_0^L \left(\int_\xi^L p(t,x)\,\mathrm{d}x\right) u_x^{\varepsilon_k}(t,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\
&= \int_0^T\!\!\int_0^L w^{\varepsilon}(t)p(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_0^L u_x^{\varepsilon_k}(t,\xi)P(t,\xi)\,\mathrm{d}\xi\,\mathrm{d}t,
\end{aligned}$$

where $P(t,\xi) = \int_\xi^L p(t,x)\,\mathrm{d}x$ is still in $L^2(0,T;L^2(0,L))$ by the Jensen inequality. Using (4.15a), (4.26), and (4.28), we find

$$u(t,x) = w(t) + \int_0^x q(t,\xi)\,\mathrm{d}\xi,$$

for a.e. $(t,x) \in (0,T)\times(0,L)$. This shows that $q = u_x$. We thus have proved that $u^{\varepsilon_k} \rightharpoonup u$ in $L^2(0,T;H^1(0,L))$.

We now prove (4.25). We employ Lemma 4.7, which can be rephrased as follows by using the open η-neighbourhood of $\mathrm{Graph}^*\ell$ and (4.23): for every $\eta \in (0,\ell_0)$ and for k sufficiently large we have

$$\mathrm{Graph}\,\ell^{\varepsilon_k} \subset (\mathrm{Graph}^*\ell)_\eta, \tag{4.29}$$

see Figure 4.2. Hence, we pick a test function $v \in H^1((0,T)\times(0,L))$ such that $v(t,0) = 0$ and $v(t,x) = 0$ whenever $(t,x) \in (\mathrm{Graph}^*\ell)_\eta$. By integration by parts in time and space, the equation solved by u^{ε_k} gives

$$\begin{aligned}
0 &= \int_0^T \int_0^L \left(\varepsilon_k^2 u_{tt}^{\varepsilon_k} - u_{xx}^{\varepsilon_k}\right) v\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_0^L \left(\varepsilon_k^2 u_t^{\varepsilon_k} v_t - u_x^{\varepsilon_k} v_x\right)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon_k^2 \int_0^L u_t^{\varepsilon_k}(T,x)v(T,x)\,\mathrm{d}x \\
&\quad - \varepsilon_k^2 \int_0^{\ell_0} u_1(x)v(0,x)\,\mathrm{d}x - \varepsilon_k^2 \int_{\ell_0}^L u_t^{\varepsilon_k}((\ell^{\varepsilon_k})^{-1}(x),x)v((\ell^{\varepsilon_k})^{-1}(x),x)\,\mathrm{d}x \\
&\quad - \int_0^T u_x^{\varepsilon_k}(t,\ell^{\varepsilon_k}(t))v(t,\ell^{\varepsilon_k}(t))\,\mathrm{d}t + \int_0^T u_x^{\varepsilon_k}(t,0)v(t,0)\,\mathrm{d}t.
\end{aligned} \tag{4.30}$$

Notice that the boundary term in the last expression makes sense since $(\ell^{\varepsilon_k})^{-1}(x)$ is defined for a.e. $x \in [0,L]$.

We now show that each summand in (4.30) converges to zero as $k \to \infty$. Using (4.21) we obtain

$$\varepsilon_k^2 \int_0^L u_t^{\varepsilon_k}(T,x)v(T,x)\,\mathrm{d}x \le \varepsilon_k \|\varepsilon_k u_t^{\varepsilon_k}(T,\cdot)\|_{L^2(0,L)} \|v(T,\cdot)\|_{L^2(0,L)} \to 0.$$

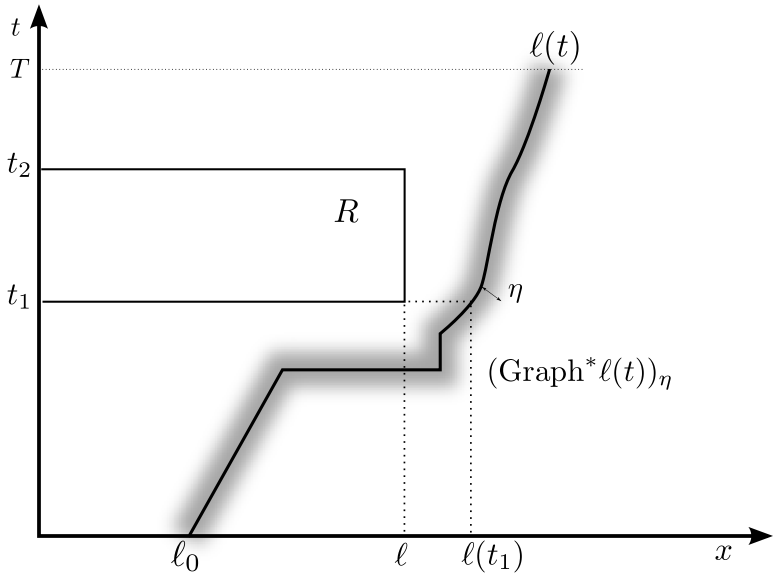

Figure 4.2: The set $(\mathrm{Graph}^*\ell)_\eta$ and the rectangle R employed in the proof of Theorem 4.8.

Integrating (4.21) in time we find that $-\int_0^T\int_0^L \varepsilon_k^2 u_t^{\varepsilon_k} v_t \,\mathrm{d}x\,\mathrm{d}t \to 0$. Moreover,

$$-\varepsilon_k^2\int_0^{\ell_0} u_1(x)v(0,x)\,\mathrm{d}x \to 0,$$

since $\varepsilon_k u_1$ is bounded by (4.15c). We also notice that

$$\varepsilon_k^2\int_{\ell_0}^{\ell^{\varepsilon_k}(T)} u_t^{\varepsilon_k}((\ell^{\varepsilon_k})^{-1}(x),x)v((\ell^{\varepsilon_k})^{-1}(x),x)\,\mathrm{d}x = -\varepsilon_k^2\int_0^T u_t^{\varepsilon_k}(t,\ell^{\varepsilon_k}(t))v(t,\ell^{\varepsilon_k}(t))\dot{\ell}^{\varepsilon_k}_k(t)\,\mathrm{d}t = 0,$$

since $v(t,x)=0$ in $(\mathrm{Graph}^*\ell)_\eta$ and (4.29). Finally

$$\int_0^T u_x^{\varepsilon_k}(t,0)v(t,0)\,\mathrm{d}t = 0$$

by assumption on v. This implies that in the limit we find

$$\int_0^T\int_0^{\ell(t)} u_x v_x = 0, \tag{4.31}$$

for every test function v in $H^1((0,T)\times(0,L))$ such that $v(t,0)=0$ and $v(t,x)=0$ whenever $(t,x)\in(\mathrm{Graph}^*\ell)_\eta$.

Finally, we prove that the limit function $u(t,\cdot)$ is affine in $[0,\ell(t)]$ for every t. We fix a rectangle $R:=(t_1,t_2)\times(0,\ell)$, with $t_1,t_2\in[0,T]$ and $0<\ell<\ell(t_1)$, see Figure 4.2. Let v be of the form $v(t,x)=\alpha(t)\beta(x)$, with $\alpha\in H_0^1(t_1,t_2)$ and $\beta\in H_0^1(0,\ell)$. Then, by (4.31) we know that

$$\int_{t_1}^{t_2}\alpha(t)\left(\int_0^\ell u_x(t,x)\dot{\beta}(x)\,\mathrm{d}x\right)\mathrm{d}t = 0.$$

Applying twice the Fundamental Lemma of Calculus of Variations, we find $a(t)$ and $b(t)$ such that

$$u(t,x) = a(t)x + b(t), \tag{4.32}$$

for a.e. $(t,x) \in R$. Then, by the arbitrariness of R, equation (4.32) is satisfied almost everywhere in $\{(t,x) : x < \ell(t)\}$.

On the other hand, in the region $\{(t,x) : x \geq \ell(t)\}$ we have $u(t,x) = u^{\varepsilon_k}(t,x) = 0$. Then we obtain the boundary condition $u(t,\ell(t)) = 0$ for a.e. t. By the weak convergence of u^{ε_k} to u and by (4.15a) we also recover the boundary condition $u(t,0) = w(t)$ for every t. This implies, together with (4.32), that

$$u(t,x) = -\frac{w(t)}{\ell(t)}x + w(t),$$

for a.e. $t \in [0,T]$ and a.e. $x \in [0,\ell(t)]$, while $u = 0$ for $x > \ell(t)$. □

4.2.3 Convergence of the stability condition

At this stage of the asymptotic analysis we have found a limit pair (u,ℓ) that describes the evolution of the debonding when the speed of the external loading tends to zero. We now investigate on the limit of Griffith's criterion (4.11) and we question whether the limit pair (u,ℓ) satisfies the quasistatic version of this criterion, i.e., whether (u,ℓ) is a rate-independent evolution according to the definition below.

Given a non-decreasing function $\lambda\colon [0,T] \to [0,L]$ and an external loading $w \in C^{0,1}([0,T])$ as above, for every $t \in [0,T]$ the internal quasistatic (potential) energy governing the process is

$$\mathcal{E}_{qs}(t;\lambda,w) := \min\left\{\frac{1}{2}\int_0^{\lambda(t)} \dot{v}(x)^2 \,\mathrm{d}x : v \in H^1(0,L),\ v(0) = w(t),\ v(\lambda(t)) = 0\right\},$$

where $\dot{v}$ denotes the derivative of v with respect to x, as always in this thesis for functions of only one variable. As in Chapter 4.1, we define the quasistatic energy release rate G_{qs} as the opposite of the derivative of $\mathcal{E}_{qs}(t;\lambda,w)$ with respect to λ, i.e.,

$$G_{qs}(t) := -\partial_\lambda \mathcal{E}_{qs}(t;\lambda,w).$$

Notice that ∂_λ has to be interpreted as a Gâteaux differential with respect to the function λ, indeed the displacement u depends on λ. The expression of $G_{qs}(t)$ is simplified by taking into account that an equilibrium displacement is affine in $(0,\lambda(t))$, see Remark 4.10.

Definition 4.9 (Rate-independent evolution)**.** *Let $\lambda\colon [0,T] \to [0,L]$ be a non-decreasing function and $v \in L^2(0,T;H^1(0,L))$. We say that (v,λ) is a rate-independent evolution if it satisfies the equilibrium equation for a.e. $t \in [0,T]$,*

$$v_{xx}(t,x) = 0, \quad \text{for } 0 < x < \lambda(t), \tag{4.33a}$$
$$v(t,0) = w(t), \tag{4.33b}$$
$$v(t,x) = 0, \qquad \text{for } x \geq \lambda(t), \tag{4.33c}$$

and the quasistic formulation of Griffith's criterion for a.e. $t > 0$,

$$\dot{\lambda}(t) \geq 0, \tag{4.34a}$$
$$G_{qs}(t) \leq \kappa(\lambda(t)), \tag{4.34b}$$
$$[G_{qs}(t) - \kappa(\lambda(t))]\,\dot{\lambda}(t) = 0. \tag{4.34c}$$

Remark 4.10. By (4.33), we know that $v(t,x) = \big[-\frac{w(t)}{\lambda(t)}x + w(t)\big] \vee 0$ for a.e. $t \in [0,T]$. Then, the quasistatic energy release rate can be explicitly computed and is given by

$$G_{qs}(t) = \frac{w(t)^2}{2\lambda(t)^2} = \frac{1}{2}v_x(t,\lambda(t))^2.$$

Moreover, under the additional assumption that $\lambda \in AC([0,T])$, (4.34c) is equivalent to the energy-dissipation balance that reads as follows:

$$\mathcal{E}_{qs}(t;\lambda,w) - \mathcal{E}_{qs}(0;\lambda,w) + \int_{\lambda_0}^{\lambda(t)} \kappa(x)\,\mathrm{d}x + \int_0^t v_x(s,0)\dot{w}(s)\,\mathrm{d}s = 0, \tag{4.35}$$

for every $t \in [0,T]$. Indeed, we use again the formula for v and differentiate (4.35) with respect to t, obtaining for a.e. $t \in [0,T]$

$$\frac{w(t)\dot{w}(t)}{\lambda(t)} + \dot{\lambda}(t)\left[-\frac{w(t)^2}{2\lambda(t)^2} + \kappa(\lambda(t))\right] - \frac{w(t)\dot{w}(t)}{\lambda(t)} = 0,$$

which is (4.34c). Therefore, Definition 4.9 complies with the usual notion of rate-independent evolution satisfying a first-order stability and an energy-dissipation balance, see [54].

Notice that (4.33)&(4.34) do not prescribe the behaviour of the system at time discontinuities. In order to determine suitable solutions, additional requirements can be imposed, e.g. requiring that the total energy is conserved also after jumps in time.

Remark 4.11. By (4.34c), we deduce that three different regimes for the evolution of a rate-independent debonding front λ are possible: λ is constant in a time subinterval, or it has a jump, or it is of the form

$$\lambda(t) = \frac{w(t)}{\sqrt{2\kappa(\lambda(t))}}.$$

Notice that, in the case of a non-decreasing local toughness κ, the quasistatic energy functional

$$\mathcal{E}_{qs}(\lambda) = \frac{w}{2\lambda} + \int_0^\lambda \kappa(x)\,\mathrm{d}x$$

is convex. This implies that a rate-independent evolution is a global minimizer for the total quasistatic energy.

We now consider the pair (u,ℓ) obtained in Proposition 4.6 and Theorem 4.8. We want to verify if (u,ℓ) satisfies Definition 4.9. First, we observe that by construction (cf. the application of Helly's theorem in Proposition 4.6) $t \mapsto \ell(t)$ is non-decreasing, thus (4.34a) automatically holds for a.e. t.

Next, we show that (4.34b) is satisfied. We first prove a few technical results.

Lemma 4.12. *Let Ω be a bounded domain in $\mathbb{R}^N$ and $g_n \to 1$ in measure, with $g_n\colon \Omega \to \mathbb{R}$ equibounded. Then, $g_n \to 1$ strongly in $L^2(\Omega)$.*

Proof. Fix $\eta, \delta > 0$. By the convergence in measure of the sequence g_n, there exists $n_0 \in \mathbb{N}$ and a set

$$A_\delta := \{x : |g_n - 1| > \delta\}$$

such that $|A_\delta| < \eta$ for every $n > n_0$. Therefore

$$\begin{aligned}\int_\Omega |g_n - 1|^2 \,\mathrm{d}x &= \int_{A_\delta} |g_n - 1|^2 \,\mathrm{d}x + \int_{\Omega\setminus A_\delta} |g_n - 1|^2 \,\mathrm{d}x \\ &\le C \int_{A_\delta} \mathrm{d}x + \int_\Omega \delta^2 \,\mathrm{d}x,\end{aligned}$$

where $C > 0$. In the last passage we have used the equiboundedness of g_n. The arbitrariness of η and δ leads to the conclusion of the proof. □

Lemma 4.13. *Let Ω be a bounded open interval, $g_n\colon \Omega \to \mathbb{R}$ a sequence of functions such that $g_n \to 1$ in measure and let $\rho_n\colon \Omega \to \Omega$ such that ρ_n^{-1} are equi-Lipschitz and $\rho_n \to 1$ uniformly in $\overline{\Omega}$. Then, $g_n \circ \rho_n \to 1$ in measure.*

Proof. For every $\delta > 0$ we have

$$\{x : |g_n \circ \rho_n - 1| > \delta\} = \rho_n^{-1}\big(\{y : |g_n(y) - 1| > \delta\}\big).$$

Since ρ_n^{-1} is equi-Lipschitz,

$$|\rho_n^{-1}\{y : |g_n(y) - 1| > \delta\}| \le C|\{y : |g_n(y) - 1| > \delta\}|,$$

where C is a positive constant. We conclude using the convergence in measure of g_n to 1. □

Theorem 4.14. *Assume* (4.2), (4.12), *and* (4.15) *and let $(u^\varepsilon, \ell^\varepsilon)$ be the solution to the coupled problem* (4.1)&(4.14). *Let L and ε_k be as in Proposition 4.6. Then, for a.e. $t \in [0,T]$ conditions* (4.34a) *and* (4.34b) *are satisfied.*

Proof. By (4.18) $\dot f^{\varepsilon_k}$ is bounded in $L^\infty(-\varepsilon_k\ell_0, T)$ uniformly with respect to ε_k. Therefore, $\dot f^{\varepsilon_k}$ is bounded in $L^2(-\varepsilon_k\ell_0, T)$ as well. Since $f^{\varepsilon_k}(0) = 0$, we have that f^{ε_k} is bounded in $H^1(-\varepsilon_k\ell_0, T)$ and thus, up to a subsequence (not relabelled), f^{ε_k} weakly converges to a function f in $H^1(0,T)$. Moreover, it is possible to characterise the limit function f in terms of w and ℓ. If we differentiate (4.4) with respect to x we find

$$u^\varepsilon_x(t,x) = -\dot f^{\varepsilon_k}(t-\varepsilon_k x) + \varepsilon_k \dot w^\varepsilon_k(t+\varepsilon_k x) - \dot f^{\varepsilon_k}(t+\varepsilon_k x).$$

By (4.24) and (4.25), we know that, up to a subsequence,

$$u^{\varepsilon_k}_x \rightharpoonup -\frac{w}{\ell} \quad \text{in } L^2(0,T;L^2(0,L)).$$

