DOCTORAL THESIS

Kinematics of 4D Conformal Field Theories

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Abstract

In this thesis we develop a framework for performing bootstrap computations in 4-dimensional conformal field theories. We use the conformal symmetry to construct generic 2-, 3- and 4-point functions and in turn generic bootstrap equations. An emphasis is made on the unification of all the obtained theoretical results and on their implementation into a Mathematica package "CFTs4D" for an easy and convenient use.

The two main conceptual problems one faces are the construction of generic n-point tensor structures and the construction of generic conformal blocks.

We address the first problem using 2 alternative methods: the covariant (embedding space) formalism and the non-covariant (conformal frame) formalism. Both have their advantages and disadvantages. We establish a precise connection between them which allows their interchangeable use depending on the situation.

We address the second problem by reducing generic conformal blocks to an (infinite) set of seed conformal blocks. This is done using the so called spinning differential operators. We first construct explicitly a suitable (finite) set of such operators. We then introduce a new formalism which provides an (infinite) set of conformally covariant differential operators. The spinning operators are obtained as their invariant products. This heavily enlarges the original list of spinning differential operators.

Finally we compute the seed conformal blocks in two different ways: by directly solving the Casimir equation and by using the shadow formalism augmented with group-theoretic properties of our new covariant differential operators.

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To my beloved teacher Natalia Valerianovna Mashutikova

Chapter 1

Introduction

Motivation Conformal Quantum Field Theories (CFTs) is an important sub-class of Quantum Field Theories (QFTs). They describe fixed points of UV-complete QFTs. The primal interest for studying CFTs in 4 space-time dimensions (4D) is to learn the IR behavior of gauge theories possessing a non-trivial fixed point. Besides being interestin per se, this studies might also find applications in physics beyond the standard model, see [1] for a discussion.

The most famous example of a 4D gauge theory is the Quantum Chromo Dynamics (QCD). It consists of N_f Weyl fermions transforming in fundamental and N_f Weyl fermions transforming in anti-fundamental representation of the $SU(N_c)$ gauge group. The theory exhibits an IR fixed point for

$$N_f^\star < N_f \leq \frac{11}{2} N_c, \tag{1.1}$$

where the upper limit can be studied in perturbation theory [2]; the lower limit lies in the strongly interacting regime and currently can only be determined with the lattice techniques. For current estimates of $N_f^\star$ with $N_c = 3$ see for example [3].[1] The range of parameters (1.1) is called the conformal window. A lot of work was also devoted to $SU(2)$ gauge theories with fermions in adjoint representation [7–10]. Gauge theories with other gauge groups and different representations were considered in [11–18]. See also [19].

Conformal Bootstrap Approach One powerful non-perturbative method to study CFTs is the conformal bootstrap [20–24] (see [25, 26] for recent introduction). In this approach CFTs are described by the local CFT data (an infinite set of parameters), which consists of scaling dimensions and Lorentz representations of local primary operators together with structure constants of the operator product expansion (OPE). The observables of the theory are correlation functions which are computed by maximally exploiting the conformal symmetry and the operator product expansion. The CFT data is heavily constrained by the associativity of the OPE, which manifests itself in the form of consistency equations called the crossing or the bootstrap equations.

To analyze the bootstrap equations and to put any constraints on the CFT data one faces a very difficult technical problem of constructing generic bootstrap equations. This task can be logically split in two steps. First, one has to construct and classify the so called tensor structures[2] for 2-, 3- and 4-point functions. Second, one should reduce 4-point functions to 3-point functions by means of the OPE. This reduction allows to rewrite any 4-point function in terms of the CFT data and the so called conformal blocks.

It should be noted right away that the bootstrap approach deals only with fixed points and is blind to any microscopic realizations of the IR fixed point. Thus, it is not straightforward to apply it for studying gauge theories. To get any constraints on a specific gauge theory one should inject all available information about the UV theory to the bootstrap set up. In some

[1] Further studies of $N_c = 3$ case were performed in [4–6].

[2] We call tensor structures the algebraic objects which carry information about the spin of operators in the correlation function.

cases the knowledge of the global symmetries might suffice. In general one could try to inject an approximate spectrum of IR operators obtained using other methods (e.g. perturbation theory, lattice).

1.1 A Brief Review of Existing Results

In recent years there has been an immense progress in the bootstrap program in $d \geq 3$ CFTs. In a seminal work [27] it was shown how constraints on a finite subset of the OPE data can be extracted numerically from the scalar bootstrap equations.[3] In 4D the approach of [27] was further developed in [1, 28–39]. In 3D a major advance came with the numerical identification of the 3D Ising [40, 41] and the $O(N)$ models [42–45]. An analytic approach for analyzing the bootstrap equations was proposed in [46, 47] and further developed in [48–57]. In addition, we have analytic results for the conformal collider bounds [58–61] and the average null energy condition [62]. Many more results were obtained and new promising directions were considered in [63–110].

Most of these studies, however, focus on correlation functions of scalar operators, and thus only have access to the scaling dimensions of traceless symmetric operators and their OPE coefficients with a pair of scalars. In order to derive constraints on the most general elements of the CFT data, one has to consider more general correlation functions. To the best of our knowledge, the only numerical studies of a 4-point function of non-scalar operators in non-supersymmetric theories up to date were done in 3D for a 4-point function of Majorana fermions [111, 112] and for a 4-point function of conserved abelian currents [113] and energy-momenutm tensors [114].

One reason for the lack of results on 4-point functions of spinning operators is that such correlators are rather hard to deal with. This difficulty increases with the dimension d due to an increasing complexity of the d-dimesnional Lorentz group. For instance, the representations of the 4D Lorentz group are already much richer than the ones in 3D.

The problem of constructing tensor structures has a long history [111, 115–123]. In 4D all the 3-point tensor structures were obtained in [124] using the covariant embedding formalism approach. Unfortunately, in this approach 4- and higher-point tensor structures are hard to analyze due to a growing number of non-linear relations between the basic building blocks. This problem is alleviated in the conformal frame approach [118, 125, 126]. In [126] a complete classification of general conformally invariant tensor structures was obtained in a non-covariant form.

The problem of computing the conformal blocks for scalar 4-point functinons was solved by a variety of methods in [40, 43, 124, 127–131].[4] Spinning conformal blocks were considered in [110, 111, 123, 124, 138–143]. Remarkably, in [138] it was found that the Lorentz representations of external operators can be changed by means of differential operators. In 3D, this relates all bosonic conformal blocks to conformal blocks with external scalars. These results were extended to 3D fermions in [111, 140] completing in principle the program of computing general conformal blocks in 3D.

Results of [138] concerning traceless symmetric operators apply also to 4D, but are not sufficient even for the analysis of an OPE of traceless symmetric operators since such an OPE also contains non-traceless symmetric operators. The expression for a 4D spinning conformal block was obtained in [139] for a special case of 2 scalars and 2 vectors.

[3] The bootstrap equations constitute an infinite system of coupled non-linear equations for the CFT data.

[4] Recently conformal blocks got some attention from the AdS prospective when a relation between them and Witten diagrams was discovered [132–137].

1.2 A Brief Review of Novel Results

In this thesis we develop and unify all necessary theoretical ingredients and tools for constructing general bootstrap equations in 4D CFTs. The main goal of this thesis is to make future studies of dynamical quantities (the OPE data) widely accessible with both numerical and analytical techniques. We leave any applications for future research.

This thesis is based on the following works [144–148]. Below we briefly summarize the main novel results and outline the structure of the thesis.

chapter 2 We describe our formalism giving precise definitions to all the main objects: n-point functions, tensor structures, spinning differential operators, Casimir operators, conformal blocks, seed conformal blocks and the bootstrap equations. We show how to reduce generic conformal blocks to an (infinite) set of seed and dual conformal block

$$H_e^{(p)}(z,\bar{z}) \quad \textbf{and} \quad \overline{H}_e^{(p)}(z,\bar{z}), \quad e=0,\ldots,p \tag{1.2}$$

by means of the so called spinning differential operators. We do not provide explicit realization of all these objects, this is done in chapter 3 instead.

chapter 3 We develop two independent formalisms for constructing tensor structures and differential operators: the covariant (embedding formalism) approach in section 3.1 and non-covariant (conformal frame) approach in section 3.2. Both approaches have their advantages and disadvantages. We unify them by providing an explicit connection in section 3.2.2, making it possible to switch between them at any time.

We construct and attempt to classify tensor structures in the embedding formalism in (3.32). We provide a set of spinning differential operators in (3.34), (3.35) and (3.36). We construct and classify tensor structures in the conformal frame in (3.61) and (3.68). Finally we explain how to take derivatives in the conformal frame allowing to translate all the differential operators found in the embedding formalism into non-covariant language.

Many important relevant details of this chapter are moved to the set of appendices A. Appendices A.1 and A.2 summarize our conventions in 4D Minkowski space and 6D embedding space, as well as cover the action of $\mathcal{P}$- and $\mathcal{T}$-symmetries. Appendix A.2 also contains details of the embedding formalism. In appendix A.3 we give details on normalization conventions for 2-point functions and seed conformal blocks. Appendices A.4 and A.5 contain details on explicitly covariant tensor structures. In appendix A.6 we describe all 3 Casimir generators of the four-dimensional conformal group. Appendices A.7 and A.8 cover conservation conditions and permutation symmetries.

chapter 4 We address the problem of computing seed and dual seed conformal blocks (1.2). In section 4.1 we derive a system of Casimir partial differential equations for them in a compact form. We solve the Casimir system in section 4.2 by introducing a proper ansatz. The solution for the seed and dual seed blocks is given in (4.33) and (4.34). Schematically it has the following form

$$\left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p+1} \sum_{m,n} c^e_{m,n} \mathcal{F}^{-\,(\ldots)}_{\ldots+m,\,\ldots+n}(z,\bar{z}), \tag{1.3}$$

where $\mathcal{F}^-$ are the eigenfunctions of the scalar Casimir operator in 2D. The coefficients $c^e_{m,n}$ are computed using the recursion relation (4.32). We find their explicit form for $p\leq 4$. The properties of the $\mathcal{F}$-functions are given in appendix B.2. An explicit $p=1$ solution for the coefficients $c^e_{m,n}$ is given in appendix B.3.

A crucial role in constructing the ansatz was played by the results of appendix B.1. There we find a compact integral representation for the seed conformal blocks. Using this expression we show how to reduce seed and dual seed blocks to the scalar Dolan and Osborn block.[5] We then compute explicitly the $p = 1, 2$ cases.

chapter 5 We introduce here a new powerful formalism based on representation theory of the conformal group. We show in section 5.1 the existence of covariant differential operators corresponding to finite-dimensional representations of the conformal group. In section 5.2 we explain that these operators satisfy a very special "crossing" equation which can be used to move an action of a covariant differential operator from one point to another.

We construct explicitly these differential operators in section 5.3 in the case of fundamental and anti-fundamental representations of the conformal group $SU(2,2)$ in 4D, the result is summarized in (5.38). We show then how the spinning differential operators (3.35) and (3.36) are related to (5.38).

We combine the "crossing" equation with the shadow techniques in section 5.4. We use this to derive recursion relations[6] for the seed and dual seed blocks of the form

$$H_e^{(p)}(z,\bar{z}) = \sum_{e'} \# H_{e'}^{(p-1)}(z,\bar{z}), \tag{1.4}$$

which allow to reduce successively any (dual) seed conformal block to the scalar Dolan and Osborn block ($p = 0$). This result is an example of changing the spin of "exchanged" (internal) operators in the OPE and can be consider as a generalization of results [138] to internal operators.

The CFTs4D Package The results of chapters 2-5 are technically complicated. For their practically efficient use we introduce a `Mathematica` package "CFTs4D". It allows to work with general 2-, 3- and 4-point functions and to construct arbitrary spin crossing equations in 4D CFTs. The package can be downloaded from https://gitlab.com/bootstrapcollaboration/CFTs4D. Once it is installed one gets an access to a comprehensive documentation and examples. We also refer to the relevant functions from the package throughout the thesis as `[function]`.

[5] The task of reducing the seeds to scalar blocks is also addressed in chapter 5. However the results of appendix B.1 are weaker since they apply only to the seeds, whereas the formalism of chapter 5. applies to generic blocks. Also in appendix B.1 we do not take care of the overall normalization which is extremely tedious to compute. This issue is solved beautifully in chapter 5.

[6] A posteriori we notice that we could have derived these relations using our results in the shadow appendix (without fixing an overall normalization), see also footnote 5.

Chapter 2

The Framework

In this paper we consider only the consequences of the conformal symmetry. In particular, we do not consider global (internal) symmetries because they commute with conformal trasformations and thus can be straightforwardly included. We also do not discuss supersymmetry.

The local operators in 4D CFT are labeled by $(\ell,\bar{\ell})$ representation of the Lorentz group $SO(1,3)$ and the scaling dimension Δ.In a CFT one can distinguish a special class of *primary* operators, the operators which transform homogeneously under conformal transformations [20]. In a unitary CFT any local operator is either a primary or a derivative of a primary, in which case it is called a descendant operator. A primary operator in representation $(\ell,\bar{\ell})$ can be written as[1]

$$\mathcal{O}_{\alpha_1\ldots\alpha_\ell}^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}(x), \tag{2.1}$$

symmetric in spinor indices α_i and $\dot{\beta}_j$. Because of the symmetry in these indices, we can equivalently represent $\mathcal{O}$ by a homogeneous polynomial in auxiliary spinors s^α and $\bar{s}_{\dot{\beta}}$ of degrees ℓ and $\bar{\ell}$ correspondingly

$$\mathcal{O}(x,s,\bar{s}) = s^{\alpha_1}\cdots s^{\alpha_\ell}\bar{s}_{\dot{\beta}_1}\cdots\bar{s}_{\dot{\beta}_{\bar{\ell}}}\mathcal{O}_{\alpha_1\ldots\alpha_\ell}^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}(x). \tag{2.2}$$

We often call the auxiliary spinors s and $\bar{s}$ the spinor polarizations. The indices can be restored at any time by using

$$\mathcal{O}_{\alpha_1\ldots\alpha_\ell}^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}(x) = \frac{1}{\ell!\,\bar{\ell}!}\prod_{i=1}^{\ell}\prod_{j=1}^{\bar{\ell}}\frac{\partial}{\partial s^{\alpha_i}}\frac{\partial}{\partial\bar{s}_{\dot{\beta}_j}}\,\mathcal{O}(x,s,\bar{s}). \tag{2.3}$$

In principle the auxiliary spinors s and $\bar{s}$ are independent quantities, however without loss of generality we can assume them to be complex conjugates of each other, $s_\alpha = (\bar{s}_{\dot{\alpha}})^*$. This has the advantage that if $\mathcal{O}$ with $\ell=\bar{\ell}$ is a Hermitian operator, e.g. for $\ell=\bar{\ell}=1$,

$$\mathcal{O}_{\alpha\dot{\beta}}(x) = \left(\mathcal{O}_{\beta\dot{\alpha}}(x)\right)^\dagger, \tag{2.4}$$

then so is $\mathcal{O}(x,s,\bar{s})$,

$$\mathcal{O}(x,s,\bar{s}) = \left(\mathcal{O}(x,s,\bar{s})\right)^\dagger. \tag{2.5}$$

More generally for non-Hermitian operators we define

$$\overline{\mathcal{O}}(x,s,\bar{s}) \equiv \left(\mathcal{O}(x,s,\bar{s})\right)^\dagger, \tag{2.6}$$

see (A.8) for the index-full version.

[1] Our conventions relevant for 3+1 dimensional Minkowski spacetime are summarized in appendix A.1.

Conformal field theories possess an operator product expansion (OPE) with a finite radius of convergence [117, 125, 149, 150]

$$\mathcal{O}_1(x_1, s_1, \overline{s}_1)\mathcal{O}_2(x_2, s_2, \overline{s}_2) = \sum_{\mathcal{O}} \sum_a \lambda^a_{\langle \mathcal{O}_1 \mathcal{O}_2 \overline{\mathcal{O}} \rangle} \mathcal{B}_a(\partial_{x_2}, \partial_s, \partial_{\overline{s}}, \ldots)\mathcal{O}(x_2, s, \overline{s}), \tag{2.7}$$

where $\mathcal{B}_a$ are differential operators in the indicated variables (depending also on $x_1 - x_2, s_j, \overline{s}_j$, where $j = 1, 2$), which are fixed by the requirement of conformal invariance of the expansion. Here λ's are the OPE coefficients which are not constrained by the conformal symmetry. In general there can be several independent OPE coefficients for a given triple of primary operators, in which case we label them by an index a.

The OPE provides a way of reducing any n-point function to 2-point functions, which have canonical form in a suitable basis of primary operators. Therefore, the set of scaling dimensions and Lorentz representations of local operators, together with the OPE coefficients, completely determines all correlation functions of local operators in conformally flat $\mathbb{R}^{1,3}$. For this reason we call this set of data the CFT data in what follows.[2] The goal of the bootstrap approach is to constrain the CFT data by using the associativity of the OPE. In practice this is done by using the associativity inside of a 4-point correlation function, resulting in the crossing equations which can be analyzed numerically and/or analytically. In the remainder of this section we describe in detail the path which leads towards these equations.

2.1 Correlation Functions of Local Operators

We are interested in studying n-point correlation functions

$$f_n(\mathbf{p}_1 \ldots \mathbf{p}_n) \equiv \langle 0|\mathcal{O}^{(\ell_1,\bar{\ell}_1)}_{\Delta_1}(\mathbf{p}_1) \ \ldots \ \mathcal{O}^{(\ell_n,\bar{\ell}_n)}_{\Delta_n}(\mathbf{p}_n)|0\rangle, \tag{2.8}$$

where for convenience we defined a combined notation for dependence of operators on coordinates and auxiliary spinors

$$\mathbf{p}_i \equiv (x_i, s_i, \overline{s}_i). \tag{2.9}$$

We have labeled the primary operators with their spins and scaling dimensions. In general these labels do not specify the operator uniquely (for example in the presence of global symmetries); we ignore this subtlety for the sake of notational simplicity. For our purposes it will be sufficient to assume that all operators are space-like separated (this includes all Euclidean configurations obtained by Wick rotation), and thus the ordering of the operators will be irrelevant up to signs coming from permutations of fermionic operators.

The conformal invariance of the system puts strong constraints on the form of (2.8). By inserting an identity operator $\mathbf{1} = UU^\dagger$, where U is the unitary operator implementing a generic conformal transformation, inside this correlator and demanding the vacuum to be invariant $U|0\rangle = 0$, one arrives at the constraint

$$\langle 0|\big(U^\dagger \mathcal{O}^{(\ell_1,\bar{\ell}_1)}_{\Delta_1} U\big) \ldots \big(U^\dagger \mathcal{O}^{(\ell_n,\bar{\ell}_n)}_{\Delta_n} U\big)|0\rangle = \langle 0|\mathcal{O}^{(\ell_1,\bar{\ell}_1)}_{\Delta_1} \ldots \mathcal{O}^{(\ell_n,\bar{\ell}_n)}_{\Delta_n}|0\rangle. \tag{2.10}$$

The algebra of infinitesimal conformal transformations, as well as their action on the primary operators are summarized in our conventions in appendix A.1.

[2] Besides the correlation functions of local operators one can consider extended operators, such as conformal defects, as well as the correlation functions on various non-trivial manifolds. In order to be able to compute these quantities one has to in general extend the notion of the CFT data.

The general solution to the above constraint has the following form,

$$f_n(x_i, s_i, \bar{s}_i) = \sum_{I=1}^{N_n} g_n^I(\mathbf{u})\, \mathbb{T}_n^I(x_i, s_i, \bar{s}_i), \tag{2.11}$$

where $\mathbb{T}_n^I$ are the conformally-invariant tensor structures which are fixed by the conformal symmetry up to a $\mathbf{u}$-dependent change of basis, and $\mathbf{u}$ are cross-ratios which are the scalar conformally-invariant combinations of the coordinates x_i. The structures $\mathbb{T}_n^I$ and their number N_n depend non-trivially on the $SO(1,3)$ representations of $\mathcal{O}_i$, but rather simply on Δ_i, so we can write

$$\mathbb{T}_n^I(x_i, s_i, \bar{s}_i) = \mathcal{K}_n(x_i)\hat{\mathbb{T}}_n^I(x_i, s_i, \bar{s}_i), \tag{2.12}$$

where all Δ_i-dependence is in the "kinematic" factor $\mathcal{K}_n$[3] and all the the Δ_i enter $\mathcal{K}_n$ through the quantity

$$\kappa \equiv \Delta + \frac{\ell + \bar{\ell}}{2}. \tag{2.13}$$

Note that $\mathbb{T}$ and $\hat{\mathbb{T}}$ are homogeneous polynomials in the auxiliary spinors, schematically,

$$\mathbb{T}_n^I, \hat{\mathbb{T}}_n^I \sim \prod_{i=1}^{n} s_i^{\ell_i} \bar{s}_i^{\bar{\ell}_i}. \tag{2.14}$$

In the rest of this subsection we give an overview of the structure of n-point correlation functions for various n, emphasizing the features specific to 4D.

2-point functions A 2-point function can be non-zero only if it involves two operators in complex-conjugate representations, $(\ell_1, \bar{\ell}_1) = (\bar{\ell}_2, \ell_2)$, and with equal scaling dimensions, $\Delta_1 = \Delta_2$. In fact, it is always possible to choose a basis for the primary operators so that the only non-zero 2-point functions are between Hermitian-conjugate pairs of operators. We always assume such a choice.

The general 2-point function `[n2CorrelationFunction]` then has an extremely simple form given by

$$\langle \overline{\mathcal{O}}_\Delta^{(\bar{\ell},\ell)}(\mathbf{p}_1) \mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{p}_2) \rangle = c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle} \underbrace{x_{12}^{-2\kappa_1}}_{=\mathcal{K}_2} \underbrace{\left[\hat{\mathbb{I}}^{12}\right]^{\ell} \left[\hat{\mathbb{I}}^{21}\right]^{\bar{\ell}}}_{=\hat{\mathbb{T}}_2}, \tag{2.15}$$

where $c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle}$ is a constant. There is a single tensor structure $\hat{\mathbb{T}}_2$, and the building blocks $\hat{\mathbb{I}}^{ij}$ are defined in appendix A.4. Changing the normalization of $\mathcal{O}$ one can rescale the coefficient $c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle}$ by a positive factor. The phase is fixed by the requirement of unitarity, see appendix A.3. We can make the following choice

$$c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle} = i^{\ell - \bar{\ell}}, \quad c_{\langle \mathcal{O}\overline{\mathcal{O}}\rangle} = (-)^{\ell - \bar{\ell}} c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle} = i^{\bar{\ell} - \ell}, \tag{2.16}$$

where the factor $(-)^{\ell - \bar{\ell}}$ appears due to the spin statistics theorem.

[3]This does not uniquely fix the factorization, and we will make a choice based on convenience later.

3-point functions A generic form of a 3-point function [n3ListStructures, n3ListStructuresAlternativeTS] is given by[4]

$$\langle \mathcal{O}^{(\ell_1,\bar{\ell}_1)}_{\Delta_1}(\mathbf{p}_1)\mathcal{O}^{(\ell_2,\bar{\ell}_2)}_{\Delta_2}(\mathbf{p}_2)\mathcal{O}^{(\ell_3,\bar{\ell}_3)}_{\Delta_3}(\mathbf{p}_3)\rangle = \sum_{a=1}^{N_3}\lambda^{(a)}_{\langle O_1O_2O_3\rangle}\langle O_1O_2O_3\rangle^{(a)}, \tag{2.17}$$

where

$$\langle O_1O_2O_3\rangle^{(a)} \equiv \mathbb{T}_3^a = \mathcal{K}_3\,\hat{\mathbb{T}}_3^a. \tag{2.18}$$

In case of a single tensor structure we will use the notation $\langle\ldots\rangle^{(\bullet)}$. The kinematic factor [n3KinematicFactor] is given by

$$\mathcal{K}_3 = \prod_{i<j}|x_{ij}|^{-\kappa_i-\kappa_j+\kappa_k}. \tag{2.19}$$

The necessary and sufficient condition for the 3-point tensor structures $\hat{\mathbb{T}}_3^a$ to exist is that the 3-point function contains an even number of fermions and the following inequalities hold,

$$|\ell_i - \bar{\ell}_i| \leq \ell_j + \bar{\ell}_j + \ell_k + \bar{\ell}_k, \quad \text{for all distinct } i,j,k. \tag{2.20}$$

A general discussion on how to construct a basis of tensor structures $\hat{\mathbb{T}}_3^a$ is given in section 3.1. For convenience we summarize this construction for 3-point functions in appendix A.5.

The fact that the OPE coefficients enter 3-point functions follows simply from using the OPE (2.7) and the form of (2.15) in the left hand side of (2.17). It is also clear that one can always choose the bases for $\mathcal{B}_a$ and $\hat{\mathbb{T}}_3^a$ to be compatible.

There is a number of relations the OPE coefficients $\lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}$ have to satisfy. The simplest one comes from applying complex conjugation to both sides of (2.17). On the left hand side one has

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle^* = \langle\overline{\mathcal{O}}_3\overline{\mathcal{O}}_2\overline{\mathcal{O}}_1\rangle. \tag{2.21}$$

Using the properties of tensor structures under conjugation summarized in appendix A.4 one obtains a relation of the form

$$\left(\lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}\right)^* = C^{ab}\,\lambda^b_{\langle\overline{\mathcal{O}}_3\overline{\mathcal{O}}_2\overline{\mathcal{O}}_1\rangle}, \tag{2.22}$$

where the matrix C^{ab} is often diagonal with ±1 entries. Other constraints arise from the possible $\mathcal{P}$- and $\mathcal{T}$-symmetries (see appendix A.1), conservation equations (see appendix A.7), and permutation symmetries (see appendix A.8). Importantly all these conditions give linear equations for λ's, which can be solved in terms of an independent set of real quantities $\hat{\lambda}$ as

$$\lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle} = \sum_{\hat{a}=1}^{\hat{N}_3} P^{a\,\hat{a}}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}\hat{\lambda}^{\hat{a}}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}, \quad \hat{N}_3 < N_3. \tag{2.23}$$

It will be important for the calculation of conformal blocks that we can actually construct all the tensor structures $\mathbb{T}_3^a$ in (2.17) by considering a simpler 3-point function with two out of three operators having canonical spins $(\ell'_1,\bar{\ell}'_1)$ and $(\ell'_2,\bar{\ell}'_2)$, chosen in a way such that the 3-point function has a single tensor structure

$$\langle\mathcal{O}^{(\ell'_1,\bar{\ell}'_1)}_{\Delta'_1}\mathcal{O}^{(\ell'_2,\bar{\ell}'_2)}_{\Delta'_2}\mathcal{O}^{(\ell_3,\bar{\ell}_3)}_{\Delta_3}\rangle = \lambda\,\mathbb{T}_{seed}. \tag{2.24}$$

[4] For notational convenience we use lowercase index a instead of capital index I to label the 3-point tensor structures.

A simple choice is to set as many spin labels to zero as possible, for example

$$\ell_1' = \bar{\ell}_1' = \ell_2' = 0, \quad \bar{\ell}_2' = |\ell_3 - \bar{\ell}_3|. \tag{2.25}$$

As we see in section 3.1.2 one can then construct a set of differential operators $\mathbb{D}^a$ acting on the coordinates and polarization spinors of the first two operators such that

$$\mathbb{T}_3^a = \mathbb{D}^a \, \mathbb{T}_{seed}. \tag{2.26}$$

We will call the canonical tensor structure $\mathbb{T}_{seed}$ a seed tensor structure in what follows. Our choice of seed structures is described in appendix A.3. When the third field is traceless symmetric, one has obviously $\bar{\ell}_2' = 0$, thus relating a pair of generic operators to a pair of scalars [138].

4-point functions and beyond In the case $n = 4$ one has

$$\langle \mathcal{O}_{\Delta_1}^{(\ell_1,\bar{\ell}_1)}(\mathbf{p}_1)\mathcal{O}_{\Delta_2}^{(\ell_2,\bar{\ell}_2)}(\mathbf{p}_2)\mathcal{O}_{\Delta_3}^{(\ell_3,\bar{\ell}_3)}(\mathbf{p}_3)\mathcal{O}_{\Delta_4}^{(\ell_3,\bar{\ell}_4)}(\mathbf{p}_4)\rangle = \sum_{I=1}^{N_4} g_4^I(u,v)\, \mathbb{T}_4^I, \tag{2.27}$$

where $g_4^I(u,v)$ are not fixed by conformal symmetry and are functions of the 2 conformally invariant cross-ratios `[formCrossRatios]`

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.28}$$

In most of the applications it will be more convenient to use another set of variables $(z, \bar{z})$ `[changeVariables]` defined as

$$u = z\bar{z}, \;\; v = (1-z)(1-\bar{z}). \tag{2.29}$$

We classify and construct all the 4-point tensor structures $\mathbb{T}_4$ `[n4ListStructures, n4ListStructuresEF]` in section 3.2. Following the literature we choose the kinematic factor `[n4KinematicFactor]` of the form[5]

$$\mathcal{K}_4 = \left(\frac{x_{24}}{x_{14}}\right)^{\kappa_1-\kappa_2}\left(\frac{x_{14}}{x_{13}}\right)^{\kappa_3-\kappa_4} \times \frac{1}{x_{12}^{\kappa_1+\kappa_2} x_{34}^{\kappa_3+\kappa_4}}. \tag{2.30}$$

The case of $n \geq 5$ point functions is similar to the $n = 4$ case with a difference that the number of conformally invariant cross-ratios is $4n - 15$. We briefly discuss the classification of tensor structures for higher-point functions in section 3.2.

In general 4- and higher-point functions are subject to the same sort of conditions as 3-point functions. Reality conditions and implications of $\mathcal{P}$- and $\mathcal{T}$-symmetries are not conceptually different from the 3-point case. However, implications of permutation symmetries and conservation equations are more involved than those for 3-point functions, see [151], due to the existence of non-trivial conformal cross-ratios (2.28). See also appendices A.8 and A.7 for details.

[5] In section 3.2 we never separate the kinematic factor which has an extremely simple form $(z\bar{z})^{-\frac{\kappa_1+\kappa_2}{2}}$ in the conformal frame.

2.2 Decomposition in Conformal Partial Waves

Since the OPE data determines all the correlation functions, the functions $g_4^I(u,v)$ entering (2.27) can also be computed. To compute $g_4^I(u,v)$ we use the s-channel OPE, namely the OPE in pairs $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$. One way to do this is to insert a complete orthonormal set of states in the correlator

$$f_4 \underset{s-OPE}{=} \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \sum_{|\Psi\rangle}\langle \mathcal{O}_1\mathcal{O}_2|\Psi\rangle\langle\Psi|\mathcal{O}_3\mathcal{O}_4\rangle. \tag{2.31}$$

By virtue of the operator-state correspondence, see for example [25, 26], the states $|\Psi\rangle$ are in one-to-one correspondence with the local primary operators $\mathcal{O}$ and their descendants $\partial^n\mathcal{O}$. This allows us to express the inner products above in terms of the 3-point functions $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle$ and $\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle$ with the primary operator $\mathcal{O}$ and its conjugate $\overline{\mathcal{O}}$, resulting in the following s-channel conformal partial wave decomposition

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \sum_{\mathcal{O}}\sum_{a,b}\lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle}W^{ab}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}\lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}. \tag{2.32}$$

The objects W^{ab} are called the Conformal Partial Waves (CPWs). The summation in (2.32) is over all primary operators $\mathcal{O}$ which appear in both 3-point functions $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle$ and $\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle$ and we can write explicitly

$$\sum_{\mathcal{O}} = \sum_{|\ell-\bar\ell|=0}^{\infty}\ \sum_{\ell=0}^{\infty}\ \sum_{\Delta,i}, \tag{2.33}$$

where i labels the possible degeneracy of operators at fixed spin and scaling dimensions (coming, for example, from a global symmetry). Note that according to properties of 3-point functions (2.20), there is a natural upper cut-off in the first summation

$$\sum_{|\ell-\bar\ell|=0}^{\infty} = \sum_{|\ell-\bar\ell|=0}^{|\ell-\bar\ell|_{max}}, \tag{2.34}$$

where

$$|\ell-\bar\ell|_{max} = \min(\ell_1+\bar\ell_1+\ell_2+\bar\ell_2,\ \ell_3+\bar\ell_3+\ell_4+\bar\ell_4). \tag{2.35}$$

Furthermore, if the operator $\mathcal{O}$ is bosonic then $|\ell-\bar\ell|$ assumes only even values; if the operator $\mathcal{O}$ is fermionic $|\ell-\bar\ell|$ assumes only odd values. The CPWs can be further rewritten in terms of Conformal Blocks (CB) and tensor structures as

$$W^{ab}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle} = \sum_{I=1}^{N_4} G^{I,ab}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}(u,v)\ \mathbb{T}_4^I, \tag{2.36}$$

inducing the conformal block expansion for g_4^I

$$g_4^I(u,v) \underset{s-OPE}{=} \sum_{\mathcal{O}}\sum_{a,b}\lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle}G^{I,ab}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}(u,v)\lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}. \tag{2.37}$$

Computation of Conformal Partial Waves The CPWs can be computed by using the shadow formalism, see (5.54) and (B.2) or the Casimir equations (see below). However computing generic case is rather difficult. Luckily there is a way of reducing them to simpler objects called the seed CPWs by means of differential operators [138, 145].

For example, the s-channel CPW appearing due to the exchange of a generic operator

$$O_{\Delta}^{(\ell,\bar{\ell})}, \quad p \equiv |\ell - \bar{\ell}| \tag{2.38}$$

by using (2.26) can be written as

$$W^{ab}_{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle \langle \overline{\mathcal{O}} \mathcal{O}_3 \mathcal{O}_4 \rangle} = \mathbb{D}^a_{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle} \mathbb{D}^b_{\langle \overline{\mathcal{O}} \mathcal{O}_3 \mathcal{O}_4 \rangle} W^{seed}_{\langle \mathcal{F}_1^{(0,0)} \mathcal{F}_2^{(p,0)} \mathcal{O} \rangle \langle \overline{\mathcal{O}} \mathcal{F}_3^{(0,0)} \mathcal{F}_4^{(0,p)} \rangle}, \tag{2.39}$$

where $\mathcal{F}_i$ are the operators with the same 4D scaling dimensions Δ_i as $\mathcal{O}_i$, see section 3.1.2. The seed CPWs are defined as the s-channel contribution of (2.38) to the seed 4-point function

$$\langle \mathcal{F}_1^{(0,0)} \, \mathcal{F}_2^{(p,0)} \, \mathcal{F}_3^{(0,0)} \, \mathcal{F}_4^{(0,p)} \rangle. \tag{2.40}$$

An important property of the seed 4-point function (2.40) is that it has only $p+1$ tensor structures. We will distinguish two dual types of seed CPWs, following the convention of [146],

$$W^{(p)}_{seed} \equiv W^{seed}_{\langle \mathcal{F}_1^{(0,0)} \mathcal{F}_2^{(p,0)} \mathcal{O} \rangle \langle \overline{\mathcal{O}} \mathcal{F}_3^{(0,0)} \mathcal{F}_4^{(0,p)} \rangle}, \quad if \ \ell - \bar{\ell} \leq 0, \tag{2.41}$$

$$\overline{W}^{(p)}_{dual\ seed} \equiv W^{seed}_{\langle \mathcal{F}_1^{(0,0)} \mathcal{F}_2^{(p,0)} \mathcal{O} \rangle \langle \overline{\mathcal{O}} \mathcal{F}_3^{(0,0)} \mathcal{F}_4^{(0,p)} \rangle}, \quad if \ \ell - \bar{\ell} \geq 0. \tag{2.42}$$

The case $W^{(0)}_{seed} = \overline{W}^{(0)}_{dual\ seed}$ reproduces the classical scalar conformal block found by Dolan and Osborn [127, 128]. The seed CPWs `[seedCPW]` can be written in terms of a set of seed Conformal Blocks $H^{(p)}_e(z,\bar{z})$ and $\overline{H}^{(p)}_e(z,\bar{z})$ as[6]

$$W^{(p)}_{seed} = \mathcal{K}_4 \sum_{e=0}^{p} (-2)^{p-e} \, H^{(p)}_e(z,\bar{z}) \left[\hat{\mathbb{I}}^{42}\right]^e \left[\hat{\mathbb{I}}^{42}_{31}\right]^{p-e}, \tag{2.43}$$

$$W^{(p)}_{dual\ seed} = \mathcal{K}_4 \sum_{e=0}^{p} (-2)^{p-e} \, \overline{H}^{(p)}_e(z,\bar{z}) \left[\hat{\mathbb{I}}^{42}\right]^e \left[\hat{\mathbb{I}}^{42}_{31}\right]^{p-e}, \tag{2.44}$$

where the tensor structures are defined in appendix A.4.

The seed Conformal Blocks $H^{(p)}_e(z,\bar{z})$ and $\overline{H}^{(p)}_e(z,\bar{z})$ will be computed in chapter 4 and chapter 5 analytically in two different ways. `[plugSeedBlocks, plugDualSeedBlocks, plugSeedRecursion, plugDualSeedRecursion]`. Other relevant functions are `[plugCoefficients, plugKFunctions, reduceKFunctionDerivatives, plugPolynomialsPQ]`.

The Casimir Equation A very important property of the CPWs is that they satisfy the conformal Casimir eigenvalue equations [128, 129][7] which have the form

$$\left(\mathfrak{C}_n - E_n\right) W^{ab}_{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle \langle \overline{\mathcal{O}} \mathcal{O}_3 \mathcal{O}_4 \rangle} = 0, \tag{2.45}$$

where $n = 2,3,4$ and $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $\mathfrak{C}_4$ are the quadratic, cubic and quartic Casimir differential operators respectively `[opCasimir`n`EF, opCasimir24D]`. They are defined in appendix A.6 together with their eigenvalues `[casimirEigenvalue`n`]`, where the conformal generators $\mathfrak{L}_{MN}$ given in appendix A.2 are taken to act on 2 different points

$$\mathfrak{L}_{MN} = \mathfrak{L}_{i\,MN} + \mathfrak{L}_{j\,MN}, \tag{2.46}$$

[6] The factors $(-2)^{p-e}$ are introduced here to match the original work [146].

[7] DK thanks Hugh Osborn for useful discussion on this topic.

with $(ij) = (12)$ or $(ij) = (34)$ corresponding to the s-channel CPWs[8].

The $n = 2$ Casimir is used in chapter 4 for constructing the seed CPWs. Given that the seed CPWs are known, in practice the Casimir equations can be used to validate the more general CPWs computed using the prescription above.

