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Explorations in AdS/CFT Correspondence

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Foreword

List of PhD Publications

This thesis contains a partial summary of my PhD research and is heavily based on the following three publications.

- M. Bertolini, D. Musso, I. Papadimitriou, and H. Raj, *A goldstino at the bottom of the cascade*, JHEP **11** (2015) 184, [arXiv:1509.0359].
- V. Bashmakov, M. Bertolini, L. Di Pietro, and H. Raj, *Scalar Multiplet Recombination at Large N and Holography*, JHEP **05** (2016) 183, [arXiv:1603.0038].
- V. Bashmakov, M. Bertolini, and H. Raj, *Broken current anomalous dimensions, conformal manifolds, and renormalization group flows*, Phys. Rev. **D95** (2017), no. 6 066011, [arXiv:1609.0982].

My other works, include calculation of sphere partition functions of p-form gauge theories and investigation of conformal manifolds using conformal perturbation theory, for which we refer to the following pre-prints.

- H. Raj, *A note on sphere free energy of p-form gauge theory and Hodge duality* (To appear in Classical and Quantum gravity), arXiv:1611.0250.
- V. Bashmakov, M. Bertolini and H. Raj, *On non-supersymmetric conformal manifolds: field theory and holography*, arXiv:1709.01749.

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To my parents

Contents

Chapter 1

Introduction and Outline

String theory is a leading candidate for a consistent theory of quantum gravity. It includes consistent interactions between gauge and gravitational forces and has given us a rather deep insight on various interconnections between them. The discovery of D-branes in 1995 [5], eventually led to the celebrated AdS/CFT correspondence [6], or more generally the gauge/gravity duality. This correspondence is a conjectured equivalence between two seemingly different theories: a quantum field theory living in d dimensional space and its 'dual' theory of quantum gravity living in $(d+1)$ dimensions. By now, there are several well studied examples that give rather strong evidence in favour of this duality. In this chapter, we give a quick tour of some key features of AdS/CFT correspondence supplementing it with well studied examples.

Quantitatively, the AdS/CFT correspondence is a statement about the generating functional of a quantum field theory and its dual theory of quantum gravity. The complete knowledge of the generating functional allows us to know all that there is to know about a given theory. In its strongest form, the AdS/CFT correspondence states that

$$\mathcal{Z}_{\mathrm{QFT}} = \mathcal{Z}_{\mathrm{Grav}} \ . \tag{1.1}$$

In general, it is very hard to evaluate the generating functional of a given theory in full detail. However, there might be regions in parameter space where it is possible to explicitly evaluate the generating functional using certain approximations. In what follows we give two concrete examples of the AdS/CFT correspondence and in each case identify a regime where the partition function on either side of (1.1) can be explicitly evaluated.

1.1 $\mathcal{N}=4$ SYM / $AdS_5 \times S^5$ duality

One the most thoroughly studied examples of the AdS/CFT duality is that of the maximally superconformal $\mathcal{N}=4$ super Yang-Mills (SYM) in 4 dimensions with gauge group $U(N)$ and Type IIB string theory on $AdS_5 \times S^5$ with N units of five-form flux threading through the

S^5 [6]. By now this duality has become the canonical example of AdS/CFT correspondence. At the heart of this conjecture lies the dual nature of D-branes, which itself is related to the open/close string duality. In the context of perturbative open string theory, a D-brane can be pictured as a hyperplane on which open strings end. The massless excitations of the open strings ending on the brane defines a gauge theory whose dynamics takes place on the brane world-volume. In another description, D-branes are non-perturbative states of the closed string spectrum. Their tension scales as $1/g_s$ where g_s is the string coupling and at low energy they are described by soliton-like solutions of the corresponding supergravity equations of motion. The $\mathcal{N}=4$, SYM / $AdS_5 \times S^5$ conjecture is motivated by the following line of reasoning (see [6] for the original argument and [7] and [8] for reviews). Consider Type IIB string theory in $\mathbb{R}^{1,9}$ in presence of N parallel D3 branes. Taking the limit $\alpha' \to 0$ (where $\sqrt{\alpha'}$ is the characteristic string length) and keeping fixed the string coupling g_s and N along with all physical scales, the open string description of D3-branes give rise to two decoupled systems: the $\mathcal{N}=4$, $U(N)$ super Yang-Mills in $\mathbb{R}^{1,3}$ and free Type IIB supergravity in $\mathbb{R}^{1,9}$. In the same limit, the closed string description also gives rise to two decoupled systems: the *full* Type IIB superstring theory on $AdS_5 \times S^5$ and free Type IIB supergravity in $\mathbb{R}^{1,9}$. Since the free Type IIB supergravity appears in both the descriptions it is natural to identity $\mathcal{N}=4$ $U(N)$ super Yang-Mills in $3+1$ dimensions with Type IIB superstring theory on $AdS_5 \times S^5$. The aforementioned limit is known as the Maldacena limit or more commonly the decoupling limit.

One of the key features of this correspondence is that the group of symmetries on both the sides of the duality match. Global symmetries of the field theory are translated into large gauge transformations in the bulk theory that leave the background invariant. In the $\mathcal{N}=4$ SYM / $AdS_5 \times S^5$ duality, the group of symmetries is the maximal superconformal group in four dimensions $PSU(2,2|4)$. All operators/states on either side of the duality lie in some unitary representation of this group. One of the statements of AdS/CFT is that there is a one-to-one correspondence between gauge invariant operators in $\mathcal{N}=4$ SYM and the spectrum of Type IIB superstring theory on $AdS_5 \times S^5$. This statement is nothing but an isomorphism between representations; indeed one can think of AdS/CFT as a map between representations of the group of symmetries.

The $\mathcal{N}=4$ theory contains the following fundamental fields: a gauge field A_μ, 6 real scalars Φ^i, and 4 Weyl fermions λ^a, each in the adjoint representation of the gauge group $U(N)$. The theory also has an $SU(4)$ R-symmetry that acts as an automorphism on the supercharges. The 4 fermions transform in the fundamental representation of $SU(4)$ whereas the scalars transform as a **6**. There are two dimensionless parameters in the gauge theory: the Yang-Mill's coupling g_{YM} and the rank of the gauge group N. On the gravity side, we have (in the bosonic sector) the metric $g_{\mu\nu}$, the axio-dilaton $\tau = C_0 + ie^{-\phi}$, the NSNS two-form B_2 and the RR forms C_2 and C_4. The theory is characterized by the dimensionless string coupling g_s and two dimensionful parameters: the string scale α' and the length scale

L of AdS (which is given by $L^4 = 4\pi g_s N(\alpha')^2$ and is the same as the radius of S^5). The $SU(2,2) \approx SO(2,4)$ conformal symmetry of the CFT is realized as isometry group of AdS_5 whereas the global $SU(4)$ R-symmetry is realized as the group of isometries of S^5. The string scale and the AdS length scale can be combined into a dimensionless quantity, the string tension $T = L^2/(2\pi\alpha')$. The AdS/CFT dictionary is then governed by two fundamental relations

$$g_{YM}^2 = 4\pi g_s \ , \qquad T = \frac{1}{2\pi}\sqrt{\lambda} \ , \tag{1.2}$$

where λ is the 't Hooft coupling, $\lambda = g_{YM}^2 N$.

In the perturbative regime $g_{YM} \to 0$, the $\mathcal{N} = 4$ theory admits a perturbative expansion that can be written as a sum over 2D surfaces of different topologies, weighted by a factor of N^{2-2g}, where g is the genus of a surface which is defined by the possibility of drawing a vacuum diagram on it without self-intersections. Consequently, the leading contribution comes from planar diagrams only, and surface of higher genus contribute only at subleading order in $1/N$. On the other side, the closed string path-integral also admits a similar genus expansion where the expansion parameter is the string coupling g_s. However, type IIB string theory on $AdS_5 \times S^5$ and RR background is quantitatively intractable for arbitrary values of the string coupling and tension. Therefore, we take the classical limit where $g_s \to 0$ (hence, no genus expansion) with T held fixed. In this regime, string theory becomes non-interacting. Since we are keeping T fixed, we have to send $N \to \infty$. In this regime, the gauge theory partition function gets contribution only from planar diagrams. However, contrary to flat space, it is difficult to study even classical string theory on $AdS_5 \times S^5$. So we *further* take the low energy limit in which we send $T \to \infty$ (or effectively small α' with respect to the AdS length squared L^2). This brings us to the so-called *supergravity approximation*. In this limit, the characteristic length scale of the space is very large. So we can replace all the complications arising because of the stringy nature (α' effects) with a point particle. On the field theory side this corresponds to $\lambda \to \infty$ and brings $\mathcal{N} = 4$ SYM at strong t'Hooft coupling.

In the supergravity approximation, we get the weaker form of the AdS/CFT correspondence which states:

$$\left\langle \exp\left(\int d^4x \ J(x)\mathcal{O}(x)\right) \right\rangle_{\text{QFT}} = \exp\left(-S_{\text{SUGRA}}[\phi(z,x)|_{z\to 0} \sim J(x)]\right) \ , \tag{1.3}$$

where the x's are the four coordinates on which the QFT lives and z is the fifth radial dimension which foliates the AdS_5 into 4D Poincaré slices. We have choosen a coordinate system where $z = 0$ marks the conformal boundary of AdS_5. In the left side of (1.3) we have a local operator $\mathcal{O}$ belonging to the spectrum of gauge invariant operators in $\mathcal{N} = 4$ SYM along with a non-zero source $J(x)$ turned on. In the right side of (1.3) we have (as a consequence of saddle-point approximation) S_{SUGRA}, the on-shell action of type IIB supergravity written as a functional of the asymptotic value of the supergravity field $\phi(z,x)$.

The asymptotic (small z) value of ϕ is identified with the source J of the operator $\mathcal{O}$ and we say that $\mathcal{O}$ and ϕ are dual to each other. We identify what field is dual to what gauge invariant operator by looking at their quantum numbers under the group of symmetries.

As already mentioned, there is an isomorphism between the set of all gauge invariant operators in the $\mathcal{N} = 4$ SYM and states in type IIB theory on $AdS_5 \times S^5$. We now shed some more light on this isomorphism. Since all fundamental fields are in the adjoint representation of $SU(N)$ (the $U(1)$ part of the full $U(N)$ gauge group can be disregarded in the large N limit), gauge invariant operators can be constructed by taking trace over the $SU(N)$ indices of (finite) product of fundamental fields. Hence, local gauge invariant operators in the $\mathcal{N} = 4$ theory organize into single-trace and multi-trace operators:

$$\mathrm{tr}\,[\Phi_1 \ldots \Phi_n]\ , \qquad \mathrm{tr}\,[\Phi_1 \ldots \Phi_n]\mathrm{tr}\,[\Phi_1 \ldots \Phi_m]\ , \qquad \ldots \tag{1.4}$$

In the large N limit, with the 't Hooft coupling λ kept finite, correlation functions of single trace operators factorize into products of two-point functions. The limit can therefore be interpreted as a classical one albeit different from the usual free field theory limit $g_{YM} \to 0$. Moreover, insertions of multiple-trace operators in the correlation functions are suppressed in this limit. The set of protected single-trace and multi-trace operators are respectively dual to single- and multi-particle states in AdS_5. There also exists unprotected operators (e.g. Konishi operator $\mathrm{tr}\,\Phi^i\Phi^i$) which are dual to massive string modes of type IIB. Correlation functions between local operators at strong 't Hooft coupling can be calculated in terms of the supergravity data from the AdS/CFT prescription specified in (1.3).

Apart from local operators, there are also non-local gauge invariant operators in the spectrum of $\mathcal{N} = 4$ SYM. These are line operators (like the Wilson loop, 't Hooft loop or the dyonic Wilson-'t Hooft loop) surface operators (characterized by singular field configurations of SYM fields) and three dimensional defects (which are characterized by varying spatial profile for the gauge coupling). The set of non-local operators can be holographically captured by strings and D-branes or supergravity solutions of the Janus type.

Finally, the $\mathcal{N} = 4$ theory admits an $SL(2,\mathbb{Z})$ Montonen-Olive duality symmetry which acts on the complexified gauge coupling τ as a modular transformation. The Wilson loop is dual to the 't Hooft loop under $SL(2,\mathbb{Z})$ duality. On the gravity side this maps to the S-duality symmetry which acts on the axio-dilaton in exactly the same way, transforms the NSNS field B_2 and RR field C_2 together but leaves the metric and the five-form flux invariant.

Other extensions

The AdS/CFT correspondence, stemming from the $\mathcal{N} = 4$ SYM / $AdS_5 \times S^5$ duality has been extended to cases with lesser supersymmetries. A class of well studied holographic duality is obtained when considering D-branes at toric Calabi-Yau singularities. These are real cones over five-dimensional Sasaki-Einstein manifolds X^5 admitting at least a $U(1)^3$ isometry. In the decoupling limit, one gets a duality between type IIB string theory on $AdS_5 \times X^5$ and a quiver gauge theory, a gauge theory involving several gauge groups and charged matter fields in diverse representations.

The simplest of such examples are orbifolds of $\mathbb{R}^6$, first considered by Kachru and Silverstein in [9]. In this case, the duality is between the gauge theory obtained by orbifolding the $\mathcal{N} = 4$ SYM by a discrete subgroup Γ of the $SU(4)$ R-symmetry and type IIB theory on $AdS_5 \times S^5/\Gamma$. This was later generalized to more general singularities, giving a plethora of new AdS/CFT dual pairs. A prototypical example is the $\mathcal{N} = 1$ conifold theory, for which $X^5 = (SU(2) \times SU(2))/U(1)$, originally proposed by Klebanov and Witten in [10]. We defer to references [11–13] for reviews.

Whenever the Calabi-Yau cone differs from the maximally symmetric case, i.e. $X^5 = S^5$, the conical singularity is a true metric singularity, and this implies that there exist topologically non-trivial two- and three-cycles collapsing at the tip of the cone. This, in turn, implies the existence of fractional (as opposed to regular) D-branes in the string spectrum, open string theories whose low energy effective dynamics is a non-conformal gauge theory. Considering combinations of regular and fractional branes at the tip of Calabi-Yau cones, the AdS/CFT duality has then also been extended to non-conformal settings, some of which we will consider in the present thesis.

AdS/CFT dual pairs have also been obtained in different dimensions. Two well studied examples are that of conformal field theories describing low energy dynamics of stacks of $M2$ and $M5$ branes and M-theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$, respectively. Again, also in these cases, the possibility of obtaining dual pairs with less supersymmetry and with broken conformal invariance has also been investigated.

1.2 Vector model / Higher spin duality

We now discuss a completely separate class of examples of AdS/CFT that do not necessarily have a stringy origin. In the theories discussed in the previous section, the fundamental degrees of freedom in the CFT were matrices. There are another class of theories where the fundamental degrees of freedom are vectors, instead. The simplest example of this class is the $O(N)$ vector model which consists of N scalars transforming in the fundamental representation of $O(N)$. The simplest such $O(N)$ invariant theory is the free theory

$$S = \frac{1}{2} \int d^3x \sum_{a=1}^{N} (\partial_\mu \phi^a)^2 \ . \tag{1.5}$$

It was first conjectured in [14] that the singlet sector of the $O(N)$ model in three dimensions is dual to the theory of bosonic higher-spin gravity in AdS_4 provided we identify the coupling constant in the higher spin theory as $G \sim 1/N$ (this is because the dynamical fields in the dual CFT are N-component vector fields rather than $N \times N$ matrices). Being free, this theory contains a tower of higher-spin $O(N)$ singlet conserved currents

$$J_{(\mu_1 \cdots \mu_s)} = \phi^a \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^a + \dots . \tag{1.6}$$

The singlet condition picks those conserved currents which have even spins only[1]. For $s = 0$ we have an $O(N)$ singlet scalar operator of dimension 1. For other even values of s we get $O(N)$ singlet conserved currents of dimension $s + 1$. Since we are truncating the spectrum of $O(N)$ vector model to the singlet sector, it is meaningful to distinguish between "single-trace" and "multi-trace" operators. In the context of vector-like theories, "single-trace" means there is *single* sum over the $O(N)$ indices. These are the analogue of single-trace operators in matrix field theories. $O(N)$ invariant operators containing more than two scalar fields are the analogue of the multi-trace operators. For example an operator of the form $(\phi^a \partial^{s_1} \phi^a)(\phi^b \partial^{s_2} \phi^b)$ should be viewed as a double-trace operator.

In this non-supersymmetric AdS/CFT duality, the spectrum of $O(N)$ singlet single-trace operators (1.6) is expected to be in one-to-one correspondence with the spectrum of single particle states in AdS_4. These are massless higher-spin gauge fields (with even spins) in AdS_4 and a scalar of mass squared $m^2 = -2$. The scalar[2] is dual to the singlet $J = \phi^a \phi^a$ and the higher spin gauge field is dual to the higher spin conserved currents. The mass of the scalar happens to fall in the double quantization range $-9/2 \leq m^2 < -5/2$ where both the leading and the sub-leading mode are normalizable and we can choose to set either one of them to zero thereby resulting in regular/irregular quantization. As we will see in section 1.3.2, for a scalar of $m^2 = -2$ in AdS_4, regular quantization leads to an operator of dimension two whereas irregular quantization leads to an operator of dimension one. Hence, for the free $O(N)$ vector model, the dual gravitational theory must contain a scalar of $m^2 = -2$ and be quantized with irregular boundary condition. The massless higher-spin gauge fields and the scalar (of the above mass) matches the spectrum of minimal bosonic Vasiliev higher spin theory in AdS_4. The correlation functions of the singlet operators can be obtained from the bulk action in AdS_4 via the usual AdS/CFT prescription where we identify the boundary value of the fields with source of the dual operators [15]. This duality has also been tested at the one-loop level by calculating and comparing the one-loop sphere-partition function on the two sides [16–19].

The duality between free vector model and HS gauge theory can be generalized to the case of interacting large-N vector models [14]. The free $O(N)$ theory when deformed by a double-trace operator $(\phi^a \phi^a)^2$, which is relevant in 3 dimensions, flows to a non-trivial IR fixed point known as the critical $O(N)$ vector model. At this fixed point, the dimension

[1] Conserved currents with odd value of the spin transforms in the anti-symmetric representation of $O(N)$. The spin-1 current which corresponds to the global $O(N)$ symmetry is in the adjoint representation and will form the centre of discussion in appendix D of this thesis.

[2] A scalar of squared mass $m^2 = -2$ in AdS_4 corresponds to a conformally coupled scalar and can therefore be viewed as a massless field in AdS_4.

of the operator $\phi^a\phi^a$ changes from one to two plus $\mathcal{O}(1/N)$ corrections. This can be seen by studying the two-point function of $\phi^a\phi^a$ in the deep IR (the analysis is given in detail in section 4.1.1 of chapter 4). The gravity dual of the double-trace deformation is well known [20–22]. The dual of the $O(N)$ singlet $\phi^a\phi^a$ is a scalar field of same mass as before but with regular boundary condition (instead of irregular boundary condition as in the free $O(N)$ model). The critical $O(N)$ vector model is dual to the *same* Vasiliev theory, but with a different choice of boundary condition on the bulk scalar field. The higher spin currents (apart from $s = 2$ that corresponds to the energy-momentum tensor) are still present in the spectrum but they are no longer conserved. They are weakly broken $\partial \cdot J_s \propto \frac{1}{\sqrt{N}}$. Consequently the anomalous dimensions start at order $1/N$, which means that the dual HS fields remain massless at the tree level, and receive masses through loop corrections. It is somewhat remarkable to note that the same Lagrangian of HS fields give the holographic description of two different fixed points; one free, while the other interacting, the only difference being in the boundary condition of the scalar field.

Another example of a vector-like theory is the free $U(N)$ vector model of complex N-component scalars. The $U(N)$ singlet spectrum now contains HS currents of all integer spins. This precisely matches the spectrum of the complete bosonic Vasiliev theory commonly referred to as Type A Vasiliev theory. Other examples include the theory of N free massless Dirac fermions (transforming under a global $U(N)$ symmetry) whose $U(N)$ singlet sector is conjectured to be dual to Type B Vasiliev theory and the theory of N free complex p-form gauge fields in $d = 2p + 2$ (having a $U(N)$ global symmetry) whose $U(N)$ singlet sector is conjectured to be dual to "Type C" Vasiliev theory.

1.3 CFTs on flat space vs. QFTs in AdS Space

In this section, we provide some essential details of CFTs and free QFTs in AdS space, formulating the AdS/CFT dictionary along the way.

1.3.1 Conformal Field Theories

Conformal Field theories are special class of Quantum field theories that has conformal symmetry. The Conformal group of $\mathbb{R}^{1,d-1}$ consists of the usual Poincaré transformations, $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$, accompanied by scaling, $x^\mu \to \lambda x^\mu$ and special conformal transformations,

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \; . \tag{1.7}$$

which can be thought of as an inversion, $x^\mu \to x^\mu/x^2$, followed by a translation by b^μ followed by another inversion. Here, $\Lambda^\mu{}_\nu$ is the general Lorentz transformation matrix, containing $d(d-1)/2$ generators, a^μ are the independent translations along the d coordinates, λ is scaling parameter, and b^μ is a fixed d-vector. In total there are $(d+1)(d+2)/2$

generators[3] that constitute the $SO(d,2)$ algebra. We denote these generators by $M_{\mu\nu}$ for Lorentz transformations, P_μ for translations, D for dilations and K_μ for special conformal transformations. These generators obey the following algebra

$$\begin{aligned}
[D, P_\mu] &= -iP_\mu \\
[D, K_\mu] &= -i(-K_\mu) \\
[M_{\mu\nu}, P_\rho] &= -i\left(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu\right) \\
[M_{\mu\nu}, K_\rho] &= -i\left(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu\right) \\
[P_\mu, K_\nu] &= -i\left(2\eta_{\mu\nu}D + 2M_{\mu\nu}\right) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\left(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right)
\end{aligned} \tag{1.8}$$

with the remaining commutators being zero. The first two commutation relations imply that the generators P_μ and K_μ have eigenvalues $+1$ and -1 respectively, under dilatations. $M_{\mu\nu}$ has eigenvalue zero. The third and the fourth relations imply that both P_μ and K_μ transform as a vector under Lorentz transformations. The last relations is the usual commutator of Lorentz generators. This algebra is isomorphic to the algebra of $SO(d,2)$, and can be recast in the standard form of the $SO(d,2)$ algebra (with signature $(-,-,+,...,+)$) with generators S_{ab}, $(a,b = -1,0,1,...,d)$ by defining

$$S_{\mu\nu} = M_{\mu\nu}\ , \qquad S_{\mu d} = \frac{1}{2}(P_\mu - K_\mu)\ , \qquad S_{\mu,-1} = \frac{1}{2}(P_\mu + K_\mu)\ , \qquad S_{-1,d} = D\ . \tag{1.9}$$

where $\mu,\nu = 0,1,...,d-1$. All operators in a CFT lie in some unitary representation of the conformal group. Since we are interested in unitary representations, we take all the generators of the conformal algebra to be Hermitian (note that this is compatible with the commutation relations). Representations of the conformal group are labelled by representations under the maximal compact subgroup $SO(d) \times SO(2)$. Therefore, states are classified by

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0)\ , \tag{1.10}$$

and representations under $SO(d)$ generated by $M_{\mu\nu}$ which are labelled by $\left[\frac{d}{2}\right]$ eigenvalues of the Cartan generators of $SO(d)$. Translations P_μ raise Δ by one unit and special conformal transformations K_μ lower Δ by one unit. Unitary representations are obtained by studying bounds on the eigenvalues Δ of operators under dilatations D. Every unitary representation has an operator of minimal conformal dimension Δ. Therefore, each representation of the conformal group must have some operator of lowest dimension, which must then be annihilated by K_μ. Such operators are called primary operators. Starting from primary operators one can act with P_μ in all possible ways and construct a tower of descendants of increasing dimension and varying spins. For scalar primaries $\mathcal{O}$, the general structure of

[3] $d = 2$ is a special case where the conformal group is larger, and is in fact infinite dimensional.

the corresponding descendants take the following form

$$\partial_{\mu_1}...\partial_{\mu_l}\Box^n\mathcal{O}\ . \tag{1.11}$$

Such a descendant has spin l and conformal dimension

$$\Delta_{\mathcal{O}}+l+2n\ .$$

A primary operator $\mathcal{O}$ along with all its descendants forms a representation of $SO(d,2)$ and is called a conformal family or a conformal multiplet.

In passing, we remark that it is often useful to study CFTs in Euclidean space $\mathbb{R}^d$. In this case the conformal group is $SO(d+1,1)$. The generators of $SO(d+1,1)$ can be obtained from the generators of $SO(d,2)$ by the following relations [23]

$$M'_{pq}=S_{pq}\ ,\quad D'=iS_{-1,0}\ ,\quad P'_p=S_{p,-1}+iS_{p,0}\ ,\quad K'_p=S_{p,-1}-iS_{p,0}\ , \tag{1.12}$$

where $p,q=1,2,...,d$. The commutation relations of these operators are those of the generators of the Euclidean conformal group $SO(d+1,1)$ (with signature $(-,+,+,...,+)$). However, unlike the Lorentzian case, these operators are not Hermitian

$$M'^{\dagger}_{pq}=M'_{pq}\ ,\quad D'^{\dagger}=-D'\ ,\quad P'^{\dagger}_p=K'_p\ ,\quad K'^{\dagger}_p=P'_p\ . \tag{1.13}$$

An equivalent way to study unitary representation of $SO(d,2)$ is to study the Wick rotated Euclideanized theory that has symmetry group $SO(d+1,1)$ implemented in a non-unitary fashion (1.13).

Unitarity restricts the conformal dimensions of primary operators from below. For scalars the unitarity bound is $(d-2)/2$ and is saturated by a free field satisfying the Klein-Gordon equation $\Box\mathcal{O}=0$. For operators that are spinors the bound is $(d-1)/2$ and is saturated by operators satisfying the free Dirac equation $\gamma^\mu\partial_\mu\mathcal{O}=0$. For spin-one operators the bound it $d-1$. This bound is not saturated by free gauge fields. The point is that a gauge field by itself is not a gauge invariant operator and so it does not act on positively normed Hilbert space. The operator that saturates this bound is instead a conserved current $J_\mu:\ \ \partial^\mu J_\mu=0$ and indeed this can be seen by noticing that the conserved charge $Q=\int d^{d-1}x\ J_0$ is dimensionless which implies the dimension of J_μ is $d-1$.

Conformal symmetry puts strong constraints on correlation functions of primary operators. For scalars operators the form is

$$\langle\mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(0)\rangle=\frac{\delta_{\Delta_1\Delta_2}}{|x|^{\Delta_1+\Delta_2}} \tag{1.14}$$

which implies that if $\Delta_1\neq\Delta_2$ the two-point function vanishes. Note that from scale invariance the two-point function would have been $\frac{1}{|x|^{\Delta_1+\Delta_2}}$. It is because of the full conformal symmetry that we have a Kronecker delta in (1.14). Three-point functions are also fixed upto an overall constant

$$\langle \mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(y)\mathcal{O}_{\Delta_3}(z)\rangle = \frac{C_{\Delta_1\Delta_2\Delta_3}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|y-z|^{\Delta_2+\Delta_3-\Delta_1}|z-x|^{\Delta_1+\Delta_3-\Delta_2}} \ . \tag{1.15}$$

The constant $C_{\Delta_1\Delta_2\Delta_3}$ is an operator product expansion (OPE) coefficient. It appears as the coefficient of $\mathcal{O}_{\Delta_i}$ in the OPE of $\mathcal{O}_{\Delta_j}$ and $\mathcal{O}_{\Delta_k}$ ($i, j, k = 1, 2, 3$ but all distinct). Four-point functions are determined upto a function of the conformally invariant cross-ratios built out of the four insertion points. A CFT can be specified by a set of operators labelled by their $SO(d, 2)$ quantum numbers (conformal dimension and spins) and the OPE coefficients. This constitutes what is often called the CFT data.

1.3.2 Quantum Field Theories in AdS

We now look at QFTs in Anti-de Sitter (AdS) space. AdS is one of the maximally symmetric space[4] which have a Riemann tensor proportional to the metric, i.e. $R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. The AdS space is a solution of Einstein's field equations with a negative cosmological constant term in Einstein's theory of General Relativity, described by the Einstein-Hilbert action,

$$S_{\rm EH} = \frac{1}{2\kappa_{d+1}^2}\int d^{d+1}x\ \sqrt{|g|}\,(R - 2\Lambda)\ , \tag{1.16}$$

where $g = \det g_{\mu\nu}$ is the determinant of the metric, R is the Ricci scalar, κ_{d+1} is related to the $(d+1)$-dimensional Newton's gravitational constant G as $\kappa_{d+1}^2 = 8\pi G$, and Λ is the cosmological constant. The field equations that follow from (1.16) are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0\ . \tag{1.17}$$

A solution to these equations is the $(d+1)$-dimensional Lorentzian AdS space, which may be defined as the following hyperboloid embedded in $\mathbb{R}^{d,2}$

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_d^2 = -L^2\ . \tag{1.18}$$

On this space

$$R_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu}\ , \qquad R = g^{\mu\nu}R_{\mu\nu} = -\frac{d(d+1)}{L^2}\ , \qquad \Lambda = -\frac{d(d-1)}{2L^2}\ , \tag{1.19}$$

where L is the radius of curvature of AdS_{d+1}. The isometries of AdS_{d+1} are given by transformations in the embedding space that leave (1.18) invariant. The group of transfor-

[4]Maximally symmetric spaces of dimension d are defined as spaces that have maximal number $d(d+1)/2$ of Killing vector fields which generates the symmetry algebra of the space.

mations that preserves the quadratic form (1.18) is $SO(d,2)$. Global coordinates on AdS_{d+1} are defined by making the $SO(d)\times\mathbb{R}$ part manifest, which corresponds to rotations and time translations in AdS space. Hence, we set

$$X_1^2+...+X_d^2=r^2\ ,\qquad X_{-1}^2+X_0^2=L^2+r^2\ . \tag{1.20}$$

The first of (1.20) can be solved by

$$X_i=r\Omega_i\ ,\qquad \text{with } i=1,...,d \quad \text{and} \quad \sum_{i=1}^{d}\Omega_i^2=1\ , \tag{1.21}$$

while the second equation in (1.20) can solved by taking

$$X_{-1}=\sqrt{L^2+r^2}\ \cos(\tau/L)\ ,\qquad X_0=\sqrt{L^2+r^2}\ \sin(\tau/L)\ . \tag{1.22}$$

The coordinates (τ,r,Ω_i) are called global coordinates and the metric in these coordinates is given by

$$ds^2=-\left(1+\frac{r^2}{L^2}\right)d\tau^2+\frac{1}{\left(1+\frac{r^2}{L^2}\right)}dr^2+r^2d\Omega_{d-1}\ . \tag{1.23}$$

The range of r is $[0,\infty)$ and τ by construction is a periodic coordinate with period $2\pi L$. However, we can discard the perodicity (since there are propagating fields in AdS that do not respect this periodicity) and unroll the τ coordinate to take values in $\mathbb{R}$. The metric (1.23) with $\tau\in\mathbb{R}$ is called the universal covering of AdS_{d+1}. ∂_τ is a time-like Killing vector and corresponds to the generator $S_{0,-1}=\frac{1}{2}\left(P_0+K_0\right)$.