For every $p \in L^2(0,T)$ we have

$$\begin{aligned}&\lim_{k\to\infty}\int_0^L\int_0^T u^{\varepsilon_k}_x(t,x)p(t)\,\mathrm{d}t\,\mathrm{d}x = -\lim_{k\to\infty}\int_0^L\int_0^T \Big[\dot f^{\varepsilon_k}(t-\varepsilon_k x) + \dot f^{\varepsilon_k}(t+\varepsilon_k x)\Big]p(t)\,\mathrm{d}t\,\mathrm{d}x \\ &= -\lim_{k\to\infty}\int_0^L\int_{-\varepsilon_k x}^{T-\varepsilon_k x} \dot f^{\varepsilon_k}(s)p(s+\varepsilon_k x)\,\mathrm{d}s\,\mathrm{d}x - \lim_{k\to\infty}\int_0^L\int_{\varepsilon_k x}^{T+\varepsilon_k x} \dot f^{\varepsilon_k}(s)p(s-\varepsilon_k x)\,\mathrm{d}s\,\mathrm{d}x \\ &= -\int_0^L\int_0^T 2\dot f(t)p(t)\,\mathrm{d}t\,\mathrm{d}x,\end{aligned}$$

by the continuity in L^2 with respect to translations and the weak convergence of $\dot{f}^{\varepsilon_k}$. Therefore,

$$\dot{f}(t) = \frac{w(t)}{2\ell(t)} \quad \text{for a.e. } t \in [0,T]. \tag{4.36}$$

Since $f^{\varepsilon_k}(0) = 0$, we have $f(0) = 0$. Therefore,

$$f(t) = \int_0^t \frac{w(s)}{2\ell(s)}\,\mathrm{d}s.$$

We now use Griffith's condition (4.11b) and (4.13) in order to find that, for every subinterval $(a,b) \subset (0,T)$,

$$\int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \geq \int_a^b \sqrt{G^{\varepsilon_k}(t)}\,\mathrm{d}t = \int_a^b \sqrt{2g_{\varepsilon_k}(t)\dot{f}^{\varepsilon_k}(\varphi_k^\varepsilon(t))}\,\mathrm{d}t, \tag{4.37}$$

where $g_{\varepsilon_k}(t) := \frac{1-\varepsilon_k\dot{\ell}^{\varepsilon_k}(t)}{1+\varepsilon_k\dot{\ell}^{\varepsilon_k}(t)}$ and $\varphi_k^\varepsilon(t)$ is as in (4.17). Since $\dot{\varphi}^{\varepsilon_k}(t) \leq 1$, we can continue (4.37) and find that

$$\begin{aligned}\int_a^b \sqrt{G^{\varepsilon_k}(t)}\,\mathrm{d}t &\geq \int_a^b \sqrt{2g_{\varepsilon_k}(t)\,\dot{f}^{\varepsilon_k}(\varphi^{\varepsilon_k}(t))\dot{\varphi}^{\varepsilon_k}(t)}\,\mathrm{d}t \\ &= \int_{-\varepsilon_k\ell_0}^T \mathbb{1}_{(\varphi^{\varepsilon_k}(a),\varphi^{\varepsilon_k}(b))}(s)\sqrt{2g_{\varepsilon_k}((\varphi^{\varepsilon_k})^{-1}(s))\,\dot{f}^{\varepsilon_k}(s)}\,\mathrm{d}s.\end{aligned} \tag{4.38}$$

By Čebyšëv's inequality and by the fact that, by (4.22), ℓ^{ε_k} is uniformly bounded, we now show that $\varepsilon\dot{\ell}^{\varepsilon_k} \to 0$ in measure. Indeed, for every $\eta > 0$ there exists a constant $C = C(\eta) > 0$ such that

$$\left|\{t \in [0,T] : \varepsilon_k\dot{\ell}^{\varepsilon_k}(t) > \eta\}\right| \leq \frac{1}{\eta}\varepsilon_k \int_0^T \dot{\ell}^{\varepsilon_k}(t)\,\mathrm{d}t \leq \varepsilon_k C.$$

This implies that g_{ε_k} converges in measure to one. Since φ^{ε_k} is equi-Lipischitz, then Lemma 4.13 ensures that $g_{\varepsilon_k} \circ (\varphi^{\varepsilon_k})^{-1} \to 1$ in measure. By Lemma 4.12, $g_{\varepsilon_k}((\varphi^{\varepsilon_k})^{-1}) \to 1$ strongly in $L^2(0,T)$. Finally, since $\mathbb{1}_{(\varphi^{\varepsilon_k}(a),\varphi^{\varepsilon_k}(b))}$ strongly converges to $\mathbb{1}_{(a,b)}$ in $L^2(0,T)$ (because $\varphi^{\varepsilon_k}(t) \to t$ uniformly) and since $\dot{f}^{\varepsilon_k} \rightharpoonup \dot{f}$ in $L^2(0,T)$, then the right hand side of (4.38) tends to

$$\int_a^b \sqrt{2}\dot{f}(s)\,\mathrm{d}s = \int_a^b \sqrt{G_{qs}(t)}\,\mathrm{d}s$$

as $k \to \infty$, where the equality follows by (4.36). Therefore,

$$\int_a^b \sqrt{G_{qs}(t)}\,\mathrm{d}t \leq \limsup_k \int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t)}\,\mathrm{d}t.$$

By the Fatou lemma and by upper semicontinuity of κ, we find that

$$\limsup_k \int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \leq \int_a^b \limsup_k \sqrt{\kappa(\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \leq \int_a^b \sqrt{\kappa(\ell(t))}\,\mathrm{d}t.$$

Using the the arbitrariness of (a,b), we obtain

$$G_{qs}(t) = \frac{w(t)^2}{2\ell(t)^2} = 2\dot{f}(t)^2 \leq \kappa(\ell(t))$$

for a.e. $t \in [0,T]$, thus (4.34b) is proved. □

Remark 4.15. It is possible to extend the result of Theorem 4.14 to the case in which κ is piecewise continuous with a finite number of discontinuities, considered in [28, 48]. Indeed, one repeats the previous arguments in every zone where κ is continuous and finds that (4.34b) still holds almost everywhere.

Remark 4.16. We recall that Theorem 4.8 guarantees only that u^{ε_k} converges to u weakly in $L^2(0,T;H^1(0,L))$. If in addition we knew that

$$u^{\varepsilon_k}(t,\cdot) \rightharpoonup u(t,\cdot) \text{ weakly in } H^1(0,L) \text{ for every } t\in[0,T], \tag{4.39}$$

then it would be possible also to pass to the limit in the activation condition (4.11c) obtaining (4.34c).

To this end, besides (4.15) we assume that w^{ε_k} converges to w strongly in $H^1(0,T)$, that $u_0^{\varepsilon_k}$ converges to u_0 strongly in $H^1(0,\ell_0)$, and that εu_1 converges to 0 strongly in $L^2(0,\ell_0)$, i.e., the initial kinetic energy tends to zero. Then, by (4.39) the lower semicontinuity of the potential energy ensures that $\mathcal{E}_{qs}(t;\ell,w)\le \liminf_{k\to\infty}\mathcal{E}^{\varepsilon_k}(t;\ell^{\varepsilon_k},w^{\varepsilon_k})$. Passing to the limit in (4.16) and using (4.36), we obtain an energy inequality; the opposite inequality derives from (4.34b) with arguments similar to Remark 4.10. We thus obtain (4.35) which is equivalent to the activation condition (at least in time intervals with no jumps).

However, conditions (4.39) and (4.35) may not hold in general, as shown in the example of the following section. The example shows that in general (4.11c) does not pass to the limit and (4.34c) is not satisfied, even in the case of a constant toughness.

4.3 Counterexample to the convergence of the activation condition

We now show an explicit case where the convergence of (4.11c) to (4.34c) fails.

A first counterexample to the convergence of the activation condition was presented in [48]. In this case, the singular behaviour is due to the choice of a toughness with discontinuities. More precisely, in [48] it is assumed that $\kappa(x)=\kappa_1$ in $(\ell_1,\ell_1+\delta)$ and $\kappa(x)=\kappa_2$ for $x\notin(\ell_1,\ell_1+\delta)$, where $\kappa_1<\kappa_2$, $\ell_1>\ell_0$, and δ is sufficiently small; this models a short defect of the glue between the film and the substrate.

In this section we show an example of singular behaviour arising even if the local toughness is constant. For simplicity we set $\kappa := \frac{1}{2}$. Moreover we fix the external loading

$$w^\varepsilon(s) := s + 2\left(\sqrt{1+\varepsilon^2} - \varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right)$$

and the initial conditions

$$\ell_0 := 2, \qquad u_1^\varepsilon(x) := 1,$$

$$u_0^\varepsilon(x) := \begin{cases} (2\varepsilon\lfloor\frac{1}{\varepsilon}\rfloor - \sqrt{1+\varepsilon^2})x + 2(\sqrt{1+\varepsilon^2}-\varepsilon\lfloor\frac{1}{\varepsilon}\rfloor), & 0\le x\le 1,\\ -\sqrt{1+\varepsilon^2}x + 2\sqrt{1+\varepsilon^2}, & 1\le x\le 2. \end{cases} \tag{4.40a}$$

Here $\lfloor\cdot\rfloor$ denotes the integer part. Notice that w^ε is a perturbation of $w(s):=s$, u_0^ε is a perturbation of a "hat function" $u_0(x):=x\wedge(2-x)$, and (4.2b) is satisfied; see Figure 4.3. Moreover, the initial kinetic energy $\frac{1}{2}\|\varepsilon u_1^\varepsilon\|^2_{L^2(0,\ell_0)} = \frac{1}{2}\|\varepsilon u_t^\varepsilon(0,\cdot)\|^2_{L^2(0,\ell_0)}$ tends to zero. The specific choice made in (4.40) simplifies the forthcoming computations; however the same qualitative behaviour can be observed even without perturbations.

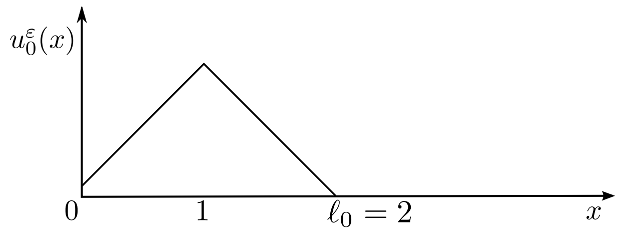

Figure 4.3: The initial datum u_0^ε in the example of Chapter 4.3. We have $u_0^\varepsilon(0) = 2(\sqrt{1+\varepsilon^2} - \varepsilon\lfloor\frac{1}{\varepsilon}\rfloor)$, $u_0^\varepsilon(1) = \sqrt{1+\varepsilon^2}$, and $u_0^\varepsilon(2) = 0$.

4.3.1 Analysis of dynamic solutions

We now study the solutions $(u^\varepsilon, \ell^\varepsilon)$ to the coupled problem (4.1)&(4.14). Using (4.4) and (4.6) we find the following expression for f^ε in $[-2\varepsilon, 2\varepsilon]$,

$$f^\varepsilon(t) = \begin{cases} f_1^\varepsilon(t) := \frac{\varepsilon+\sqrt{1+\varepsilon^2}}{2}t + \varepsilon^2\lfloor\frac{1}{\varepsilon}\rfloor, & a_0^\varepsilon \le t \le a_1^\varepsilon, \\ f_2^\varepsilon(t) := \frac{\varepsilon+\sqrt{1+\varepsilon^2}}{2}t - \varepsilon\lfloor\frac{1}{\varepsilon}\rfloor s, & a_1^\varepsilon \le t \le a_2^\varepsilon, \\ f_3^\varepsilon(t) := \frac{\varepsilon+\sqrt{1+\varepsilon^2}}{2}t - \varepsilon^2\lfloor\frac{1}{\varepsilon}\rfloor, & a_2^\varepsilon \le t \le a_3^\varepsilon, \end{cases} \tag{4.41}$$

where $a_0^\varepsilon := -\varepsilon\ell_0 = -2\varepsilon$, $a_1^\varepsilon := -\varepsilon$, $a_2^\varepsilon := \varepsilon$, and $a_3^\varepsilon = 2\varepsilon$. Notice that $\dot{f}^\varepsilon$ is constant in every interval $(a_{i-1}^\varepsilon, a_i^\varepsilon)$, for $i = 1, 2, 3$.

By (4.14) we have

$$\dot{\ell}^\varepsilon(t) = \begin{cases} \dot{\ell}_1^\varepsilon := \frac{1}{\varepsilon}\frac{2(\dot{f}_1^\varepsilon)^2-\kappa}{2(\dot{f}_1^\varepsilon)^2+\kappa} \vee 0 = \frac{1}{\sqrt{1+\varepsilon^2}}, & b_0^\varepsilon < t < b_1^\varepsilon, \\ \dot{\ell}_2^\varepsilon := \frac{1}{\varepsilon}\frac{2(\dot{f}_2^\varepsilon)^2-\kappa}{2(\dot{f}_2^\varepsilon)^2+\kappa} \vee 0 = 0, & b_1^\varepsilon < t < b_2^\varepsilon, \\ \dot{\ell}_3^\varepsilon := \frac{1}{\varepsilon}\frac{2(\dot{f}_3^\varepsilon)^2-\kappa}{2(\dot{f}_3^\varepsilon)^2+\kappa} \vee 0 = \frac{1}{\sqrt{1+\varepsilon^2}} & b_2^\varepsilon < t < b_3^\varepsilon, \end{cases} \tag{4.42}$$

where $b_0^\varepsilon := 0$ and

$$b_i^\varepsilon := b_{i-1}^\varepsilon + \frac{1}{1-\varepsilon\dot{\ell}_i^\varepsilon}(a_i^\varepsilon - a_{i-1}^\varepsilon). \tag{4.43}$$

Since $\dot{f}^\varepsilon$ is constant in $(a_{i-1}^\varepsilon, a_i^\varepsilon)$ for $i = 1, 2, 3$, also $\dot{\ell}^\varepsilon$ is constant in the intervals $(b_{i-1}^\varepsilon, b_i^\varepsilon)$. We obtain $b_1^\varepsilon = \frac{1}{1-\varepsilon\dot{\ell}_i^\varepsilon}\varepsilon$, $b_2^\varepsilon = b_1^\varepsilon + 2\varepsilon$, and $b_3^\varepsilon = b_2^\varepsilon + \frac{1}{1-\varepsilon\dot{\ell}_i^\varepsilon}\varepsilon$.

We remark that in (4.42) $\dot{\ell}_2^\varepsilon = 0$ because of (4.11). Indeed, for every $\varepsilon > 0$ we have $2(\dot{f}_2^\varepsilon)^2 \le \kappa$ since

$$\left|\varepsilon + \sqrt{1+\varepsilon^2} - 2\varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right| \le 1. \tag{4.44}$$

Then, (4.11b) is satisfied as a strict inequality, by (4.13), and therefore (4.11c) implies that the debonding speed in this second interval is zero.

We now determine f^ε for $t \ge a_3^\varepsilon$ and ℓ^ε for $t \ge b_3^\varepsilon$ by using (4.5) and (4.14) recursively, cf. e.g. the proof of Proposition 4.4 for a similar construction. Because of (4.41) and (4.42), we can immediately see that $\dot{f}^\varepsilon$ and $\dot{\ell}^\varepsilon$ are piecewise constant. More precisely, $\dot{f}^\varepsilon(t) = \dot{f}_i^\varepsilon$ in each interval $(a^\varepsilon_{i-1}, a^\varepsilon_i)$, $i > 0$, while $\dot\ell^\varepsilon(t) = \dot\ell^\varepsilon_i$ in each interval $(b^\varepsilon_{i-1}, b^\varepsilon_i)$, $i > 0$, where

$$a^\varepsilon_{i+3} = a^\varepsilon_{i+2} + \frac{1+\varepsilon\dot\ell^\varepsilon_i}{1-\varepsilon\dot\ell^\varepsilon_i}(a^\varepsilon_i - a^\varepsilon_{i-1}) \tag{4.45}$$

and b^ε_i is given by (4.43). Notice that we have used (4.5) to obtain (4.45). Using (4.20) and recalling that $\dot w^\varepsilon = 1$, we get

$$\dot f^\varepsilon_{i+3} = \varepsilon + \frac{1-\varepsilon\dot\ell^\varepsilon_i}{1+\varepsilon\dot\ell^\varepsilon_i}\dot f^\varepsilon_i. \tag{4.46}$$

Whenever $\dot\ell^\varepsilon_i = 0$, then $\dot f^\varepsilon_{i+3} = \varepsilon + \dot f^\varepsilon_i$. On the other hand, when $\dot\ell^\varepsilon_i > 0$, we can plug (4.14) in (4.46), which gives

$$\dot f^\varepsilon_{i+3} = \varepsilon + \frac{1-\frac{2(\dot f^\varepsilon_i)^2-\kappa}{2(\dot f^\varepsilon_i)^2+\kappa}}{1+\frac{2(\dot f^\varepsilon_i)^2-\kappa}{2(\dot f^\varepsilon_i)^2+\kappa}}\dot f^\varepsilon_i = \varepsilon + \frac{\kappa}{2\dot f^\varepsilon_i}. \tag{4.47}$$

This suggests us to study the map $h\colon x \mapsto \varepsilon + \frac{\kappa}{2x}$, which has a fixed point for $\bar x = \frac{\varepsilon+\sqrt{2\kappa+\varepsilon^2}}{2} = \frac{\varepsilon+\sqrt{1+\varepsilon^2}}{2}$. Notice that $\bar x = \dot f^\varepsilon_1$. This implies that $\dot f^\varepsilon_{3i} = \dot f^\varepsilon_{3i+1} = \dot f^\varepsilon_1$ and $\dot\ell^\varepsilon_{3i} = \dot\ell^\varepsilon_{3i+1} = \dot\ell^\varepsilon_1$ for $i \geq 1$. In fact, the choice of the initial datum u^ε_0 as in (4.40a) has been made in order to satisfy these conditions and to simplify such formulas.