Conserved and Identical Operators, $\mathcal{P}-$ and $\mathcal{T}-$symmetries As noted in section 2.1, in general there are various constraints imposed on 3- and 4-point functions, such as reality conditions, permutation symmetries, conservation, and $\mathcal{P}-$ and $\mathcal{T}-$ symmetries. Recall that the most general CPW decomposition is given by (2.37),

$$g_4^I(u,v) \underset{s-OPE}{=} \sum_{\mathcal{O}} \sum_{a,b} \lambda^a_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle} G^{I,ab}_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}(u,v)\lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}. \tag{2.47}$$

According to the discussion around (2.23), the general solution to these constraints relevant for this expansion is

$$\lambda^a_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle} = \sum_{\hat{a}} P^{a\,\hat{a}}_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle}\hat{\lambda}^{\hat{a}}_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle} \quad \text{and} \quad \lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle} = \sum_{\hat{b}} P^{b\,\hat{b}}_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}\hat{\lambda}^{\hat{b}}_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}. \tag{2.48}$$

Besides that, if the pair of operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is the same as the pair of operatirs $\mathcal{O}_3$ and $\mathcal{O}_4$, there has to exist relations of the form

$$\lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle} = \sum_b N^{b\,c}_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}\lambda^c_{\langle \mathcal{O}_1\mathcal{O}_2\overline{\mathcal{O}}\rangle}. \tag{2.49}$$

Once the relations (2.48) and (2.49) are inserted in the general expression (2.47), the resulting 4-point function will satisfy all the required constraints which preserve the s-channel.[9] In particular, the "reduced" CPWs corresponding to the coefficients $\hat{\lambda}$ will also satisfy these constraints automatically. Note that by construction the reduced CPWs are just the linear combinations of the generic CPWs.

2.3 The Bootstrap Equations

The conformal bootstrap equations are the equations which must be satisfied by the consistent CFT data. They arise as follows. The s-channel OPE (2.31) is not the only option to compute 4-point functions, there are in fact two other possibilities. One can use the t-channel OPE expansion

$$\begin{aligned} f_4 &\underset{t-OPE}{=} \langle \mathcal{O}_1(\mathbf{p}_1)\mathcal{O}_2(\mathbf{p}_2)\mathcal{O}_3(\mathbf{p}_3)\mathcal{O}_4(\mathbf{p}_4)\rangle = \\ &\pm \langle \mathcal{O}_3(\mathbf{p}_1)\mathcal{O}_2(\mathbf{p}_2)\mathcal{O}_1(\mathbf{p}_3)\mathcal{O}_4(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3} = \pm \langle \mathcal{O}_1(\mathbf{p}_1)\mathcal{O}_4(\mathbf{p}_2)\mathcal{O}_3(\mathbf{p}_3)\mathcal{O}_2(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_2\leftrightarrow\mathbf{p}_4} \end{aligned} \tag{2.50}$$

[8]Notice that the eigenvalue of $\mathfrak{C}_3$ taken at $(ij) = (34)$ will differ by a minus sign from the eigenvalue of $\mathfrak{C}_3$ taken at $(ij) = (12)$.

[9]Possible constraints which do not preserve s-channel are permutations of the form (13), etc. Such permutations, if present, are equivalent to the crossing equations discussed below.

or the u-channel OPE expansion

$$f_4 \underset{u-OPE}{=} \langle \mathcal{O}_1(\mathbf{p}_1)\mathcal{O}_2(\mathbf{p}_2)\mathcal{O}_3(\mathbf{p}_3)\mathcal{O}_4(\mathbf{p}_4)\rangle =$$
$$\pm \langle \mathcal{O}_4(\mathbf{p}_1)\mathcal{O}_2(\mathbf{p}_2)\mathcal{O}_3(\mathbf{p}_3)\mathcal{O}_1(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_4} = \pm \langle \mathcal{O}_1(\mathbf{p}_1)\mathcal{O}_3(\mathbf{p}_2)\mathcal{O}_2(\mathbf{p}_3)\mathcal{O}_4(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_2\leftrightarrow\mathbf{p}_3}. \quad (2.51)$$

In the above relations we permuted operators in the second and third equalities to get back the s-channel configuration. Minus signs are inserted for odd permutation of fermion operators.

In a consistent CFT the function f_4 is unique and does not depend on the channel used to compute it, leading to the requirement that the expressions (2.31), (2.50) and (2.51) must be equal. These equalities are the bootstrap equations. To be concrete we write the s-t consistency equation using (2.32) and (2.50)

$$f_4 \underset{s-OPE}{=} \sum_{\mathcal{O}} \lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle} W^{ab}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle} \lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}, \quad (2.52)$$

$$f_4 \underset{t-OPE}{=} \pm \sum_{\mathcal{O}} \lambda^a_{\langle\mathcal{O}_3\mathcal{O}_2\mathcal{O}\rangle} W^{ab}_{\langle\mathcal{O}_3\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_1\mathcal{O}_4\rangle} \lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_1\mathcal{O}_4\rangle}\Bigg|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3}. \quad (2.53)$$

In this example the tensor structures $\hat{\mathbb{T}}^I_n$ transform under permutation of points $\mathbf{p}_i \leftrightarrow \mathbf{p}_j$ as

$$\hat{\mathbb{T}}^I_{\langle\mathcal{O}_3\mathcal{O}_2\mathcal{O}_1\mathcal{O}_4\rangle}\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3} = M^{IJ}_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3}\, \hat{\mathbb{T}}^J_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}, \quad (2.54)$$

since they form a basis. Further decomposing these expressions using the basis of tensor structures one can compute the unknown $g^I_4(z,\bar{z})$

$$g^I_4(z,\bar{z}) \underset{s-OPE}{=} \sum_{\mathcal{O}}\sum_{a,b} \lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle} G^{I,ab}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}(z,\bar{z}) \lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_3\mathcal{O}_4\rangle}, \quad (2.55)$$

$$g^I_4(z,\bar{z}) \underset{t-OPE}{=} \pm M^{IJ}_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3} \sum_{\mathcal{O}}\sum_{a,b} \lambda^a_{\langle\mathcal{O}_3\mathcal{O}_2\mathcal{O}\rangle} G^{J,ab}_{\langle\mathcal{O}_3\mathcal{O}_2\mathcal{O}\rangle\langle\overline{\mathcal{O}}\mathcal{O}_1\mathcal{O}_4\rangle}(1-z,1-\bar{z}) \lambda^b_{\langle\overline{\mathcal{O}}\mathcal{O}_1\mathcal{O}_4\rangle}. \quad (2.56)$$

Equating (2.55) and (2.56) we get N_4 independent equations. In a presence of additional constraints discussed in appendices A.1, A.7 and A.8, not all the N_4 equations are independent, and one should chose only those equations which correspond to the independent degrees of freedom. In the conventional numerical approach to conformal bootstrap, when Taylor expanding the crossing equations around $z = \bar{z} = 1/2$, one should also be careful to understand which Taylor coefficients are truly independent. Among other things, this depends on the analyticity properties of tensor structures $\mathbb{T}_4$, see appendix A of [126] for a discussion.

Chapter 3

Embedding Formalism and Conformal Frame

3.1 Embedding Formalism

The EF is based on a key observation that the 4D conformal group is isomorphic to $SO(4,2)$, the linear Lorentz group in 6D. It is then convenient to *embed* the 4D space into the 6D space where the group acts linearly, lifting the 4D operators to 6D operators [124]. In particular, the linearity of the action of the conformal group in 6D allows one to easily build conformally invariant objects. However, non-trivial relations between these exist, posing problems for constructing the basis of tensor structures already in the case of 4-point functions. This motivates the introduction of a different formalism described in section 3.2.

The details of the 6D EF, its connection to the usual 4D formalism, and the relevant conventions are reviewed in appendix A.2. In this section we discuss only the construction of n-point tensor structures and the spinning differential operators. Our presentation focuses on the EF as a practical realization of the framework discussed in section 2.[1]

Embedding Let us first review the very basics of the EF. We label the points in the 6D space by $X^M = \{X^\mu,\, X^+,\, X^-\}$, with the metric given by

$$X^2 = X^\mu X_\mu + X^+ X^-. \tag{3.1}$$

The 4D space is then identified with the $X^+ = 1$ section of the lightcone $X^2 = 0$, and the coordinates on this section are chosen to be $x^\mu = X^\mu$.

A generic 4D operator $\mathcal{O}_{\alpha_1\ldots\alpha_\ell}^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}(x)$ in spin-$(\ell,\bar\ell)$ representation can be uplifted according to (A.66) to a 6D operator $O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X)$ defined on the lightcone $X^2 = 0$ and totally symmetric in its both sets of indices. We can define an index-free operator $O(X, S, \overline{S})$ using the 6D polarizations S_a and $\overline{S}^b$ by

$$O(X, S, \overline{S}) \equiv O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X) S_{a_1}\ldots S_{a_\ell}\overline{S}^{b_1}\ldots\overline{S}^{b_{\bar\ell}}. \tag{3.2}$$

The 6D operators are homogeneous in X and the 6D polarizations,

$$O(X, S, \overline{S}) \sim X^{-\kappa}\, S^\ell\, \overline{S}^{\bar\ell}, \quad \kappa = \Delta + \frac{\ell+\bar\ell}{2}. \tag{3.3}$$

[1]Note that most of the results discussed in section 2, like the explicit construction of 2- and 3-point tensor strucutures [121, 124, 144] and the existence of the spinning differential operators [138, 145] were originally obtained within the EF.

It is sometimes useful to assign the 4D scaling dimensions to the basic 6D objects as

$$\Delta[X] = -1 \quad \text{and} \quad \Delta[S] = \Delta[\overline{S}] = -\frac{1}{2}. \tag{3.4}$$

According to (A.69) there is a lot of freedom in choosing the lift $O(X,S,\overline{S})$. We can express this freedom by saying that the operators differing by gauge terms proportional to $S\overline{\mathbf{X}}$, $\overline{S}\mathbf{X}$ or $\overline{S}S$ are equivalent. Note that $O(X,S,\overline{S})$ is a priori defined only on the lightcone $X^2=0$, but it is convenient to extend it arbitrarily to all values of X. This gives an additional redundancy that the operators differing by terms proportional to X^2 are equivalent.

The 4D field can be recovered via a projection operation defined in appendix A.2,

$$\mathcal{O}(x,s,\overline{s}) = O(X,S,\overline{S})\Big|_{proj}, \tag{3.5}$$

which essentially substitutes X, S, $\overline{S}$ with some expressions depending on x, s, $\overline{s}$ only. All the gauge terms proportional to $S\overline{\mathbf{X}}$, $\overline{S}\mathbf{X}$, $\overline{S}S$ or X^2 vanish under this operation.

Sometimes it is convenient to work with index-full form $O^{a_1\ldots a_\ell}_{b_1\ldots b_{\overline{\ell}}}(X)$ and to fix part of the gauge freedom by requiring it to be traceless. We can restore the traceless form from the index-free expression $O(X,S,\overline{S})$ by

$$O^{a_1\ldots a_\ell}_{b_1\ldots b_{\overline{\ell}}}(X) = \frac{2}{\ell!\,\overline{\ell}!\,(2+\ell+\overline{\ell})!}\left(\prod_{i=1}^{\ell}\partial^{a_i}\right)\left(\prod_{j=1}^{\overline{\ell}}\partial_{b_j}\right)O(X,S,\overline{S}), \tag{3.6}$$

where[2]

$$\partial^a \equiv \left(S\cdot\frac{\partial}{\partial S}+\overline{S}\cdot\frac{\partial}{\partial\overline{S}}+3\right)\frac{\partial}{\partial S_a} - \overline{S}^a\left(\frac{\partial}{\partial S\cdot\partial\overline{S}}\right), \tag{3.7}$$

$$\partial_b \equiv \left(S\cdot\frac{\partial}{\partial S}+\overline{S}\cdot\frac{\partial}{\partial\overline{S}}+3\right)\frac{\partial}{\partial \overline{S}^b} - S_b\left(\frac{\partial}{\partial S\cdot\partial\overline{S}}\right). \tag{3.8}$$

Correlation functions A correlation function of 6D operators on the light cone must be SO(4,2) invariant and obey the homogeneity property (3.3). Consequently, it has the following generic form

$$\langle O^{(\ell_1,\overline{\ell}_1)}_{\Delta_1}(\mathbf{P}_1)\ldots O^{(\ell_n,\overline{\ell}_n)}_{\Delta_n}(\mathbf{P}_n)\rangle = \sum_{I=1}^{N_n} g_I(\mathbf{U})T^I(X,S,\overline{S}), \tag{3.9}$$

where $T^I(X,S,\overline{S})$ are the 6D homogeneous $SU(2,2)$ invariant tensor structures and $g_I(\mathbf{U})$ are functions of 6D cross-ratios, i.e. homogeneous with degree zero SO(4,2) invariant functions of coordinates on the projective light cone. We also defined a short-hand notation

$$\mathbf{P} \equiv (X,S,\overline{S}). \tag{3.10}$$

[2]These operators are constructed to map terms proportional to $\overline{S}S$ to other terms proportional to $\overline{S}S$. In the equivalence class of uplifts, given an operator $O(X,S,\overline{S})$ one can find another operator $O'(X,S,\overline{S}) = O(X,S,\overline{S}) + (\overline{S}S)(\ldots)_O$ which differs from O by terms proportional to $\overline{S}S$ and encodes a traceless operator $O^{a_1\ldots a_\ell}_{b_1\ldots b_{\overline{\ell}}}(X)$. Since after taking the maximal number of derivatives the $\overline{S}S$ terms can only map to zero, we can safely replace O by O'. The action on $O'(X,S,\overline{S})$ is proportional to the action of $\frac{\partial}{\partial S_a}$ and $\frac{\partial}{\partial \overline{S}^a}$ and thus provides an inverse operation to (3.2).

Tensor structures split in a scaling-dependent and in a spin-dependent parts as

$$T^I(X,S,\overline{S}) = K_n \hat{T}^I(X,S,\overline{S}), \quad T^I, \hat{T}^I_n \sim \prod_{i=1}^{n} S_i^{\ell_i} \overline{S}_i^{\bar{\ell}_i}. \tag{3.11}$$

The object K_n is the 6D kinematic factor and $\hat{T}^I$ are the $SO(4,2)$ invariants of degree zero in each coordinate. The main invariant building block is the scalar product[3]

$$X_{ij} \equiv -2\,(X_i \cdot X_j), \tag{3.12}$$

The 6D kinematic factors `[n3KinematicFactor, n4KinematicFactor]` are given by

$$K_2 \equiv X_{12}^{-\frac{\kappa_1}{2}}, \quad K_3 \equiv \prod_{i<j} X_{ij}^{-\frac{\kappa_i+\kappa_j-\kappa_k}{2}}, \tag{3.13}$$

and

$$K_4 \equiv \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\kappa_1-\kappa_2}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\kappa_3-\kappa_4}{2}} \times \frac{1}{X_{12}^{\frac{\kappa_1+\kappa_2}{2}} X_{34}^{\frac{\kappa_3+\kappa_4}{2}}}. \tag{3.14}$$

We also define the 6D cross-ratios by taking products of X_{ij} factors. For $n=4$ only two cross ratios can be formed

$$U \equiv \frac{X_{12}^2 X_{34}^2}{X_{13}^2 X_{24}^2}, \quad V \equiv \frac{X_{14}^2 X_{23}^2}{X_{13}^2 X_{24}^2}. \tag{3.15}$$

With these definitions, under projection we recover the usual 4D expressions:

$$X_{ij}\Big|_{proj} = x_{ij}^2, \quad K_n\Big|_{proj} = \mathcal{K}_n, \quad U\Big|_{proj} = u, \quad V\Big|_{proj} = v. \tag{3.16}$$

Finally, given a correlator in the embedding space one can recover the 4D correlator

$$\langle \mathcal{O}_{\Delta_1}^{(\ell_1,\bar{\ell}_1)}(\mathbf{p}_1)\ldots\mathcal{O}_{\Delta_n}^{(\ell_n,\bar{\ell}_n)}(\mathbf{p}_n)\rangle = \langle O_{\Delta_1}^{(\ell_1,\bar{\ell}_1)}(\mathbf{P}_1)\ldots O_{\Delta_n}^{(\ell_n,\bar{\ell}_n)}(\mathbf{P}_n)\rangle\Big|_{proj}, \tag{3.17}$$

with the projections of the 6D invariants entering the 6D correlator given in the formula (3.16) and appendix A.4.

3.1.1 Construction of Tensor Structures

Let us discuss the construction of tensor structures $\hat{T}^I_n(X,S,\overline{S})$. In index-free notation, this is equivalent to finding all $SU(2,2)$ invariant homogeneous polynomials in S, $\overline{S}$. All $SU(2,2)$ invariants are built fully contracting the indices of the following objects:

$$\delta^a_b,\ \epsilon_{abcd},\ \bar{\epsilon}^{abcd},\ \mathbf{X}_{i\,ab},\ \overline{\mathbf{X}}_j^{ab},\ S_{k\,a},\ \overline{S}_l^a. \tag{3.18}$$

With the exception of taking traces over the coordinates $\mathrm{tr}[\mathbf{X}_i\overline{\mathbf{X}}_j\ldots\mathbf{X}_k\overline{\mathbf{X}}_l]$, [4] all other tensor structures are built out of simpler invariants of degree two or four in S and $\overline{S}$.

List of non-normalized invariants By taking into account eq. (A.50) and the relations (A.68) and (A.72), it is possible to identify a set of invariants with the properties discussed above. These can be conveniently divided in five classes. The number of possible invariants

[3] Notice a difference in the definition of X_{ij} compared to [144–146]: $X_{ij}^{\text{here}} = -2X_{ij}^{\text{there}}$.
[4] All such traces can be reduced to the scalar product $X_{ij} = -\mathrm{Tr}[\mathbf{X}_i\overline{\mathbf{X}}_j]/2$.

increases with the number of points n. Below we provide a complete list of them for $n \leq 5$ and indicate their transformation property under the 4D parity. In what follows the indices $i, j, k, l, \ldots$ are assumed to label different points.

Class I constructed from $\overline{S}_i$ and S_j belonging to two different operators.

$$\begin{array}{llll} n \geq 2: & I^{ij} & \equiv (\overline{S}_i S_j) & \xrightarrow{\mathcal{P}} -I^{ji}, \\ n \geq 4: & I^{ij}_{kl} & \equiv (\overline{S}_i \mathbf{X}_k \overline{\mathbf{X}}_l S_j) & \xrightarrow{\mathcal{P}} -I^{ji}_{lk}, \\ n \geq 6: & \ldots & \ldots & \ldots \end{array} \tag{3.19}$$

Class II constructed from $\overline{S}_i$ and S_i belonging to the same operator.

$$\begin{array}{llll} n \geq 3: & J^i_{jk} & \equiv (\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_i) & \xrightarrow{\mathcal{P}} -J^i_{kj} = J^i_{jk}, \\ n \geq 5: & J^i_{jklm} & \equiv (\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k \mathbf{X}_l \overline{\mathbf{X}}_m S_i) & \xrightarrow{\mathcal{P}} -J^i_{mlkj}, \\ n \geq 7: & \ldots & \ldots & \ldots \end{array} \tag{3.20}$$

Class III constructed from S_i and S_j belonging to two different operators.

$$\begin{array}{llllll} n \geq 3: & K^{ij}_k & \equiv (S_i \overline{\mathbf{X}}_k S_j) & \xleftrightarrow{\mathcal{P}} & \overline{K}^{ij}_k & \equiv (\overline{S}_i \mathbf{X}_k \overline{S}_j), \\ n \geq 5: & K^{ij}_{klm} & \equiv (S_i \overline{\mathbf{X}}_k \mathbf{X}_l \overline{\mathbf{X}}_m S_j) & \xleftrightarrow{\mathcal{P}} & \overline{K}^{ij}_{klm} & \equiv (\overline{S}_i \mathbf{X}_k \overline{\mathbf{X}}_l \mathbf{X}_m \overline{S}_j), \\ n \geq 7: & \ldots & \ldots & & \ldots & \ldots \end{array} \tag{3.21}$$

Class IV constructed from S_i and S_i belonging to the same operator.

$$\begin{array}{llllll} n \geq 4: & L^i_{jkl} & \equiv (S_i \overline{\mathbf{X}}_j \mathbf{X}_k \overline{\mathbf{X}}_l S_i) & \xleftrightarrow{\mathcal{P}} & \overline{L}^i_{jkl} & \equiv (\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k \mathbf{X}_l \overline{S}_i), \\ n \geq 6: & \ldots & \ldots & & \ldots & \ldots \end{array} \tag{3.22}$$

Class V constructed from four S or four $\overline{S}$ belonging to different operators.

$$n \geq 4: \quad M^{ijkl} \equiv \epsilon(S_i S_j S_k S_l) \xleftrightarrow{\mathcal{P}} \overline{M}^{ijkl} \equiv \epsilon(\overline{S}_i \overline{S}_j \overline{S}_k \overline{S}_l). \tag{3.23}$$

Basic linear relations Simple properties `[applyEFProperties]` arise due to the relation (A.50). For instance

$$J^i_{jk} = -J^i_{kj}, \quad K^{ij}_k = -K^{ji}_k, \quad \overline{K}^{ij}_k = -\overline{K}^{ji}_k \tag{3.24}$$

for $n \geq 3$. Consequently not all these invariants are independent and it is convenient to work only with a subset of them, for instance $J^i_{j<k}$, $K^{i<j}_k$, $\overline{K}^{i<j}_k$. For $n \geq 4$ other properties must be taken into account:

$$I^{ij}_{kl} + I^{ij}_{lk} = -X_{kl} I^{ij}, \quad L^i_{jkl} = L^i_{[jkl]}, \quad M^{ijkl} = M^{[ijkl]}, \quad \overline{M}^{ijkl} = \overline{M}^{[ijkl]}. \tag{3.25}$$

These can be used in analogous manner to work only with a subset of invariants, for instance $I^{i<j}_{k<l}$, $I^{i>j}_{k>l}$, $L^i_{j<k<l}$, M^{1234} and $\overline{M}^{1234}$. Another important linear relation is

$$J^i_{[jk} X_{l]m} = 0 \tag{3.26}$$

where m is allowed to be equal to i.

Non-linear relations Unfortunately, even after taking into account all the linear relations above, many non-linear relations between products of invariant are present, see equations (A.122) - (A.125) for $n \geq 3$ relations [applyJacobiRelations] and appendix A in [145] for some $n \geq 4$ relations.[5] We expect that they all arise from (A.73).[6] As an example consider the following set of relations

$$M^{ijkl} = -2\, X_{ij}^{-1} \big(K_i^{jk}\, K_j^{il} - K_i^{jl}\, K_j^{ik} \big), \tag{3.27}$$
$$\overline{M}^{ijkl} = -2\, X_{ij}^{-1} \big(\overline{K}_i^{jk}\, \overline{K}_j^{il} - \overline{K}_i^{jl}\, \overline{K}_j^{ik} \big). \tag{3.28}$$

They show that M^{ijkl} and $\overline{M}^{ijkl}$ can be rewritten in terms of other invariants; hence class V objects are never used. All the relations obtained by fully contracting (3.18) with (A.73) in all possible ways, involve at most products of two invariants in class $I - IV$. In fact, we will see in section 3.2.2 that all non-linear relations have a quadratic nature. However, these quadratic relations can be combined together to form relations involving products of three or more invariants.[7] See appendix A.5 for an example of such phenomena in the $n = 3$ case.

Normalization of invariants The $\hat{T}_n^I(X, S, \overline{S})$ are required to be of degree zero in all coordinates. It is then convenient to introduce the following normalization factors

$$N_{ij} \equiv X_{ij}^{-1}, \quad N_k^{ij} \equiv \sqrt{\frac{X_{ij}}{X_{ik} X_{kj}}}, \quad N_{ijk} \equiv \frac{1}{\sqrt{X_{ij} X_{jk} X_{ki}}}. \tag{3.29}$$

Using these factors [normalizeInvariants, denormalizeInvariants] it is possible to define normalized type I and type II tensor structures

$$\hat{I}^{ij} \equiv I^{ij}, \quad \hat{I}^{ij}_{kl} \equiv N_{kl} I^{ij}_{kl}, \quad \hat{J}^i_{jk} \equiv N_{jk} J^i_{jk}, \quad \hat{J}^i_{jklm} \equiv N_{jk} N_{lm} J^i_{jklm}, \tag{3.30}$$

and normalized type III and type IV tensor structures

$$\hat{K}_k^{ij} \equiv N_k^{ij} K_k^{ij}, \quad \hat{K}^{ij}_{klm} \equiv N_{klm} K^{ij}_{klm}, \quad \hat{L}^i_{jkl} \equiv N_{jkl} L^i_{jkl}, \tag{3.31}$$

with the analogous expressions for parity conjugated invariants $\hat{\overline{K}}{}_k^{ij}$, $\hat{\overline{K}}{}^{ij}_{klm}$ and $\hat{\overline{L}}{}^i_{jkl}$. In appendix A.4 we provide an explicit 4D form of these invariants after projection. Notice the slight change of notation from previous works[8].

Basis of tensor structures Given an n-point function, one can construct a set of tensor structures [n3ListStructures, n3ListStructuresAlternativeTS, n4ListStructuresEF]

[5] Mind the difference in notation, see footnote 8 for details.

[6] In principle the Schouten identities might also contribute, see the footnote at page 26 of [124]; we found however that the Schouten identities, when contracted, give relations equivalent to (A.73) for $n \leq 4$.

[7] In other words, we have a graded ring of invariants and an ideal I of relations between them. The goal is to find a basis of independent invariants of a given degree modulo I. In principle, I is generated by a quadratic basis, but it is not trivial to reduce invariants modulo this basis. One would like to find a better basis, e.g. a Gröbner basis, which then will contain higher-order relations.

[8] The correspondence with the notation of [144–146] is as follows: $\hat{I}^{ij} \sim I_{ij}$, $-2\,\hat{I}^{ij}_{kl} \sim \hat{J}_{ij,\,kl}$, $-2\,\hat{J}^i_{jk} \sim J_{i,\,jk}$, $\sqrt{-2}\,\hat{K}^{ij}_k \sim K_{k,\,ij}$, $\sqrt{-2}\,\hat{\overline{K}}{}^{ij}_k \sim \overline{K}_{k,\,ij}$, $\sqrt{-8}\,\hat{L}^i_{jkl} \sim K_{i,jkl}$, $\sqrt{-8}\,\hat{\overline{L}}{}^i_{jkl} \sim \overline{K}_{i,jkl}$, where the expressions in the l.h.s. represent our notation and the expressions in the r.h.s. represent their notation.

by taking products of basic invariants as

$$\hat{T}^I_n = \Big\{ \prod_{i,j,\dots} \underbrace{[\hat{I}^{ij}]^{\#}}_{n\geq 2} \underbrace{[\hat{J}^i_{jk}]^{\#} [\hat{K}^{jk}_i]^{\#} [\hat{\overline{K}}{}^{jk}_i]^{\#}}_{n\geq 3} \underbrace{[\hat{I}^{ij}_{kl}]^{\#} [\hat{L}^i_{jkl}]^{\#} [\hat{\overline{L}}{}^i_{jkl}]^{\#}}_{n\geq 4} \underbrace{[\hat{J}^i_{jklm}]^{\#} [\hat{K}^{jk}_{ilm}]^{\#} [\hat{\overline{K}}{}^{jk}_{ilm}]^{\#}}_{n\geq 5} \dots \Big\}. \tag{3.32}$$

The subscripts stress that for a given number of points n not all the invariants are defined. The non-negative exponents # are determined by requiring $\hat{T}^I_n$ to be of degree $(\ell_i, \bar{\ell}_i)$ in $(S_i, \overline{S}_i)$. Generally, not all tensor structures obtained in this way are independent, due to the properties and relations discussed above. The number of relations to take into account increase rapidly with n. For $n \leq 3$ the problem of constructing a basis of independent tensor structures has been succesfully solved in [124, 144]; we review the construction for $n = 3$ in appendix A.5. However the increasing number of relations makes this approach inefficient to study general correlators for $n \geq 4$, mainly because many relations which are cubic or higher order in invariants can be written. In section 3.2 an alternative method of identifying all the independent structures is provided. Using this method we will also prove in section 3.2.2 that any n-point function tensor structure is constructed out of $n \leq 5$ invariants, namely the invariants involving five or less points in the formula (3.32).

3.1.2 Spinning Differential Operators

Let us now discuss the EF realization of the *spinning differential operators* used in (2.26) which allow to relate 3-point tensor structures of correlators with different spins[9]

$$\langle O^{(\ell_i,\bar{\ell}_i)}_{\Delta_{\mathcal{O}_i}} O^{(\ell_j,\bar{\ell}_j)}_{\Delta_{\mathcal{O}_j}} O^{(\ell,\bar{\ell})}_{\Delta_{\mathcal{O}}} \rangle \sim \mathbf{D}_{ij} \, \langle O^{(\ell'_i,\bar{\ell}'_i)}_{\Delta'_{\mathcal{O}_i}} O^{(\ell'_j,\bar{\ell}'_j)}_{\Delta'_{\mathcal{O}_j}} O^{(\ell,\bar{\ell})}_{\Delta_{\mathcal{O}}} \rangle. \tag{3.33}$$

The operators[10] $\mathbf{D}_{ij}$ are written as a product of basic differential operators

$$\mathbf{D}_{ij} = \Big\{ \prod_{i,j=1,2} \nabla^{\#}_{ij} I^{\#}_{ij} \overline{d}^{\#}_{ij} d^{\#}_{ij} D^{\#}_{ij} \widetilde{D}^{\#}_{ij} \Big\}. \tag{3.34}$$

The exponents are determined by matching the spins on both sides of (3.33). The basic spinning differential operators are constructed to be insensitive to pure gauge modifications and different extensions of fields outside of the light cone as stressed in (A.74). The action of these operators in 4D can be deduced by using the projection rules given in (A.76).

We provide here the list of basic differential operators[11] entering (3.34) arranging them in two sets according to the value of $\Delta\ell = |\ell_i + \ell_j - \bar{\ell}_i - \bar{\ell}_j| = 0, 2$. For $\Delta\ell = 0$ we have

$$\begin{aligned}
D_{ij} &\equiv \frac{1}{2}\overline{S}_i \Sigma^M \overline{\Sigma}^N S_i \Big(X_{jM} \frac{\partial}{\partial X_i^N} - X_{jN} \frac{\partial}{\partial X_i^M} \Big) && \sim \overline{S}_i S_i, \\
\widetilde{D}_{ij} &\equiv \overline{S}_i \mathbf{X}_j \overline{\Sigma}^N S_i \frac{\partial}{\partial X_j^N} + 2 I^{ij} \, S_{ia} \frac{\partial}{\partial S_{ja}} - 2 I^{ji} \, \overline{S}^a_i \frac{\partial}{\partial \overline{S}^a_j} && \sim \overline{S}_i S_i, \\
I^{ij} &\equiv \overline{S}_i S_j && \sim \overline{S}_i S_j, \\
\nabla_{ij} &\equiv [\mathbf{X}_i \overline{\mathbf{X}}_j]_a^b \, \frac{\partial^2}{\partial S_{i\,a} \, \partial \overline{S}^b_j} && \sim S_i^{-1} \overline{S}_j^{-1}.
\end{aligned} \tag{3.35}$$

[9]This relation is of course purely kinematic, it holds only at the level of tensor structures and does not hold at the level of the full correlator.

[10]We distinguish the operators $\mathbf{D}$ here and the operators $\mathbb{D}$ described in section 2.1 because acting on the seed tensor structures they generate different bases. The basis spanned by $\mathbf{D}$ is often called the differential basis.

[11]Notice a change in the normaliztion of the basic spinning differential operators compared to [145].

For $\Delta\ell = 2$ we have

$$
\begin{aligned}
d_{ij} &\equiv S_j \overline{X}_i \frac{\partial}{\partial \overline{S}_i} &\sim \overline{S}_i^{-1} S_j \\
\overline{d}_{ij} &\equiv \overline{S}_j X_i \frac{\partial}{\partial S_i} &\sim S_i^{-1} \overline{S}_j .
\end{aligned}
\tag{3.36}
$$

Note that for any differential operator $\mathbf{D}_{ij}$ we necessarily have $\Delta\ell$ even, since it has to preserve the total Fermi/Bose statistics of the pair of local operators.

The basic spinning differential operators described above carry the 4D scaling dimension according to (3.4), thus it is convenient to introduce an operator Ξ which formally shifts the 4D dimensions of external operators in a way that effectively makes the 4D scaling dimensions of $\mathbf{D}_{ij}$ vanish. The action of Ξ on basic spinning differential operators is defined as

$$
\Xi[D_{ij}] f_n = (D_{ij} f_n)\Big|_{\Delta_j \to \Delta_j + 1}, \quad \Xi[\widetilde{D}_{ij}] f_n = (\widetilde{D}_{ij} f_n)\Big|_{\Delta_i \to \Delta_i + 1}
\tag{3.37}
$$

and

$$
\Xi[op] f_n = (op\, f_n)\Big|_{\Delta_i \to \Delta_i + 1/2}\Big|_{\Delta_j \to \Delta_j + 1/2},
\tag{3.38}
$$

where op denotes any of the remaining spinning differential operators.[12] These formal shifts of course make sense only if the scaling dimensions appear as variables in f_n. The use of the dimension-shifting operator Ξ allows to keep the same scaling dimensions in the seed CPWs and the CPW related by (2.39).

The relevant functions in the package are `[opDEF, opDtEF, opdEF, opdbEF, opIEF, opNEF]` and Ξ.

3.2 Conformal Frame

For sufficiently complicated correlation functions one finds a lot of degeneracies in the embedding space construction of tensor structures. There exists an alternative construction [118, 126] which provides better control under degeneracies. More precisely, it reduces the problem of constructing tensor structures to the well studied problem of finding invariant tensors of orthogonal groups of small rank.

Our aim is to describe the correlation function $f_n(x, s, \overline{s})$ whose generic form is given in (2.11). The conformal symmetry relates the values of $f_n(x, s, \overline{s})$ at different values of x. There is a classical argument, usually applied to 4-point correlation functions, saying that it is sufficient to know only the value $f_n(x_{CF}, s, \overline{s})$ for some standard choices of x_{CF} such that all the other values of x can be obtained from some x_{CF} by a conformal transformation. This conformal transformation then allows one to compute $f_n(x, s, \overline{s})$ from $f_n(x_{CF}, s, \overline{s})$. The standard configurations x_{CF} are chosen in such a way that there are no conformal transformations relating two different standard configurations, so that the values $f_n(x_{CF}, s, \overline{s})$ can be specified independently. Following [126], we call the set of standard configurations x_{CF} the conformal frame (CF).

The usefulness of this construction lies in the fact that the values $f_n(x_{CF}, s, \overline{s})$ have to satisfy only a few constraints. In particular, these values have to be invariant only under the conformal transformations which do not change x_{CF} [126]. For the conformal group SO$(1, d+1)$ such conformal transformations form a group $\mathcal{G}_n$ which we call the "little group".[13] which

[12] The shift in the last formula can alternatively be implemented with multiplication by a factor $X_{ij}^{-1/2}$.

[13] The little group or the stabilizer group is defined as follows. Given a group G which acts on space X, the little (stabilizer) group H for an element $x \subset X$ is a subgroup of G which leaves the element x invariant.

is

$$\mathcal{G}_n = \begin{cases} \mathrm{SO}(1,1)\times\mathrm{SO}(d) & n=2, \\ \mathrm{SO}(d+2-m) & n\geq 3, \end{cases} \tag{3.39}$$

where $m=\min(n,d+2)$ for n-point functions in d dimensions.[14] For example, for 4-point functions in 4D it is $SO(2)\simeq U(1)$. One can already see a considerable simplification offered by this construction for 4-point functions in 4D, since the invariants of $SO(2)$ are extremely easy to classify.

We use the following choice for the conformal frame configurations x_{CF} for $n\geq 3$,

$$x_1^\mu = (0,0,0,0), \tag{3.40}$$
$$x_2^\mu = ((\bar{z}-z)/2,0,0,(z+\bar{z})/2), \tag{3.41}$$
$$x_3^\mu = (0,0,0,1), \tag{3.42}$$
$$x_4^\mu = (0,0,0,L), \tag{3.43}$$
$$x_5^\mu = (x_5^0,x_5^1,0,x_5^3), \tag{3.44}$$

where if $n=3$ we can set $z=\bar{z}=1/2$ and if we have more than 5 operators, the unspecified positions $x_{\geq 6}$ are completely unconstrained.

Here L is a fixed number, and we always take the limit $L\to+\infty$ to place the corresponding operator "at inifinity". In this limit one should use the rescaled operator $\mathcal{O}_4$

$$\mathcal{O}_4 \to \mathcal{O}_4\, L^{2\Delta_4} \tag{3.45}$$

inside all correlators to get a finite and non-zero result.

The variables z, $\bar{z}$, x_5^0, x_5^1, x_5^3 and the 4-vectors x_6, x_7, ... are the coordinates on the conformal frame and thus are essentially the conformal cross-ratios. Note that we have 2 conformal cross-ratios for 4 points, and $4\,n-15$ for n points with $n\geq 5$. Notice also that for 4-point functions the analytic continuation with $z=\bar{z}^*$ corresponds to Euclidean kinematics. It is easy to check that there are no conformal generators which take the conformal frame configuration (3.40) - (3.44) to another nearby conformal frame configuration.

3.2.1 Construction of Tensor Structures

Three-point Functions As shown in appendix A.5, an independent basis for general 3-point tensor structures is relatively easy to construct in EF, and there is no direct need for the conformal frame construction. Nonetheless, in this section we employ the CF to construct 3-point tensor structures in order to illustrate how the formalism works in a familiar case.[15]

The little group algebra $\mathfrak{so}(1,2)$ which fixes the points x_1,x_2,x_3 is defined by the following generators

$$M^{01},\quad M^{02},\quad M^{12}, \tag{3.46}$$

see appendix A.1 for details. According to our conventions, the corresponding generators acting on polarizations s_α are

$$\mathcal{S}^{01} = -\frac{1}{2}\sigma^1,\quad \mathcal{S}^{02} = -\frac{1}{2}\sigma^2,\quad \mathcal{S}^{12} = \frac{i}{2}\sigma^3, \tag{3.47}$$

[14] For $n\geq 3$ and generic x. The little group is trivial for $n\geq d+2$.

[15] The CF construction of 3-point functions is not implemented in the package.

and the generators acting on $\bar{s}^{\dot{\alpha}}$ are

$$\overline{\mathcal{S}}^{01} = \frac{1}{2}\sigma^1, \quad \overline{\mathcal{S}}^{02} = \frac{1}{2}\sigma^2, \quad \overline{\mathcal{S}}^{12} = \frac{i}{2}\sigma^3. \tag{3.48}$$

It is easy to see that if we introduce $t_\alpha \equiv s_\alpha$ and $\tilde{t}_\alpha \equiv \sigma^3_{\alpha\dot{\beta}}\bar{s}^{\dot{\beta}}$, then t and $\tilde{t}$ transform in the same representation of $\mathfrak{so}(1,2)$.

General 3-point structures are put in one-to-one correspondence with the $\mathfrak{so}(1,2) \simeq \mathfrak{su}(2)$ conformal frame invariants built out of t_i and $\tilde{t}_i$, $i = 1, 2, 3$. This gives an explicit implementation of the rule [118, 125, 126] which states that 3-point structures correspond to the invariants of $SO(d-1) = SO(3)$ group

$$\left((\ell_1, \bar{\ell}_1) \otimes (\ell_2, \bar{\ell}_2) \otimes (\ell_3, \bar{\ell}_3)\right)^{SO(3)} = \left(\ell_1 \otimes \bar{\ell}_1 \otimes \ell_2 \otimes \bar{\ell}_2 \otimes \ell_3 \otimes \bar{\ell}_3\right)^{SO(3)}. \tag{3.49}$$

Using this rule, we can immediately build independent bases of 3-point structures, for example by first computing the tensor product decompositions

$$\ell_i \otimes \bar{\ell}_i = \bigoplus_{j_i = |\ell_i - \bar{\ell}_i|}^{\ell_i + \bar{\ell}_i} j_i, \qquad (j_i + \ell_i + \bar{\ell}_i \text{ even}) \tag{3.50}$$

and then for every set of j_i constructing the unique singlet in $j_1 \otimes j_2 \otimes j_3$ when it exists.