Another way to parametrize AdS space is through Poincaré coordinates. Here we make $SO(d-1,1)$ symmetry manifest. Therefore, we set

$$\begin{aligned}
X_{-1}&=\frac{z}{2}\left(1+\frac{1}{z^2}\left(L^2+\vec{x}^2-t^2\right)\right)\ ,\\
X_0&=\frac{Lt}{z}\ ,\\
X_i&=\frac{Lx^i}{z}\ ,\qquad \text{with } i=1,2,...,d-1\ ,\\
X_d&=\frac{z}{2}\left(1+\frac{1}{z^2}\left(-L^2+\vec{x}^2-t^2\right)\right)\ ,
\end{aligned} \tag{1.24}$$

where $z>0$, $\vec{x}\in\mathbb{R}^{d-1}$ and the boundary is located at $z=0$. Unlike global coordinates, the coordinates (z,t,x^i) cover only half of AdS space as can be seen from $X_{-1}-X_d=L^2/z>0$. These coordinates are referred to as Poincaré coordinates or Poincaré patch. The metric in this coordinate system reads

$$ds^2=\frac{L^2}{z^2}\left(dz^2-dt^2+d\vec{x}^2\right)\ . \tag{1.25}$$

where now the $SO(d-1,1) \times SO(1,1)$ symmetry is manifest.

It is useful to also mention the properties of Euclidean AdS_{d+1} space (also simply known as hyperbolic space $\mathbb{H}^{d+1}$). It is given by the hyperboloid

$$-X_{-1}^2 + X_0^2 + X_1^2 + ... + X_d^2 = -L^2 \ , \tag{1.26}$$

embedded in $\mathbb{R}^{d+1,1}$. The isometry group of Euclidean AdS_{d+1} is $SO(d+1,1)$ as is manifest from the above quadratic form. We solve this constraint by setting

$$X_1^2 + ... + X_d^2 = r^2 \ , \quad X_{-1}^2 - X_0^2 = L^2 + r^2 \ , \tag{1.27}$$

which can be solved by

$$\begin{aligned} &X_i = r\Omega_i \ , \quad \text{with } i = 1, ..., d \quad \text{and} \quad \sum_{i=1}^{d} \Omega_i^2 = 1 \ , \\ &X_{-1} = \sqrt{L^2 + r^2} \ \cosh(\tau/L) \ , \quad X_0 = \sqrt{L^2 + r^2} \ \sinh(\tau/L) \ . \end{aligned} \tag{1.28}$$

The metric in global coordinates becomes

$$ds^2 = \left(1 + \frac{r^2}{L^2}\right) d\tau^2 + \frac{1}{\left(1 + \frac{r^2}{L^2}\right)} dr^2 + r^2 d\Omega_{d-1} \ , \tag{1.29}$$

which has a Killing vector field ∂_τ that corresponds to the generator $S_{0,-1} = iD'$. Similarly the metric in the Poincaré coordinates (now Euclidean coordinates) becomes

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dt^2 + d\vec{x}^2\right) \ . \tag{1.30}$$

The Lorentzian τ and t coordinates become the Wick rotated Euclidean time coordinates.

To study the global structure of AdS space we use global coordinates. AdS is a space of infinite volume. The $r = \infty$ point is at infinite proper distance from any point in the interior since the integral $\int_{r_0}^{\infty} \frac{dr}{\sqrt{1 + r^2/L^2}}$ diverges. However, radially outgoing light rays can reach $r = \infty$ in finite global time τ since

$$ds^2 = 0 \Rightarrow \left(1 + \frac{r^2}{L^2}\right) d\tau^2 = \frac{1}{\left(1 + \frac{r^2}{L^2}\right)} dr^2 \Rightarrow \int d\tau = \int^{\infty} \frac{dr}{1 + r^2/L^2} = \text{finite} \ . \tag{1.31}$$

Therefore, it is appropriate to say that $r = \infty$ is an asymptotic boundary in the sense that we need to specify boundary conditions at $r = \infty$ in order to have well defined τ evolution. It is often convenient to think of AdS space as a box, in which energy levels (eigenvalues of the Hamiltonian conjugate to the global time τ) have a discrete spacing. In the next section, we demonstrate this by proving the discreteness in the frequency spectrum for the case of a free scalar in AdS.

The ensuing sections also clarify some basic aspects of the AdS/CFT dictionary.

Free scalar in AdS_{d+1}

Consider a free scalar field ϕ of squared mass $m^2 = -d^2/4 + \nu^2$, $(\nu > 0)$ propagating in AdS_{d+1} of unit radius (1.23). It satisfies the Klein-Gordon equation

$$\left(\Box - m^2\right)\phi = 0 \ . \tag{1.32}$$

We look for solutions of the form $\phi(\tau, r, \Omega_i) = e^{-i\omega\tau} Y_l(\Omega) f(r)$ where Y_l is a spherical harmonic on S^{d-1} satisfying $\Delta_{\Omega_{d-1}} Y_l = -l(l+d-2)Y_l$, with $\Delta_{\Omega_{d-1}}$ being the scalar Laplacian on a $(d-1)$-dimensional sphere. Near the boundary (large r) the KG equation (1.32) gets leading contribution from the following term

$$\frac{1}{r^{d-1}}\partial_r\left(r^{d+1}\partial_r\phi\right) - m^2\phi = 0 \ , \tag{1.33}$$

Assuming ϕ behaves like $\phi \sim r^\lambda$ near the AdS boundary one gets

$$\lambda(\lambda+d) - m^2 = 0 \ , \qquad \Rightarrow \lambda_\pm = -\frac{d}{2} \pm \frac{1}{2}\sqrt{d^2+4m^2} \ . \tag{1.34}$$

For stable solutions $\lambda_\pm$ should be real. This imposes a lower bound on the mass of the scalar $m^2 > -\frac{d^2}{4}$ which is famously known in the literature as the Breitenlohner-Freedman stability bound. From (1.34) we see that a general solution behaves as $\phi \sim r^{\lambda_+}$ near the boundary. However, in general this behaviour is not normalizable. To see this let us expand the notion of normalizability. The quantization of the scalar field proceeds by expanding out the field operator in a complete set of normalizable solutions of the wave equation $\hat{\phi}(x) = \sum_n \left(\phi_n(x)\hat{a}_n + \phi_n^*(x)\hat{a}_n^\dagger\right)$. Given a spacelike surface Σ and a unit normal vector n^μ to Σ, the modes ϕ_n satisfy the Klein-Gordon norm

$$\int_\Sigma d^dx\sqrt{h}\; n^\mu\left(\phi_n^*\partial_\mu\phi_m - \phi_m\partial_\mu\phi_n^*\right) = \delta_{m,n} \ . \tag{1.35}$$

Then, by imposing $[a_n, a_m^\dagger] = \delta_{m,n}$, we satisfy the equal time canonical commutation relation $[\hat{\phi}(\vec{x},t), \dot{\hat{\phi}}(\vec{y},t)] = \delta^{(d)}(\vec{x}-\vec{y})$. For AdS_{d+1} in global coordinates we have

$$\sqrt{h} = \frac{r^d}{\sqrt{1+r^2}}\sqrt{g_{\Omega_{d-1}}} \sim r^{d-2} \ , \qquad n^t = \frac{1}{\sqrt{g_{tt}}} \sim \frac{1}{r} \ . \tag{1.36}$$

Therefore, the integrand in (1.35) behaves as $r^{d-3}r^{2\lambda}$. Hence, normalizability requires that $d - 3 + 2\lambda < -1$ or $\pm\sqrt{d^2+4m^2} < 2$. From this we see that λ_- is always a normalizable mode. However, we find that λ_+ is normalizable only in the range

$$-\frac{d^2}{4} \leq m^2 < 1 - \frac{d^2}{4} \ . \tag{1.37}$$

This range is often referred to as the double quantization window where both the leading and sub-leading modes are normalizable.

To proceed with obtaining the frequency spectrum we first make a coordinate change $r = \tan\rho$. This brings the metric into the following form

$$ds^2 = \sec^2\rho\left(-dt^2 + dz^2\right) + \tan^2\rho\ d\Omega_{d-1} \ . \tag{1.38}$$

The boundary is then located at $\rho = \pi/2$ and the origin at $\rho = 0$. Changing further the radial coordinate $u = \cos\rho$, such that the boundary is at $u = 0$ and the origin is at $u = 1$, we get that for generic frequency ω the solution at small u will behave as

$$\phi \sim u^{d/2+\nu}\phi^{(+)}(\omega, l) + u^{d/2-\nu}\phi^{(-)}(\omega, l) \ . \tag{1.39}$$

The solution of (1.32) which is regular at the origin is given by

$$f = u^{d/2-\nu}\left(1-u^2\right)^{l/2}\ {}_2F_1\left(c - \frac{\omega}{2}, c + \frac{\omega}{2}; \frac{d}{2} + l; 1 - u^2\right) \ . \tag{1.40}$$

Here $c = (d + 2l - 2\nu)/4$. The solutions which are irregular at the origin $u = 1$ can be obtained from (1.40) by the replacement $l \to 2 - d - l$. From (1.40) we find that $\phi^{(\pm)}(\omega, l) = e^{-i\omega\tau}Y_l(\Omega)\, f_\pm$, where

$$f_\pm = \mp\frac{\pi\csc(\pi\nu)\Gamma\left(\frac{d}{2} + l\right)}{\Gamma(1 \pm \nu)\Gamma\left(\frac{1}{4}(d + 2l \mp 2\nu - 2\omega)\right)\Gamma\left(\frac{1}{4}(d + 2l \mp 2\nu + 2\omega)\right)} \ . \tag{1.41}$$

Outside the double quantization window, normalizability requires that we set $\phi^{(-)}(\omega, l) = 0$ or $f_- = 0$. Since Γ-function has poles at negative integers $-n$ including 0 we get that $f_- = 0$ implies the following condition on ω

$$\omega = \Delta + l + 2n \ , \tag{1.42}$$

provided we identify

$$\Delta = \frac{d}{2} + \nu \ , \qquad \text{or conversely} \ \ m^2 = \Delta(\Delta - d) \ . \tag{1.43}$$

The formula (1.42) tells us that when we quantize a scalar field we obtain single particle states of energy $\omega_{n,l} = \Delta + l + 2n$ which matches the spectrum of descendents of a scalar primary operator of conformal dimension Δ. Hence, provided we identify $\frac{d}{2} + \nu$ as the conformal dimension of the primary operator $\mathcal{O}$, we learn that a quantized scalar field in global AdS space along with all its excited states is dual to the conformal multiplet of $\mathcal{O}$. In particular, descendants are dual to excited states of scalar fields in global AdS.

Figure 1.1 is a plot of the mass/dimension relation (1.43), that gives a pictorial summary of the relation between various quantizations of the bulk scalar and operator dimensions.

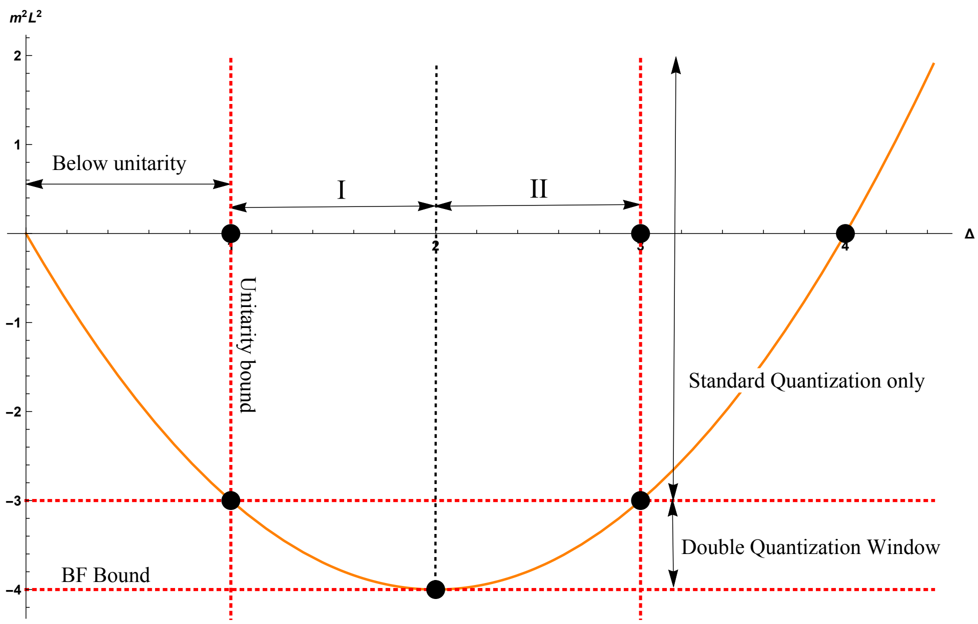

Figure 1.1: Scalar mass/dimension relation in $d = 4$. The double quantization window is the range $-4 \leq m^2L^2 < -3$. Operators in region I have dimension $\Delta_- = 2 - \nu$ (where $\nu > 0$) and dual scalars are always quantized with alternative quantization. Operators in region II have dimension $\Delta_+ = 2 + \nu$ and the dual scalars are always quantized with standard quantization.

For calculation of correlation functions it is often convenient to make use of the Poincaré coordinate system. Consider the Euclidean AdS in the Poincaré coordinates (1.30). To solve the KG equation on this metric it is convenient to Fourier transform the scalar field in x coordinates

$$\phi(z, x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} \phi(z, k) \ . \tag{1.44}$$

After Fourier transforming, the Klein-Gordon equation becomes

$$z^2 \frac{d^2\chi(z, k)}{dz^2} + z\frac{d\chi(z, k)}{dz} - \left(k^2z^2 + \Delta(\Delta - d) + \frac{d^2}{4}\right)\chi(z, k) = 0, \tag{1.45}$$

where we have defined $\phi(z, k) = z^{d/2}\chi(z, k)$. This is nothing but the modified Bessel's equation which has the general solution

$$\phi(z, k) = z^{d/2} \left(K_\nu(kz)C_1(k) + I_\nu(kz)C_2(k)\right) \ , \qquad \nu = \Delta - \frac{d}{2} \ . \tag{1.46}$$

In the interior of AdS, $I_\nu(kz)$ blows up so we pick-up only the regular part and set $C_2 = 0$. The near boundary expansion of this solution is

$$\phi(z,k) \underset{z\to 0}{\sim} (\phi^-(k) z^{\frac{d}{2}-\nu} + \phi^+(k) z^{\frac{d}{2}+\nu})(1 + O(z^2)) \ , \tag{1.47}$$

where the leading mode ϕ^- and the subleading mode ϕ^+ are related as

$$\phi^+(k) = \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2}\right)^{2\nu} \phi^-(k) \ . \tag{1.48}$$

This is an important relation in calculating correlation functions of dual operators. The mode ϕ^- is identified as the source for the dual operator. The action of the scalar when evaluated on the solution takes the form

$$\begin{aligned} S &= \frac{1}{2}\int dz\ d^d x \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \Delta(\Delta - d)\phi^2\right) \ , \\ &= -\frac{1}{2}\int d^d x \sqrt{g} g^{zz} \phi \partial_z \phi|_{z=\epsilon} \ . \end{aligned} \tag{1.49}$$

Plugging (1.47) in the Fourier transform of the above equation we get following non-zero terms

$$S = -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{\epsilon^{2\nu}}\left(\frac{d}{2} - \nu\right)\phi^-(k)\phi^-(-k) + d\ \phi^+(k)\phi^-(-k)\right] \ , \tag{1.50}$$

where the first term is divergent in the limit $z \to 0$. Two-point correlation functions can now be computed through the AdS/CFT prescription (1.3). We simply differentiate this action twice with respect to ϕ^-. Since the divergent part is local in the source ϕ^- it will not contribute to the correlation function at separated points. The non-local information is contained only in the finite term. The two-point function in momentum space then reads

$$\begin{aligned} \langle \mathcal{O}(k)\mathcal{O}(-k)\rangle &= d\ \frac{\delta\phi^+(k)}{\delta\phi^-(k)} \ , \\ &= d\ \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{k}{2}\right)^{2\nu} \delta^d(0) \ . \end{aligned} \tag{1.51}$$

Upon Fourier transformation, the position space two-point function becomes

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{d}{\pi^{d/2}} \frac{\Gamma(\frac{d}{2}+\nu)}{\Gamma(\nu)} \frac{1}{|x|^{d+2\nu}} \ . \tag{1.52}$$

This result is almost correct. The normalization factor is slightly inconsistent with Ward identities due to global symmetries (if any) [24]. This inconsistency can be cured by doing a proper holographic renormalization [25–27] which also systematically removes the divergent piece in (1.50) instead of brutally ignoring it. There exists a local covariant counterterm that cancels the divergence in (1.50) and gives rise to correctly normalized two-point functions.

The counterterm action is

$$S_{\rm ct} = \frac{1}{2} \int d^d x \sqrt{|\gamma|} \left((d-\Delta)\phi^2 + ... \right) \ . \tag{1.53}$$

Consequently, the correctly normalized two-point function is obtained to be

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{2\nu}{\pi^{d/2}} \frac{\Gamma(\frac{d}{2}+\nu)}{\Gamma(\nu)} \frac{1}{|x|^{d+2\nu}} \ . \tag{1.54}$$

Free spin-$1/2$ field in AdS_{d+1}

A free spin-1/2 field obeys the Dirac equation in any spacetime

$$\left(\not{\nabla} - |m| \right) \Psi(z,x) = 0 \ . \tag{1.55}$$

The information about the metric is in the spin connection inside the covariant derivative. On the Poincaré patch of AdS, the Dirac operator $\not{\nabla}$ takes the following form

$$\not{\nabla} = \gamma^{d+1} z \partial_z - i z \not{k} - \frac{d}{2} \gamma^{d+1} \tag{1.56}$$

where we have performed Fourier transformation along the x directions and the vector k^μ is the Fourier transform of $i\partial_\mu$. γ^{d+1} is the gamma matrix with component along the radial direction z. We can use γ^{d+1} to project the Dirac equation (1.55) into the following two coupled equations:

$$\left(z\partial_z - |m| - \frac{d}{2} \right) \Psi^+ - i \not{k} z \Psi^- = 0 \tag{1.57}$$

$$\left(z\partial_z + |m| - \frac{d}{2} \right) \Psi^- + i \not{k} z \Psi^+ = 0, \tag{1.58}$$

where $\Psi = \Psi^+ + \Psi^-$ with the property that $\gamma^{d+1}\Psi^+ = \Psi^+$ and $\gamma^{d+1}\Psi^- = -\Psi^-$. This coupled set of equations can be solved exactly and the solution is:

$$\Psi^+(z,k) = -i \not{k} z^{(d+1)/2} K_{\nu-1}(kz) C(k) \tag{1.59}$$

$$\Psi^-(z,k) = k z^{(d+1)/2} K_{\nu}(kz) C(k), \tag{1.60}$$

where $C(k)$ is an arbitrary smooth spinor that satisfies $\gamma^{d+1} C = -C$. The full exact solution is therefore

$$\Psi(z,k) = z^{(d+1)/2} \Big(k K_\nu(kz) - i \not{k} K_{\nu-1}(kz) \Big) C(k) \ , \qquad \nu = |m| + \frac{1}{2}, \tag{1.61}$$

and is regular in the interior of AdS. Since $K_\nu(z) \sim z^{-\nu}$ for small z, we see that the leading mode behaves as $z^{\frac{d+1}{2}-\nu}$ and that Ψ^- corresponds to the source of the dual operator $\mathcal{O}^+$

which satisfies $\gamma^{d+1}\mathcal{O}^+ = \mathcal{O}^+$. The scaling behaviour of the leading mode indicates that Ψ^- at the boundary corresponds to a spinor of conformal dimension $\frac{d+1}{2} - \nu$ which in turn implies that the conformal dimension of $\mathcal{O}^+$ is $d - \frac{d+1}{2} + \nu$. Hence, the mass/dimension relation of spin 1/2 representations read

$$\Delta = \frac{d-1}{2} + \nu \ . \tag{1.62}$$

If the mass term in (1.55) flips, sign then $\Psi^+ \to \Psi^-$ and $\Psi^- \to -\Psi^+$. In this case Ψ^+ is the leading mode and becomes the source of the dual operator $\mathcal{O}^-$ which satisfies $\gamma^{d+1}\mathcal{O}^- = -\mathcal{O}^-$. Therefore, the sign of the mass term in (1.55) is important because it differentiates two spin-1/2 operators of the same conformal dimension by its γ^{d+1} eigenvalue. For $d = 4$, γ^{d+1} is the usual chirality matrix.

Free vector field in AdS_{d+1}

A free vector field of mass m satisfies the following equation

$$\frac{1}{\sqrt{-g}}\partial_\nu \left(\sqrt{-g}F^{\nu\mu}\right) - m^2 A^\mu = 0 \ , \tag{1.63}$$

which (after gauge fixing $A_z = 0$) can be rewritten as follows

$$z^{d+1}\partial_z \left(z^{-d+3}\partial_z A_j\right) + z^4 \partial_i \left(\partial_i A_j - \partial_j A_i\right) - m^2 z^2 A_j = 0 \ . \tag{1.64}$$

Decompose the gauge field into transverse and longitudinal parts $A_i = A_i^t + \partial_i A^l$, $\partial_i A_i^t = 0$ and construct the following projectors

$$A^l = \frac{\partial_i}{\Box} A_i, \qquad A_i^t = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Box}\right) A_j \ . \tag{1.65}$$

Projecting onto the longitudinal and the transverse part we get

$$z^{d+1}\partial_z \left(z^{-d+3}\partial_z A^l\right) - m^2 z^2 A^l = 0 \ , \tag{1.66}$$

$$z^{d+1}\partial_z \left(z^{-d+3}\partial_z A_j^t\right) - z^4 k^2 A_j^t - m^2 z^2 A_j^t = 0 \ , \tag{1.67}$$

where we have performed Fourier transformation over the i coordinates. These equations admit the following exact solution:

$$A^l(z,k) = z^{\frac{1}{2}(d-2-\delta)} C_1(k) + z^{\frac{1}{2}(d-2+\delta)} C_2(k) \ , \tag{1.68}$$

$$A_i^t(z,k) = z^{\frac{d}{2}-1} \left(K_\nu(kz) C_{3i}(k) + I_\nu(kz) C_{4i}(k)\right) \ , \qquad \nu = \frac{\delta}{2}. \tag{1.69}$$

where $C_{1,2,3,4}$ are four integration constants that in general depend on k and

$$\delta = \sqrt{(d-2)^2 + 4m^2} \ . \tag{1.70}$$

Imposing regularity in the interior of AdS sets $C_{2,4} = 0$. The asymptotic behaviour of the transverse part is:

$$A_i^t(z,k) \sim z^{\frac{1}{2}(d-2-\delta)} \widetilde{C}_{3i}(k) + z^{\frac{1}{2}(d-2+\delta)} \widetilde{C}_{4i}(k) \ , \tag{1.71}$$

which is the same as that of the longitudinal part. The leading mode behaves as $z^{\frac{1}{2}(d-2-\delta)}$. This indicates that A_μ at the boundary corresponds to a 1-form of conformal dimension $1 + \frac{1}{2}(d-2-\delta) = \frac{d-\delta}{2}$ and couples to an operator of conformal dimension $d - \frac{d-\delta}{2}$. Thus the mass/dimension relation for spin-1 representations read

$$\Delta = \frac{1}{2}\left(d + \sqrt{(d-2)^2 + 4m^2}\right) \ . \tag{1.72}$$

For $m = 0$ we have $\Delta = d - 1$ which is the conformal dimension of a conserved current. Hence, a massless gauge field in the bulk corresponds to a conserved current in the boundary CFT. If $m \neq 0$ then we see that the current operator gets an anomalous dimension which is proportional to the mass of the dual gauge field

$$\Delta = (d-1) + \frac{m^2}{d-2} + \mathcal{O}(m^4) \ , \qquad \Rightarrow \ \gamma_J = \frac{m^2}{d-2} + \mathcal{O}(m^4) \ . \tag{1.73}$$

Therefore, in holographic CFTs, anomalous dimensions of spin-1 operators can be equivalently calculated from the mass of the dual gauge field. Anomalous global symmetries in the boundary CFTs correspond to Higgsing of the corresponding gauge symmetry in the gravity dual.

1.4 An invitation to the thesis

In this chapter, we have portrayed some universal features of AdS/CFT to show how this correspondence captures the equivalence between a quantum field theory and its dual theory of quantum gravity. We have described very few selected topics in this subject that will be relevant for the rest of the thesis. It is often the case that there exists regions in parameter space where AdS/CFT correspondence takes the form of a strong / weak duality. In these cases AdS/CFT turns out to be quite useful because it gives access to strongly coupled regimes of field theories which otherwise cannot be accessed by usual tools of perturbation theory.

The rest of the thesis is devoted to studying deformations of the AdS/CFT correspondence in various set-up. In chapter 2, we construct the gravity dual of a strongly coupled $\mathcal{N} = 1$ superconformal field theory (SCFT) perturbed by a supersymmetric relevant deformation. The theory can then either be in a supersymmetry preserving or a supersymmetry breaking vacuum. If it is in the supersymmetry breaking vacuum, then we expect a Goldstino in the spectrum which can be identified by the presence of a massless pole in the two-point function of the supercurrent. This massless pole manifests as contact terms in supersymmetry Ward identities. In chapter 2, we derive such Ward identities holographically and present the fingerprints of the Goldstino.

In chapter 3, we will study supersymmetry breaking aspects of a particularly well known example where the strong / weak nature of AdS/CFT duality is realized to the extreme. The field theory model is a non-conformal extension of the well known $\mathcal{N}=1$ superconformal Klebanov-Witten (KW) theory [10]. This theory has $SU(N)\times SU(N)$ gauge group and $SU(2)\times SU(2)\times U(1)_R$ global symmetry with bi-fundamental matter carrying non-trivial representations of the global symmetry. The theory also has an $SU(2)\times SU(2)$ invariant *quartic* superpotential. The presence of the quartic superpotential is an indication of the fact that the KW CFT is inherently strongly coupled (since the bi-fundamental matter fields appearing in the superpotential have acquired order one anomalous dimension). There is no region in the space of the three couplings (the two gauge coupling and the superpotential coupling) where the KW CFT admits a perturbative description. The conformal vacuum of this theory has a gravitational description in terms of type IIB string theory on $AdS_5\times T^{1,1}$, where $T^{1,1}$ is a Sasaki-Einstein manifold with topology $S^2\times S^3$.

The non-conformal extension that we previously alluded to is an $\mathcal{N}=1$ gauge theory with gauge group $SU(N)\times SU(N+M)$, known as the Klebanov-Strassler (KS) theory [28]. This theory has a huge moduli space of supersymmetric vacua [29]. For $N=kM$, the moduli space consists of both mesonic and baryonic branches. Instead, for $N=kM+p$, with $p<M$, the baryonic branch is lifted and it was conjectured, by Kachru, Pearson and Verlinde (KPV) in [30], that for $p\ll M$ there exists a metastable vacuum on the would-be baryonic branch. Since the meta-stable state breaks supersymmetry spontaneously, a Goldstino is expected in the spectrum. As the KS theory is always at strong coupling, a quantitative investigation for the presence of the Goldstino can be made *only* through AdS/CFT or more generally gauge/gravity techniques. In chapter 3, we use the techniques developed in chapter 2 and prove the existence of the Goldstino in the KPV vacuum by deriving supersymmetry Ward identities holographically.

In chapter 4, we shift gears and address multi-trace deformations in the context of AdS/CFT correspondence particularly focusing on the double-trace case. We consider relevant deformations of a CFT by double-trace operators and study the IR fixed point of the resulting RG flow both from the point of view of field theory and holography. Later in the chapter, we use double-trace deformations to address the phenomenon of *multiplet recombination* where two distinct conformal multiplets in the undeformed CFT merge and become a single conformal multiplet in the deformed CFT. It is well known that for Higher Spin operators ($s\geq 1$) that undergo multiplet recombination, the holographic counterpart of this phenomenon is a generalized Higgs mechanism in AdS. In chapter 4, using double trace deformations of large-N CFT, we will demonstrate the holographic counterpart of the $s = 0$ case, i.e., multiplet recombination for scalar operators. We show that also in this case a Higgs-like mechanism is at work, albeit of unconventional type, which exists only in AdS (as it should, if AdS/CFT correspondence were correct).

In the final chapter, we study exactly marginal deformations of a large class of $\mathcal{N} = 1$ SCFTs. In particular, we focus on exactly marginal deformations that break some of the global symmetries of the SCFT, more specifically on deformations that break the global symmetry group G down to its Cartan subgroup H. Such deformations are dubbed β-deformations, and exist for a large class of SCFTs, including $\mathcal{N} = 4$ SYM as well as the KW theory. The gravity dual of β-deformed theories corresponds to a warped product of AdS_5 and a particular H-preserving deformation of corresponding compact spaces (i.e., S^5 in the case of $\mathcal{N} = 4$ SYM and $T^{1,1}$ in the case of KW theory). An interesting question to ask is: What are the conformal dimensions of the spin-one flavour currents which are no longer conserved in the deformed CFT? One could make an attempt to obtain directly the masses of the dual gauge fields in AdS_5. However, computing the mass spectrum of the corresponding supergravity theories on such warp product spaces is highly non-trivial. We will provide a quantitative answer to the above question in a rather simple manner by using AdS/CFT and constraints from conformal symmetry.

Five appendices contain necessary material to reproduce the main formulae and results presented in the main text.

Chapter 2

Spontaneous SUSY breaking in AdS/CFT

In this chapter, we present a concise treatment of supersymmetry breaking in AdS/CFT correspondence by means of a concrete bottom-up toy example. For this purpose we take the model of [31] and rederive their main results from a slightly different route. This will serve as a warmup for the ensuing chapter where we consider in detail the KPV vacuum that appears in 4d, $\mathcal{N}=1$ cascading Klebanov-Strassler gauge theory.

2.1 Field theory description

Let us consider a strongly coupled 4-dimensional $\mathcal{N}=1$ supersymmetric quantum field theory described by an RG flow from a UV fixed point deformed by a relevant operator. Schematically, the action can be written as

$$S = S_{\rm SCFT} + \lambda \int d^4x \; d^2\theta \; \mathcal{O} + h.c. \; , \tag{2.1}$$

where S_{SCFT} is an $\mathcal{N}=1$ superconformal field theory and $\mathcal{O}$ is chiral operator of dimension in the range $1 \leq \Delta_{\mathcal{O}} < 3$. Its operator components are specified as $\mathcal{O} = \mathcal{O}_S + \sqrt{2}\,\theta\,\mathcal{O}_\psi + \theta^2 \mathcal{O}_F$. Strictly speaking, deforming an SCFT with just one operator might not be consistent. This is because at the quantum level there could be mixing with other relevant operators. We make the simplifying assumption that there is no such mixing and consider a consistent deformation with just one relevant operator.