We still have to determine $\dot f^\varepsilon_{3i+2}$ and $\dot\ell^\varepsilon_{3i+2}$ for $i \geq 1$. To this end, we start by showing the explicit expression of ℓ^ε in the interval $(b^\varepsilon_3, b^\varepsilon_6)$. By (4.42) and (4.45), we find

$$a^\varepsilon_4 = a^\varepsilon_3 + c_\varepsilon\varepsilon, \quad a^\varepsilon_5 = a^\varepsilon_4 + 2\varepsilon, \quad a^\varepsilon_6 = a^\varepsilon_5 + c_\varepsilon\varepsilon,$$

where

$$c_\varepsilon := \frac{1+\varepsilon\dot\ell^\varepsilon_1}{1-\varepsilon\dot\ell^\varepsilon_1} = 1 + \frac{2\varepsilon}{\sqrt{1+\varepsilon^2}-\varepsilon}.$$

We have already observed that $\dot f^\varepsilon_6 = \dot f^\varepsilon_4 = \dot f^\varepsilon_1$. Moreover, by (4.46) and since $\dot\ell^\varepsilon_2 = 0$, we find $\dot f^\varepsilon_5 = \dot f^\varepsilon_2 + \varepsilon$. It easily follows that

$$\dot\ell^\varepsilon(t) = \begin{cases} \dot\ell^\varepsilon_4 = \dot\ell^\varepsilon_1 = \frac{1}{\sqrt{1+\varepsilon^2}}, & b^\varepsilon_3 < t < b^\varepsilon_4, \\ \dot\ell^\varepsilon_5 = \dot\ell^\varepsilon_2 = 0, & b^\varepsilon_4 < t < b^\varepsilon_5, \\ \dot\ell^\varepsilon_6 = \dot\ell^\varepsilon_1 = \frac{1}{\sqrt{1+\varepsilon^2}} & b^\varepsilon_5 < t < b^\varepsilon_6. \end{cases}$$

Notice that $\dot\ell^\varepsilon_5 = 0$ holds for ε small enough, since

$$\left|3\varepsilon + \sqrt{1+\varepsilon^2} - 2\varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right| \leq 1 \tag{4.48}$$

and therefore $2(\dot f^\varepsilon_5)^2 = 2(\dot f^\varepsilon_2 + \varepsilon)^2 \leq \kappa$, cf. (4.44).

We can iteratively repeat this argument as long as the following condition, analog of (4.44) and (4.48), is satisfied:

$$\left|(2i+1)\varepsilon + \sqrt{1+\varepsilon^2} - 2\varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right| \leq 1. \tag{4.49}$$

Let

$$n^\varepsilon = \min\{n \in \mathbb{N} : \left|(2n+1)\varepsilon + \sqrt{1+\varepsilon^2} - 2\varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor\right| > 1\}.$$

Notice that (4.11) implies that $2(\dot f^\varepsilon_{3i+2})^2 = 2(\dot f^\varepsilon_2 + i\varepsilon)^2 \le \kappa$ and $\dot\ell^\varepsilon_{3i-1} = 0$ for every $i < n^\varepsilon$. Condition (4.49) is a threshold condition that fails after n^ε iterations of this process. Direct computations show that $n^\varepsilon = \lfloor \frac{1}{\varepsilon} \rfloor$. (In fact, the choice of the initial datum u^ε_0 has been made in order to obtain this equality.) In conclusion, for ε sufficiently small and $1 \le i < n^\varepsilon$, we have

$$\dot f^\varepsilon(t) = \begin{cases} \dot f^\varepsilon_{3i+1} = \dot f^\varepsilon_1, & a^\varepsilon_{3i} < t < a^\varepsilon_{3i+1}, \\ \dot f^\varepsilon_{3i+2} = \dot f^\varepsilon_2 + i\varepsilon, & a^\varepsilon_{3i+1} < t < a^\varepsilon_{3i+2}, \\ \dot f^\varepsilon_{3i+3} = \dot f^\varepsilon_1, & a^\varepsilon_{3i+2} < t < a^\varepsilon_{3i+3}, \end{cases} \tag{4.50}$$

and

$$\dot\ell^\varepsilon(t) = \begin{cases} \dot\ell^\varepsilon_{3i+1} = \dot\ell^\varepsilon_1, & b^\varepsilon_{3i} < t < b^\varepsilon_{3i+1}, \\ \dot\ell^\varepsilon_{3i+2} = 0, & b^\varepsilon_{3i+1} < t < b^\varepsilon_{3i+2}, \\ \dot\ell^\varepsilon_{3i+3} = \dot\ell^\varepsilon_1, & b^\varepsilon_{3i+2} < t < b^\varepsilon_{3i+3}, \end{cases} \tag{4.51}$$

where

$$\begin{cases} a^\varepsilon_{3i+1} = a^\varepsilon_{3i} + c_\varepsilon(a^\varepsilon_{3i-2} - a^\varepsilon_{3i-3}) = 2\varepsilon(i-1) + 2\varepsilon\frac{1-c^i_\varepsilon}{1-c_\varepsilon} + c_\varepsilon\varepsilon, \\ a^\varepsilon_{3i+2} = a^\varepsilon_{3i+1} + 2\varepsilon = 2\varepsilon i + 2\varepsilon\frac{1-c^i_\varepsilon}{1-c_\varepsilon} + c_\varepsilon\varepsilon, \\ a^\varepsilon_{3i+3} = a^\varepsilon_{3i+2} + c_\varepsilon(a^\varepsilon_{3i} - a^\varepsilon_{3i-1}) = 2\varepsilon i + 2\varepsilon\frac{1-c^{i+1}_\varepsilon}{1-c_\varepsilon}, \\ b^\varepsilon_{3i-2} = b^\varepsilon_{3i-3} + \frac{1}{1-\varepsilon\dot\ell^\varepsilon_1}(a^\varepsilon_{3i-2} - a^\varepsilon_{3i-3}) = 2\varepsilon i + \frac{c^i_\varepsilon - 1}{\dot\ell^\varepsilon_1} + 2\varepsilon\frac{1}{1-\varepsilon\dot\ell^\varepsilon_1}c^i_\varepsilon, \\ b^\varepsilon_{3i-1} = b^\varepsilon_{3i-2} + 2\varepsilon = 2\varepsilon(i+1) + \frac{c^i_\varepsilon - 1}{\dot\ell^\varepsilon_1} + 2\varepsilon\frac{1}{1-\varepsilon\dot\ell^\varepsilon_1}c^i_\varepsilon, \\ b^\varepsilon_{3i} = b^\varepsilon_{3i-1} + \frac{1}{1-\varepsilon\dot\ell^\varepsilon_1}(a^\varepsilon_{3i} - a^\varepsilon_{3i-1}) = 2\varepsilon i + \frac{c^{i+1}_\varepsilon - 1}{\dot\ell^\varepsilon_1}. \end{cases} \tag{4.52}$$

This means that there is a first phase, corresponding to the time interval $[0, b^\varepsilon_{3n^\varepsilon}]$, where the material debonds according to a "stop and go" process and the speed oscillates between 0 and $\dot\ell^\varepsilon_1$ (see Fig. 4.4).

Let us now consider the evolution for times larger than $b^\varepsilon_{3n^\varepsilon}$. Arguing as above, we obtain

$$\dot f^\varepsilon_{3n^\varepsilon+2} = \frac{\varepsilon + \sqrt{1+\varepsilon^2}}{2} = \dot f^\varepsilon_1 \quad \text{and} \quad \dot\ell^\varepsilon_{3n^\varepsilon+2} = \dot\ell^\varepsilon_1.$$

We employ (4.47) and recall that the map $h\colon x \mapsto \varepsilon + \frac{\kappa}{2x}$ has a fixed point at $\bar x = \dot f^\varepsilon_1$. Therefore, for every $i \ge n^\varepsilon$,

$$\begin{cases} \dot f^\varepsilon_{3i+1} = \dot f^\varepsilon_{3i+2} = \dot f^\varepsilon_{3i+3} = \dot f^\varepsilon_1, \\ \dot\ell^\varepsilon_{3i+1} = \dot\ell^\varepsilon_{3i+2} = \dot\ell^\varepsilon_{3i+3} = \dot\ell^\varepsilon_1. \end{cases}$$

This shows that in this second phase the debonding proceeds at constant speed $\dot\ell^\varepsilon_1$ for every time.

Remark 4.17. By (4.7), (4.41), and (4.50), the displacement's derivatives are piecewise constant; in the (t,x) plane, their discontinuities lie on some shock waves originating from $(0, \ell_0/2)$ (where the initial datum has a kink), travelling backword and forward in the debonded film, and reflecting at boundaries; they are represented by thick dashed lines in Figure 4.4. Notice that the

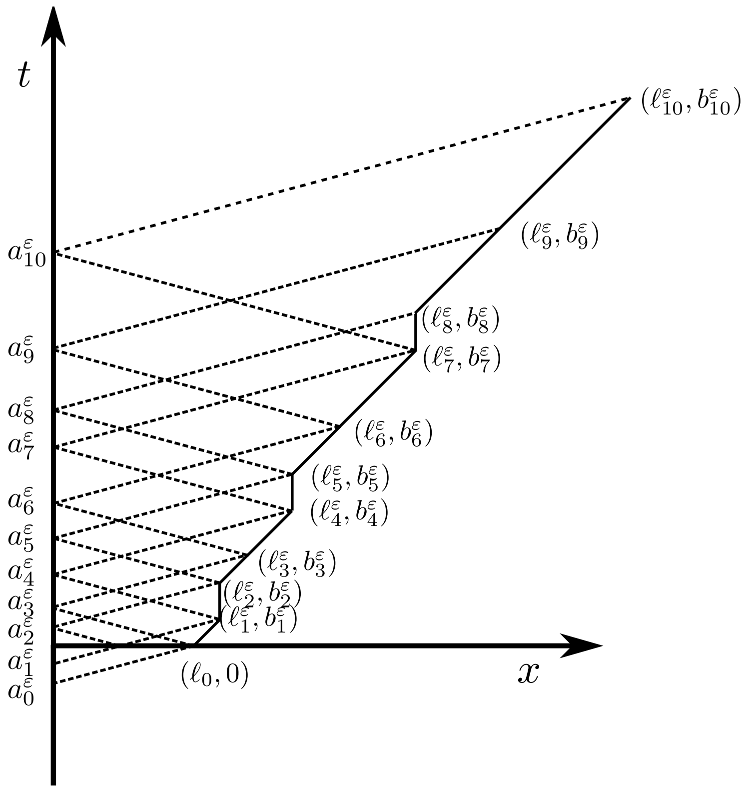

Figure 4.4: Evolution of ℓ^ε with a "stop and go" process.

lines originating from $(0,\ell_0)$, employed in the construction above and marked in Figure 4.4 by thin dashed lines, are *not* discontinuity lines, since $\dot f^\varepsilon_{3i} = \dot f^\varepsilon_{3i+1}$ for every $i \geq 1$. This is actually a consequence of the compatibility among u^ε_0, u^ε_1, and $\dot\ell^\varepsilon$ at $(0,\ell_0)$; namely, $\dot u^\varepsilon_0(\ell_0)\dot\ell^\varepsilon(0)+u^\varepsilon_1(\ell_0)=0$. We refer to [23, Remark 1.12] for more details on the regularity of the solutions.

4.3.2 Limit for vanishing inertia

We now study the limit ℓ of the evolutions ℓ^ε as $\varepsilon \to 0$. Notice that the initial conditions are not at equilibrium; in particular the initial position $u_0(x)$ is not of the form $\big[-\frac{w(0)}{\ell_0}x + w(0)\big] \vee 0$. Because of (4.25), there must be a time discontinuity at $t = 0$, i.e., the limit displacement u jumps to an equilibrium configuration. Nonetheless, we will show that ℓ is continuous even at $t = 0$. In order to determine ℓ, the main point is to study the limit evolution of the debonding during the first phase characterised by the "stop and go" process. Afterwards, during the second phase, the evolution of the debonding will proceed at constant speed, given by $\lim_{\varepsilon\to 0}\dot\ell^\varepsilon_1 = 1$.

We first compute the instant at which the first phase ends. By (4.52), we have

$$b^\varepsilon_{3n^\varepsilon} = 2\varepsilon\Big\lfloor\frac{1}{\varepsilon}\Big\rfloor + \frac{c_\varepsilon^{\lfloor\frac{1}{\varepsilon}\rfloor+1}-1}{\dot\ell^\varepsilon_1} \quad\stackrel{\varepsilon\to 0}{\longrightarrow}\quad e^2+1.$$

On the other hand, at time $b^\varepsilon_{3n^\varepsilon}$ the position of ℓ^ε is given by

$$\ell^\varepsilon(b^\varepsilon_{3n^\varepsilon}) = \ell_0 + \left(b^\varepsilon_{3n^\varepsilon} - 2\varepsilon\Big\lfloor\frac{1}{\varepsilon}\Big\rfloor\right)\dot\ell^\varepsilon_1 = \ell_0 + c_\varepsilon^{\lfloor\frac{1}{\varepsilon}\rfloor+1} - 1 \quad\stackrel{\varepsilon\to 0}{\longrightarrow}\quad e^2+1. \tag{4.53}$$

Indeed, in $[0, b^\varepsilon_{3n^\varepsilon}]$ the speed $\dot\ell^\varepsilon$ is either zero or $\dot\ell^\varepsilon_1$, and the total length of the intervals where $\dot\ell^\varepsilon = 0$ is $2\varepsilon\lfloor\frac{1}{\varepsilon}\rfloor$.

Therefore, for $t \geq e^2+1$ we have $\ell(t) = t$. In the time interval $[e^2+1,+\infty)$, corresponding to the second phase, the quasistatic limit ℓ is a rate-independent evolution in the sense of Definition 4.9, see also Remark 4.11.

We now explicitly find the law of the evolution of ℓ in the first phase. Rather than finding an expression for $t \mapsto \ell(t)$, it is more convenient to determine the inverse map $\ell \mapsto t(\ell)$, cf. [48] for a similar computation in another example. We consider the map

$$i \mapsto \ell^\varepsilon(b^\varepsilon_{3i}) = \ell_0 + c_\varepsilon^{i+1} - 1, \tag{4.54}$$

where $1 \leq i < n^\varepsilon$. Notice that the last equality follows as in (4.53). We now take the inverse of (4.54) and define

$$i^\varepsilon(\ell) := \left\lfloor \frac{\log\left(\frac{1+\ell-\ell_0}{c_\varepsilon}\right)}{\log c_\varepsilon} \right\rfloor.$$

Since $c_\varepsilon^{\frac{1}{\varepsilon}} \to e^2$ as $\varepsilon \to 0$, then we have

$$\varepsilon i^\varepsilon(\ell) \quad\stackrel{\varepsilon\to 0}{\longrightarrow}\quad \frac{\log(\ell-1)}{2}. \tag{4.55}$$

Therefore,

$$t(\ell) = \lim_{\varepsilon\to 0} b^\varepsilon_{i^\varepsilon(\ell)} = \lim_{\varepsilon\to 0}\left(2\varepsilon i^\varepsilon(\ell) + \frac{c_\varepsilon^{i^\varepsilon(\ell)+1}-1}{\dot\ell^\varepsilon_1}\right) = \log(\ell-1)+\ell-2 \tag{4.56}$$

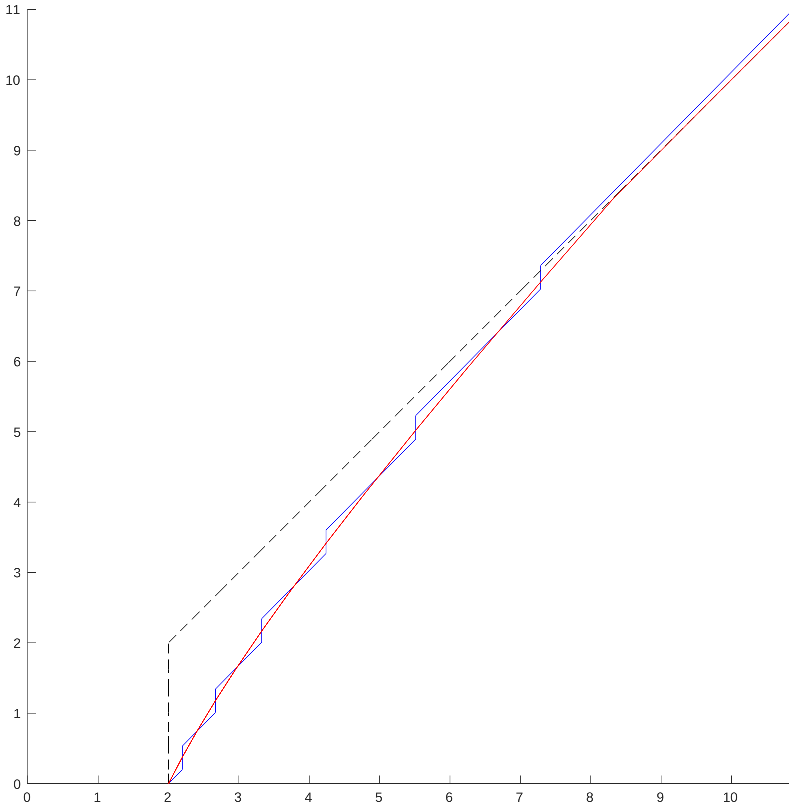

Figure 4.5: Limit solution for $\varepsilon = 0.25$, $\varepsilon = 0.05$, asymptotic limit (continuous lines), and a rate-independent evolution (dotted line).

denotes the trajectory followed by the debonding during the first phase (see Figure 4.5). Notice that $t(\ell)$ is the sum of a strictly concave and an affine function, thus $\ell(t)$ is strictly convex in the first phase. It is interesting that the first phase features a strictly positive debonding acceleration.

We can give first and second order laws characterising the first phase. By (4.56) we obtain

$$\dot{t}(\ell) = \frac{\ell}{\ell - 1} \quad \text{for } \ell \in (2, e^2{+}1),$$

hence

$$\dot{\ell}(t) = \frac{\ell(t) - 1}{\ell(t)} \quad \text{for } t \in (0, e^2{+}1)$$

and

$$\ddot{\ell}(t) = \frac{\ell(t) - 1}{\ell(t)^3} \quad \text{for } t \in (0, e^2{+}1).$$

As already observed, we have $\dot{\ell}, \ddot{\ell} > 0$ in the first phase. Both $\dot{\ell}$ and $\ddot{\ell}$ are discontinuous at $t = e^2{+}1$.