A more direct way, which does not however automatically avoid degeneracies, is to use the basic building blocks for $SO(3)$ invariants, which are the contractions of the form $t_i^\alpha t_{j\,\alpha}$, $t_i^\alpha \tilde{t}_{j\,\alpha}$ and $\tilde{t}_i^\alpha \tilde{t}_{j\,\alpha}$. It is then straightforward to establish the correspondence with the embedding formalism invariants

$$I^{ij} \propto \tilde{t}_i t_j, \quad J^i_{jk} \propto \tilde{t}_i t_i, \quad K^{ij}_k \propto t_i t_j, \quad \overline{K}^{ij}_k \propto \tilde{t}_i \tilde{t}_j, \tag{3.51}$$

where it is understood that i, j, k are all distinct. Up to the coefficients, this dictionary is fixed completely by matching the degrees of s and $\bar{s}$ on each side.

Correspondingly, as in the embedding space formalism, we have relations between these building blocks, which now come from the Schouten identity[16]

$$(AB)C_\alpha + (BC)A_\alpha + (CA)B_\alpha = 0. \tag{3.52}$$

For example we can take $A = t_i$, $B = t_k$, $C = \tilde{t}_j$ and contract (3.52) with $\tilde{t}_k$ to find

$$(t_i t_k)(\tilde{t}_j \tilde{t}_k) + (t_k \tilde{t}_j)(t_i \tilde{t}_k) + (\tilde{t}_j t_i)(t_k \tilde{t}_k) = 0, \tag{3.53}$$

which corresponds via the dictionary (3.51) to an identity of the form

$$\# K^{ik}_j \overline{K}^{jk}_i + \# I^{jk} I^{ki} + \# I^{ji} J^k_{ij} = 0. \tag{3.54}$$

This gives precisely the structure of the relation (A.122). We thus effectively reproduce the EF construction.

Finally, let us briefly comment on the action of $\mathcal{P}$ in the 3-point conformal frame. The parity transformation of operators (A.26) induces the following transformation of polarizations

$$s_\alpha \to i\bar{s}^{\dot{\alpha}}, \quad \bar{s}^{\dot{\beta}} \to i s_\beta \quad \Longrightarrow \quad t \to i\sigma^3 \tilde{t}, \quad \tilde{t} \to i\sigma^3 t. \tag{3.55}$$

[16] Which itself follows from contracting $\epsilon^{\beta\gamma}$ with the identity $A_{[\alpha} B_\beta C_{\gamma]} = 0$ valid for two-component spinors.

The full parity transformation does not however preserve the conformal frame since it reflects all three spatial axes and thus moves the points x_2 and x_3. We can reproduce the correct parity action in the conformal frame by supplementing the full parity transformation with $i\pi$ boost in the 03 plane given by $e^{-i\pi\mathcal{S}^{03}} = i\sigma_3$ on t and by $\sigma^3 e^{-i\pi\mathcal{S}^{03}}\sigma^3 = -i\sigma_3$ on $\tilde{t}$. This leads to

$$t \to \tilde{t}, \quad \tilde{t} \to -t. \tag{3.56}$$

Note that according to (3.56) the transformations properties of (3.51) under parity match precisely the ones found in (3.19) - (3.21).

Four-point Functions In the $n=4$ case the little group algebra $\mathfrak{so}(2) \simeq \mathfrak{u}(1)$ which fixes the points x_1, x_2, x_3, x_4 is given by the generator

$$M^{12}. \tag{3.57}$$

Note that the algebra $\mathfrak{so}(2)$ is a subalgebra of the 3-point little group algebra $\mathfrak{so}(1,2)$ discussed above. According to (3.47), its action on both t and $\tilde{t}$ is given by

$$\mathcal{S}^{12} = \frac{i}{2}\sigma^3. \tag{3.58}$$

This generator acts diagonally on t and $\tilde{t}$, so that we can decompose

$$s_\alpha \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \bar{s}_{\dot\beta} \equiv \begin{pmatrix} \bar\xi \\ \bar\eta \end{pmatrix} \quad \Longrightarrow \quad t \equiv s_\alpha = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \tilde{t} \equiv \sigma^3_{\alpha\dot\beta}\bar{s}^{\dot\beta} = \begin{pmatrix} \bar\eta \\ \bar\xi \end{pmatrix}. \tag{3.59}$$

Note that our convention $\bar{s}_{\dot\alpha} = (s_\alpha)^*$ implies that $\bar\xi = \xi^*$ and $\bar\eta = \eta^*$. Appropriately defining the $\mathfrak{u}(1)$ charge Q we can say that

$$Q[\xi] = Q[\bar\eta] = +1 \quad \textbf{and} \quad Q[\eta] = Q[\bar\xi] = -1. \tag{3.60}$$

Tensor structures of 4-point functions are just the products of $\xi, \bar\xi, \eta, \bar\eta$ of total charge $Q=0$. These are given by [CF4pt,n4ListStructures]

$$\begin{gathered}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 & \bar{q}_4 \end{bmatrix} \equiv \prod_{i=1}^4 \xi_i^{-q_i+\ell_i/2}\eta_i^{q_i+\ell_i/2}\bar\xi_i^{-\bar{q}_1+\bar\ell_i/2}\bar\eta_i^{\bar{q}_i+\bar\ell_i/2}, \\ q_i \in \{-\ell_i/2, \ldots, \ell_i/2\},\, \bar{q}_i \in \{-\bar\ell_i/2, \ldots, \bar\ell_i/2\},\end{gathered} \tag{3.61}$$

subject to

$$\sum_{i=1}^4 (q_i - \bar{q}_i) = 0. \tag{3.62}$$

It is clear from the construction that these 4-point structures are all independent, i.e. there are no relations between them. It is in contrast with the embedding space formalism, where there are a lot of relations between various 4 point building blocks.

As a simple example, consider a 4-point function of a $(1,0)$ fermion at position 1, a $(0,1)$ fermion at position 2 and two scalars at position 3 and 4. The allowed 4-point tensor structures are then

$$\begin{bmatrix} +\frac{1}{2} & 0 & 0 & 0 \\ 0 & +\frac{1}{2} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}. \tag{3.63}$$

To compute the action of space parity, we need to supplement the full spatial parity (3.55) with a π rotation in, say, the 13 plane in order to make sure that parity preserves the

4-point conformal frame (3.40) - (3.43). In this case the combined transformation is simply a reflection in the 2'nd coordinate direction. It is easy to compute that this gives the action

$$\xi \to -i\overline{\xi}, \quad \overline{\xi} \to i\xi, \quad \eta \to -i\overline{\eta}, \quad \overline{\eta} \to i\eta. \tag{3.64}$$

Note that this does not commute with the action of $\mathfrak{u}(1)$ since the choice of the 13 plane was arbitrary – we could have also chosen the 23 plane, and $\mathfrak{u}(1)$ rotates between these two choices. It is only important that this reflection reverses the charges of $\mathfrak{u}(1)$ and thus maps invariants into invariants.

From (3.64) we find that the parity acts as

$$\mathcal{P}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix} = i^{-\sum_i \ell_i - \overline{\ell}_i} \begin{bmatrix} \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}. \tag{3.65}$$

From the definition (3.61) we also immediately find the complex conjugation rule

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix}^* = \begin{bmatrix} \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}. \tag{3.66}$$

According to (A.36), by combining these two transformations we find the action of time reversal

$$\mathcal{T}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix} = i^{\sum_i \ell_i - \overline{\ell}_i} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix}. \tag{3.67}$$

Five-point Functions and Higher In the $n \geq 5$ case there are no conformal generators which fix the conformal frame. It means that all $\xi, \overline{\xi}, \eta, \overline{\eta}$ are invariant by themselves.[17] This allows us to construct the n-point tensor structures

$$\begin{bmatrix} q_1 & q_2 & \dots & q_n \\ \overline{q}_1 & \overline{q}_2 & \dots & \overline{q}_n \end{bmatrix} \equiv \prod_{i=1}^{n} \xi_i^{-q_i+\ell_i/2} \eta_i^{q_i+\ell_i/2} \overline{\xi}_i^{-\overline{q}_1+\overline{\ell}_i/2} \overline{\eta}_i^{\overline{q}_i+\overline{\ell}_i/2}, \tag{3.68}$$

with the only restriction

$$q_i \in \{-\ell_i/2, \dots \ell_i/2\}, \quad \overline{q}_i \in \{-\overline{\ell}_i/2, \dots \overline{\ell}_i/2\}. \tag{3.69}$$

3.2.2 Relation with the EF

In practical applications, 3- and 4-point functions are the most important objects. It is possible to treat 3-point functions in the CF or the EF. Since the latter is explicitly covariant, it is often more convenient. On the other hand, 4-point functions are treated most easily in the conformal frame approach. This creates a somewhat unfortunate situation when we have two formalisms for closely related objects. To remedy this, let us discuss how to go back and forth between the EF and the CF.

Embedding formalism to conformal frame It is relatively straightforward to find the map `[toConformalFrame]` from the embedding formalism tensor structures to the conformal frame ones. First one needs to project the 6D elements to the 4D ones and then to substitute the appropriate values of coordinates according to the choice of the conformal frame.

[17] More precisely, there is still the $\mathbb{Z}_2$ kernel of the projection $Spin(1,3) \to SO(1,3)$, which gives the selection rule that the full correlator should be bosonic (in this sense $\xi, \overline{\xi}, \eta, \overline{\eta}$ are not individually invariant).

For 6D coordinates according to (A.65) and the definition of the conformal frame (3.40) - (3.43) one has

$$\begin{aligned} X_1 &= (0,0,0,0,1,0), \\ X_2 &= ((\bar{z}-z)/2, 0, 0, (z+\bar{z})/2, 1, -z\bar{z}), \\ X_3 &= (0,0,0,1,1,-1), \\ X_4 &= (0,0,0,L,1,-L^2), \end{aligned} \tag{3.70}$$

and for the 6D polarizations according to (A.71) one has

$$(S_i)_a = \begin{pmatrix} (s_i)_\alpha \\ -x_i^\mu \bar{\sigma}_\mu^{\dot{\alpha}\beta}(s_i)_\beta \end{pmatrix}, \quad (\overline{S}_i)^a = \begin{pmatrix} (\bar{s}_i)_{\dot{\beta}} \bar{\sigma}_\mu^{\dot{\beta}\alpha} x_i^\mu \\ (\bar{s}_i)_{\dot{\alpha}} \end{pmatrix}. \tag{3.71}$$

In the last expression it is understood that all the coordinates x belong to the conformal frame x_{CF} (3.40) - (3.43).

The final step is to perform the rescaling (3.45) and to take the limit $L \to +\infty$. There is a very neat way to do it by recalling that 6D operators O according to (3.3) are homogeneous in 6D coordinates and 6D polarizations, thus

$$O(S_4, \overline{S}_4, X_4) L^{2\Delta_4} = O(S_4, \overline{S}_4, X_4) L^{2\kappa_O - \ell_4 - \bar{\ell}_4} = O(S_4/L, \overline{S}_4/L, X_4/L^2). \tag{3.72}$$

It is then clear that the final step is equivalent to the following substitution of the 6D coordinates at the 4th position

$$X_4 \to \lim_{L\to+\infty} X_4/L^2 = (0,0,0,0,0,-1) \tag{3.73}$$

and for the 6D polarizations

$$(S_4)_a \to \lim_{L\to+\infty} (S_4)_a/L = \begin{pmatrix} 0 \\ -\bar{\sigma}_3^{\dot{\alpha}\beta}(s_4)_\beta \end{pmatrix}, \quad (\overline{S}_4)^a \to \lim_{L\to+\infty} (\overline{S}_4)^a/L = \begin{pmatrix} (\bar{s}_4)_{\dot{\beta}} \bar{\sigma}_3^{\dot{\beta}\alpha} \\ 0 \end{pmatrix}. \tag{3.74}$$

Conformal frame to embedding formalism As discussed in section 3.2.1, 4-point tensor structures are given by products of $\xi_i, \bar{\xi}_i, \eta_i, \bar{\eta}_i$ with vanishing total $U(1)$ charge. It is easy to convince oneself that any such product can be represented (not uniquely) by a product of $U(1)$-invariant bilinears

$$\bar{\xi}_i \xi_j, \quad \bar{\eta}_i \eta_j, \quad \xi_i \eta_j, \quad \bar{\xi}_i \bar{\eta}_j, \tag{3.75}$$

where $i, j = 1 \dots 4$. For $n \geq 5$-point a general tensor structure is still represented by a product of bilinears, see footnote 17, but since there is no $U(1)$-invariance condition, the following set of bilinears should also be taken into account

$$\xi_i \xi_j, \quad \eta_i \eta_j, \quad \bar{\xi}_i \bar{\xi}_j, \quad \bar{\eta}_i \bar{\eta}_j, \quad \bar{\eta}_i \xi_j, \quad \bar{\xi}_i \eta_j, \tag{3.76}$$

where $i, j = 1 \dots n$.

These bilinears themselves are tensor structures with low spin. Noticing that the EF invariants are also naturally bilinears in polarizations we can write a corresponding set of EF invariants with the same spin signatures. Translating these invariants to conformal frame via the procedure described above `[toConformalFrame]`, one can then invert the result and express the bilinears (3.75) and (3.76) in terms of covariant expressions. We could call this procedure covariantization `[toEmbeddingFormalism]`. The basis of EF structures is over-complete so the inversion procedure is ambiguous and one is free to choose one out of many options.

Since there is a finite number of bilinears (3.75) and (3.76) there will be a finite number of covariant tensor structures they can be expressed in terms of after the covariantization procedure. It is then very easy to see that one needs only the class of $n = 4$ tensor structures to cover all the bilinears (3.75) and the class of $n = 5$ tensor structures to cover all the bilinears (3.76).

The ambiguity of the inversion procedure mentioned above is related to the linear relations between EF structures. Non-linear relations between EF structures arise due to the tautologies such as

$$(\bar{\xi}_i\xi_j)(\bar{\eta}_k\eta_l) = (\bar{\xi}_i\bar{\eta}_k)(\xi_j\eta_l). \tag{3.77}$$

This observation in principle allows to classify all relations between $n \geq 4$ EF invariants.

Example. By going to the conformal frame we get

$$\hat{J}^1_{23} = \frac{z}{z-1}\,\bar{\xi}_1\xi_1 - \frac{\bar{z}}{\bar{z}-1}\,\bar{\eta}_1\eta_1, \quad \hat{J}^1_{24} = -z\,\bar{\xi}_1\xi_1 + \bar{z}\,\bar{\eta}_1\eta_1, \quad \hat{J}^1_{34} = -\bar{\xi}_1\xi_1 + \bar{\eta}_1\eta_1. \tag{3.78}$$

Inverting these relation one gets

$$\bar{\xi}_1\xi_1 = -\frac{z-1}{z\,(z-\bar{z})}\Big((\bar{z}-1)\,\hat{J}^1_{23} + \hat{J}^1_{24}\Big), \quad \bar{\eta}_1\eta_1 = -\frac{\bar{z}-1}{\bar{z}\,(z-\bar{z})}\Big((z-1)\,\hat{J}^1_{23} + \hat{J}^1_{24}\Big). \tag{3.79}$$

We see right away that the invariants J^1_{23}, J^1_{24} and J^1_{34} must be dependent. One can easily get a relation between them by plugging (3.79) to the third expression (3.78). The obtained relation will match perfectly the linear relation (3.26).

Note that there is a factor $1/(z-\bar{z})$ in (3.79), which suggests that the structure $\bar{\xi}_1\xi_1$ blows up at $z = \bar{z}$. This is not the case simply by the definition of ξ and $\bar{\xi}$; instead, it is the combination of structures on the right hand side which develops a zero giving a finite value at $z = \bar{z}$. However, this value will depend on the way the limit is taken. This is related to the enhancement of the little group from $U(1) = SO(2)$ to $SO(1,2)$ at $z = \bar{z}$. At $z = \bar{z}$ it is no longer true that $\bar{\xi}_1\xi_1$ is a little group invariant. This enhancement implies certain boundary conditions for the functions which multiply the conformal frame invariants. See appendix A of [126] for a detailed discussion of this point.

3.2.3 Differentiation in the Conformal Frame

Now we would like to understand how to implement the action of the embedding formalism differential operators such as (3.35) and (3.36) directly in the conformal frame. We need to make two steps. First, to understand the form of these differential operators in 4D space. This is done by using the projection of 6D differential operators to 4D given in appendix A.2. Second, to understand how to act with 4D differential operators directly in the conformal frame. We focus on this step in the remainder of this section. For simplicity, we restrict the discussion to the most important case of four points.

A correlation function in the conformal frame is obtained by restricting its coordinates x to the conformal frame configurations x_{CF}. The action of the derivatives $\partial/\partial s$ and $\partial/\partial\bar{s}$ in polarizations on this correlation function is straightforward, since nothing happens to polarizations during this restriction. The only non-trivial part is the coordinate derivatives $\partial/\partial x_i$: in the conformal frame a correlator only depends on the variables z and $\bar{z}$ which describe two degrees of freedom of the second operator and it is not immediately obvious how to take say the $\partial/\partial x_1$ derivatives.

The resolution is to recall that 4-point functions according to (2.10) are invariant under generic conformal transformation spanned by 15 conformal generators L_{MN}. By using (A.57) one can see that it is equivalent to 15 differential equations

$$(\mathfrak{L}_{1\,MN} + \mathfrak{L}_{2\,MN} + \mathfrak{L}_{3\,MN} + \mathfrak{L}_{4\,MN})\, f_4(x_i, s_i, \bar{s}_i) = 0. \tag{3.80}$$

The differential operators $\mathfrak{L}_{i\,MN}$ defined in (A.58) together with (A.76) and (A.77) are given by linear combinations of derivatives $\partial/\partial x_i$, $\partial/\partial s_i$ and $\partial/\partial \bar{s}_i$. Out of 15 differential equations (3.80) one equation (for L_{12}) expresses the little group invariance under rotations in the 12 plane and thus when restricted to the 4-point conformal frame (3.40) - (3.43) does not contain derivatives $\partial/\partial x_i$. The remaining 14 equations allow to express the 14 unknown derivatives $\partial/\partial x_i^\mu$ restricted to the conformal frame in terms of $\partial/\partial x_2^0$, $\partial/\partial x_2^3$, $\partial/\partial s_i$ and $\partial/\partial \bar{s}_i$. Higher-order derivatives can be obtained in a similar way by differentiating (3.80).

Computation of general derivatives can be cumbersome, but in practice it is easily automated with `Mathematica`. We provide a conformal frame implementation of the differential operators (3.35) - (3.36) `[opD4D, opDt4D, opd4D, opdb4D, opI4D, opN4D]` as well as of the quadratic Casimir operator `[opCasimir24D]` acting on 4-point functions. As a simple example (although it does not require differentiation in x), we display here the action of ∇_{12} on a generic conformal frame structure

$$\begin{aligned}\nabla_{12}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 & \bar{q}_4 \end{bmatrix} g(z,\bar{z}) = &-\frac{(\ell_1+2q_1)(\bar{\ell}_2+2\bar{q}_2)}{4}\begin{bmatrix} q_1-\frac{1}{2} & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2-\frac{1}{2} & \bar{q}_3 & \bar{q}_4 \end{bmatrix} z g(z,\bar{z}) \\ &+\frac{(\ell_1-2q_1)(\bar{\ell}_2-2\bar{q}_2)}{4}\begin{bmatrix} q_1+\frac{1}{2} & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2+\frac{1}{2} & \bar{q}_3 & \bar{q}_4 \end{bmatrix} \bar{z} g(z,\bar{z}).\end{aligned} \tag{3.81}$$

Other operators, e.g. (3.35), give rise to more complicated expressions which however can still be efficiently applied to the seed CPWs.

Chapter 4

Solving the Casimir Equation

We would like to address now the problem of computing the seed conformal blocks $H_e^{(p)}(z,\bar{z})$ and $\overline{H}_e^{(p)}(z,\bar{z})$ defined in (2.43). We start by deriving a system of linear partial differential equations for them. We then solve this system by using a proper ansatz found in appendix B.1.

4.1 The System of Casimir Equations

The second order casimir equation (2.45) for the seed CPWs reads as

$$(\mathfrak{C}_2 - E_\ell^p)\, W_{seed}(p) = 0, \tag{4.1}$$

where the casimir eigenvalue (A.134) is

$$E_\ell^p = \Delta\,(\Delta-4) + \ell^2 + (2+p)(\ell + \frac{p}{2}). \tag{4.2}$$

The identical equation holds for the dual seed CPWs.[1] Splitting up the CPW in (4.1) into conformal blocks one can find the system of Casimir equations for conformal blocks H

$$\mathcal{K}_4 \sum_{e=0}^{p} Cas_e^{(p)}(H) \big[I^{42}\big]^e \big[I^{42}_{31}\big]^{p-e} = 0 \quad \Rightarrow \quad Cas_e^{(p)}(H) = 0, \tag{4.3}$$

where $Cas_e^{(p)}(H)$ are the $p+1$ Casimir equations. An identical system of equations holds for the dual seed blocks $\overline{H}$. The system $Cas_e^{(p)}(H)$ can be cast into the following remarkably compact form

$$\begin{aligned} Cas_e^{(p)}(H) = \Big(\Delta_{2+p}^{(a_e^{(p)},b_e^{(p)};\, c_e^{(p)})} - \frac{1}{2}\left(E_\ell^p - \varepsilon_e^p\right)\Big)\, H_e^{(p)} + \\ A_e^p\, z\bar{z}\, L(a_{e-1}^{(p)})\, H_{e-1}^{(p)} + B_e\; L(b_{e+1}^{(p)})\, H_{e+1}^{(p)} = 0\,, \end{aligned} \tag{4.4}$$

where the coefficients are given by

$$\varepsilon_e^p \equiv \tfrac{3}{4}\, p^2 - (1+2e)\, p + 2e\,(2+e), \qquad A_e^p \equiv 2(p-e+1), \qquad B_e \equiv \frac{e+1}{2}. \tag{4.5}$$

The differential operators entering (4.5) are split into linear operators and second order operators. The second order operator is

$$\Delta_\epsilon^{(a,b;c)} = D_z^{(a,b;c)} + D_{\bar{z}}^{(a,b;c)} + \epsilon\, \frac{z\bar{z}}{z-\bar{z}} \Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)\,. \tag{4.6}$$

[1] seed and the dual seed blocks are solutions to the same system of equations, they differ however by their asymptotic behavior.

It consists of two separate hypergeometric differential operator

$$D_x^{(a,b;c)} \equiv x^2(1-x)\partial_x^2 - \big((a+b+1)x^2 - cz\big)\partial_x - abx \tag{4.7}$$

and a mixing part. The linear operator is given by

$$L(\mu) \equiv -\frac{1}{z-\bar{z}}\left(z(1-z)\partial_z - \bar{z}(1-\bar{z})\partial_{\bar{z}}\right) + \mu. \tag{4.8}$$

The parameters describing the conformal blocks are defined as

$$a^{(p)} \equiv -\frac{\Delta_1 - \Delta_2 - p/2}{2}, \quad b^{(p)} \equiv +\frac{\Delta_3 - \Delta_4 - p/2}{2} \tag{4.9}$$

and

$$a_e^{(p)} \equiv a^{(p)}, \;\; b_e^{(p)} \equiv b^{(p)} + (p-e), \;\; c_e^{(p)} \equiv p - e. \tag{4.10}$$

We might sometimes drop the upper index (p) to simplify notation.

In (4.4) it is understood that $H_{-1}^{(p)} = H_{p+1}^{(p)} = 0$. A remarkable property of the Casimir system (4.4) is that, for each given e and p, at most three conformal blocks mix with each other in a sort of "nearest-neighbour interaction": H_e mixes only with H_{e+1} and H_{e-1}. The Casimir equations at the "boundaries" $Cas_0^{(p)}$ and $Cas_p^{(p)}$ involve just two blocks. For $p=0$, the second and third terms in (4.4) vanish and the system trivially reduces to the single equation found by Dolan and Osborn.

4.2 Solving the System of Casimir Equations

The goal of this section is to find an explicit form of the conformal blocks $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ by solving the Casimir system (4.4). In doing it we adopt and expand the methods introduced by Dolan and Osborn in refs. [128, 129] to obtain the 6D scalar conformal blocks.

Before jumping into details let us outline the main logical steps of our derivation. In section 4.2.1 we start with an ansatz for the seed and dual seed conformal blocks motivated by the computations done in shadow formalism given in appendix B.1. Plugging this ansatz into (4.4) in section 4.2.2 we reduce the problem of solving the system of linear partial differential equations of second order in two variables to a system of linear algebraic equations for the unknown coefficients entering the ansatz. Then in section 4.2.3 we show that the non-zero coefficients in the ansatz admit a geometric interpretation. They form a two-dimensional lattice with an octagon shape structure. This interpretation allows us to precisely predict which non-zero coefficients enter our ansatz for any value of p. Finally, in section 4.2.4 we show that the linear algebraic system admits a recursive solution and we discuss the complexity of deriving the full solutions for higher values of p. In section 4.2.5 we draw an analogy between our seed bloks $H_e^{(p)}$, $\overline{H}_e^{(p)}$ and the symmetric scalar blocks in d even dimensions.

4.2.1 The Ansatz

The key ingredient of the ansatz is the function $k_\rho^{(a,b;c)}(z)$ defined as[2]

$$k_\rho^{(a,b;c)}(z) \equiv z^\rho \, {}_2F_1(a+\rho,\, b+\rho;\, c+2\rho;\, z)\,. \tag{4.11}$$

[2]We adopt here the notation first used in ref.[27] for this function, but notice the slight difference in the definition: $k_\rho^{there} = k_{\rho/2}^{here}$.

which is an eigenfunction of the hyper-geometric like operator (4.7)

$$D_z^{(a,b;c)}\, k_\rho^{(a,b;c)}(z) = \rho\,(\rho+c-1)\; k_\rho^{(a,b;c)}(z). \tag{4.12}$$

Using (4.12) one can define an eigenfunction of the operator $\Delta_0^{(a,b;c)}$ defined in (4.6) as the product of two k-functions

$$\begin{aligned} \mathcal{F}_{\rho_1,\,\rho_2}^{(a,b;c)}(z,\bar z) &\equiv k_{\rho_1}^{(a,b;c)}(z) k_{\rho_2}^{(a,b;c)}(\bar z), && (4.13)\\ \mathcal{F}_{\rho_1,\,\rho_2}^{\pm\,(a,b;c)}(z,\bar z) &\equiv \mathcal{F}_{\rho_1,\,\rho_2}^{(a,b;c)}(z,\bar z) \pm \mathcal{F}_{\rho_1,\,\rho_2}^{(a,b;c)}(\bar z,z). && (4.14) \end{aligned}$$

These functions played an important role in ref.[128] for the derivation of an analytic closed expression of the scalar CBs in even space-time dimensions. In our case, the situation is much more complicated, because we have different blocks appearing in the Casimir equations. We notice, however, that the second order operator Δ in each equation $Cas_e^{(p)}$ acts only on the block $H_e^{(p)}$, while the blocks $H_{e-1}^{(p)}$ and $H_{e+1}^{(p)}$ are multiplied by first order operators only. Since, as we will shortly see, first order derivatives and factors of z and $\bar z$ acting on the functions $\mathcal{F}$ can always be expressed in terms of functions $\mathcal{F}$ with shifted parameters, a reasonable ansatz for the CBs is to take each H_e proportional to a sum of functions of the kind $\mathcal{F}_{\rho_1,\,\rho_2}^{(a_e,b_e;c_e)}(z,\bar z)$ for some ρ_1 and ρ_2. Taking also into account (B.34), the form of the ansatz for the blocks $H_e^{(p)}$ should be[3]

$$H_e^{(p)}(z,\bar z) = \left(\frac{z\bar z}{z-\bar z}\right)^{2\,p+1} g_e^{(p)}(z,\bar z), \qquad g_e^{(p)}(z,\bar z) \equiv \sum_{m,n} c_{m,n}^{e} \mathcal{F}_{\rho_1+m,\,\rho_2+n}^{-\,(a_e^{(p)},b_e^{(p)};c_e^{(p)})}(z,\bar z), \tag{4.15}$$

where $c_{m,n}^e$ are coefficients to be determined and the sum over the two integers m and n in (4.15) is so far unspecified. Notice that all the functions $\mathcal{F}$ entering the sum over m and n have the same values of $a_e^{(p)}$, $b_e^{(p)}$ and $c_e^{(p)}$.

4.2.2 Reduction to a Linear System

The eigenfunctions $\mathcal{F}_{\rho_1,\,\rho_2}^{\pm\,(a,b;c)}(z,\bar z)$ have several properties that would allow us to find a solution to the system (4.4). In order to exploit such properties, we first have to express the system (4.4) for $H_e^{(p)}$ in terms of the functions $g_e^{(p)}(z,\bar z)$ defined in (4.15). We plug the ansatz (4.15) into (4.4) and use the following relations

$$\begin{aligned} \Delta_\epsilon^{(a,b;c)}\Big(\frac{z\bar z}{z-\bar z}\Big)^k &= \Big(\frac{z\bar z}{z-\bar z}\Big)^k\left(\Delta_{\epsilon-2k}^{(a,b;c)} + k\,(k-\epsilon+c-1) - k\,(k-\epsilon+1)\,\frac{z\bar z(z+\bar z)-2z\bar z}{(z-\bar z)^2}\right),\\ L(\mu)\Big(\frac{z\bar z}{z-\bar z}\Big)^k &= \Big(\frac{z\bar z}{z-\bar z}\Big)^k\left(L(\mu) + k\,\frac{z+\bar z-2z\bar z}{(z-\bar z)^2}\right), \end{aligned} \tag{4.16}$$

to obtain the system of Casimir equations for $g_e^{(p)}$:

$$\widetilde{Cas}_e^{(p)}(g) \equiv Cas^0\, g_e^{(p)} + Cas^+\, g_{e+1}^{(p)} + Cas^-\, g_{e-1}^{(p)} = 0\,. \tag{4.17}$$

We have split each Casimir equation in terms of three differential operators Cas^0, Cas^+, Cas^-, that act on $g_e^{(p)}$, $g_{e+1}^{(p)}$ and $g_{e-1}^{(p)}$, respectively. In order to avoid cluttering, we have

[3] Recall that the conformal blocks are even under $z \leftrightarrow \bar z$ exchange, that leaves u and v unchanged.

omitted the obvious e and p dependences of such operators. Their explicit form is as follows:

$$
\begin{aligned}
Cas^0 &= \left(\frac{z-\bar z}{z\bar z}\right)^2\left(\Delta_0^{(a_e,b_e;c_e)} + (1+2p)(2p-2-e) - \frac{1}{2}\left(E^p_\ell - \varepsilon^p_e\right)\right) \\
&\quad -3p\,\frac{z-\bar z}{z\bar z}\times\Big((1-z)\partial_z - (1-\bar z)\partial_{\bar z}\Big) - p\,(1+2p)\,\frac{z+\bar z-2}{z\bar z}, && (4.18)\\
Cas^+ &= B_e\,\frac{z-\bar z}{z\bar z}\times\frac{z-\bar z}{z\bar z}L(b_{e+1}) + (1+2p)\,B_e\,\frac{z+\bar z-2z\bar z}{z\bar z}\frac{1}{z\bar z}, && (4.19)\\
Cas^- &= A^p_e\,\frac{z-\bar z}{z\bar z}\times(z-\bar z)L(a_{e-1}) + (1+2p)\,A^p_e\,\frac{z+\bar z-2z\bar z}{z\bar z}. && (4.20)
\end{aligned}
$$

Notice that the action of $\Delta_0^{(a_e,b_e;c_e)}$ in (4.18) on $g_e^{(p)}$ is trivial and gives just the sum of the eigenvalues of the $\mathcal{F}^{-\,(a,b;c)}_{\rho_1,\rho_2}(z,\bar z)$ entering $g_e^{(p)}$. It is clear from the form of the ansatz (4.15) that the system (4.17) involves three different kinds of functions $\mathcal{F}^-$, with different values of a, b and c (actually only b and c differ, recall (4.10)).

Using properties of hypergeometric functions, however, we can bring the Casimir system (4.17) into an algebraic system involving functions $\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+r,\,\rho_2+t}(z,\bar z)$ only, with different values of r and t, but crucially with the same values of a_e, b_e and c_e. In order to do that, it is useful to interpret each of the terms entering the definitions of Cas^0, Cas^+ and Cas^- as an operator acting on the functions $\mathcal{F}^-$ shifting their parameters. Their action can be reconstructed from the more fundamental operators provided in the appendix B.2. For each function $\mathcal{F}^-$ appearing in the ansatz (4.15), we have

$$
\begin{aligned}
Cas^0\,\mathcal{F}^{-(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar z) &= \sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar z)\,, && (4.21)\\
Cas^+\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar z) &= \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b+1;c+1)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar z)\,, && (4.22)\\
Cas^-\,\mathcal{F}^{-\,(a,b;c)}_{\rho_1+m,\,\rho_2+n}(z,\bar z) &= \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m,n)\,\mathcal{F}^{-\,(a,b-1;c-1)}_{\rho_1+m+r,\,\rho_2+n+t}(z,\bar z)\,, && (4.23)
\end{aligned}
$$

where A^0, A^- and A^+ are coefficients that in general depend on all the parameters involved: a, b, Δ, ℓ, e and p but not on z and $\bar z$, namely they are just constants. For future purposes, in (4.21)-(4.23) we have only made explicit the dependence of A^0, A^- and A^+ on the integers m and n. The sum over (r,t) in each of the above terms runs over a given set of pairs of integers. We report in fig. 4.1 the values of (r,t) spanned in each of the three regions $\mathcal{R}_0$, $\mathcal{R}_+$ and $\mathcal{R}_-$. We do not report the explicit and quite lengthy expression of the coefficients $A^0_{r,t}$, $A^+_{r,t}$ and $A^-_{r,t}$, but we refer the reader again to appendix B.2 where we provide all the necessary relations needed to derive them. Using (4.15) and (4.21)-(4.23), the Casimir system (4.17) can be rewritten in terms of the functions $\mathcal{F}^-$ only, with the same set of coefficients a_e, b_e and c_e:[4]

$$
\sum_{m,n}\Bigg(\sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m,n)\,c^e_{m,n} + \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m,n)\,c^{e+1}_{m,n} + \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m,n)\,c^{e-1}_{m,n}\Bigg)\mathcal{F}^{-\,(a_e,b_e;c_e)}_{\rho_1+m+r,\,\rho_2+n+t} = 0\,. \tag{4.24}
$$

The functions $\mathcal{F}^-$ appearing in (4.24) are linearly independent among each other, since they all have a different asymptotic behaviour as $z,\bar z\to 0$. Hence the only way to satisfy (4.24) is

[4]It is understood that $c^{-1}_{m,n} = c^{p+1}_{m,n} = 0$ in (4.24).

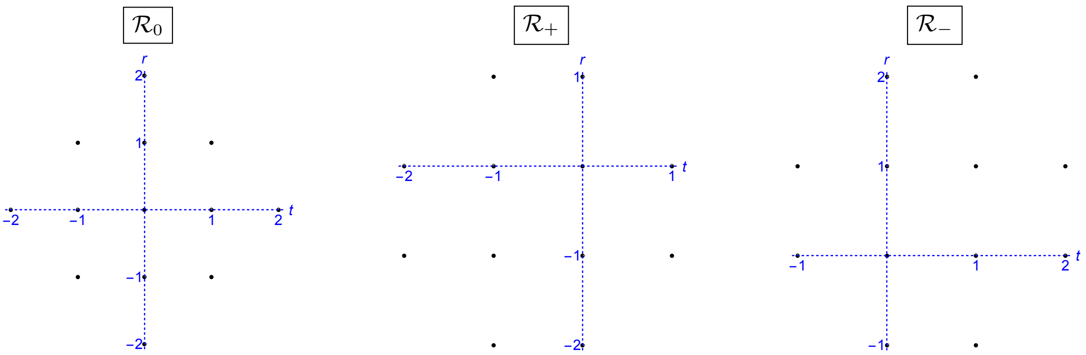

FIGURE 4.1: Set of points in the (r, t) plane forming the regions $\mathcal{R}_0$ (13 points), $\mathcal{R}_+$ (12 points) and $\mathcal{R}_-$ (12 points) defined in (4.21)-(4.23).

to demand that terms multiplying different $\mathcal{F}^-$ vanish on their own:

$$\sum_{(r,t)\in\mathcal{R}_0} A^0_{r,t}(m'-r,n'-t)c^e_{m'-r,n'-t} + \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m'-r,n'-t)c^{e+1}_{m'-r,n'-t}$$
$$+ \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m'-r,n'-t)c^{e-1}_{m'-r,n'-t} = 0\,, \qquad \forall m', n', \quad e = 0, \dots p\,, \tag{4.25}$$

where $m' = m + r$, $n' = n + t$. The Casimir system is then reduced to the over-determined *linear algebraic system of equations* (4.25).

4.2.3 Octagons and Recursion Relation for the Coefficients

In order to solve the system (4.25), we have to determine the range of values of (m, n) entering the ansatz (4.15), that also determines the size of the linear system. By rewriting the known $p = 1$ and $p = 2$ CBs found using the shadow formalism in the form of (4.15), we have deduced the range in (m, n) of the coefficients $c^e_{m,n}$ for any p (a posteriori proved using the results below). For each value of e, the non-trivial coefficients $c^e_{m,n}$ span a two-dimensional lattice in the (m, n) plane. For each e, the shape of the lattice is an octagon, with p and e dependent edges. The position and shape of the generic octagon in the (m, n) plane is depicted in fig. 4.2. One has

$$n_{min} = -p, \quad n_{max} = e + p, \quad m_{min} = e - 2p, \quad m_{max} = p\,. \tag{4.26}$$

For $e = 0$ and $e = p$, the octagons collapse to hexagons. The number N^e_p of points inside a generic octagon is

$$N^e_p = 2p\,(2p - e) + (1 + e)\,(3p + 1 - e) \tag{4.27}$$

and correspond to the number of non-trivial coefficients $c^e_{m,n}$ entering the ansatz (4.15). The total number N_p of coefficients to be determined at level p is then

$$N_p \equiv \sum_{e=0}^{p} N^e_p = (1 + p)\left(1 + \frac{17}{6}\,p + \frac{25}{6}\,p^2\right). \tag{4.28}$$

The size of the linear system grows as p^3. The first values are $N_1 = 16$, $N_2 = 70$, $N_3 = 188$, $N_4 = 395$. For illustration, we report in fig. 4.3 the explicit lattice of non-trivial coefficients $c^e_{m,n}$ for $p = 3$.