In any supersymmetric quantum field theory there always exists the supercurrent multiplet which contains the energy-momentum tensor $T_{\mu\nu}$ and the supersymmetry current $S_{\mu\alpha}$. For non-conformal theories, this multiplet is described by two superfields $(\mathcal{J}_\mu, X)$ that satisfy the following on-shell relation

$$-2\bar{D}^{\dot{\alpha}} \sigma^{\mu}_{\alpha\dot{\alpha}} \mathcal{J}_\mu = D_\alpha X \; , \tag{2.2}$$

with $\mathcal{J}_\mu$ a real superfield and X some chiral superfield. On solving the constraint (2.2) we get

$$\begin{aligned}\mathcal{J}_\mu &= j_\mu + \theta^\alpha \left(S_{\mu\alpha} + \frac{1}{3}(\sigma_\mu \bar{\sigma}^\rho S_\rho)_\alpha \right) + \bar{\theta}_{\dot{\alpha}} \left(\bar{S}_\mu^{\dot{\alpha}} + \frac{1}{3}\epsilon^{\dot{\alpha}\dot{\beta}} (\bar{S}_\rho \bar{\sigma}^\rho \sigma_\mu)_{\dot{\beta}} \right) \\ &\quad + (\theta\sigma^\nu\bar{\theta}) \left(2T_{\nu\mu} - \frac{2}{3}\eta_{\nu\mu} T - \frac{1}{4}\epsilon_{\nu\mu\rho\sigma}\partial^{[\rho} j^{\sigma]} \right) + \frac{i}{2}\theta^2 \partial_\mu \bar{x} - \frac{i}{2}\bar{\theta}^2 \partial_\mu x + \cdots \\ \partial^\mu T_{\mu\nu} &= \partial^\mu S_{\mu\alpha} = 0 \ , \qquad T_{\mu\nu} = T_{\nu\mu} \ . \end{aligned} \tag{2.3}$$

The ellipses denote terms with more θs. In the last line we have the conservation law for the energy-momentum tensor and the supercurrent plus the fact that the energy momentum tensor is symmetric. In terms of these fields the chiral superfield X is given by

$$\begin{aligned} X &= x(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \ , \\ \psi_\alpha &= \frac{\sqrt{2}}{3}\sigma^\mu_{\alpha\dot{\alpha}} \bar{S}^{\dot{\alpha}}_\mu \ , \qquad F = \frac{2}{3}T + i\partial_\mu j^\mu \ , \end{aligned} \tag{2.4}$$

In eq. (2.4) we see that X contains the trace of the energy-momentum tensor and the σ-trace of the supercurrent. Therefore, in a superconformal field theory X vanishes identically. In a non-conformal theory as in (2.1), the superfield X is non-zero and is sourced by the operator $\mathcal{O}$ as

$$X = \frac{4}{3}(3 - \Delta_{\mathcal{O}})\lambda\, \mathcal{O} \ . \tag{2.5}$$

For definiteness we take $\Delta_{\mathcal{O}} = 2$ and hence, $X = \frac{4}{3}\,\lambda\,\mathcal{O}$. For the discussion that follows, this choice is not particularly important.

The theory can be in a phase where supersymmetry is either preserved or broken, depending on whether or not the operator $\mathcal{O}$ acquires a non-vanishing VEV for its F-term component. This can be seen from the structure of the one- and two-point functions of operators belonging to the FZ multiplet. Indeed, regardless of the vacua, the supersymmetry algebra implies the following Ward identities

$$\langle \partial^\mu S_{\mu\alpha}(x)\, \bar{S}_{\nu\dot{\beta}}(0)\rangle = -\delta^4(x)\langle \delta_\alpha \bar{S}_{\nu\dot{\beta}}\rangle = -2\sigma^\mu{}_{\alpha\dot{\beta}}\,\langle T_{\mu\nu}\rangle\,\delta^4(x) \tag{2.6a}$$

$$\langle \partial^\mu S_{\mu\alpha}(x)\, \mathcal{O}_{\psi\,\beta}(0)\rangle = -\delta^4(x)\langle \delta_\alpha \mathcal{O}_{\psi\,\beta}\rangle = \sqrt{2}\,\langle \mathcal{O}_F\rangle\,\varepsilon_{\alpha\beta}\,\delta^4(x) \ , \tag{2.6b}$$

The two Ward identities above imply the presence of the following structures in the two-point functions of the supercurrent with itself and with the fermionic operator $\mathcal{O}_\psi$

$$\langle S_{\mu\alpha}(x)\, \bar{S}_{\nu\dot{\beta}}(0)\rangle = -\frac{i}{4\pi^2}\langle T\rangle\,(\sigma_\mu\bar{\sigma}^\rho\sigma_\nu)_{\alpha\dot{\beta}}\frac{x_\rho}{x^4} + \ldots \tag{2.7a}$$

$$\langle S_{\mu\alpha}(x)\, \mathcal{O}_{\psi\,\beta}(0)\rangle = -\frac{i}{2\pi^2}\sqrt{2}\,\langle \mathcal{O}_F\rangle\,\varepsilon_{\alpha\beta}\frac{x_\mu}{x^4} + \ldots \ , \tag{2.7b}$$

where $\langle T \rangle = \eta^{\mu\nu}\langle T_{\mu\nu}\rangle$. On Fourier transformation, the expressions above give rise to the massless pole associated to the goldstino, which is the lowest energy excitation in both S_μ and $\mathcal{O}_\psi$. Indeed, in the deep IR, one can write $S_\mu = \sigma_\mu \bar{G}$, where G is the goldstino field. Plugging this relation in (2.7a), one recovers, up to an overall normalization, the goldstino propagator.

Furthermore from Eq. (2.5) we have the following identities for the traces

$$\langle {\sigma^\mu}_{\alpha\dot{\alpha}}\, \bar{S}^{\dot{\alpha}}_\mu \rangle = 2\sqrt{2}\,\lambda\,\langle \mathcal{O}_{\psi_\alpha} \rangle \ , \tag{2.8a}$$

$$\eta_{\mu\nu}\langle T^{\mu\nu}\rangle = 2\lambda \ \mathrm{Re}\,\langle \mathcal{O}_F \rangle \ , \tag{2.8b}$$

where the angular brackets indicate that these equalities are supposed to hold inside correlation functions. The one-point functions of the fermionic operators are to be seen as being computed at generic non-vanishing sources.

In the next section we will demonstrate how to reproduce these identities from the dual holographic description.

2.2 Holographic description

To capture the essential features of the field theory model (2.1) holographically, we consider the theory of 5-dimensional $\mathcal{N}=2$ gauged supergravity coupled to one hypermultiplet. The operator $\mathcal{O}$ in (2.1) is dual to a hypermultiplet. Turning on a relevant deformation in the quantum field theory corresponds to a non-trivial profile for the corresponding scalar in the hypermultiplet. The backreaction of the scalar deforms the AdS space to a domain-wall geometry which is the holographic dual of the RG flow in the QFT. Correlation functions in the QFT can then be obtained by studying linearized perturbations of supergravity fields on the domain-wall geometry. Since the gravity theory we will use to describe holographic SUSY breaking is 5-dimensional $\mathcal{N}=2$ gauged supergravity, we now briefly review the aspects of the this theory that we will need and defer the detailed account to the literature [32, 33].

2.2.1 $\mathcal{N}=2$, 5-dimensional gauged supergravity model

Let us first state the field content of various 5D $\mathcal{N}=2$ supermultiplets. The field content is determined by irreducible representations of the Poincaré superalgebra [34]. Besides the translations P_a and Lorentz transformations M_{ab}, the Poincaré superalgebra consists of the supercharges $Q^{\alpha i}$, the generators of the automorphism group T^A and the central elements Z^{ij}. The automorphism group and the form of anticommutators $\{Q, Q\}$ depend on the spinor type of Q^i. In five dimensions the supercharges are symplectic Majorana spinors Q^i $(i = 1, 2, ..., \mathcal{N})$ where the number $\mathcal{N}$ is even. These spinors satisfy the relation $Q^i = \Omega^{ij}(Q^j)^c$, where $(Q^j)^c$ is the charge conjugated spinor and Ω^{ij} is an anti-symmetric matrix. The $2\mathcal{N}$ symplectic Majorana spinors are equivalent to $\mathcal{N}$ Dirac spinors. Since the minimal spinor in 5 dimensions is a Dirac spinor we have that the smallest value of $\mathcal{N}$ is 2. In this case the automorphism group is $USp(2) = SU(2)$ and the anti-commutator of the supercharges reads

$$\{Q^i, Q^{jT}\} = \gamma^a C P_a \Omega^{ij} + C Z^{ij} \ , \tag{2.9}$$

where C is a charge conjugation matrix. Supergravity fields belong to irreducible representation of this super Poincaré algebra. The representations of interest for constructing a theory of supergravity theory will contain the metric, gauge fields, scalars and their fermionic partners. These representations (also called multiplets) are

$$\begin{aligned} \text{gravity multiplet:} \quad & \left(e^a_\mu, \psi^i_{\mu\alpha}, A_\mu\right) \\ \text{vector multiplet:} \quad & \left(A_\mu, \lambda^i, \phi\right) \\ \text{hyper multiplet:} \quad & \left(\zeta^A, q^X\right) \end{aligned}$$

The gravity multiplet consists of the graviton (with 5 on-shell real degrees of freedom), two $SU(2)_R$ symplectic Majorana gravitino (with 8 on-shell real degrees of freedom) and one vector (called the graviphoton having 3 on-shell real degrees of freedom). Here, a is a flat spacetime index, $i = 1, 2$ is an $SU(2)_R$ fundamental index and α is a spinor index. Each vector multiplet consists of vector field, a doublet of $SU(2)_R$ spin 1/2 symplectic Majorana fermion, called the gaugino (having 4 on-shell real degrees of freedom) and one real scalar field. Each hypermultiplet contains an $SU(2)$ symplectic Majorana fermion, called the hyperino (with $A = 1, 2$ being an $SU(2)$ index) and four real scalars ($X = 1, ..., 4$). It is worth mentioning that the $SU(2)$ used to implement the symplectic Majorana condition in this case is different from the $SU(2)_R$ and is inherent to the hypermultiplet representation. If there are n copies of hypermultiplets, then the $SU(2)^n$ can get enhanced upto $USp(2n)$.

In the following we consider 5-dimensional $\mathcal{N} = 2$ gauged supergravity (studied in [33]) coupled to one hypermultiplet and no vector multiplets. To simplify notations we write the gravitino and the hyperino as one Dirac fermion instead of two symplectic Majorana fermions. The scalars of the hypermultiplet (which can be thought of two complex scalars ρ and ϕ) parametrize the homogenous space $\mathcal{M} = SU(2, 1)/(U(1) \times SU(2))$. The graviphoton gauges a proper $U(1)$ subgroup of the isometry group of $\mathcal{M}$. In a theory of gauged supergravity a choice of the gauging is dictated by the choice for the dimension of the dual operator $\mathcal{O}$.

Since $\mathcal{O}$ is a dimension 2 superfield, the bottom component of $\mathcal{O}$ is a scalar operator of dimension $\Delta(\mathcal{O}_S) = 2$ and the F-term component is a operator of dimension $\Delta(\mathcal{O}_F) = 3$. From the mass/dimension relation $m^2 = \Delta(\Delta - 4)$ this translates to masses of the scalar fields $m^2_\rho = -4$, $m^2_\phi = -3$, where ρ and ϕ are scalar fields dual to the operators $\mathcal{O}_S$ and $\mathcal{O}_F$ respectively. A source for ρ corresponds to a non-supersymmetric deformation in the dual quantum field theory (since supersymmetric deformations are either F-term or D-term deformations and $\mathcal{O}_S$ is neither). Therefore, we consider the case where the source of ρ is switched off. A-priory one can allow a vev for ρ because it is susy preserving. However, without affecting the main aspects of the holographic model we will switch off the vev of ρ and hence, the scalar ρ completely.

A convenient metric on the coset $\mathcal{M}$ is

$$\mathrm{d}q^X \mathrm{d}q_X = 2\left(\mathrm{d}\phi^2 + \mathrm{d}\rho^2\right) + 2\sinh^2(\rho)\left(\mathrm{d}\phi^2 + \mathrm{d}\alpha^2\right) + \frac{1}{2}\left(\mathrm{e}^{-2\phi}\cosh^2(\rho)\mathrm{d}\sigma + 2\sinh^2(\rho)\mathrm{d}\alpha\right)^2 .$$

The full isometry group of the metric above is SU(2,1) of which we choose to gauge a U(1) subgroup [33]. The gauging procedure, besides promoting the partial derivatives to their gauge-covariant counterparts, introduces a potential for the scalar fields as well as interaction terms for the fermions. Since we are interested in a single scalar background we will consistently truncate $(\rho, \sigma, \alpha) = 0$. The action of the truncated gauged supergravity (ignoring terms containing the graviphoton and four-fermion interactions) is completely fixed by supersymmetry and is shown below

$$\begin{aligned} S_{5\mathrm{D}} = \int \mathrm{d}^5x\sqrt{-G}\Big(&\frac{1}{2}R - \partial_M\phi\,\partial^M\phi - \mathcal{V}(\phi) - \bar{\Psi}_M\Gamma^{MNP}D_N\Psi_P - 2\,\bar{\zeta}\Gamma^M D_M\zeta \\ &+ 4\bar{\zeta}\Gamma^M\mathcal{F}_-\Psi_M + 4\bar{\Psi}_M\mathcal{F}_+\Gamma^M\zeta + m(\phi)\,\bar{\Psi}_M\Gamma^{MN}\Psi_N - 2\,\mathcal{M}(\phi)\,\bar{\zeta}\zeta\Big) , \end{aligned} \tag{2.10}$$

where the scalar potential is given by

$$\mathcal{V}(\phi) = \frac{1}{12}\left(10 - \cosh(2\phi)\right)^2 - \frac{51}{4} , \tag{2.11}$$

and can be obtained from a superpotential

$$W(\phi) = \frac{1}{6}\left(5 + \cosh(2\phi)\right) \tag{2.12}$$

by the equation

$$\mathcal{V} = \frac{9}{4}\,\partial_\phi W \partial_\phi W - 6W^2 . \tag{2.13}$$

The other quantities which appear in (2.10) are given by

$$m(\phi) = \frac{3}{2}W(\phi) = \frac{1}{4}\left(5 + \cosh(2\phi)\right) , \tag{2.14a}$$

$$\mathcal{M}(\phi) = \frac{9}{2}W(\phi) - 5 = -\frac{1}{4}\left(5 - 3\cosh(2\phi)\right) , \tag{2.14b}$$

$$\mathcal{F}_\pm(\phi) = -\frac{1}{4}\left(2\mathcal{N} \mp \mathrm{i}\partial\!\!\!/\phi\right) , \tag{2.14c}$$

where,

$$\mathcal{N}(\phi) = -\frac{3}{4}\mathrm{i}\partial_\phi W(\phi) = -\frac{\mathrm{i}}{4}\sinh(2\phi) . \tag{2.15}$$

The supersymmetry transformation of the fermions are

$$\delta_\epsilon \Psi_M = D_M \epsilon + \frac{1}{3} m(\phi) \Gamma_M \epsilon \ , \tag{2.16a}$$

$$\delta_\epsilon \zeta = 2\mathcal{F}_-(\phi)\epsilon \ , \tag{2.16b}$$

while those of the bosons are

$$\delta_\epsilon e^A_{\ M} = \frac{1}{2}\bar{\epsilon}\Gamma^A \Psi_M \ + \text{h.c.}, \tag{2.16c}$$

$$\delta_\epsilon \phi = -\frac{i}{2}\bar{\epsilon}\zeta + \text{h.c.} \tag{2.16d}$$

2.2.2 Supersymmetric and non-supersymmetric solutions

We look for both supersymmetric and non-supersymmetric flat domain-wall solutions of the model (2.10). We take the following ansatz for the domain-wall geometry supported by a non-trivial profile for the scalar

$$ds^2 = \frac{1}{z^2}\left(dz^2 + F(z)dx^2\right) \ , \tag{2.17a}$$

$$\phi \equiv \phi(z) \ . \tag{2.17b}$$

with the condition that near the timelike boundary, the limit $z \to 0$, which corresponds to the UV limit in the dual field theory, the warp factor $F \to 1$ and the scalar $\phi \to 0$ or at most a constant and the geometry becomes that of pure AdS_5. Such bulk geometries which are only asymptotically AdS_5 near the timelike boundary are dual to either relevant deformations of the CFT or to non-conformal vacua.

The equation of motion for ϕ is

$$z^2\phi'' - \left(3 - 2\frac{zF'}{F}\right) z\phi' = \frac{1}{2}\partial_\phi \mathcal{V}(\phi) \, , \tag{2.18}$$

whereas the Einstein's equations give

$$6\left(1 - \frac{zF'}{2F}\right)^2 = z^2\phi'^2 - \mathcal{V}(\phi) \ , \qquad \left(1 - \frac{zF'}{2F}\right)' = \frac{2}{3} z\phi'^2 \ . \tag{2.19}$$

The prime denotes differentiation with respect to the z coordinate. The two Einstein's equations are redundant and can shown to be equivalent to a single first-order non-linear differential equation in $F(z)$. A generic solution to these equations will be non-supersymmetric. To find supersymmetric solutions we should analyze the following BPS system of first order differential equations which results from the SUSY variation of the fermions (2.16a, 2.16b)

$$1-\frac{zF'}{2F}=W(\phi)\ , \tag{2.20a}$$

$$z\phi'=\frac{3}{2}\partial_\phi W(\phi)\,. \tag{2.20b}$$

Pure AdS_5 where $F=1$ and $\phi=0$ is a solution to these first order equations. Hence, this solution is supersymmetric and corresponds to the conformal vacuum of S_{SCFT} appearing in (2.1). Around this solution the mass of the scalar ϕ is

$$m_\phi^2=\frac{1}{2}\partial_\phi^2\mathcal{V}(0)=-3\,, \tag{2.21}$$

By mass/dimension relation ϕ is dual to a scalar operator of dimension 3. Therefore, ϕ is the correct dual of $\mathcal{O}_F$. Another supersymmetric solution is the domain-wall geometry which can be analytically obtained

$$\phi(z)=\frac{1}{2}\log\left(\frac{1+az}{1-az}\right)\ ,\qquad F(z)=\left(1-a^2z^2\right)^{1/3}\ , \tag{2.22}$$

where a is an integration constant which is identified as the source for the dual operator. Pure AdS_5 solution is recovered for $a=0$. This solution corresponds to the relevant deformation in (2.1).

Next we turn to the second order equations of motion. The system of equations cannot be solved analytically but can be integrated numerically. The general solution has two integration constants and its expression for small z is given by the expansions

$$\phi(z)=a\,z+b\,z^3+O(z^5)\ ,\qquad F(z)=1-\frac{a^2}{3}\,z^2+\frac{a^4-9ab}{18}\,z^4+O(z^6)\,. \tag{2.23}$$

This solution reduces to the supersymmetric case for $b=a^3/3$. For different value of b the solution is non-supersymmetric. Therefore, we define the offset $\mathcal{S}=\frac{a^3}{3}-b$ as the supersymmetry breaking order parameter which we use to discriminate between supersymmetric solutions ($\mathcal{S}=0$) and non-supersymmetric ones ($\mathcal{S}\neq 0$). We expect this solution to correspond to a supersymmetry breaking vacua of (2.1). In section 2.3.3 we will demonstrate that this is indeed the case by calculating the vacuum expectation value of $\mathcal{O}_F$ and showing it to be proportional to $\mathcal{S}$.

Both the supersymmetric and non-supersymmetric solutions presented here suffer from an IR singularity. These solutions are presented merely as an existence proof, and in the following we will not need to discuss their properties in any detail. The nature of the IR singularity (either good or bad [35]) does not affect our analysis of the Ward identities.

To conclude this subsection, we see that the model presented here is in fact a concrete holographic candidate dual to the QFT (2.1). The scalar ϕ is dual to a relevant operator of dimension 3 which triggers a supersymmetric RG-flow out of some given UV fixed point. The general solution (2.22) is the holographic dual of this RG-flow. The dual QFT can find itself in a supersymmetric vacuum, $\langle \mathcal{O}_F \rangle = 0$, or a non-supersymmetric one $\langle \mathcal{O}_F \rangle \neq 0$. Correspondingly, the background solution can preserve bulk supersymmetry, $\mathcal{S} = 0$, or break it, $\mathcal{S} \neq 0$. One is then led to identify $\mathcal{S}$ with the VEV of the QFT operator $\mathcal{O}_F$.

2.3 Holographic Ward Identities

Since the theory in (2.1) is an $\mathcal{N} = 1$ QFT, supersymmetric Ward identities, like (3.2), should hold in any of its vacua. In this section we provide a holographic derivation of these and few other identities (related to traces of the stress-tensor and the supercurrent).

Fields in the bulk are dual to gauge invariant operators in the QFT. In the bosonic sector we have a scalar and a metric. The scalar field is dual to a dimension 3 operator $\mathcal{O}_F$ and the metric is dual to the stress-tensor. In the fermionic sector we have a spin 1/2 fermion, the hyperino and a spin 3/2 fermion, the gravitino. The hyperino is dual to a dimension 5/2 fermionic operator $\mathcal{O}_\psi$, the superpartner of $\mathcal{O}_F$, whereas gravitino is dual to dimension 7/2 supercurrent operator, the superpartner of the stress-tensor. The mass spectrum and the mass-dimension relations are summarized in the table 2.1

$\mathcal{N} = 2$ multiplet	field fluctuations	mass	Δ	dual operators
gravity	Ψ_M	$m = \frac{3}{2}$	$\frac{7}{2}$	$S_{\mu\alpha}$
	g_{MN}	$m^2 = 0$	4	$T_{\mu\nu}$
hypermultiplet	ζ	$m = -\frac{1}{2}$	$\frac{5}{2}$	$\mathcal{O}_\psi$
	ϕ	$m^2 = -3$	3	$\mathcal{O}_F$

Table 2.1: Mass spectrum of the $\mathcal{N} = 2$ gauged supergravity model.

As a first step towards the derivation of the Ward identities, we have to define holographically the renormalized one-point functions in the presence of sources. The one-point functions of various operators are computed by functionally differentiating the renormalized on-shell action with respect to the source

$$\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\partial S_{\text{ren}}}{\partial \gamma_{\mu\nu}}\bigg|_{\phi,\Psi^+,\zeta^-}, \qquad \langle \overline{S}^{-\mu} \rangle = \frac{-2i}{\sqrt{-\gamma}} \frac{\partial S_{\text{ren}}}{\partial \Psi^+_\mu}\bigg|_{\gamma,\phi,\zeta^-},$$
$$\langle \mathcal{O}_F \rangle = \frac{1}{2\sqrt{-\gamma}} \frac{\partial S_{\text{ren}}}{\partial \phi}\bigg|_{\gamma,,\Psi^+,\zeta^-}, \qquad \langle \overline{\mathcal{O}}^+_\psi \rangle = \frac{1}{\sqrt{-\gamma}} \frac{i}{\sqrt{2}} \frac{\partial S_{\text{ren}}}{\partial \zeta^-}\bigg|_{\gamma,\phi,\Psi^+}. \tag{2.24}$$

The subscripts indicate the variables that are held fixed while performing the differentiation. S_{ren} denotes the renormalized on-shell action $S_{\text{ren}} = S_{\text{reg}} + S_{\text{ct}}$ where the regularized action

$S_{\rm reg}$ stands for the bulk on-shell action plus the Gibbons-Hawking term (together with its supersymmetric completion), and the covariant boundary counterterms $S_{\rm ct}$ contain both bosonic and fermionic terms. The counterterms, by construction, ensure that $S_{\rm ren}$ admits a smooth limit as the radial cut-off is removed. From the asymptotic behavior of the fields (see (A.8)) this implies that the renormalized one-point functions with the cut-off removed correspond to the limits

$$\begin{aligned}
&\langle T^\mu_\nu\rangle_{\rm QFT} = \lim_{z\to 0} z^{-4}\langle T^\mu_\nu\rangle \ , \quad \langle \overline{S}^{-\mu}\rangle_{\rm QFT} = \lim_{z\to 0} z^{-9/2}\langle \overline{S}^{-\mu}\rangle \ , \\
&\langle \mathcal{O}_\phi\rangle_{\rm QFT} = \lim_{z\to 0} z^{-3}\langle \mathcal{O}_\phi\rangle \ , \quad \langle \overline{\mathcal{O}}^{+}_\psi\rangle_{\rm QFT} = \lim_{z\to 0} z^{-5/2}\langle \overline{\mathcal{O}}^{+}_\psi\rangle \ .
\end{aligned} \tag{2.25}$$

Although the explicit expression for the local counterterms was constructed in [31], they are not required for the holographic derivation of the Ward identities. It is sufficient to assume that there exist local and covariant counterterms that render the on-shell action finite, while preserving the symmetries of the dual QFT, particularly, supersymmetry. Of course, explicit knowledge of the counterterms is necessary in order to evaluate the one-point functions (2.25). In the next section we will present the boundary counterterms required to evaluate the bosonic VEVs in domain wall background.

Given the holographic identification of the sources with the one-point functions, the derivation of the Ward identities proceeds exactly as in standard QFT textbooks. We turn on sources for all gauge invariant operators. Using the transformation of all the sources under the local (gauged) symmetries together with the invariance of the generating functional, leads to the Ward identities at the level of one-point functions in the presence of arbitrary sources.

To proceed with the holographic derivation of the Ward identities we need the transformation of the sources under the local symmetries. These are presented in Eqns. (A.11), (A.21), (A.22) of Appendix A.

In the bulk these symmetries correspond to infinitesimal local supersymmetry transformations and bulk diffeomorphisms generated respectively by a 4-component Dirac spinor ϵ and a 5-vector ξ^M, preserving the gauge-fixing conditions (A.3) and (A.7). The spinor ϵ has 8 real components which correspond to the 8 real supercharges of the $\mathcal{N}=2$, 5d supergravity. This can be written as $\epsilon = \epsilon^+ + \epsilon^-$. Since ϵ^+ and ϵ^- are linearly independent supersymmetry transformation parameters, the renormalized on-shell action is not only invariant under ϵ but also under ϵ^+ and ϵ^- independently. The spinor ϵ^+ generates (local) boundary supersymmetry transformations, while ϵ^- generates superWeyl transformations. Invariance under ϵ^+ and ϵ^- leads respectively to the supersymmetry Ward identities and the operator identity involving the gamma-trace of the supercurrent. Similarly, the infinitesimal bulk diffeomorphisms ξ^M preserving the gauge-fixing conditions (A.3) are parameterized by two independent parameters, a scalar $\sigma(x)$ generating boundary Weyl transformations and an infinitesimal boundary diffeomorphism $\xi^\mu_o(x)$. Invariance under these leads respectively to the trace Ward identity and the Ward identity involving the divergence of the stress tensor.

2.3.1 Supersymmetry Ward identities

The supersymmetry Ward identities are obtained by requiring the invariance of the renormalized action under the local spinor ϵ^+, $\delta_{\epsilon^+} S_{\rm ren} = 0$. However, to calculate $\delta_{\epsilon^+} S_{\rm ren}$, we need the transformation properties of the covariant sources under ϵ^+, which are given in appendix A, eq. (A.21) . Using the one-point functions (2.24) the variation of the renormalized action under ϵ^+ gives

$$\begin{aligned}
\delta_{\epsilon^+} S_{\rm ren} &= \int d^4x\sqrt{-\gamma}\left(\frac{i}{2}\langle \overline{S}^{-\mu}\rangle \delta_{\epsilon^+}\Psi^+_\mu + \frac{1}{2}\langle T^{\mu\nu}\rangle \delta_{\epsilon^+}\gamma_{\mu\nu} + 2\langle \mathcal{O}_F\rangle \delta_{\epsilon^+}\phi\right) \\
&= \int d^4x\sqrt{-\gamma}\left(-\frac{i}{2}\partial_\mu\langle \overline{S}^{-\mu}\rangle - \frac{1}{2}\langle T^{\mu\nu}\rangle \overline{\Psi}^+_\mu \Gamma_\nu + i\langle \mathcal{O}_F\rangle \bar{\zeta}^-\right)\epsilon^+ = 0\ .
\end{aligned} \tag{2.26}$$

which implies the following identity between one-point functions at non-zero sources

$$\frac{i}{2}\partial_\mu\langle \overline{S}^{-\mu}\rangle = -\frac{1}{2}\langle T^{\mu\nu}\rangle \overline{\Psi}^+_\mu \Gamma_\nu + i\langle \mathcal{O}_F\rangle \bar{\zeta}^-\ . \tag{2.27}$$

where $\Gamma_\mu = e^a_\mu \gamma_a$. The two-point functions $\langle \partial_\mu \overline{S}^{-\mu} S^{-\nu}\rangle$ and alike, can be obtained from the one-point functions as

$$\langle \partial_\mu \overline{S}^{-\mu} S^{-\nu}\rangle = -\frac{-2i}{\sqrt{-\gamma}}\frac{\delta}{i\delta \overline{\Psi}^+_\nu}\langle \partial_\mu \overline{S}^{-\mu}\rangle. \tag{2.28}$$

The extra factor of i in the denominator is because of the Lorentzian signature of the spacetime and the overall minus sign is because the functional derivative is with respect to a Grassmann variable. Using this, we can now differentiate this identity with respect to the various fermionic sources, and then put all sources to zero to obtain

$$\langle \partial_\mu \overline{S}^{-\mu}(x) S^-_\nu(0)\rangle = 2i\,\Gamma_\mu \langle T^\mu_\nu\rangle\,\delta^4(x,0)\ , \tag{2.29a}$$

$$\langle \partial_\mu \overline{S}^{-\mu}(x) \mathcal{O}^+_{\zeta_\phi}(0)\rangle = -\sqrt{2}\,\langle \mathcal{O}_\phi\rangle\,\delta^4(x,0)\ , \tag{2.29b}$$

where $\delta^4(x,y) = \delta^4(x-y)/\sqrt{-\gamma}$ is the covariant 4d Dirac delta function. The last step is to take the cut-off all the way to infinity, which can be done using the limits (2.25). All these limits can be easily evaluated using the asymptotic expansions of the fields given in appendix A. Notice that all fermionic operators here are in the Dirac representation. In order to match with the field theory expressions, it is better to convert them into Weyl notation. The conversion rules are

$$\psi^+ = \psi_\alpha, \quad \psi^- = \overline{\psi}^{\dot\alpha}, \quad \overline{\psi}^+ = \overline{\psi}_{\dot\alpha}, \quad \overline{\psi}^- = \psi^\alpha\ , \quad (\gamma_\mu)_{\alpha\dot\beta} = i\,(\sigma_\mu)_{\alpha\dot\beta}\,. \tag{2.30}$$

Adopting the above dictionary and upon sending the cut-off to infinity, we eventually get

$$\langle \partial^\mu S_{\mu\alpha}(x)\ \bar{S}_{\nu\dot{\beta}}(0)\rangle_{\rm QFT} = -2\,\sigma^\mu_{\alpha\dot{\beta}}\langle T_{\mu\nu}\rangle_{\rm QFT}\ \delta^4(x)\ , \tag{2.31a}$$

$$\langle \partial^\mu S^\alpha_\mu(x)\ \mathcal{O}_{\zeta_\phi\alpha}(0)\rangle_{\rm QFT} = -\sqrt{2}\ \langle \mathcal{O}_F\rangle_{\rm QFT}\ \delta^4(x)\ . \tag{2.31b}$$

The identity (2.31a) reproduces the supercurrent Ward identity (2.6a). The identity (2.31b) reproduces (2.6b), the supersymmetry Ward identity involving the operator $\mathcal{O}_F$.

2.3.2 Trace identities

In this section we derive the operator relation between the trace of the energy-momentum tensor and $\mathcal{O}_F$ which is the consequence of Weyl invariance. We also derive the operator identities between the trace of the energy-momentum tensor and the σ-trace of the supercurrent which is a consequence of superWeyl invariance.

Let us consider the latter first. SuperWeyl transformation is generated by ϵ^-. From the ϵ^- supersymmetry transformations in (A.22), we get the following variation in $S_{\rm ren}$

$$\delta_{\epsilon^-}S_{\rm ren} = \int d^4x\sqrt{-\gamma}\left(\frac{i}{2}\langle\overline{S}^{-\mu}\Gamma_\mu\rangle + \sqrt{2}\phi\ \langle\overline{\mathcal{O}}^+_\psi\rangle\right)\epsilon^- = 0\ , \tag{2.32}$$

which yields the following identity between the one-point functions of the gamma-trace of the supercurrent and of the operator $\mathcal{O}_\psi$ at non-vanishing sources at the cut-off surface

$$\frac{i}{2}\langle\overline{S}^{-\mu}\Gamma_\mu\rangle = -\sqrt{2}\phi\ \langle\overline{\mathcal{O}}^+_\psi\rangle\,. \tag{2.33}$$

Again, from this identity one can compute relations between various correlation functions by further differentiating. Using the limits (2.25), we can remove the cut-off to obtain the relation

$$\langle\sigma^\mu_{\alpha\dot{\beta}}\bar{S}^{\dot{\beta}}_\mu\rangle_{\rm QFT} = 2\sqrt{2}\phi_0\ \langle\mathcal{O}_{\psi\,\alpha}\rangle_{\rm QFT}\ , \tag{2.34}$$

where ϕ_0 is the coefficient of leading term in the asymptotic expansion of ϕ. Since, by AdS/CFT, the coefficient ϕ_0 is a source for the boundary operator $\mathcal{O}_F$ [26], we identify it with the coupling λ. This reproduces the first of (2.8).