Notice that during the first phase the quasistatic limit ℓ does not satisfy (4.34c), thus it does not comply with the notion of rate-independent evolution given in Definition 4.9. Indeed, since the local toughness is constant, Remark (4.11) implies that a rate-independent evolution must be piecewise affine (with possible jumps); in contrast, (4.56) is not the equation of a line. This result is similar to the one obtained in [48] with a discontinuous local toughness: here we showed that a singular behaviour can be observed even if the local toughness is constant.

Remark 4.18. We recall that the initial displacement u_0^ε chosen in (4.40a) has a kink at $\frac{\ell_0}{2} = 1$. In this section, we showed that the interaction between the two slopes generates the "stop and go" process, which gives as a result the convergence to an evolution that does not satisfy Defition 4.9. However, this singular behaviour can be obtained even for a smooth initial datum. Indeed, let us consider a regularisation of u_0^ε, coinciding with the original profile outside of $(1 - \frac{\delta}{2}, 1 + \frac{\delta}{2})$, where $\delta \in (0, 1)$ is fixed. As a consequence of this choice, the function $\dot{\ell}^\varepsilon$ differs from (4.51) only in a portion of the order $\varepsilon\delta$ of each interval $(b_i^\varepsilon, b_{i+1}^\varepsilon)$. The resulting evolution of the debonding front ℓ^ε is smooth. However, in the limit we observe the same qualitative behaviour described above, due to the interaction of the different slopes of the initial datum. This shows that the singular behaviour is not due to the choice of a initial datum with a kink.

4.3.3 Analysis of the kinetic energy

The striking behaviour observed in the previous example can be explained by computing the oscillations of the kinetic energy

$$K^\varepsilon(t) := \frac{\varepsilon^2}{2} \int_0^{\ell^\varepsilon(t)} u_t^\varepsilon(t, x)^2 \, \mathrm{d}x. \tag{4.57}$$

We recall that the displacement's derivatives are piecewise constant, with discontinuity lines given by shock waves originating at $\ell_0/2$ (where the initial datum has a kink) and travelling backward and forward in the debonded film (cf. Remark 4.17).

Let us introduce some notation. The sectors determined by shock waves (Figure 4.6) are divided into three families: T_i denotes a triangular sector adjacent to the time axis (i.e., the vertical axis in figure), S_i a triangular sector adjacent to the graph of ℓ^ε, and R_i a rhomboidal

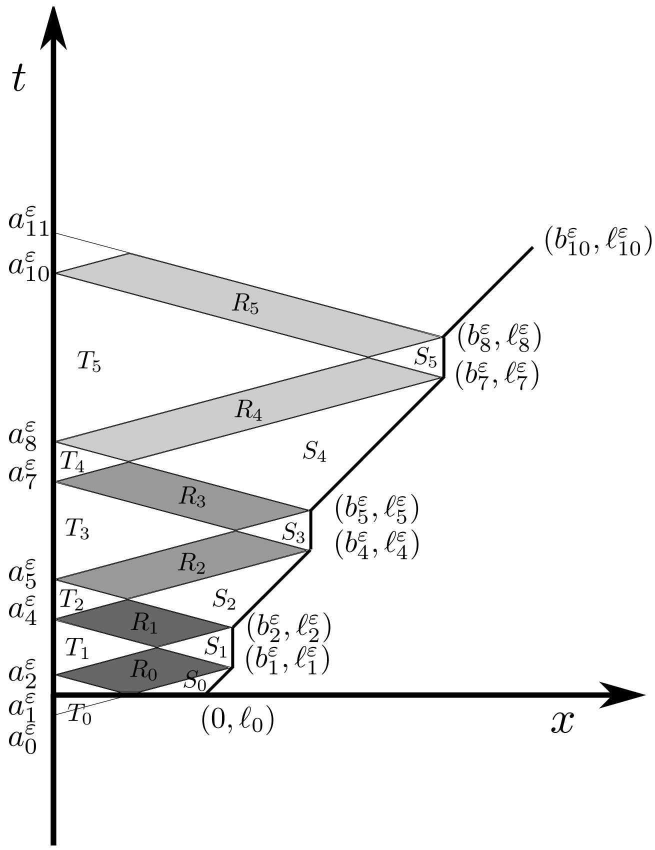

Figure 4.6: The sectors composing Ω^{ε} give different contributions to the kinetic energy K^{ε}. The darker the shade of grey, the larger is $u_t^{\varepsilon}(t,x)^2$ in that region. White sectors give a negligible contribution.

sector; T_0 contains the segment $\{0\}\times[a_1^\varepsilon, a_2^\varepsilon]$, S_0 contains the segment $[(0,\ell_0),(b_1^\varepsilon,\ell_1^\varepsilon)]$, and R_0 is adjacent to T_0 and S_0; the families are indexed increasingly in the direction of the time axis.

It is easy to see that the boundary conditions imply that $u_t^\varepsilon = 1$ in the sectors T_i and $u_t^\varepsilon = 0$ in the sectors S_i with i odd, i.e., those triangles corresponding to a stop phase of the debonding front. Moreover, by (4.7a), (4.41), and (4.50) we obtain that $u_t^\varepsilon = 1$ in the sectors S_i with i even. In all triangular sectors we thus have $u_t^\varepsilon(t,x)$ of order at most one, so that their contribution to the kinetic energy (4.57) is of order at most ε^2. More precisely,

$$\frac{\varepsilon^2}{2}\int_{(\{t\}\times[0,\ell^\varepsilon(t)])\cap T_i} u_t^\varepsilon(t,x)^2\,\mathrm{d}x \le C\varepsilon^2 \quad \text{for every } t, i,$$

and the same holds for S_i. In particular,

$$K^\varepsilon(t) = O(\varepsilon^2) \quad \text{for } t = \frac{a_{3i-1}^\varepsilon + a_{3i+1}^\varepsilon}{2} \text{ and for } t = \frac{a_{3i+1}^\varepsilon + a_{3i+2}^\varepsilon}{2} \tag{4.58}$$

for every $i \ge 1$.

We now show that the remaining rhomboids R_i give a relevant contribution to the kinetic energy. By (4.7a), (4.41), and (4.50) we obtain for every i

$$u_t^\varepsilon(t,x) = 1 - \frac{1}{\varepsilon}\dot f_1^\varepsilon + \frac{1}{\varepsilon}\left(\dot f_1^\varepsilon - \varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor + i\varepsilon\right) = 1 - \left\lfloor\frac{1}{\varepsilon}\right\rfloor + i \quad \text{in } R_{2i},$$
$$u_t^\varepsilon(t,x) = 1 - \frac{1}{\varepsilon}\left(\dot f_1^\varepsilon - \varepsilon\left\lfloor\frac{1}{\varepsilon}\right\rfloor + (i+1)\varepsilon\right) + \frac{1}{\varepsilon}\dot f_1^\varepsilon = 1 + \left\lfloor\frac{1}{\varepsilon}\right\rfloor - (i+1) \quad \text{in } R_{2i+1}.$$

To obtain the kinetic energy (4.57), we observe that the maximal t-section of each rhomboid has length $\ell_0 = 2$. Therefore,

$$K^\varepsilon(t) = \varepsilon^2\left(\left\lfloor\frac{1}{\varepsilon}\right\rfloor - i\right)^2 + O(\varepsilon) \quad \text{for } t \in [a_{3i+2}^\varepsilon, b_{3i+1}^\varepsilon] \cup [b_{3i+2}^\varepsilon, a_{3i+4}^\varepsilon]. \tag{4.59}$$

This gives the maximal asymptotic amount of kinetic energy; we do not detail the computation of the kinetic energy in other intervals. We recall that by (4.58) the minimal asymptotic amount is zero, so the energy is oscillating.

Moreover, since (4.59) holds for $i = 0,\dots,n^\varepsilon$ and since $n^\varepsilon = \lfloor\frac{1}{\varepsilon}\rfloor$, we observe that the maximal oscillations of the kinetic energy decrease as time increases, until the kinetic energy is close to zero for $i = n^\varepsilon$, i.e., when the non-quasistatic phase finishes and the second phase starts. In fact, since in the second phase $\dot f^\varepsilon(t) = \dot f_1^\varepsilon$ for a.e. t, then u_t^ε is constantly equal to one, so the kinetic energy is negligible by (4.57). We can also give an asymptotic expression for the maximal (resp., minimal) oscillations by plugging (4.55) in (4.59) (resp., by (4.58)):

$$\Gamma\text{-}\lim_{\varepsilon\to0}(-K^\varepsilon)(t) = -\left(1 - \frac{\log(\ell(t)-1)}{2}\right)^2, \qquad \Gamma\text{-}\lim_{\varepsilon\to0}(K^\varepsilon)(t) = 0.$$

We refer to [10] for the notion of Γ-convergence. A similar phenomenon was observed in [48] for a discontinuous toughness.

Summarising,

- the non-quasistatic phase, where Griffith's quasistatic criterion fails in the limit, is characterised by the presence of a relevant kinetic energy (of order one as $\varepsilon \to 0$, at each fixed time);

- during such first phase, kinetic energy oscillates and is exchanged with potential energy at a time scale of order ε;
- overall, the total mechanical energy decreases and is transferred to energy dissipated in the debonding growth;
- as time increases, the maximal oscillations of the kinetic energy decrease and approach zero as $t \to e^2 + 1$, i.e., all of the kinetic energy is converted into potential and dissipated energy;
- in the second (stable) phase, for $t \geq e^2 + 1$, the kinetic energy is of order ε^2 and does not influence the limit behaviour of the debonding evolution as $\varepsilon \to 0$.

4.4 Quasistatic limit in the case of a speed-dependent local toughness

In this section we consider the quasistatic limit in the case of a speed-dependent local toughness introduced in Section 1.4. We recall that the local toughness κ satisfies (1.62). Moreover, we shall assume that

$$\kappa \text{ is upper semicontinuous.}$$

We fix $\varepsilon > 0$ and consider again u_0^ε, u_1^ε, and w^ε as in (4.2) and such (4.15) is satisfied.

If $t \mapsto \ell^\varepsilon(t)$ is as in (4.3), then by Proposition 4.2 there exists $u^\varepsilon \in \widetilde{C}^{0,1}(\Omega^\varepsilon)$ unique solution to (4.1) and it is represented through (4.4) by $f^\varepsilon \in \widetilde{C}^{0,1}(0, +\infty)$ such that (4.5) is satisfied.

In the case of $t \mapsto \ell^\varepsilon(t)$ unknown, we formulate Griffith's criterion as in Section 4.1 noting that here the scaling (0.14) affects the local toughness as follows:

$$\kappa(\ell_\varepsilon(t), \dot{\ell}_\varepsilon(t)) = \kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t)),$$

so that Griffith's criterion reads now as

$$\dot{\ell}^\varepsilon(t) \geq 0, \tag{4.60a}$$
$$G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t) \leq \kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t)), \tag{4.60b}$$
$$\left[G^\varepsilon_{\dot{\ell}^\varepsilon(t)}(t) - \kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t))\right] \dot{\ell}^\varepsilon(t) = 0. \tag{4.60c}$$

As above we employ its equivalent form

$$\begin{cases} \dot{\ell}^\varepsilon(t) = \dfrac{1}{\varepsilon} \dfrac{2\dot{f}^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 - \kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t))}{2\dot{f}^\varepsilon(t - \varepsilon\ell^\varepsilon(t))^2 + \kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t))} \vee 0, \quad \text{for a.e. } t > 0, \\ \ell^\varepsilon(0) = \ell_0. \end{cases} \tag{4.61}$$

The following result generalises Proposition 4.6, Theorem 4.8, and Theorem 4.14 to the case of a speed-dependent local toughness. This extension requires only minor modifications in the proof. In Remark 4.20 we highlight the only step where the toughness plays a role.

Theorem 4.19. *Let $T > 0$. Assume that the toughness κ satisfies* (1.62) *and is upper semi-continuous. Assume* (4.2) *and* (4.15). *Let $(u^\varepsilon, \ell^\varepsilon)$ be the solution to the coupled problem* (4.1)

& (4.61). Then, there exists $L > 0$ such that $\ell^\varepsilon(T) \le L$, and there exists a non-decreasing evolution $\ell\colon [0,T] \to [0,L]$ and a sequence ε_k such that

$$\ell^{\varepsilon_k}(t) \to \ell(t)$$

for every $t \in [0,T]$. Moreover,

$$u^{\varepsilon_k} \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(0,L)),$$

where

$$u(t,x) = \begin{cases} -\frac{w(t)}{\ell(t)}x + w(t) & \text{for a.e. } (t,x)\colon x < \ell(t), \\ 0 & \text{for a.e. } (t,x)\colon x \ge \ell(t). \end{cases}$$

Finally,

$$G_{qs}(t) \le \kappa(\ell(t),0) \tag{4.62}$$

and the quasistatic energy release rate is given by $G_{qs}(t) = \frac{w(t)^2}{2\ell(t)^2}$.

Remark 4.20. We highlight that in the quasistatic limit the toughness appearing in Griffith's criterion is evaluated at debonding speed zero. Indeed, following the proof of Theorem 4.14 we see that

$$\int_a^b \sqrt{G_{qs}(t)}\,\mathrm{d}t \le \limsup_k \int_a^b \sqrt{G^{\varepsilon_k}(t)}\,\mathrm{d}t \le \limsup_k \int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t), \varepsilon_k \dot\ell^{\varepsilon_k}(t)}\,\mathrm{d}t.$$

for every interval $(a,b) \subset [0,T]$, where the second inequality follows by (0.16b). By the Fatou lemma and the upper semicontinuity of κ, we find

$$\limsup_k \int_a^b \sqrt{\kappa(\ell^{\varepsilon_k}(t), \varepsilon_k \dot\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \le \int_a^b \limsup_k \sqrt{\kappa(\ell^{\varepsilon_k}(t), \varepsilon_k \dot\ell^{\varepsilon_k}(t))}\,\mathrm{d}t \le \int_a^b \sqrt{\kappa(\ell(t),0)}\,\mathrm{d}t,$$

which yields (4.62)

In this work we observe a particular behaviour of the quasistatic limit by providing two examples. The first example shows that (0.16c) does not pass to the limit as $\varepsilon \to 0$, i.e.,

$$[G_{qs}(t) - \kappa(\lambda(t),0)]\,\dot\lambda(t) = 0$$

does *not* hold in general. The second example shows that brutal propagation is possible in the quasistatic limit even if the dynamic toughness penalises high-speed debonding.

We will employ (4.5) in the following form:

$$f^\varepsilon(s) = w^\varepsilon(s) + f^\varepsilon(\omega_\varepsilon(s)), \tag{4.63}$$

recalling that

$$\omega_\varepsilon := \varphi_\varepsilon \circ \psi_\varepsilon^{-1}, \quad \varphi_\varepsilon(s) := s - \varepsilon\ell^\varepsilon(s), \quad \psi_\varepsilon(s) := s + \varepsilon\ell^\varepsilon(s).$$

Notice that

$$\dot\omega_\varepsilon^{-1}(s) = \frac{1 + \varepsilon\ell^\varepsilon(\psi_\varepsilon^{-1}(s))}{1 - \varepsilon\ell^\varepsilon(\psi_\varepsilon^{-1}(s))}. \tag{4.64}$$

(See Figure 4.7.)

4.4.1 Example 1: the activation condition fails

We now show that the presence of a viscous term in Griffith's criterion, given by the local toughness explicitly depending on the debonding speed, is not in general sufficient to guarantee the convergence of (4.60c).

We consider here a local toughness

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot\ell^\varepsilon(t)) = \frac{1}{2} + c_3\varepsilon\dot\ell^\varepsilon(t), \tag{4.65}$$

with $c_3 > 0$. Notice that the choice $\kappa = \tilde\kappa := \frac{1}{2}$ was precisely the one employed in Section 4.3 and we will henceforth refer to $(v^\varepsilon, \lambda^\varepsilon)$ as the dynamic solutions analysed in that section and to (v, λ) as their limit as $\varepsilon \to 0$. Using (4.65), we write (4.61) in normal form, obtaining

$$\begin{cases} \dot\ell^\varepsilon(t) = \frac{1}{\varepsilon}\frac{4\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2-1}{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2+\frac12+c_3+\sqrt{(2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2+\frac12-c_3)^2+16c_3\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2}} \vee 0, & \text{for a.e. } t>0, \\ \ell(0) = \ell_0. \end{cases} \tag{4.66}$$

We set $\ell_0 = 2$, $w^\varepsilon(t) = t$, $u_1^\varepsilon = 1$, and

$$u_0^\varepsilon(x) := \begin{cases} (2\varepsilon\lfloor\frac{1}{\varepsilon}\rfloor - \sqrt{1+\varepsilon^2})x + 2(\sqrt{1+\varepsilon^2} - \varepsilon\lfloor\frac{1}{\varepsilon}\rfloor), & 0 \le x \le 1, \\ -\sqrt{1+\varepsilon^2}x + 2\sqrt{1+\varepsilon^2}, & 1 \le x \le 2. \end{cases}$$

These are the same data that we used in Section 4.3. As above, we have f^ε of the form

$$\dot f^\varepsilon(t) = \begin{cases} \dot f_1^\varepsilon := \dfrac{\varepsilon + \sqrt{1+\varepsilon^2}}{2}, & -2\varepsilon \le t \le -\varepsilon, \\ \dot f_2^\varepsilon := \dfrac{\varepsilon + \sqrt{1+\varepsilon^2}}{2} - \varepsilon\lfloor\dfrac{1}{\varepsilon}\rfloor, & -\varepsilon \le t \le \varepsilon, \\ \dot f_3^\varepsilon := \dot f_1^\varepsilon, & \varepsilon \le t \le 2\varepsilon. \end{cases}$$

For every $i \ge 1$ we call ℓ_i^ε the solution of (4.66) when $\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t)) = \dot f_i^\varepsilon$.