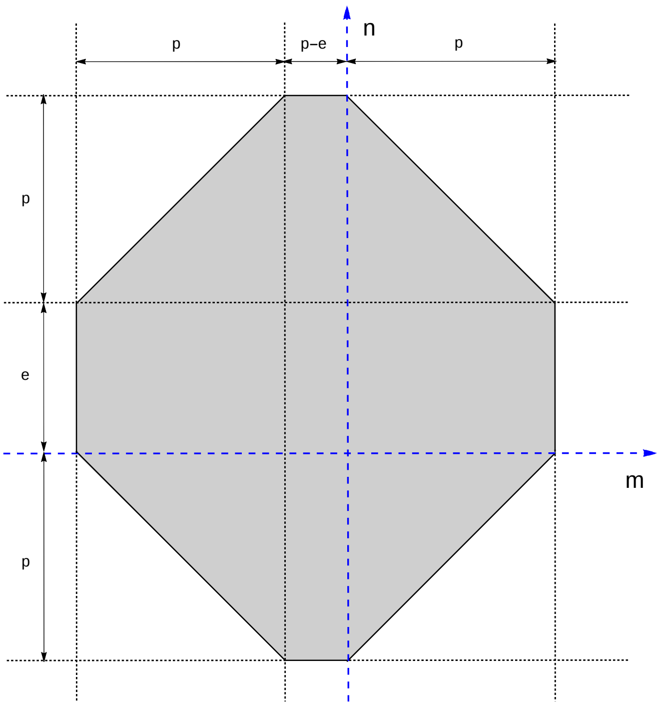

FIGURE 4.2: The dimensions of the generic octagon enclosing the lattice of non-vanishing coefficients $c^e_{m,n}$ entering the ansatz for mixed tensor CBs in (4.33).

The system (4.25) is always over-determined, since it is spanned by the values (m', n') whose range is bigger than the range of $(m, n) \in Oct_e^{(p)}$ (spanning all the coefficients to be determined) due to the presence of $(r, t) \in [-2, 2]$. There are only $N_p - 1$ linearly independent equations, because the system of Casimir equations can only determine conformal blocks up to an overall factor. The most important property of the system (4.25) is the following: while the number of equations grows with p, the total number of coefficients $c^e_{m,n}$ entering any given equation in the system (4.25) does not. This is due to the "local nearest-neighbour" nature of the interaction between the blocks, for which at most three conformal blocks can enter the Casimir system (4.4), independently of the value of p. More precisely, all the equations (4.25) involve from a minimum of one coefficient $c^e_{m,n}$ up to a maximum of 37 ones. Thirty seven corresponds to the total number of coefficients A^0, A^+ and A^- entering (4.21)-(4.23), see fig.4.1. The only coefficients that enter alone in some equations are the ones corresponding to the furthermost vertices of the hexagons, namely

$$c^p_{0,-p},\; c^p_{0,2p},\; c^0_{p,0},\; c^0_{-2p,0}\,. \tag{4.29}$$

For instance, let us take $n' = -2 - p$ and $e = p$ in (4.25), with m' generic. Since $n_{min} = -p$, a non-vanishing term can be obtained only by taking $t = -2$. Considering that $c^{p+1} = 0$ and $\mathcal{R}_-$ does not include $t = -2$ (see fig.4.1), this equation reduces to

$$A^0_{0,-2}(m, -p)|_{e=p}\, c^p_{m,-p} = 0\,, \quad \forall m\,, \tag{4.30}$$

where $m' = m$, since the point in $\mathcal{R}_0$ with $t = -2$ has $r = 0$. This equation forces all the coefficients $c^p_{m,-p}$ to vanish, unless the factor $A^0_{0,-2}(m, -p)$ vanishes on its own. One has

$$A^0_{0,-2}(m, n)|_{e=p} \propto (m + n + p)\Delta + (m - n - p)\ell + m^2 + \frac{1}{2}m(p - 2) + (n + p)(n + \frac{3}{2}p - 2)\,.$$

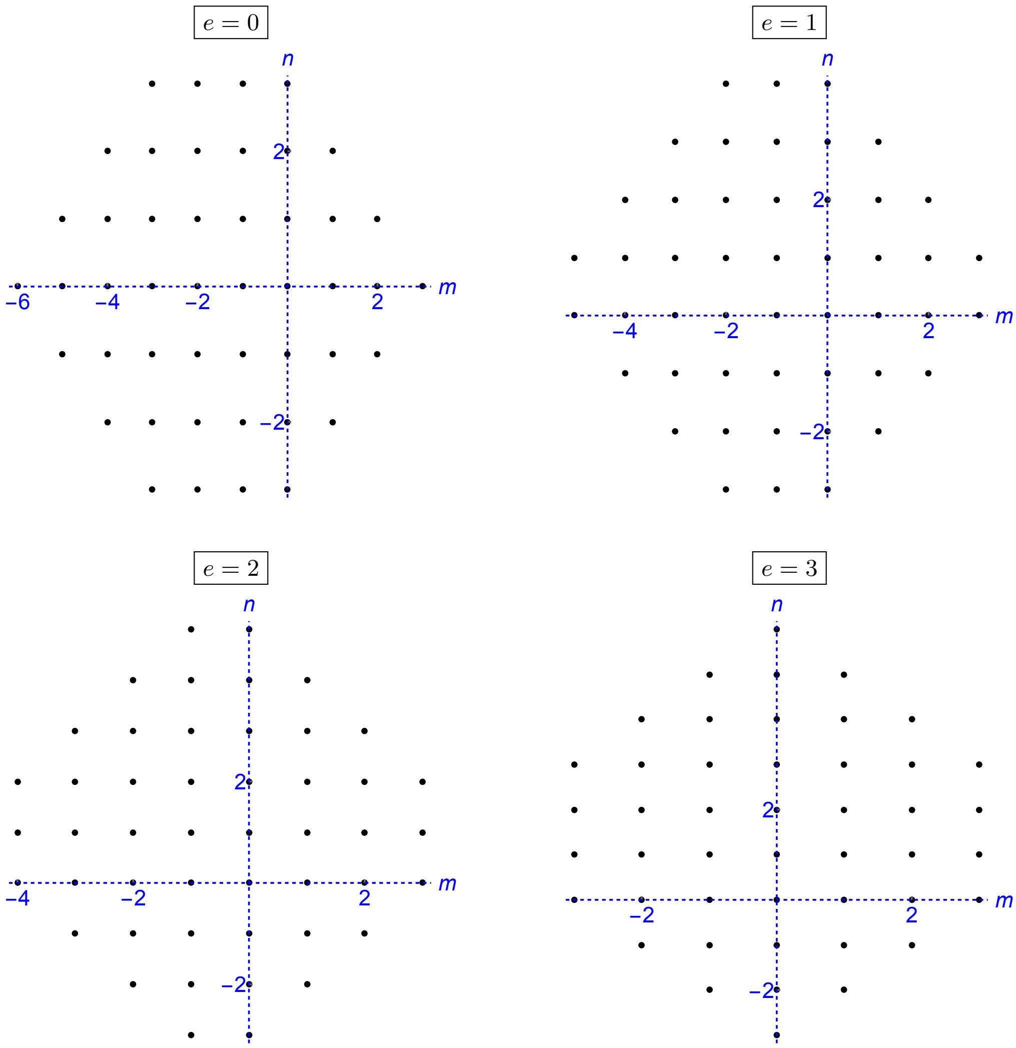

FIGURE 4.3: Set of non-vanishing coefficients $c^e_{m,n}$ (represented as black dots) entering the ansatz for mixed tensor CBs in (4.33) for $p = 3$ and $e = 0, 1, 2, 3$. For $e = 0$ and $e = p$ the octagons collapse to hexagons.

This factor is generally non-vanishing, unless $m = 0$ and $n = -p$, in which case it vanishes for any Δ, ℓ and p. In this way (4.30) selects $c^p_{0,-p}$ as the only non-vanishing coefficient at level $n = -p$ for $e = p$. Notice that it is crucial that $A^0_{0,-2}(m,n)|_{e=p}$ vanishes automatically for a given pair (m,n), otherwise either the whole set of equations would only admit the trivial solution $c^e_{m,n} = 0$, or the system would be infinite dimensional. A similar reasoning applies for the other three coefficients. One has in particular

$$
\begin{aligned}
A^0_{0,2}(0,2p)|_{e=p} \qquad & c^p_{0,2p} = 0\,, \\
A^0_{2,0}(p,0)|_{e=0} \qquad & c^0_{p,0} = 0\,, \\
A^0_{-2,0}(-2p,0)|_{e=0} \qquad & c^0_{-2p,0} = 0\,,
\end{aligned}
\tag{4.31}
$$

that are automatically satisfied because the three coefficients $A^0_{0,2}$, $A^0_{2,0}$ and $A^0_{-2,0}$ vanish when evaluated for the specific values reported in (4.31) for any Δ, ℓ and p.

The system (4.25) is efficiently solved by extracting a subset of $N_p - 1$ linearly independent equations. This can be done by fixing the values $(r,t) = (r^*, t^*)$ entering the definitions of (m', n'). There are 4 very special subsets of the $N_p - 1$ equations (corresponding to very specific values (r^*, t^*)) which allows us to determine the solution iteratively starting from (4.25). They correspond to a solution where one of the four coefficients (4.35) is left undetermined, in other words (r^*, t^*) can be set to be $(0, -2)$, $(0, 2)$, $(2, 0)$ or $(-2, 0)$. For instance, if we choose $c_0 \equiv c^p_{0,-p}$ as the undetermined coefficient, a recursion relation is found from (4.25) by just singling out the term with $t = -2$ in A^0 and setting $(r^*, t^*) = (0, -2)$. Such a choice leads to $m' = m$, $n' = n - 2$, and one finally gets

$$\begin{aligned} -A^0_{0,-2}(m,n)c^e_{m,n} &= \sum_{\substack{(r,t)\in\mathcal{R}_0 \\ (r,t)\neq(0,-2)}} A^0_{r,t}(m-r, n-2-t)c^e_{m-r,n-2-t} \\ &+ \sum_{(r,t)\in\mathcal{R}_+} A^+_{r,t}(m-r, n-2-t)c^{e+1}_{m-r,n-2-t} \\ &+ \sum_{(r,t)\in\mathcal{R}_-} A^-_{r,t}(m-r, n-2-t)c^{e-1}_{m-r,n-2-t}\,. \end{aligned} \tag{4.32}$$

It is understood in (4.32) that $c^e_{m,n} = 0$ if the set (m, n) lies outside the e-octagon of coefficients. The recursion (4.32) allows us to determine all the coefficients $c^e_{m,n}$ at a given $e = e_0$ and $n = n_0$ in terms of the ones $c^e_{m,n}$ with $n < n_0$ and c^e_{m,n_0} with $e > e_0$. Hence, starting from c_0, one can determine all $c^e_{m,n}$ as a function of c_0 for any p. The overall normalization of the CBs is clearly irrelevant and can be reabsorbed in a redefinition of the OPE coefficients. From (4.25) one can easily write the three other relations similar to (4.32) to determine recursively $c^e_{m,n}$ starting from $c^p_{0,2p}$, $c^0_{p,0}$ or $c^0_{-2p,0}$.

4.2.4 The Solution

We write down the full analytic solution for the CBs $H^{(p)}_e$:

$$H^{(p)}_e(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p+1} \sum_{(m,n)\in Oct^{(p)}_e} c^e_{m,n} \mathcal{F}^{-\,(a_e,b_e;c_e)}_{\frac{\Delta+\ell+\frac{p}{2}}{2}+m,\ \frac{\Delta-\ell+\frac{p}{2}}{2}-(p+1)+n}(z,\bar{z}), \tag{4.33}$$

where $c^e_{m,n}$ satisfy the recursion relation (4.32) (or any other among the four possible ones) and (m, n) runs over the points within the e-octagon depicted in fig.4.2. By convention we have chosen the coefficient $c^p_{0,-p}$ to be undetermined.

A similar expression holds for the dual blocks $\overline{H}^{(p)}_e$:

$$\overline{H}^{(p)}_e(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p+1} \sum_{(m,n)\in Oct^{(p)}_{p-e}} \bar{c}^e_{m,n} \mathcal{F}^{-\,(a_e,b_e;c_e)}_{\frac{\Delta+\ell-\frac{p}{2}}{2}+e+m,\ \frac{\Delta-\ell-\frac{p}{2}}{2}+e-(p+1)+n}(z,\bar{z}), \tag{4.34}$$

where $\bar{c}^e_{m,n}$ satisfy the recursion relation similar to (4.32); the hexagon vertices's are given by

$$\bar{c}^0_{0,-p},\ \bar{c}^0_{0,2p},\ \bar{c}^p_{p,0},\ \bar{c}^p_{-2p,0} \tag{4.35}$$

and we have chosen the undetermined coefficient to be $\bar{c}^0_{0,-p}$.

One can find that expressions (4.33) and (4.34) in the limit $z \sim \bar{z} \sim 0$ give (A.92) and (A.94). Comparing them to the direct OPE evaluation, one fixes the undetermined coefficients

$$c^p_{0,-p} = (-1)^\ell\, i^p \quad \text{and} \quad \bar{c}^0_{0,-p} = 2^{-p}\,(-1)^\ell\, i^p. \tag{4.36}$$

See appendix A.3 for details. It is also interesting to note, that the coefficients c and $\bar{c}$ are relate by

$$\bar{c}^{e}_{m,n}(a^{(p)},\, b^{(p)},\, \Delta,\, l,\, p) = 4^{e-\frac{p}{2}} c^{p-e}_{m,n}\Big(-a^{(p)}+\frac{p}{2},\, -b^{(p)}-\frac{p}{2},\, \Delta,\, l,\, p\Big)\,. \tag{4.37}$$

Generating the full explicit solution from (4.32) can be computationally quite demanding for large values of p. For concreteness, we only report in appendix B.3 the explicit form of the 16 coefficients $c^{e}_{m,n}$ for $p=1$ and $a=-b=1/2$. The general form of $c^{e}_{m,n}$ for $p=1,2,3,4$ and any a, b, Δ and ℓ are implemented in the "CFTs4D" package.

It is important to remind the reader that the CBs $H^{(p)}_e$ computed here are supposed to be the seed blocks for possibly other 4-point correlation functions, whose CBs are determined by acting with given differential operators (3.34) on $H^{(p)}_e$. The complexity of the form of the blocks $H^{(p)}_e$ at high p is somehow compensated by the fact that the operators one has to act with become simpler and simpler, the higher is p. An example should clarify the point. Let us consider a 4-point function of spin two operators. In this case, one has to determine conformal blocks associated to the exchange of operators $\mathcal{O}^{(\ell,\ell+p)}$ (and $\overline{\mathcal{O}}^{(\ell+p,\ell)}$) for $p=0,2,4,6,8$ (and any ℓ). The conformal blocks associated to the traceless symmetric operators are obtained by applying up to 8 derivative operators in several different combinations to the scalar CB $H^{(0)}_0$. Despite the seed block is very simple, the final blocks are given by (many) complicated sum of derivatives of $H^{(0)}_0$. The $p=8$ CBs, instead, are essentially determined by the very complicated $H^{(8)}_e$ (and $\overline{H}^{(8)}_e$) blocks, but no significant extra complications come from the external operators.

4.2.5 Analogy with Scalar Conformal Blocks in Even Dimensions

It is worth pointing out some similarities between the CBs $H^{(p)}_e$ for mixed symmetry tensors computed above and the scalar conformal blocks H_d in $d>2$ even space-time dimensions ($H_4 = H^{(0)}_0$ in our previous notation). The quadratic Casimir equation for scalar CBs in any number of dimensions is

$$\Delta^{(a,b;0)}_{d-2} H_d(z,\bar{z}) = \frac{1}{2} E_\ell(d) H_d(z,\bar{z})\,, \tag{4.38}$$

where

$$E_\ell(d) = \Delta\,(\Delta-d) + \ell(\ell+d-2) \tag{4.39}$$

is the quadratic Casimir eigenvalue for traceless symmetric tensors. The explicit analytical form of scalar blocks in $d=2,4,6$ dimensions has been found in refs.[127, 128]. The same authors also found a relation between scalar blocks in any even space-time dimensionality, (5.4) of ref.[128] (see also the more elegant (4.36) of ref.[129]), that allows us to iteratively determine H_d for any d, starting from H_2. The $d=4$ and $d=6$ solutions found in ref.[128] have the form

$$H_d(z,\bar{z}) = \Big(\frac{z\bar{z}}{z-\bar{z}}\Big)^{d-3} g_d(z,\bar{z})\,, \qquad g_d(z,\bar{z}) = \sum_{m,n} x_{m,n} \mathcal{F}^{-\,(a,b;0)}_{\frac{\Delta+\ell}{2}+m,\, \frac{\Delta-\ell+2-d}{2}+n}(z,\bar{z}), \tag{4.40}$$

where a and b are defined in (4.9) with $p=0$ and $x_{m,n}$ are coefficients that in general depend on Δ, l, a and b. In $d=4$ there is only one non-vanishing coefficient centered at $(m,n)=(0,0)$, while in $d=6$ there are five of them. They are at $(m,n)=(0,-1)$, $(-1,0)$, $(0,0)$, $(1,0)$ and $(0,1)$. These five points form a slanted square in the (m,n) plane, centered at the origin. The explicit form of the coefficients $x_{m,n}$ is known, but it will not be needed

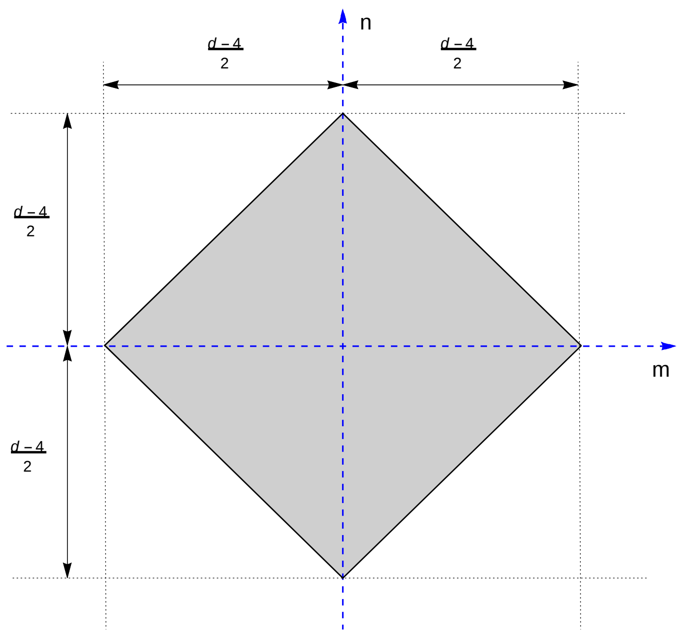

FIGURE 4.4: The dimensions of the generic slanted square enclosing the lattice of non-vanishing coefficients $x_{m,n}$ entering the ansatz for scalar symmetric CBs in (4.40).

in what follows. It is natural to expect that (4.40) should apply for any even $d \geq 4$, with a number of non-vanishing coefficients that increases with d.[5] This is not difficult to prove. From the first relation in (4.16) we can get the form of the Casimir equation for the function $g_d(z, \bar{z})$ defined in (4.40), that can be written as

$$\left(\frac{1}{\bar{z}} - \frac{1}{z}\right)\left(\Delta_0^{(a,b;0)} + 6 - 2d - \frac{1}{2}E_\ell(d)\right)g_d = (d-4)\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)g_d\,. \tag{4.41}$$

Using the techniques explained in subsection 4.2.2 and the results of appendix B.2, it is now straightforward to identify which is the range of (m, n) of the non-vanishing coefficients $x_{m,n}$ for any d (see fig.4.4).[6] In d dimensions, the minimum and maximum values of m and n are given by

$$n_{min} = \frac{4-d}{2}\,, \quad n_{max} = \frac{d-4}{2}\,, \quad m_{min} = \frac{4-d}{2}\,, \quad m_{max} = \frac{d-4}{2}\,. \tag{4.42}$$

The number $\widetilde{N}_d$ of coefficients $x_{m,n}$ entering the ansatz (4.40) for scalar blocks in d even space-time dimensions is easily computed by counting the number of lattice points enclosed in the slanted square. We have

$$\widetilde{N}_d = \frac{d^2}{2} - 3d + 5\,. \tag{4.43}$$

For large d, $\widetilde{N}_d \propto d^2$ and matches the behavior of $N_e^p \propto p^2$ for large p in (4.27).

Let us finally emphasize a technical, but relevant, point where the analogy between H_d in d dimensions and $H_e^{(p)}$ in 4 dimensions does not hold. A careful reader might have noticed that in the Casimir equation for g_d the term proportional to $(z+\bar{z})-2$, namely the third

[5] See also ref.[152], where similar considerations were conjectured.
[6] Alternatively, one might use (4.36) of ref.[129] to compute H_d and then recast it in the form (4.40).

term in the r.h.s. of the first equation in (4.16), automatically vanishes. Indeed, if we did not know the power $d-3$ in the ansatz (4.40), we could have guessed it by demanding that term to vanish. On the contrary, no such simple guess seems to be possible for the power $2p+1$ entering $H_e^{(p)}$, given also the appearance of the operator L defined in (4.16). As discussed, we have fixed the power $2p+1$ by means of the shadow formalism.

Chapter 5

Covariant Differential Operators Formalism

We introduce here a new powerful formalism based on representation theory of the conformal group. In section 5.1 we show the existence of covariant differential operators (5.17) corresponding to finite-dimensional representations of the conformal group. In section 5.2 we explain that these operators satisfy a very special "crossing" equation (5.23). In these sections we make only a short summary of the arguments, see [148] for detailed and more precise discussion. All the ingredients of sections 5.1 and 5.2 are purely mathematical and do not have a physical meaning. The formalism however turns out to be very convenient for obtaining (5.17) and (5.23).

We proceed with explicit construction of the conformal differential operators in section 5.3. We systematically construct covariant differential operators in fundamental and anti-fundamental representations (5.38) and also discuss briefly higher-dimensional representations. Finally we make connection with spinning differential operators of section 3.1.2.

We discuss the shadow formalism in section 5.4. Combining it with the "crossing" equation one arrives at a very powerful "integration-by-parts" formula (5.55). We proceed with computation of seeds (2.41). We derive the recursion relations (5.81) and (5.96) which allow to reduce them recursively to the known scalar Dolan and Osborn blocks. These procedure is an example of a general idea of changing spin of internal "exchanged" operators, see [148] for more details. One can consider it as a generalization of the results [138] (applicable only to external operators) to internal operators.

5.1 Existence of Covariant Differential Operators

We discuss here the conformal group $\mathrm{SO}(2,d)$. In what follows we consider primary operators and their descendants. As already briefly discussed in chapter 2 these operators are labeled by representations $(\rho)_\Delta$ of the subgroup

$$\mathrm{SO}(2,d) \supset \mathrm{SO}(1,1)\times\mathrm{SO}(1,\,d-1). \tag{5.1}$$

For future purposes it will be convenient to write an explicit decomposition of a finite-dimensional representation, denoted by W from now on, under the sub-group (5.1)

$$W = \bigoplus (\lambda)_\Delta = \bigoplus_{k=-j}^{j} (\lambda_k)_{\Delta=k}, \qquad j\in \tfrac{1}{2}\mathbb{N}. \tag{5.2}$$

Here k is a label. We will see explicit examples of this decomposition in (5.27)-(5.30). The scaling dimensions Δ in (5.2) are integer-spaced and must be integers or half-integers.

Verma module Consider a primary operator[1]

$$\mathcal{O}^{\{\mu\}}(0), \tag{5.3}$$

which has a scaling dimension Δ and the collective spin index[2] $\{\mu\}$ in representation ρ of $\mathrm{SO}(1,\, d-1)$ in (5.1). We can then construct an infinite-dimensional conformal multiplet or generalized (parabolic) Verma module,[3] denoted by $V_{(\Delta,\,\rho)}$ from now on, spanned by the basis of operators

$$\mathcal{O}^{\{\mu\}}(0), \quad \partial_{\nu_1}\mathcal{O}^{\{\mu\}}(0), \quad \partial_{\nu_1}\partial_{\nu_2}\mathcal{O}^{\{\mu\}}(0), \quad \dots \tag{5.4}$$

The operator

$$\mathcal{O}^{\{\mu\}}(x) \tag{5.5}$$

is an element in the space spanned by (5.4) since it can be written in terms of the basis (5.4) using the Taylor expansion around $x = 0$.

Finite-dimensional representations Consider now a very special situation when starting from some N we set all the derivatives in (5.4) to zero

$$w^{\{\mu\}}(0), \quad \partial_{\nu_1} w^{\{\mu\}}(0), \quad \dots, \quad \partial_{\nu_1}\dots\partial_{\nu_N} w^{\{\mu\}}(0). \tag{5.6}$$

In the above expression it is understood that some derivatives inside (5.6) can also and generically do vanish. The multiplet (5.6) can only be consistent with the conformal symmetry if it defines some finite-dimensional representation of the conformal group W. The dimension of this representation $\dim W$ is equal to the number of non-zero elements in (5.6).

The operator

$$w^{\{\mu\}}(x). \tag{5.7}$$

is an element in the linear space spanned by (5.6). Using the Taylor expansion around $x = 0$ it can be written in terms of the basis elements (5.6). Since there are only finite number of non-vanishing derivatives, (5.7) is a polynomial in x of degree N.

The operator (5.7) compared to the operator (5.5) is extremely degenerate and thus should satisfy some conformally invariant equation ("shortening" condition). It can be decomposed in some basis of the finite-dimensional representation $\{e\}$ as

$$w^{\{\mu\}}(x) = w^{\{\mu\}}_A(x) e^A, \quad A = 1, \dots, \dim W. \tag{5.8}$$

The functions $w^{\{\mu\}}_A(x)$ are the coefficients in the expansion (5.8). Finally according to (5.2) the scaling dimensions of the elements (5.6) are not generic and only take integer or half-integer values. For instance the primary state has the dimension

$$w^{\{\mu\}}_{\Delta=-j}(0), \tag{5.9}$$

where j is the lowest weight of $\mathrm{SO}(1,1)$ representation in (5.2).

[1] In what follows we will not deal with a physical interpretation of primary operators, quantization procedure or the operator-state correspondence. We will rather treat then as some abstract objects.

[2] We stress that we work with generic representations of the conformal group, thus the index $\{\mu\}$ denotes not only tensor but also spinor representations.

[3] See section 6.2 in [141] for more details.

Example Consider the adjoint representation $\substack{\square\\\square}$ of the conformal group $\mathrm{SO}(2,d)$. Under (5.1) it decomposes as

$$\substack{\square\\\square} = (\square)_{-1} \oplus (\bullet \oplus \substack{\square\\\square})_0 \oplus (\square)_1. \tag{5.10}$$

The operator $w^\mu(0)$ is thus a vector with dimension $\Delta = -1$. A basis for $W = \substack{\square\\\square}$ is given by $\{e\} = \{K^\mu, D, M^{\mu\nu}, P^\mu\}$ and the coefficients $w^\mu_A(x)$ in this basis are the usual conformal Killing vectors on $\mathbb{R}^d$

$$w^\mu(x) = K^\mu - 2x^\mu D + (x_\rho \delta^\mu_\nu - x_\nu \delta^\mu_\rho) M^{\nu\rho} + (2x^\mu x_\nu - x^2 \delta^\mu_\nu) P^\nu. \tag{5.11}$$

In this case the differential equation satisfied by $w^\mu(x)$ is the usual conformal Killing equation

$$\partial^\mu w^\nu(x) + \partial^\nu w^\mu(x) - \text{trace} = 0. \tag{5.12}$$

Tensor products of representations We consider now a tensor product of the Verma module and the finite-dimensional representation. It can be shown that

$$W \otimes V_{(\Delta,\,\rho)} = \bigoplus_{k=-j}^{j} \bigoplus_{\tau \in \lambda_k \otimes \rho} V_{(\Delta+k,\,\tau)} = \bigoplus_{(k,s)\in\Pi(W)} V_{\Delta+k,\rho+s}, \tag{5.13}$$

which follows intuitively from (5.2). The last equality holds only when the Dynkin indices of the representation ρ are sufficiently large.[4] Here, $\Pi(W)$ denotes the weights of W (with multiplicity). At the level of operators it is equivalent to taking a tensor product

$$w^{\{\mu\}}(x) \otimes O^{\{\nu\}}(x). \tag{5.14}$$

We would like to build all primary operators out of the product (5.14). Their number should be equal to the number of Verma modules (with multiplicities) in the right-hand side of (5.13). The ansatz for theses primaries is[5]

$$\mathcal{O}'^{\{\sigma\}}(0) = \pi^{\{\sigma\}\{\kappa\}}_{\{\mu\}\{\nu\}} \Big(c_1 \partial_{\kappa_1} \cdots \partial_{\kappa_m} w^{\{\mu\}}(0) \otimes \mathcal{O}^{\{\nu\}}(0) + c_2 \partial_{\kappa_1} \cdots \partial_{\kappa_{m-1}} w^{\{\mu\}}(0) \otimes \partial_{\kappa_m} \mathcal{O}^{\{\nu\}}(0) + \dots \Big) \tag{5.15}$$

with the coefficients c_i determined from the requirement that the primary is annihilated by $1 \otimes K_\mu + K_\mu \otimes 1$. Finally one defines a covariant differential operator $\mathcal{D}_A$ via

$$\mathcal{O}'^{\{\rho\}}(0) \equiv \mathcal{O}'^{\{\rho\}}_A(0)\, e^A \equiv e^A\, (\mathcal{D}_A)^{\{\rho\}}{}_{\{\sigma\}} \mathcal{O}^{\{\sigma\}}(0). \tag{5.16}$$

The expression for $\mathcal{D}_A$ can be read off from (5.15) using (5.8).

We can see that there exist operators $\mathcal{D}_A$ which transform a generic primary[6] $\mathcal{O}$ to a primary $\mathcal{O}'$ by shifting its weight according to (5.13) as

$$\mathcal{D}^{(n)}_A : \quad [\Delta,\, \rho] \to [\Delta + k_n,\, \rho + s_n], \quad n = 1, \dots, \dim W. \tag{5.17}$$

We thus call these operators the weight-shifting operators. Their total number corresponds to the dimension of the representation W. To indicate this we have add an extra index (n).

[4] This relation is called the Brauer's formula or the Klimyk's rule [153, 154].

[5] This ansatz is written in a schematic form. For instance the projector π can be different for each term in (5.15).

[6] In case of a non-generic primary the last equality in (5.13) does not hold and we have to use the first one instead. In section 5.3 we are interested in building all possible operators and thus we deal with a generic situation.

The labels k_n and s_n are all the weights of the representation W.

The operators $\mathcal{D}_A$ correspond to the finite-dimensional representation W and transforms in the dual representation W^*. We indicate this by the lower position of its index A. Analogously there exist operators $\mathcal{D}^A$ corresponding to the finite-dimensional dual representation W^* and transforming in the representation W.

5.2 Covariant Tensor Structures and the $6j$ Symbols

Let us consider an $(n+1)$-point function of n primary operators $\mathcal{O}_i$ and one highly degenerate operator w associated to a finite-dimensional representation of the conformal group

$$\langle \mathcal{O}_1^{\{\mu_1\}}(x_1)\cdots \mathcal{O}_n^{\{\mu_n\}}(x_n) w^{\{\nu\}}(y)\rangle = \langle \mathcal{O}_1^{\{\mu_1\}}(x_1)\cdots \mathcal{O}_n^{\{\mu_n\}}(x_n)\, e^A\rangle\; w_A^{\{\nu\}}(y). \tag{5.18}$$

The right-hand side of (5.18) contains a conformally-covariant n-point function[7]

$$\langle \mathcal{O}_1^{\{\mu_1\}}(x_1)\cdots \mathcal{O}_n^{\{\mu_n\}}(x_n)\, e^A\rangle = \sum_I \mathbb{T}_I^{\{\mu_1\}\ldots\{\mu_n\},\,A}(x_i) g^I(\mathbf{u}). \tag{5.19}$$

Here we make a distinction between conformally-covariant and conformally-invariant objects. For us, the former carry finite-dimensional $\mathrm{SO}(2,d)$ labels A, whereas the latter do not. As in the usual case of invariant n-point functions (2.11), $\mathbf{u}$ are the conformal cross-ratios of points x_i and $\mathbb{T}_I^{\{\mu_1\}\ldots\{\mu_n\},\,A}$ are the conformally-covariant tensor structures. One can classify these structures in the conformal frame approach of [125, 126]. Here we discuss only how they can be constructed in practice. It can be done by using covariant differential operators $\mathcal{D}^A$ (which transforms $\mathcal{O}'$ to $\mathcal{O}_A$) acting on invariant n-point tensor structures.

Let us focus on the $n=3$ case, then

$$\mathbb{T}_I^{\{\mu_1\}\{\mu_2\}\{\mu_3\},\,A} = \sum_{m,p} \mathbf{M}_{Imp}\, \mathcal{D}_{x_3}^{(m)A} \langle \mathcal{O}_1^{\{\mu_1\}}(x_1)\; \mathcal{O}_2^{\{\mu_2\}}(x_2)\; \mathcal{O}'^{\{\mu_3\}}_3(x_3)\rangle^{(p)}, \tag{5.20}$$

$$\mathbb{T}_I^{\{\mu_1\}\{\mu_2\}\{\mu_3\},\,A} = \sum_{m,p} \mathbf{N}_{Imp}\, \mathcal{D}_{x_1}^{(m)A} \langle \mathcal{O}'^{\{\mu_1\}}_1(x_1)\; \mathcal{O}_2^{\{\mu_2\}}(x_2)\; \mathcal{O}_3^{\{\mu_3\}}(x_3)\rangle^{(p)}. \tag{5.21}$$

In (5.20) and (5.21) it is understood that for every $\mathcal{D}^{(m)A}$ (in the sum over m) there is its own $\mathcal{O}'$. We do not add a label (m) to $\mathcal{O}'$ in order not to clutter the notation. In the right-hand side of (5.20) and (5.21) we use the notation for tensor structures as in (2.18). The symbols $\mathbf{M}$ and $\mathbf{N}$ denote some constant matrices. One can convince him or herself in the validity of (5.20) and (5.21) by proving that[8]

$$\dim(\{I\}) = \dim(\{m,p\}), \tag{5.22}$$

so $\mathbf{M}$ and $\mathbf{N}$ are the square matrices. An analogous statement can also be proved for $n=2$.[9]

[7] This correlator is a purely abstract construction and does not have any physical meaning.

[8] Using conformal frame approach one should count independently the number of tensor structures in (5.19) and in the right-hand side of (5.20) or (5.21). Turns out that they match precisely.

[9] Using the conformal frame approach as usually the cases $n=2$ and $n=3$ should be considered separately due to the different structure of the "little" group (3.39).

As we can see from (5.20) and (5.21) there are two bases related by a linear transformation. This fact can be expressed in a form of the relation

$$\mathcal{D}_{x_3}^{(r)A}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3'(x_3)\rangle^{(p)} = \sum_{m,n}\begin{Bmatrix}\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & W & \mathcal{O}_3'\end{Bmatrix}^{pr}_{mn}\mathcal{D}_{x_1}^{(n)A}\langle\mathcal{O}_1'(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle^{(m)} \tag{5.23}$$

which we call the "crossing" equation. The coefficients $\{\ldots\}$ in (5.23) are called the $6j$ symbols or the Racah-Wigner coefficients.[10,11,12] Exactly as in equations (5.20) and (5.21) for each operator $\mathcal{D}_{x_1}^{(n)A}$ (in the sum over n) in the right-hand side of (5.23) there is its own operator $\mathcal{O}'$. One uses (5.13) to understand what $\mathcal{D}_{x_1}^{(n)A}$ and $\mathcal{O}'$ can contribute. In simple words the sum of weight of operators at every point in the left- and right-hand side of (5.23) should match

$$\forall n: \quad \mathcal{O}_1(x_1) \sim \mathcal{D}_{x_1}^{(n)A}\,\mathcal{O}_1'(x_1) \quad \textbf{and} \quad \mathcal{D}_{x_3}^{(r)A}\mathcal{O}_3'(x_3) \sim \mathcal{O}_3(x_3), \tag{5.24}$$

where r is fixed. For explicit examples see (5.59) and (5.62).

The equation (5.23) also holds for $n = 2$. In that case $\mathcal{O}_2$ should be just replaced by the identity operator $\mathbf{1}$ and since there is only a single structure in 2-point functions, $(p) = (\bullet)$ and $(m) = (\bullet)$.

The $6j$ symbols for operator representations (generalized Verma modules) of the conformal group have seen some recent interest for their role in the crossing equations for CFT four-point functions [70–72]. Here, we have a degenerate form of these objects, where one of the representations appearing is finite-dimensional. These degenerate $6j$ symbols enter in a degenerate crossing equation (5.23) where the objects on both sides live in a finite dimensional space.

5.3 Construction of Covariant Differential Operators

For practical construction[13] of the covariant weight-shifting operators it is convenient to use the embedding formalism 3. From now on we focus solely on the 4D case. The branching (5.1) takes the following form then

$$\mathrm{SU}(2,2) \supset U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{5.25}$$

We consider few smallest finite-dimensional representations of the conformal group $SU(2,2)$: fundamental, anti-fundamental, anti-symmetric and adjoint. They are denoted as

$$\boldsymbol{\mathcal{S}} \equiv \square, \quad \overline{\boldsymbol{\mathcal{S}}} \equiv \overline{\square}, \quad \boldsymbol{A} \equiv \ydiagram{1,1}, \quad \boldsymbol{Adj} \equiv \ydiagram{2,1,1}. \tag{5.26}$$

[10]Technically, Racah and Wigner coefficients are sometimes defined to differ by various normalization factors. We will not distinguish between them and use both terms to refer to the coefficients.

[11]$6j$ symbols depend only on a set of representations and three-point structures. However, for brevity, we often label them with operators $\mathcal{O}_i$ transforming in those representations, as in (5.23).

[12]One can define $3(k-1)-j$ symbols as coefficients in a decomposition of tensor product $\rho_1 \otimes \ldots \otimes \rho_k$ into irreducible representations. The case $k = 2$ was first considered by Clebsh and Gordon. This coefficients are well studied in the case of SU(2) group, since they play an important role in the theory of angular momentum in Quantum Mechanics.

[13]One could of course use (5.15) and (5.16) directly. But this procedure is tedious.

These representations decompose[14] under (5.25) as

$$\square = (1,0)_{-\frac{1}{2}} \oplus (0,1)_{\frac{1}{2}}, \tag{5.27}$$

$$\overline{\square} = (0,1)_{-\frac{1}{2}} \oplus (1,0)_{\frac{1}{2}}, \tag{5.28}$$

$$\ydiagram{1,1} = (0,0)_{-1} \oplus (0,0)_{1} \oplus (1,1)_{0}, \tag{5.29}$$

$$\ydiagram{2,1,1} = (0,0)_{0} \oplus (1,1)_{-1} \oplus (1,1)_{1} \oplus (2,0)_{0} \oplus (0,2)_{0}. \tag{5.30}$$

The subscript denotes the $U(1)$ charge (the scaling dimension). It is normalized in such a way to be consistent with (3.4). The tensor products of finite-dimensional representations with Verma modules for the representations (5.26) are easily obtained from (5.27)-(5.30)

$$V_{(\Delta,\ell,\bar{\ell})} \otimes \square = V_{(\Delta-\frac{1}{2},\ell\pm1,\bar{\ell})} \oplus V_{(\Delta+\frac{1}{2},\ell,\bar{\ell}\pm1)}, \tag{5.31}$$

$$V_{(\Delta,\ell,\bar{\ell})} \otimes \overline{\square} = V_{(\Delta-\frac{1}{2},\ell,\bar{\ell}\pm1)} \oplus V_{(\Delta+\frac{1}{2},\ell\pm1,\bar{\ell})}, \tag{5.32}$$

$$V_{(\Delta,\ell,\bar{\ell})} \otimes \ydiagram{1,1} = V_{(\Delta\pm1,\ell,\bar{\ell})} \oplus V_{(\Delta,\ell\pm1,\bar{\ell}\pm1)}, \tag{5.33}$$

$$V_{(\Delta,\ell,\bar{\ell})} \otimes \ydiagram{2,1,1} = 3\,V_{(\Delta,\ell,\bar{\ell})} \oplus V_{(\Delta\pm1,\ell\pm1,\bar{\ell}\pm1)} \oplus V_{(\Delta,\ell\pm2,\bar{\ell})} \oplus V_{(\Delta,\ell,\bar{\ell}\pm2)}. \tag{5.34}$$

Each shift in the weight of the Verma module $V_{(\Delta,\ell,\bar{\ell})}$ according to section5.1 corresponds to a covariant differential operator.