Next, let us derive the Ward identity following from local shifts in the radial coordinate, which correspond to local Weyl transformations on the boundary. Using the transformation of the covariant sources given in eq. (A.11), we get

$$\begin{aligned}\delta_\sigma S_{\rm ren} &= \int d^4x\sqrt{-\gamma}\left(\frac{1}{2}\delta_\sigma\gamma_{\mu\nu}\langle T^{\mu\nu}\rangle + 2\delta_\sigma\phi\,\langle\mathcal{O}_F\rangle + \frac{i}{2}\langle\overline{S}^{-\mu}\rangle\delta_\sigma\Psi^+_\mu - \sqrt{2}i\langle\overline{\mathcal{O}}^+_\psi\rangle\delta_\sigma\zeta^-\right)\\ &= \int d^4x\sqrt{-\gamma}\left(\gamma_{\mu\nu}\langle T^{\mu\nu}\rangle - 2\phi_0\,\langle\mathcal{O}_F\rangle + \frac{i}{4}\langle\overline{S}^{-\mu}\rangle\Psi^+_{(0)\mu} + \frac{3}{\sqrt{2}}i\langle\overline{\mathcal{O}}^+_\psi\rangle\zeta^-_0\right)\sigma\end{aligned} \tag{2.35}$$

This leads to the following identity between various one-point functions at the cut-off

$$\langle T^{\mu}_{\mu}\rangle - 2\phi_0 \langle \mathcal{O}_F\rangle + \frac{i}{4}\langle \overline{S}^{-\mu}\rangle \Psi^{+}_{(0)\mu} + \frac{3}{\sqrt{2}} i \langle \overline{\mathcal{O}}^{+}_{\psi}\rangle \zeta^{-}_{0} = 0 \tag{2.36}$$

Removing the cut-off (and setting the fermionic sources to zero), we finally obtain

$$\langle T^{\mu}_{\mu}\rangle_{\mathrm{QFT}} = 2\phi_0 \langle \mathcal{O}_F\rangle_{\mathrm{QFT}} \tag{2.37}$$

This is the bosonic partner of the fermionic trace identity (2.34) and the two are in perfect agreement, numerical coefficients included. This concludes the holographic derivation of the field theory identities (2.8).

2.3.3 Bosonic one-point functions and the holographic Goldstino

The Ward identities derived in the previous section are satisfied in any vacua of the theory (2.1). In supersymmetric vacua they are satisfied trivially whereas they are satisfied non-trivially in supersymmetry breaking vacua. The vacuum expectation value of the energy-momentum tensor that appears in the r.h.s of (2.31a) is associated to the residue of the goldstino pole as discussed towards the end of section 2.1. Therefore, a non-vanishing vev for the energy-momentum tensor would imply the existence of a goldstino in the IR spectrum and therefore, spontaneous breaking of supersymmetry. This non-zero vev for the stress-tensor is supplied by a vev for $\mathcal{O}_F$ and we should find that for consistency (2.37) is obeyed. We now calculate these one-point functions.

In order to evaluate the bosonic one-point functions in (2.25) explicitly, we compute separately the contributions coming respectively from the regularized action and the counterterms in $S_{\mathrm{ren}} = S_{\mathrm{reg}} + S_{\mathrm{ct}}$. The contribution coming from S_{reg} is the radial canonical momentum associated with the corresponding field at a radial cut-off. Using the expressions for the radial canonical momenta corresponding to the fields $\gamma_{\mu\nu}$ and ϕ in the coordinate system(A.3) (see e.g. [36]), the bosonic VEVs in (2.25) become

$$\langle T^{\mu\nu}\rangle = \left(-\left(K\gamma^{\mu\nu} - K^{\mu\nu}\right) + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{\mathrm{ct}}}{\delta \gamma_{\mu\nu}} \right) , \tag{2.38}$$

$$\langle \mathcal{O}_F\rangle = -\dot{\phi} + \frac{1}{2\sqrt{-\gamma}}\frac{\delta S_{\mathrm{ct}}}{\delta \phi} , \tag{2.39}$$

where the dot represents derivatives with respect to the radial coordinate r, which is defined in eq. (A.3), while $K_{\mu\nu}$ is the extrinsic curvature of the radial slices which, for the metric (A.3), takes the form

$$K_{\mu\nu} = \frac{1}{2}\dot{\gamma}_{\mu\nu} = -\frac{1}{2} z\partial_z \left(\frac{F}{z^2}\right) \eta_{\mu\nu} , \tag{2.40}$$

which gives

$$K\gamma^{\mu\nu} - K^{\mu\nu} = 3\left(1 - \frac{zF'(z)}{2F(z)}\right) \ . \tag{2.41}$$

The contribution to the bosonic VEVs from $S_{\rm ct}$ requires to know the explicit form of the (bosonic part of the) boundary counterterms. For backgrounds enjoying 4D Poincaré invariance it turns out that the bosonic counterterms in a supersymmetric scheme [27] are simply given by the superpotential (2.12), namely

$$S_{\rm ct} = -\int d^4x\sqrt{-\gamma}\,3W \ . \tag{2.42}$$

Evaluating the limits in (2.25) using the asymptotic behavior of the fields we finally get

$$\langle T^\mu_\nu\rangle_{\rm QFT} = -\phi_0\mathcal{S}\ \delta^\mu_\nu \ , \tag{2.43}$$

$$\langle\mathcal{O}_\phi\rangle_{\rm QFT} = -2\mathcal{S} \ , \tag{2.44}$$

in agreement with the corresponding expressions in [31]. Note that the value of these expectation values satisfy the identity (2.37). Since the vacuum energy is positive, the sign of $\mathcal{S}$ is fixed from (2.43).

Few comments are in order. On the supersymmetric solution $\mathcal{S} = 0$, these VEVs vanish; i.e., the vacuum energy is zero and there is no non-trivial VEV for $\mathcal{O}_F$. The supersymmetry Ward identities are trivially satisfied, and there is no massless pole in the supercurrent two-point function (2.7a). This is perfectly consistent with supersymmetry being preserved. On the supersymmetry breaking solution $\mathcal{S} \neq 0$ and there is non-vanishing vacuum energy, which is supplied by a non-zero VEV for $\mathcal{O}_F$. From the supercurrent Ward identities (2.31), which hold non-trivially in this vacuum, we see that a goldstino mode is present in the supercurrent two-point function (2.7a). From the operator identity (2.34) it follows that the goldstino eigenstate is

$$G \sim \langle\mathcal{O}_F\rangle\sigma^\mu\bar{S}_\mu \sim \langle\mathcal{O}_F\rangle\phi_0\mathcal{O}_\psi \tag{2.45}$$

All these properties are consistent with a vacuum where supersymmetry is spontaneously broken.

Chapter 3

AdS/CFT on the Conifold and SUSY breaking

In this chapter we apply the techniques developed in chapter 2 in a more interesting setup. We study a five-dimensional consistent truncation of type IIB supergravity dimensionally reduced on $T^{1,1}$. The consistent truncation we study contains the supersymmetric Klebanov-Strassler [28] solution and its non-supersymmetric deformations. The non-supersymmetric solutions are parametrized by two integration constants φ and $\mathcal{S}$. The solution parametrized by φ is associated to the usual independent fluctuation of the dilaton and as we will show, corresponds to explicit breaking of supersymmetry. The solution parametrized by $\mathcal{S}$ is related to anti D-branes at the tip of the conifold and correspond the to dual field theory vacua where a goldstino mode is present. These findings do not depend on the IR singularity of the solution, nor on its resolution. As such, they constitute an independent check for the existence of supersymmetry breaking vacua in the conifold cascading gauge theory. The analysis presented in this chapter relies on the holographic derivation of the Ward identities introduced in chapter 2. It is somewhat remarkable (modulo few caveats) that an identical derivation goes through irrespective of the fact that the Klebanov-Strassler gauge theory has no UV fixed point.

3.1 Introduction

Since the early days of the AdS/CFT correspondence [6, 37, 38], the new tools that have become available to understand field theory dynamics in the strong coupling regime have opened-up new promising avenues to study supersymmetric theories where supersymmetry is broken dynamically.

There is by now rather strong evidence that a large class of supersymmetric field theories admitting supersymmetry-breaking vacua can be constructed in string theory. We are thinking in particular of quiver gauge theories obtained by placing stacks of D-branes at Calabi-Yau singularities. This can be interesting in view of phenomenological applications within string compactification scenarios, but can also be instrumental within the gauge/gravity duality. Indeed, in the decoupling limit, one can have a way to describe, at least in principle, strongly coupled supersymmetry breaking vacua by means of dual gravitational backgrounds. This is promising, but in general more work is needed to have precise control on these vacua, understand their stability properties, dynamics and spectrum.

A concrete proposal to construct supersymmetry-breaking vacua in string theory was put forward time ago in [30] (from now on KPV) for the $\mathcal{N}=1$ theory obtained by placing N regular and M fractional D3-branes at a conifold singularity. This is a quiver gauge theory with $SU(N+M)\times SU(N)$ gauge group, four bi-fundamental fields A_i, B_j $(i,j=1,2)$ and a quartic superpotential $W=\lambda\,\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\,(A_iB_kA_jB_l)$ [28,39,40] (henceforth KS model). The proposal, based on the idea of adding anti D-branes at the tip of the deformed conifold, suggests that besides supersymmetric vacua, like the one described by the KS solution [28], the dual field theory admits also supersymmetry-breaking, metastable vacua. If correct, this is likely not to be a specific phenomenon of the KS model, but rather a generic fact in D-brane/string constructions, see for instance [41,42]. As a consequence, an understanding of the non-supersymmetric dynamics of the conifold theory has a more general relevance and it is not just interesting *per se*.

In the gauge/gravity duality framework, a vacuum of the QFT is described by a (four-dimensional Poincaré invariant) five-dimensional solution of the dual gravitational system. Solutions sharing the same asymptotics correspond, in general, to different vacua of the same QFT. A supergravity solution describing, asymptotically, the KPV vacuum was obtained in [43]. This solution, as well as the original one found in [28], asymptotes to the Klebanov-Tseytlin (KT) solution [40] near the boundary. The latter, in fact, furnishes a UV-regulator for any gravitational background describing a vacuum of the KS theory.

In a QFT, whenever a global symmetry is spontaneously broken, a massless particle appears in the spectrum. In the case of supersymmetry, this is a fermionic mode, the goldstino. Hence, a natural question to try to answer is whether the supergravity mode dual to the goldstino field is present in the non-supersymmetric background of [43].

When one deals with a supergravity solution which breaks supersymmetry, two obvious questions arise:

1. Is the solution (meta)stable, gravitationally?

2. Is the supergravity mode dual to the goldstino present?

A positive answer to the first question guarantees that the solution is describing holographically an actual QFT vacuum. The second ensures that in such a vacuum supersymmetry is broken spontaneously.

From a QFT perspective, it is obvious that these two questions can be answered independently. As we mentioned in the previous chapter, the goldstino is the lowest energy excitation in the supercurrent operator $S_{\mu\alpha}$, and as such it appears as a massless pole in the two-point function

$$\langle S_{\mu\alpha}\, \bar{S}_{\nu\dot{\beta}} \rangle \ . \tag{3.1}$$

This correlator has in general a very complicated structure, which depends on the vacuum that one is considering. However, in order to display the goldstino pole, one does not need to compute (3.1) fully. The information is encoded just in the term implied by the supersymmetry Ward identity

$$\langle \partial^\mu S_{\mu\alpha}(x)\, \bar{S}_{\nu\dot{\beta}}(0) \rangle = -2\sigma^\mu{}_{\alpha\dot{\beta}}\, \langle T_{\mu\nu} \rangle\, \delta^4(x) \ , \tag{3.2}$$

which is a (quasi-local) contact term. (Upon integration, this identity relates the vacuum energy $E \sim \eta^{\mu\nu} T_{\mu\nu}$ to the residue of the goldstino pole in the two-point function (3.1) [44].) Ward identities hold in any vacuum of a QFT, and depend on UV data only. On the contrary, vacuum stability is an IR property.

For theories with a gravity dual, this disentanglement should emerge from a holographic analysis, too. In [31] a rather general class of holographic supersymmetric RG-flows was considered, Ward identities as (3.2) were derived holographically, and it was shown that, indeed, they hold regardless of the detailed structure of the bulk solution in the deep interior, the presence of IR singularities and their possible resolution mechanism. Similar results were obtained for bosonic global symmetries in [45].

Whenever one has sufficient control on the QFT, this result can be seen (just) as a consistency check of the AdS/CFT correspondence. But it may become instrumental when one has to deal with field theories for which a satisfactory understanding of the dynamics and vacuum structure is lacking. The KS theory falls in this class, at least as far as supersymmetry-breaking vacua are concerned. There has been a lively discussion in the last few years, initiated in [46], regarding the stability properties of the dual supersymmetry-breaking backgrounds and the mechanism to resolve the IR singularity.[1] In this work we do not offer any new insight on this issue. What we do, instead, will be to apply the analysis of [31] to the KS model, and try to give a definite answer to the second question. The answer will be affirmative. In particular, we derive via holography the supercurrent Ward identities (3.2) for the KS cascading theory, and, by computing explicitly eq. (3.2) both in supersymmetric and supersymmetry-breaking vacua, we find the goldstino pole whenever expected. Our results confirm the possibility that spontaneous supersymmetry-breaking vacua may exist in the KS model, specifically that a goldstino mode is indeed present in the asymptotic solution of [43]. As an interesting outcome of our analysis, we show that some recently-found non-supersymmetric supergravity solutions [50], which have an

[1] See the citation list for [46] for a complete account of the many contributions since then. Suggestive results in favor of (meta)stability of the KPV vacuum were recently obtained in [47] working within an effective field theory approach. For a discussion regarding the possibility to cloak the singularity beyond an event horizon, instead, which according to the criterion of [35] would make it acceptable, see [48, 49].

asymptotic compatible with the KS theory, do not accommodate a goldstino mode. Hence they correspond to explicit, rather than spontaneous, supersymmetry breaking.

Holographic renormalization for cascading theories is known to be trickier than for asymptotically AdS (AAdS) backgrounds, and we will clarify a couple of issues which are instrumental to holographically renormalize the theory in these cases. In particular, we will argue that to treat the log-divergent structure of cascading backgrounds properly, it is appropriate to define the renormalized action in terms of induced fields instead of the sources, define the renormalized correlators as functions of induced fields at the cut-off [51], and take the cut-off to infinity only at the very end of the calculation [24].

We start in section 3.2 by presenting the relevant five-dimensional supergravity Lagrangian, and derive supersymmetric and non-supersymmetric solutions with correct KS asymptotics. The latter are a two-parameter family. Although these are known results, we re-derive them from a consistently truncated 5d supergravity, as a preliminary step for the subsequent analysis. In section 3.3, which contains the main results of this chapter, we derive holographically all the supersymmetry Ward identities that we need, showing that they hold independently of the vacuum one considers. The derivation, which relies on the existence of local covariant counterterms that renormalize the on-shell action, as well as on a renormalization scheme respecting boundary diffeomorphisms and supersymmetry transformations, is general enough to account for any Ward and operator identities one expects to hold.[2] Finally, in section 3.4 we evaluate explicitly the supersymmetry Ward identities for those vacua described by the solutions derived in section 3.2. The requirement of non-vanishing vacuum energy selects only a one-dimensional subspace within the space of supersymmetry-breaking solutions, in agreement with the analysis of [43], where evidence was also given that this corresponds to the set of (asymptotic) solutions generated by antiD-branes at the tip of the conifold. The field/operator map will offer a simple explanation of these results from the dual field theory perspective, including the absence of a goldstino mode for the complementary set of solutions. In section 3.5 we present our conclusions and outlook. Several appendices contain a number of technical details that we omitted from the chapter.

3.2 Cascading gauge theories from $\mathcal{N} = 2$, 5D gauged supergravity

The 5d $\mathcal{N} = 2$ supergravity that we need is obtained by reducing 10d type IIB supergravity on $T^{1,1}$, the conifold basis. The supergravity theory (presented in appendix B) that one should consider in order to analyze the full KS cascading theory (namely, to describe its

[2] The renormalization scheme, though, will generically break Weyl and superWeyl invariance, leading to a trace and a supertrace anomaly, respectively.

complete set of vacua) is rather complicated; almost intractable, in fact. However, there are a number of simplifications that our analysis allows.

First, we will focus on an $SU(2)\times SU(2)$-invariant truncation (the dimensional reduction was performed in [52] and [53]; we use the notations of [52]). This truncation cannot capture all possible vacua of the KS theory, but it is general enough to admit the original KS solution as one of its supersymmetric solutions. This solution describes the most symmetric point in the baryonic branch of the $SU(N+M)\times SU(N)$ KS model, with $N = kM$ and k an integer number. The same bulk Lagrangian admits also supersymmetry-breaking solutions, some of which should describe, according to the KPV construction, a metastable vacuum of the $SU(N+M)\times SU(N)$ cascading theory with vacuum energy $E \sim p$, where now $N = kM - p$, and $p \ll M$ (in the KPV vacuum p corresponds to the number of antiD-branes; from a ten-dimensional viewpoint, keeping the $SU(2) \times SU(2)$ symmetry amounts to smearing the antiD-branes over the compact space). In fact, we will work with a simplified ansatz, which preserves an extra $U(1)$ symmetry [54] and which can just accommodate KT-like solutions. This simplification further reduces the number of active fields and, in particular, it excludes the mode related to the conifold deformation parameter.

A second simplification occurs at the level of the solutions themselves. As already stressed, in order to prove the presence of the goldstino, one does not need to consider the full solution but just its asymptotic expansion up to the order where the supersymmetry-breaking deformation appears. This simplifies the analysis considerably, and allows one to consider the backgrounds only to order z^4, z being the holographic coordinate. This may sound inconsistent, at first sight. Indeed, as noticed in [43, 55], the KS and KT solutions, which are one and the same to leading order in a near-boundary expansion, differ already at order z^3 by terms proportional to ε, the conifold deformation parameter, which is zero in the KT solution. These effects are dominant against z^4, the order at which supersymmetry-breaking effects enter. However, being a supersymmetric deformation, it is possible to see that ε does not affect the supersymmetry-breaking dynamics in any dramatic manner, modifying, at most, the numerical values of some quantities, but not the possible existence of supersymmetry-breaking vacua and of the associated massless fermionic mode.

Table 3.1 contains all the fields entering the truncation and the multiplet structure, including, for future reference, the AdS masses obtained in the conformal limit [10], $M = 0$. We refer to appendix C.1 for more details on the simplified five-dimensional σ-model.

To search for domain wall solutions, we can truncate the Lagrangian to its scalar field content only (plus the graviton). Moreover, the extra $U(1)$ symmetry reduces the number of active scalar fields to just four, which, without loss of generality, we can take to be real. The end result is

$$S = \int d^5x\sqrt{-g_5}\left(R - \frac{8}{15}dU^2 - \frac{4}{5}dV^2 - e^{-\frac{4}{5}(U+V)-\phi}\left(db^\Phi\right)^2 - \frac{1}{2}d\phi^2 - \mathcal{V}\right) , \tag{3.3}$$

$\mathcal{N}=2$ multiplet	field fluctuations	AdS mass	spin	Δ
gravity	V_A	$m^2=0$	1	3
	Ψ_A	$m=\frac{3}{2}$	$\frac{3}{2}$	$\frac{7}{2}$
	g_{AB}	$m^2=0$	2	4
universal hyper	$b^\Omega - i\,c^\Omega$	$m^2=-3$	0	3
	ζ_ϕ	$m=-\frac{3}{2}$	$\frac{1}{2}$	$\frac{7}{2}$
	$\tau = C_0 + ie^{-\phi}$	$m^2=0$	1	4
Betti hyper	$t\,e^{i\theta}$	$m^2=-3$	0	3
	ζ_b	$m=-\frac{3}{2}$	$\frac{1}{2}$	$\frac{7}{2}$
	$b^\Phi, \;\; c^\Phi$	$m^2=0$	1	4
massive vector	V	$m^2=12$	0	6
	ζ_V	$m=\frac{9}{2}$	$\frac{1}{2}$	$\frac{13}{2}$
	V_A^1	$m^2=24$	1	7
	$b^\Omega + i\,c^\Omega$	$m^2=21$	0	7
	ζ_U	$m=-\frac{11}{2}$	$\frac{1}{2}$	$\frac{15}{2}$
	U	$m^2=32$	0	8

Table 3.1: Spectrum of bosons and fermions in the $\mathcal{N}=2$ truncation of [52] (5d indices are dubbed A, B).

where we have set the five-dimensional Newton constant $G_5 = 1/16\pi$ and R is the Ricci scalar. The scalar potential $\mathcal{V}$ is given by

$$\mathcal{V} = \frac{1}{2}(27\pi N - 9M\, b^\Phi)^2 e^{-\frac{8}{3}U} + \frac{81}{4}M^2 e^{-\frac{4}{15}(7U-3V)+\phi} - 24 e^{-\frac{2}{15}(8U+3V)} + 4 e^{-\frac{4}{15}(4U+9V)} \, . \tag{3.4}$$

The parameters N and M are continuous quantities in supergravity, but should be thought of as integers, since they correspond to type IIB higher-form fluxes integrated over the non-trivial cycles of $T^{1,1}$ and are thus quantized. Upon uplifting, they are related respectively to the number of regular and fractional D3-branes at the conifold singularity.

3.2.1 Supersymmetric and non-supersymmetric solutions

The solutions we are after should correspond to vacua of the KS dual field theory and, as such, they should satisfy given boundary conditions. First, due to four-dimensional Poincaré invariance, we should focus on domain wall solutions, where all scalars depend on the radial coordinate only and where the ansatz for the metric reads

$$ds^2 = \frac{1}{z^2}\left(e^{2Y(z)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2X(z)}dz^2\right) \, , \tag{3.5}$$

with $\mu, \nu = 0, \dots 3$. The function $X(z)$ can be eliminated by a redefinition of the radial coordinate, while the function $Y(z)$ is the only dynamical variable parameterizing the domain wall metric. We have written the metric ansatz in this form for later convenience. From now on we split the 5d indices as $A = (z, \mu)$. The AdS metric is recovered for $X = Y = 0$, the conformal boundary being at $z = 0$. Another requirement is that for $M = 0$ we should recover the Klebanov-Witten (KW) AdS solution [10].

The solutions we derive below were already obtained working in a ten-dimensional setting in [43] (see also [56] whose normalization for the metric is the same as ours). In this section we re-obtain the same solutions within the truncated five-dimensional model (3.3).

Imposing that the fields satisfy the BPS equations (see appendix C.1) one finds the supersymmetric solution

$$\begin{aligned} e^{2Y} &= h^{\frac{1}{3}}(z) \ , \ e^{2X} = h^{\frac{4}{3}}(z) \ , \ e^{2U} = h^{\frac{5}{2}}(z) \ , \\ b^{\Phi}(z) &= -\frac{9}{2} g_s M \ \log(z/z_0) \ , \\ \phi(z) &= \log \, g_s \ , \ V = 0 \ , \end{aligned} \tag{3.6}$$

where the warp factor $h(z)$ is

$$h(z) = \frac{27\pi}{4g_s} \left(g_s N + \frac{1}{4} a (g_s M)^2 - a (g_s M)^2 \log(z/z_0) \right) , \tag{3.7}$$

with $a = 3/2\pi$, and z_0 is a scale introduced to make the arguments of the log's dimensionless (in the dual QFT, z_0 corresponds to a renormalization scale). The parameter g_s, which in 5d supergravity is an integration constant, has been dubbed as the 10d string coupling, to which it actually gets matched upon uplifting. The characteristic features of this solution are a constant dilaton ϕ and a vanishing V field. This solution is nothing but the five-dimensional formulation of the KT-solution [40]. The KW pure AdS solution [10] is recovered upon setting $M = 0$.

We now look for solutions of the second order equations of motion descending from the action (3.3) (see again appendix C.1). We should require that the solutions reduce to the supersymmetric solution (3.6)-(3.7) in the far UV, that is as $z \to 0$. Up to the order z^4, which is our focus here, the general solution depends on two additional parameters only which, adapting to the notation of [43], we denote with $\mathcal{S}$ and φ. The result is

$$\begin{aligned}
e^{2Y} &= h^{\frac{1}{3}}(z)\, h_2^{\frac{1}{2}}(z)\, h_3^{\frac{1}{2}}(z) \quad , \quad e^{2X} = h^{\frac{4}{3}}(z)\, h_2^{\frac{1}{2}}(z) \ , \\
e^{2U} &= h^{\frac{5}{2}}(z)\, h_2^{\frac{3}{2}}(z) \quad , \quad e^{2V} = h_2^{-\frac{3}{2}}(z) \ , \\
b^{\Phi}(z) &= -\frac{9}{2} g_s M \ \log\left(z/z_0\right) \\
&\quad + z^4 \left[\left(\frac{9\pi N}{4M} + \frac{99}{32} g_s M - \frac{27}{4} g_s M \log\left(z/z_0\right) \right) \mathcal{S} - \frac{9}{8} g_s M \varphi \right] + \mathcal{O}(z^8) \ , \\
\phi(z) &= \log g_s + z^4 \left(3\mathcal{S} \log\left(z/z_0\right) + \varphi \right) + \mathcal{O}(z^8) \ ,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
h(z) &= \frac{27\pi}{4 g_s} \left(g_s N + \frac{1}{4} a (g_s M)^2 - a (g_s M)^2 \log\left(z/z_0\right) \right) \qquad (3.9) \\
&+ \frac{z^4}{g_s} \left[\left(\frac{54\pi g_s N}{64} + \frac{81}{4}\frac{13}{64} \left(g_s M\right)^2 - \frac{81}{16} \left(g_s M\right)^2 \log\left(z/z_0\right) \right) \mathcal{S} - \frac{81}{64} \left(g_s M\right)^2 \varphi \right] + \mathcal{O}(z^8) \ , \\
h_2(z) &= 1 + \frac{2}{3} \mathcal{S} z^4 + \mathcal{O}(z^8) \quad , \quad h_3(z) = 1 + \mathcal{O}(z^8) \ . \qquad (3.10)
\end{aligned}$$

This two-parameter family breaks supersymmetry, in general, but reduces to the supersymmetric KT solution of (3.6)-(3.7) for $\mathcal{S} = \varphi = 0$. Furthermore, as anticipated, supersymmetry-breaking effects enter at order z^4 relative to the KT solution, so for $z \to 0$ the generic solution within the two-parameter family asymptotes to KT. Note, moreover, that the dilaton now runs. In [43] evidence was given that the branch $\varphi = 0$ describes (the large distance asymptotics of) the solution generated by p antiD3-branes at the tip of the conifold, $\mathcal{S}$ being proportional to p. On the contrary, the branch $\mathcal{S} = 0$, which in the AdS limit $M = 0$ corresponds to the usual independent fluctuation of the dilaton [57, 58], was recently extended to all orders in z and a full (still singular) solution was found [50].[3] As we will see later, this branch describes a vacuum where supersymmetry is explicitly broken in the dual field theory and hence does not correspond to a vacuum of the KS field theory. Let us finally notice, in passing, that an ansatz with constant $h_3(z)$ is inconsistent with the equations of motion. Although $h_3(z)$ does not affect the solution at order z^4, one can check that it is necessary to have $h_3(z)$ non-trivial at order z^8 in order to extend the solution deeper in the bulk.

3.3 Holographic Ward identities

The KS theory is an $\mathcal{N} = 1$ QFT and supersymmetry Ward identities like (3.2) should hold in any of its vacua. In this section we provide a holographic derivation of these identities.

[3] The matching between the branch $\mathcal{S} = 0$ and the solution of [50] can be seen upon the following relation between the parameters $\varphi = -\sqrt{10}\, r_s^4$, while the holographic coordinates are inverse to one another, $z = 1/r$.

In the next section, we will test them against the supersymmetric and non-supersymmetric solutions that we have found in section 3.2.1.

Fields in the bulk are dual to QFT gauge invariant operators. In the present case, the bosonic bulk sector consists of four real scalars and the metric. In particular, the fields $e^{-\phi}$ and $\widetilde{b}^{\Phi} = e^{-\phi} b^{\Phi}$ are dual to dimension 4 operators, $\mathcal{O}_{\phi}$ and $\mathcal{O}_{\widetilde{b}}$ (see e.g. [39]), which are respectively related to the sum and the difference of the inverse of the two gauge couplings squared.[4] In the conformal limit they are exact moduli. The scalars V and $\widetilde{U} = (4qb^{\Phi} - 4k + q^2 e^{\phi})e^{-\frac{4}{5}U}/8$ are dual to dimension 6 and 8 operators, respectively. The constants k and q are defined in (C.7). As explained in appendix C.3, the composite field $\widetilde{U}$ is the unique combination of bosonic fields that is sourced solely by the dimension 8 operator. Moreover, although not necessary, it is natural to define the covariant source of the energy-momentum tensor as the field that couples only to metric fluctuations, namely $\widetilde{\gamma}_{\mu\nu} = e^{-4U/15}\gamma_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the four-dimensional induced metric at the radial cut-off. The fermionic sector contains four spin 1/2 fermions and the spin 3/2 gravitino. The field $\widetilde{\Psi}^{+}_{\mu} = e^{-\frac{2}{15}U}\left(\Psi^{+}_{\mu} - \frac{2i}{15}\Gamma_{\mu}\zeta^{-}_{U}\right)$ is dual to the supercurrent, the supersymmetric partner of the energy-momentum tensor, while the fields ζ_{ϕ} and $\widetilde{\zeta}_{b} = e^{-\phi}\left(\zeta_{b} - b^{\Phi}\zeta_{\phi}\right)$, are dual to dimension 7/2 operators, the supersymmetric partners of $\mathcal{O}_{\phi}$ and $\mathcal{O}_{\widetilde{b}}$, respectively. Finally, ζ_{V} and $\widetilde{\zeta}_{U} = -\frac{4}{5}\widetilde{U}\zeta_{U} + \frac{1}{8}e^{-\frac{4}{5}U}\left(4q\zeta_{b} + q^2 e^{\phi}\zeta_{\phi}\right)$ are dual to irrelevant operators as their supersymmetric partners V and $\widetilde{U}$. More details on the identification of the bulk fields dual to gauge-invariant operators can be found in appendix C.3. In what follows, we will switch off the sources of bulk fields that are dual to irrelevant operators. Moreover, the asymptotic supersymmetry breaking solution we presented in section 3.2.1 is given just to order z^4, and this is sufficient for calculating VEVs of relevant or marginal operators only.[5]

The procedure for deriving the Ward Identities is identical to the one presented in the previous chapter. As a first step, we have to define holographically the renormalized one-point functions in the presence of sources. The former are defined as derivatives of the renormalized on-shell action at a radial cut-off with respect to the induced fields at the cut-off and read (care is required here since, as we have already noticed, the supergravity

[4] In fact, the precise correspondence involves also the quartic superpotential coupling [10, 28].