We notice that, by plugging $\dot f_2^\varepsilon$ in (4.66), we have $\dot\ell_2^\varepsilon = 0$. As a consequence of (4.63), it results that $\dot f^\varepsilon$ and $\dot\ell^\varepsilon$ are piecewise constant in $[0, +\infty)$; we denote by $\dot f_i^\varepsilon$, $\dot\ell_i^\varepsilon$ their values, indexed increasingly with respect to time, see Figure 4.7. The rule for the update of $\dot f^\varepsilon$ is again (4.46). Hence, $\dot f_5^\varepsilon = \dot f_2^\varepsilon + \varepsilon$. By direct computation it is possible to prove that $\dot\ell_5^\varepsilon = 0$ and that for every $0 \le i < \lfloor\frac{1}{\varepsilon}\rfloor =: n^\varepsilon$ we have $\dot f_{3i+2}^\varepsilon = \dot f_2^\varepsilon + i\varepsilon$ and $\dot\ell_{3i+2}^\varepsilon = 0$. Thus, the indices $3i+2$ correspond to stop phases with no propagation of the debonding front until a certain threshold is reached.

In contrast, we have propagation phases for the indices $3i+1$ and $3i+3$. Indeed, starting from (4.66), we deduce that

$$\frac{1}{\varepsilon}\frac{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 - \tilde\kappa}{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 + \tilde\kappa + 2\sqrt{c_3}\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))} \le \dot\ell^\varepsilon(t) \le \frac{1}{\varepsilon}\frac{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 - \tilde\kappa}{2\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2 + \tilde\kappa}. \tag{4.67}$$

In the previous chain of inequalities, the first is obtained from (4.66) by using the fact that $\sqrt{a^2+b^2} \le a+b$ for $a, b \ge 0$; the second is obtained by ignoring the term $16c_3\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))^2$ in the denominator of (4.66). Therefore, from the first inequality of (4.67) we obtain

$$\dot\ell_3^\varepsilon(t) = \dot\ell_1^\varepsilon(t) \ge \frac{1}{\varepsilon}\frac{2(\dot f_1^\varepsilon)^2 - \tilde\kappa}{2(\dot f_1^\varepsilon)^2 + \tilde\kappa + 2\sqrt{c_3}\dot f_1^\varepsilon} = \frac{\varepsilon + \sqrt{1+\varepsilon^2}}{\varepsilon^2 + \varepsilon\sqrt{1+\varepsilon^2} + 1 + \sqrt{c_3}(\varepsilon + \sqrt{1+\varepsilon^2})} \ge \frac{1}{2+\sqrt{c_3}},$$

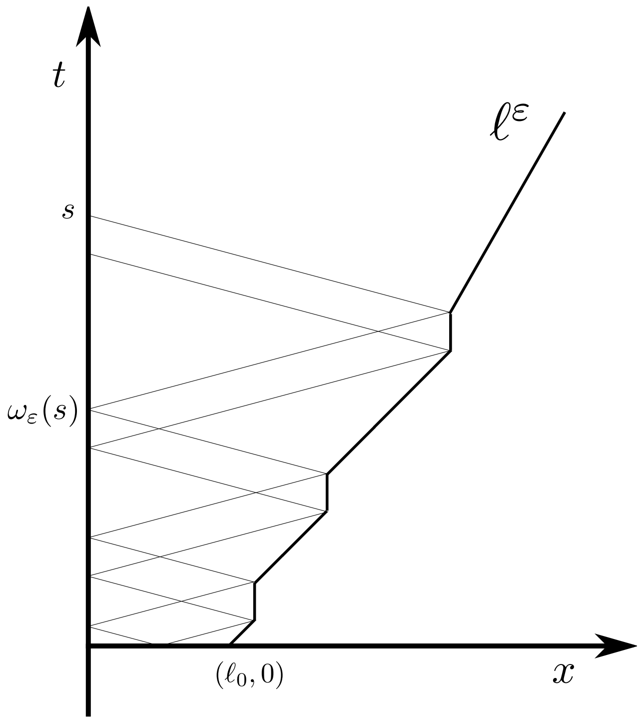

Figure 4.7: The evolution of ℓ^ε is represented by a zig-zag in which there is alternation between phases of propagation of the debonding front and stop phases.

where the last inequality holds for ε sufficiently small. On the other hand, the second inequality of (4.67) implies that the debonding speed is controlled by the one of the evolution for $\kappa = \tilde{\kappa}$:

$$\dot{\ell}^\varepsilon(t) \le \dot{\lambda}^\varepsilon(t) \quad \text{for a.e. } t > 0.$$

Since the function $x \mapsto \frac{1-\varepsilon x}{1+\varepsilon x}$ is non-increasing, then

$$\dot{f}_6^\varepsilon = \dot{f}_4^\varepsilon = \varepsilon + \frac{1-\varepsilon\dot{\ell}_1^\varepsilon}{1+\varepsilon\dot{\ell}_1^\varepsilon}\dot{f}_1^\varepsilon \ge \varepsilon + \frac{1-\varepsilon\dot{\lambda}_1^\varepsilon}{1+\varepsilon\dot{\lambda}_1^\varepsilon}\dot{f}_1^\varepsilon = \dot{f}_1^\varepsilon,$$

where the last equality follows from the explicit expression of the debonding speed for $\kappa = \tilde{\kappa}$ in the first interval, $\dot{\lambda}_1^\varepsilon = 1/\sqrt{1+\varepsilon^2}$, obtained by plugging $\kappa = \tilde{\kappa}$ into (4.61). We iterate this argument and obtain for every $i \ge 1$

$$\dot{f}_{3i+3}^\varepsilon = \dot{f}_{3i+1}^\varepsilon = \varepsilon + \frac{1-\varepsilon\dot{\ell}_{3i-2}^\varepsilon}{1+\varepsilon\dot{\ell}_{3i-2}^\varepsilon}\dot{f}_{3i-2}^\varepsilon \ge \varepsilon + \frac{1-\varepsilon\dot{\lambda}_{3i-2}^\varepsilon}{1+\varepsilon\dot{\lambda}_{3i-2}^\varepsilon}\dot{f}_1^\varepsilon = \dot{f}_1^\varepsilon. \tag{4.68}$$

Moreover, by (4.67) we recall that $\dot{\ell}^\varepsilon(t) \ge \frac{1}{\varepsilon}H(\dot{f}^\varepsilon(t - \varepsilon\ell^\varepsilon(t)))$, where $H(x) := \frac{2x^2 - \tilde{\kappa}}{2x^2 + \tilde{\kappa} + 2\sqrt{c_3}x}$. We have

$$H'(x) = \frac{1}{\varepsilon}\frac{4\sqrt{c_3}x^2 + 4x + \sqrt{c_3}}{(2x^2 + \frac{1}{2} + 2\sqrt{c_3}x)^2} \ge 0 \quad \text{for } x \ge 0.$$

(Recall that $\tilde{\kappa} = \frac{1}{2}$ and notice that the second order polynomial at its numerator has negative roots.) Therefore, by (4.68), we get

$$\dot{\ell}^{\varepsilon}_{3i+3} = \dot{\ell}^{\varepsilon}_{3i+3} \geq H(\dot{f}^{\varepsilon}_{3i+1}) \geq H(\dot{f}^{\varepsilon}_{1}) \geq \frac{1}{2+\sqrt{c_3}} =: \nu,$$

for ε small enough.

Summarising, in the first $3n^{\varepsilon}$ iterations we observe the alternation of two phases:

- stop phases, where the debonding speed is zero,
- propagation phases, where the debonding speed is uniformly bounded from below.

So far, we have not insisted on detailing the time intervals where $\dot{\ell}^{\varepsilon}$ is zero or positive. Let us just notice that the length of those intervals is determined by the rule for the update of f^{ε}, see also (4.63). By (4.64), we obtain that in the iterative scheme outlined above the length of the time intervals is dilated by a factor $\frac{1+\varepsilon\dot{\ell}^{\varepsilon}_{i}}{1-\varepsilon\dot{\ell}^{\varepsilon}_{i}}$, see Figure 4.8. This shows that the intervals where $\dot{\ell}^{\varepsilon} = 0$ have all the same length $2\varepsilon = \varepsilon\ell_0$. In contrast, the length of the intervals where $\dot{\ell}^{\varepsilon} \neq 0$ is increasing, since at the i-th iteration those intervals are dilated by a factor $\frac{1+\varepsilon\dot{\ell}^{\varepsilon}_{3i+1}}{1-\varepsilon\dot{\ell}^{\varepsilon}_{3i+1}} = \frac{1+\varepsilon\dot{\ell}^{\varepsilon}_{3i+3}}{1-\varepsilon\dot{\ell}^{\varepsilon}_{3i+3}} \geq \frac{1+\varepsilon\nu}{1-\varepsilon\nu}$.

Following a similar iterative scheme, we now construct a fictitious zig-zag evolution $\gamma^{\varepsilon}(t)$ such that $\gamma^{\varepsilon}(0) = \ell_0 = 2$ and $\dot{\gamma}^{\varepsilon} \in \{0, \nu\}$. More precisely, imitating the construction of ℓ^{ε}, we set

$$\dot{\gamma}^{\varepsilon}(t) = \begin{cases} \nu & \text{if } t - \gamma^{\varepsilon}(t) \in (-2\varepsilon, -\varepsilon), \\ 0 & \text{if } t - \gamma^{\varepsilon}(t) \in (-\varepsilon, \varepsilon), \\ \nu & \text{if } t - \gamma^{\varepsilon}(t) \in (\varepsilon, 2\varepsilon). \end{cases}$$

This defines γ^{ε} in $[0, b^{\varepsilon}_3]$, where b^{ε}_3 is as in (4.43) and denotes the time such that $b^{\varepsilon}_3 - \gamma^{\varepsilon}(b^{\varepsilon}_3) = 2\varepsilon$. It turns out that $b^{\varepsilon}_3 = 2\varepsilon(2-\varepsilon\nu)/(1-\varepsilon\nu)$, see Figure 4.8. Next we repeat this pattern with the following rule: at each iteration the intervals where $\dot{\ell}^{\varepsilon} = 0$ maintain the same length 2ε; the two intervals where $\dot{\ell}^{\varepsilon} \neq 0$ are dilated by the fixed factor $\frac{1+\varepsilon\nu}{1-\varepsilon\nu}$. By construction we obtain

$$\gamma^{\varepsilon}(t) \leq \ell^{\varepsilon}(t) \leq \lambda^{\varepsilon}(t), \tag{4.69}$$

where the latter inequality follows by (4.67). More precisely, let us denote by b^{ε}_{3i} the extremum of the interval where γ^{ε} is defined after the $(i-1)$-th iteration, obtained replicating b^{ε}_3. For every $i = 1, \dots, n^{\varepsilon}$,

$$b^{\varepsilon}_{3i} = 2\varepsilon i + \frac{2\varepsilon}{1-\varepsilon\nu}\sum_{j=0}^{i-1} d^{j}_{\varepsilon} = 2\varepsilon i + \frac{2\varepsilon}{1-\varepsilon\nu}\frac{1-d^{i}_{\varepsilon}}{1-d_{\varepsilon}}, \tag{4.70}$$

where

$$d_{\varepsilon} := \frac{1+\varepsilon\nu}{1-\varepsilon\nu}.$$

The first summand in (4.70) corresponds to the total length of all intervals where $\dot{\gamma}^{\varepsilon} = 0$ up to b^{ε}_{3i}, while the second accounts for the intervals where $\dot{\gamma}^{\varepsilon} = \nu$. The position of the debonding front at time b^{ε}_{3i} is

$$\gamma^{\varepsilon}(b^{\varepsilon}_{3i}) = 2 + \frac{2\varepsilon\nu}{1-\varepsilon\nu}\frac{1-d^{i}_{\varepsilon}}{1-d_{\varepsilon}}.$$

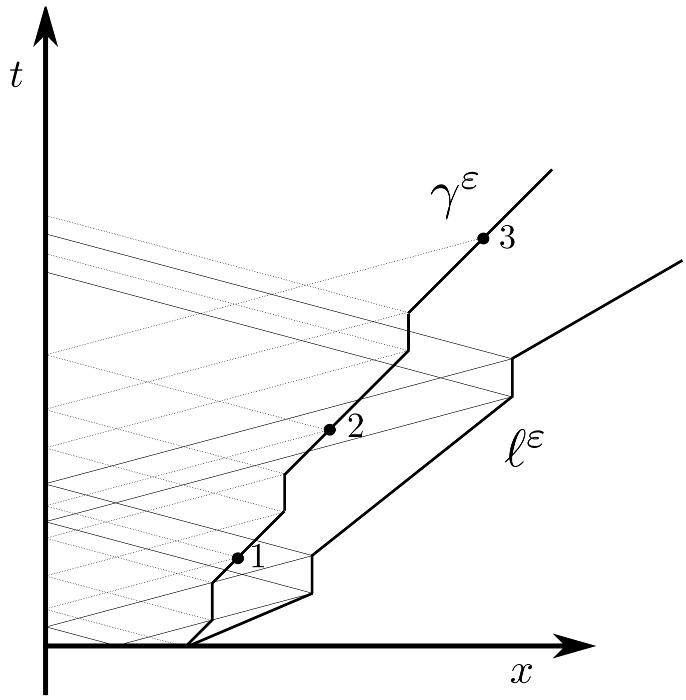

Figure 4.8: Evolution of γ^ε and ℓ^ε. The points $i = 1, 2, 3$ in figure represent the points $(b^\varepsilon_{3i}, \gamma^\varepsilon(b^\varepsilon_{3i}))$ in the construction of γ^ε.

We consider the map $i \mapsto x = \gamma^\varepsilon(b^\varepsilon_{3i})$ for $i = 1, \ldots, n^\varepsilon$ and its inverse

$$i^\varepsilon(x) = \left\lfloor \frac{\log(x-1)}{\log d_\varepsilon} \right\rfloor .$$

In order to understand the limit behaviour of ℓ^ε, we study the limit of γ^ε. (Notice that their pointwise limits are both uniform limits.) A straightforward computation shows that

$$\varepsilon i^\varepsilon(x) \quad \stackrel{\varepsilon\to 0}{\longrightarrow} \quad \frac{1}{2\nu}\log(x-1) = \frac{2+\sqrt{c_3}}{2}\log(x-1).$$

Moreover,

$$d_\varepsilon^{i^\varepsilon(x)} \stackrel{\varepsilon\to 0}{\longrightarrow} x-1.$$

We now let $\varepsilon \to 0$ in the expression for $b^\varepsilon_{3i^\varepsilon(x)}$ and find the expression for the inverse $t \mapsto \gamma(t)$:

$$\gamma^{-1}(x) = \lim_{\varepsilon\to 0} b^\varepsilon_{3i^\varepsilon(x)} = (2+\sqrt{c_3})\left[\log(x-1) + x - 2\right]. \tag{4.71}$$

Notice also that

$$b^\varepsilon_{3n^\varepsilon} \stackrel{\varepsilon\to 0}{\longrightarrow} 2 + \frac{e^{2\nu}-1}{\nu}, \quad \gamma^\varepsilon(b^\varepsilon_{3n^\varepsilon}) \stackrel{\varepsilon\to 0}{\longrightarrow} 1 + e^{2\nu}.$$

Finally, passing to the limit in (4.69), we obtain

$$\lambda^{-1}(x) \le \ell^{-1}(x) \le \gamma^{-1}(x) \quad \text{for } x \le 1 + e^{2\nu}. \tag{4.72}$$

The explicit expression for λ^{-1} derived in (4.56) gives

$$\lambda^{-1}(x) = \frac{1}{2+\sqrt{c_3}}\gamma^{-1}(x). \tag{4.73}$$

This shows that $t \mapsto \ell(t)$ cannot satisfy Griffith's quasistatic criterion. Indeed, by (4.67), (4.71), and (4.73), the debonding speed is uniformly bounded by one, so ℓ has no jumps. Moreover, since a Griffith evolution must satisfy (4.34c), we would have

$$\frac{t^2}{2\ell(t)^2} = \kappa(\ell, 0) = \frac{1}{2} \quad \text{if } \dot{\ell} > 0,$$

whence $\ell(t) = t$. This is incompatible with (4.72) and therefore the limit evolution $t \mapsto \ell$ does not satisfy Griffith's activation condition (4.34c).

Notice also that the same behaviour may be observed even with a toughness such that

$$\lim_{\mu\to 1^-} \kappa(x,\mu) = +\infty \quad \text{for every } x. \tag{4.74}$$

Indeed, in the previous example $\dot{\ell}^\varepsilon$ is uniformly bounded by one, thus $\mu = \varepsilon\dot{\ell}^\varepsilon \le \varepsilon$. If we consider a toughness satisfying (4.74) and such that it coincides with (4.65) for $\mu = \varepsilon\dot{\ell}^\varepsilon \le 1$, we obtain the same counterexample.

4.4.2 Example 2: brutal propagation

A further question arises when studying a local toughness $\kappa(x,\mu)$ satisfying property (4.74): are high speed propagations penalised by such a local toughness κ? Does assuming (4.74) prevent jumps in the limit?