Operators in representations $\boldsymbol{\mathcal{S}}$ and $\overline{\boldsymbol{\mathcal{S}}}$ We construct systematically such operators for the representations $\boldsymbol{\mathcal{S}}$ and $\overline{\boldsymbol{\mathcal{S}}}$. As in section 3 we denote the fundamental and anti-fundmantal indices of $SU(2,2)$ by a. Thus the index of finite-dimensional representation of the conformal group is $A = a$.[15] In the notation (5.13) the decompositions (5.31) and (5.32) read

$$\Pi(\boldsymbol{\mathcal{S}}) = \{(-,+,0),\ (-,-,0),\ (+,0,+),\ (+,0,-)\}, \tag{5.35}$$

$$\Pi(\overline{\boldsymbol{\mathcal{S}}}) = \{(-,0,+),\ (-,0,-),\ (+,+,0),\ (+,-,0)\}. \tag{5.36}$$

The operators $\mathcal{D}^a$ are then labeled by the weights (5.36) of $\overline{\boldsymbol{\mathcal{S}}}$,and the operators $\overline{\mathcal{D}}_a$ are labeled by the weights (5.35) of $\overline{\boldsymbol{\mathcal{S}}}^* = \boldsymbol{\mathcal{S}}$.

We use the following shorthand notation

$$\partial_{\bar{S},a} \equiv \frac{\partial}{\partial \bar{S}^a}, \qquad \partial_S^a \equiv \frac{\partial}{\partial S_a},$$
$$\partial_{ab} \equiv \Sigma^m_{ab}\frac{\partial}{\partial X^m}, \qquad \bar{\partial}^{ab} \equiv \overline{\Sigma}^{m\,ab}\frac{\partial}{\partial X^m}. \tag{5.37}$$

[14] An excellent tool for working with finite-dimensional representations of Lie group is the Mathematica package “LieART” [155].

[15] That basis vectors for $\boldsymbol{\mathcal{S}}$ are e^a (so that we can contract them with S_a) and for $\overline{\boldsymbol{\mathcal{S}}}$ the basis vectors are e_a (so that we can contract them with $\bar{S}^a$).

The differential operators corresponding to (5.35) and (5.36) have the following explicit expressions

$$
\begin{aligned}
&\mathcal{D}^a_{-0+} \equiv \overline{S}^a, \\
&\mathcal{D}^a_{-0-} \equiv \overline{\mathbf{X}}^{ab}\partial_{\overline{S},b}, \\
&\mathcal{D}^a_{++0} \equiv \overline{a}\overline{\partial}^{ab} S_b + \overline{S}^a(S\overline{\partial}\partial_{\overline{S}}), \\
&\mathcal{D}^a_{+-0} \equiv \overline{b}c\partial^a_S + \overline{b}\overline{S}^a(\partial_S\partial_{\overline{S}}) + c\mathbf{X}_{bc}\overline{\partial}^{ab}\partial^c_S - \overline{S}^a(\mathbf{X}_{bc}\overline{\partial}^{bd}\partial^c_S\partial_{\overline{S},d}), \\
&\overline{\mathcal{D}}_a^{-+0} \equiv S_a, \\
&\overline{\mathcal{D}}_a^{--0} \equiv \mathbf{X}_{ab}\partial^b_S, \\
&\overline{\mathcal{D}}_a^{+0+} \equiv a\partial_{ab}\overline{S}^b + S_a(\overline{S}\partial\partial_S), \\
&\overline{\mathcal{D}}_a^{+0-} \equiv bc\partial_{\overline{S},a} + bS_a(\partial_{\overline{S}}\partial_S) + c\overline{\mathbf{X}}^{bc}\partial_{ab}\partial_{\overline{S},c} - S_a(\overline{\mathbf{X}}^{bc}\partial_{bd}\partial_{\overline{S},c}\partial^d_S),
\end{aligned} \tag{5.38}
$$

where

$$
\begin{aligned}
&a = 1 - \Delta + \tfrac{\ell}{2} - \tfrac{\overline{\ell}}{2}, && \overline{a} = 1 - \Delta - \tfrac{\ell}{2} + \tfrac{\overline{\ell}}{2}, \\
&b = 2(\overline{\ell}+1), && \overline{b} = 2(\ell+1), \\
&c = -2 + \Delta - \tfrac{\ell+\overline{\ell}}{2}.
\end{aligned} \tag{5.39}
$$

The coefficients above come from requiring that the operators preserve the gauge choice (A.72) together with $X^2 = 0$. We have added these operators to the `CFTs4D` Mathematica package as `[spinorD,spinorDb]` functions.

Operators in representations $\boldsymbol{A}$ and $\boldsymbol{Adj}$ We do not consider systematically differential operators corresponding to representations $\boldsymbol{A}$ and $\boldsymbol{Adj}$ and instead construct only few of them. For $\boldsymbol{A}$ one can have

$$
\overline{\mathcal{D}}_{ab}^{-100} \equiv \mathbf{X}_{ab}, \quad \overline{\mathcal{D}}_{ab}^{0++} \equiv \left(S_a\,\partial_{bc} - S_b\,\partial_{ac}\right)\overline{S}^c \tag{5.40}
$$

with their dual ones. For $\boldsymbol{Adj}$ one for example has

$$
\mathcal{D}^{000}{}_a{}^c \equiv \mathbf{X}_{ab}\,\overline{\partial}^{bc} + 2\left(S_a\overline{\mathfrak{d}}^c - \overline{S}^c\mathfrak{d}_a\right) + (\Delta+\overline{\ell})\,\delta^c_a. \tag{5.41}
$$

Differential operators in higher-representations can always be obtained from the $\boldsymbol{\mathcal{S}}$ and $\overline{\boldsymbol{\mathcal{S}}}$ operators. For example

$$
a\,\overline{\mathcal{D}}_{ab}^{0++} = \overline{\mathcal{D}}_a^{-+0}\,\overline{\mathcal{D}}_b^{+0+} - \overline{\mathcal{D}}_b^{-+0}\,\overline{\mathcal{D}}_a^{+0+}. \tag{5.42}
$$

Relation with spinning differential operators By forming invariant products of the operators (5.38), (5.40) and (5.41) one can form numerous invariant operators which change spin and/or scaling-dimensions. We will use them extensively in section 5.4. Here will only provide connection with the spinning differential operators obtained in chapter 3 in (3.35) and (3.36). For (3.35) one has

$$\begin{aligned} D_{ij} &= \frac{1}{2}\,\mathcal{D}_{i}{}^{ab}{}_{0++} \cdot \overline{\mathcal{D}}_{j}{}_{ab}{}^{-100}, && (5.43)\\ \widetilde{D}_{ij} &= \mathcal{D}^{a}_{i\,-0+} \cdot \mathcal{D}^{000}_{j}{}_{a}{}^{c} \cdot \overline{\mathcal{D}}_{i\,c}{}^{-+0}, && (5.44)\\ I^{ij} &= \mathcal{D}^{a}_{i\,-0+} \cdot \overline{\mathcal{D}}_{j\,a}{}^{-+0}, && (5.45)\\ \nabla_{ij} &= \overline{\mathcal{D}}_{i\,a}{}^{--0} \cdot \mathcal{D}^{a}_{j\,-0-}. && (5.46) \end{aligned}$$

For (3.36) one has

$$d_{ij} = \overline{\mathcal{D}}_{j\,a}{}^{-+0} \cdot \mathcal{D}^{a}_{i\,-0-}, \qquad \bar{d}_{ij} = \mathcal{D}^{a}_{j\,-0+} \cdot \overline{\mathcal{D}}_{i\,a}{}^{--0}. \tag{5.47}$$

5.4 Conformal Blocks

A general CPW can be expressed as the integral of a product of three-point functions. For simplicity, consider the case where the external and internal operators are scalars. Given three-point functions $\langle\phi_1(x_1)\phi_2(x_3)\phi(x)\rangle$ and $\langle\phi(y)\phi_3(x_3)\phi_4(x_4)\rangle$, the following object is a solution to the conformal Casimir equation with the correct transformation properties to be a scalar CPW

$$\frac{1}{\mathcal{N}_\Delta}\int d^dx\, d^dy \langle\phi_1(x_1)\phi_2(x_3)\phi(x)\rangle \frac{1}{(x-y)^{2(d-\Delta)}}\langle\phi(y)\phi_3(x_3)\phi_4(x_4)\rangle, \tag{5.48}$$

where $\Delta = \Delta_\phi$. This can be understood, for example, by writing the integral in a manifestly conformally-invariant way [124].[16,17]

Let us denote the operation which glues two ϕ-correlators by[18]

$$|\phi\rangle \bowtie \langle\phi| \equiv \frac{1}{\mathcal{N}_\Delta}\int d^dx\, d^dy |\phi(x)\rangle \frac{1}{(x-y)^{2(d-\Delta)}}\langle\phi(y)| \tag{5.49}$$

We should choose the normalization $\mathcal{N}_\Delta$ by demanding that

$$\langle\phi|\; |\phi\rangle \bowtie \langle\phi| = \langle\phi|. \tag{5.50}$$

That is, we demand that the shadow integral acting on a two-point function $\langle\phi\phi\rangle$ gives the identity transformation. In the case of scalars, this fixes the normalization factor to be [124, 157]

$$\mathcal{N}_\Delta = \frac{\pi^d\Gamma(\Delta-\frac{d}{2})\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)\Gamma(d-\Delta)}. \tag{5.51}$$

[16] In Euclidean signature, we take the range of integration of x, y to be all of $\mathbb{R}^d$. In this case (5.48) produces a solution to the conformal Casimir equation with the wrong boundary conditions to be a conformal block. However, the conformal block can be extracted by taking a suitable linear combination of analytic continuations of the integral [124]. One can alternatively isolate the conformal block by performing the integral in Lorentzian signature over a domain defined by the lightcones of the four points x_1, x_2, x_3, x_4 [156]. Calculations involving differential operators are insensitive to these issues because the differential operators always transform trivially under monodromy. Thus, our methods allow us to study spinning versions of any of the solutions to the Casimir equation.

[17] We expect that (5.48) only converges when Δ lies on the principal series $\Delta \in \frac{d}{2} + i\mathbb{R}$. We obtain a general conformal block by analytically continuing in Δ.

[18] Instead of thinking of the gluing operation (5.49) in terms of shadow integrals, we can alternatively think of it as simply a sum over normalized descendants of ϕ. The only properties of the gluing procedure that we use in this work are that it is bilinear, conformally-invariant, and satisfies the normalization condition (**??**).

For spinning operators, $\mathcal{O}$ glues to its dual-reflected representation $\mathcal{O}^\dagger$ — i.e. the representation with which $\mathcal{O}$ has a nonzero two-point function,

$$|\mathcal{O}_{\Delta,\rho}\rangle \bowtie \langle\mathcal{O}^\dagger_{\Delta,\rho^\dagger}| \equiv \frac{1}{\mathcal{N}_{\Delta,\rho}}\int d^dx\, d^dy |\mathcal{O}_{\Delta,\alpha}(x)\rangle \frac{t^{\alpha\overline{\alpha}}(x-y)}{(x-y)^{2(d-\Delta)}}\langle\mathcal{O}^\dagger_{\Delta,\overline{\alpha}}(y)|. \tag{5.52}$$

Here, $t^{\alpha\overline{\alpha}}(x-y)$ is the tensor structure appearing in the two point function of the shadow operators $\langle\widetilde{\mathcal{O}}\widetilde{\mathcal{O}}^\dagger\rangle$. The operation (5.52) can be realized in the shadow formalism, see (B.2). Here we will not need the explicit expression of (5.52), but simply the normalization condition

$$\langle\mathcal{O}|\ |\mathcal{O}\rangle \bowtie \langle\mathcal{O}| = \langle\mathcal{O}|. \tag{5.53}$$

A general CPW is given by

$$W^{ab} \equiv \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle^{(a)} \bowtie {}^{(b)}\langle\mathcal{O}^\dagger\mathcal{O}_3\mathcal{O}_4\rangle \tag{5.54}$$

We will perform some explicit calculations using (5.54) and (5.52) in appendix B.1.

In what follows we will only need the formula which explains how to move covariant differential operators from one side of a shadow integration (5.52) to another

$$|\mathcal{D}^{(c)A}\mathcal{O}\rangle \bowtie \langle\mathcal{O}'^\dagger| = \sum_m \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}'^\dagger \\ \mathcal{O}' & W & \mathcal{O} \end{Bmatrix}^{\bullet c}_{\bullet m} |\mathcal{O}\rangle \bowtie \langle\mathcal{D}^{(m)A}\mathcal{O}'^\dagger|. \tag{5.55}$$

Equation (5.55) essentially implements two integrations by parts in the double integral (5.52).

To derive this formula we consider a two-point function. Moving a differential operator past a two-point vertex is a special case of the definition of a $6j$ symbol

$$\mathcal{D}^{(b)A}_{x_2}\langle\mathcal{O}^\dagger\mathcal{O}\rangle^{(\bullet)} = \sum_m \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}'^\dagger \\ \mathcal{O}' & W & \mathcal{O} \end{Bmatrix}^{\bullet b}_{\bullet c} \mathcal{D}^{(c)A}_{x_1}\langle\mathcal{O}'^\dagger\mathcal{O}'\rangle^{(\bullet)}. \tag{5.56}$$

Adding shadow integrals onto both $\mathcal{O}$ and $\mathcal{O}'$ in the above diagram and using (5.53), we find (5.55).

According to (5.54) the seed CPWs are given by

$$W^{(p)}_{\ell,\overline{\ell}} \equiv \langle F^{(0,0)}_{\Delta_1} F^{(p,0)}_{\Delta_2}\mathcal{O}^{(\ell,\overline{\ell})}_{\Delta}\rangle^{(\bullet)} \bowtie {}^{(\bullet)}\langle\overline{\mathcal{O}}^{(\overline{\ell},\ell)}_{\Delta} F^{(0,0)}_{\Delta_3} F^{(0,p)}_{\Delta_4}\rangle, \tag{5.57}$$

with $|\ell-\overline{\ell}| = p$. Depending on the values ℓ and $\overline{\ell}$ the seed blocks split into primal seed blocks and dual seed block, see the convention (2.41). In what follows we address the computation of seed and dual seed blocks separately.

5.4.1 Seed Blocks

We first rewrite the right three-point function entering (5.57) as

$$\langle\overline{\mathcal{O}}^{(\ell+p,\ell)}_{\Delta} F^{(0,0)}_{\Delta_3} F^{(0,p)}_{\Delta_4}\rangle^{(\bullet)} = (\overline{\mathcal{D}}_0^{-+0}\cdot\mathcal{D}_{4,-0+})\,\langle\overline{\mathcal{O}}^{(\ell+p-1,\ell)}_{\Delta+1/2} F^{(0,0)}_{\Delta_3} F^{(0,p-1)}_{\Delta_4+1/2}\rangle^{(\bullet)}. \tag{5.58}$$

The subscript 0 indicates that $\overline{\mathcal{D}}_0^{-+0}$ acts on the internal operator $\overline{\mathcal{O}}$. We would like to move it across $\bowtie$ (integrate by parts) using the rule (5.55).

Crossing of 2-point functions The definition of the $6j$ symbol entering (5.55) in the present case is

$$\begin{aligned}&\overline{\mathcal{D}}_{2\,a}^{+0-}\langle\overline{\mathcal{O}}_{\Delta}^{(\ell+p,\ell)}(X_1,S_1,\overline{S}_1)\mathcal{O}_{\Delta}^{(\ell,\ell+p)}(X_2,S_2,\overline{S}_2)\rangle^{(\bullet)}\\&=\mathcal{A}\,\overline{\mathcal{D}}_{1\,a}^{-+0}\langle\overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)}(X_1,S_1,\overline{S}_1)\mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)}(X_2,S_2,\overline{S}_2)\rangle^{(\bullet)},\end{aligned}\tag{5.59}$$

where

$$\mathcal{A}\equiv\begin{Bmatrix}\overline{\mathcal{O}}_{\Delta}^{(\ell+p,\ell)} & \mathbf{1} & \overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)}\\ \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} & \boldsymbol{\mathcal{S}} & \mathcal{O}_{\Delta}^{(\ell,\ell+p)}\end{Bmatrix}^{\bullet(+0-)}_{\bullet(-+0)}=2i(\ell+p)(\Delta-\tfrac{p}{2}-1)(\Delta-\ell-\tfrac{p}{2}-2).\tag{5.60}$$

Applying (5.55) and (5.58) to (5.57) we arrive at

$$W_{seed}^{(p)}=\mathcal{A}^{-1}\,(\overline{\mathcal{D}}_0^{+0-}\cdot\mathcal{D}_{4,-0+})\langle F_{\Delta_1}^{(0,0)}F_{\Delta_2}^{(p,0)}\mathcal{O}_{\Delta}^{(\ell,\ell+p)}\rangle^{(\bullet)}\bowtie^{(\bullet)}\langle\overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)}F_{\Delta_3}^{(0,0)}F_{\Delta_4}^{(0,p-1)}\rangle,\tag{5.61}$$

where $\overline{\mathcal{D}}_0^{+0-}$ now acts on the left three-point function.

Crossing of 3-point functions We now use the crossing equation for the 3-point function

$$\begin{aligned}\overline{\mathcal{D}}_{0\,a}^{+0-}\langle F_{\Delta_1}^{(0,0)}F_{\Delta_2}^{(p,0)}\mathcal{O}_{\Delta}^{(\ell,\ell+p)}\rangle^{(\bullet)}=&\sum_{n=1}^{2}\mathcal{B}^{(n)}\overline{\mathcal{D}}_{1\,a}^{--0}\langle F_{\Delta_1+1/2}^{(1,0)}F_{\Delta_2}^{(p,0)}\mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)}\rangle^{(n)}+\\&\sum_{n=1}^{2}\mathcal{C}^{(n)}\overline{\mathcal{D}}_{1\,a}^{+0-}\langle F_{\Delta_1-1/2}^{(0,1)}F_{\Delta_2}^{(p,0)}\mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)}\rangle^{(n)},\end{aligned}\tag{5.62}$$

where $\mathcal{B}^{(n)}$ and $\mathcal{C}^{(n)}$ denote the $6j$ symbols

$$\begin{aligned}\mathcal{B}^{(n)}&\equiv\begin{Bmatrix}F_{\Delta_1}^{(0,0)} & F_{\Delta_2}^{(p,0)} & F_{\Delta_1+1/2}^{(1,0)}\\ \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} & \boldsymbol{\mathcal{S}} & \mathcal{O}_{\Delta}^{(\ell,\ell+p)}\end{Bmatrix}^{\bullet(+0-)}_{(n)(--0)},\\ \mathcal{C}^{(n)}&\equiv\begin{Bmatrix}F_{\Delta_1}^{(0,0)} & F_{\Delta_2}^{(p,0)} & F_{\Delta_1-1/2}^{(0,1)}\\ \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} & \boldsymbol{\mathcal{S}} & \mathcal{O}_{\Delta}^{(\ell,\ell+p)}\end{Bmatrix}^{\bullet(+0-)}_{(n)(+0-)}.\end{aligned}\tag{5.63}$$

The 3-point functions in the right-hand side of (5.62) have the following form

$$\begin{aligned}\langle F_{\Delta_1+1/2}^{(1,0)}F_{\Delta_2}^{(p,0)}\mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)}\rangle^{(n)}&=\mathcal{K}_3[\hat{I}^{32}]^{p-1}[\hat{J}_{12}^{3}]^{\ell-1}\begin{pmatrix}\hat{I}^{32}\hat{K}_2^{13}\\ \hat{I}^{31}\hat{K}_1^{23}\end{pmatrix},\\ \langle F_{\Delta_1-1/2}^{(0,1)}F_{\Delta_2}^{(p,0)}\mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)}\rangle^{(n)}&=\mathcal{K}_3'[\hat{I}^{32}]^{p-1}[\hat{J}_{12}^{3}]^{\ell-1}\begin{pmatrix}\hat{I}^{13}\hat{I}^{32}\\ \hat{I}^{12}\hat{J}_{12}^{3}\end{pmatrix}.\end{aligned}\tag{5.64}$$

We can find the $6j$ symbols $\mathcal{B}^{(n)}$ and $\mathcal{C}^{(n)}$ by an explicit calculation

$$
\begin{aligned}
\mathcal{B}^{(1)} &= \mathcal{B}^{(2)} - \frac{\ell(\Delta_1+\Delta_2+\Delta-\ell-p-6)}{4(\Delta_1-2)}\times \\
&\quad \Big(4(\ell+p+1)(\Delta_1-\Delta_2+\ell+\frac{p}{2}+1)+(\Delta_1-\Delta_2+\Delta+\ell)(2\Delta-4\ell-3p-6)\Big)\,, \\
\mathcal{B}^{(2)} &= -\frac{p(\Delta_1-\Delta_2+\Delta+\ell)(2\Delta-2\ell-p-4)(\Delta_1+\Delta_2+\Delta-\ell-p-6)}{4(\Delta_1-2)}, \\
\mathcal{C}^{(1)} &= -\frac{\ell(2\Delta+p-2)(\Delta_1-\Delta_2-\Delta+\ell+p+2)}{4(\Delta_1-3)(\Delta_1-2)}, \\
\mathcal{C}^{(2)} &= \frac{p(-2\Delta+2\ell+p+4)(\Delta_1-\Delta_2-\Delta+\ell+p+2)}{4(\Delta_1-3)(\Delta_1-2)}.
\end{aligned}
\tag{5.65}
$$

Differential basis The last step is to relate the 3-point functions entering (5.62) to the seed 3-point functions $\langle F^{(0,0)}_{\Delta_1'} F^{(p-1,0)}_{\Delta_2'} \mathcal{O}^{(\ell,\ell+p-1)}_{\Delta+1/2}\rangle$ with shifted dimensions by using the differential basis trick

$$
\langle F^{(1,0)}_{\Delta_1+1/2}\, F^{(p,0)}_{\Delta_2}\, O^{(\ell,\,\ell+p-1)}_{\Delta+1/2}\rangle^{(n)} = M^{nr}_{\mathbf{1}} D^r_{\mathbf{1}}\, \langle F^{(0,0)}_{\Delta_1^r}\, F^{(p-1,\,0)}_{\Delta_2^r}\, O^{(\ell,\,\ell+p-1)}_{\Delta+1/2}\rangle^{(\bullet)}, \tag{5.66}
$$
$$
\langle F^{(0,1)}_{\Delta_1-1/2}\, F^{(p,0)}_{\Delta_2}\, O^{(\ell,\,\ell+p-1)}_{\Delta+1/2}\rangle^{(n)} = M^{nr}_{\mathbf{2}} D^r_{\mathbf{2}}\, \langle F^{(0,0)}_{\Delta_1'^j}\, F^{(p-1,\,0)}_{\Delta_2'^j}\, O^{(\ell,\,\ell+p-1)}_{\Delta+1/2}\rangle^{(\bullet)}, \tag{5.67}
$$

where the differential operators are defined as

$$
D^{r=1}_{\mathbf{1}} \equiv (\overline{\mathcal{D}}_1^{-+0}\cdot \mathcal{D}_{2,++0}) \quad \textbf{and} \quad D^{r=2}_{\mathbf{1}} \equiv (\mathcal{D}_{1,++0}\cdot \overline{\mathcal{D}}_2^{-+0}), \tag{5.68}
$$
$$
D^{r=1}_{\mathbf{2}} \equiv (\overline{\mathcal{D}}_1^{+0+}\cdot \mathcal{D}_{2,++0}) \quad \textbf{and} \quad D^{r=2}_{\mathbf{2}} \equiv (\mathcal{D}_{1,-0+}\cdot \overline{\mathcal{D}}_2^{-+0}). \tag{5.69}
$$

One can read off easily the dimension in the right hand side of (5.66)

$$
\Delta_1^{r=1} = \Delta_1+1, \quad \Delta_2^{r=1} = \Delta_2-1/2, \quad \Delta_1^{r=2} = \Delta_1, \quad \Delta_2^{r=2} = \Delta_2+1/2, \tag{5.70}
$$
$$
\Delta_1'^{r=1} = \Delta_1-1, \quad \Delta_2'^{r=1} = \Delta_2-1/2, \quad \Delta_1'^{r=2} = \Delta_1, \quad \Delta_2'^{r=2} = \Delta_2+1/2. \tag{5.71}
$$

The matrices $M^{nr}_{\mathbf{1}}$ and $M^{nr}_{\mathbf{2}}$ are found to be

$$
M^{nr}_{\mathbf{1}} = \frac{1}{\ell(2\Delta+p-2)}\begin{pmatrix} \frac{\Delta_1-\Delta_2+\Delta+\ell}{2\Delta_2+p-4} & \frac{-\Delta_1+\Delta_2+\Delta-\ell+p-2}{2(\Delta_1-1)} \\ \frac{-\Delta_1+\Delta_2-\Delta+\ell}{2\Delta_2+p-4} & -\frac{-\Delta_1+\Delta_2+\Delta+\ell+p-2}{2(\Delta_1-1)} \end{pmatrix} \tag{5.72}
$$

and

$$
M^{nr}_{\mathbf{2}} = \begin{pmatrix} -\frac{1}{(\Delta_1-2)\ell(2\Delta_2+p-4)(2\Delta+p-2)} & -\frac{(\Delta_1+\Delta_2-\Delta+\ell-2)(-\Delta_1-\Delta_2-\Delta+\ell-p+6)}{2\ell(2\Delta+p-2)} \\ 0 & 1 \end{pmatrix}. \tag{5.73}
$$

The recursion relation Combining the expressions (5.61), (5.62), and the differential basis (5.66) we find the following recursion relation

$$
\begin{aligned}
&W^{(p)}_{\Delta,\ell;\,\Delta_1,\Delta_2,\Delta_3,\Delta_4} = \\
&\qquad \mathcal{A}^{-1}\Bigg(v_1(\overline{\mathcal{D}}_1^{--0}\cdot\mathcal{D}_{4,-0+})(\overline{\mathcal{D}}_1^{-+0}\cdot\mathcal{D}_{2,++0})\, W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\,\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\qquad\quad +v_2(\overline{\mathcal{D}}_1^{--0}\cdot\mathcal{D}_{4,-0+})(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}}_2^{-+0})\, W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\qquad\quad +v_3(\overline{\mathcal{D}}_1^{+0-}\cdot\mathcal{D}_{4,-0+})(\overline{\mathcal{D}}_1^{+0+}\cdot\mathcal{D}_{2,++0})\, W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\,\Delta_1-1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\qquad\quad +v_4(\overline{\mathcal{D}}_1^{+0-}\cdot\mathcal{D}_{4,-0+})(\mathcal{D}_{1,-0+}\cdot\overline{\mathcal{D}}_2^{-+0})\, W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}\Bigg),
\end{aligned}
\tag{5.74}
$$

where the coefficients v_i are given by

$$
\begin{aligned}
v_1 &\equiv \mathcal{B}^{(1)}\,M^{11}_{\mathbf{1}} + \mathcal{B}^{(2)}\,M^{21}_{\mathbf{1}}, \quad v_2 \equiv \mathcal{B}^{(1)}\,M^{12}_{\mathbf{1}} + \mathcal{B}^{(2)}\,M^{22}_{\mathbf{1}},\\
v_3 &\equiv \mathcal{C}^{(1)}\,M^{11}_{\mathbf{2}} + \mathcal{C}^{(2)}\,M^{21}_{\mathbf{2}}, \quad v_4 \equiv \mathcal{C}^{(1)}\,M^{11}_{\mathbf{2}} + \mathcal{C}^{(2)}\,M^{21}_{\mathbf{2}}
\end{aligned}
$$

and according to (5.65), (5.72) and (5.73) have the following explicit form

$$
\begin{aligned}
v_1 &= \frac{(\Delta+\Delta_1-\Delta_2+\ell)(-\Delta-\Delta_1+\Delta_2+\ell+2)(\Delta+\Delta_1+\Delta_2-\ell-p-6)}{4(\Delta_1-2)(2\Delta_2+p-4)},\\
v_2 &= \frac{(-\Delta+\Delta_1-\Delta_2+\ell+p+2)(\Delta+\Delta_1-\Delta_2-\ell-2p-2)(\Delta+\Delta_1+\Delta_2-\ell-p-6)}{8(\Delta_1-2)(\Delta_1-1)},\\
v_3 &= \frac{-\Delta+\Delta_1-\Delta_2+\ell+p+2}{4(\Delta_1-3)(\Delta_1-2)^2(2\Delta_2+p-4)},\\
v_4 &= -\frac{(-\Delta+\Delta_1-\Delta_2+\ell+p+2)(-\Delta+\Delta_1+\Delta_2+\ell+2p-2)(\Delta+\Delta_1+\Delta_2-\ell-p-6)}{8(\Delta_1-3)(\Delta_1-2)}.
\end{aligned}
\tag{5.75}
$$

Decomposition into conformal blocks By using (2.43) one can write the recursion relation (5.74) at the level of seed conformal blocks $H^{(p)}_e(z,\overline{z})$.

First we notice that according to chapter 4 the components $H^{(p)}_e(z,\overline{z})$ of the seed blocks depend on the external scaling dimensions Δ_i only via the quantities

$$
a^p_e \equiv a^{(p)}, \quad b^p_e \equiv b^{(p)} + p - e, \quad c^p_e \equiv p - e, \tag{5.76}
$$

where

$$
a^{(p)} \equiv -\frac{\Delta_1-\Delta_2-p/2}{2}, \quad b^{(p)} \equiv +\frac{\Delta_3-\Delta_4-p/2}{2}. \tag{5.77}
$$

Let us now analyze the expression (5.74). Almost all the conformal blocks entering the right hand side of (5.74) correspond to the same parameters $a^{(p)}$ and $b^{(p)}$ (the difference in p is compensated by a difference in Δ_i). The only exception is the conformal block

$$
W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\,\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \tag{5.78}
$$

which contains $a^{(p)}-1$ and $b^{(p)}$. We can use a dimension shifting operator to simplify the structure of the recursion relation (5.74). The only difference is that we need to shift the external dimensions of a general seed block. We find

$$W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\ \Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} = \mathcal{E}^{-1}(\mathcal{D}_{1,+-0}\cdot\overline{\mathcal{D}_2}^{--0})(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}_2}^{-+0})W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\ \Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}, \tag{5.79}$$

where

$$\mathcal{E} \equiv -(p+1)(\Delta_1-1)(\Delta_1-2)(\Delta+\Delta_1-\Delta_2+\ell)(\Delta+\Delta_1-\Delta_2-\ell-2). \tag{5.80}$$

Note that this is in fact completely analogous to the differential basis trick, except that instead of changing the external spins, we change the external dimensions.

Plugging the relation (5.79) in (5.74), stripping off the kinematic factor and decomposing this relation into components according to (2.43) one obtains a recursion relation for the seed blocks of the form

$$H^{(p)}_e(z,\bar z) = -\frac{\mathcal{A}^{-1}}{z-\bar z}\left(D_0\ H^{(p-1)}_e(z,\bar z) - 2D_1\ H^{(p-1)}_{e-1}(z,\bar z) + 4c^{p-1}_{e-2} z\bar z D_2\ H^{(p-1)}_{e-2}(z,\bar z)\right), \tag{5.81}$$

where the conformal block in the l.h.s depends on $[\Delta,\ell;\ \Delta_1,\Delta_2,\Delta_3,\Delta_4]$ while the conformal blocks in the r.h.s. depend on $[\Delta+\frac{1}{2},\ell;\ \Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}]$. The differential operators D_i are given by

$$\begin{aligned}
D_0 \equiv& \nabla_{\bar z}[b^{p-1}_e]D^{(p-1,e)}_z - \nabla_z[b^{p-1}_e]D^{(p-1,e)}_{\bar z} \\
&+ k\left(D^{(p-1,e)}_z - D^{(p-1,e)}_{\bar z}\right) - (c^{p-1}_e+1)L[b^{p-1}_e]B\left[-\frac{k(k-2)}{1+c^{p-1}_e}\right], \qquad (5.82)\\
D_1 \equiv& z\nabla_{\bar z}[b^{p-1}_{e-1}+c^{p-1}_{e-1}]D^{(p-1,e-1)}_z - \bar z\nabla_z[b^{p-1}_{e-1}+c^{p-1}_{e-1}]D^{(p-1,e-1)}_{\bar z} \\
&+ k\left(zD^{(p-1,e-1)}_z - \bar z D^{(p-1,e-1)}_{\bar z}\right) \\
&+ (2c^{p-1}_{e-1}+1)z\bar z L[b^{p-1}_{e-1}](z-\bar z)^{-1}L[a] - (k-2)(k-c^{p-1}_{e-1}-1)(z-\bar z)B[k], \qquad (5.83)\\
D_2 \equiv& D^{(p-1,e-2)}_z - D^{(p-1,e-2)}_{\bar z} - L[a]B\left[k-c^{p-1}_{e-2}-1\right], \qquad (5.84)
\end{aligned}$$

where the coefficient k is

$$k \equiv \frac{4-\Delta+\ell}{2} + \frac{3p}{4}. \tag{5.85}$$

The elementary differential operators[19] used here are

$$\begin{aligned}
D^{(a,b;c)}_x &\equiv x^2(1-x)\partial^2_x - \left((a+b+1)x^2 - cx\right)\partial_x - abx, &(5.86)\\
\nabla_x[\mu] &\equiv -x(1-x)\partial_x + \mu\, x, &(5.87)\\
L[\mu] &\equiv \nabla_z[\mu] - \nabla_{\bar z}[\mu], &(5.88)\\
B[\mu] &\equiv \frac{z\bar z}{z-\bar z}\left((1-z)\partial_x - (1-\bar z)\partial_{\bar z}\right) + \mu, &(5.89)
\end{aligned}$$

and we also use the following short-hand notation

$$D^{(p,e)}_x \equiv D^{(a^p_e,b^p_e;c^p_e)}_x. \tag{5.90}$$

5.4.2 Dual Seed Blocks

In this appendix we provide the final expression for the dual seed conformal blocks recursion relation omitting all the derivations. All the quantities below carry a bar to distinguish them

[19] Exactly the same differential operators (except for $\nabla_x[\mu]$) enter the quadratic Casimir equation for the seed blocks in chapter 4. Note that here the definition of L differs by a factor of $z-\bar z$.

from their analogous in the seed case.

By performing calculation completely analogous to the previous section, we find that the dual seed conformal blocks obey the following recursion relation

$$\begin{aligned}
&\overline{W}^{(p)}_{\Delta,\ell;\,\Delta_1,\Delta_2,\Delta_3,\Delta_4} = \\
&\qquad \overline{\mathcal{A}}^{-1}\Big(\overline{v}_1\,(\overline{\mathcal{D}}_1^{--0}\cdot\mathcal{D}_{4,-0+})\,(\overline{\mathcal{D}}_1^{-+0}\cdot\mathcal{D}_{2,++0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\qquad\quad +\overline{v}_2\,(\overline{\mathcal{D}}_1^{--0}\cdot\mathcal{D}_{4,-0+})\,(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}}_2^{-+0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\qquad\quad +\overline{v}_3\,(\overline{\mathcal{D}}_1^{+0-}\cdot\mathcal{D}_{4,-0+})\,(\overline{\mathcal{D}}_1^{+0+}\cdot\mathcal{D}_{2,++0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1-1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\qquad\quad +\overline{v}_4\,(\overline{\mathcal{D}}_1^{+0-}\cdot\mathcal{D}_{4,-0+})\,(\mathcal{D}_{1,-0+}\cdot\overline{\mathcal{D}}_2^{-+0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}\Big), \qquad (5.91)
\end{aligned}$$

where the coefficients are[20]

$$\overline{\mathcal{A}} = -\frac{i\,(\ell+p)(\Delta+\Delta_3-\Delta_4+\ell+p-2)}{2\Delta+2\ell+p-2} \qquad (5.92)$$

and

$$\begin{aligned}
\overline{v}_1 &= \frac{(\Delta-\Delta_1-\Delta_2+\ell+p+2)(-\Delta-\Delta_1+\Delta_2+\ell+p+2)}{2(\Delta_1-2)(2\Delta+p-4)(2\Delta_2+p-4)}, \\
\overline{v}_2 &= -\frac{(\Delta-\Delta_1-\Delta_2+\ell+p+2)(\Delta-\Delta_1+\Delta_2+\ell+2p-2)}{4(\Delta_1-2)(\Delta_1-1)(2\Delta+p-4)}, \\
\overline{v}_3 &= -\frac{1}{2(\Delta_1-3)(\Delta_1-2)^2(2\Delta+p-4)(2\Delta_2+p-4)}, \\
\overline{v}_4 &= -\frac{(\Delta-\Delta_1-\Delta_2+\ell+p+2)(\Delta+\Delta_1+\Delta_2+\ell+2p-6)}{4(\Delta_1-3)(\Delta_1-2)(2\Delta+p-4)}. \qquad (5.93)
\end{aligned}$$

Analogously to the primal seed case, we replace one of the conformal blocks on the right hand side of (5.91) by using the dimension-shifting operator

$$\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} = \overline{\mathcal{E}}^{-1}\,(\mathcal{D}_{1,+-0}\cdot\overline{\mathcal{D}}_2^{--0})(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}}_2^{-+0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}, \qquad (5.94)$$

where

$$\overline{\mathcal{E}} \equiv (p+1)(\Delta_1-2)(\Delta_1-1)(\Delta+\Delta_1-\Delta_2+l+p-2)(-\Delta-\Delta_1+\Delta_2+l+p+2). \qquad (5.95)$$

Decomposition into components Plugging the relation (5.94) in (5.91), stripping off the kinematic factor and decomposing this relation into four-point tensor structures according to (2.44) one obtains a recursion relation for the dual seed blocks of the form analogous to (5.81)

$$\overline{H}^{(p)}_e(z,\overline{z}) = -\frac{\overline{\mathcal{A}}'^{-1}}{z-\overline{z}}\left(D_0\,\overline{H}^{(p-1)}_e(z,\overline{z}) - 2D_1\,\overline{H}^{(p-1)}_{e-1}(z,\overline{z}) + 4c^{p-1}_{e-2}z\overline{z}D_2\,\overline{H}^{(p-1)}_{e-2}(z,\overline{z})\right), \qquad (5.96)$$

where the blocks in the l.h.s depend on $[\Delta,\ \ell;\ \Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4]$ and the blocks in the r.h.s. depend on $[\Delta-\frac{1}{2},\ \ell;\ \Delta_1,\ \Delta_2+\frac{1}{2},\ \Delta_3,\ \Delta_4+\frac{1}{2}]$.

[20] Here $\overline{\mathcal{A}}$ is not the $6j$ symbol analogous to $\mathcal{A}$, but simply an overall coefficient.