[5] We could turn on a (perturbative) source for the irrelevant operators and calculate their VEVs once we obtain an asymptotic solution to order z^8.

field basis is not diagonal with respect to the basis of the field theory operators)

$$\begin{aligned}
\langle T^{\mu\nu}\rangle &= \frac{2}{\sqrt{-\widetilde{\gamma}}}\left.\frac{\partial S_{\rm ren}}{\partial\widetilde{\gamma}_{\mu\nu}}\right|_{\phi,\widetilde{b}^{\Phi},\widetilde{U},\widetilde{\Psi}^{+},\zeta_{\phi}^{-},\widetilde{\zeta}_{b}^{-},\widetilde{\zeta}_{U}^{-}}, &\quad \langle\overline{S}^{-\mu}\rangle &= \frac{-2i}{\sqrt{-\widetilde{\gamma}}}\left.\frac{\partial S_{\rm ren}}{\partial\widetilde{\Psi}^{+}_{\mu}}\right|_{\widetilde{\gamma},\phi,\widetilde{b}^{\Phi},\widetilde{U},\zeta_{\phi}^{-},\widetilde{\zeta}_{b}^{-},\widetilde{\zeta}_{U}^{-}}, \\
\langle\mathcal{O}_{\phi}\rangle &= \frac{1}{2\sqrt{-\widetilde{\gamma}}}\left.\frac{\partial S_{\rm ren}}{\partial\phi}\right|_{\widetilde{\gamma},\widetilde{b}^{\Phi},\widetilde{U},\widetilde{\Psi}^{+},\zeta_{\phi}^{-},\widetilde{\zeta}_{b}^{-},\widetilde{\zeta}_{U}^{-}}, &\quad \langle\overline{\mathcal{O}}^{+}_{\zeta_{\phi}}\rangle &= \frac{1}{\sqrt{-\widetilde{\gamma}}}\frac{i}{\sqrt{2}}\left.\frac{\partial S_{\rm ren}}{\partial\zeta_{\phi}^{-}}\right|_{\widetilde{\gamma},\phi,\widetilde{b}^{\Phi},\widetilde{U},\widetilde{\Psi}^{+},\widetilde{\zeta}_{b}^{-},\widetilde{\zeta}_{U}^{-}}, \\
\langle\mathcal{O}_{\widetilde{b}}\rangle &= \frac{1}{2\sqrt{-\widetilde{\gamma}}}\left.\frac{\partial S_{\rm ren}}{\partial\widetilde{b}^{\Phi}}\right|_{\widetilde{\gamma},\phi,\widetilde{U},\widetilde{\Psi}^{+},\zeta_{\phi}^{-},\widetilde{\zeta}_{b}^{-},\widetilde{\zeta}_{U}^{-}}, &\quad \langle\overline{\mathcal{O}}^{+}_{\widetilde{\zeta}_{b}}\rangle &= \frac{1}{\sqrt{-\widetilde{\gamma}}}\frac{i}{\sqrt{2}}\left.\frac{\partial S_{\rm ren}}{\partial\widetilde{\zeta}_{b}^{-}}\right|_{\widetilde{\gamma},\phi,\widetilde{b}^{\Phi},\widetilde{U},\widetilde{\Psi}^{+},\zeta_{\phi}^{-},\widetilde{\zeta}_{U}^{-}},
\end{aligned} \tag{3.11}$$

where the subscripts in the partial functional derivatives indicate the variables held fixed, which is crucial for evaluating correctly these one-point functions. The resulting expressions in terms of derivatives with respect to the supergravity fields are given in appendix C.3. The quantity $\widetilde{\gamma}$ is the determinant of $\widetilde{\gamma}_{\mu\nu}$, while the normalization of the one-point functions has been chosen in accordance with the conventions for organizing these operators in $\mathcal{N}=1$ superfields.

Several comments are in order here. Firstly, $S_{\rm ren}$ denotes the renormalized on-shell action

$$S_{\rm ren} = S_{\rm reg} + S_{\rm ct} \ , \tag{3.12}$$

where the regularized action $S_{\rm reg}$ stands for the bulk on-shell action plus the Gibbons-Hawking term (together with its supersymmetric completion [31]), and the covariant boundary counterterms $S_{\rm ct}$ contain both bosonic and fermionic terms. The counterterms, by construction, ensure that $S_{\rm ren}$ admits a smooth limit as the radial cut-off is removed. Given the asymptotic behavior of the induced fields given in appendix C.2, this implies that the renormalized one-point functions with the cut-off removed correspond to the limits

$$\begin{aligned}
\langle T^{\mu}_{\nu}\rangle_{\rm QFT} &= \lim_{z\to0} z^{-4}\langle T^{\mu}_{\nu}\rangle \ , \quad & \langle\overline{S}^{-\mu}\rangle_{\rm QFT} &= \lim_{z\to0} z^{-9/2}e^{-X(z)/8}\langle\overline{S}^{-\mu}\rangle \ , \\
\langle\mathcal{O}_{\phi}\rangle_{\rm QFT} &= \lim_{z\to0} z^{-4}\langle\mathcal{O}_{\phi}\rangle \ , \quad & \langle\overline{\mathcal{O}}^{+}_{\zeta_{\phi}}\rangle_{\rm QFT} &= \lim_{z\to0} z^{-7/2}e^{-X(z)/8}\langle\overline{\mathcal{O}}^{+}_{\zeta_{\phi}}\rangle \ , \\
\langle\mathcal{O}_{\widetilde{b}}\rangle_{\rm QFT} &= \lim_{z\to0} z^{-4}\langle\mathcal{O}_{\widetilde{b}}\rangle \ , \quad & \langle\overline{\mathcal{O}}^{+}_{\widetilde{\zeta}_{b}}\rangle_{\rm QFT} &= \lim_{z\to0} z^{-7/2}e^{-X(z)/8}\langle\overline{\mathcal{O}}^{+}_{\widetilde{\zeta}_{b}}\rangle \ .
\end{aligned} \tag{3.13}$$

Note that one of the indices of the stress tensor has been lowered with the field theory metric $\widetilde{\gamma}_{\mu\nu}$, and not $\gamma_{\mu\nu}$. The explicit expression for the local boundary counterterms is not required in order to derive the Ward identities holographically. It suffices that there exist local and covariant boundary counterterms that render the on-shell action finite, while preserving the symmetries of the dual QFT –most importantly for us, supersymmetry– up to possible anomalies. Of course, explicit knowledge of the counterterms is necessary in order to evaluate the one-point functions (3.13) for any given solution. In the next section we will present the boundary counterterms required to evaluate the bosonic VEVs in domain wall backgrounds of the form (3.8).

Another point worth mentioning is that the one-point functions of the bosonic operators are given by the derivative of the renormalized action with respect to the corresponding induced field on the radial cut-off, which is therefore identified with the covariant source. However, the covariant sources for the fermionic operators are given by the corresponding induced field –which is a four-dimensional spinor– projected onto a definite chirality. As a consequence, the dual operators have definite (and opposite) chirality.[6] The chirality that corresponds to the covariant fermionic source is determined by the leading asymptotics which in turn are fixed by the sign of the their masses (see Table 3.1 and appendix C.2 for details).

Given the holographic identification of the covariant sources and one-point functions at the radial cut-off, the derivation of the Ward identities proceeds exactly as in chapter 2. Namely, global symmetries are gauged, giving rise to generic sources for all global symmetry currents. In addition, sources are manually turned on for all other operators, such as scalar and fermion operators. Using the transformation of all the sources under the local (gauged) symmetries together with the invariance (up to anomalies) of the generating functional, leads to the Ward identities at the level of one-point functions in the presence of arbitrary sources. In the bulk description all symmetries are already gauged and all sources are turned on, so the only other ingredient we need in order to derive holographically the Ward identities is the transformation of the covariant sources under the local symmetries. These are given explicitly in appendix C.4. In the bulk these symmetries correspond to infinitesimal local supersymmetry transformations and bulk diffeomorphisms generated respectively by a 4-component Dirac spinor ϵ and a 5-vector ξ^A, preserving the gauge-fixing conditions (C.25). The spinor ϵ has 8 real components which correspond to the 8 real supercharges of the $\mathcal{N}=2$ 5d supergravity. This can be written as $\epsilon=\epsilon^{+}+\epsilon^{-}$. Since ϵ^{+} and ϵ^{-} are linearly independent supersymmetry transformation parameters, the renormalized on-shell action is not only invariant under ϵ but also under ϵ^{+} and ϵ^{-} independently. The spinor ϵ^{+} generates (local) boundary supersymmetry transformations, while ϵ^{-} generates superWeyl transformations. Invariance under ϵ^{+} and ϵ^{-} leads respectively to the supersymmetry Ward identities and the operator identity involving the gamma-trace of the supercurrent. Similarly, the infinitesimal bulk diffeomorphisms ξ^A preserving the gauge-fixing conditions (C.25) are parameterized by two independent parameters, a scalar $\sigma(x)$ generating boundary Weyl transformations[7] and an infinitesimal boundary diffeomorphism $\xi_o^\mu(x)$. Invariance under these leads respectively to the trace Ward identity and the Ward identity involving the divergence of the stress

[6]This difference reflects the structure of the radial Hamiltonian phase space for bosonic and fermionic fields. The holographic one-point functions (3.11) are in either case the renormalized radial canonical momenta [51].

[7]The corresponding bulk diffeomorphisms are known as Penrose-Brown-Henneaux (PBH) diffeomorphisms and are discussed in detail in [59].

tensor. If the theory has a trace anomaly, then supersymmetry implies that there will also be an anomaly in the operator identity involving the gamma-trace of the supercurrent.

3.3.1 Supersymmetry Ward identities

The supersymmetry Ward identities are obtained by requiring the invariance of the renormalized action under the local spinor ϵ^+, $\delta_{\epsilon^+}S_{\rm ren}=0$. However, to calculate $\delta_{\epsilon^+}S_{\rm ren}$, we need the transformation properties of the covariant sources under ϵ^+, which are given in appendix C.4, eq. (C.41) . Using the one-point functions (3.11) the variation of the renormalized action under ϵ^+ gives[8]

$$\begin{aligned}\delta_{\epsilon^+}S_{\rm ren}&=\int d^4x\sqrt{-\widetilde{\gamma}}\left(\frac{i}{2}\langle\overline{S}^{-\mu}\rangle\delta_{\epsilon^+}\widetilde{\Psi}^+_\mu+\frac{1}{2}\langle T^{\mu\nu}\rangle\delta_{\epsilon^+}\widetilde{\gamma}_{\mu\nu}+2\langle\mathcal{O}_\phi\rangle\delta_{\epsilon^+}\phi+2\langle\mathcal{O}_{\widetilde{b}}\rangle\delta_{\epsilon^+}\widetilde{b}^\Phi\right)\\&=\int d^4x\sqrt{-\widetilde{\gamma}}\left(-\frac{i}{2}e^{-\frac{2}{15}U}\langle\partial_\mu\overline{S}^{-\mu}\rangle-\frac{1}{2}\langle T^{\mu\nu}\rangle\overline{\widetilde{\Psi}}{}^+_\mu\widetilde{\Gamma}_\nu+i\langle\mathcal{O}_\phi\rangle\overline{\zeta}^-_\phi+i\langle\mathcal{O}_{\widetilde{b}}\rangle\overline{\widetilde{\zeta}}{}^-_b\right)\epsilon^+=0\ ,\end{aligned}\tag{3.14}$$

which implies the following identity between one-point functions at non-zero sources

$$\frac{i}{2}e^{-\frac{2}{15}U}\langle\partial_\mu\overline{S}^{-\mu}\rangle=-\frac{1}{2}\langle T^{\mu\nu}\rangle\overline{\widetilde{\Psi}}{}^+_\mu\widetilde{\Gamma}_\nu+i\langle\mathcal{O}_\phi\rangle\overline{\zeta}^-_\phi+i\langle\mathcal{O}_{\widetilde{b}}\rangle\overline{\widetilde{\zeta}}{}^-_b\ ,\tag{3.15}$$

where $\widetilde{\Gamma}_\mu=\widetilde{e}^a_\mu\gamma_a=e^{-\frac{2}{15}U}e^a_\mu\gamma_a$. We can now differentiate this identity with respect to the various fermionic fields, i.e. the covariant sources, and then put all sources to zero to obtain [9]

$$e^{-\frac{2}{15}U}\langle\partial_\mu\overline{S}^{-\mu}(x)S^-_\nu(0)\rangle=2i\,\widetilde{\Gamma}_\mu\langle T^\mu_\nu\rangle\,\delta^4(x,0)\ ,\tag{3.16}$$

$$e^{-\frac{2}{15}U}\langle\partial_\mu\overline{S}^{-\mu}(x)\mathcal{O}^+_{\zeta_\phi}(0)\rangle=-\sqrt{2}\,\langle\mathcal{O}_\phi\rangle\,\delta^4(x,0)\ ,\tag{3.17}$$

$$e^{-\frac{2}{15}U}\langle\partial_\mu\overline{S}^{-\mu}(x)\mathcal{O}^+_{\widetilde{\zeta}_b}(0)\rangle=-\sqrt{2}\,\langle\mathcal{O}_{\widetilde{b}}\rangle\,\delta^4(x,0)\ ,\tag{3.18}$$

where $\delta^4(x,y)=\delta^4(x-y)/\sqrt{-\widetilde{\gamma}}$ is the covariant 4d Dirac delta function. The last step is to take the cut-off all the way to infinity, which can be done using the limits (3.13). All these limits can be easily evaluated using the asymptotic expansions of the induced fields given in appendix C.2. Notice that all fermionic operators here are in the Dirac representation. In order to match with the field theory expressions, it is better to convert them into Weyl

[8] Note that there are no contributions to the Ward identities from the irrelevant operators dual to V and $\widetilde{U}$, as well as their fermionic superpartners, because their sources can be consistently set to zero.

[9] Notice that the two-point functions in (3.16) (and the ensuing equations) are defined in terms of the one-point functions as: $\langle\partial_\mu\overline{S}^{-\mu}S^{-\nu}\rangle=-\frac{-2i}{\sqrt{-\widetilde{\gamma}}}\frac{\delta}{i\delta\overline{\widetilde{\Psi}}{}^+_\nu}\langle\partial_\mu\overline{S}^{-\mu}\rangle$. The extra factor of i in the denominator is because of the Lorentzian signature and the overall minus sign is because the functional derivative is with respect to a Grassmann variable.

notation. This can be done easily using the following conversion rules

$$\psi^+ = \psi_\alpha, \quad \psi^- = \overline{\psi}^{\dot\alpha}, \quad \overline{\psi}^+ = \overline{\psi}_{\dot\alpha}, \quad \overline{\psi}^- = \psi^\alpha \ , \quad (\gamma_\mu)_{\alpha\dot\beta} = i\,(\sigma_\mu)_{\alpha\dot\beta} \,. \tag{3.19}$$

Adopting the above dictionary and upon sending the cut-off to infinity, we eventually get

$$\langle \partial^\mu S_{\mu\alpha}(x)\ \bar S_{\nu\dot\beta}(0)\rangle_{\rm QFT} = -2\,\sigma^\mu_{\alpha\dot\beta}\langle T_{\mu\nu}\rangle_{\rm QFT}\ \delta^4(x)\ , \tag{3.20}$$

$$\langle \partial^\mu S^\alpha_\mu(x)\ \mathcal{O}_{\zeta_\phi\alpha}(0)\rangle_{\rm QFT} = -\sqrt{2}\ \langle \mathcal{O}_\phi\rangle_{\rm QFT}\ \delta^4(x)\ , \tag{3.21}$$

$$\langle \partial^\mu S^\alpha_\mu(x)\ \mathcal{O}_{\widetilde\zeta_b\alpha}(0)\rangle_{\rm QFT} = -\sqrt{2}\ \langle \mathcal{O}_{\widetilde b}\rangle_{\rm QFT}\ \delta^4(x)\ . \tag{3.22}$$

The identity (3.20) reproduces exactly the supercurrent Ward identity (3.2). Eqs. (3.21) and (3.22) are analogous Ward identities for the supermultiplets where the operators $\mathcal{O}_\phi$ and $\mathcal{O}_{\widetilde b}$ sit. Since $\mathcal{O}_\phi$ and $\mathcal{O}_{\widetilde b}$ are higher-component operators, a non-vanishing r.h.s. in eqs. (3.21) and (3.22) signals that supersymmetry is broken in the corresponding vacuum. The supersymmetric partner of these identities is the Ward identity involving the divergence of the stress tensor. This can be easily derived holographically by considering the invariance of the renormalized action under boundary diffeomorphisms, but we will not discuss it here.

3.3.2 Trace identities

In this section we derive the trace operator identities associated respectively with the energy-momentum tensor and the supercurrent. Let us consider the latter first. From the ϵ^- supersymmetry transformations (C.42), and using (C.7), for the variation of $S_{\rm ren}$ we get[10]

$$\delta_{\epsilon^-} S_{\rm ren} = \int d^4x \sqrt{-\widetilde\gamma}\left(\frac{i}{2}\langle \overline{S}^{-\mu}\widetilde\Gamma_\mu\rangle - \frac{9M}{\sqrt2}\ \langle \overline{\mathcal{O}}^+_{\widetilde\zeta_b}\rangle\right) e^{-\frac{8}{15}U}\epsilon^- = 0\ , \tag{3.23}$$

which yields the following identity between the one-point functions of the gamma-trace of the supercurrent and of the operator $\mathcal{O}_{\widetilde\zeta_b}$ at non-zero sources and at the cut-off

$$\frac{i}{2}\langle \overline{S}^{-\mu}\widetilde\Gamma_\mu\rangle = \frac{9M}{\sqrt2}\ \langle \overline{\mathcal{O}}^+_{\widetilde\zeta_b}\rangle\,. \tag{3.24}$$

Again, from this identity one can compute relations between various correlation functions by further differentiating. Using the limits (3.13), we can remove the cut-off to obtain the relation

$$\langle \sigma^\mu_{\alpha\dot\beta}\bar S^{\dot\beta}_\mu\rangle_{\rm QFT} = -9\sqrt2 M\ \langle \mathcal{O}_{\widetilde\zeta_b\,\alpha}\rangle_{\rm QFT}\ . \tag{3.25}$$

[10]As we pointed out already, there is a potential anomaly on the r.h.s. of this equation, as well as on the r.h.s. of (3.26). To compute these anomalies an explicit computation of the local counterterms $S_{\rm ct}$ is required. However, the anomalies only contribute ultralocal contact terms in the Ward identities, which are not relevant for the present discussion.

Next, let us derive the Ward identity following from local shifts in the radial coordinate, which correspond to local Weyl transformations on the boundary. Using the transformation of the covariant sources given in eq. (C.30), we get

$$
\begin{aligned}
\delta_\sigma S_{\text{ren}} =& \int d^4x \sqrt{-\widetilde{\gamma}} \left(\frac{1}{2}\delta_\sigma \widetilde{\gamma}_{\mu\nu} \langle T^{\mu\nu}\rangle + 2\delta_\sigma \phi \, \langle \mathcal{O}_\phi \rangle + 2\delta_\sigma \widetilde{b}^\Phi \langle \mathcal{O}_{\widetilde{b}} \rangle \right. \\
& \left. + \left[\frac{i}{2} \langle \overline{S}^{-\mu} \rangle \delta_\sigma \widetilde{\Psi}^+_\mu - \sqrt{2} i \langle \overline{\mathcal{O}}^+_{\zeta^\phi} \rangle \delta_\sigma \zeta^-_\phi - \sqrt{2} i \langle \overline{\mathcal{O}}^+_{\widetilde{\zeta}^b} \rangle \delta_\sigma \widetilde{\zeta}^-_b + \text{h.c.} \right] \right) \\
=& \int d^4x \sqrt{-\widetilde{\gamma}} \left(\langle T^\mu_\mu \rangle + 9M \langle \mathcal{O}_{\widetilde{b}} \rangle \right. \\
& \left. + \left[\frac{i}{4} \langle \overline{S}^{-\mu} \rangle \widetilde{\Psi}^+_\mu + \frac{i}{\sqrt{2}} \langle \overline{\mathcal{O}}^+_{\zeta^\phi} \rangle \zeta^-_\phi + \frac{i}{\sqrt{2}} \langle \overline{\mathcal{O}}^+_{\widetilde{\zeta}^b} \rangle \widetilde{\zeta}^-_b + \text{h.c.} \right] \right) e^{-\frac{8}{15}U} \sigma \; .
\end{aligned}
\tag{3.26}
$$

This leads to the following identity between bosonic one-point functions at the cut-off

$$
\langle T^\mu_\mu \rangle + 9M \langle \mathcal{O}_{\widetilde{b}} \rangle + \left[\frac{i}{4} \langle \overline{S}^{-\mu} \rangle \widetilde{\Psi}^+_\mu + \frac{i}{\sqrt{2}} \langle \overline{\mathcal{O}}^+_{\zeta^\phi} \rangle \zeta^-_\phi + \frac{i}{\sqrt{2}} \langle \overline{\mathcal{O}}^+_{\widetilde{\zeta}^b} \rangle \widetilde{\zeta}^-_b + \text{h.c.} \right] = 0 \, . \tag{3.27}
$$

Removing the cut-off (and setting all sources to zero), we finally obtain

$$
\langle T^\mu_\mu \rangle_{\text{QFT}} = -9M \langle \mathcal{O}_{\widetilde{b}} \rangle_{\text{QFT}} \, . \tag{3.28}
$$

This is the bosonic partner of the fermionic trace identity (3.25) and the two are in perfect agreement, numerical coefficients included.

Notice that only the VEV of $\mathcal{O}_{\widetilde{b}}$ and not that of $\mathcal{O}_\phi$ enters eq. (3.28). From the general formula $T^\mu_\mu = -\frac{1}{2}\sum_i \beta_i \, \mathcal{O}_i$ this suggests that in the KS theory the operator $\mathcal{O}_\phi$ remains marginal, at least in the supergravity regime, while $\mathcal{O}_{\widetilde{b}}$ has non-trivial β-function. This is indeed the case, as shown in [28], in perfect agreement with the field theory answer in the large-N limit. We will further comment on this point later.

3.4 Bosonic one-point functions and the Goldstino

Our goal here is to see how the supersymmetry Ward identities (3.20)-(3.22) are realized differently in the backgrounds (3.6) and (3.8). Given the derivation of subsection 3.3.1, it suffices to evaluate the bosonic one-point functions of $T_{\mu\nu}, \mathcal{O}_\phi$ and $\mathcal{O}_{\widetilde{b}}$.

The calculation of the bosonic VEVs in the background (3.8) was already performed in [56]. The authors of that paper took the most general compactification of the normalizable deformations of the 10d KT solution. In particular, their solution contains transverse dependence and is obtained from an ansatz which is gauge-redundant because of radial diffeomorphisms. This makes the calculation of the VEVs technically involved. However, if we focus just on flat domain wall solutions and fix radial diffeomorphisms, we can obtain the one-point functions in a simpler manner. With this simplification in mind, we provide below an independent derivation of the VEVs of $T_{\mu\nu}, \mathcal{O}_\phi$ and $\mathcal{O}_{\widetilde{b}}$, and find agreement with the results of [43, 56].

In order to evaluate the bosonic one-point functions in (3.11) explicitly, we compute separately the contributions coming respectively from the regularized action and the counterterms in (3.12). The contribution coming from $S_{\rm reg}$ is the radial canonical momentum associated with the corresponding induced field, as follows from Hamilton-Jacobi theory.[11] Using the expressions for the radial canonical momenta corresponding to the fields $\widetilde{\gamma}_{\mu\nu}$, ϕ, and $\widetilde{b}^\Phi$ in the coordinate system (C.25) (see e.g. [36]) and using the identities (C.24), the bosonic VEVs in (3.11) become

$$
\begin{aligned}
\langle T^{\mu\nu}\rangle &= e^{\frac{4}{15}U}e^X\left(-2\left(K\gamma^{\mu\nu}-K^{\mu\nu}\right)+\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{\rm ct}}{\delta\gamma_{\mu\nu}}\right)\,, && (3.29)\\
\langle\mathcal{O}_\phi\rangle &= -e^X\left(\mathcal{G}_{\phi\phi}\dot\phi+b^\Phi\mathcal{G}_{b^\Phi b^\Phi}\dot b^\Phi+\left(1+\frac{k}{2}e^{-\frac{4}{5}U}\right)\left(\frac{5}{4}\mathcal{G}_{UU}\dot U-\frac{1}{2}K\right)\right)\\
&\quad+\frac{1}{2\sqrt{-\gamma}}e^X\left(\frac{\delta S_{\rm ct}}{\delta\phi}+b^\Phi\frac{\delta S_{\rm ct}}{\delta b^\Phi}+\left(1+\frac{k}{2}e^{-\frac{4}{5}U}\right)\left(\frac{5}{4}\frac{\delta S_{\rm ct}}{\delta U}+\frac{1}{3}\gamma_{\mu\nu}\frac{\delta S_{\rm ct}}{\delta\gamma_{\mu\nu}}\right)\right), && (3.30)\\
\langle\mathcal{O}_{\widetilde{b}}\rangle &= e^\phi e^X\left(-\mathcal{G}_{b^\Phi b^\Phi}\dot b^\Phi-\frac{q}{2}e^{-\frac{4}{5}U}\left(\frac{5}{4}\mathcal{G}_{UU}\dot U-\frac{1}{2}K\right)\right.\\
&\quad\left.+\frac{1}{2\sqrt{-\gamma}}\left(\frac{\delta S_{\rm ct}}{\delta b^\Phi}+\frac{q}{2}e^{-\frac{4}{5}U}\left(\frac{5}{4}\frac{\delta S_{\rm ct}}{\delta U}+\frac{1}{3}\gamma_{\mu\nu}\frac{\delta S_{\rm ct}}{\delta\gamma_{\mu\nu}}\right)\right)\right), && (3.31)
\end{aligned}
$$

where the dot represents derivatives with respect to the radial coordinate r, which is defined in eq. (C.25), while $K_{\mu\nu}$ is the extrinsic curvature of the radial slices which, for the metric (C.25), takes the form

$$
K_{\mu\nu}=\frac{1}{2}\dot\gamma_{\mu\nu}=-\frac{1}{2}ze^{-X}\partial_z\left(\frac{e^{2Y}}{z^2}\right)\eta_{\mu\nu}\,. \tag{3.32}
$$

The contribution to the bosonic VEVs from $S_{\rm ct}$ requires to know the explicit form of the (bosonic part of the) boundary counterterms, at least for the case of Poincaré domain wall solutions. Both the bosonic and fermionic counterterms can be derived systematically for general cascading solutions. For backgrounds enjoying 4D Poincaré invariance it turns out that the bosonic counterterms in a supersymmetric scheme [27] are simply given by the superpotential (C.8), namely

$$
S_{\rm ct}=-\int d^4x\sqrt{-\gamma}\,2\mathcal{W}\,. \tag{3.33}
$$

[11]As an elementary example consider the canonical momentum of a point particle described by the Lagrangian $L=\frac{1}{2}\dot x^2$, given by $p=\partial L/\partial\dot x=\dot x$. Invoking the equations of motion it follows that this canonical momentum can also be expressed as $p=\partial S_{\rm reg}/\partial x$, where the on-shell action is identified with Hamilton's principal function.

Putting the two contributions together, the VEVs (3.29)-(3.31) at the radial cut-off take the form

$$\begin{aligned}
\langle T^{\mu}_{\nu}\rangle &= -2\left[3z\partial_z \log\left(\frac{e^Y}{z}\right) + e^X \mathcal{W}\right]\delta^{\mu}_{\nu}\ , \\
\langle \mathcal{O}_\phi\rangle &= \frac{1}{2}z\partial_z\phi + e^{-\phi}b^\Phi\left(e^{-\frac{4}{5}(U+V)}z\partial_z b^\Phi - e^\phi e^X \partial_{b^\Phi}\mathcal{W}\right) \\
&\quad + \left(1+\frac{k}{2}e^{-\frac{4}{5}U}\right)\left[\frac{5}{4}\left(\frac{8}{15}z\partial_z U - e^X\partial_U\mathcal{W}\right) - 2z\partial_z\log\left(\frac{e^Y}{z}\right) - \frac{2}{3}e^X\mathcal{W}\right]\ , \\
\langle \mathcal{O}_{\widetilde{b}}\rangle &= e^{-\frac{4}{5}(U+V)}z\partial_z b^\Phi - e^\phi e^X\partial_{b^\Phi}\mathcal{W} \\
&\quad + \frac{q}{2}e^{-\frac{4}{5}U+\phi}\left[\frac{5}{4}\left(\frac{8}{15}z\partial_z U - e^X\partial_U\mathcal{W}\right) - 2z\partial_z\log\left(\frac{e^Y}{z}\right) - \frac{2}{3}e^X\mathcal{W}\right]\ .
\end{aligned} \tag{3.34}$$

Evaluating the limits in (3.13) using the asymptotic behavior of the induced fields we finally get

$$\langle T^{\mu}_{\mu}\rangle_{\rm QFT} = -12\,\mathcal{S}\ , \tag{3.35}$$

$$\langle \mathcal{O}_\phi\rangle_{\rm QFT} = \frac{(3\mathcal{S}+4\varphi)}{2}\ , \tag{3.36}$$

$$\langle \mathcal{O}_{\widetilde{b}}\rangle_{\rm QFT} = \frac{4}{3\,M}\mathcal{S}\ , \tag{3.37}$$

in agreement with the corresponding expressions in [43, 56] (note that the sign of $\mathcal{S}$ is univocally fixed from (3.35), by unitarity).

Let us elaborate on the above result, utilizing the Ward identities (3.20)-(3.22), which we derived holographically. On the supersymmetric solution (3.6), for which $\mathcal{S} = \varphi = 0$, all the above VEVs vanish, i.e. the vacuum energy is zero and there are no non-trivial VEVs for higher component operators. The supersymmetry Ward identities are trivially satisfied, and there is no massless pole in the supercurrent two-point function (3.1). This is all consistent with supersymmetry being preserved.

More interestingly, let us now look at non-supersymmetric branches, and start with the branch $\mathcal{S} = 0,\ \varphi \neq 0$. Here supersymmetry is broken in the dual field theory, since a higher-component operator, $\mathcal{O}_\phi$, has a non-vanishing VEV. Since $\langle T^{\mu}_{\mu}\rangle = 0$, however, the vacuum energy vanishes and the goldstino mode is absent in (3.1). This is an indication of explicit supersymmetry breaking, meaning that this branch does not describe vacua of the KS model. This agrees with the fact that the β-function of the sum of the inverse gauge coupling squared, the coupling dual to $\mathcal{O}_\phi$, actually vanishes [28] and hence $\mathcal{O}_\phi$ remains exactly marginal. As such, it cannot trigger spontaneous supersymmetry breaking (the dynamics along this branch is basically the same as in the case of the dilaton background of [57, 58], though in a non-conformal theory).