In fact, we now prove that even in this case we have limit evolutions with jumps. We consider

$$\kappa(x,\mu) = \tilde{\kappa}(x)\frac{1+\mu}{1-\mu}, \tag{4.75}$$

where

$$\tilde{\kappa}(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le \bar{x}, \\ \frac{1}{8}, & \text{if } x > \bar{x}, \end{cases}$$

and $\bar{x} > \ell_0 = 2$. Notice that $\tilde{\kappa}(x)$ takes only two values and it is non-increasing. In [28] was proved that such a discontinuity generates a jump in the limit in the case where κ depends only on the position x. Moreover, we fix $0 < \varepsilon \ll 1$ and take initial data similar to those considered in Section 4.3:

$$u_0^\varepsilon(x) = -\sqrt{1+\varepsilon^2}x + 2\sqrt{1+\varepsilon^2},$$

$u_1^\varepsilon(x) = 1$, and $w^\varepsilon(t) = t + 2\sqrt{1+\varepsilon^2}$. By the results showed in Section 4.4.1 and in analogy with [28], we know that the evolution is characterised by different phases. Indeed, by explicit computation, we find that

$$\dot{f}^\varepsilon(t) = \frac{\varepsilon + \sqrt{1+\varepsilon^2}}{2} =: \dot{f}_1^\varepsilon$$

for a.e. $-2\varepsilon < t < 2\varepsilon$ and, by (4.75), we write (4.61) as

$$\dot{\ell}_1^\varepsilon = \frac{1}{\varepsilon}\frac{2(\dot{f}_1^\varepsilon)^2 - \frac{1}{2}\frac{1+\varepsilon\dot{\ell}_1^\varepsilon}{1+\varepsilon\dot{\ell}_1^\varepsilon}}{2(\dot{f}_1^\varepsilon)^2 + \frac{1}{2}\frac{1+\varepsilon\dot{\ell}_1^\varepsilon}{1+\varepsilon\dot{\ell}_1^\varepsilon}}.$$

By explicitly solving this equation as a second order polynomial equation in $\varepsilon\dot{\ell}_1^\varepsilon$, we find

$$
\begin{aligned}
\dot{\ell}_1^\varepsilon &= \frac{1}{\varepsilon}\frac{(2\dot{f}_1^\varepsilon - 1)^2}{4(\dot{f}_1^\varepsilon)^2 - 1} \\
&= \frac{1}{\varepsilon}\frac{2 + 2\varepsilon + 2\varepsilon^2 - 2(1+\varepsilon)\sqrt{1+\varepsilon^2}}{2\varepsilon^2 + 2\varepsilon\sqrt{1+\varepsilon^2}} \\
&\sim \frac{1}{\varepsilon}\frac{2 + 2\varepsilon + 2\varepsilon^2 - 2(1+\varepsilon)(1+\frac{\varepsilon^2}{2})}{2\varepsilon^2 + 2\varepsilon(1+\frac{\varepsilon^2}{2})}.
\end{aligned}
$$

Notice that $\dot{\ell}_1^\varepsilon \sim \frac{1}{2}$. We have thus found the evolution of $t \mapsto \ell^\varepsilon(t)$ for a.e. $t \in (0, s_1^\varepsilon)$, where s_1^ε is defined as in Section 4.3 by $\varphi_\varepsilon(s_1^\varepsilon) := s_1^\varepsilon - \ell^\varepsilon(s_1^\varepsilon) = 2\varepsilon =: t_1^\varepsilon$. We extend f^ε using (4.20): for a.e. $t \in (t_1^\varepsilon, t_2^\varepsilon)$, where $t_2^\varepsilon = \omega_\varepsilon^{-1}(t_1^\varepsilon) := \varphi_\varepsilon(\psi_\varepsilon^{-1}(t_1^\varepsilon))$, we have

$$
\begin{aligned}
\dot{f}_2^\varepsilon &= \varepsilon + \frac{1 - \varepsilon\dot{\ell}_1^\varepsilon}{1 + \varepsilon\dot{\ell}_1^\varepsilon}\dot{f}_1^\varepsilon = \varepsilon + \frac{1 - \frac{(2\dot{f}_1^\varepsilon - 1)^2}{4(\dot{f}_1^\varepsilon)^2 - 1}}{1 + \frac{(2\dot{f}_1^\varepsilon - 1)^2}{4(\dot{f}_1^\varepsilon)^2 - 1}}\dot{f}_1^\varepsilon \\
&= \varepsilon + \frac{4\dot{f}_1^\varepsilon - 2}{8(\dot{f}_1^\varepsilon)^2 - 4\dot{f}_1^\varepsilon}\dot{f}_1^\varepsilon = \varepsilon + \frac{1}{2}.
\end{aligned}
$$

Again, one solves (4.61) to find $\dot{\ell}_2^\varepsilon$ in $(s_1^\varepsilon, s_2^\varepsilon)$, where $\varphi_\varepsilon(s_2^\varepsilon) = t_2^\varepsilon$. We obtain

$$
\dot{\ell}_2^\varepsilon = \frac{1}{\varepsilon}\frac{(2\dot{f}_2^\varepsilon - 1)^2}{4(\dot{f}_2^\varepsilon)^2 - 1} > 0,
$$

because $\dot{f}_2^\varepsilon > \frac{1}{2}$ and, arguing as above, $\dot{\ell}_2^\varepsilon \sim \frac{1}{2}$. This argument is repeated as long as $\ell(t) \leq \bar{x}$ and we always find $\dot{f}_i^\varepsilon = \varepsilon + \frac{1}{2}$ and $\dot{\ell}_i^\varepsilon = \dot{\ell}_2^\varepsilon$ for every $i \geq 2$. The first phase is thus characterised by an evolution with speed that tends to $\frac{1}{2}$ as $\varepsilon \to 0$.

When the evolution of the debonding front reaches the discontinuity $\bar{x}$ the process features an abrupt change. Indeed, there exists n_ε such that $\bar{x} \in [\ell^\varepsilon(s_{n_\varepsilon - 1}^\varepsilon), \ell^\varepsilon(s_{n_\varepsilon}^\varepsilon)]$ and, according to (4.75), the equation for ℓ^ε is now given by

$$
\dot{\ell}^\varepsilon(t) = \dot{\ell}_{n_\varepsilon}^\varepsilon = \frac{1}{\varepsilon}\frac{2(\dot{f}_{n_\varepsilon}^\varepsilon)^2 - \frac{1}{8}\frac{1+\varepsilon\dot{\ell}_{n_\varepsilon}^\varepsilon}{1+\varepsilon\dot{\ell}_{n_\varepsilon}^\varepsilon}}{2(\dot{f}_{n_\varepsilon}^\varepsilon)^2 + \frac{1}{8}\frac{1+\varepsilon\dot{\ell}_{n_\varepsilon}^\varepsilon}{1+\varepsilon\dot{\ell}_{n_\varepsilon}^\varepsilon}},
$$

for a.e. t such that $\ell^\varepsilon(t) \in [\bar{x}, \ell^\varepsilon(s_{n_\varepsilon}^\varepsilon)]$. As before, one explicitly solves this equation as a second order polynomial in $\varepsilon\dot{\ell}_{n_\varepsilon}^\varepsilon$ to find

$$
\dot{\ell}_{n_\varepsilon}^\varepsilon = \frac{1}{\varepsilon}\frac{(4\dot{f}_{n_\varepsilon}^\varepsilon - 1)^2}{16(\dot{f}_{n_\varepsilon}^\varepsilon)^2 - 1}.
$$

Since $\dot{f}_{n_\varepsilon}^\varepsilon = \varepsilon + \frac{1}{2}$ by the argument above, direct computations shows that in this region $\dot{\ell}^\varepsilon \sim \frac{5}{7}\frac{1}{\varepsilon}$. This implies that there is a fast propagation for the evolution of the debonding front $t \mapsto \ell^\varepsilon(t)$ that leads to a jump in the limit evolution. This proves that that limit jumps are still possible when $\kappa(x, \mu)$ satisfies (4.74).

Moreover, we notice that, during this fast propagation, we have

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t)) = \frac{1}{8}\frac{1+\frac{5}{7}}{1-\frac{5}{7}} = \frac{3}{4},$$

while, in the previous phase we had

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t)) = \frac{1}{2}\frac{1-\varepsilon\dot{\ell}_i^\varepsilon}{1+\varepsilon\dot{\ell}_i^\varepsilon} \to \frac{1}{2},$$

as $\varepsilon \to 0$ for every $i = 1, \dots, n_\varepsilon - 1$. Notice that, if we replace $\tilde{\kappa}(x)$ with

$$\bar{\kappa}(x) := \begin{cases} \frac{1}{2}, & \text{if } 0 \le x \le \bar{x}, \\ \frac{1}{c}, & \text{if } x > \bar{x}, \end{cases}$$

with $c > 2$ then, using the same argument, we find that the speed of the fast propagation is

$$\frac{1}{\varepsilon}\frac{2c(\varepsilon+\frac{1}{2})^2 + 1 - 2\sqrt{2c}(\varepsilon+\frac{1}{2}}{2c(\varepsilon+\frac{1}{2})^2 - 1} \sim \frac{1}{\varepsilon}\frac{c+1-\sqrt{2c}}{c-1},$$

and the corresponding local toughness

$$\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t); c) \sim \frac{1}{c}\frac{2c-\sqrt{2c}}{\sqrt{2c}-2},$$

so that $\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t); c) \to 0$ as $c \to +\infty$. In particular, for every $c > 2$ the local toughness $\kappa(\ell^\varepsilon(t), \varepsilon\dot{\ell}^\varepsilon(t); c)$ is bounded for every $\varepsilon > 0$ and the limit evolution features a jump.

The same analysis can be carried out even in the case in which $\tilde{\kappa}$ is not discontinuous (e.g. in the case where κ oscillates strongly in a small interval). Indeed, one considers a continuous approximation of $\tilde{\kappa}$ and by the previous argument, since there is a decrease of the local toughness, then fast propagations are expected.

CHAPTER 5

Singular perturbations of second order potential-type equations

In this chapter we study the convergence as $\varepsilon \to 0$ of the dynamic solutions u_ε of the equation

$$\varepsilon^2 \ddot{u}_\varepsilon(t) + V_x(t, u_\varepsilon(t)) = 0, \tag{5.1}$$

to the solution u of the limit equation

$$V_x(t, u(t)) = 0, \tag{5.2a}$$
$$V_{xx}(t, u(t)) > 0, \tag{5.2b}$$

where V is a time-dependent energy. The results presented in this chapter have been published in the paper [55].

5.1 Setting of the problem

Let $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $V(t, \cdot) \in C^2(\mathbb{R}^n)$. It will play the role of a time-dependent energy. We assume that there exists a function $u_0 \in C^0([0, T]; \mathbb{R}^n)$ such that the following properties are satisfied:

$$V_x(t, u_0(t)) = 0, \quad \text{for every } t \in [0, T], \tag{5.3}$$

$$\exists\, \alpha > 0 : V_{xx}(t, u_0(t))\xi \cdot \xi \geq \alpha|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n. \tag{5.4}$$

Furthermore, for a.e. $t \in [0, T]$ and for every $x \in \mathbb{R}^n$, we assume that there is a constant $A > 0$ such that

$$|V_x(t, x)|,\ |V_{xx}(t, x)| \leq A. \tag{5.5}$$

We also assume that there exists a C^1-Carathéodory function $V_t : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, i.e., a Carathéodory function such that $V_t(t, \cdot) \in C^1(\mathbb{R}^n)$, satisfying

$$V(t_2, x) - V(t_1, x) = \int_{t_1}^{t_2} V_t(t, x)\,\mathrm{d}t, \tag{5.6}$$

for a.e. $t_1, t_2 \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. Moreover, for every $R > 0$, we require that there exists $a_R \in L^1(\mathbb{R})$ such that

$$|V_t(t,x)|,\ |V_{tx}(t,x)| \leq a_R(t), \tag{5.7}$$

for a.e. $t \in \mathbb{R}$ and all $x \in B_R(0)$. We notice that, by condition (5.7), it is possible to prove that V_x is continuous in both variables.

We consider, for fixed $\varepsilon > 0$, the Cauchy problem

$$\varepsilon^2 \ddot{u}_\varepsilon + V_x(t, u_\varepsilon(t)) = 0, \tag{5.8a}$$

$$u_\varepsilon(0) = u_\varepsilon^0, \tag{5.8b}$$

$$\dot{u}_\varepsilon(0) = v_\varepsilon^0, \tag{5.8c}$$

where we assume that

$$u_\varepsilon^0 \to u_0(0) = 0 \text{ and } \varepsilon v_\varepsilon(0) \to 0, \tag{5.9}$$

as $\varepsilon \to 0$. Global existence and uniqueness of the solutions u_ε to (5.8) are consequences of standard theorems on ordinary differential equations by the continuity of V_x and (5.5). Our goal is to study when convergence, as $\varepsilon \to 0$, of solutions u_ε to (5.8) satisfying conditions (5.9) to u_0 is possible.

Using (5.5) and (5.7), we now study the dependence on x of the set of Lebesgue points for a function $t \mapsto f(t,x)$ which will then play the role of V_t and V_{tx}.

Lemma 5.1. *Let $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$ be a Carathéodory function such that, for every $R > 0$, there exists $a_R \in L^1(\mathbb{R})$ with $f(t,x) \leq a_R(t)$ for every $x \in B_R(0)$. Fix $x \in \mathbb{R}^n$, then for a.e. $t \in \mathbb{R}$ we have*

$$\lim_{y\to x}\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}[f(\tau,y) - f(t,y)]\,\mathrm{d}\tau = 0.$$

Proof. Let $t \in \mathbb{R}$ be a right Lebesgue point for $\tau \mapsto f(\tau,x)$, i.e.,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}[f(\tau,x) - f(t,x)]\,\mathrm{d}\tau = 0.$$

Let $\delta > 0$ and define

$$\omega_R^\delta(\tau) := \sup_{\substack{x,y\in B_R(0)\\ |x-y|<\delta}} |f(\tau,x) - f(\tau,y)|. \tag{5.10}$$

By assumption we have that $\omega_R^\delta(\tau) \leq 2a_R(\tau)$; moreover ω_R^δ is measurable because the supremum can be taken over all rational points and along a sequence $\delta = 1/n$. Therefore $\omega_R^\delta(\cdot) \in L^1(\mathbb{R})$. If t is also a right Lebesgue point for $\tau \mapsto \omega_R^\delta(\tau)$ for every $\delta \in \mathbb{Q}$, $\delta > 0$ and $|x-y| < \delta$, then

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}|f(\tau,x) - f(t,x) - (f(\tau,y) - f(t,y))|\,\mathrm{d}\tau\\ \leq\ &\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\left[\omega_R^\delta(\tau) + \omega_R^\delta(t)\right]\,\mathrm{d}\tau = 2\omega_R^\delta(t).\end{aligned} \tag{5.11}$$

Since $f(t,\cdot)$ is uniformly continuous in $\overline{B_R(0)}$, the last term in (5.11) tends to zero as $\delta \to 0$ for a.e. $t \in \mathbb{R}$. ☐

Remark 5.2. Given any $u \in W^{1,1}(0,T;\mathbb{R}^n)$, we are now able to get a chain rule for a.e. $t \in [0,T]$, by differentiating $z(t) := V(t,u(t))$. Indeed, by the Mean Value Theorem and (5.6), if t is a Lebesgue point for $\tau \mapsto V_t(\tau,u(t))$, we have

$$\begin{aligned}\frac{z(t+h)-z(t)}{h} &= \frac{V(t+h,u(t+h))-V(t,u(t+h))}{h} + \frac{V(t,u(t+h))-V(t,u(t))}{h} \\ &= \frac{1}{h}\int_t^{t+h} V_t(\tau,u(t+h))\,\mathrm{d}\tau + V_x(t,\xi)\frac{u(t+h)-u(t)}{h},\end{aligned} \tag{5.12}$$

for some point ξ belonging to the segment $[u(t),u(t+h)]$. We now re-write the first summand of (5.12) in the following form:

$$\begin{aligned}&\frac{1}{h}\int_t^{t+h} V_t(\tau,u(t+h))\,\mathrm{d}\tau \\ &= \frac{1}{h}\int_t^{t+h} [V_t(\tau,u(t+h)) - V_t(t,u(t+h))]\,\mathrm{d}\tau + V_t(t,u(t+h).\end{aligned} \tag{5.13}$$

By Lemma 5.1, the integral in (5.13) tends to zero, for a.e. $t \in [0,T]$ as $h \to 0$. This result is obtained using a diagonal argument and the fact that $u(t+h) \to u(t)$ as $h \to 0$, because u is an absolutely continuous function. Moreover, by the continuity of $V_t(t,u(\cdot))$, the second summand in (5.13) tends to $V_t(t,u(t))$. Therefore, as $h \to 0$ in (5.12), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,u(t)) = \dot{z}(t) = V_t(t,u(t)) + V_x(t,u(t))\dot{u}(t), \tag{5.14}$$

for a.e. $t \in [0,T]$, because $V_x(t,\cdot)$ is continuous.

We now argue similarly for $V_x(t,u(t))$ and get a chain rule again. Since $V_t(t,\cdot) \in C^1(\mathbb{R}^n)$ for a.e. $t > 0$ and by (5.7), we apply the Dominated Convergence Theorem and obtain

$$V_x(t_2,x) - V_x(t_1,x) = \int_{t_1}^{t_2} V_{tx}(t,x)\,\mathrm{d}t. \tag{5.15}$$

Therefore,

$$\begin{aligned}&\frac{V_x(t+h,u(t+h))-V_x(t,u(t))}{h} \\ &= \frac{V_x(t+h,u(t+h))-V_x(t,u(t+h))}{h} + \frac{V_x(t,u(t+h))-V_x(t,u(t))}{h} \\ &= \frac{1}{h}\int_t^{t+h} V_{tx}(\tau,u(t+h))\,\mathrm{d}\tau + \frac{V_x(t,u(t+h))-V_x(t,u(t))}{h}.\end{aligned} \tag{5.16}$$

Since $V_{tx}(t,\cdot)$ is continuous and $V_{tx}(\cdot,x)$ is measurable (indeed, it can be obtained as the limit along a sequence of measurable difference quotients), then V_{tx} is a Carathéodory function controlled by an integrable function $a_R(t)$. Arguing as before and recalling that $V(t,\cdot) \in C^2(\mathbb{R}^n)$, we have that for a.e. $t \in [0,T]$

$$\lim_{h\to 0}\frac{V_x(t+h,u(t+h))-V_x(t,u(t))}{h} = V_{tx}(t,u(t)) + V_{xx}(t,u(t))\dot{u}(t).$$

In particular, since $V_x(t,u_0(t)) = 0$, we have

$$V_{tx}(t,u_0(t)) + V_{xx}(t,u_0(t))\dot{u_0}(t) = 0. \tag{5.17}$$

The following result will enable us to restrict to the case of absolutely continuous functions throughout the sequel.