The overall coefficient is

$$\overline{\mathcal{A}}' \equiv -(\Delta + \frac{p}{2} - 2)(\Delta + \Delta_1 - \Delta_2 + l + p - 2)\,\overline{\mathcal{A}}. \tag{5.97}$$

The differential operators D_i are given by the expression (5.82)-(5.84) with the parameter k replaced by $\overline{k}$

$$\overline{k} \equiv \frac{\Delta + \ell}{2} + \frac{3p}{4}. \tag{5.98}$$

Chapter 6

Conclusions and Discussions

In this thesis we have developed a formalism for working with arbitrary spin 2-, 3- and 4-point functions and for constructing general bootstrap equations in 4D. Here we discuss further the obtained results. In the end we comment on their applications.

chapter 2 We have made an overview of our framework for constructing generic 2-, 3-, 4-point functions and the bootstrap equations. The precise formalism for constructing tensor structures and differential operators is addressed in chapter 3 instead. We have explained how to reduce generic conformal blocks to the set of simpler seed conformal blocks. Depending on the spin they split into 2 dual types: the seed $H_e^{(p)}$ and the dual seed $\overline{H}_e^{(p)}$ blocks. We have computed them in chapters 4 and 5.

chapter 3 We have developed and unified two alternative formalism for performing computations in 4D CFTs: covariant (embedding space) approach and non-covariant (conformal frame) approach.

In the embedding formalism we have explained the recipe for constructing tensor structures of n-point functions in the 6D embedding space. We have also summarized the so called spinning differential operators relating generic CPWs to the seed CPWs. The conformally covariant expressions in 4D are easily obtained from the 6D expressions by using the so called projection operation. For the objects like kinematic factors and 2-, 3-, and 4-point tensor structures we have performed the projection operation explicitly.

The construction of a *basis* of tensor structures in the embedding formalism requires however the knowledge of a complete set of non-linear relations between products of the basic conformal invariants. Starting from $n = 4$ it is rather difficult to find such a set of relations and thus the embedding formalism turns out to be practically inefficient for $n \geq 4$. This problem is solved using the conformal frame approach.

In the conformal frame we have provided a complete basis for ($n \geq 4$)-point tensor structures in a remarkably simple form. For instance in the $n = 4$ case the tensor structures are simply monomials in polarization spinors with vanishing total charge under the $U(1)$ little group. In the $n < 4$ cases the little group is larger and constructing its singlets becomes harder whereas the embedding formalism is easily manageable. Since the embedding formalism is also explicitly covariant it becomes preferable for working with 2- and 3-point functions.

With practical applications in mind, we have found the action of various differential operators on 4-point functions in the conformal frame formalism. We have also shown how to apply permutations in the conformal frame. These results allow one to work with the 4-point functions (and, consequently, the crossing equations) entirely within the conformal frame formalism.

We have established a connection between the tensor structures constructed in the embedding and the conformal frame formalisms. The embedding formalism to conformal frame transition is straightforward and amounts to performing the 4D projection of the 6D structures and setting all the coordinates to the conformal frame. The conformal frame to the embedding formalism transition is slightly more complicated since it is not uniquely defined due to redundancies among the allowed 6D structures. After "translating" all the basic 6D structures to the conformal frame one inverts these relations by choosing only the independent 6D structures.

In the appendices we made our best effort to establish consistent conventions; we have provided a proper normalization of 2-point functions and the seed conformal blocks and summarized all the Casimir differential operators available in 4D. We have also given some extra details on permutation symmetries and conserved operators.

chapter 4 We have computed the seed conformal blocks by solving the system of Casimir equations (4.4) using an ansatz. The final solution is given in (4.33) and (4.34). The coefficients $c^e_{m,n}$ can be determined recursively, by means of (4.32). An explicit example of the $p = 1$ coefficients is given in (B.53). For each conformal block, the coefficients $c^e_{m,n}$ span a 2D octagon-shape lattice in the (m,n) plane with the size depending on p and e.

We have computed explicitly the coefficients $c^e_{m,n}$ for completely generic values of ℓ, Δ and external dimensions for $p \leq 4$. Unfortunately these explicit solutions have a very complicated form and become unmanageable for $p > 5$ in a generic situation.

Things improve when studying the blocks in the light-cone limit. In that case the seed blocks either contain (for exchanged operators with finite spin) only very few terms with very simple coefficients $c^e_{m,n}$[1] or (for exchanged operators with large spin) the form of generic $c^e_{m,n}$ simplify considerably in the large ℓ limit.

Unfortunately even the $p \leq 4$ solutions are not applicable for performing numerical bootstrap since their present form does not admit an efficient polynomial approximation. New insights on the seed blocks discussed in chapter 5 are needed.

chapter 5 We introduced new mathematical tools for performing computations in conformal representation theory. We show the existence of covariant (weight-shifting) differential operators corresponding to finite-dimensional representations of the conformal group. We construct explicitly a list of such operators in (5.38) corresponding to fundamental and anti-fundamental representations of the conformal group SU(2, 2) in 4D.

We then make a connection with spinning differential operators 3.35 and 3.36. We show that they essentially are the invariant product of covariant differential operators corresponding to fundamental, anti-fundamental, anti-symmetric and adjoint representations of SU(2, 2).

We discuss group-theoretic property (5.23) of covariant differential operators which allows to move their action from one point to another. Finally we combine this property with the shadow techniques to obtain an "integration by parts" formula 5.55.

The main application of this formalism is a reduction of any spinning conformal block to the scalar Dolan and Osborn conformal block. We apply this to the case of seed blocks. The result is the recursion relations (5.81) and (5.96). These recursion relations match precisely the solution obtained in chapter 4: (4.33) and (4.34). We will discuss their practical importance below.

The CFTs4D Package We have implemented our framework in a Mathematica package "CFTs4D" freely available at https://gitlab.com/bootstrapcollaboration/CFTs4D. It can perform any manipulations with generic 2-, 3- and 4-point functions in both covariant and non-covariant formalisms. It also allows to switch between formalism when needed. The explicit form of conformal blocks is implemented for $p \leq 4$ together with their recursion

[1] These coefficients always lie on the boundary of the octagon and have the simplest form among all, see for instance (B.53) for a $p = 1$ example.

relations. We have also implemented spinning differential operators together with more fundamental weight-shifting operators. A detailed documentation is incorporated in the package with many explicit examples.

Future Applications It is our hope that the results of this thesis will facilitate the bootstrap studies of 4D CFTs using spinning correlators such as 4-point functions of fermionic operators, global symmetry currents and stress-energy tensors.

Analytic Bootstrap The formalism of weight-shifting operators will be very useful for generalization of the scalar inversion formula [69] to the spinning case: using the integration by parts procedure one can obtain a spinning inversion formula by reducing it to the scalar one. It can then be used for a systematic analytic studies of spinning 4-point functions.

Numerical Bootstrap For performing numerical analysis the full analytic expressions for spinning conformal blocks are not suitable and not needed. Instead one needs an approximate rational form of them and their derivatives at the crossing-symmetric $z = \overline{z} = \frac{1}{2}$ point. The recursion relations (5.81) and (5.96) provide an efficient way for finding such an approximation.

The idea is to rewrite the relations (5.81) and (5.96) for the seeds and their derivatives at the crossing symmetric point and then use them to successively obtain a polynomial approximation[2] of the p-seeds starting from the polynomial approximation of the $p = 0$ blocks (which is well studied).

We hope to perform an analysis of 4-fermion correlator which provides an access to composite "baryons" of gauge theories in the near future.

[2] Instead of dealing with rational approximations we will work only with polynomials keeping track of the denominators separately.

Appendix A

Details of the Framework

A.1 Details of the 4D Formalism

We work in the signature $-+++$ and denote the diagonal 4D Minkowski metric by $h_{\mu\nu}$. We mostly follow the conventions of Wess and Bagger [158].

The representations of the connected Lorentz group in 4D are labeled by a pair of non-negative integers $(\ell, \bar{\ell})$. These representations can be constructed as the highest-weight irreducible components in a tensor product of the two basic spinor representations $(1,0)$ and $(0,1)$.

We denote the objects in the left-handed spinor representation $(1,0)$ as ψ_α, $\alpha = 1,2$, and the objects in its dual representation as ψ^α. The original and the dual representations are equivalent via the identification

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \tag{A.1}$$

where

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = +1. \tag{A.2}$$

Because of the equivalence between $(1,0)$ and its dual representation, we will not be careful to distinguish them in the text, the distinction in formulas will be clear from the location of indices.

The right-handed spinor representation $(0,1)$ is the complex conjugate of the left-handed spinor representation, and the objects transforming in $(0,1)$ representation will be denoted as $\chi_{\dot{\alpha}}$. Here the dot should not be considered as part of the index, but rather as an indication that this index transforms in $(0,1)$ and not in $(1,0)$ representation. For example, the definition of $(0,1)$ representation is essentially

$$\psi^\dagger_{\dot{\alpha}} = (\psi_\alpha)^\dagger. \tag{A.3}$$

The dual of $(0,1)$ is equivalent to $(0,1)$ via the conjugation of (A.1)

$$\chi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\chi^{\dot{\beta}}, \quad \chi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\beta}}, \tag{A.4}$$

where $\epsilon_{\dot{\alpha}\dot{\beta}} \equiv \epsilon_{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}} \equiv \epsilon^{\alpha\beta}$. We use the contraction conventions

$$\psi_1\psi_2 = \psi_1^\alpha\psi_{2\alpha}, \quad \chi_1\chi_2 = \chi_{1\dot{\alpha}}\chi_2^{\dot{\alpha}}. \tag{A.5}$$

The tensor product $(1,0)\otimes(0,1) = (1,1)$ is equivalent to the vector representation, and the equivalence is established by the 4D sigma matrices $\sigma^\mu_{\alpha\dot{\beta}}$ and $\overline{\sigma}^{\mu\dot{\alpha}\beta}$, which we define as

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.6}$$

and $\overline{\sigma}^0 = \sigma^0$, $\overline{\sigma}^1 = -\sigma^1$, $\overline{\sigma}^2 = -\sigma^2$, $\overline{\sigma}^3 = -\sigma^3$. For a convenient summary of relations involving sigma-matrices see for example [159].[1]

For primary operators we adopt the convention to write them out with dotted indices upstairs and the undotted indices downstairs

$$\mathcal{O}^{\dot\alpha_1\ldots\dot\alpha_{\overline{\ell}}}_{\beta_1\ldots\beta_\ell}, \quad \left(\mathcal{O}^{\dot\alpha_1\ldots\dot\alpha_{\overline{\ell}}}_{\beta_1\ldots\beta_\ell}\right)^\dagger = \mathcal{O}^{\dagger\,\alpha_1\ldots\alpha_{\overline{\ell}}}_{\dot\beta_1\ldots\dot\beta_\ell}. \tag{A.7}$$

In this notation the index-full version of (2.6) is

$$\overline{\mathcal{O}}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_{\overline{\ell}}} \equiv (-1)^{\overline{\ell}-\ell}\,\epsilon_{\alpha_1\alpha'_1}\ldots\epsilon_{\alpha_{\overline{\ell}}\alpha'_{\overline{\ell}}}\epsilon^{\dot\beta_1\dot\beta'_1}\ldots\epsilon^{\dot\beta_\ell\dot\beta'_\ell}\mathcal{O}^{\dagger\,\alpha'_1\ldots\alpha'_{\overline{\ell}}}_{\dot\beta'_1\ldots\dot\beta'_\ell}. \tag{A.8}$$

Action of Conformal Generators We denote the conformal generators by P, K, D, M. We choose to work with anti-Hermitian generators (related to the Hermitian ones by a factor of i)

$$D^\dagger = -D, \quad P^\dagger = -P, \quad K^\dagger = -K, \quad M^\dagger = -M, \tag{A.9}$$

which allow us to avoid many factors of i in the formulas below (note that even though D is anti-Hermitian, its adjoint action has real eigenvalues). These generators satisfy the following algebra

$$[D, D] = 0, \quad [D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \tag{A.10}$$

$$[P_\mu, P_\nu] = 0, \quad [K_\mu, K_\nu] = 0, \quad [K_\mu, P_\nu] = 2h_{\mu\nu}D - 2M_{\mu\nu}, \tag{A.11}$$

$$[M_{\mu\nu}, D] = 0, \quad [M_{\mu\nu}, P_\rho] = h_{\nu\rho}P_\mu - h_{\mu\rho}P_\nu, \quad [M_{\mu\nu}, K_\rho] = h_{\nu\rho}K_\mu - h_{\mu\rho}K_\nu, \tag{A.12}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = h_{\nu\rho}M_{\mu\sigma} - h_{\mu\rho}M_{\nu\sigma} - h_{\nu\sigma}M_{\mu\rho} + h_{\mu\sigma}M_{\nu\rho}. \tag{A.13}$$

The action of the conformal generators on primary fields is given by

$$[D, \mathcal{O}(x, s, \overline{s})] = (x^\mu\partial_\mu + \Delta)\,\mathcal{O}(x, s, \overline{s}), \tag{A.14}$$

$$[P_\mu, \mathcal{O}(x, s, \overline{s})] = \partial_\mu\,\mathcal{O}(x, s, \overline{s}), \tag{A.15}$$

$$[K_\mu, \mathcal{O}(x, s, \overline{s})] = (2x_\mu x^\sigma - x^2\delta^\sigma_\mu)\partial_\sigma\mathcal{O}(x, s, \overline{s}) + 2(\Delta\, x_\mu - x^\sigma\mathcal{M}_{\mu\sigma})\mathcal{O}(x, s, \overline{s}), \tag{A.16}$$

$$[M_{\mu\nu}, \mathcal{O}(x, s, \overline{s})] = (x_\nu\partial_\mu - x_\mu\partial_\nu)\mathcal{O}(x, s, \overline{s}) + \mathcal{M}_{\mu\nu}\mathcal{O}(x, s, \overline{s}), \tag{A.17}$$

where the spin generators are

$$\mathcal{M}_{\mu\nu}\mathcal{O}(x, s, \overline{s}) = \left(-s^\alpha(SS_{\mu\nu})_\alpha{}^\beta\frac{\partial}{\partial s^\beta} - \overline{s}_{\dot\alpha}(\overline{SS}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\frac{\partial}{\partial \overline{s}_{\dot\beta}}\right)\mathcal{O}(x, s, \overline{s}). \tag{A.18}$$

We have defined here the generators of the left- and right-handed spinor representations

$$(SS_{\mu\nu})_\alpha{}^\beta = -\frac{1}{4}(\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu)_\alpha{}^\beta, \tag{A.19}$$

$$(\overline{SS}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{4}(\overline{\sigma}_\mu\sigma_\nu - \overline{\sigma}_\nu\sigma_\mu)^{\dot\alpha}{}_{\dot\beta}, \tag{A.20}$$

[1] One should download and compile the version with mostly plus metric. Notice also a factor of i difference between their $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ and ours $SS^{\mu\nu}$ and $\overline{SS}^{\mu\nu}$.

which satisfy the same commutation relations as $M_{\mu\nu}$. Notice that as usual the differential operators in the right hand side of (A.14)-(A.17) have the commutation relations opposite to those of the Hilbert space operators in the left hand side. This is because if the Hilbert space operators A and B act on fields by differential operators $\mathfrak{A}$ and $\mathfrak{B}$, then their product AB acts by $\mathfrak{B}\mathfrak{A}$.

Action of Space Parity If a theory preserves parity, there exists a unitary operator $\mathcal{P}$ with the following commutation rule with Lorentz generators

$$\mathcal{P}M_{0i}\mathcal{P}^{-1} = -M_{0i}, \quad \mathcal{P}M_{ij}\mathcal{P}^{-1} = M_{ij}, \tag{A.21}$$

where $i,j = 1,2,3$. Applying this to (A.17) at $x=0$, we see that

$$[M_{\mu\nu}, \mathcal{P}\mathcal{O}_\alpha(0)\mathcal{P}^{-1}] = (\overline{S}S_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\mathcal{P}\mathcal{O}_\beta(0)\mathcal{P}^{-1}. \tag{A.22}$$

This implies that we can define an operator $\widetilde{\mathcal{O}}$ as

$$\widetilde{\mathcal{O}}^{\dot\alpha}(x) \equiv -i\mathcal{P}\mathcal{O}_\alpha(\mathcal{P}x)\mathcal{P}^{-1} \tag{A.23}$$

which transform as a primary operator in the representation $(0,1)$. We also have $\mathcal{P}x^0 = x^0$, $\mathcal{P}x^k = -x^k$, $k = 1,2,3$. More generally, it is easy to check that we can consistently define

$$\widetilde{\mathcal{O}}^{\dot\alpha_1\ldots\dot\alpha_\ell}_{\beta_1\ldots\beta_{\bar\ell}}(x) \equiv (-i)^{\ell+\bar\ell}\mathcal{P}\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(\mathcal{P}x)\mathcal{P}^{-1}. \tag{A.24}$$

The factor of i was introduced to reproduce the standard parity action on traceless symmetric operators in the $\widetilde{\mathcal{O}} = \mathcal{O}$ case.

The above definition provides the most generic action of parity on the operators $\mathcal{O}$ which can be slightly rewritten as

$$\mathcal{P}\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x)\mathcal{P}^{-1} = i^{\ell+\bar\ell}\widetilde{\mathcal{O}}^{\dot\alpha_1\ldots\dot\alpha_\ell}_{\beta_1\ldots\beta_{\bar\ell}}(\mathcal{P}x), \tag{A.25}$$

or equivalently in index-free notation

$$\mathcal{P}\mathcal{O}(x,s,\bar s)\mathcal{P}^{-1} = \widetilde{\mathcal{O}}(\mathcal{P}x,\mathcal{P}s,\mathcal{P}\bar s), \quad (\mathcal{P}\bar s)_{\dot\alpha} = is^\alpha, \quad (\mathcal{P}s)^\alpha = i\bar s_{\dot\alpha}. \tag{A.26}$$

Notice that if $\mathcal{O}$ transforms in the $(\ell,\bar\ell)$ representation then the operator $\widetilde{\mathcal{O}}$ transforms in $(\bar\ell,\ell)$ and may or may not be related to the operator $\overline{\mathcal{O}}$ defined in (2.6) or to $\mathcal{O}$ itself if $\ell = \bar\ell$. This depends on a specific theory. What is important for us is that in a theory which preserves $\mathcal{P}$ there is a relation between correlators involving $\mathcal{O}_i$ and $\widetilde{\mathcal{O}}_i$

$$\begin{aligned}\langle 0|\mathcal{O}_1(x_1,s_1,\bar s_1)\cdots\mathcal{O}_n(x_n,s_n,\bar s_n)|0\rangle =& \\ &= \langle 0|\mathcal{P}\mathcal{O}_1(x_1,s_1,\bar s_1)\mathcal{P}^{-1}\cdots\mathcal{P}\mathcal{O}_n(x_n,s_1,\bar s_1)\mathcal{P}^{-1}|0\rangle \\ &= \langle 0|\widetilde{\mathcal{O}}_1(\mathcal{P}x_1,\mathcal{P}s_1,\mathcal{P}\bar s_1)\cdots\widetilde{\mathcal{O}}_n(\mathcal{P}x_n,\mathcal{P}s_n,\mathcal{P}\bar s_n)|0\rangle.\end{aligned} \tag{A.27}$$

Written in terms of tensor structures this equality reads as

$$\sum_I \mathbb{T}^I_n g^I_n = \sum_I (\mathcal{P}\widetilde{\mathbb{T}}^I_n)\widetilde{g}^I_n, \tag{A.28}$$

where $\mathcal{P}\widetilde{\mathbb{T}}^I_n$ is given by $\widetilde{\mathbb{T}}^I_n$ with $x \to \mathcal{P}x$, $s \to \mathcal{P}s$, $\bar s \to \mathcal{P}\bar s$ and $\widetilde{\mathbb{T}}^I_n$ are the tensor structures appropriate to the correlators with the operators $\widetilde{\mathcal{O}}_i$.[2] We provide the rules for the action of

[2] If there are any parity-odd cross-ratios (i.e. $n \geq 6$) then $\widetilde{g}$ should have these with reversed signs.

$\mathcal{P}$ on various tensor structures in equations (A.114), (A.115) and (3.65) `[applyPParity]`.

Action of Time Reversal If a theory has time reversal symmetry, there exists an anti-unitary operator $\mathcal{T}$ with the following commutation rule with Lorentz generators

$$\mathcal{T}M_{0i}\mathcal{T}^{-1} = -M_{0i}, \quad \mathcal{T}M_{ij}\mathcal{T}^{-1} = M_{ij}, \tag{A.29}$$

where $i,j = 1,2,3$. Applying it to (A.17) at $x = 0$, we see that

$$[M_{\mu\nu}, \mathcal{T}\mathcal{O}_\alpha(0)\mathcal{T}^{-1}] = \left[(\overline{S}S_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\right]^* \mathcal{T}\mathcal{O}_\beta(0)\mathcal{T}^{-1}. \tag{A.30}$$

This implies that $\mathcal{T}\mathcal{O}_\beta(0)\mathcal{T}^{-1}$ transforms as ψ^β and we can define the operator $\widehat{\mathcal{O}}$ as

$$\widehat{\mathcal{O}}_\alpha(x) \equiv -i\epsilon_{\alpha\beta}\mathcal{T}\mathcal{O}_\beta(\mathcal{T}x)\mathcal{T}^{-1}, \tag{A.31}$$

where $\mathcal{T}x^0 = -x^0$, $\mathcal{T}x^k = x^k$, $k = 1,2,3$. One can similarly define

$$\widehat{\mathcal{O}}^{\dot\alpha}(x) \equiv i\epsilon^{\dot\alpha\dot\beta}\mathcal{T}\mathcal{O}^{\dot\beta}(\mathcal{T}x)\mathcal{T}^{-1} \tag{A.32}$$

and extend the above definitions to arbitrary representations in an obvious way. For traceless symmetric operators in the $\widehat{\mathcal{O}} = \mathcal{O}$ case, this reproduces the standard time reversal action. In index-free notation we can write[3]

$$\mathcal{T}\mathcal{O}(x,s,\overline{s})\mathcal{T}^{-1} = \widehat{\mathcal{O}}(\mathcal{T}x, \mathcal{T}s, \mathcal{T}\overline{s}), \quad (\mathcal{T}s)^\alpha = is^*_{\dot\alpha},\ (\mathcal{T}\overline{s})_{\dot\alpha} = -i(\overline{s}^*)^\alpha. \tag{A.33}$$

Again, $\widehat{\mathcal{O}}$ may or may not be related to $\mathcal{O}$ depending on a theory. The only important point is that there is a relation between correlators with $\mathcal{O}_i$ and $\widehat{\mathcal{O}}_i$ in a theory preserving the time reversal symmetry

$$\begin{aligned}\langle 0|\mathcal{O}_1(x_1,s_1,\overline{s}_1)\cdots\mathcal{O}_n(x_n,s_n,\overline{s}_n)|0\rangle &= \\ &= \left[\langle 0|\mathcal{T}\mathcal{O}_1(x_1,s_1,\overline{s}_1)\mathcal{T}^{-1}\cdots\mathcal{T}\mathcal{O}_n(x_n,s_1,\overline{s}_1)\mathcal{T}^{-1}|0\rangle\right]^* \\ &= \left[\langle 0|\widehat{\mathcal{O}}_1(\mathcal{T}x_1,\mathcal{T}s_1,\mathcal{T}\overline{s}_1)\cdots\widehat{\mathcal{O}}_n(\mathcal{T}x_n,\mathcal{T}s_n,\mathcal{T}\overline{s}_n)|0\rangle\right]^*,\end{aligned} \tag{A.34}$$

where the conjugation happens because of the anti-unitarity of $\mathcal{T}$.[4] Written in terms of tensor structures this equality reads as

$$\sum_I \mathbb{T}^I_n g^I_n = \sum_I (\mathcal{T}\widehat{\mathbb{T}}^I_n)(\widehat{g}^I_n)^*, \tag{A.35}$$

where $\mathcal{T}\widehat{\mathbb{T}}^I_n$ is given by $(\widehat{\mathbb{T}}^I_n)^*$ with the replacements $x \to \mathcal{T}x$, $s \to \mathcal{T}s$, $\overline{s} \to \mathcal{T}\overline{s}$ made before the conjugation and $\widehat{\mathbb{T}}^I_n$ are the structures appropriate for the operators $\widehat{\mathcal{O}}_i$.

Computing $\mathcal{T}\widehat{\mathbb{T}}^I_n$ is easy, since we can construct $\mathcal{T}$ conjugation from $\mathcal{P}$ and the rotation $e^{i\pi M^{03}+\pi M^{12}}$. The latter rotation sends $s \to s$, $\overline{s} \to -\overline{s}$, which takes $\mathcal{T}s$ and $\mathcal{T}\overline{s}$ to $\mathcal{P}s$ and $\mathcal{P}\overline{s}$. The end result is

$$\mathcal{T}\widehat{\mathbb{T}}^I_n = \left(\mathcal{P}\widehat{\mathbb{T}}^I_n\right)^*. \tag{A.36}$$

We list the rules for the action of $\mathcal{T}$ on tensor structures in equations (A.116), (A.117) and (3.67) `[applyTParity]`.

[3] Note that $\mathcal{T}s$ and $\mathcal{T}\overline{s}$ are not complex conjugates of each other even if s and $\overline{s}$ are, so to avoid confusion here we do not assume that s and $\overline{s}$ are complex-conjugate. There is always a second complex conjugation (see below), so this is only intermediate.

[4] As an extreme example $\mathcal{T}i\mathcal{T}^{-1} = -i$, so we have $i = \langle 0|i|0\rangle = [\langle 0|\mathcal{T}i\mathcal{T}^{-1}|0\rangle]^* \neq \langle 0|\mathcal{T}i\mathcal{T}^{-1}|0\rangle$.

A.2 Details of the 6D Formalism

In this appendix we describe our conventions for the 6D embedding space. We mostly follow [124, 144].

We work in the signature $\{-++++-\}$, and we denote the 6D metric by h_{MN}. We often use the lightcone coordinates

$$X^{\pm} \equiv X^4 \pm X^5, \tag{A.37}$$

and write the components of 6D vectors as

$$X^M = \{X^\mu, X^+, X^-\}. \tag{A.38}$$

The metric in lightcone coordinates has the components

$$h_{+-} = h_{-+} = \frac{1}{2}, \quad h^{+-} = h^{-+} = 2. \tag{A.39}$$

The 6D Lorentz group $Spin(2,4)$ is isomorphic to the $SU(2,2)$ group. The latter can be defined as the group of 4 by 4 matrices U which act on 4-component complex vectors V_a and preserve the sesquilinear form

$$\langle V, W\rangle = g^{\overline{a}b}(V_a)^* W_b, \quad \langle UV, UW\rangle = \langle V, W\rangle. \tag{A.40}$$

Here the metric tensor $g^{\overline{a}b}$ is a Hermitian matrix with eigenvalues $\{+1,+1,-1,-1\}$, which we choose to be

$$g^{\overline{a}b} \equiv g^{b\overline{a}} \equiv \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}_{ab}. \tag{A.41}$$

The bar over the index $\overline{a}$ indicates that this index transforms in a complex conjugate representation. In other words, we say that V_a transforms in the fundamental representation while

$$V^*_{\overline{a}} \equiv (V_a)^* \tag{A.42}$$

transforms in the complex conjugate of the fundamental representation (that is, by matrices U^*). The metric $g^{\overline{a}b}$ establishes an isomorphism between the complex conjugate representation and the dual representation

$$\overline{V}^a \equiv g^{a\overline{b}}\overline{V}_{\overline{b}}. \tag{A.43}$$

We say that $\overline{V}^a$ transforms in the anti-fundamental representation (that is, the anti-fundamental representation is the dual of the fundamental representation). The inverse isomorphism is established by the tensor

$$g_{\overline{a}b} \equiv g_{b\overline{a}} \equiv -g^{\overline{a}b}. \tag{A.44}$$

We have the relations

$$g_{a\overline{b}}g^{\overline{b}c} = g^{c\overline{b}}g_{\overline{b}a} = \delta^c_a, \quad (g^{a\overline{b}})^* = g^{\overline{a}b}. \tag{A.45}$$

The isomorphism between $Spin(2,4)$ and $SU(2,2)$ can be established by identifying the vector representation of $Spin(2,4)$ with the exterior square of the fundamental or anti-fundamental representations of $SU(2,2)$.[5] This equivalence is provided by the invariant

[5] The fundamental and anti-fundamental representations themselves are the two spinor representations of $Spin(2,4)$.

tensors Σ^M_{ab} and $\overline{\Sigma}^{M\,ab}$ defined by

$$\Sigma^\mu_{ab} = \begin{pmatrix} 0 & -(\sigma^\mu\epsilon)_\alpha{}^{\dot\beta} \\ (\overline{\sigma}^\mu\epsilon)^{\dot\alpha}{}_\beta & 0 \end{pmatrix}, \quad \Sigma^+_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \quad \Sigma^-_{ab} = \begin{pmatrix} -2\,\epsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.46}$$

and

$$\overline{\Sigma}^{\mu\,ab} = \begin{pmatrix} 0 & -(\epsilon\sigma^\mu)^\alpha{}_{\dot\beta} \\ (\epsilon\overline{\sigma}^\mu)_{\dot\alpha}{}^\beta & 0 \end{pmatrix}, \quad \overline{\Sigma}^{+\,ab} = \begin{pmatrix} -2\,\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{\Sigma}^{-\,ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}. \tag{A.47}$$

These tensors have the following simple conjugation properties,

$$\left(\Sigma^M_{ab}\right)^* = g_{\bar{a}a'}g_{\bar{b}b'}\overline{\Sigma}^{M\,a'b'} \qquad \left(\overline{\Sigma}^{M\,ab}\right)^* = g^{\bar{a}a'}g^{\bar{b}b'}\Sigma^M_{a'b'}. \tag{A.48}$$

The above sigma-matrices satisfy many useful relations, for an incomplete list of them see appendix A in [144]. Using the sigma matrices we define the coordinate matrices

$$\mathbf{X}_{ab} \equiv X_M\Sigma^M_{ab} = -\mathbf{X}_{ba}, \qquad \overline{\mathbf{X}}^{ab} \equiv X_M\overline{\Sigma}^{M\,ab} = -\overline{\mathbf{X}}^{ba}, \tag{A.49}$$

which satisfy the algebra

$$_a(\mathbf{X}_i\overline{\mathbf{X}}_j)^b + {}_a(\mathbf{X}_j\overline{\mathbf{X}}_i)^b = 2\,(X_i\cdot X_j)\delta^b_a. \tag{A.50}$$

We can now identify the $SU(2,2)$ generators corresponding to the standard 6D Lorentz generators

$$\Sigma^{MN} \equiv \frac{1}{4}\,(\Sigma^M\overline{\Sigma}^N - \Sigma^N\overline{\Sigma}^M), \quad \overline{\Sigma}^{MN} \equiv \frac{1}{4}\,(\overline{\Sigma}^M\Sigma^N - \overline{\Sigma}^N\Sigma^M), \tag{A.51}$$

satisfying the commutation relations

$$[\Sigma^{MN},\Sigma^{PQ}] = h^{NP}\Sigma^{MQ} - h^{MP}\Sigma^{NQ} - h^{NQ}\Sigma^{MP} + h^{MQ}\Sigma^{NP}, \tag{A.52}$$
$$[\overline{\Sigma}^{MN},\overline{\Sigma}^{PQ}] = h^{NP}\overline{\Sigma}^{MQ} - h^{MP}\overline{\Sigma}^{NQ} - h^{NQ}\overline{\Sigma}^{MP} + h^{MQ}\overline{\Sigma}^{NP}, \tag{A.53}$$

thus establishing the isomorphism $Spin(2,4) \simeq SU(2,2)$ at Lie algebra level.

By comparing the expressions for $\Sigma_{\mu\nu}$ and $\overline{\Sigma}^{\mu\nu}$ with $SS^{\mu\nu}$ and $\overline{SS}^{\mu\nu}$, we find that under the Lorentz $Spin(1,3)$ subgroup of $Spin(2,4)$ the fundamental and anti-fundamental representations of $SU(2,2)$ decompose as

$$V_a = \begin{pmatrix} V_\alpha \\ V^{\dot\alpha} \end{pmatrix}, \quad \overline{W}^a = \begin{pmatrix} \overline{W}^\alpha \\ \overline{W}_{\dot\alpha} \end{pmatrix}. \tag{A.54}$$

In other words, we write V_α or $V^{\dot\alpha}$ to refer to first two or second two components of V_a, and analogously for $\overline{W}^a$.

Conformal algebra in 6D notation We can identify explicitly the conformal generators with the 6D Lorentz algebra

$$M_{\mu\nu} = L_{\mu\nu}, \quad D = L_{45}, \quad P_\mu = L_{5\mu} - L_{4\mu}, \quad K_\mu = -L_{4\mu} - L_{5\mu}. \tag{A.55}$$

With these conventions, the generators L_{MN} satisfy the algebra

$$[L_{MN},L_{PQ}] = h_{NP}L_{MQ} - h_{MP}L_{NQ} - h_{NQ}L_{MP} + h_{MQ}L_{NP}. \tag{A.56}$$

These generators act on the 6D primary operators as

$$[L_{MN}, O(X,S,\overline{S})] = \mathfrak{L}_{MN} O(X,S,\overline{S}), \tag{A.57}$$

where the differential 6D generator is defined as

$$\mathfrak{L}_{MN} \equiv -(X_M\partial_N - X_N\partial_M) - S\overline{\Sigma}_{MN}\partial_S - \overline{S}\Sigma_{MN}\partial_{\overline{S}}. \tag{A.58}$$

It is sometimes convenient to work with the conformal generators in $SU(2,2)$ notation

$$L_a{}^b \equiv \left[\Sigma^{MN}\right]_a{}^b L_{MN}, \quad L_{i\,MN} = -\frac{1}{2} L_a{}^b \left[\Sigma_{MN}\right]_b{}^a. \tag{A.59}$$

In this notation the conformal generators obey the commutation relations

$$\left[L_a{}^b, L_c{}^d\right] = 2\delta_c^b\, L_a{}^d - 2\delta_a^d\, L_c{}^b. \tag{A.60}$$

We also have the following action on the primary operators

$$[L_a{}^b, O(X,S,\overline{S})] = \mathfrak{L}_a{}^b\, O(X,S,\overline{S}), \tag{A.61}$$

where $\mathfrak{L}_a{}^c$ is the differential operator associated to the 6D generator $L_a{}^c$ in Hilbert space

$$\mathfrak{L}_a{}^b \equiv -\frac{1}{2}\left[\left(\mathbf{X}\overline{\Sigma}^M\right)_a{}^b \partial_M - \left(\Sigma^M\overline{\mathbf{X}}\right)_a{}^b \partial_M\right] + \frac{1}{2}\delta_a^b\left(S\cdot\partial_S - \overline{S}\cdot\partial_{\overline{S}}\right) - 2\left(S_a\partial_S^b - \overline{S}^b\partial_{\overline{S}\,a}\right). \tag{A.62}$$

Embedding formalism In the embedding formalism the flat 4D space is identified with a particular section of the 6D light cone $X^2 = 0$. Namely, we take the Poincare section $X^+ = 1$, which then implies

$$X^- = -X^\mu X_\mu. \tag{A.63}$$

The 4D coordinates x_μ are identified on this section as

$$x^\mu = X^\mu. \tag{A.64}$$

In particular, on the Poincare section we have

$$X^M\Big|_{\text{Poincare}} = \{x^\mu,\, 1,\, -x^2\}. \tag{A.65}$$

Consider an operator $O^{a_1\ldots a_l}_{b_1\ldots b_{\bar{l}}}(X)$, defined on the light cone $X^2=0$, symmetric in its two sets of indices. Following [124], it can be projected down to a 4D operator $\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x)$ as

$$\mathcal{O}^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x) = \mathbf{X}_{\alpha_1 a_1}\ldots\mathbf{X}_{\alpha_\ell a_\ell}\overline{\mathbf{X}}^{\dot\beta_1 b_1}\ldots\overline{\mathbf{X}}^{\dot\beta_{\bar\ell} b_{\bar\ell}} O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X)\Big|_{\text{Poincare}}. \tag{A.66}$$

If the 6D operator satisfies the homogeneity property

$$O^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(\lambda X) = \lambda^{-\kappa_{\mathcal{O}}} O^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(X), \tag{A.67}$$

where $\kappa_{\mathcal{O}}$ is defined in (2.13), then the resulting 4D operator will transform as a primary operator of dimension $\Delta_{\mathcal{O}}$ under conformal transformations. We call O a 6D uplift of $\mathcal{O}$.

Notice that the 6D uplift O is not uniquely defined. Indeed as a consequence of the light cone condition in terms of the matrices in (A.50),

$$X^2 = 0 \quad \Longrightarrow \quad {}_a(\mathbf{X}\overline{\mathbf{X}})^b = 0 \quad \text{and} \quad {}^a(\overline{\mathbf{X}}\mathbf{X})_b = 0, \tag{A.68}$$

the 6D operator is defined up to terms which vanish in (A.66), leading to the following equivalence relation

$$O^{a_1 \dots a_\ell}_{b_1 \dots b_{\bar\ell}} \sim O^{a_1 \dots a_\ell}_{b_1 \dots b_{\bar\ell}} + \overline{\mathbf{X}}^{a_1\, c} A^{a_2 \dots a_\ell}_{c\, b_1 \dots b_{\bar\ell}} + \mathbf{X}_{b_1 c}\, B^{c\, a_1 \dots a_\ell}_{b_2 \dots b_{\bar\ell}} + \delta^{a_1}_{b_1}\, C^{a_2 \dots a_\ell}_{b_2 \dots b_{\bar\ell}}. \tag{A.69}$$

Furthermore, in order to simplify the treatment of derivatives in the embedding space, it is convenient to arbitrarily extend $O(X)$ away from the light cone $X^2 = 0$ and treat all the extensions as equivalent. This means that we can also add to $O(X)$ terms proportional to X^2. Following the terminology of [138], we refer to this possibility as a gauge freedom and the terms proportional to $\mathbf{X}_{ab}$, $\overline{\mathbf{X}}^{ab}$, δ^a_b or X^2 will be called pure gauge terms.

It is convenient to use the index-free notation (3.2). Contracting the 4D auxiliary spinors with (A.66), we find that

$$\mathcal{O}(x, s, \bar s) = O(X, S, \overline{S})\Big|_{proj}, \tag{A.70}$$

where we introduced the formal operation $|_{proj}$ defined as

$$X^M\Big|_{proj} \equiv X^M\Big|_{\text{Poincare}}, \qquad S_a\Big|_{proj} \equiv s^\alpha \mathbf{X}_{\alpha a}\Big|_{\text{Poincare}}, \qquad \overline{S}^a\Big|_{proj} \equiv \bar s_{\dot\beta} \overline{\mathbf{X}}^{\dot\beta b}\Big|_{\text{Poincare}}. \tag{A.71}$$

As a consequence of the gauge freedom, the index-free 6D uplift $O(X, S, \overline{S})$ is defined up to pure gauge terms proportional to $S\overline{\mathbf{X}}$, $\overline{S}\mathbf{X}$, $\overline{S}S$ or X^2. Note that they all vanish under the operation of projection (A.70) due to (A.68)

$$\overline{\mathbf{X}}^{ab} S_b\Big|_{proj} = 0, \quad \overline{S}^b \mathbf{X}_{ba}\Big|_{proj} = 0, \quad \overline{S}^a S_a\Big|_{proj} = 0, \quad X^2\Big|_{proj} = 0, \tag{A.72}$$

We will always work modulo the gauge terms (A.72). In practice this is taken into account by treating (A.72) as explicit relations in the embedding formalism even before the projection. Note then that as a consequence of the relations (A.68), (A.72), the anti-symmetric properties (A.49) and the relations (A.7) in appendix A of [144], the following identities hold[6] which we call the 6D Jacobi identities

$$S_{[a}\mathbf{X}_{bc]} = 0, \quad \overline{S}^{[a}\overline{\mathbf{X}}^{bc]} = 0, \quad \mathbf{X}_{[ab}\mathbf{X}_{c]d} = 0, \quad \overline{\mathbf{X}}^{[ab}\overline{\mathbf{X}}^{c]d} = 0. \tag{A.73}$$

Differential operators In section 2 we commented upon the importance of some differential operators, such as the conservation operator (A.142), spinning differential operators (3.35), (3.36) and the Casimir operators entering (2.45). To consistently define these operators in embedding space, we require their action to be insensitive to different extensions of fields outside the light cone and the other gauge terms in (A.72). This results in the requirement[7]

$$D\left(\frac{\partial}{\partial X^M}, \frac{\partial}{\partial S_a}, \frac{\partial}{\partial \overline{S}^a}, X, S, \overline{S}\right) \cdot O(X^2, S\overline{\mathbf{X}}, \overline{S}\mathbf{X}, \overline{S}S) = O(X^2, S\overline{\mathbf{X}}, \overline{S}\mathbf{X}, \overline{S}S). \tag{A.74}$$

[6] We thank Emtinan Elkhidir for showing this simple derivation.