Finally, let us consider the branch $\varphi = 0,\ \mathcal{S} \neq 0$. This was suggested in [43] to correspond to the (asymptotic description of the) metastable state obtained by placing

$p \sim \mathcal{S}$ antiD3-branes at the tip of the deformed conifold. Along this branch we see that the vacuum energy (3.35) is non-vanishing, this being triggered by the VEV of the operator $\mathcal{O}_{\widetilde{b}}$, eq. (3.37). Indeed, these two quantities exactly satisfy the relation $T^\mu_\mu = -\frac{1}{2}\beta_{\widetilde{b}^\Phi}\mathcal{O}_{\widetilde{b}}$ (the difference with respect to the normalization of [28] is just due to a different normalization of the operator $\mathcal{O}_{\widetilde{b}}$). From the supercurrent Ward identities (3.20) and (3.22), which hold non-trivially in this vacuum, we see that a goldstino mode is present in the supercurrent two-point function (3.1). From the operator identity (3.25) it follows that the goldstino eigenstate is

$$G \sim \langle \mathcal{O}_{\widetilde{b}} \rangle \, \sigma^\mu \bar{S}_\mu \sim \langle \mathcal{O}_{\widetilde{b}} \rangle \, \mathcal{O}_{\widetilde{\zeta}_b} \, . \tag{3.38}$$

All these properties are consistent with a vacuum where supersymmetry is spontaneously broken and suggest that (if it exists, cf. the discussion in the Introduction of this chapter) the KPV vacuum is in fact a vacuum of the KS theory. Before concluding this section, there is one remark worth mentioning. From field theory viewpoint, there are no obvious symmetries protecting the dimension of $\mathcal{O}_\phi$. Hence, one would expect its dimension to get corrections, at least beyond the supergravity regime. Evidence for this was given in [60], where α'^3-corrections were computed suggesting that the otherwise marginal operator gets contributions to its anomalous dimension at order $\sim (\frac{M}{N})^4 (g_s N)^{-1/2}$ (recall that the supergravity limit is $g_s N \to \infty$). So, given that in this branch $\langle \mathcal{O}_\phi \rangle_{\text{QFT}} \neq 0$, the goldstino eigenstate could get a (very much suppressed) contribution from $\mathcal{O}_{\zeta_\phi}$, too, in the KPV vacuum.

3.5 Conclusions

In this chapter, we have focused on deriving holographically the supersymmetry Ward identities of the conifold cascading gauge theory, and to evaluate them explicitly in supersymmetric and supersymmetry-breaking dual backgrounds. Within the consistent truncation we have considered, a two-parameter family of supersymmetry-breaking solutions exists with the correct asymptotics. We have shown that only a one-dimensional branch respects the supersymmetry Ward identities and displays the expected goldstino mode. This branch was conjectured in [43] to describe, asymptotically, the state constructed by placing antiD-branes at the tip of the deformed conifold, which is a metastable state in the probe approximation [30]. In this sense, our results provide evidence that the KS cascading theory can admit vacua where supersymmetry is broken at strong coupling, and also that antiD-brane states, if they exist beyond the probe approximation, are valuable candidates for such vacua.[12]

[12] It would be interesting to repeat our computation for the solution of [55], which includes also the conifold deformation parameter. The computation is more involved, since the truncation one should consider includes more fields. However, as already argued, we do not expect any qualitative changes in the end result.

The derivation of the supersymmetry Ward identities we performed is quite general and does not rely very much on the specific structure of the conifold theory, nor on the explicit form of the solutions. This suggests that supersymmetry breaking vacua might be generic in quiver gauge theories with running couplings driven by fractional branes, the KS model being just a prototype example (superconformal theories cannot break supersymmetry spontaneously, hence fractional branes are a necessary ingredient in the construction). Considering this larger class of theories, in terms of more general 5d sigma-models than the one presented in appendix C.1, could be instructive.[13]

Our results are consistent with previous findings [56, 64, 65], where it was suggested that cascading theories, although being rather unconventional from the field theory point of view, are in fact renormalizable holographically (see also [66–68]). There are however several remaining open questions. The derivation of the counterterms we pursued is all one needs to renormalize bosonic one-point functions, but this is not the full story. In fact, the approach we used, where correlators are defined in terms of induced fields at a finite cut-off rather than in terms of sources, seems robust and general enough to let one compute the full counterterm action, including all bosonic and fermionic counter-terms. This could make the analysis initiated in [56, 65] more rigorous and possibly far reaching.

Working in terms of induced fields looks also as an efficient approach to try and answer the question on how to derive, from first principles, counterterms respecting supersymmetry in generic setups. In fact, this could also provide a technically and conceptually promising way to attack the problem of holographically renormalize supersymmetric theories on curved manifolds.

[13] In the probe approximation, where the goldstino is a massless excitation on the antiD3-brane world-volume, this was shown to be the case in generalizations of the KPV construction on conifold-like geometries with orientifolds [61], see also [62, 63].

Chapter 4

Multi-trace deformations in AdS/CFT

In this chapter, we study some aspects of renormalization group flows triggered in a large-N conformal field theory by multi-trace deformation in the context of AdS/CFT correspondence, focusing concretely on the double-trace case. We consider a d-dimensional conformal field theory perturbed by a relevant double-trace deformation of the form $f\mathcal{O}^2$, where $\mathcal{O}$ is a single trace operator of dimension $\Delta_- < d/2$. The CFT flows to a non-trivial conformal fixed point in the IR. In the large-N limit, the IR theory is related to the UV theory by a Legendre transformation with respect to the source for the operator $\mathcal{O}$. In particular, the scaling dimension of $\mathcal{O}$ flips from Δ_- in the UV to $\Delta_+ = d - \Delta_-$ in the IR. As briefly noted in chapter 1, in the context of $O(N)$ vector model, double-trace deformations are implemented in AdS by a change of boundary conditions on the scalar field ϕ dual to $\mathcal{O}$. The boundary conditions preserve symmetries of AdS space only at the fixed points of the flow, where they correspond to the two roots $\Delta_\pm$ in the usual AdS/CFT mass/dimension relation $\Delta(\Delta - d) = m^2$. This feature is quite generic and is independent of the spacetime dimension.

Later in the chapter we use double-trace deformations to study holographic aspects of CFT phenomenon called *multiplet recombination* for the case when $\mathcal{O}$ is a scalar primary. We will consider the coupling of a free scalar to a single-trace operator of a large N CFT in d dimensions. This is equivalent to a double-trace deformation coupling two primary operators of the CFT, in the limit when one of the two saturates the unitarity bound. At leading order, the RG-flow has a non-trivial fixed point where multiplets recombine. We show this phenomenon in field theory, and provide the holographic dual description. Free scalars correspond to singleton representations of the AdS algebra. The double-trace interaction is mapped to a boundary condition mixing the singleton with the bulk field dual to the single-trace operator. In the IR, the singleton and the bulk scalar merge, providing just one long representation of the AdS algebra.

4.1 Double-trace deformation in AdS/CFT

In this section we present some details of double-trace deformations in a large-N CFT and its AdS dual. This will serve as a warmup for the following sections. We proceed by studying, both in field theory and holography, two examples. The first example is that of a relevant deformation of the type $f\mathcal{O}^2$ where $\Delta_{\mathcal{O}} = \Delta_-$, $\mathcal{O}$ being a single-trace primary. The second example is that of a classically marginal deformation of the type $f\mathcal{O}_1\mathcal{O}_2$ where $\Delta_{\mathcal{O}_1} = \Delta_-$ and $\Delta_{\mathcal{O}_1} = \Delta_+$ where both $\mathcal{O}_1$ and $\mathcal{O}_2$ are single-trace primary operators. This model was first studied in [20].

4.1.1 CFT analysis

Let us consider a deformation of a large-N CFT by a relevant double trace operator $\mathcal{O}^2$. The partition function of the theory in the presence of a source J for the single trace operator $\mathcal{O}$ is given by

$$\mathcal{Z}_f[J] = \left\langle e^{-\frac{f}{2}\int \mathcal{O}^2 + \int J\mathcal{O}} \right\rangle_0 , \tag{4.1}$$

where the subscript 0 means that the expectation value is evaluated in the undeformed CFT. In the large-N limit this partition function can be simplified by performing a Hubbard-Stratonovich transformation to decouple the interaction term by introducing an auxiliary field σ. We get

$$\begin{aligned}
\mathcal{Z}_f[J] &= \sqrt{\det\left(-\frac{1}{f}\mathbf{1}\right)} \int \mathcal{D}\sigma \left\langle e^{\frac{1}{2f}\int \sigma^2 + \int \sigma\mathcal{O} + \int J\mathcal{O}} \right\rangle_0 , \\
&\simeq \sqrt{\det\left(-\frac{1}{f}\mathbf{1}\right)} \int \mathcal{D}\sigma \; e^{\frac{1}{2f}\int \sigma^2} e^{\frac{1}{2}\left\langle \left(\int (J+\sigma)\mathcal{O}\right)^2 \right\rangle_0} , \\
&= \sqrt{\det\left(-\frac{1}{f}\mathbf{1}\right)} \int \mathcal{D}\sigma \; e^{\frac{1}{2f}\int \sigma^2} e^{\frac{1}{2}\int \left(\sigma\hat{G}\sigma + 2\sigma\hat{G}J + J\hat{G}J\right)} , \\
&= \sqrt{\det\left(-\frac{1}{f}\mathbf{1}\right)} \int \mathcal{D}\sigma \; e^{\frac{1}{2}\int \sigma\left(\hat{G}+\frac{1}{f}\right)\sigma + \int \sigma\hat{G}J + \frac{1}{2}\int J\hat{G}J} , \\
&= \frac{1}{\sqrt{\det\hat{K}}} e^{-\frac{1}{2}\int J\hat{G}\frac{f}{K}\hat{G}J + \frac{1}{2}\int J\hat{G}J} , \\
&= \frac{1}{\sqrt{\det\hat{K}}} \exp\left(\frac{1}{2}\int J\frac{\hat{G}}{1+f\hat{G}}J\right) .
\end{aligned} \tag{4.2}$$

where we have introduced the following shorthand notations

$$(\hat{G}\sigma)(x) = \int d^d y \; \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_0 \; \sigma(y) , \quad \text{and} \quad \hat{K} = 1 + f\hat{G} . \tag{4.3}$$

In the above manipulation we have used the identities

$$\int \mathcal{D}\varphi \; e^{-\frac{1}{2}\int \varphi A\varphi + \int \rho\varphi} = \frac{1}{\sqrt{\det A}} e^{\frac{1}{2}\int \rho A^{-1}\rho} \;, \tag{4.4}$$

and

$$\left\langle e^{\int \sigma\mathcal{O} + \int J\mathcal{O}} \right\rangle_0 \simeq e^{\frac{1}{2}\left\langle \left(\int (J+\sigma)\mathcal{O}\right)^2\right\rangle_0} \;. \tag{4.5}$$

The last identity is a result of large N factorization of correlation functions. It can be proved as follows. In the large-N limit, all odd-point functions vanish whereas all even-point functions decomposes into products of two-point functions. This implies that

$$\left\langle \left(\int \mathcal{O}\right)^{2n} \right\rangle = (2n-1)!! \left\langle \left(\int \mathcal{O}\right)^{2} \right\rangle^{n} + \mathcal{O}(1/N) \;, \tag{4.6}$$

where $(2n-1)!!$ is the number of ways of partitioning $2n$ objects into n pairs. Then, using the identity $(2n)! = 2^n \; n! \; (2n-1)!!$, one can easily derive (4.5).

From the last line in (4.2) we obtain the generating functional (defined as $W_f[J] = \log \mathcal{Z}_f[J]$). From this we can compute the two-point function in the presence of the double-trace perturbation

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_f = \frac{\delta^2 W_f[J]}{\delta J(x) J(0)} \;. \tag{4.7}$$

It is easier to calculate this by expressing the kernel $\hat{G}$ in momentum space

$$G(k) = \int d^d x \frac{e^{ik\cdot x}}{x^{d/2-\nu}} = 2^{2\nu}\pi^{d/2} \frac{\Gamma(\nu)}{\Gamma\left(\frac{d}{2}-\nu\right)} \frac{1}{k^{2\nu}} \equiv \frac{A_\nu}{k^{2\nu}} \;. \tag{4.8}$$

using which we can obtain the two-point function in momentum space

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_f = \frac{G(k)}{1+fG(k)} \; \delta^d(0) = \frac{A_\nu}{k^{2\nu}+fA_\nu} \; \delta^d(0) \;. \tag{4.9}$$

Now we can explore the UV ($k\to\infty$) and IR ($k\to 0$) limits. In the UV limit we have

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_{f=0} = \frac{A_\nu}{k^{2\nu}} \; \delta^d(0) \;, \tag{4.10}$$

which corresponds to an operator of scaling dimension $\Delta_- = d/2-\nu$, as expected. In the IR limit, we expand (4.9), and pick the leading non-local term in the momentum

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_{f=\infty} = \frac{1}{f}\left(1-\frac{k^{2\nu}}{fA_\nu}+\ldots\right) \; \delta^d(0). \tag{4.11}$$

From the leading non-local contribution we read that this two-point function corresponds to an operator of scaling dimension $\Delta_+ = d/2+\nu$. Although the sign of the leading non-local term is negative in momentum space, it is indeed positive in position space for $D_- = d/2-\nu$,

$0 \leq \nu \leq 1$. Few remarks are in order. First of all, the power law behaviour is the prime evidence for the existence of a non-trivial IR fixed point. Second, we see that the dimension of the operator $\mathcal{O}$ has changed from Δ_- to $\Delta_+ = d - \Delta_-$. Moreover, $\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_{f=0}$ in the UV and $\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_{f=\infty}$ in the IR are related by a Legendre transformation. This can be seen by making use of the Hubbard-Stratonovich auxiliary field σ. From the first line in (4.2) we see that the double-trace perturbation can be written in terms of σ as follows

$$\frac{1}{2f}\int d^dx\ \sigma^2 + \int d^dx\ \sigma\mathcal{O}\ . \tag{4.12}$$

From the equation of motion of σ one sees that it is related to the operator $\mathcal{O}$ by: $\sigma = f\mathcal{O}$. Next, by performing the path integral in the undeformed CFT, we can derive an effective action for σ. At large N we have

$$\left\langle e^{\int \sigma\mathcal{O}}\right\rangle_0 \approx \exp\left(\frac{1}{2}\int d^dx\ d^dy\ \sigma(x)\sigma(y)\langle\mathcal{O}(x)\mathcal{O}(y)\rangle_0 + O(\sigma^3)\right)\ , \tag{4.13}$$

so the quadratic term in the effective action for σ is

$$\begin{aligned} S[\sigma] &= -\frac{1}{2f}\int d^dx\ \left(\sigma^2 - \frac{1}{2}\int d^dx\ d^dy\ \sigma(x)\sigma(y)\langle\mathcal{O}(x)\mathcal{O}(y)\rangle_0\right)\ , \\ &= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\ \sigma(k)\sigma(-k)\left(\frac{1}{f} + G(k)\right)\ . \end{aligned} \tag{4.14}$$

From this effective action one can study the two-point function of σ in the small k limit and conclude that the Hubbard-Stratonovich auxiliary field σ is an operator of dimension Δ_+. We can view the field σ as an IR image of the UV operator under the RG evolution and $S[\sigma]$ as the effective action of IR correlators. Next, define a field $\widetilde{\sigma} = \sigma/f$, rewrite $S[\sigma]$ in terms of $\widetilde{\sigma}$ and consider the Legendre transform of S w.r.t. $\widetilde{\sigma}$

$$S'[\widetilde{\sigma}] = S[\widetilde{\sigma}] + \int \frac{d^dk}{(2\pi)^d} J(p)\widetilde{\sigma}(-p)\ , \qquad J(p) = -\frac{\delta S[\widetilde{\sigma}]}{\delta\widetilde{\sigma}}\ . \tag{4.15}$$

To proceed we write S' in terms of the sources J (by introducing the source term and integrating out σ) and add a term $\frac{1}{2f}J^2$ which contributes only a contact term that doesn't affect correlators at separated points. The resulting expression is identical to the exponent in last line of (4.2) where J was the source of the UV operator $\mathcal{O}$. Hence, we can view $S'[\widetilde{\sigma}]$ as the effective action of UV correlators. Then from (4.15) one can make the statement that the effective action of the UV correlators is a Legendre transform of the IR one.

Next we consider the second example of double-trace deformations by an operator of the type $f\mathcal{O}_1\mathcal{O}_2$. At the quantum level, deformation by this operator alone may not be consistent because it may generate $g_1\mathcal{O}_1$ and $g_2\mathcal{O}_2$. Therefore, we consider the most general double-trace deformation that one can write down with two single-trace operators,

$$f \int d^d x \; \mathcal{O}_1 \mathcal{O}_2 + \frac{g_1}{2} \int d^d x \; \mathcal{O}_1^2 + \frac{g_2}{2} \int d^d x \; \mathcal{O}_2^2, \tag{4.16}$$

The path integral of the deformed CFT under the presence of the sources reads:

$$\mathcal{Z}_f[J_1, J_2] = \left\langle e^{-f \int \mathcal{O}_1 \mathcal{O}_2 - \frac{g_1}{2} \int \mathcal{O}_1^2 - \frac{g_2}{2} \int \mathcal{O}_2^2 + \int J_1 \mathcal{O}_1 + \int J_2 \mathcal{O}_2} \right\rangle_0 . \tag{4.17}$$

We perform a Hubbard-Stratonovich transformation on this expression to decouple the interaction terms by introducing again auxiliary fields. First we write the interacting term as follows:

$$f \mathcal{O}_1 \mathcal{O}_2 \equiv f_1 f_2 \mathcal{O}_1 \mathcal{O}_2 = \frac{1}{2} \left((f_1 \mathcal{O}_1 + f_2 \mathcal{O}_2)^2 - f_1^2 \mathcal{O}_1^2 - f_2^2 \mathcal{O}_2^2 \right), \tag{4.18}$$

where f_1 and f_2 are such that

$$[f_1] = \frac{d}{2} - \Delta_1 \;, \quad [f_2] = \frac{d}{2} - \Delta_2. \tag{4.19}$$

So the path integral becomes:

$$\mathcal{Z}_f[J_1, J_2] = \left\langle e^{-\frac{1}{2} \int \left((f_1 \mathcal{O}_1 + f_2 \mathcal{O}_2)^2 - (f_1^2 - g_1) \mathcal{O}_1^2 - (f_2^2 - g_2) \mathcal{O}_2^2 \right) + \int J_1 \mathcal{O}_1 + \int J_2 \mathcal{O}_2} \right\rangle_0 . \tag{4.20}$$

Now we decouple the quadratic interactions by introducing three auxiliary fields σ_1, σ_2 and σ_3 in the following manner:

$$\begin{aligned}
\mathcal{Z}_f[J_1, J_2] = & \sqrt{\det \left(-\frac{1}{h_1 h_2} \mathbf{1} \right)} \int \mathcal{D}\sigma_1 \mathcal{D}\sigma_2 \mathcal{D}\sigma_3 \Big\langle \exp \left(\frac{1}{2} \int \left(\sigma_3^2 - \frac{\sigma_1^2}{h_1} - \frac{\sigma_2^2}{h_2} \right) \right. \\
& \left. + \int \sigma_3 (f_1 \mathcal{O}_1 + f_2 \mathcal{O}_2) + \int \sigma_1 \mathcal{O}_1 + \int \sigma_2 \mathcal{O}_2 + \int J_1 \mathcal{O}_1 + \int J_2 \mathcal{O}_2 \right) \Big\rangle_0 , \\
\simeq & \sqrt{\det \left(-\frac{1}{h_1 h_2} \mathbf{1} \right)} \int \mathcal{D}\sigma_1 \mathcal{D}\sigma_2 \mathcal{D}\sigma_3 \; \exp \left(\frac{1}{2} \int \left(\sigma_3^2 - \frac{\sigma_1^2}{h_1} - \frac{\sigma_2^2}{h_2} \right) \right) \times \\
& \exp \left[\frac{1}{2} \Big\langle \left(\int (\sigma_3 f_1 + \sigma_1 + J_1) \, \mathcal{O}_1 \right)^2 \Big\rangle_0 + \frac{1}{2} \Big\langle \left(\int (\sigma_3 f_2 + \sigma_2 + J_2) \, \mathcal{O}_2 \right)^2 \Big\rangle_0 \right] ,
\end{aligned} \tag{4.21}$$

where $h_i = f_i^2 - g_i$. The result of the path integral over $\sigma_1, \sigma_2, \sigma_3$ is

$$\mathcal{Z}_f[J_1, J_2] = \frac{1}{\sqrt{\det K_1 K_2 K_3}} \exp \left(\frac{1}{2} \int J_1 P_1 J_1 + \frac{1}{2} \int J_2 P_2 J_2 - \int J_1 P_3 J_2 \right) , \tag{4.22}$$

where we have defined the following kernels (below $i = 1, 2$ and G_i is defined as in (4.8))

$$K_i = 1 - h_i G_i \ , \qquad K_3 = 1 + f_1^2 Q_1 + f_2^2 Q_2 \ , \qquad Q_i = \frac{G_i}{1 - h_i G_i} \ , \tag{4.23}$$

and

$$P_1 = \frac{Q_1 + f_2^2 Q_1 Q_2}{K_3} = \frac{G_1 + g_2 G_1 G_2}{1 + g_1 G_1 + g_2 G_2 + (g_1 g_2 - f^2)\, G_1 G_2} \ , \tag{4.24}$$

$$P_2 = \frac{Q_2 + f_1^2 Q_1 Q_2}{K_3} = \frac{G_2 + g_1 G_1 G_2}{1 + g_1 G_1 + g_2 G_2 + (g_1 g_2 - f^2)\, G_1 G_2} \ , \tag{4.25}$$

$$P_3 = \frac{f Q_1 Q_2}{K_3} = \frac{f G_1 G_2}{1 + g_1 G_1 + g_2 G_2 + (g_1 g_2 - f^2)\, G_1 G_2} \ . \tag{4.26}$$

The product appearing in the determinant of (4.22) is

$$K_1 K_2 K_3 = 1 + g_1 G_1 + g_2 G_2 + \left(g_1 g_2 - f^2\right) G_1 G_2. \tag{4.27}$$

We consider the special case of classically marginal deformation where $g_1 = g_2 = 0$ and $\Delta_{\mathcal{O}_1} = \frac{d}{2} - \nu$ and $\Delta_{\mathcal{O}_2} = \frac{d}{2} + \nu$. In this case the product $G_1 G_2 = A_\nu A_{-\nu}$ (see (4.8) for the definition of A_ν). The denominator in the kernels $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ simplifes to $1 - f^2 G_1 G_2 = 1 - A_\nu A_{-\nu} f^2$. Define $\kappa^2 = -A_\nu A_{-\nu}$ (since the product $A_\nu A_{-\nu}$ is negative). Then, in terms of the quantity $\widetilde{f} = \kappa f$, the generating functional becomes

$$W[J_1, J_2] = \frac{1}{2} \frac{1}{1 + \widetilde{f}^2} \int \frac{d^d k}{(2\pi)^2} \Big(J_1(k) J_1(-k) G_1(k) + J_2(k) J_2(-k) G_2(k) \\ + 2 \widetilde{f} \kappa J_1(k) J_2(-k) \Big) \ , \tag{4.28}$$

which allows us to obtain correlation functions for generic f. The last term, being a contact term, can be ignored. The pre-factor $\frac{1}{1+\widetilde{f}^2}$ is a smoking gun of the non-perturbative duality under $\widetilde{f} \to 1/\widetilde{f}$ pointed out in [20], provided we rescale the sources by f. The physics at $f = 0$ and $f = \infty$ are identical. As a further evidence of this duality, we note that the partition function (4.22) (at vanishing sources), which depends upon $\widetilde{f}$ through the following function

$$-\log \mathcal{Z}[0, 0] = \frac{1}{2} \mathrm{tr} \, \log \left(-\frac{\kappa \widetilde{f}}{1 + \widetilde{f}^2} \mathbb{I} \right) \ , \tag{4.29}$$

is invariant under $\widetilde{f} \to 1/\widetilde{f}$.

Before we conclude this subsection it is worth mentioning that the generation function (4.22) can be studied for non-zero g_1 and g_2 and other operator dimension. Interesting RG flows and non-trivial IR fixed points can then be investigated for various choices of $\Delta_1, \Delta_2, g_1, g_1, f$.

In the next subsection we turn to the holographic aspects of the two examples discussed so far.

4.1.2 Holographic analysis

Let us first recall Witten's prescription for incorporating multi-trace deformations in AdS/CFT. Consider a scalar field ϕ of mass squared $m^2 = -d^2/4 + \nu$ propagating in the Poincaré patch of AdS_{d+1} with z being the radial coordinate. Near the boundary ($z = 0$), the scalar field behaves as (1.47)

$$\phi(z,k) \underset{z\to 0}{\sim} (\phi^-(k) z^{\frac{d}{2}-\nu} + \phi^+(k) z^{\frac{d}{2}+\nu})(1 + O(z^2)) \ .$$

In AdS/CFT we relate the mode ϕ^+ to the expectation value of the dual operator $\mathcal{O}$ of dimension $\Delta_+ = \frac{d}{2} + \nu$. In the presence of sources the precise relation can be written down as

$$\langle \mathcal{O} \rangle = (d - 2\Delta)\phi^+ \equiv \pi^+ \ , \tag{4.30}$$

where we have taken the contribution of the counterterm into the expectation value of $\mathcal{O}$. On the other hand, the mode ϕ^- is related to the source for $\mathcal{O}$. For later convenience we reiterate the relation between π^+ and ϕ^- (c.f. Eq. (1.48))

$$\pi^+ = \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \left(\frac{1}{2}\right)^{2\nu-1} \kappa^{2\nu}\phi^- \equiv a_\nu k^{2\nu}\phi^- \equiv G(k)\phi^- \ . \tag{4.31}$$

In large-N CFTs, where $\mathcal{O}$ is a single-trace operator, the computation of the expectation value of $\exp\left(-\int d^dx J(x)\mathcal{O}(x)\right)$ proceeds by specifying the following boundary condition on ϕ

$$\phi^-(x) = J(x) \ . \tag{4.32}$$

ϕ^- and π^+ are canonically conjugate variables. Computing the expectation value of $\exp\left(-\int d^dx J(x)\mathcal{O}(x)\right)$ is the same as computing the partition functional of the boundary CFT in the presence of the coupling $W = \int d^dx J(x)\mathcal{O}(x)$. Since the mode ϕ^+ is related, by the AdS/CFT correspondence, to the expectation value of $\mathcal{O}$, we can symbolically write the boundary coupling as $W(\pi) = \int d^dx \ J\pi^+$. Then it follows that the boundary condition (4.33) can be written as

$$\phi^- = \frac{\delta W}{\delta \pi^+} \ . \tag{4.33}$$

Multi-trace interactions arise if we take $W(\mathcal{O})$ to be a local but nonlinear functional of $\mathcal{O}$ and its derivatives. Witten's prescription for incorporating multi-trace interactions in the AdS/CFT correspondence is the following: replace $\mathcal{O}$ with π^+ in the non-linear function W and impliment the boundary condition (4.33). For instance, if we have double-trace deformation (4.1), where $W = \frac{f}{2}\int d^dx \mathcal{O}^2(x)$, we obtain the following boundary condition from (4.33)

$$\phi^- = f\pi^+ \ . \tag{4.34}$$

If we add a source term for $\mathcal{O}$ (as in (4.1)), $W = \frac{f}{2}\int d^dx\mathcal{O}^2(x) + \int d^dx J(x)\mathcal{O}(x)$, we obtain the following boundary condition from (4.33)

$$\phi^- = f\pi^+ + J \ . \tag{4.35}$$

We recall that if $0 \leq \nu < 1$, then it is possible to relate the mode π^+ to the source of the dual operator $\mathcal{O}$ of dimension $\Delta_- = \frac{d}{2} - \nu$ as

$$-\pi^+(x) = J(x) \ , \qquad \text{for} \qquad \Delta = \Delta_- \ , \tag{4.36}$$

and ϕ^- as its vacuum expectation value. The prescription for incorporating multi-trace deformations by such an operator is the same as (4.33) with $\phi^- \to -\pi^+$ and $\pi^+ \to \phi^-$. For instance, if we consider the double-trace deformation (4.1), where $W = \frac{f}{2}\int d^dx\mathcal{O}^2(x)$, we obtain the following boundary condition from (4.33)

$$-\pi^+ = f\phi^- \ , \tag{4.37}$$

and if we add a source term for $\mathcal{O}$ (as in (4.1)), $W = \frac{f}{2}\int d^dx\mathcal{O}^2(x) + \int d^dx J(x)\mathcal{O}(x)$, we obtain the following boundary condition from (4.33)

$$-\pi^+ = f\phi^- + J \ . \tag{4.38}$$

The renormalized on-shell action with Dirichlet boundary condition (4.32) on ϕ reads (c.f. (1.50, 1.53))

$$S_D[J = \phi^-] = \frac{1}{2}\int d^dx \ \pi^+\phi^- \ . \tag{4.39}$$

For a scalar with Neuman boundary condition (4.36) we have to add an additional boundary term to ensure that the bulk action is stationary. This gives

$$S_N[J = -\pi^+] = -\frac{1}{2}\int d^dx \ \pi^+\phi^- \ . \tag{4.40}$$

The two-point function that one obtains from S_N is

$$\begin{aligned} \langle\mathcal{O}(k)\mathcal{O}(-k)\rangle &= \frac{\delta\phi^-}{\delta\pi^+} = \frac{\Gamma(\nu)}{\Gamma(1-\nu)}\frac{2^{2\nu-1}}{k^{2\nu}}\delta^d(0) \ , \\ \Rightarrow \langle\mathcal{O}(x)\mathcal{O}(0)\rangle &= \frac{1}{2\pi^{\frac{d}{2}}}\frac{\Gamma\left(\frac{d}{2}-\nu\right)}{\Gamma(1-\nu)}\frac{1}{x^{d-2\nu}} \ . \end{aligned} \tag{4.41}$$

Hence, we see that the choice of the boundary condition (4.36) gives rise to a correlation function of an operator of scaling dimension $\frac{d}{2} - \nu$. Next we consider adding a double-trace deformation $\frac{1}{2}f\mathcal{O}^2$to this CFT. This gets implemented in (4.40) by the following modification

$$S_N^f[J] = -\frac{1}{2}\int d^dx \,\left(\pi^+\phi^- + f\left(\phi^-\right)^2\right) \,. \tag{4.42}$$

Making use of (4.38) we can recast this action into the following simple form

$$S_N^f[J] = \frac{1}{2}\int d^dx\ \phi^-(x)J(x)\ , \tag{4.43}$$

which gives rise to the two-point function $\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle = -\frac{\delta\phi^-}{\delta J}$. To evaluate this we need to know ϕ^- in terms of J. This relation can be obtained from (4.38) and is found to be

$$-J = \left(f + \frac{\Gamma(1-\nu)}{\Gamma(\nu)}\left(\frac{1}{2}\right)^{2\nu-1}k^{2\nu}\right)\phi^-\ , \tag{4.44}$$

and we finally obtain the generating functional

$$S_N^f[J] = -\frac{1}{2}\int d^dx\ J(x)\left(f + \frac{\Gamma(1-\nu)}{\Gamma(\nu)}\left(\frac{1}{2}\right)^{2\nu-1}k^{2\nu}\right)^{-1}J(x)\ , \tag{4.45}$$

which reproduces the same non-local behaviour in k as the one obtained from CFT analysis (c.f. last line of (4.2)).