Proposition 5.3. *Let $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $V(t,\cdot) \in C^2(\mathbb{R}^n)$ for a.e. $t \in \mathbb{R}$. Let V_t fulfill conditions (5.6) and (5.7), and let $u_0\colon [0,T] \to \mathbb{R}^n$ be a continuous function such that there exists $\alpha > 0$:*

$$V_{xx}(t, u_0(t))\xi \cdot \xi \geq \alpha|\xi|^2, \tag{5.18}$$

for every $\xi \in \mathbb{R}^n$ and for a.e. $t \in [0,T]$. Then, u_0 is absolutely continuous in $[0,T]$.

Proof. We show that, if ε is small enough, there exists $\delta > 0$ such that, for a.e. $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| < \delta$, there exists $M_\varepsilon > 0$ and an integrable function g such that

$$|u_0(t_2) - u_0(t_1)| \leq M_\varepsilon \int_{t_1}^{t_2} g(t)\,\mathrm{d}t. \tag{5.19}$$

We know that

$$\begin{aligned} 0 &= V_x(t_2, u_0(t_2)) - V_x(t_1, u_0(t_1)) \\ &= V_x(t_2, u_0(t_2)) - V_x(t_1, u_0(t_2)) + V_x(t_1, u_0(t_2)) - V_x(t_1, u_0(t_1)) \\ &= V_x(t_2, u_0(t_2)) - V_x(t_1, u_0(t_2)) + V_{xx}(t_1, y)(u_0(t_2) - u_0(t_1)), \end{aligned}$$

where y is in the segment $[u_0(t_1), u_0(t_2)]$. Therefore, we have

$$u_0(t_2) - u_0(t_1) = -V_{xx}(t_1, y)^{-1}[V_x(t_2, u_0(t_2)) - V_x(t_1, u_0(t_2))]. \tag{5.20}$$

Since $V_{xx}(t_1, \cdot)$ is continuous and it satisfies the coercivity condition (5.18), there exists $\varepsilon > 0$ such that, if $y \in B_\varepsilon(u_0(t_1))$ then

$$V_{xx}(t_1, y)\xi \cdot \xi \geq \frac{\alpha}{2}|\xi|^2.$$

We can thus invert $V_{xx}(t_1, \cdot)$ in a neighborhood of $u_0(t_1)$. Let λ_y be the minimum eigenvalue of $V_{xx}(t_1, y)$. Therefore, the norm of $V_{xx}(t_1, y)^{-1}$ is controlled by $1/\lambda_y$. If v_y is an eigenvector of $V_{xx}(t_1, y)$ with eigenvalue λ_y, then we have

$$\lambda_y |v_y|^2 = V_{xx}(t_1, y)v_y \cdot v_y \geq \frac{\alpha}{2}|v_y|^2,$$

from which we deduce that $\lambda_y \geq \frac{\alpha}{2}$ and therefore

$$\left|V_{xx}(t_1, y)^{-1}\right| \leq \frac{2}{\alpha}. \tag{5.21}$$

We now plug (5.21) in (5.20) and, arguing as in (5.15) of the previous Remark, we get

$$|u_0(t_2) - u_0(t_1)| \leq \frac{2}{\alpha} \int_{t_1}^{t_2} |V_{tx}(t, u_0(t_2))\,\mathrm{d}t| \leq \frac{2}{\alpha} \int_{t_1}^{t_2} a_R(t)\,\mathrm{d}t$$

and obtain (5.19). □

5.2 Convergence of solutions

This section is devoted to the study of the convergence of the solutions u_ε to problem (5.8). We will show that u_ε uniformly converges to u_0, which is the equilibrium for the potential V introduced in the previous section, provided that the initial conditions (5.8b)&(5.8c) satisfy (5.9).

We recall here the standard Gronwall lemma which will be used as a main tool in the proof of the convergence.

Lemma 5.4 (Gronwall). *Let $\varphi \in L^\infty(\mathbb{R})$, $\varphi(t) \geq 0$ for a.e. $t \in \mathbb{R}$ and $a \in L^1(\mathbb{R})$, $a(t) \geq 0$ for a.e. $t \in \mathbb{R}$. We assume that there exists a constant $C > 0$ such that*

$$\varphi(t) \leq \int_0^t a(s)\varphi(s)\,\mathrm{d}s + C, \text{ for a.e. } t \in \mathbb{R}.$$

Then,

$$\varphi(t) \leq C \exp\left(\int_0^t a(s)\,\mathrm{d}s\right), \text{ for a.e. } t \in \mathbb{R}.$$

Remark 5.5. From now on we assume that there exists $\psi\colon [0, +\infty) \to \mathbb{R}$ such that

$$\lim_{t \to +\infty} \psi(t) = +\infty \text{ and } V(t,x) \geq \psi(|x|), \tag{5.22}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^n$, and there exist $a(\cdot), b(\cdot) \in L^1(0,T)$ such that

$$V_t(t,x) \leq a(t) + b(t)V(t,x), \tag{5.23}$$

for a.e. $t \in [0,T]$ and for all $x \in \mathbb{R}^n$. Then, it is easy to deduce uniform boundedness for the sequence $\{u_\varepsilon\}$, by applying Lemma 5.4 to the following energy estimate:

$$V(t, u_\varepsilon(t)) \leq \|a\|_{L^1(0,T)} + \int_0^T b(t)V(t, u_\varepsilon(t))\,\mathrm{d}t.$$

We remark that conditions (5.22) and (5.23), which are standard in this context, are not necessary for establishing our result if we already knew that the sequence $\{u_\varepsilon\}$ is uniformly bounded.

We are now in a position to state the main result of this section.

Theorem 5.6. *Let V be a function fulfilling the assumptions of Proposition 5.3 and let $u_0 \in C^0([0,T];\mathbb{R}^n)$ be such that $V_{xx}(t, u_0(t)) = 0$ for every $t \in [0,T]$. Assume also that conditions (5.22) and (5.23) are satisfied and that $V_{xx}(t,x)$ and $V_{tx}(t,x)$ are locally equi-Lipschitz in x, uniformly in t, i.e. for every $x \in \mathbb{R}^n$ there exists $\delta > 0$ and constants $C_1, C_2 > 0$ (which may depend on x), such that, for every $|h| < \delta$*

$$\begin{aligned} |V_{xx}(t, x+h) - V_{xx}(t,x)| &\leq C_1|h|, \\ |V_{tx}(t, x+h) - V_{tx}(t,x)| &\leq C_2|h|, \end{aligned} \tag{5.24}$$

for a.e. $t \in [0,T]$. Let u_ε be a solution of the Cauchy problem (5.8) and assume (5.9). Then,

$$u_\varepsilon \to u_0 \text{ uniformly in } [0,T]$$

and

$$\varepsilon\|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1} \to 0,$$

as $\varepsilon \to 0$.

Proof. We fix a sequence $\varepsilon_j \to 0$ and we prove convergence for u_{ε_j}: this will show convergence for the whole family $\{u_\varepsilon\}$ to u_0, by the arbitrariness of ε_j. However, we shall keep writing u_ε for the sake of simplicity of notation.

By Proposition 5.3, we have that $u_0 \in W^{1,1}(0,T;\mathbb{R}^n)$. Since $C^2([0,T];\mathbb{R}^n)$ is dense in $W^{1,1}(0,T;\mathbb{R}^n)$, for every $k \in \mathbb{N}$ there exists a sequence $\{u_0^k\} \subset C^2([0,T];\mathbb{R}^n)$ such that

$$\|u_0 - u_0^k\|_{W^{1,1}} < \frac{1}{k}. \tag{5.25}$$

A suitable choice of k will take place in due course. However, we can already notice that, since $W^{1,1}(0,T;\mathbb{R}^n) \subset C^0([0,T];\mathbb{R}^n)$, then u_0^k uniformly converges to u_0 in $[0,T]$ and therefore they are all contained in a compact set containing $\{u_0(t),\, t \in [0,T]\}$. We now introduce a surrogate of energy estimate, multiplying (5.8a) by $\dot u_\varepsilon(t) - \dot u_0^k(t)$. After an integration, we get

$$\begin{aligned}
&\frac{\varepsilon^2}{2}|\dot u_\varepsilon(t) - \dot u_0^k(t)|^2 + V(t, u_\varepsilon(t)) \\
&= \frac{\varepsilon^2}{2}|\dot u_\varepsilon(0) - \dot u_0^k(0)|^2 + V(0, u_\varepsilon(0)) - \int_0^t \varepsilon^2 \ddot u_0^k(s)(\dot u_\varepsilon(s) - \dot u_0^k(s))\,\mathrm{d}s \\
&\quad + \int_0^t \Big[V_t(s, u_\varepsilon(s)) + V_x(s, u_\varepsilon(s))\dot u_0^k(s)\Big]\,\mathrm{d}s
\end{aligned} \tag{5.26}$$

Our aim is thus to infer some lower and upper bounds for (5.26) in order to get, by Lemma 5.4, convergence of $u_\varepsilon - u_0^k$ and then deduce convergence to u_0. It is thus convenient to consider the following "shifted" potential $\tilde V$ defined as

$$\tilde V(t,x) := V(t,x) - V(t, u_0^k(t)). \tag{5.27}$$

Since u_0^k is of class C^2, then all regularity assumptions on V are inherited by $\tilde V$. We have, in particular, that

$$\tilde V_t(t,x) = V_t(t,x) - V_t(t, u_0^k(t)) - V_x(t, u_0^k(t))\dot u_0^k(t). \tag{5.28}$$

Moreover it is easy to show that

$$\tilde V_x(t, u_0(t)) = 0, \quad \text{for every } t \in [0,T]. \tag{5.29}$$

We also notice that (5.26) is equivalent to

$$\begin{aligned}
&\frac{\varepsilon^2}{2}|\dot u_\varepsilon(t) - \dot u_0^k(t)|^2 + \tilde V(t, u_\varepsilon(t)) \\
&= \frac{\varepsilon^2}{2}|\dot u_\varepsilon(0) - \dot u_0^k(0)|^2 + \tilde V(0, u_\varepsilon(0)) - \int_0^t \varepsilon^2 \ddot u_0^k(s)(\dot u_\varepsilon(s) - \dot u_0^k(s))\,\mathrm{d}s \\
&\quad + \int_0^t \tilde V_t(s, u_\varepsilon(s)) + \tilde V_x(s, u_\varepsilon(s))\dot u_0^k(s)\,\mathrm{d}s.
\end{aligned} \tag{5.30}$$

We set $A_\varepsilon := \frac{\varepsilon^2}{2}|\dot u_\varepsilon(0) - \dot u_0^k(0)|^2 + \tilde V(0, u_\varepsilon(0))$, which tends to 0 as $\varepsilon \to 0$, by (5.9) and because $u_0^k \to u_0$ uniformly in $[0,T]$.

We now subdivide the proof into parts obtaining estimates which will then be used in the final Gronwall argument.

Lower estimate. We look for a lower bound for the summand $\tilde{V}(t, u_\varepsilon)$ in the left hand side of (5.30). We have that, by first order expansion, there exists y in the segment $[0, x]$ such that

$$\begin{aligned} V(t, x + u_0(t)) &= V(t, u_0(t)) + V_x(t, u_0(t))x + V_{xx}(t, y)x \cdot x \\ &= V(t, u_0(t)) + V_{xx}(t, y)x \cdot x, \end{aligned} \tag{5.31}$$

because $V_x(t, u_0(t)) = 0$ for every $t \in [0, T]$. We now compute twice (5.31), once for $x = u_\varepsilon(t) - u_0(t)$ and once for $x = u_0^k(t) - u_0(t)$, and then we make the difference between the two results. Therefore, for suitable y_1 between $u_0(t)$ and $u_\varepsilon(t)$, and y_2 between $u_0(t)$ and $u_0^k(t)$, we have

$$\begin{aligned} \tilde{V}(t, u_\varepsilon(t)) &= V_{xx}(t, y_1)(u_\varepsilon(t) - u_0(t)) \cdot (u_\varepsilon(t) - u_0(t)) \\ &\quad - V_{xx}(t, y_2)(u_0^k(t) - u_0(t)) \cdot (u_0^k(t) - u_0(t)). \end{aligned} \tag{5.32}$$

By a continuity argument and the coercivity assumption for V_{xx} (5.18), there exists $\delta > 0$ such that, if $|z| < \delta$, then

$$V_{xx}(t, z + u_0(t))\xi \cdot \xi \geq \frac{\alpha}{2}|\xi|^2.$$

We apply this estimate in the first summand of the right hand side of (5.32), while for the other one we use boundedness of $V_{xx}(t, \cdot)$. We thus get

$$\tilde{V}(t, u_\varepsilon(t)) \geq \frac{\alpha}{2}|u_\varepsilon(t) - u_0(t)|^2 - c|u_0^k(t) - u_0(t)|^2, \tag{5.33}$$

for a suitable $c > 0$, provided that

$$|u_\varepsilon(t) - u_0(t)| < \delta \text{ for every } t \in [0, T] \text{ and for } \varepsilon \text{ small enough.} \tag{5.34}$$

For the moment we assume that this bound holds and postpone its proof to the end.

Since

$$\frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 \leq \frac{\alpha}{2}|u_\varepsilon(t) - u_0(t)|^2 + \frac{\alpha}{2}|u_0(t) - u_0^k(t)|^2,$$

we deduce, by (5.33), that

$$\tilde{V}(t, u_\varepsilon(t)) \geq \frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 - (c + \frac{\alpha}{2})|u_0^k(t) - u_0(t)|^2, \tag{5.35}$$

where the last summand on the right hand side of (5.35) is small by the uniform convergence of u_0^k to u_0.

Upper estimate. We now switch our attention to the estimate on the right hand side of (5.30), which we now write as

$$\begin{aligned} &A_\varepsilon - \int_0^t \varepsilon^2 \ddot{u}_0^k(s)(\dot{u}_\varepsilon(s) - \dot{u}_0^k(s))\,\mathrm{d}s + \int_0^t \Big[\tilde{V}_t(s, u_\varepsilon(s)) + \tilde{V}_x(s, u_\varepsilon(s))\dot{u}_0^k(s)\Big]\,\mathrm{d}s \\ &=: A_\varepsilon - A_1 + A_2. \end{aligned}$$

<u>*Estimate of A_1*</u>. We first apply the Cauchy inequality and obtain

$$|A_1| = \left|\int_0^t \varepsilon^2 \ddot{u}_0^k(\dot{u}_\varepsilon - \dot{u}_0^k)\,\mathrm{d}s\right| \leq \frac{\varepsilon^2}{2}\int_0^t |\ddot{u}_0^k|^2\,\mathrm{d}s + \frac{\varepsilon^2}{2}\int_0^t |\dot{u}_\varepsilon - \dot{u}_0^k|^2\,\mathrm{d}s \tag{5.36}$$

The second summand in (5.36) will enter the final estimate via the Gronwall lemma, while for the first one we argue as follows. We have no information about how big $\|\ddot{u}_0^k\|_{L^2(0,T)}$ is, nevertheless we can find, for every $k \in \mathbb{N}$, an $\varepsilon > 0$ such that

$$\|\ddot{u}_0^k\|^2_{L^2(0,T)} \leq \frac{1}{\varepsilon}. \tag{5.37}$$

Then, we can invert the function which associates ε to k and get $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$, though this convergence may be very slow. This is done by recalling that $\varepsilon = \varepsilon_j$ and then defining

$$k(\varepsilon_j) := \min\left\{k \in \mathbb{N} : \|\ddot{u}_0^k\|^2_{L^2(0,T)} > \frac{1}{\varepsilon_j}\right\} - 1.$$

With a little abuse of notation, we write k instead of $k(\varepsilon_j)$ with this peculiar construction. Combining (5.36) with (5.37), we have

$$|A_1| \leq A_1^\varepsilon + \frac{\varepsilon^2}{2}\int_0^t |\dot{u}_\varepsilon - \dot{u}_0^k|^2 \, ds, \tag{5.38}$$

where $A_1^\varepsilon \to 0$ as $\varepsilon \to 0$.