[7] In this equation O stands for the usual big-O notation and not the 6D operator.

To go from 6D differential operators to 4D differential operators, we need to find an explicit uplift of the 4D operators $\mathcal{O}(x,s,\overline{s})$ to the 6D operators $O(X,S,\overline{S})$. As noted above, there are infinitely many such uplifts differing by gauge terms, but all lead to the same result for 4D differential operators if the 6D operator satisfies (A.74). For example, we can choose the uplift

$$O(X,S,\overline{S}) = (X^+)^{-\kappa_{\mathcal{O}}}\mathcal{O}(X^\mu/X^+, S_\alpha, \overline{S}_{\dot\alpha}). \tag{A.75}$$

In particular, X^-, $S^{\dot\beta}$, $\overline{S}^\beta$ derivatives of this uplift of $\mathcal{O}$ vanish. By applying 6D derivatives to this expression we automatically obtain the required 4D derivatives on the right hand side. For instance, we find for the first order derivatives after the 4D projection

$$\partial/\partial X^M\Big|_{proj} = \{\partial/\partial x^\mu, -\kappa_{\mathcal{O}} - x^\nu\partial/\partial x^\nu, 0\}\,, \tag{A.76}$$

$$\partial/\partial S_a\Big|_{proj} = \{\partial/\partial s_\alpha, 0\}\,, \qquad \partial/\partial \overline{S}^a\Big|_{proj} = \{0, \partial/\partial \overline{s}_{\dot\alpha}\}\,. \tag{A.77}$$

Reality properties of the basic invariants Using the reality properties (A.48) of the sigma matrices, the projection rules (A.71) for S and $\overline{S}$, and the reality convention for 4D auxiliary polarizations $s_\alpha = (\overline{s}_{\dot\alpha})^*$, we can find the following reality properties for the basic objects hold

$$(\mathbf{X}_{ab})^* = \overline{\mathbf{X}}_{\overline{a}\overline{b}}, \quad \left(\overline{\mathbf{X}}^{ab}\right)^* = \mathbf{X}^{\overline{a}\overline{b}}, \quad (S_a)^* = i\overline{S}_{\overline{a}}, \quad (\overline{S}^a)^* = iS^{\overline{a}}. \tag{A.78}$$

Due to the relations such as $Y^aW_a = Y_{\overline{a}}W^{\overline{a}}$, we have an extremely simple conjugation rule for the expressions such as $\left(\overline{S}_i\mathbf{X}_j\overline{\mathbf{X}}_kS_l\right)$: replace $\mathbf{X}\leftrightarrow\overline{\mathbf{X}}$, $S\leftrightarrow\overline{S}$ and add a factor of i for each S and $\overline{S}$.

Action of Space Parity To analyze space parity, let us denote by P_N^M the 6x6 matrix which relfects the spacial components of X^μ. We also denote by $\hat{a}$ indices transforming in the representation reflected relative to the one of a.[8] Note that the reflection of the fundamental representation is equivalent to anti-fundamental and vice versa and this equivalence should be implemented by some matrices $p^{\hat{a}b}$ and $p_{\hat{a}b}$. In terms of these matrices we then have

$$P_N^M\Sigma^N_{ab} = \Sigma^N_{\hat{a}\hat{b}} = p_{\hat{a}a'}p_{\hat{b}b'}\overline{\Sigma}^{M\,a'b'}, \tag{A.79}$$
$$P_N^M\overline{\Sigma}^{N\,ab} = \Sigma^{N\,\hat{a}\hat{b}} = p^{\hat{a}a'}p^{\hat{b}b'}\Sigma^M_{a'b'}. \tag{A.80}$$

It is easy to check that these identities (as well as the equivalence between the representations) are achieved by choosing

$$p^{\hat{a}b} = p^{b\hat{a}} = -p_{\hat{a}b} = -p_{b\hat{a}} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}_{ab}. \tag{A.81}$$

From the above we deduce the action of parity on on $\mathbf{X}$ and $\overline{\mathbf{X}}$

$$\mathbf{X}_{ab} \mapsto \overline{\mathbf{X}}_{\hat{a}\hat{b}}, \quad \overline{\mathbf{X}}^{ab} \mapsto \mathbf{X}^{\hat{a}\hat{b}}. \tag{A.82}$$

[8] The reflected representation is the representation with the Lorentz generators $\mathcal{M}^{\text{refl}}_{MN}$ given by $\mathcal{M}^{\text{refl}}_{MN} = P_M^{M'}P_N^{N'}\mathcal{M}_{M'N'}$, where $\mathcal{M}$ are the original generators.

We can also check, based on 4D projections of S and $\overline{S}$, that

$$S_a \mapsto -\overline{S}_{\hat{a}}, \quad \overline{S}^a \mapsto S^{\hat{a}}. \tag{A.83}$$

Due to the identities such as $Y^a W_a = Y_{\hat{a}} W^{\hat{a}}$, we have the following parity conjugation rule for the products like $\big(\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_l\big)$: replace $\mathbf{X} \leftrightarrow \overline{\mathbf{X}}$, $S \leftrightarrow \overline{S}$ and a factor of -1 for each S in the original expression.

Action of Time Reversal As discussed in appendix A.1, see equation (A.36), the time reversal transformation can be implemented by combining the space parity with complex conjugation. Using the above rule, $\mathcal{T}$ acts simply as a multiplication by $i^{\sum_i \ell_i - \bar{\ell}_i}$ on each structure.

A.3 Normalization of Two-point Functions and Seed CPWs

In this appendix our goal is to fix the normalization constants of 2-point functions (2.16) and the seed CPWs (4.36).

The phase of 2-point functions is constrained by unitarity. A simple manifestation of the unitarity is the requirement that all the states in a theory have non-negative norms

$$\langle \Psi | \Psi \rangle \geq 0. \tag{A.84}$$

Our strategy is to define a state whose norm is related to 2-point functions (2.15) and use this relation to fix the phase (2.16). In particular, we set

$$|\mathcal{O}(s,\bar{s})\rangle \equiv \mathcal{O}(x_0, s, \bar{s})\,|0\rangle, \quad x_0^\mu \equiv \{i\epsilon, 0, 0, 0\}, \tag{A.85}$$

where $\epsilon > 0$. Here we are working in the standard Lorentzian quantization where the states are defined on spacelike hyperplanes. The state $|\mathcal{O}(s,\bar{s})\rangle$ can then be interpreted as a NS-quantization state in a Euclidean CFT [26]. Note that we have

$$|\mathcal{O}(s,\bar{s})\rangle = e^{-\epsilon H} \mathcal{O}(0, s, \bar{s})|0\rangle. \tag{A.86}$$

Here $H = -iP_0$ is the Hamiltonian[9] of the theory, and thus its spectrum is bounded from below. Therefore, we need $\epsilon > 0$ in order for $|\mathcal{O}(s,\bar{s})\rangle$ to have a finite norm. To compute this norm, we first consider the conjugate state

$$\langle \mathcal{O}(s,\bar{s})| = \langle 0|(\mathcal{O}(x_0, s, \bar{s}))^\dagger = \langle 0|\overline{\mathcal{O}}(-x_0, s, \bar{s}), \tag{A.87}$$

where we used $x_0^* = -x_0$. Then the norm is given by

$$\langle \mathcal{O}(s,\bar{s})|\mathcal{O}(s,\bar{s})\rangle = \langle 0|\overline{\mathcal{O}}(-x_0, s, \bar{s})\mathcal{O}(x_0, s, \bar{s})|0\rangle. \tag{A.88}$$

By using (2.15) to further rewrite (A.88), with the invariants x_{12}^2, $\mathbb{I}^{21}$ and $\mathbb{I}^{12}$ taking the form

$$x_{12}^2 = 4\epsilon^2, \quad \mathbb{I}^{21} = 2i\epsilon\; s^\dagger s, \quad \mathbb{I}^{12} = -2i\epsilon\; s^\dagger s, \tag{A.89}$$

we find

$$\langle 0|\overline{\mathcal{O}}(-x_0, s, \bar{s})\mathcal{O}(x_0, s, \bar{s})|0\rangle = c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle}(2\epsilon)^{-2\Delta}(s^\dagger s)^{\ell+\bar{\ell}} i^{\bar{\ell}-\ell} \geq 0, \tag{A.90}$$

[9] Recall that in our conventions P is anti-Hermitian.

where $s^\dagger s = |s_1|^2 + |s_2|^2 \geq 0$. This equation fixes the phase of $c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle}$, and we can consistently set

$$c_{\langle \overline{\mathcal{O}}\mathcal{O}\rangle} = i^{\ell-\bar{\ell}}. \tag{A.91}$$

Normalization of seed CPWs One can find the leading OPE behavior of the seed and the dual seed conformal blocks by taking the limit $z, \bar{z} \to 0$, $z \sim \bar{z}$, of the solutions (4.33) and (4.34). In particular, for the seed blocks we find

$$\lim_{z,\bar{z}\to 0} H_e^{(p)} = c^p_{0,-p} \, \frac{(-2)^{e-p}\, p!\, (p-e+1)_e}{e!\, (\ell+1)_p} \, (z\bar{z})^{\frac{\Delta+e-p/2}{2}} \, C^{(p+1)}_{\ell-p+e}\left(\frac{z+\bar{z}}{2\,(z\bar{z})^{1/2}}\right), \tag{A.92}$$

and for the dual seed blocks

$$\lim_{z,\bar{z}\to 0} \overline{H}_e^{(p)} = (-2)^p \, \bar{c}^0_{0,-p} \, \frac{(-2)^{e-p}\, p!\, (p-e+1)_e}{e!\, (\ell+1)_p} \, (z\bar{z})^{\frac{\Delta+e-p/2}{2}} \, C^{(p+1)}_{\ell-e}\left(\frac{z+\bar{z}}{2\,(z\bar{z})^{1/2}}\right), \tag{A.93}$$

where $C_j^{(\nu)}(x)$ are the Gegenbauer polynomials, which in the limit $0 < z \ll \bar{z} \ll 1$ read as

$$C_s^{(p+1)}\left(\frac{z+\bar{z}}{2\,(z\bar{z})^{1/2}}\right) \approx \frac{(p+1)_s}{s!} \, z^{-\frac{s}{2}} \, \bar{z}^{\frac{s}{2}}. \tag{A.94}$$

We remind that $c^p_{0,-p}$ and $\bar{c}^0_{0,-p}$ are some undetermined overall normalization coefficients. The purpose of this paragraph is to find the values of these coefficients appropriate for our conventions for 2- and 3-point functions.

In order to fix these coefficients, it suffices to consider the leading term in the s-channel OPE in the seed 4-point functions. We have checked that the OPE exactly reproduces the form of (A.92) and (A.93) if one sets

$$c^p_{0,-p} = 2^p \, \bar{c}^0_{0,-p} = (-1)^\ell \, i^p. \tag{A.95}$$

Let us stress that this normalization factor is fixed by the convention (2.15) and (2.16) for the 2-point functions, and the definitions of the seed 3-point functions

$$\langle \mathcal{F}_1^{(0,0)}(\mathbf{p}_1)\, \mathcal{F}_2^{(p,0)}(\mathbf{p}_2)\, \mathcal{O}_\Delta^{(\ell,\,\ell+p)}(\mathbf{p}_3)\rangle^{(\bullet)} = [\hat{\mathbb{I}}^{32}]^p \, [\hat{\mathbb{J}}^3_{12}]^\ell \mathcal{K}_3, \tag{A.96}$$

$$\langle \overline{\mathcal{O}}_\Delta^{(\ell+p,\,\ell)}(\mathbf{p}_2)\, \mathcal{F}_3^{(0,0)}(\mathbf{p}_3)\, \mathcal{F}_4^{(0,p)}(\mathbf{p}_4)\,\rangle^{(\bullet)} = [\hat{\mathbb{I}}^{42}]^p \, [\hat{\mathbb{J}}^2_{34}]^\ell \mathcal{K}_3, \tag{A.97}$$

and the dual seed 3-point functions

$$\langle \mathcal{F}_1^{(0,0)}(\mathbf{p}_1)\, \mathcal{F}_2^{(p,0)}(\mathbf{p}_2)\, \overline{\mathcal{O}}_\Delta^{(\ell+p,\,\ell)}(\mathbf{p}_3)\rangle^{(\bullet)} = [\hat{\mathbb{K}}_1^{23}]^p \, [\hat{\mathbb{J}}^3_{12}]^\ell \mathcal{K}_3, \tag{A.98}$$

$$\langle \mathcal{O}_\Delta^{(\ell,\,\ell+p)}(\mathbf{p}_2)\, \mathcal{F}_3^{(0,0)}(\mathbf{p}_3)\, \mathcal{F}_4^{(0,p)}(\mathbf{p}_4)\,\rangle^{(\bullet)} = [\hat{\overline{\mathbb{K}}}_3^{24}]^p \, [\hat{\mathbb{J}}^2_{34}]^\ell \mathcal{K}_3, \tag{A.99}$$

where in each equation $\mathcal{K}_3$ has to be replaced with the appropriate 3-point kinematic factor as defined in (2.19).

Equation (A.95) can be derived from these three-point functions and the corresponding leading OPE terms

$$\mathcal{F}_1^{(0,0)}(0)\mathcal{F}_2^{(p,0)}(x_2,s_2) = \frac{(-i)^p}{\ell!(\ell+p)!} |x_2|^{\Delta-\Delta_1-\Delta_2-\ell} (s_2\partial_s)^p (x_2^\mu \partial_s \sigma_\mu \partial_{\bar{s}})^\ell \overline{\mathcal{O}}_\Delta^{(\ell+p,\ell)}(0,s,\bar{s}) + \ldots, \tag{A.100}$$

$$\mathcal{F}_1^{(0,0)}(0)\mathcal{F}_2^{(p,0)}(x_2,s_2) = \frac{i^p}{\ell!(\ell+p)!} |x_2|^{\Delta-\Delta_1-\Delta_2-\ell-p} (x_2^\mu s_2 \sigma_\mu \partial_{\bar{s}})^p (x_2^\mu \partial_s \sigma_\mu \partial_{\bar{s}})^\ell \mathcal{O}_\Delta^{(\ell,\ell+p)}(0,s,\bar{s}) + \ldots, \tag{A.101}$$

where we have defined

$$(\partial_s)^\alpha \equiv \frac{\partial}{\partial s_\alpha}, \quad (\partial_{\bar{s}})^{\dot{\alpha}} \equiv \frac{\partial}{\partial \bar{s}_{\dot{\alpha}}}. \tag{A.102}$$

The normalization coefficients in these OPEs can be computed by substituting the OPEs into (A.96) and (A.98) and using the two-point function (2.16). The normalization coefficients for the CPWs are then obtained by using these OPEs in the seed four-point function

$$\langle \mathcal{F}_1^{(0,0)}\, \mathcal{F}_2^{(p,0)}\, \mathcal{F}_3^{(0,0)}\, \mathcal{F}_4^{(0,p)} \rangle \tag{A.103}$$

and utilizing the 3-point function definitions (B.6) and (B.8). In practice, when comparing the normalization coefficients, we found it convenient to use the conformal frame (3.40) - (3.43) in the limit $0 < z \ll \bar{z} \ll 1$ and further set $\eta_2 = 0$ and $e = p$ for the seed CPWs or $\xi_2 = 0$ and $e = 0$ for the dual seed CPWs.

A.4 4D Form of Basic Tensor Invariants

Here we provide the form of basic tensor invariants in 4D for $n \le 4$ point functions. They are obtained by applying the projection operation (A.71) to the basic 6D tensor invariants constructed in section 3.1.1

$$(\hat{\mathbb{I}}^{ij},\, \hat{\mathbb{I}}^{ij}_{kl},\, \hat{\mathbb{J}}^{k}_{ij},\, \hat{\mathbb{K}}^{ij}_{k},\, \hat{\overline{\mathbb{K}}}{}^{ij}_{k},\, \hat{\mathbb{L}}^{i}_{jkl},\, \hat{\overline{\mathbb{L}}}{}^{i}_{jkl}) \equiv (\hat{I}^{ij},\, \hat{I}^{ij}_{kl},\, J^{k}_{ij},\, K^{ij}_{k},\, \overline{K}^{ij}_{k},\, L^{i}_{jkl},\, \overline{L}^{i}_{jkl})\Big|_{proj}, \tag{A.104}$$

where

$$\hat{\mathbb{I}}^{ij} = x^\mu_{ij}\,(\bar{s}_i \bar{\sigma}_\mu s_j), \tag{A.105}$$

$$\hat{\mathbb{I}}^{ij}_{kl} = \frac{1}{2\,x^2_{kl}} \times \Big((x^2_{ik} x^\mu_{jl} - x^2_{il} x^\mu_{jk}) + (x^2_{jk} x^\mu_{il} - x^2_{jl} x^\mu_{ik}) - x^2_{ij} x^\mu_{kl} - x^2_{kl} x^\mu_{ij} \\ - 2i\epsilon^{\mu\nu\rho\sigma} x_{ik\,\nu} x_{lj\,\rho} x_{lk\,\sigma} \Big) \times (\bar{s}_i \bar{\sigma}_\mu s_j), \tag{A.106}$$

$$\hat{\mathbb{J}}^{k}_{ij} = \frac{x^2_{ik} x^2_{jk}}{x^2_{ij}} \times \left(\frac{x^\mu_{ik}}{x^2_{ik}} - \frac{x^\mu_{jk}}{x^2_{jk}} \right) \times (\bar{s}_k \bar{\sigma}_\mu s_k), \tag{A.107}$$

$$\hat{\mathbb{K}}^{ij}_{k} = \frac{1}{2} \frac{|x_{ij}|}{|x_{ik}||x_{jk}|} \times \Big((x^2_{ik} + x^2_{jk} - x^2_{ij})(s_i s_j) - 4 x^\mu_{ik} x^\nu_{jk}\,(s_i \sigma_{\mu\nu} s_j) \Big), \tag{A.108}$$

$$\hat{\overline{\mathbb{K}}}{}^{ij}_{k} = \frac{1}{2} \frac{|x_{ij}|}{|x_{ik}||x_{jk}|} \times \Big((x^2_{ik} + x^2_{jk} - x^2_{ij})(\bar{s}_i \bar{s}_j) - 4 x^\mu_{ik} x^\nu_{jk}\,(\bar{s}_i \bar{\sigma}_{\mu\nu} \bar{s}_j) \Big), \tag{A.109}$$

$$\hat{\mathbb{L}}^{i}_{jkl} = \frac{2}{|x_{jk}||x_{kl}||x_{lj}|} \times \Big(x^2_{ij} x^\mu_{kl} x^\nu_{il} + x^2_{ik} x^\mu_{lj} x^\nu_{ij} + x^2_{il} x^\mu_{jk} x^\nu_{ik} \Big) \times (s_i \sigma_{\mu\nu} s_i)\,, \tag{A.110}$$

$$\hat{\overline{\mathbb{L}}}{}^{i}_{jkl} = \frac{2}{|x_{jk}||x_{kl}||x_{lj}|} \times \Big(x^2_{ij} x^\mu_{kl} x^\nu_{il} + x^2_{ik} x^\mu_{lj} x^\nu_{ij} + x^2_{il} x^\mu_{jk} x^\nu_{ik} \Big) \times (\bar{s}_i \bar{\sigma}_{\mu\nu} \bar{s}_i)\,. \tag{A.111}$$

We recall that $x^\mu_{ij} \equiv x^\mu_i - x^\mu_j$ and $\epsilon_{0123} = -1$ in our conventions. From these expressions it is possible to derive the conjugation properties of the invariants. They read as follows

$$\left(\hat{\mathbb{I}}^{ij}\right)^* = -\hat{\mathbb{I}}^{ji}\,, \qquad \left(\hat{\mathbb{I}}^{ij}_{kl}\right)^* = -\hat{\mathbb{I}}^{ji}_{lk}\,, \qquad \left(\hat{\mathbb{J}}^{k}_{ij}\right)^* = \hat{\mathbb{J}}^{k}_{ij}\,, \tag{A.112}$$

$$\left(\hat{\mathbb{K}}^{ij}_{k}\right)^* = -\hat{\overline{\mathbb{K}}}{}^{ij}_{k}\,, \qquad \left(\hat{\mathbb{L}}^{i}_{jkl}\right)^* = -\hat{\overline{\mathbb{L}}}{}^{i}_{jkl}. \tag{A.113}$$

Their parity transformation can be deduced from (A.26)

$$\mathcal{P}\,\hat{\mathbb{I}}^{ij} = -\hat{\mathbb{I}}^{ji}\,, \qquad \mathcal{P}\,\hat{\mathbb{I}}^{ij}_{kl} = -\hat{\mathbb{I}}^{ji}_{lk}\,, \qquad \mathcal{P}\,\hat{\mathbb{J}}^{k}_{ij} = \hat{\mathbb{J}}^{k}_{ij}\,, \tag{A.114}$$

$$\mathcal{P}\,\hat{\mathbb{K}}^{ij}_{k} = \hat{\overline{\mathbb{K}}}^{ij}_{k}\,, \qquad \mathcal{P}\,\hat{\mathbb{L}}^{i}_{jkl} = \hat{\overline{\mathbb{L}}}^{i}_{jkl}. \tag{A.115}$$

Finally, according to (A.36) one gets transformations under time reversal

$$\mathcal{T}\,\hat{\mathbb{I}}^{ij} = \hat{\mathbb{I}}^{ij}\,, \qquad \mathcal{T}\,\hat{\mathbb{I}}^{ij}_{kl} = \hat{\mathbb{I}}^{ij}_{kl}\,, \qquad \mathcal{T}\,\hat{\mathbb{J}}^{k}_{ij} = \hat{\mathbb{J}}^{k}_{ij}\,, \tag{A.116}$$

$$\mathcal{T}\,\hat{\mathbb{K}}^{ij}_{k} = -\hat{\mathbb{K}}^{ij}_{k}\,, \qquad \mathcal{T}\,\hat{\mathbb{L}}^{i}_{jkl} = -\hat{\mathbb{L}}^{i}_{jkl}. \tag{A.117}$$

The same properties follow from the discussion of $\mathcal{P}$-, $\mathcal{T}$-symmetries, and conjugation in appendix A.2.

A.5 Covariant Bases of Three-point Tensor Structures

Let us review the construction `[n3ListStructures]` of 3-point function tensor structures [144]. According to the discussion below (3.32) one has

$$\hat{\mathbb{T}}^{a}_{3} = \Big\{ \prod_{i\neq j} [\hat{\mathbb{I}}^{ij}]^{m_{ij}} \times \prod_{i,\,j<k} [\hat{\mathbb{J}}^{i}_{jk}]^{n_i}\, [\hat{\mathbb{K}}^{jk}_{i}]^{k_i}\, [\hat{\overline{\mathbb{K}}}^{jk}_{i}]^{\overline{k}_i} \Big\}, \tag{A.118}$$

where the exponents satisfy the following system

$$\ell_i = \sum_{l\neq i} m_{li} + \sum_{l\neq i} k_l + n_i, \tag{A.119}$$

$$\overline{\ell}_i = \sum_{l\neq i} m_{il} + \sum_{l\neq i} \overline{k}_l + n_i. \tag{A.120}$$

Let us also define the quantity

$$\Delta\ell \equiv \sum_i (\ell_i - \overline{\ell}_i). \tag{A.121}$$

Due to relations among products of invariants, not all the structures obtained this way are independent and constraints on possible values of the exponents in (A.118) must be imposed. Theses relations come from the Jacobi identities (A.73) by contracting them with 6D polarizations and 6D coordinate matrices in all possible ways.

The first set of relations reads

$$\hat{\mathbb{K}}^{ik}_{j}\hat{\overline{\mathbb{K}}}^{jk}_{i} = -\,\hat{\mathbb{I}}^{ki}\hat{\mathbb{I}}^{jk} - \hat{\mathbb{I}}^{ji}\hat{\mathbb{J}}^{k}_{ij}\,, \tag{A.122}$$

$$\hat{\mathbb{K}}^{ij}_{k}\hat{\overline{\mathbb{K}}}^{ij}_{k} = \hat{\mathbb{I}}^{ij}\hat{\mathbb{I}}^{ji} - \hat{\mathbb{J}}^{jk}_{i}\hat{\mathbb{J}}^{ik}_{j}\,. \tag{A.123}$$

If $\Delta\ell \neq 0$ we use these relations to set $\overline{k}_i = 0$ or $k_i = 0$ for $\forall\, i$ in the expression (A.118); if $\Delta\ell = 0$ we set instead $k_i = \overline{k}_i = 0\ \ \forall\, i$.

The second set of relations reads

$$\hat{\mathbb{J}}^{j}_{ik}\hat{\mathbb{K}}^{ik}_{j} = \hat{\mathbb{I}}^{ji}\hat{\mathbb{K}}^{jk}_{i} - \hat{\mathbb{I}}^{jk}\hat{\mathbb{K}}^{ij}_{k}\,, \tag{A.124}$$

$$\hat{\mathbb{J}}^{j}_{ik}\hat{\overline{\mathbb{K}}}^{ik}_{j} = \hat{\mathbb{I}}^{ij}\hat{\overline{\mathbb{K}}}^{kj}_{i} + \hat{\mathbb{I}}^{kj}\hat{\overline{\mathbb{K}}}^{ij}_{k}\,. \tag{A.125}$$

This allows to set either $n_i = 0$ or $k_i = 0$ if $\Delta\ell > 0$ and either $n_i = 0$ or $\overline{k}_i = 0$ if $\Delta\ell < 0$ in (A.118).

If $\Delta\ell = 0$ it might seem that the relations (A.124) and (A.125) do not play any role, since all K and $\overline{K}$ are removed by mean of (A.122) and (A.123). However it is not the case, by combining (A.124) and (A.125) with (A.122) and (A.123) one gets a third order relation

$$\hat{\mathbb{J}}^1_{23}\hat{\mathbb{J}}^2_{13}\hat{\mathbb{J}}^3_{12} = \left(\hat{\mathbb{I}}^{23}\hat{\mathbb{I}}^{32}\hat{\mathbb{J}}^1_{23} - \hat{\mathbb{I}}^{13}\hat{\mathbb{I}}^{31}\hat{\mathbb{J}}^2_{13} + \hat{\mathbb{I}}^{12}\hat{\mathbb{I}}^{21}\hat{\mathbb{J}}^3_{12}\right) - \left(\hat{\mathbb{I}}^{21}\hat{\mathbb{I}}^{13}\hat{\mathbb{I}}^{32} - \hat{\mathbb{I}}^{12}\hat{\mathbb{I}}^{31}\hat{\mathbb{I}}^{23}\right). \tag{A.126}$$

This allows to set in (A.118) either $n_1 = 0$ or $n_2 = 0$ or $n_3 = 0$ when $\Delta\ell = 0$[10]. It can be verified that no other independent relations exist.

In the case when all operators are trace-less symmetric, i.e. $\ell_i = \bar{\ell}_i$ for each field, it is convenient to work in terms of structures manifestly even or odd under parity. Following [145], the most general parity definite tensor structure reads as

$$\hat{\mathbb{T}}^a_3 = \left\{\left(\hat{\mathbb{I}}^{21}\hat{\mathbb{I}}^{13}\hat{\mathbb{I}}^{32} + \hat{\mathbb{I}}^{12}\hat{\mathbb{I}}^{31}\hat{\mathbb{I}}^{23}\right)^p \times \prod_{i,j}\left(\hat{\mathbb{I}}^{ij}\hat{\mathbb{I}}^{ji}\right)^{m_{ij}} \times \prod_{i,j<k}\left[\hat{\mathbb{J}}^i_{jk}\right]^{n_i}\right\}, \tag{A.127}$$

where the structure is even if $p = 0$ and the structure is odd if $p = 1$. The form of this basis is structurally identical to the one found in [121]. This basis has extremely simple properties under complex conjugation, parity and time reversal

$$\left(\hat{\mathbb{T}}^a_3\right)^* = (-1)^p\,\hat{\mathbb{T}}^a_3, \quad \mathcal{P}\,\hat{\mathbb{T}}^a_3 = (-1)^p\,\hat{\mathbb{T}}^a_3, \quad \mathcal{T}\,\hat{\mathbb{T}}^a_3 = \hat{\mathbb{T}}^a_3. \tag{A.128}$$

This basis can be constructed using `[n3ListStructuresAlternativeTS]`.

A.6 Casimir Differential Operators

The Lie algebra of the 4D conformal group is a real form of the simple rank-3 algebra $\mathfrak{so}(6)$. Therefore, it has three independent Casimir operators, which can be defined using the 6D Lorentz generators (A.57) as follows

$$C_2 \equiv \frac{1}{2}\,L_{MN}\,L^{NM}, \tag{A.129}$$

$$C_3 \equiv \frac{1}{24i}\,\epsilon^{MNPQRS}\,L_{MN}\,L_{PQ}\,L_{RS}, \tag{A.130}$$

$$C_4 \equiv \frac{1}{2}\,L_{MN}\,L^{NP}\,L_{PQ}\,L^{QM}, \tag{A.131}$$

where $\epsilon^{012345} = \epsilon_{012345} = +1$.

To write out the Casimir eigenvalues for primary operators, it is convenient to introduce also the $SO(1,3)$ Casimir operators using the 4D Lorentz generator (A.17). There are two such Casimirs

$$c_2^+ \equiv -\frac{1}{2}L_{\mu\nu}L^{\mu\nu}, \quad c_2^- \equiv \frac{1}{4i}\epsilon^{\mu\nu\rho\sigma}L_{\mu\nu}L_{\rho\sigma}, \tag{A.132}$$

with the eigenvalues

$$e_2^+ = \frac{1}{2}\ell(\ell+2) + \frac{1}{2}\bar{\ell}(\bar{\ell}+2), \quad e_2^- = \frac{1}{2}\ell(\ell+2) - \frac{1}{2}\bar{\ell}(\bar{\ell}+2). \tag{A.133}$$

[10]Notice that for $\Delta\ell \neq 0$ at least one n_i is always 0 and hence (A.126) does not give new constraints.

The conformal Casimir eigenvalues are then given by

$$E_2 \equiv \Delta(\Delta-4) + e_2^+, \tag{A.134}$$
$$E_3 \equiv \big(\Delta-2\big)\, e_2^-, \tag{A.135}$$
$$E_4 \equiv \Delta^2(\Delta-4)^2 + 6\,\Delta(\Delta-4) + \big(e_2^+\big)^2 - \frac{1}{2}\,\big(e_2^-\big)^2. \tag{A.136}$$

Note that c_2^- is parity-odd and therefore e_2^- changes the sign under $\ell \leftrightarrow \bar\ell$. The same comment applies to C_3 and E_3.

It is convenient to write the Casimir Operators in the $SU(2,2)$ language by plugging (A.59) into the expression (A.129), (A.130) and (A.131)

$$C_2 = \frac{1}{4}\ \mathrm{tr}\, L^2, \tag{A.137}$$
$$C_3 = \frac{1}{12}\ \big(\mathrm{tr}\, L^3 - 16\, C_2\big)\,, \tag{A.138}$$
$$C_4 = -\,\frac{1}{8}\ \big(\mathrm{tr}\, L^4 - 8\ \mathrm{tr}\, L^3 - 12\, C_2^2 + 16\, C_2\big)\,. \tag{A.139}$$

Let us emphasize that the Casimir operators C_n are the Hilbert space operators. Their differential form $\mathfrak{C}_n$ can be obtained by replacing the Hilbert space operators L_{MN} and ${L_a}^c$ with their differential representations $\mathfrak{L}_{MN}$ and ${\mathfrak{L}_a}^c$ given in (A.58) and (A.62) together with reverting[11] the order of operators L_{MN} and ${L_a}^c$ in equations (A.129) - (A.131) and (A.137) - (A.139).

A.7 Conserved Operators

By conserved operators we mean primary operators in short representations of the conformal group, i.e. those possessing null descendants and thus satisfying differential equations. In a unitary 4D CFT all local primary operators satisfy the unitarity bounds [160, 161][12]

$$\Delta \geq 1 + \frac{\ell+\bar\ell}{2},\ \ \ell = 0 \textbf{ or } \bar\ell = 0, \tag{A.140}$$
$$\Delta \geq 2 + \frac{\ell+\bar\ell}{2},\ \ \ell \neq 0 \textbf{ and } \bar\ell \neq 0, \tag{A.141}$$

and unitary null states can only appear when these bounds are saturated.

The operators of the type $\ell = 0$ **or** $\bar\ell = 0$ with $\Delta = 1 + (\ell+\bar\ell)/2$ satisfy the free wave equation[13] $\partial^2 \mathcal{O}_\Delta^{(\ell,\bar\ell)} = 0$ [162], which immediately implies that such operators can only come from a free subsector of the CFT. The operators of the second type, $\ell\bar\ell \neq 0$, $\Delta = 2 + (\ell+\bar\ell)/2$, are the conserved currents which satisfy the following operator equation[14]

$$\partial \cdot \mathcal{O}_\Delta^{(\ell,\bar\ell)}(x,s,\bar s) = 0, \quad \partial \equiv (\epsilon\sigma^\mu)^{\alpha}_{\dot\beta}\, \partial_\mu\, \frac{\partial^2}{\partial s^\alpha\, \partial \bar s_{\dot\beta}}. \tag{A.142}$$

Of particular importance are the spin-1 currents J^μ in representation $(1,1)$, the stress tensor $T^{\mu\nu}$ in representation $(2,2)$ and the supercurrents J^μ_α and $J^\mu_{\dot\alpha}$ in representations $(2,1)$ and

[11] See the discussion below (A.20).

[12] An operator with $\ell = \bar\ell = 0$ has an extra option $\Delta = 0$. This is the identity operator.

[13] This is not the conformally-invariant differential equation satisfied by these operators, but rather its consequence.

[14] The operator ∂ can be applied in the conformal frame `[opConservation4D]` or in the embedding formalism `[opConservationEF]`.

$(1,2)$. Note that an appearance of traceless symmetric higher-spin currents is known to imply an existence of a free subsector [163, 164].

The conservation condition results in the following Ward identity for n-point functions

$$\partial \cdot \langle \dots \mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}(x,s,\bar{s}) \dots \rangle = 0 + \text{contact terms}, \tag{A.143}$$

where the contact terms encode charges of operators under the symmetry generated by the conserved current $\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}$. Note that since $\partial\cdot\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}$ is itself a primary operator in representation $(\ell-1,\bar{\ell}-1)$, $\Delta=3+(\ell+\bar{\ell})/2$, the left hand side of the above equation has the transformation properties of a correlation function of primary operators and thus can be expanded in a basis of appropriate tensor structures.

For 3-point functions, the Ward identities imply two kind of constraints. First, the validity of (A.143) at generic configurations of points x_i implies homogeneous linear relations between the OPE coefficients entering 3-point functions. Second, the validity of (A.143) at coincident points relates some of the OPE coefficients to the charges of the other two operators in a given 3-point function (this happens only if special relations between scaling dimensions of these operators are satisfied). The solution of these constraints is of the form (2.23), where some of $\hat{\lambda}$ can be related to the charges.

For 4-point functions the situation is more complicated, since (A.143) at non-coincident points leads to a system of first order differential equations for the functions $g_4^I(u,v)$ of the form

$$B^{AJ}(u,v,\partial_u,\partial_v)\; g_4^J(u,v) = 0, \tag{A.144}$$

where A runs through the number of tensor structures for the correlator in the left hand side of (A.143). The constraints implied by these equations were analysed in [151]. It turns out that one can solve these equations by aribtrarily specifying a smaller number N_4' of the functions $g_4^I(u,v)$ and a number of boundary conditions for the remaining $g_4^I(u,v)$.[15] It is generally important to take this into account when formulating an independent set of crossing symmetry equations. We refer the reader to [151] for details. In [151] the value N_4' was found for 4 identical conserved spin 1 and spin 2 operators. The same values N_4' were found later by other means in [145] and a general counting rule was proposed in [126].

Conservation operator in the Embedding Formalism The conservation condition (A.142) can be consistently reformulated in the embedding space `[opConservationEF]` as follows

$$D\, O_{\Delta_{\mathcal{O}}}^{(\ell,\bar{\ell})}\left(X,S,\overline{S}\right) = 0, \quad \Delta_{\mathcal{O}} = 2 + \frac{\ell+\bar{\ell}}{2} \tag{A.145}$$

and the differential operator originally found in [144] is given by[16]

$$D \equiv \frac{2}{\ell\,\bar{\ell}\,\left(2+\ell+\bar{\ell}\right)}\left(X_M \Sigma^{MN}\partial_N\right)_a^{\ b}\ \partial_b^a, \tag{A.146}$$

where we have defined

$$\partial_b^a \equiv \frac{1}{1+\ell+\bar{\ell}}\,\partial^a\partial_b = \left(4 + S\cdot\frac{\partial}{\partial S} + \overline{S}\cdot\frac{\partial}{\partial \overline{S}}\right)\frac{\partial}{\partial S_a}\frac{\partial}{\partial \overline{S}^b} - S_b\frac{\partial}{\partial S_a}\frac{\partial^2}{\partial S\cdot\partial\overline{S}} - \overline{S}^a\frac{\partial}{\partial \overline{S}^b}\frac{\partial^2}{\partial S\cdot\partial\overline{S}}. \tag{A.147}$$

[15] DK thanks Anatoly Dymarsky, João Penedones and Alessandro Vichi for discussions on this issue.

[16] We note that there is a mistake in the original paper [144] due to a wrong choice of the analogue of (3.6).

In this identity we dropped the terms which project to zero upon contraction with $\left(X_M \Sigma^{MN} \partial_N\right)_a^{\ b}$.

A.8 Permutations Symmetries

When the points in (2.8) are space-like separated, the ordering of operators is not important up to signs coming from permutations of fermions. In particular, if some operator enters the expectation value more than once, say at points $\mathbf{p}_i$ and $\mathbf{p}_j$, the function f_n enjoys the permutation symmetry

$$f_n(\ldots, \mathbf{p}_i, \ldots, \mathbf{p}_j, \ldots) = [(ij) f_n](\ldots, \mathbf{p}_i, \ldots, \mathbf{p}_j, \ldots) \equiv \pm f_n(\ldots, \mathbf{p}_j, \ldots, \mathbf{p}_i, \ldots). \tag{A.148}$$

Here we used the cycle notation for permutations, for instance (123) denotes $1 \to 2$, $2 \to 3$, $3 \to 1$. In general, there may be more identical operators in the right hand side of (2.8) in which case f_n is invariant under some subgroup of permutations $\Pi \subseteq S_n$.