Next we consider the holographic dual of the second example, where we deform the CFT by an operator to the type $W = \int f\mathcal{O}_1\mathcal{O}_2$. Since $\Delta_{\mathcal{O}_1} = \frac{d}{2} - \nu$ and $\Delta_{\mathcal{O}_2} = \frac{d}{2} + \nu$ and $\nu > 0$, we impose Dirichlet boundary conditions on the scalar ϕ_2 dual to $\mathcal{O}_2$ and Neumann boundary condition on the scalar ϕ_1 dual to $\mathcal{O}_1$. From Witten's prescription, we get the following boundary conditions (in the presence of sources)

$$\phi_2^- = f\phi_1^- + J_2\ , \qquad -\pi_1^+ = f\pi_2^+ + J_1\ . \tag{4.46}$$

On the other hand, the renormalized on-shell action is

$$\begin{aligned} S_f[J_1, J_2] &= \frac{1}{2}\int d^dx\left(\pi_2^+\phi_2^- - \pi_1^+\phi_1^- - 2f\phi_1^-\pi_2^+\right) \\ &= \frac{1}{2}\int d^dx\left(\pi_2^+J_2 + \phi_1^-J_1\right)\ , \end{aligned} \tag{4.47}$$

To proceed we need to know π_2^+ and ϕ_1^- in terms of the sources J_1 and J_2. Again this can be obtained from Eqs. (4.46) and (4.31). The expressions are

$$\pi_2^+ = -\frac{fJ_1G_2 - G_1G_2J_2}{G_1 + f^2G_2}\ , \qquad \phi_1^- = -\frac{J_1 - fG_2J_2}{G_1 + f^2G_2}\ , \tag{4.48}$$

where $G_1(k) = G_2(k) = a_\nu k^{2\nu} \equiv G(k)$. The final expression for the generating functional is

$$S_f[J_1, J_2] = -\frac{1}{2}\frac{1}{1+f^2}\int \frac{d^dk}{(2\pi)^d}\left(-J_2(k)G(k)J_2(-k) + J_1(k)\frac{1}{G(k)}J_1(-k) + 2fJ_1(k)J_2(-k)\right) . \tag{4.49}$$

This on-shell action is structurally almost the same as the generating functional derived from the field theory analysis (4.28). In particular, the f dependence is exactly the same. Moreover, for vanishing f this generating functional reproduce the two-point functions (1.54, 4.41). The boundary condition (4.46) is invariant under $f \to 1/f$ and $\phi_2^- \leftrightarrow \pi_2^+, \phi_1^- \leftrightarrow -\pi_1^+$. The latter is not a symmetry of the bulk theory because it is not a symmetry in the asymptotic expansion (1.47). Let us suppose that the bulk theory has a symmetry that exchanges ϕ_1 and ϕ_2. This symmetry was then broken by the boundary conditions (4.46). However, when accompanied with $f \leftrightarrow 1/f$, $\phi_1 \leftrightarrow \phi_2$ is indeed a symmetry, that scales the sources by f: $J_2 \to -fJ_2,\ J_1 \to fJ_1$. Also notice that with this rescaling of the sources, the on-shell action is invariant under $f \to 1/f$. Hence, the physics at $f \to 0$ and $f \to \infty$ are identical except for the exchange $\phi_1 \leftrightarrow \phi_2$. In the $f \to \infty$ limit, the scalar ϕ_1 acquires Dirichilet boundary condition and ϕ_2 acquires Neumann boundary condition. Hence, ϕ_1 has the quantization condition for it to be dual to operator $\mathcal{O}_2$ of dimension $\frac{d}{2} + \nu$ and ϕ_2 has the quantization condition for it to be dual to operator $\mathcal{O}_1$ of dimension $\frac{d}{2} - \nu$. This flip is possible because f is order 1 and the fact that the two operators are mixed by the deformation $f\mathcal{O}_1\mathcal{O}_2$.

4.2 Multiplet recombination in the context of AdS/CFT

Renormalization Group (RG) flow in Quantum Field Theory usually falls outside the regime of validity of perturbation theory. However, if an expansion parameter is available, like in the small-ϵ or the large-N expansion, it may become possible to follow operators from the UV to the IR fixed point, and have direct access to interesting phenomena induced by the RG-flow. One such example is *multiplet recombination*: a primary operator that saturates the unitarity bound at the UV fixed point recombines with another primary operator, i.e. the latter flows to a descendant of the first at the IR fixed point, and the two distinct conformal families get mapped into a single one.

Recently, multiplet recombination was used to reproduce, via simple CFT arguments, perturbative calculations of anomalous dimensions in the ϵ-expansion. This was first done for $O(N)$ scalar models in $4 - \epsilon$ dimensions [69], and later extended to the Gross-Neveu model in $2+\epsilon$ dimensions [70, 71]. In these examples, the short operator is a boson/fermion saturating the unitarity bound, which becomes long at the interacting fixed point due to its equations of motion.

We will consider multiplet recombination in large-N theories having a gravity dual description. Intuitively, holography should map multiplet recombination to a Higgs mechanism in the bulk. Indeed, when the protected operator is a conserved current that recombines due to a deformation that breaks the symmetry, the dual bulk gauge field is Higgsed and gets a mass. Here, we will discuss the case in which the protected operator is a free scalar ϕ, and couple it to a single-trace operator O of the large-N CFT via the interaction $\int d^dx\, \phi\, O$. We will see that also in this case there is a Higgs-like mechanism at work, albeit of a different kind, which exists only in AdS.

In order to study this problem, we find it useful to start considering two CFT single-trace operators (O_1, O_2) of dimension (Δ_1, Δ_2) with $\Delta_1 + \Delta_2 < d$, and thereafter take the decoupling limit $\Delta_1 \to \frac{d}{2} - 1$. The relevant double-trace deformation

$$\int d^dx\, f\, O_1 O_2 \ , \tag{4.50}$$

leads to an IR fixed point where (O_1, O_2) are replaced by two operators $(\tilde{O}_1, \tilde{O}_2)$ of dimension $(d - \Delta_1, d - \Delta_2)$, respectively. In the limit $\Delta_1 \to \frac{d}{2} - 1$, many terms in the low energy limit of the correlators become analytical in the momentum, and can be removed by appropriate counterterms. Focusing on the physical part of the correlators, we will find that in this case multiplets recombine. In particular, $\tilde{O}_1 \propto \Box \tilde{O}_2$, the IR dimensions being related as

$$\Delta_1^{IR} = \Delta_2^{IR} + 2 \ , \tag{4.51}$$

with $\Delta_2^{IR} = d - \Delta_2$.

In the bulk the interaction (4.50) gets mapped into a non scale-invariant boundary condition for the scalar fields (Φ_1, Φ_2) dual to (O_1, O_2) [20, 72] (see also [21, 22, 73–75]). These bulk scalars are free at leading order in $1/N$ expansion. The presence of the coupling f implies that the boundary modes of Φ_1 and Φ_2 get mixed. For O_1 and O_2 above the unitarity bound, the holographic analysis is standard, and the results agree with the field theory analysis. The limit $\Delta_1 \to \frac{d}{2} - 1$ should instead be treated with some care. One needs to rescale the field Φ_1, otherwise the normalization of the two-point correlator of O_1 would vanish. Doing so, one sees that the on-shell action for Φ_1 reduces to the action of a free scalar field living on the boundary of AdS, i.e. a singleton [76–80]. In the IR limit of the holographic RG-flow triggered by (4.50), the singleton gets identified with a boundary mode of Φ_2 corresponding to the VEV of the dual operator, i.e. the singleton becomes a long multiplet by eating-up the degrees of freedom of the bulk scalar.

The rest of the chapter is organized as follows. In section 4.3 we perform the large-N field theory analysis, and show that recombination takes place in the limit $\Delta_1 \to \frac{d}{2} - 1$. In section 4.4 we review the singleton limit in the bulk, and derive the holographic dual of the multiplet recombination flow. Section 4.5 contains a calculation of the variation of the quantity $\tilde{F}$ [81] induced by the flow (4.50), which shows that $\delta \tilde{F} = \tilde{F}_{UV} - \tilde{F}_{IR} > 0$, in agreement with the generalized F-theorem advocated in [81,82]. In section 4.6, we give our conclusions and make a few more comments related to previous works.

4.3 Large-N Multiplet Recombination: Field Theory

Consider a free scalar ϕ coupled to a large-N CFT through the interaction

$$\int d^dx f \phi O \ , \tag{4.52}$$

where O is a single-trace primary operator of dimension $\Delta < \frac{d}{2}+1$, so that the deformation (4.52) is relevant and triggers an RG-flow.

At leading order in the large-N expansion, one can integrate out the CFT sector and get the following non-local kinetic term for the scalar ϕ

$$\int d^dx\, f^2 \phi(-\Box)^{\Delta-\frac{d}{2}}\phi \ . \tag{4.53}$$

This term is dominant in the IR, indicating that ϕ flows to an operator of dimension $\Delta_\phi^{IR} = d-\Delta$. In fact, the equation of motion for ϕ tells that in the IR $O = f^{-1}\Box\phi$ becomes a descendant of ϕ with dimension $\Delta_O^{IR} = d-\Delta+2$. Therefore, in the IR O disappears from the spectrum of primary operators, multiplets recombine, and the short multiplet of ϕ becomes long.

As we will see, in order to understand the holographic dual phenomenon, it is useful to consider this flow as the limit of a double-trace flow induced by fO_1O_2 when the dimension of O_1 saturates the unitarity bound. In the following subsections we review this double-trace flow and show that multiplet recombination emerges in the limit.

4.3.1 Double-trace flow

Let us consider a large-N CFT deformed by the double-trace interaction

$$\int d^dx\, f\, O_1O_2 \ , \tag{4.54}$$

where (O_1, O_2) are single-trace primary operators of dimensions (Δ_1, Δ_2), with $\frac{d}{2}-1 < \Delta_{1,2} < \frac{d}{2}$. Without loss of generality we will take $\Delta_1 < \Delta_2$ in what follows. One can conveniently analyze the perturbed CFT

$$S = S_{CFT} + \int d^dx\, f\, O_1O_2 \ , \tag{4.55}$$

by introducing two Hubbard-Stratonovich auxiliary fields σ_1 and σ_2, and rewrite S as

$$S = S_{CFT} + \int d^d x \, \left(-f^{-1} \, \sigma_1 \sigma_2 + \sigma_1 O_1 + \sigma_2 O_2 \right) \ . \tag{4.56}$$

Integrating σ_1 and σ_2 out gives the following relations

$$\sigma_1 = f O_2 \, , \quad \sigma_2 = f O_1 \ , \tag{4.57}$$

which, once substituted back into (4.56), give the original action (4.55).

By performing the path integral in the CFT, one can derive an effective action for σ_1 and σ_2. To leading order at large N all correlators of O_1 and O_2 factorize in a product of two-point functions. The resulting non-local effective action for the auxiliary fields is

$$\begin{aligned} &S[\sigma_1, \sigma_2] = \\ &- \int d^d x \left(f^{-1} \, \sigma_1(x) \sigma_2(x) + \frac{1}{2} \sigma_1(x) \int \frac{d^d y}{(x-y)^{2\Delta_1}} \sigma_1(y) + \frac{1}{2} \sigma_2(x) \int \frac{d^d y}{(x-y)^{2\Delta_2}} \sigma_2(y) \right) \ . \end{aligned} \tag{4.58}$$

Given that Δ_1 and Δ_2 are smaller than $\frac{d}{2}$, the latter two terms dominate over the first, in the infrared. When only these terms are retained, σ_1 and σ_2 have IR correlators corresponding to operators with scaling dimension $d - \Delta_1$ and $d - \Delta_2$, respectively.

Substituting (4.57), we hence obtain the following operators at the IR fixed point

$$\tilde{O}_1 = f O_2, \quad \Delta_1^{IR} = d - \Delta_1 \ , \tag{4.59a}$$

$$\tilde{O}_2 = f O_1, \quad \Delta_2^{IR} = d - \Delta_2 \ . \tag{4.59b}$$

The above result shows that the IR fixed point is the same as the one reached via the double-trace deformation $g_1 O_1^2 + g_2 O_2^2$ [20]. This will be confirmed by the computation of the quantity $\tilde{F}$ [81] we do in section 4.5, where we show that the difference between the UV and IR values of $\tilde{F}$ induced by the flow (4.54) coincides with the one induced by the double-trace deformation $g_1 O_1^2 + g_2 O_2^2$.

4.3.2 Multiplet recombination

We now take $\Delta_1 = \frac{d}{2} - 1$, which means that O_1 decouples and becomes a free scalar, and consider again the perturbation (4.54) and the corresponding effective action (4.58). The kernel of the non-local quadratic action for σ_1 is now $\frac{1}{(x-y)^{d-2}}$, which is the inverse of the Laplace operator. By the local change of variable

$$\sigma_1' = \sigma_1 - f^{-1} \Box \sigma_2 \, , \quad \sigma_2' = \sigma_2 \ , \tag{4.60}$$

one can cancel the mixing term in the action (4.58), getting for the two-point function of σ'_1 just the contact term $\Box\delta^d(x-y)$. Therefore, the following operator equation holds

$$\sigma'_1 = 0 \Rightarrow \sigma_1 = f^{-1}\Box\sigma_2 \ . \tag{4.61}$$

Using (4.57), (4.59a) and (4.59b), we obtain the following operator relation at the IR fixed point

$$\tilde{O}_1 = f^{-1}\Box\tilde{O}_2 \ , \tag{4.62}$$

signaling that multiplets recombine, i.e. $\tilde{O}_1$ becomes a descendant of $\tilde{O}_2$. Recall from eq. (4.59b) that $\tilde{O}_2 = fO_1$ has dimension $d-\Delta_2$ and, by (4.62), $\tilde{O}_1 = fO_2$ has now dimension $d-\Delta_2+2$.

4.3.3 A more general flow

One might like to consider a more general double-trace deformation constructed out of O_1 and O_2, namely

$$\int d^dx \left(f\, O_1\, O_2 + \frac{g_1}{2}\, O_1^2 + \frac{g_2}{2}\, O_2^2 \right) \ , \tag{4.63}$$

and analyze the corresponding RG-flow.[1] Introducing again Hubbard-Stratonovich auxiliary fields one can recast the above action as

$$S = S_{CFT} + \int d^dx \left[-\frac{1}{2(f^2-g_1g_2)} \left(2f\sigma_1\sigma_2 - g_2\sigma_1^2 - g_1\sigma_2^2 \right) + \sigma_1 O_1 + \sigma_2 O_2 \right] \ . \tag{4.64}$$

Following the same steps as those of section 4.3.1, one ends-up with the following primary operators in the IR

$$\tilde{O}_1 = g_1 O_1 + fO_2, \quad \Delta_1^{IR} = d-\Delta_1 \ , \tag{4.65a}$$
$$\tilde{O}_2 = g_2 O_2 + fO_1, \quad \Delta_2^{IR} = d-\Delta_2 \ . \tag{4.65b}$$

This shows that the IR fixed point is the same one reaches via the simpler deformation (4.54), just the UV/IR operator dictionary is modified. So nothing qualitatively changes with respect to the previous analysis.

Here again, one can safely take the decoupling limit $\Delta_1 \to \frac{d}{2}-1$, getting a relation similar to (4.62), the proportionality coefficient being now a function of f, g_1 and g_2

$$\tilde{O}_1 = \frac{f}{(f^2-g_1g_2)}\, \Box\tilde{O}_2 \ . \tag{4.66}$$

So multiplets recombine also for this more general deformation. Notice that here the free operator is a massive one, its mass being proportional to g_1. Not suprisingly, for $f=0$

[1] A similar quadratic interaction involving several single-trace operators was studied recently in the context of large-N field theory in presence of disorder [83].

the deformation (4.63) does not trigger any multiplet recombination, as it is also clear from eq. (4.66). The (massive) free operator simply gets integrated out, while O_2 flows to an operator of dimension $d - \Delta_2$.

Let us note that had we chosen $\frac{d}{2} < \Delta_2 < d$, O_2^2 would have been an irrelevant deformation. This would not change much the story. Since one can always connect a CFT with $\Delta_2 > \frac{d}{2}$ to one with $\Delta_2 < \frac{d}{2}$ via an RG-flow with only g_2 turned on, there is no loss of generality in taking g_2 to be a relevant coupling, as we did from the outset.

As it is clear from eq.(4.64), the hypersurface in the parameter space described by the equation $f^2 - g_1 g_2 = 0$ needs a separate treatment. It is not difficult to see that in this case only one linear combination of the operators renormalizes, the IR dimensions of $\tilde{O}_1$ and $\tilde{O}_2$ being $d - \Delta_1$ and Δ_2, respectively (the symmetry in the exchange $g_1 \leftrightarrow g_2$ is broken by the fact that we have chosen $\Delta_1 < \Delta_2$). This is a different IR fixed point with respect to previous cases. Actually, the same fixed point one reaches by deforming the CFT by g_1 only. Also in this special case one can take the decoupling limit, $\Delta_1 \to \frac{d}{2} - 1$. Proceeding the same way as before, one can see that the dimensions of $\tilde{O}_1$ and $\tilde{O}_2$ are now $\Delta_2 + 2$ and Δ_2, respectively, indicating that multiplet recombination again holds. The IR fixed point is the same one reaches with a g_1 deformation only, which makes the free field disappearing from the IR spectrum, leaving only one primary of dimension Δ_2.

4.4 Large-N Multiplet Recombination: Holography

In this section we will analyze the large-N flows considered in the previous section from a dual holographic perspective. As already noticed, free operators of the QFT are dual to singleton representations of the AdS isometry group [76, 77] and some care is needed in dealing with them in the context of AdS/CFT. In particular, singletons do not enjoy any dynamics in the bulk. They correspond to propagating degrees of freedom only at the AdS boundary, and therefore the usual field/operator map should be properly interpreted. In what follows, we will first review how singletons can actually arise as a specific limit of ordinary bulk fields and how QFT correlators involving operators saturating the unitarity bound can then be consistently computed holographically using ordinary AdS/CFT techniques. This will enable us to provide a holographic realization of QFT RG-flows enjoying multiplet recombination at large N.

4.4.1 Singleton Limit

Consider a scalar Φ in AdS_{d+1} with mass $m^2 = \Delta(\Delta - d)$, and $\Delta = \frac{d}{2} - 1 + \eta$. Eventually, we will be interested in the limit $\eta \to 0$. To leading order at large N the scalar is free, and solving the Klein-Gordon equation we have the leading modes at the boundary

$$\Phi(z, x) \underset{z \to 0}{\sim} (\Phi^-(x) z^\Delta + \Phi^+(x) z^{d-\Delta})(1 + O(z^2)) \,, \tag{4.67}$$

where z is the radial coordinate that vanishes at the boundary and $x \in \mathbb{R}^d$. Since $\Delta < \frac{d}{2}$, the correct boundary condition is that $(d-2\Delta)\Phi^+(x)$ is fixed to coincide with the source of the operator of the boundary theory: $J(x) \equiv -(d-2\Delta)\Phi^+(x)$. This implies, in turn, that one needs to include an additional boundary term to ensure that the bulk action is stationary [84]. After this is done, the renormalized on-shell action consists of the following boundary term in momentum space

$$S^{\text{ren}}_{\text{on-shell}} = -\frac{1}{2}\int_{z=0} \frac{d^d k}{(2\pi)^d}\Phi^-[J(k)]J(-k) \ . \tag{4.68}$$

The solution to the Klein-Gordon equation with the prescribed boundary condition and regular for $z \to \infty$ is

$$\begin{aligned}\Phi(k,z)_{\text{on-shell}} &= \frac{1}{\Gamma(1-\frac{d}{2}+\Delta)}\left(\frac{k}{2}\right)^{\Delta-\frac{d}{2}} J(k) z^{\frac{d}{2}} K_{\frac{d}{2}-\Delta}(kz) \\ &\underset{\eta\to 0}{\sim} 2\eta\ k^{-1} J(k) z^{\frac{d}{2}} K_1(kz) \ ,\end{aligned} \tag{4.69}$$

where $K_{\frac{d}{2}-\Delta}(kz)$ is the modified Bessel's function of the second kind. From the form of the solution we see that

$$\begin{aligned}\Phi^-[J(k)] &= \frac{1}{2}\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(1-\frac{d}{2}+\Delta)}\left(\frac{k}{2}\right)^{2\Delta-d} J(k) \\ &\underset{\eta\to 0}{\sim} \frac{\eta}{2}\left(\frac{k}{2}\right)^{-2} J(k) \ .\end{aligned} \tag{4.70}$$

Recalling that the two-point function is minus the second derivative of the effective action with respect to the source, we find that

$$\langle O(k) O(-k)\rangle = \frac{1}{2}\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(1-\frac{d}{2}+\Delta)}\left(\frac{k}{2}\right)^{2\Delta-d} \underset{\eta\to 0}{\sim} \frac{2\eta}{k^2} \ . \tag{4.71}$$

This shows that in order to get a finite result in the limit $\eta \to 0$ we need to rescale the source $J(x)$ of the operator as $J(x) = \frac{1}{\sqrt{2\eta}}\hat{J}(x)$, with $\hat{J}(x)$ finite in the limit. In terms of the bulk scalar field, this amounts to rescaling $\Phi(x,z) = \sqrt{2\eta}\,\hat{\Phi}(x,z)$ with $\hat{\Phi}$ kept fixed. In this limit, the solution (4.69) goes to zero everywhere in the bulk, while the boundary term stays finite and becomes

$$S^{\text{ren}}_{\text{on-shell}} \underset{\eta\to 0}{\to} -\frac{1}{2}\int_{z=0} \frac{d^d k}{(2\pi)^d}\hat{J}(k)\frac{1}{k^2}\hat{J}(-k) \ . \tag{4.72}$$

This is the generating functional of a free scalar operator living on the boundary. Note that for $\eta \to 0$ we get $\hat{\Phi}^- = k^{-2}\hat{J}(k)$. We can identify the free scalar operator ϕ on the boundary as $\phi \equiv \hat{\Phi}^-$. In fact, if we Legendre-transform back from $\hat{J}$ to ϕ the boundary term becomes $\frac{1}{2}\int_{z=0}\frac{d^dk}{(2\pi)^d}\phi(k)k^2\phi(-k)$, i.e. the action of a free scalar.

4.4.2 Holographic Recombination Flow

We have now all ingredients to provide the holographic description of the large-N flows discussed in section 4.3. We start considering two primary operators of the CFT with dimensions $\Delta_{1,2} < \frac{d}{2}$. The CFT operators are dual to two scalar bulk fields Φ_1, Φ_2 having the following near boundary expansions

$$\Phi_1(z,x) \underset{z\to 0}{\sim} (\Phi_1^-(x)z^{\Delta_1} + \Phi_1^+(x)z^{d-\Delta_1})(1+O(z^2)) \ , \tag{4.73a}$$

$$\Phi_2(z,x) \underset{z\to 0}{\sim} (\Phi_2^-(x)z^{\Delta_2} + \Phi_2^+(x)z^{d-\Delta_2})(1+O(z^2)) \ . \tag{4.73b}$$

The deformation (4.54) is implemented by imposing the boundary condition [20]

$$J_1 \equiv (d-2\Delta_1)\Phi_1^+ + f\Phi_2^- \ , \tag{4.74a}$$

$$J_2 \equiv (d-2\Delta_2)\Phi_2^+ + f\Phi_1^- \ , \tag{4.74b}$$

where J_1 and J_2 are the sources for the field theory operators O_1 and O_2, respectively.

The solutions which are regular in the interior and have subleading boundary modes $\Phi_{1,2}^+$ are

$$\Phi_1(k,z)_{\text{on-shell}} = -N_{\Delta_1}k^{\Delta_1-\frac{d}{2}}(d-2\Delta_1)\Phi_1^+ z^{\frac{d}{2}}K_{\frac{d}{2}-\Delta_1}(kz) \ , \tag{4.75a}$$

$$\Phi_2(k,z)_{\text{on-shell}} = -N_{\Delta_2}k^{\Delta_2-\frac{d}{2}}(d-2\Delta_2)\Phi_2^+ z^{\frac{d}{2}}K_{\frac{d}{2}-\Delta_2}(kz) \ , \tag{4.75b}$$

where

$$N_\Delta = \frac{2^{\frac{d}{2}-\Delta}}{\Gamma(1-\frac{d}{2}+\Delta)} \ . \tag{4.76}$$

From the explicit expressions (4.75), we can read-off the coefficients $\Phi_{1,2}^-$, and obtain a linear relation between $\Phi_{1,2}^+$ and $\Phi_{1,2}^-$. We can plug this in (4.74) and solve for (Φ_1^-, Φ_2^-) as linear functions of (J_1, J_2). The solution is

$$\Phi_1^-[J_1,J_2] = \frac{J_1 - fJ_2G_2}{1-f^2G_1G_2}G_1, \qquad \Phi_2^-[J_1,J_2] = \frac{J_2 - fJ_1G_1}{1-f^2G_1G_2}G_2, \tag{4.77}$$

where

$$G_i(k) = -\frac{1}{2}\frac{\Gamma(\frac{d}{2}-\Delta_i)}{\Gamma(1-\frac{d}{2}+\Delta_i)}\left(\frac{k}{2}\right)^{2\Delta_i-d} . \tag{4.78}$$

Using standard techniques, one gets the following renormalized on-shell boundary action consistent with boundary conditions (4.74)

$$S^{\text{ren}}_{\text{on-shell}} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\ \left((d-2\Delta_1)\Phi_1^+\Phi_1^- + (d-2\Delta_2)\Phi_2^+\Phi_2^- + 2f\Phi_1^-\Phi_2^-\right)\ . \tag{4.79}$$

Using (4.77) this can be rewritten in terms of the sources as follows

$$\begin{aligned} S^{\text{ren}}_{\text{on-shell}}[J_1,J_2] = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\ \Bigg(J_1(k)\frac{G_1}{1-f^2G_1G_2}J_1(-k) + J_2(k)\frac{G_2}{1-f^2G_1G_2}J_2(-k) \\ -2J_1(k)\frac{fG_1G_2}{1-f^2G_1G_2}J_2(-k)\Bigg)\ . \end{aligned} \tag{4.80}$$

This expression is equivalent to the the field theory result (4.58). In order to see this, one needs to add the following local term to the above generating functional

$$S_{\text{local}} = -\int \frac{d^dk}{(2\pi)^d}\ J_1(k)\frac{1}{f}J_2(-k)\ , \tag{4.81}$$

and Legendre-transform. Identifying the Legendre-transformed fields with $(\frac{1}{f}\sigma_2, \frac{1}{f}\sigma_1)$ one gets precisely the Fourier transform of (4.58), provided we identify the field theory coupling defined in section 4.3 and the holographic coupling in the following manner

$$f^2_{\text{hol}} = 4\pi^d\frac{\Gamma(1-\frac{d}{2}+\Delta_1)\Gamma(1-\frac{d}{2}+\Delta_2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)}f^2_{\text{ft}}\ , \tag{4.82}$$

and pick the negative root for f_{hol} (it is generic in AdS/CFT that field theory couplings differ from holographic ones by such overall normalizations). After these identifications, one can repeat the analysis of section 4.3.1 and obtain eqs. (4.59).

In order to describe the phenomenon of multiplet recombination holographically, one has just to repeat the above analysis taking the singleton limit on the field Φ_1, first. One should hence set $\Delta_1 = \frac{d}{2} - 1 + \eta$, rescale the source of O_1 as $J_1 = \frac{1}{\sqrt{2\eta}}\hat{J}_1$, rescale also the coupling as $f = \frac{1}{\sqrt{2\eta}}\hat{f}$, and eventually take the limit $\eta \to 0$, with the hatted quantities kept fixed. Doing so, and repeating previous steps one gets, eventually, equation (4.62). Below, we find it instructive to adopt yet another (but equivalent) point of view. Instead of working with the effective action for (σ_1, σ_2) we will work with the generating functional (4.80) itself. After the singleton limit the on-shell action, analogous to (4.80), reads

$$\begin{aligned} S^{\text{ren}}_{\text{on-shell}} = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\ \Bigg(\hat{J}_1(k)\frac{-k^{-2}}{1+\hat{f}^2k^{-2}G_2}\hat{J}_1(-k) + J_2(k)\frac{G_2}{1+\hat{f}^2k^{-2}G_2}J_2(-k) \\ +2\hat{J}_1(k)\frac{\hat{f}k^{-2}G_2}{1+\hat{f}^2k^{-2}G_2}J_2(-k)\Bigg)\ . \end{aligned} \tag{4.83}$$

This action can be recast in the following way

$$S^{\text{ren}}_{\text{on-shell}} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\,\left((\hat{J}_1(k) + \frac{k^2}{\hat{f}}J_2(k)\right)\frac{k^{-2}}{1+\hat{f}^2k^{-2}G_2}\left(\hat{J}_1(-k) + \frac{k^2}{\hat{f}}J_2(-k)\right) \ , \qquad (4.84)$$

where certain contact terms have been dropped. We see that we are left with just one effective source $J_{eff}(x) = \hat{J}_1(x) - \frac{1}{\hat{f}}\Box J_2(x)$. Equivalently, the VEVs are related as

$$k^2\,\frac{\delta S^{\text{ren}}_{\text{on-shell}}}{\delta \hat{J}_1} = \hat{f}\,\frac{\delta S^{\text{ren}}_{\text{on-shell}}}{\delta J_2} \ . \qquad (4.85)$$

This equation shows that, as a result of the interaction f, the VEV mode of the bulk scalar gets identified with the singleton $\hat{\Phi}^-_1$, and its original VEV mode is now obtained by applying $\Box$ to $\hat{\Phi}^-_1$. This is the hallmark signature of multiplet recombination. Notice, finally, that in the IR the behavior of the two-point function of the leftover primary operator is $\langle \tilde{O}_2\tilde{O}_2\rangle \propto k^{d-2\Delta_2}$, implying that at the IR fixed point we have a primary of dimension $\Delta^{IR} = d - \Delta_2$, in agreement with field theory analysis.

Summarizing, when $f = 0$ there are two independent modes, i.e. the singleton $\hat{\Phi}^-_1$, which is just a boundary degree of freedom, and an ordinary bulk scalar, Φ_2. They are associated to two independent sources, $\hat{J}_1(k)$ and $J_2(k)$. In contrast, at the IR AdS point, there exists only one independent source, $\hat{J}_1(k) - \frac{1}{\hat{f}}\Box J_2(k)$ and in turn only one scalar. Φ_2 and the singleton merge into one bulk scalar whose VEV mode is $\hat{\Phi}^-_1$.

4.5 Calculation of $\delta\widetilde{F}$ for the double-trace flow fO_1O_2

In this section, we investigate the physicality of the RG flow resulting from the double-trace deformation (4.55)[2]. It is well known that if an RG connects a unitary UV Conformal Field Theory (CFT) to a unitary IR CFT, then a certain positive quantity defined on the space of CFTs, decreases monotonically. This is the well known c- , F- and a-theorem in $d = 2, 3, 4$ respectively [86–90].

Motivated by the similarity of such monotonicity theorems in even and odd dimensions (both of them can be formulated in terms of sphere free energy) it was conjectured in [81] that these theorems are special cases of a more general one, valid in continuous range of dimension for which

$$\widetilde{F}_{\text{UV}}(d) > \widetilde{F}_{\text{IR}}(d) \ , \qquad (4.86)$$

where $\widetilde{F}(d) = -\sin(\pi d/2)F(d)$ and $F(d)$ is the sphere free energy which is defined as $F(d) = -\log Z^{\mathbb{S}^d}$. The quantity $\widetilde{F}$ which turns out to be a smooth function of d, interpolates between the a-anomaly coefficient in even dimensions and the sphere free energy in odd dimensions. Thus, the inequality (4.86) (dubbed as the "Generalized F-theorem" in [82])

[2] See also [85] for related discussion.

smoothly interpolates between the corresponding inequalities in even and odd dimensions. In [81] several examples were provided for which $\tilde{F}$ decreases towards the IR, suggesting a generalization of the a- and F-theorems to continuous dimensions. In particular, it was proven that the generalized F-theorem holds for double-trace deformations.