Estimate of A_2. Using a variable x which will play the role of $u_\varepsilon(t) - u_0^k(t)$, we have that, for a.e. $t \in [0,T]$,

$$\begin{aligned}
&|\tilde{V}_t(t, x + u_0^k) + \tilde{V}_x(t, x + u_0^k)\dot{u}_0^k| \\
&\leq |\tilde{V}_t(t, x + u_0^k) - \tilde{V}_t(t, x + u_0)| + |\tilde{V}_x(t, x + u_0^k)\dot{u}_0^k - \tilde{V}_x(t, x + u_0^k)\dot{u}_0| \\
&\quad + |\tilde{V}_x(t, x + u_0^k)\dot{u}_0 - \tilde{V}_x(t, x + u_0)\dot{u}_0| + |\tilde{V}_t(t, x + u_0) + \tilde{V}_x(t, x + u_0)\dot{u}_0|
\end{aligned} \tag{5.39}$$

The first three summands on the right hand side are easy to deal with, by using Lipschitz and boundedness assumptions. They are estimated, independently of x, by

$$C(1 + |\dot{u}_0|)(|u_0^k - u_0| + |\dot{u}_0^k - \dot{u}_0|).$$

As for the fourth summand, we call

$$f(x) := \tilde{V}_t(t, x + u_0) + \tilde{V}_x(t, x + u_0)\dot{u}_0$$

If we set $g(x) := f(x) - f_x(0)x$, then there exists y in the segment $[0, x]$ such that $g(x) - g(0) = g_x(y)x$. Therefore we have, for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\tilde{V}_t(t, x + u_0) + \tilde{V}_x(t, x + u_0) \cdot \dot{u}_0 - \tilde{V}_t(t, u_0) \\
&\quad - \tilde{V}_x(t, u_0) \cdot \dot{u}_0 - \tilde{V}_{tx}(t, u_0) \cdot x - \tilde{V}_{xx}(t, u_0)\dot{u}_0 \cdot x \\
&\leq \left|\tilde{V}_{tx}(t, y + u_0) + \tilde{V}_{xx}(t, y + u_0)\dot{u}_0 - \tilde{V}_{tx}(t, u_0) - \tilde{V}_{xx}(t, u_0)\dot{u}_0\right| |x| \\
&\leq c(1 + |\dot{u}_0|)|y||x| \\
&\leq c|x|^2(1 + |\dot{u}_0|),
\end{aligned} \tag{5.40}$$

since $\tilde{V}_{tx}$ and $\tilde{V}_{xx}$ are locally equi-Lipschitz in x uniformly in t, by condition (5.24), and the constant $c > 0$ is independent of $x = u_\varepsilon - u_0^k$ because the functions u_ε are bounded in ε, as we pointed out in Remark 5.5. Moreover, by (5.28),

$$\begin{aligned}
|\tilde{V}_t(t, u_0)| &= |V_t(t, u_0) - V_t(t, u_0^k) - V_x(t, u_0^k) \cdot \dot{u}_0^k| \\
&\leq |V_t(t, u_0) - V_t(t, u_0^k)| + |V_x(t, u_0) \cdot \dot{u}_0^k - V_x(t, u_0^k) \cdot \dot{u}_0^k| \\
&\leq C|u_0 - u_0^k|(1 + |\dot{u}_0^k|),
\end{aligned} \tag{5.41}$$

for a.e. $t \in [0,T]$. As in (5.17), we have that for a.e. $t \in [0,T]$,

$$\tilde{V}_{tx}(t, u_0(t)) + \tilde{V}_{xx}(t, u_0(t))\dot{u}_0(t) = 0. \tag{5.42}$$

Therefore, plugging (5.29), (5.40), (5.41), and (5.42) in (5.39), we get

$$\begin{aligned} &|\tilde{V}_t(t, x + u_0^k) + \tilde{V}_x(t, x + u_0^k) \cdot \dot{u}_0^k| \\ &\leq c_1(1 + |\dot{u}_0|)(|u_0^k - u_0| + |\dot{u}_0^k - \dot{u}_0|) + c_2|x|^2(1 + |\dot{u}_0|) + c_3|u_0 - u_0^k|(1 + |\dot{u}_0^k|), \end{aligned} \tag{5.43}$$

for a.e. $t \in [0,T]$, where c_1, c_2, and $c_3 > 0$. We may therefore compute (5.43) with $x = u_\varepsilon(t) - u_0^k(t)$ and, by integrating between 0 and t, we find that

$$c_1 \int_0^t (1 + |\dot{u}_0|)(|u_0^k - u_0| + |\dot{u}_0^k - \dot{u}_0|) \,\mathrm{d}s \to 0,$$

as $k \to \infty$ by $W^{1,1}$−convergence of u_0^k to u_0. Also

$$c_3 \int_0^t |u_0 - u_0^k|(1 + |\dot{u}_0^k|) \,\mathrm{d}s \to 0,$$

as $k \to \infty$, using this time the uniform convergence of u_0^k to u_0 and the fact that $\dot{u}_0^k \in L^1(0,T)$. Therefore the second integral in (5.30) is estimated by

$$c_2 \int_0^t |u_\varepsilon - u_0^k|^2(1 + |\dot{u}_0|) \,\mathrm{d}s + A_k, \tag{5.44}$$

where $A_k \to 0$ as $k \to \infty$.

The Gronwall argument. Using the previous estimates, we conclude as follows. We set

$$B_\varepsilon := A_\varepsilon + A_1^\varepsilon + A_{k(\varepsilon)} + (c + \frac{\alpha}{2})|u_0^{k(\varepsilon)}(t) - u_0(t)|^2,$$

which tends to zero as $\varepsilon \to 0$, and we plug (5.35), (5.36), (5.37), and (5.44) into (5.30). We therefore have, for every $t \in [0,T]$,

$$\begin{aligned} &\frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)|^2 + \frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 \\ &\leq B_\varepsilon + \frac{\varepsilon^2}{2}\int_0^t |\dot{u}_\varepsilon - \dot{u}_0^k|^2 + c_2 \int_0^t |u_\varepsilon - u_0^k|^2(1 + |\dot{u}_0|). \end{aligned} \tag{5.45}$$

With some further manipulations we are in a position to apply Lemma 5.4. Therefore, there exists $C > 0$ such that

$$\frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)|^2 + \frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 \leq B_\varepsilon \exp\left(C \int_0^t (1 + |\dot{u}_0(s)|) \,\mathrm{d}s\right). \tag{5.46}$$

Since $u_0 \in W^{1,1}(0,T)$, we have that the right hand side of (5.46) tends to zero as $\varepsilon \to 0$, for every $t \in [0,T]$. In particular, since $|u_\varepsilon(t) - u_0(t)| \leq |u_\varepsilon(t) - u_0^k(t)| + |u_0^k(t) - u_0(t)|$, we obtain that $u_\varepsilon(t) \to u_0(t)$ uniformly in $[0,T]$ as $\varepsilon \to 0$. We also have

$$\begin{aligned} \varepsilon\|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1(0,T)} &\leq \varepsilon\|\dot{u}_\varepsilon - \dot{u}_0^k\|_{L^1(0,T)} + \varepsilon\|\dot{u}_0^k - \dot{u}_0\|_{L^1(0,T)} \\ &\leq \varepsilon^2 T \int_0^T |\dot{u}_\varepsilon - \dot{u}_0^k|^2 + \varepsilon\|\dot{u}_0^k - \dot{u}_0\|_{L^1(0,T)}, \end{aligned}$$

from which we deduce

$$\varepsilon\|\dot u_\varepsilon - \dot u_0\|_{L^1} \to 0,$$

as $\varepsilon \to 0$, because $\|\dot u_0^k - \dot u_0\|_{L^1(0,T)}$ is bounded.

Proof of (5.34). In order to conclude, we only need to prove that $|u_\varepsilon(t) - u_0(t)| < \delta$, for every $t \in [0,T]$ and for ε small enough. We define, for every $\varepsilon > 0$

$$t_\varepsilon = \inf\{t \in [0,T] : |u_\varepsilon(t) - u_0(t)| > \delta\},$$

with the convention that $\inf \emptyset = T$. Notice that the continuity of $u_\varepsilon(\cdot) - u_0(\cdot)$ and the initial condition $u_\varepsilon(0) \to u_0(0)$ as $\varepsilon \to 0$, implies that $t_\varepsilon > 0$. We thus have that (5.34) is satisfied for every $t \in [0, t_\varepsilon)$. We now assume, by contradiction, that $t_\varepsilon < T$. Then, with the previous Gronwall argument, we can find $\bar\varepsilon$ so small such that $|u_\varepsilon(t) - u_0(t)| < \frac{\delta}{2}$ for every $\varepsilon \in (0, \bar\varepsilon)$ and $t \in [0, t_\varepsilon]$. However, this contradicts the continuity of $u_\varepsilon - u_0$ in $t = t_\varepsilon$. Therefore $t_\varepsilon = T$ and this concludes the proof of the theorem. □

5.3 An example in which convergence fails

In the previous section we proved that, under certain assumptions on V, the solutions u_ε of problem (5.8) converge in $W^{1,1}(0,T)$ to u_0, whenever u_0 is continuous and the initial conditions satisfy (5.9). We now prove that assumptions on V can not be further relaxed in order to get the same result.

We consider the sample case

$$\varepsilon^2 \ddot u(t)_\varepsilon + u_\varepsilon(t) - u_0(t) = 0, \tag{5.47a}$$
$$u_\varepsilon(0) = u_\varepsilon^0, \tag{5.47b}$$
$$u_\varepsilon(0) = v_\varepsilon^0, \tag{5.47c}$$

where we assume that $u_\varepsilon^0 \to u_0(0) = 0$ and $\varepsilon v_\varepsilon^0 \to 0$ as $\varepsilon \to 0$. In this case the function V is given by

$$V(t,x) := \frac{1}{2}|x - u_0(t)|^2.$$

We have that $V_x(t, u_0(t)) = 0$ for every t and $V_{xx}(t, u_0(t))$ is the identity matrix. We notice that, if we only assume continuity of u_0, then a chain rule similar to (5.14) can not be established. We can, nevertheless, find an explicit solution of (5.47) with standard methods of ordinary differential equations:

$$u_\varepsilon(t) = \left(-\frac{1}{\varepsilon}\int_0^t u_0(s)\sin\tfrac{s}{\varepsilon}\,\mathrm{d}s + u_\varepsilon^0\right)\cos\tfrac{t}{\varepsilon} + \left(\frac{1}{\varepsilon}\int_0^t u_0(s)\cos\tfrac{s}{\varepsilon}\,\mathrm{d}s + \varepsilon v_\varepsilon^0\right)\sin\tfrac{t}{\varepsilon}. \tag{5.48}$$

If we assume that $u_0 \in W^{1,1}(0,T)$, then the assumptions of Theorem 5.6 are satisfied and therefore $u_\varepsilon \to u_0$ uniformly for every $t \in [0,T]$ and $\varepsilon\dot u_\varepsilon(t) \to 0$ for a.e. $t \in \mathbb{R}$. This result can be equivalently obtained by direct computation through the explicit formula (5.48). We may remark the fact that, in the presence of a dissipative term as in [1], the convergence of the solutions to the approximated problems is satisfied with weaker assumptions on the initial conditions. More precisely if the equation is

$$\varepsilon^2 \ddot u_\varepsilon(t) + \varepsilon\dot u_\varepsilon(t) + u_\varepsilon(t) - u_0(t) = 0, \tag{5.49}$$

then it is sufficient to assume

$$u_\varepsilon^0 e^{-\frac{1}{2\varepsilon}} \to 0 \text{ and } \varepsilon v_\varepsilon^0 e^{-\frac{1}{2\varepsilon}} \to 0 \text{ as } \varepsilon \to 0,$$

which is a condition weaker than (5.9).

We now show that convergence for the problem (5.47) fails if we only assume that u_0 is continuous. This gives a counterexample to the convergence result of Theorem 5.6 when the regularity assumptions on V are not satisfied. Indeed, there is at least a continuous function that can not be approximated by solutions to second order perturbed problems, as we show in the next proposition; we will exhibit one of these functions in Example 5.9. Furthermore in $W^{1,1}$ there is a dense set of C_0^0 functions with this property (see Remark 5.8).

Proposition 5.7. *There exists $u_0 \in C_0^0([0,T])$ such that the functions u_ε, defined in* (5.48)*, do not converge uniformly to u_0 as $\varepsilon \to 0$.*

Proof. We argue by contradiction. Assume that for every $u_0 \in C_0^0([0,T])$ u_ε uniformly converges to u_0 as $\varepsilon \to 0$. Without loss of generality, we can assume that $T \geq 1$ and we show that the convergence fails at $t = 1$. Let us fix $\varepsilon_k \to 0$. Then we have, from (5.48),

$$u_{\varepsilon_k}(1) = -\frac{1}{\varepsilon_k}\int_0^1 u_0(s)[\sin\tfrac{s}{\varepsilon_k}\cos\tfrac{1}{\varepsilon_k} - \cos\tfrac{s}{\varepsilon_k}\sin\tfrac{1}{\varepsilon_k}]\,\mathrm{d}s + u_{\varepsilon_k}^0\cos\tfrac{1}{\varepsilon_k} + \varepsilon_k v_{\varepsilon_k}^0 \sin\tfrac{1}{\varepsilon_k}.$$

Since $u_{\varepsilon_k}^0, \varepsilon_k v_{\varepsilon_k}^0 \to 0$ by assumption, we have convergence of $u_\varepsilon(1)$ to $u_0(1)$ if and only if the operator $F_{\varepsilon_k} : C_0^0([0,T]) \to \mathbb{R}$, defined as

$$F_{\varepsilon_k}(u_0) := -\frac{1}{\varepsilon_k}\int_0^1 u_0(s)[\sin\tfrac{s}{\varepsilon_k}\cos\tfrac{1}{\varepsilon_k} - \cos\tfrac{s}{\varepsilon_k}\sin\tfrac{1}{\varepsilon_k}]\,\mathrm{d}s,$$

converges. We thus have pointwise convergence of F_{ε_k} to F_0 defined by $F_0(u_0) = u_0(1)$. By the Banach-Steinhaus Theorem, this implies uniform equiboundedness. On the other hand, we notice that

$$F_{\varepsilon_k}(u_0) = \int_0^1 u_0(s)\,\mathrm{d}\mu_{\varepsilon_k}(s),$$

where $\mathrm{d}\mu_{\varepsilon_k}(s) = -\frac{1}{\varepsilon_k}[\sin\frac{s}{\varepsilon_k}\cos\frac{1}{\varepsilon_k} - \cos\frac{s}{\varepsilon_k}\sin\frac{1}{\varepsilon_k}]\,\mathrm{d}s$. However,

$$\sup_k |\mu_{\varepsilon_k}|(0,1) = \sup_k\left(\frac{1}{\varepsilon_k}\int_0^1 |\sin\tfrac{s-1}{\varepsilon_k}|\,\mathrm{d}s\right) = \sup_k \int_{-\frac{1}{\varepsilon_k}}^0 |\sin\tau|\,\mathrm{d}\tau = +\infty$$

which contradicts the uniform equiboundedness. □

Remark 5.8. The Banach-Steinhaus Theorem also implies that the set

$$R := \{u_0 \in C_0^0([0,T]) : \sup_\varepsilon |F_\varepsilon(u_0)| = +\infty\}$$

is dense. Therefore, there are indeed infinitely many functions for which u_ε can not converge to u_0.

Example 5.9. We now give an explicit example of a continuous function that is not approximated by solutions to second order perturbed problems. We consider as u_0 the Cantor-Vitali function $\hat{u}\colon [0,1] \to [0,1]$. Plugging $u_0 = \hat{u}$ into (5.48) and through integration by parts, we get

$$u_\varepsilon(t) = \hat{u}(t) - \cos\tfrac{t}{\varepsilon}\int_0^t \cos\tfrac{s}{\varepsilon}\,\mathrm{d}\mu(s) - \sin\tfrac{t}{\varepsilon}\int_0^t \sin\tfrac{s}{\varepsilon}\,\mathrm{d}\mu(s) + u_\varepsilon^0\cos\tfrac{t}{\varepsilon} + \varepsilon v_\varepsilon^0 \sin\tfrac{t}{\varepsilon}, \tag{5.50}$$

where μ is intended to be the distributional derivative of u_0. We now choose $\varepsilon_k = \frac{1}{2k\pi}$ and remark that

$$\int \cos(2k\pi s)\,\mathrm{d}\mu(s) = \int e^{-i2k\pi s}\,\mathrm{d}\mu(s),$$

where $\hat{u}$ and μ have been extended to $\mathbb{R}$ by setting $\hat{u} = 0$ in the complement of $[0,1]$. By using the well-known expression for the Fourier Tranform of the Cantor measure, we can compute (5.50) in $t = 1$ and get

$$u_{\varepsilon_k}(1) = \hat{u}(1) + u_{\varepsilon_k}^0 - \int_0^1 \cos(2k\pi s)\,\mathrm{d}\mu(s) = \hat{u}(1) + u_{\varepsilon_k}^0 - (-1)^k \prod_{h=1}^{\infty} \cos\tfrac{2k\pi}{3^h}.$$

Since $u_{\varepsilon_k}^0 \to 0$ by the assumptions on the initial conditions, we focus our attention on the term

$$(-1)^k \prod_{h=1}^{\infty} \cos\tfrac{2k\pi}{3^h} = (-1)^k f(2k\pi),$$

where we have defined $f\colon [0,+\infty) \to [-1,1]$ by

$$f(x) = \prod_{h=1}^{\infty} \cos\tfrac{x}{3^h}.$$

We prove that there exists a sequence k_n such that $(-1)^{k_n} f(2k_n\pi)$ does not converge to 0.

By definition, f satisfies

$$f(3x) = f(x)\cos(x).$$

In particular, this implies

$$f(6\pi) = f(3 \cdot 2\pi) = f(2\pi).$$

Inductively, one gets

$$f(3^n \cdot 2\pi) = f(2\pi)$$

and similarly

$$f(2 \cdot 3^n \cdot 2\pi) = f(4\pi).$$

Therefore we choose as k_n the sequence

$$\{3,\, 2\cdot 3,\, 3^2,\, 2\cdot 3^2, \dots, 3^n,\, 2\cdot 3^n, \dots\}.$$

Along this sequence $(-1)^{k_n} f(2k_n\pi)$ tends to $-f(2\pi)$ for the odd indeces and to $f(4\pi)$ for the even ones. We now prove that $f(2\pi)$ and $f(4\pi)$ are real non-zero numbers with the same sign. This implies that $(-1)^{k_n} f(2k_n\pi)$ does not converge and therefore $u_{\varepsilon_k}(1)$ does not converge to $\hat{u}(1)$. We have that (using the convention that $\log 0 = -\infty$)

$$\log f(x) = \sum_{h=1}^{\infty} \log\left|\cos\tfrac{x}{3^h}\right| \geq -\sum_{h=1}^{\infty} \tfrac{x^2}{3^{2h}},$$

if $y \in (0,1)$, because in this interval

$$\log|\cos y| = \log\cos y \geq \frac{\cos y - 1}{\cos y}.$$

Moreover

$$1 - \frac{1}{\cos y} \geq -x^2 \iff \cos y(1+y^2) \geq 1,$$

which is verified in $(0,1)$, using the fact that $\cos y \geq 1 - \frac{y^2}{2}$. Since $\frac{2\pi}{3^h}$ and $\frac{4\pi}{3^h}$ are in the interval $(0,1)$ for h large enough, then $f(2\pi)$ and $f(4\pi)$ are controlled by the geometric series and therefore $f(2\pi), f(4\pi) \neq 0$. This is enough in order to prove that along the sequences k_{2n} or k_{2n+1} convergence of $u_{\varepsilon_{k_n}}(1)$ to $\hat{u}(1)$ is not satisfied. Moreover, we notice that $f(2\pi)$ and $f(4\pi)$ have the same sign because $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$, while $\cos(\frac{2\pi}{3^n}) \geq 0$ for every $n \geq 3$. Therefore, we have found more than we claimed, since $u_{\varepsilon_{k_n}}(1)$ does not converge at all.

We have thus shown an explicit example in which convergence of solutions u_ε of (5.48) to a particular continuous function u_0 fails.

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