The degrees of freedom in f_n are described by the functions g_n^I defined via (2.11)

$$f_n(x_i, s_i, \bar{s}_i) = \sum_{I=1}^{N_n} g_n^I(\mathbf{u})\, \mathbb{T}_n^I(x_i, s_i, \bar{s}_i). \tag{A.149}$$

One can then find the implications of the permutation symmetries directly for g_n^I. Note that since the exchanged operators are identical, a permutation $\pi \in \Pi$ acting on a tensor structure gives a tensor structure of the same kind, and thus we can expand it in the same basis

$$\pi \mathbb{T}_n^I = \sum_J \pi_I^J(\mathbf{u}) \mathbb{T}_n^J. \tag{A.150}$$

This means that in general the consequence of a permutation symmetry is

$$g_n^I(\mathbf{u}) = \sum_J \pi_J^I(\mathbf{u}) g_n^J(\pi \mathbf{u}). \tag{A.151}$$

At this point we should divide all the permutations into two classes. We call the permutations which preserve the cross-rations ($\pi \mathbf{u} = \mathbf{u}$) the kinematic permutations and all the other permutations will be referred to as non-kinematic. The group of kinematic permutations Π_n^{kin} is S_n for $n \leq 3$ since there are no non-trivial cross-ratios in these cases. We also have $\Pi_4^{\text{kin}} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ and Π_n^{kin} is trivial for $n \geq 5$.

This distinction is important because for kinematic permutations the constraint (A.151) becomes a simple local linear constraint,

$$g_n^I(\mathbf{u}) = \sum_J \pi_J^I(\mathbf{u}) g_n^J(\mathbf{u}), \tag{A.152}$$

which we can be solved as

$$g_n^I(\mathbf{u}) = \sum_A P_A^I(\mathbf{u}) \hat{g}_n^A(\mathbf{u}). \tag{A.153}$$

In the case of 3-point functions the solution (A.153) has a particularly simple form (2.23).

Applying permutation `[permutePoints]` and computing $\pi_J^I(\mathbf{u})$ is straightforward in the EF – we simply need to permute the coordinates X_i and the polarizations $S_i, \overline{S}_i$. It is somewhat trickier to figure out the permutations in the CF [126], and we describe the case $n = 4$ in the remainder of this section. We also comment on how to permute non-identical operators, which is required, for example, in order to exchange s- and t-channels.

Appendix B

Constructing the Ansatz

B.1 Shadow Formalism

A straightforward method to obtain CBs in closed analytical form uses the so called shadow formalism. It was first introduced by Ferrara, Gatto, Grillo, and Parisi [115, 165–167] and used in [127] to get closed form expressions for the scalar CBs. In this section we apply the shadow formalism, using the recent formulation given in [124], to get compact expressions for $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in an integral form for any p and ℓ.[1] Using these expressions we first compute the CBs $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ for $\ell = 0$ and generic p. We then show how to reduce the integral expressions for the seeds to scalar conformal blocks. We use this method to compute $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ for $p = 1$ and $H_e^{(p)}$ for $p = 2$ explicitly. A much better use of this formalism would be to obtain a recursion relations for the seeds as done in chapter 5. We missed this obvious possibility at the time.[2]

B.1.1 CPW in Shadow Formalism

We recall (5.57) and write the CPW associated to an exchange of a given operator $\mathcal{O}_\Delta^{(\ell,\bar\ell)}$ inside the seed 4-point function

$$W_{\ell,\bar\ell}^{(p)} \equiv \langle F_{\Delta_1}^{(0,0)} F_{\Delta_2}^{(p,0)} \mathcal{O}_\Delta^{(\ell,\bar\ell)} \rangle^{(\bullet)} \bowtie {}^{(\bullet)} \langle \overline{\mathcal{O}}_\Delta^{(\bar\ell,\ell)} F_{\Delta_3}^{(0,0)} F_{\Delta_4}^{(0,p)} \rangle , \tag{B.1}$$

where $|\ell - \bar\ell| = p$. In the embedding formalism

$$\begin{aligned} &\langle \ldots O_\Delta^{(\ell,\bar\ell)} \rangle^{(a)} \bowtie {}^{(b)} \langle \overline{O}_\Delta^{(\bar\ell,\ell)} \ldots \rangle = \\ &\qquad \nu \int d^4X_0 \, \langle \ldots O_\Delta^{(\ell,\bar\ell)}(X_0, S, \overline{S}) \, \rangle^{(a)} \, \overleftrightarrow{\Pi}_{\ell,\bar\ell} {}^{(b)} \langle \widetilde{O}_{4-\Delta}^{(\ell,\bar\ell)}(X_0, T, \overline{T}) \, \ldots \rangle \Big|_M , \end{aligned} \tag{B.2}$$

where ν is a normalization factor, the projector gluing two 3-point functions is given by

$$\overleftrightarrow{\Pi}_{\ell,\bar\ell} = (\overleftarrow{\partial}_S X_0 \overrightarrow{\partial}_T)^\ell (\overleftarrow{\partial}_{\overline{S}} \overline{X}_0 \overrightarrow{\partial}_{\overline{T}})^{\bar\ell} , \tag{B.3}$$

and $\widetilde{O}$ is the shadow operator

$$\widetilde{O}_{4-\Delta}^{(\ell,\bar\ell)}(X, T, \overline{T}) \equiv \int d^4Y \frac{1}{(-2X \cdot Y)^{4-\Delta+\ell+\bar\ell}} \overline{\mathcal{O}}_\Delta^{(\bar\ell,\ell)}(Y, Y\overline{T}, \overline{Y}T) . \tag{B.4}$$

[1] The shadow formalism given in an index-free 6D embedding twistor space has also been used in refs.[168, 169] to compute CBs in supersymmetric CFTs.

[2] In what follows we do not pay attention to the overall normalization of CPWs. Fixing normalizations with the standard techniques used here is a tedious task.

The integral in (B.2) would actually determine the CPW associated to the operator $O_\Delta^{(\ell,\bar\ell)}$ plus its unwanted shadow counterpart, that corresponds to the exchange of a similar operator but with the scaling dimension $\Delta \to 4-\Delta$. The two contributions can be distinguished by their different behaviour under the monodromy transformation $X_{12} \to e^{4\pi i}X_{12}$. In particular, the physical CPW should transform with the phase $e^{2i\pi(\Delta-\Delta_1-\Delta_2)}$, independently of the Lorentz quantum numbers of the external and exchanged operators. This projection on the correct monodromy component explains the subscript M in the bar at the end of (B.2).

The seed tensor structures were given in (A.96) and (A.98). We applying the shadow transformation if the seed structure appears to the right in (B.2). Summarizing we have

$$\langle \mathcal{F}_1^{(0,0)}(\mathbf{p}_1)\,\mathcal{F}_2^{(p,0)}(\mathbf{p}_2)\,\mathcal{O}_\Delta^{(\ell,\,\ell+p)}(\mathbf{p}_0)\rangle^{(\bullet)} = [\hat{\mathbb{I}}^{02}]^p\,[\hat{\mathbb{J}}_{12}^0]^\ell\mathcal{K}_3, \tag{B.5}$$

$$\langle \widetilde{\mathcal{O}}_{4-\Delta}^{(\ell,\,\ell+p)}(\mathbf{p}_0)\,\mathcal{F}_3^{(0,0)}(\mathbf{p}_3)\,\mathcal{F}_4^{(0,p)}(\mathbf{p}_4)\,\rangle^{(\bullet)} \propto [\hat{\mathbb{K}}_3^{04}]^p\,[\hat{\mathbb{J}}_{34}^2]^\ell\mathcal{K}_3\big|_{\Delta\to 4-\Delta}, \tag{B.6}$$

and

$$\langle \mathcal{F}_1^{(0,0)}(\mathbf{p}_1)\,\mathcal{F}_2^{(p,0)}(\mathbf{p}_2)\,\overline{\mathcal{O}}_\Delta^{(\ell+p,\,\ell)}(\mathbf{p}_0)\rangle^{(\bullet)} = [\hat{\mathbb{K}}_1^{20}]^p\,[\hat{\mathbb{J}}_{12}^0]^\ell\mathcal{K}_3, \tag{B.7}$$

$$\langle \widetilde{\overline{\mathcal{O}}}{}_{4-\Delta}^{(\ell+p,\,\ell)}(\mathbf{p}_2)\,\mathcal{F}_3^{(0,0)}(\mathbf{p}_3)\,\mathcal{F}_4^{(0,p)}(\mathbf{p}_4)\,\rangle^{(\bullet)} \propto [\hat{\mathbb{I}}^{40}]^p\,[\hat{\mathbb{J}}_{34}^2]^\ell\mathcal{K}_3\big|_{\Delta\to 4-\Delta}, \tag{B.8}$$

Using the above relations, after a bit of algebra, one can write

$$W^{seed}(p) = \frac{\nu}{X_{12}^{a_{12}+\frac{\ell}{2}}X_{34}^{a_{34}+\frac{\ell+p}{2}}}\int D^4X_0\frac{\mathcal{N}_\ell(p)}{X_{01}^{a_{01}+\frac{\ell}{2}}X_{02}^{a_{02}+\frac{\ell+p}{2}}X_{03}^{a_{03}+\frac{\ell+p}{2}}X_{04}^{a_{04}+\frac{\ell}{2}}}\bigg|_{M=1}, \tag{B.9}$$

$$\overline{W}^{seed}(p) = \frac{\overline{\nu}}{X_{12}^{a_{12}+\frac{\ell+p}{2}}X_{34}^{a_{34}+\frac{\ell}{2}}}\int D^4X_0\frac{\overline{\mathcal{N}}_\ell(p)}{X_{01}^{a_{01}+\frac{\ell+p}{2}}X_{02}^{a_{02}+\frac{\ell}{2}}X_{03}^{a_{03}+\frac{\ell}{2}}X_{04}^{a_{04}+\frac{\ell+p}{2}}}\bigg|_{M=1}, \tag{B.10}$$

where

$$a_{01} = \frac{\Delta}{2}+\frac{p}{4}-a,\qquad a_{02} = \frac{\Delta}{2}-\frac{p}{4}+a,\qquad a_{12} = \frac{\Delta_1+\Delta_2}{2}-\frac{\Delta}{2},$$
$$a_{03} = \frac{4-\Delta}{2}+\frac{p}{4}+b,\qquad a_{04} = \frac{4-\Delta}{2}-\frac{p}{4}-b,\qquad a_{34} = \frac{\Delta_3+\Delta_4}{2}-\frac{4-\Delta}{2}, \tag{B.11}$$

and

$$\mathcal{N}_\ell(p) \equiv (\overline{S}S_2)^p(\overline{S}X_2\overline{X}_1S)^\ell\,\overleftrightarrow{\Pi}_{\ell,\ell+p}(\overline{S}_4X_3\overline{T})^p(\overline{T}X_4\overline{X}_3T)^\ell, \tag{B.12}$$

$$\overline{\mathcal{N}}_\ell(p) \equiv (\overline{S}_4S)^p(\overline{S}X_3\overline{X}_4S)^\ell\,\overleftrightarrow{\Pi}_{\ell+p,\ell}(S_2\overline{X}_1T)^p(\overline{T}X_1\overline{X}_2T)^\ell. \tag{B.13}$$

We will not need to determine the normalization factors ν and $\overline{\nu}$ in (B.9) and (B.10). Notice that the correct behaviour of the seed CPWs under $X_{12} \to e^{4\pi i}X_{12}$ is saturated by the factor X_{12} multiplying the integrals in (B.9) and (B.10). Hence the latter should be projected to their trivial monodromy components $M=1$, as indicated. Notice that (B.12) and (B.13) are related by a simple transformation:

$$\overline{\mathcal{N}}_\ell(p) = (-1)^p\,\mathcal{P}\mathcal{N}_\ell(p)\Big|_{1\leftrightarrow 3,\,2\leftrightarrow 4}, \tag{B.14}$$

where $\mathcal{P}$ is the parity operator acting on the 6D structures as found in (A.82) and (A.83).

We can recast the expression (B.12) in a compact and convenient form using some manipulations. We first define 3 variables

$$s \equiv \frac{1}{2^6} X_{12} X_{34} \prod_{n=1}^{4} X_{0n}, \tag{B.15}$$

$$t \equiv -\frac{1}{2^4\sqrt{s}}\Big(X_{02}X_{03}X_{14} - X_{01}X_{03}X_{24} - (3 \leftrightarrow 4)\Big), \tag{B.16}$$

$$u \equiv -\frac{X_{02}X_{03}X_{34}}{2^3\sqrt{s}}. \tag{B.17}$$

Then we look for a relation expressing the generic $\mathcal{N}_\ell(p)$ in terms of the known $\mathcal{N}_\ell(0)$:

$$\mathcal{N}_\ell(0) = (-1)^\ell (\ell!)^4\, s^{\ell/2} C^1_\ell(t)\,, \tag{B.18}$$

where C^p_ℓ are Gegenbauer polynomials of rank p. Starting from (B.12), after acting with the S and T derivatives, one gets

$$\mathcal{N}_\ell(p) = (\ell!)^2 (\overrightarrow{\partial}_{\overline{S}} \overline{X}_0 \overrightarrow{\partial}_{\overline{T}})^{\ell+p} \left((\overline{S}S_2)^p (\overline{S}_4 X_3 \overline{T})^p (\overline{S}\Omega\overline{T})^\ell\right)\,, \tag{B.19}$$

where we have defined $\Omega_{ab} \equiv (X_2\overline{X}_1 X_0 \overline{X}_3 X_4)_{ab}$. In order to relate $\mathcal{N}_\ell(p)$ above to $\mathcal{N}_{\ell+p}(0)$ in (B.18), we look for an operator $\widetilde{\mathcal{D}}$ satisfying

$$\widetilde{\mathcal{D}}^p\, (\overrightarrow{\partial}_{\overline{S}} \overline{X}_0 \overrightarrow{\partial}_{\overline{T}})^{\ell+p} (\overline{S}\Omega\overline{T})^{\ell+p} = (\overrightarrow{\partial}_{\overline{S}} \overline{X}_0 \overrightarrow{\partial}_{\overline{T}})^{\ell+p} \left((\overline{S}S_2)^p (\overline{S}_4 X_3 \overline{T})^p (\overline{S}\Omega\overline{T})^\ell\right)\,. \tag{B.20}$$

We deduce that $\widetilde{\mathcal{D}}$ should be bilinear in $\overline{S}_4$ and S_2 and should commute with $(\overrightarrow{\partial}_{\overline{S}} \overline{X}_0 \overrightarrow{\partial}_{\overline{T}})$. In addition to that, it should have the correct scaling in X's and should be gauge invariant. It is not difficult to see that the choice of $\widetilde{\mathcal{D}}$ defined below

$$\widetilde{\mathcal{D}} = \frac{\mathcal{D}}{2\, X_{01} X_{04}}, \quad \mathcal{D} \equiv (\overline{S}_4 X_0 \overline{\Sigma}^N S_2)\frac{\partial}{\partial X_2^N} \tag{B.21}$$

fulfills all the requirements. One has $\widetilde{\mathcal{D}}(\overline{S}\Omega\overline{T}) = (\overline{S}S_2)(\overline{S}_4 X_3 \overline{T})$. Iterating it p times gives the desired relation:

$$\mathcal{N}_\ell(p) \propto \widetilde{\mathcal{D}}^p \mathcal{N}_{\ell+p}(0)\,. \tag{B.22}$$

The operator $\mathcal{D}$ annihilates all the scalar products with the exception of X_{12}, in which case we have $\mathcal{D}X_{12} = -2\,\mathbb{I}^{42}_{01}$. The action on the s, t, and u variables is

$$\mathcal{D}\, s = -2\, X_{12}^{-1}\, s\, I^{42}_{01}, \;\; \mathcal{D}\, t = X_{12}^{-1}\,(u^{-1}\, I^{42}_{30} + t\, I^{42}_{01}), \;\; \mathcal{D}\, u^{-1} = -X_{12}^{-1}\, u^{-1}\, I^{42}_{01}\,, \tag{B.23}$$

on Gegenbauer polynomials is

$$\mathcal{D}\, C^\lambda_n(t) = 2\lambda\, C^{\lambda+1}_{n-1}(t)\, \mathcal{D}\, t\,, \tag{B.24}$$

and vanishes on I^{42}_{01} and I^{42}_{30}. Using recursively the identity for Gegenbauer polynomials

$$\frac{n}{2\lambda}\, C^\lambda_n(t) - t\, C^{\lambda+1}_{n-1}(t) = -C^{\lambda+1}_{n-2}(t)\,, \tag{B.25}$$

we can write the following expression for $\mathcal{N}_\ell(p)$:

$$\mathcal{N}_\ell(p) \propto s^{\frac{\ell}{2}} \sum_{w=0}^{p} \binom{p}{w} u^w\, C_{\ell-w}^{p+1}(t)\, [I_{30}^{42}]^{p-w}[I_{01}^{42}]^w, \tag{B.26}$$

where $\binom{p}{w}$ is the binomial coefficient. Combining together (B.9), (B.10), (B.14), (B.17) and (B.26) we can finally write

$$\begin{aligned} W^{seed}(p) &= \nu' \sum_{w=0}^{p} \binom{p}{w} \frac{1}{X_{12}^{a_{12}+\frac{w}{2}} X_{34}^{a_{34}+\frac{p-w}{2}}} \int D^4X_0 \frac{C_{\ell-w}^{p+1}(t)\, [I_{30}^{42}]^{p-w}[I_{01}^{42}]^w}{X_{01}^{a_{01}+\frac{w}{2}} X_{02}^{a_{02}+\frac{p-w}{2}} X_{03}^{a_{03}+\frac{p-w}{2}} X_{04}^{a_{04}+\frac{w}{2}}}\Bigg|_{M=1}, \\ \overline{W}^{seed}(p) &= \overline{\nu}' \sum_{w=0}^{p} \binom{p}{w} \frac{1}{X_{12}^{a_{12}+\frac{p-w}{2}} X_{34}^{a_{34}+\frac{w}{2}}} \int D^4X_0 \frac{C_{\ell-w}^{p+1}(t)\, [I_{30}^{42}]^{w}[I_{01}^{42}]^{p-w}}{X_{01}^{a_{01}+\frac{p-w}{2}} X_{02}^{a_{02}+\frac{w}{2}} X_{03}^{a_{03}+\frac{w}{2}} X_{04}^{a_{04}+\frac{p-w}{2}}}\Bigg|_{M=1} \end{aligned} \tag{B.27}$$

where ν' and $\overline{\nu}'$ are undetermined normalization factors.

B.1.2 Seed Conformal Blocks and Their Explicit Form for $\ell = 0$

The computation of the CBs $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ starting form (B.27) is a non-trivial task for generic ℓ and p, since we are not aware of a general formula for an integral that involves $C_{\ell-w}^{p+1}(t)$ for $p \neq 0$. For any given ℓ, one can however expand the Gegenbauer polynomial, in which case the CBs $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ can be computed. In this subsection we first discuss the structure of CBs for generic ℓ and then compute them for $\ell = 0$ and generic p.

Recalling the definition of t in (B.17), one realizes that the Gegenbauer polynomials in (B.27), when expanded, do not give rise to intrinsically new integrals but just amounts to shifting the exponents in the denominator. The tensor structures in the numerators bring p open indices in the form $X_0^{N_1} \ldots X_0^{N_p}$, which can be removed by using (3.21) in ref. [124]. In this way the problem is reduced to the computation of scalar integrals in $d \equiv 2h = 2(2+p)$ effective dimensions, of the form:

$$I^{(h)}_{A_{02},\,A_{03},\,A_{04}} \equiv \int D^{2h} X_0 \frac{1}{X_{01}^{A_{01}} X_{02}^{A_{02}} X_{03}^{A_{03}} X_{04}^{A_{04}}}\Bigg|_{M=1}, \tag{B.28}$$

where $A_{01} + A_{02} + A_{03} + A_{04} = 2h$. The capital A_{0i} are used for the exponents in the denomentaor with all possible shifts introduced by the Gegenbaur polynomials. This integral is given by

$$I^{(h)}_{A_{02},\,A_{03},\,A_{04}} \propto X_{13}^{A_{04}-h} X_{14}^{A_{02}+A_{03}-h} X_{24}^{-A_{02}} X_{34}^{h-A_{03}-A_{04}} \times R^{(h)}(z,\overline{z};\, A_{02}, A_{03}, A_{04}), \tag{B.29}$$

where

$$R^{(h)}(z,\overline{z};\, A_{02}, A_{03}, A_{04}) \equiv \Big(-\frac{\partial}{\partial v}\Big)^{h-1} f(z;\, A_{02}, A_{03}, A_{04}) f(\overline{z};\, A_{02}, A_{03}, A_{04}), \tag{B.30}$$

$$f(z;\, A_{02}, A_{03}, A_{04}) \equiv {}_2F_1(A_{02} - h + 1, -A_{04} + 1; -A_{03} - A_{04} + h + 1; z) \tag{B.31}$$

The derivative $-\partial/\partial v$ in $(z, \overline{z})$ coordinates equals to

$$-\frac{\partial}{\partial v} = \frac{1}{z-\overline{z}}\Big(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\Big). \tag{B.32}$$

In the case of $\ell = 0$, all the above manipulations simplify drastically. The Gegenbauer polynomials $C_{\ell-w}^{p+1}(t)$ vanishe for all the values w except for $w = 0$, leaving only one type of tensor structure: $[I^{42}_{30}]^p$ for $W^{seed}(p)$ and $[I^{42}_{01}]^p$ for $\overline{W}^{seed}(p)$. This leads to a one-to-one correspondence between CBs and integrals:

$$H_e^{(p)} \propto X_{13}^{p-e}X_{34}^{e}\,\mathcal{K}_4^{-1}\,I^{(2+p)}_{a_{02}+\frac{p}{2},\,a_{03}+\frac{p}{2},\,a_{04}+e} \propto (z\overline{z})^{\frac{\Delta+\frac{p}{2}}{2}}R^{(2+p)}(z,\overline{z};\,a_{02}+\frac{p}{2},a_{03}+\frac{p}{2},a_{04}+e), \quad \text{(B.33)}$$
$$\overline{H}_e^{(p)} \propto X_{12}^{e}X_{13}^{p-e}\,\mathcal{K}_4^{-1}\,I^{(2+p)}_{a_{02}+e,\,a_{03}+p-e,\,a_{04}+\frac{p}{2}} \propto (z\overline{z})^{\frac{\Delta-\frac{p}{2}}{2}+e}R^{(2+p)}(z,\overline{z};\,a_{02}+e,a_{03}+p-e,a_{04}+\frac{p}{2}).$$

We have omitted here the relative factors between different CBs. They must be restored if one wants to check that $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ in (B.33) satisfy the Casimir system (4.4). For generic ℓ the CBs are a sum of expressions like (B.33) with different shifts of the parameters A_{0i}, weighted by the relative constants and powers of v (coming from the Gegenbauer polynomial). Since all these terms have $p+1$ derivatives with respect to v, the highest power in $1/(z-\overline{z})$ appearing in $H_e^{(p)}$ and $\overline{H}_e^{(p)}$ is

$$\left(\frac{1}{z-\overline{z}}\right)^{1+2p}. \quad \text{(B.34)}$$

The asymptotic behaviour of the CBs when $z,\overline{z}\to 0$ ($u\to 0$, $v\to 1$) for $\ell=0$ is easily obtained from (B.33) by noticing that $R^{(h)}(z,\overline{z};\,A_{02},A_{03},A_{04})$ is constant in this limit. Then we have

$$\lim_{z\to 0,\,\overline{z}\to 0} H_e^{(p)} \propto (z\overline{z})^{\frac{\Delta}{2}+\frac{p}{4}}\,, \qquad \lim_{z\to 0,\,\overline{z}\to 0} \overline{H}_e^{(p)} \propto (z\overline{z})^{\frac{\Delta}{2}-\frac{p}{4}+e}\,. \quad \text{(B.35)}$$

Notice that the behavior (B.35) of the CBs for $z,\overline{z}\to 0$ when $\ell=0$ is not guaranteed to be straightforwardly extended for any $\ell\neq 0$. Indeed, we see from (B.27) that for a given p, the generic CPW is obtained when $\ell\geq p$, in which case all terms in the sum over w are present. It might also seem like all the values of $\ell<p$ should be treated separately.

B.1.3 Computing the Conformal Blocks for $\ell \neq 0$

A useful expression of the CBs for generic values of ℓ can be obtained using (B.22) and the known closed form of $W^{seed}(0)$. Recall that

$$W^{seed}(0) = \left(\frac{X_{14}}{X_{13}}\right)^{b^{(0)}}\left(\frac{X_{24}}{X_{14}}\right)^{-a^{(0)}}\frac{H_0^{(0)}(z,\overline{z})}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}}X_{34}^{\frac{\Delta_3+\Delta_4}{2}}}\,, \quad \text{(B.36)}$$

where $a^{(0)}$ and $b^{(0)}$ are as in (4.9) for $p=0$ and $H^{(0)}(z,\overline{z})$ are the known scalar CBs computed in [127, 128]

$$H_0^{(0)}(z,\overline{z}) = H_0^{(0)}(z,\overline{z};\Delta,l,a,b) = (-1)^{\ell}\frac{z\overline{z}}{z-\overline{z}}\left(k^{(a,b;0)}_{\frac{\Delta+\ell}{2}}(z)k^{(a,b;0)}_{\frac{\Delta-\ell-2}{2}}(\overline{z}) - (z\leftrightarrow\overline{z})\right). \quad \text{(B.37)}$$

Comparing (B.36) with (B.27) for $p=0$, one can extract the value of the shadow integral in closed form for generic spin ℓ [124]:

$$I_\ell \equiv \int D^4X_0\frac{C^1_\ell(t)}{X_{01}^{a_{01}}X_{02}^{a_{02}}X_{03}^{a_{03}}X_{04}^{a_{04}}}\Big|_{M=1} \propto \left(\frac{X_{14}}{X_{13}}\right)^{b^{(0)}}\left(\frac{X_{24}}{X_{14}}\right)^{-a^{(0)}}\frac{H_0^{(0)}(z,\overline{z};\Delta,\ell,a,b)}{X_{12}^{\frac{\Delta}{2}}X_{34}^{\frac{4-\Delta}{2}}}\,. \quad \text{(B.38)}$$

Using the relations (B.18) and (B.22) one can recast $W^{seed}(p)$ and $\overline{W}^{seed}(p)$ in the form

$$
\begin{aligned}
W^{seed}(p) &\propto \frac{\mathcal{D}_{N_1}\ldots\mathcal{D}_{N_p}}{X_{12}^{a_{12}+\frac{\ell}{2}}X_{34}^{a_{34}}}\, X_{12}^{\frac{\ell+p}{2}}\int D^4X_0 \frac{C^1_{\ell+p}(t)X_0^{N_1}\ldots X_0^{N_p}}{X_{01}^{a_{01}+\frac{p}{2}}X_{02}^{a_{02}}X_{03}^{a_{03}}X_{04}^{a_{04}+\frac{p}{2}}}\Bigg|_{M=1}\,, \\
\overline{W}^{seed}(p) &\propto \frac{\overline{\mathcal{D}}_{N_1}\ldots\overline{\mathcal{D}}_{N_p}}{X_{12}^{a_{12}}X_{34}^{a_{34}+\frac{\ell}{2}}}\, X_{34}^{\frac{\ell+p}{2}}\int D^4X_0 \frac{C^1_{\ell+p}(t)X_0^{N_1}\ldots X_0^{N_p}}{X_{01}^{a_{01}}X_{02}^{a_{02}+\frac{p}{2}}X_{03}^{a_{03}+\frac{p}{2}}X_{04}^{a_{04}}}\Bigg|_{M=1}\,, \qquad \text{(B.39)}
\end{aligned}
$$

where $\overline{\mathcal{D}} = \mathcal{P}\mathcal{D}|_{1\leftrightarrow 3, 2\leftrightarrow 4}$, as follows from (B.14), $\mathcal{D} = \mathcal{D}_M X_0^M$, $\overline{\mathcal{D}} = \overline{\mathcal{D}}_M X_0^M$. The tensor integral is evaluated using the $SO(4,2)$ Lorentz symmetry. One writes

$$
\int D^4X_0 \frac{C^1_{\ell+p}(t)X_0^{M_1}\ldots X_0^{M_p}}{X_{01}^{a_{01}+\frac{p}{2}}X_{02}^{a_{02}}X_{03}^{a_{03}}X_{04}^{a_{04}+\frac{p}{2}}} = \sum_n A_n(X_i)\ \tau_n^{M_1\ldots M_p}(X_i)\,, \qquad \text{(B.40)}
$$

where n runs over all possible rank p traceless symmetric tensors τ_n which can be constructed from X_1, X_2, X_3, X_4 and η_{MN}'s, with arbitrary scalar coefficients A_n to be determined. Performing all possible contractions, which do not change the monodromy of the integrals, the A_n coefficients can be solved as linear combinations of the scalar block integrals I_ℓ defined in (B.38), with shifted external dimensions.

In this way, we have computed the CBs $H_e^{(p)}$ with $p = 1, 2$ and $\overline{H}_e^{(p)}$ with $p = 1$ for general Δ, ℓ, a, b. In all cases the CBs satisfy the Casimir system (4.4).

The expression (B.39), (B.39) and B.40 allow to reduce the seed and the dual seed CPWs to the scalar Dolan and Osborn CPWs. Instead of this one could have related the unknown (p)-seeds with unknown $(p-1)$-seeds. We missed this obvious possibility at the time.

B.2 Properties of the $\mathcal{F}$ Functions

In this Appendix we provide all the properties of the functions $\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)}$ needed for the system of Casimir equations and more specifically to derive (4.21)-(4.23). We will not consider the functions $\mathcal{F}_{\rho_1,\rho_2}^{\pm\,(a,b;c)}$ here, since their properties can be trivially deduced from the ones below by demanding both sides to be symmetric/anti-symmetric under the exchange $z \leftrightarrow \bar{z}$.

The fundamental identities to be considered can be divided in two sets, depending on whether the values (a, b, c) of the functions $\mathcal{F}$ are left invariant or not. The former identities read

$$
\begin{aligned}
\Big(\frac{1}{z}-\frac{1}{2}\Big)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1-1,\rho_2}^{(a,b;c)} - D_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} + B_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b;c)} && \text{(B.41)}\\
\Big(\frac{1}{\bar{z}}-\frac{1}{2}\Big)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1,\rho_2-1}^{(a,b;c)} - D_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} + B_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b;c)} && \text{(B.42)}\\
L_0\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \rho_2\mathcal{F}_{\rho_1,\rho_2-1}^{(a,b;c)} - \rho_1\mathcal{F}_{\rho_1-1,\rho_2}^{(a,b;c)} - (\rho_2+c-1)B_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b;c)} + && \text{(B.43)}\\
&\quad (\rho_1+c-1)B_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b;c)} + \frac{1}{2}(2-c)(D_{\rho_1}^{(a,b,c)} - D_{\rho_2}^{(a,b,c)})\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)},
\end{aligned}
$$

where

$$
L_0 = \Big((1-\bar{z})\partial_{\bar{z}} - (1-z)\partial_z\Big), \qquad \text{(B.44)}
$$

and we have defined

$$\begin{aligned} C_\rho^{(a,b,c)} &= \frac{(a+\rho)(b-c-\rho)}{(c+2\rho)(c+2\rho-1)}\,, \qquad (B.45)\\ B_\rho^{(a,b,c)} &= C_\rho^{(a,b,c)}\, C_{\rho+1}^{(b-1,a,c-1)} = \frac{(\rho+a)(\rho+b)(\rho+c-b)(\rho+c-a)}{(2\rho+c)^2(c+2\rho+1)(c+2\rho-1)}\,,\\ D_\rho^{(a,b,c)} &= \frac{(2a-c)(2b-c)}{2(c+2\rho)(c+2\rho-2)}\,. \qquad (B.46) \end{aligned}$$

The latter identities read

$$\begin{aligned} \mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1,\rho_2}^{(a,b-1;c-1)} - C_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b-1;c-1)} - \qquad (B.47)\\ &\quad C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b-1;c-1)} + C_{\rho_1}^{(a,b,c)}C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2+1}^{(a,b-1;c-1)}\,,\\ \mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1,\rho_2}^{(a-1,b;c-1)} - C_{\rho_1}^{(b,a,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a-1,b;c-1)} - \qquad (B.48)\\ &\quad C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a-1,b;c-1)} + C_{\rho_1}^{(b,a,c)}C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1+1,\rho_2+1}^{(a-1,b;c-1)}\,,\\ \frac{1}{z\bar z}\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= \mathcal{F}_{\rho_1-1,\rho_2-1}^{(a+1,b+1;c+2)}\,, \qquad (B.49)\\ (z-\bar z)L(a)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= (\rho_2-\rho_1)\mathcal{F}_{\rho_1,\rho_2}^{(a,b-1;c-1)} - (\rho_1+\rho_2+c-1)C_{\rho_1}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2}^{(a,b-1;c-1)} + \qquad (B.50)\\ &\quad (\rho_1+\rho_2+c-1)C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1,\rho_2+1}^{(a,b-1;c-1)} - (\rho_2-\rho_1)C_{\rho_1}^{(a,b,c)}C_{\rho_2}^{(a,b,c)}\mathcal{F}_{\rho_1+1,\rho_2+1}^{(a,b-1;c-1)}\,,\\ \frac{z-\bar z}{z\bar z}L(b)\mathcal{F}_{\rho_1,\rho_2}^{(a,b;c)} &= (\rho_2-\rho_1)\mathcal{F}_{\rho_1-1,\rho_2-1}^{(a,b+1;c+1)} - (\rho_1+\rho_2+c-1)C_{\rho_1}^{(b,a,c)}\mathcal{F}_{\rho_1,\rho_2-1}^{(a,b+1;c+1)} + \qquad (B.51)\\ &\quad (\rho_1+\rho_2+c-1)C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1-1,\rho_2}^{(a,b+1;c+1)} - (\rho_2-\rho_1)C_{\rho_1}^{(b,a,c)}C_{\rho_2}^{(b,a,c)}\mathcal{F}_{\rho_1,\rho_2}^{(a,b+1;c+1)}\,. \end{aligned}$$

The relations (B.41)-(B.43) were first derived in ref.[128] (see also ref.[129]), while the relations (B.50) and (B.51) are novel. It is straightforward to see that (4.21)-(4.23) can be derived using proper combinations of (B.41)-(B.51). For instance, the action of the first term appearing in the r.h.s. of (4.20) is reproduced (modulo a trivial constant factor) by taking the combined action given by ((B.42)−(B.41))× (B.50)× (B.47). All other terms in (4.18)-(4.20) are similarly deconstructed.

B.3 The Conformal Blocks for $p = 1$

We report in this appendix the full explicit solution for the two conformal blocks $H_0^{(1)}$ and $H_1^{(1)}$ associated to the exchange of fermion operators $\mathcal{O}^{(\ell,\ell+1)}$ for the specific values

$$a^{(1)} = \frac{1}{2}\,, \qquad b^{(1)} = -\frac{1}{2}\,. \qquad (B.52)$$

We choose as undetermined coefficient $c^1_{0,-1}$ and report below the values of the coefficients normalized to $c^1_{0,-1}$. We have

$$c^0_{-2,0} = \frac{(2+\ell)}{2\,(1+\ell)}\,,\quad c^0_{-1,-1} = -\frac{\ell}{2\,(1+\ell)}\,,\quad c^1_{-1,0} = -\frac{(3+\ell)}{1+\ell}\,. \qquad (B.53)$$

$$
\begin{aligned}
c^0_{-1,0} &= \frac{(3+\ell)(-1+2\Delta)(-1+2\ell+2\Delta)}{8(1+\ell)(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{-1,1} &= -\frac{(2+\ell)(5+2\ell-2\Delta)^2(-7+2\Delta)}{32(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^0_{0,-1} &= -\frac{(-1+2\Delta)(-1+2\ell+2\Delta)}{8(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{0,0} &= \frac{\ell(-7+2\Delta)(-1+2\ell+2\Delta)^2}{32(1+\ell)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{0,1} &= -\frac{(3+\ell)(5+2\ell-2\Delta)^2(-5+2\Delta)(-1+2\ell+2\Delta)}{128(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)(1+2\ell+2\Delta)},\\
c^0_{1,0} &= \frac{(-5+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)^2}{128(-3+2\Delta)(1+2\ell+2\Delta)^2(5+2\ell+2\Delta)},\\
c^1_{-1,1} &= -\frac{(2+\ell)(5+2\ell-2\Delta)(-1+2\Delta)}{4(1+\ell)(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^1_{0,2} &= \frac{(2+\ell)(1+2\ell-2\Delta)(5+2\ell-2\Delta)^2(-5+2\Delta)}{64(1+\ell)(3+2\ell-2\Delta)^2(7+2\ell-2\Delta)(-3+2\Delta)},\\
c^1_{1,0} &= -\frac{(-7+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)}{16(-3+2\Delta)(1+2\ell+2\Delta)^2},\\
c^1_{1,1} &= -\frac{\ell(5+2\ell-2\Delta)(-5+2\Delta)(-1+2\ell+2\Delta)(3+2\ell+2\Delta)}{64(1+\ell)(7+2\ell-2\Delta)(-3+2\Delta)(1+2\ell+2\Delta)^2}
\end{aligned}
$$

$$
\begin{aligned}
c^1_{0,0} &= \frac{1}{4(1+\ell)(11+2\ell-2\Delta)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)}\times\\
&\Big(576-384\Delta+\ell\Big(627-2\ell(-29+2\ell(7+2\ell))-472\Delta+4\ell(-47+4\ell(3+\ell))\Delta\\
&+8(-9+\ell(19+2\ell))\Delta^2-16(-6+\ell)\Delta^3-16\Delta^4\Big)\Big),\\
c^1_{0,1} &= \frac{(5+2\ell-2\Delta)}{16(1+\ell)(3+2\ell-2\Delta)(7+2\ell-2\Delta)(-3+2\Delta)(-3+2\ell+2\Delta)(1+2\ell+2\Delta)}\times\\
&\Big(\ell(643-14\ell(-3+2\ell(9+2\ell)))+4\ell(-232+\ell(-115+4\ell(1+\ell)))\Delta+8(3+\ell)\\
&(-24+\ell(17+2\ell))\Delta^2-16(-7+\ell)(3+\ell)\Delta^3-16(3+\ell)\Delta^4+27(9+4\Delta)\Big).
\end{aligned}
$$

The asymptotic behaviour of the CBs for $z, \bar{z} \to 0$ is dominated by the coefficients with $n = -1$ and the lowest value of m, i.e. $c^0_{-1,-1}$ and $c^1_{0,-1}$. For $\ell = 0$, the asymptotic behaviour of $H_0^{(1)}$ is given by the next term $c^0_{0,-1}$, since $c^0_{-1,-1}$ in (B.53) vanishes. Notice how the complexity of the $c^e_{m,n}$ varies from coefficient to coefficient. In general the most complicated ones are those in the "interior" of the octagons (hexagons only for $p = 1$).

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