We want to compute the leading large-N variation of $\tilde{F}$ induced by the flow (4.50). We follow the methods of [74,81,91]. At leading order at large-N the sphere partition function depends on the deformation fO_1O_2 as

$$Z_f^{\mathbb{S}^d} = Z_0^{\mathbb{S}^d} \times \frac{1}{\sqrt{\det(1^{\mathbb{S}^d} - f^2 G_1^{\mathbb{S}^d} \star G_2^{\mathbb{S}^d})}} \ . \tag{4.87}$$

This can be derived from the equivalent of action (4.58) for the theory on $\mathbb{S}^d$, by performing the path integral over σ_1 and σ_2. $G_i^{\mathbb{S}^d}$ is the two-point function of O_i on the sphere of radius R

$$G_i^{\mathbb{S}^d}(x,y) = \frac{1}{(R\, s(x,y))^{2\Delta_i}} \ , \tag{4.88}$$

where s is the distance between the two points x, y induced by the round metric g on the sphere of radius 1. Moreover $1^{\mathbb{S}^d}_{x,y} = \frac{1}{R^d\sqrt{g(x)}}\delta^d(x-y)$ and $\star$ is the product

$$(G_1^{\mathbb{S}^d} \star G_2^{\mathbb{S}^d})(x,y) = \int_{\mathbb{S}^d} d^d z R^d \sqrt{g(z)} G_1^{\mathbb{S}^d}(x,z) G_2^{\mathbb{S}^d}(z,y) \ . \tag{4.89}$$

Taking the logarithm of (4.87) we have

$$\tilde{F}_f - \tilde{F}_0 = -\sin\left(\frac{\pi d}{2}\right)\frac{1}{2}\log\det(1 - (fR^{d-\Delta_1-\Delta_2})^2 s^{-2\Delta_1} \star s^{-2\Delta_2}) \ . \tag{4.90}$$

We want to compute the difference between the values in the deep UV and in the deep IR. Those are obtained by taking $fR^{d-\Delta_1-\Delta_2}$ to be 0 or ∞, respectively. We obtain

$$\begin{aligned} \delta_f \tilde{F} = \tilde{F}_f^{UV} - \tilde{F}_f^{IR} &= \sin\left(\frac{\pi d}{2}\right)\frac{1}{2}\log\det(s^{-2\Delta_1} \star s^{-2\Delta_2}) \\ &= \sin\left(\frac{\pi d}{2}\right)\frac{1}{2}\left(\log\det(s^{-2\Delta_1}) + \log\det(s^{-2\Delta_2})\right) \\ &\equiv \delta\tilde{F}_{\Delta_1} + \delta\tilde{F}_{\Delta_2} \ . \end{aligned} \tag{4.91}$$

Comparing (4.91) with eq. (3.4) in [81], we see that this coincides with the variation of $\tilde{F}$ induced by the deformation $g_1O_1^2 + g_2O_2^2$. This agrees with the fact that the deformations fO_1O_2 and $g_1O_1^2 + g_2O_2^2$ connect the same UV and IR fixed points. In [81,91] the logarithm of the functional determinant was evaluated via an appropriate regularization of the infinite sum, and the end result shown to be positive whenever $\frac{d}{2} - 1 < \Delta_i < \frac{d}{2}$. We refer to these papers for an explicit expression (see also [82]).

In the limit $\Delta_1 \to \frac{d}{2} - 1$, the part of $\delta_f \tilde{F}$ that depends on Δ_1 equals the value of $\tilde{F}$ for the CFT of a free scalar, and we have

$$\delta_f \tilde{F} = \tilde{F}_{\text{scalar}} + \delta \tilde{F}_{\Delta_2} \ , \tag{4.92}$$

which is again a positive quantity if $\frac{d}{2} - 1 < \Delta_2 < \frac{d}{2}$, since $\tilde{F}_{\text{scalar}} > 0$ [81]. This equation reflects the fact that along the flow the free scalar and the primary single-trace operator of dimension Δ_2 recombine, giving in the IR one primary single-trace operator of dimension $d - \Delta_2$.

The upshot is then that the generalized F-theorem holds for the double-trace deformation (4.55), and it does so also when multiplets recombine, which is further support of the physicality of the RG flow that leads to multiplet recombination.

4.6 Comments

So far we have described multiplet recombination involving scalar primaries induced by coupling a large-N CFT in d dimensions to a free sector. Working at leading order in $1/N$, we have described this phenomenon in field theory, and provided the holographic dual description. Let us comment on the relation with previous work on multiplet recombination in holography.

In the context of AdS$_5$/CFT$_4$, multiplet recombination involving spin-one primaries was studied in [92–95]. In that case the recombination is not due to an RG-flow, rather it occurs as one moves away from the free point $g_{YM} = 0$ of $\mathcal{N} = 4$ SYM on the line of the marginal coupling, and the higher-spin currents of the free theory get broken. Another instance of higher-spin multiplet recombination is the case of $O(N)$ vector models in AdS$_4$/CFT$_3$ [14, 96, 97]. The holographic dual description consists of a Higgs mechanism for the higher-spin gauge fields dual to the higher-spin currents of the free theory. The Higgs mechanism happens at tree-level in the example of $\mathcal{N} = 4$ SYM, while it is a $1/N$ effect for the $O(N)$ vector models.

The crucial difference between these examples and our setting is that in these examples one starts with $N \gg 1$ free fields with a singlet condition, while we are considering only one free field. For this reason, in our setting there are no higher-spin gauge fields in the bulk. We only have higher spin currents associated to the singleton and supported on the boundary, and those are broken by the boundary condition.

A natural follow-up of our work would be to consider fermionic operators, along the lines of [98, 99], and study the analogous singleton limit and recombination in the bulk due to the boundary condition.

The idea of multiplet recombination has been applied extensively in the literature in various contexts, to compute anomalous dimensions [69–71, 100–104], to constrain the form of three-point functions [105], or to find exactly marginal deformations [106]. In these examples one works perturbatively in a small parameter that controls the breaking of the shortening condition. In the case we consider, instead, the recombination happens at leading order in $1/N$, so we cannot apply these methods to obtain more information about the IR fixed point. It would be interesting to consider a set-up in which the shortening is violated by a multi-trace operator with a suppressed coupling at large N, as would follow for instance from an interaction $\int d^dx \phi O^2$, and see if similar techniques could instead be used in that case. Another open problem is to try to use multiplet recombination to compute anomalous dimensions in the IR fixed point of QED in $d = 4 - 2\epsilon$ [107, 108].

In appendix D we give some applications of multiplet recombination to calculate anomalous dimensions in ϕ^4 theory and symmetry-breaking deformations of the $O(N)$ vector model in $d = 4 - \epsilon$ dimensions. In the next chapter instead we study multiplet recombination involving spin-one single-trace conformal primary operators and use it to calculate anomalous dimensions of broken currents.

Chapter 5

On Conformal deformations in AdS/CFT

Consider a d-dimensional conformal field theory (CFT) denoted by $\mathcal{P}_0$. Suppose that the theory contains a non-empty set of exactly marginal operators O_i. By exactly marginal we mean that if we deform $\mathcal{P}_0$ by

$$\int d^d x \; g^i O_i \; , \tag{5.1}$$

then there are no β-functions whatsoever for the couplings g_i. The theory obtained after the deformation is a new CFT $\mathcal{P}_1$ different from $\mathcal{P}_0$. The family of CFTs obtained in this way is referred to as the conformal manifold. It is a true manifold in the sense that it can be endowed with a topology. The maximal dimension of this topological space is equal to the total number of exactly marginal operators present in the spectrum of primary operators at the point $\mathcal{P}_0$.

Let us endow the CFT at the point $\mathcal{P}_0$ with a global symmetry group G. Then among the set of primary operators there exists spin-one conserved currents J^a_μ that lie in the adjoint representation of G and constitute a short multiplet[1] of the conformal group with the shortening condition being the conservation law

$$\partial^\mu J^a_\mu = 0 \; , \tag{5.2}$$

From this equation it follows that J^a_μ is at the unitarity bound, its dimension being $\Delta_J = d - 1$.

If there are exactly marginal operators that carry non-trivial representations of G then there exists regions of the conformal manifold where G is broken explicitly. In this region the spin-one operators J^a_μ are no longer conserved and constitute a long multiplet of the conformal group. The dimension of J^a_μ is lifted along with the marginally irrelevant operator

[1] By short multiplet of the conformal group we mean that some of the descendants (here $\partial^\mu J^a_\mu$) are absent in the conformal family of J^a_m

$T^a_{ij}g^iO^j$ which becomes a descendant of J^a_μ in the $\mathcal{P}_1$ CFT. This is evident from the operator relation

$$\partial^\mu J^a_\mu = T^a_{ij}g^iO^j \ . \tag{5.3}$$

Contrast this with the situation in the $\mathcal{P}_0$ CFT where both $T^a_{ij}g^iO^j$ and J^a_μ are primary operators and therefore sit in different representations of the conformal group. This is an instance of the phenomenon of multiplet recombination discussed in the previous chapter, applied to the case of spin-one operator.

As already mentioned in section 4.6, this phenomenon can be used to determine properties of the $\mathcal{P}_1$ CFT in terms of the data at $\mathcal{P}_0$. For small deformations, one can calculate the leading order corrections to the anomalous dimension of J^a_μ from the knowledge of correlators in the $\mathcal{P}_0$ CFT [100]. From eq. (5.3) it follows that the scaling dimension of the current in the deformed CFT is $\Delta_J > d-1$, meaning that some anomalous dimension has been generated by the exactly marginal deformation:

$$\Delta_J = d - 1 + \gamma \ , \tag{5.4}$$

where γ is positive by unitarity. If the $\mathcal{P}_0$ CFT is a free theory then the leading order anomalous dimension can be computed using just Wick contractions. An example of this sort is the $\mathcal{N}=4$ SYM where the free point is a part of the conformal manifold. However, for interacting, possibly strongly coupled CFTs, field theory methods do not suffice. This is where AdS/CFT techniques find their virtue. This will allow us to compute the anomalous dimension of broken currents in a class of $\mathcal{N}=1$ superconformal field theories (SCFT) arising from D-branes at toric Calabi-Yau singularities which admit symmetry breaking exactly marginal deformations known as β-deformations [109, 110].

In the next section we explain the method used to calculate the leading order anomalous dimension. We will be quite general in our exposition and include both marginal and relevant deformation in the discussion. In section 5.2, we will review some basic facts regarding the nature of exactly marginal deformations and global symmetries in 4-dimensional $\mathcal{N}=1$ SCFTs. We also comment upon its holographic dual. In section 5.3 we present a simple toy example to demonstrate the methods developed in section 5.1. We then discuss β-deformed $\mathcal{N}=1$ SCFTs in section 5.4. We conclude in section 5.5 with a list more examples of current multiplet recombination.

5.1 Methods - Field theory and holography

Computing exact anomalous dimensions of an arbitrary operator is, in principle, very difficult. However, for operators (in a CFT) that undergo multiplet recombination, the constraints imposed by conformal symmetry can be exploited.

Let us first suppose that the global symmetry is broken weakly (either by exactly marginal or relevant deformation). By weak breaking we mean that the CFT with broken symmetries can be made parametrically near to the symmetry-preserving one. Let us rewrite eq. (5.3) (ignoring the global symmetry indices) as

$$\partial^\mu J_\mu = g\, O \ . \tag{5.5}$$

For the case of exactly marginal deformations, we can make g as small as we like (because g does not run). For the case of relevant deformations g should be understood as g_*, the value of the coupling at the IR fixed point.

In such a situation the anomalous dimension of J can be determined, to *leading* order in g, by computing the two-point functions of O and J_μ in the unperturbed CFT. This idea is quite old [100, 101] but has been used and studied in relatively recent works [69, 102–104]. The basic idea goes as follows.

In a CFT, the structure of two-point functions of primary operators is fixed, up to an overall normalization, by conformal invariance. In particular, we have for a spin-one operator

$$\langle J_\mu(x) J_\nu(y)\rangle \ = \ C_J\, \frac{I_{\mu\nu}}{(2\pi)^d\,(x-y)^{2\Delta_J}}, \qquad I_{\mu\nu} = \delta_{\mu\nu} - 2\frac{(x-y)_\mu\,(x-y)_\nu}{(x-y)^2} \ . \tag{5.6}$$

This relation is a consequence of conformal symmetry and holds for any spin-one operator in a CFT. The operator dimension Δ_J as well as C_J will be different in the undeformed and the deformed CFT, the difference being proportional to g.

Differentiating twice the correlator (5.6) one gets

$$\langle \partial^\mu J_\mu(x) \partial^\nu J_\nu(y)\rangle = C_J\, \frac{2(2\Delta_J+2-d)(\Delta_J+1-d)}{(2\pi)^d (x-y)^{2\Delta_J+2}} \ . \tag{5.7}$$

By the operator identity (5.5) the same two-point function is given by

$$\langle \partial^\mu J_\mu(x) \partial^\nu J_\nu(y)\rangle = g^2 \langle O(x) O(y)\rangle \ . \tag{5.8}$$

By taking the ratio with (5.6), using (5.4), one gets

$$g\,(x-y)^2 I_{\mu\nu} \frac{\langle O(x) O(y)\rangle}{\langle J_\mu(x) J_\nu(y)\rangle} = 2\gamma\,(d+2\gamma) \ . \tag{5.9}$$

The above equation shows that in computing current anomalous dimension one needs to know the correlators to one order less in perturbation theory in g. In particular, to get γ to leading order in g one needs the value of the two-point functions of J_μ and of O at zeroth order, namely in the undeformed theory where O is not a descendant of J_μ but is a primary operator and its two-point function has the usual structure

$$\langle O(x)O(y)\rangle \ = \ \frac{C_O}{(2\pi)^d\,(x-y)^{2\Delta_O}}\ , \tag{5.10}$$

where C_O is some normalization. Plugging this expression into eq. (5.9) and using (5.6) one gets, upon expanding in powers of the coupling g (note that eq. (5.5) implies that $\Delta_O = \Delta_J + 1$)

$$\gamma = \frac{1}{2d}\,g^2\,\frac{C_O}{C_J} + O(g^4)\ , \tag{5.11}$$

with C_J and C_O evaluated in the undeformed theory, namely at $g = 0$.

This method is powerful because it allows to get information on the deformed CFT by just doing computations in the undeformed one. In practice, however, there are two limitations. First, as already emphasized, the perturbative expansion (5.11) makes sense only if the symmetry is weakly broken. If this is not the case, the above strategy cannot be applied and one should resort to some other method. Second, computing the two-point functions of O and J_μ, and hence the exact proportionality coefficient in eq. (5.11), is straightforward only if the undeformed CFT is a weakly coupled theory. In such a case one has to deal with correlators at tree-level and there are no issues of regularization and renormalization. A different story is if the original CFT is an interacting, possibly strongly coupled theory, e.g. emerging from some non-trivial gauge theory dynamics. In this situation AdS/CFT techniques can be employed (for field theories with a holographic dual).

In AdS/CFT, global currents in the boundary CFT are dual to gauge fields in AdS and the corresponding mass/dimension relation, in units of the AdS radius is

$$m^2 = d - 1 + \Delta_J(\Delta_J - d)\ . \tag{5.12}$$

From Eq. (5.12) it follows that massless gauge fields are dual to conserved currents, and massive ones to non-conserved currents. Therefore, when two CFTs are related by a symmetry-breaking deformation the gauge field dual to the (broken) current is massless in the vacuum dual to the undeformed CFT, and massive in that dual to the deformed CFT. Indeed, as known since the early days of the AdS/CFT correspondence, the breaking of a field theory global symmetry (be it explicit, like in the present case, or spontaneous) corresponds to a Higgs mechanism in the bulk, by which a massless vector eats-up a scalar and becomes massive. This is the bulk counterpart of the dynamics which governs current multiplet recombination [2]. Therefore, to compute current anomalous dimensions holographically, one has to calculate the mass of the dual gauge field and plug the result into eq. (5.12). Note

[2]See [2] for a holographic description of scalar multiplet recombination, and [14,92–97] for that of higher spin currents. These are both described by a Higgs-like mechanism in the bulk, though of a different nature in the two cases.

that this provides the anomalous dimension at face value so it also applies to long RG flows, i.e. when g_* cannot be tuned to zero. In section D.3 of Appendix D, we will discuss an instance of this kind in the context of 5-dimensional $\mathcal{N}=2$ gauged supergravity. Another situation in which AdS/CFT techniques can be useful is when the breaking is weak but the undeformed CFT is itself at strong coupling, and therefore computing at $g=0$ is itself non-trivial. In this case, one can evaluate the two-point functions $\langle J_\mu(x)J_\nu(y)\rangle$ and $\langle O(x)O(y)\rangle$ entering eq. (5.9) holographically. The β-deformed SCFTs we will discuss later are one such examples.

5.2 On exactly marginal deformations and global symmetries

The existence of exactly marginal deformations is difficult to establish, and for a generic CFT they do not exist, in general. However, as shown originally by Leigh and Strassler [109], and further elaborated by e.g. [106, 110–112], four-dimensional $\mathcal{N}=1$ SCFTs often enjoy non-trivial conformal manifolds.

Suppose we have a SCFT with some global symmetry group G and a bunch of marginal chiral operators $\mathcal{O}_i$ carrying some non-trivial representation of G. Deforming the theory by a G-breaking marginal superpotential $\mathcal{W}=\sum_i g^i\mathcal{O}_i$, an RG flow is induced since, generically, the operators $\mathcal{O}_i$ acquire an anomalous dimension.[3] In fact, marginal operators may either remain marginal or become marginally irrelevant, but never marginally relevant [106]. A space of exactly marginal operators exists, in general, and near the origin, namely around $g_i=0$, it is described by the quotient

$$\mathcal{M}_c=\{g_i|D^a=0\}/G \quad \text{with} \quad D^a=g^iT^a_{ij}\bar{g}^j \ . \tag{5.13}$$

Equivalently, $\mathcal{M}_c=\{g_i\}/G^{\mathbb{C}}$, where $G^{\mathbb{C}}$ is the complexified broken symmetry group. To summarize, the conformal manifold is parametrized by all uncharged operators (which trivially satisfy the constraint $D^a=0$ and are hence exactly marginal by themselves) plus all G-inequivalent linear combinations of charged, classically marginal operators $\mathcal{O}_i$ satisfying the constraint (5.13).

There can exist submanifolds of $\mathcal{M}_c$ where only a subgroup $H\subset G$ of the global symmetries is preserved. Along such submanifolds, current multiplets belonging to the complement of H in G recombine. These are the submanifolds we will be interested in.

The holographic dual of $\mathcal{M}_c$ is the moduli space of AdS vacua parametrized by constant scalars [113–116]. To see this we we recall that exactly marginal operators are dual to massless scalar fields in the bulk (which is evident from the mass/dimension relation $m^2=\Delta(\Delta-d)$). The near-boundary expansion of a massless scalar in Poincaré coordinates is $\phi(z,x)\sim\phi_0(x)+\phi_2(x)z^2+\mathcal{O}\left(z^4\right)$. The non-normalizable mode ϕ_0 of the scalar field is

[3] In a SCFT there do not exist marginal Kähler deformations [106]. Therefore, marginal deformations are described by superpotential deformations.

the source for the marginal operator, and corresponds to a deformation (5.1) in the dual field theory. In other words ϕ_0 is mapped to g under the AdS/CFT duality. The conformal manifold $\mathcal{M}_c$ is hence mapped into the moduli space $\mathcal{M}$ of AdS vacua which are AdS solutions of bulk equations of motion parametrized by massless, constant scalar fields.

5.3 Abelian toy model

As discussed above, the existence of exactly marginal deformation, and in turn of a conformal manifold, is a generic property of supersymmetric field theories. Hence we consider a four-dimensional $\mathcal{N}=1$ SCFT admitting a $U(1)$ global symmetry, and assume there exists n chiral primary (classically) marginal operators $\mathcal{O}_i$ with charge q_i under $U(1)$. A generic symmetry breaking deformation can be described by the following action

$$S = S_{\text{SCFT}} + \sum_i \int d^4x\, g_i O_i + h.c.\ , \tag{5.14}$$

where O_i are the F-components of the chiral superfields $\mathcal{O}_i$ and g_i are complex couplings.

The submanifold of $\mathcal{M}_c$ along which the $U(1)$ symmetry is broken is described by the D-term-like equation

$$\sum_{i,=1}^{n} q_i\, g_i \bar{g}_i = 0\ , \tag{5.15}$$

modulo $U(1)$ transformations. There exist $n-1$ non-trivial solutions of the above equation, in general. Let us dub O a linear combination of operators O_i which solves eq. (5.15)

$$O = g_1 O_1 + g_2 O_2 + \cdots + g_n O_n\ . \tag{5.16}$$

This is an exactly marginal deformation. Hence, if perturbing the original SCFT with $\mathcal{W} = g\,\mathcal{O}$, one describes yet another SCFT, which is parametrically near to the original one as $g \to 0$. In the deformed SCFT the $U(1)$ symmetry is broken and the $U(1)$ current is not conserved

$$\text{SCFT}_0 : \quad \partial_\mu J^\mu = 0 \quad , \quad \text{SCFT}_g : \quad \partial_\mu J^\mu \neq 0\ . \tag{5.17}$$

The minimal number of marginal operators which can provide non-trivial solutions of eq. (5.15) is two. In the following, we will then consider, for definiteness, $i = 1, 2$. In this case there exists a one-dimensional subspace in the space of couplings which corresponds to an exactly marginal deformation, described by the equation

$$q_1\,|g_1|^2 + q_2\,|g_2|^2 = 0\ , \tag{5.18}$$

modulo $U(1)$ transformations. The general solution is $g_1 = \sqrt{-q_2/q_1}e^{i\phi}g_2 \equiv g$, with ϕ an arbitrary phase.[4] Within this set we can choose a convenient representative. Upon a $U(1)$ rotation

$$O_1 \to e^{iq_1\alpha}O_1 \quad , \quad O_2 \to e^{iq_2\alpha}O_2 \ . \tag{5.19}$$

Choosing $\alpha = \phi/(q_2 - q_1)$, and fixing for definiteness $q_1 = -q_2 \equiv q$ we get for the representative

$$O_+ \equiv O_1 + O_2 \ , \tag{5.20}$$

and the symmetry breaking SCFT is described by the action

$$S'_{\text{SCFT}} = S_{\text{SCFT}} + \int d^4x \, gO_+ + h.c. \ . \tag{5.21}$$

Note that once this parametrization is chosen, any combinations of O_1 and O_2 not proportional to O_+ itself, will be marginally irrelevant (in particular the operator $O_- \equiv O_1 - O_2$).

By Noether method one can compute the current (non) conservation equation which reads

$$\partial_\mu J^\mu = iq \ g \, O_- + h.c. \ . \tag{5.22}$$

The fact that O_- is (marginally) irrelevant nicely agrees with Δ_J being bigger than 3 whenever $g \neq 0$.

To leading order in g the anomalous dimension of the current J_μ can be computed following the approach reviewed in section 5.1. The result is

$$\gamma = \frac{1}{4}q^2|g|^2\,\frac{C_{O_-}}{C_J} + O(g^4) \ , \tag{5.23}$$

where C_{O_-} is the normalization of the two point function $\langle O_- O_-^\dagger \rangle$.[5] Here C_{O-} and C_J are to be evaluated at $g = 0$, so are data of the undeformed SCFT.

For interacting CFTs it may happen that the coupling λ governing their dynamics is itself exactly marginal and the free limit, $\lambda = 0$, is part of the conformal manifold (this is the case for $\mathcal{N} = 4$ SYM, which we will consider later). If a holographic description is available, one could then compute eq. (5.23) for small and large values of λ, and compare. In general, one should expect different answers for γ at small and large λ. A simplification is that the coefficients entering eq. (5.23) are to be evaluated at $g = 0$. At any λ, the symmetry is preserved for $g = 0$ and, for a conserved current, the coefficient C_J of the two-point function does not renormalize.[6] On the contrary, nothing like this is expected

[4] Note that from eq. (5.18) it follows that q_1 and q_2 should have opposite sign.

[5] The discrepancy in the numerical coefficient with eq. (5.11) is because the deformation considered here is complex, compare eq. (5.22) with eq. (5.5).

[6] This is because in a SCFT the coefficient C_J of the two-point function of a conserved non-R current is nothing but the cubic 't Hooft anomaly between the superconformal R-current and the current $\text{Tr}\,(T_R T_J T_J)$ itself [117, 118]. As such, it does not depend on λ.

to hold for the operator O_- and so for C_{O_-}, in principle. In fact, supersymmetry can also protect C_{O_-}, sometime, as we will see later.

5.4 β-deformed superconformal field theories

D3-branes at conical Calabi-Yau (CY) singularities, that is real cones over Sasaki-Einstein manifolds X_5, provide a large class of $\mathcal{N} = 1$ SCFT with holographic duals, the dual geometry being $AdS_5 \times X_5$. The most studied examples are toric CY, which are CY for which X_5 admits at least a $U(1)^3$ isometry group. Of these three abelian factors, one (that associated to the Reeb vector) corresponds to the superconformal R-symmetry. The other two are flavor symmetries of the dual field theory.

For any toric CY singularity there always exists a supersymmetric, exactly marginal deformation preserving the $U(1)^3$ symmetry [110]. This is known as β-deformation. It may happen that X_5 has an enlarged isometry group $H \supset U(1)^3$. In this case, the β-deformation triggers current multiplet recombination since by β-deforming the theory the flavor group H is broken to $U(1)_R \times U(1)^2$ and several currents are not conserved anymore.[7] This is the class of models of interest to our present analysis.

In what follows we will discuss three such examples: the β-deformed $\mathcal{N} = 4$ SYM, the β-deformed conifold theory and the β-deformed $Y^{p,q}$ theories. In the first case $H = U(1)_R \times SU(3)$.[8] For the conifold theory $H = U(1)_R \times SU(2) \times SU(2)$, while for $Y^{p,q}$ singularities $H = U(1)_R \times SU(2) \times U(1)$.

These models share many similarities, but there is one sharp difference: for $\mathcal{N} = 4$ the free theory is part of the conformal manifold. For the conifold and $Y^{p,q}$ theories, it is not [13]. Therefore, in the latter cases the only available tool to compute current anomalous dimension is AdS/CFT. In the β-deformed $\mathcal{N} = 4$ theory, instead, one can compute current anomalous dimensions both at weak and strong coupling.

In preparation for what we do next, let us recall some basic results about the structure of the conformal manifold for these theories.

On conformal manifolds of toric Calabi-Yau singularities

The space of exactly marginal deformations of $\mathcal{N} = 4$ SYM is three-dimensional [109]. Besides the one associated to the complex gauge coupling, which preserves all flavor symmetries, there exist two $\mathcal{N} = 1$ preserving deformations: the β-deformation, which preserves a $U(1)^2$ of the flavor symmetry group, and the so-called cubic deformation, which breaks the flavor symmetry group fully. We will be interested in the β-deformation, which is generated

[7] Note that exactly marginal deformations do not break conformal symmetry and therefore always preserve the superconformal R-current.

[8] From a $\mathcal{N} = 1$ perspective the $SU(4)$ R-symmetry group of $\mathcal{N} = 4$ SYM should be seen as $U(1)_R \times SU(3)$, with the abelian factor being the $\mathcal{N} = 1$ R-symmetry and $SU(3)$ a flavor symmetry.

by the superpotential

$$\mathcal{W}_\beta = \lambda_\beta \,\mathrm{Tr}\left(\Phi_1\Phi_2\Phi_3 + \Phi_1\Phi_3\Phi_2\right) \;, \tag{5.24}$$

where Φ_i are the three adjoint chiral superfields of the $\mathcal{N}=4$ vector multiplet and transform in the $\mathbf{3}$ of $SU(3)$.

The SCFT describing the dynamics of D3-branes at the tip of the conifold (a CY with $X_5 = T^{1,1}$ whose topology is $S^3 \times S^2$) [10] is a four-dimensional $\mathcal{N}=1$ superconformal gauge theory with gauge group $SU(N)\times SU(N)$, a flavor symmetry group $SU(2)\times SU(2)$, bi-fundamental matter and a quartic superpotential

$$\mathcal{W} = \lambda_{KW}\, \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\mathrm{Tr}\left(A_\alpha B_{\dot\alpha} A_\beta B_{\dot\beta}\right) \;, \tag{5.25}$$

where α and $\dot\alpha$ are flavor indices, corresponding to the two $SU(2)$ factors, respectively. The fields A_α transform in the $(\frac{1}{2},0)$ of the flavor symmetry group $SU(2)\times SU(2)$. The $B_{\dot\alpha}$ transform instead in the $(0,\frac{1}{2})$.

The conformal manifold of the conifold theory is a five-dimensional space [110]. Two exactly marginal deformations, parametrized by suitable functions of the superpotential coupling λ_{KW} and the sum and difference of the inverse gauge coupling squared [10], are invariant under $SU(2)\times SU(2)$. The other three break the flavor symmetry group. As already emphasized, an important difference with respect to $\mathcal{N}=4$ SYM is that the free theory, $g_1 = g_2 = 0$, is not part of the conformal manifold [13]. This means that in computing eq. (5.11), there is no regime where a field theory, perturbative analysis applies.

Holographically, each exactly marginal deformation is associated to a massless excitation in the bulk. The dilaton and the B_2-flux over S^2 are dual to the flavor-singlet deformations. The flavor-breaking deformations are instead associated to excitations of KK modes. Of these, the β-deformation, which preserves a $U(1)^2$ flavor symmetry, corresponds to the following superpotential coupling

$$\mathcal{W}_\beta \;\;=\;\; \lambda_\beta \,\mathrm{Tr}\left(A_1 B_{\dot 1} A_2 B_{\dot 2} + A_1 B_{\dot 2} A_2 B_{\dot 1}\right) \;. \tag{5.26}$$

The conifold theory is in fact part of an infinite class of $\mathcal{N}=1$ SCFT which arises by considering D3-branes at CY singularities whose bases are the so-called $Y^{p,q}$ manifolds [119, 120]. These are Sasaki-Einstein manifolds with the same topology of the conifold (the conifold is nothing but a real cone over $Y^{1,0}$), but with different properties for generic p, q, *e.g.* the R-charges are irrationals [121, 122]. The flavor symmetry group is $SU(2)\times U(1)$, there are $2p$ $SU(N)$ gauge groups and $4p+2q$ bi-fundamental fields of four different types, U^α, V^α, Y and Z, with α an $SU(2)$ flavor index. The properties of these fields are summarized in appendix E.1. Finally, there is a superpotential with cubic and quartic couplings

$$\mathcal{W} = \sum_{i=1}^{q} \epsilon_{\alpha\beta} \operatorname{Tr}\left(U_i^\alpha V_i^\beta Y_{2i-1} + V_i^\alpha U_{i+1}^\beta Y_{2i}\right) + \sum_{j=q+1}^{p} \epsilon_{\alpha\beta} \operatorname{Tr}\left(Z_j U_{j+1}^\alpha Y_{2j-1} U_j^\beta\right) \ . \tag{5.27}$$

The conformal manifold is three-dimensional [110]. Two exactly marginal deformations are flavor singlets and correspond to the dilaton and the B_2-flux, as for the conifold. The third breaks the flavor group to $U(1)^2$ and is described by the superpotential coupling

$$\mathcal{W}_\beta = \lambda_\beta \operatorname{Tr}\left(\sum_{i=1}^{q} \sigma_{3\alpha}^{\ \beta} \left(U_i^\alpha V_{i\beta} Y_{2i+2} + V_i^\alpha U_{i+1\beta} Y_{2i+3}\right) + \sum_{j=q+1}^{p} \sigma_{3\alpha}^{\ \beta} Z_j U_{j+1}^\alpha Y_{2j+3} U_{j\beta}\right) \ , \tag{5.28}$$

where σ_3 is a Pauli matrix. As for the conifold theory, the free theory is not part of the conformal manifold.

By performing a β-deformation in the $\mathcal{N}=4$, conifold and $Y^{p,q}$ theories, several global currents acquire an anomalous dimension. Our aim will be to compute the leading correction to γ, eq. (5.11), where g here is λ_β and $\mathcal{O}$ are chiral primaries obtained acting with a flavor symmetry transformation on the operators (5.24), (5.26) and (5.28) respectively, at $\lambda_\beta = 0$. To this aim, we need to compute the two-point functions of these scalar operators (actually of their F-components) and of the corresponding broken currents at $\lambda_\beta = 0$. For the conifold and the $Y^{p,q}$ series this is a computation inherently at strong coupling, hence the only available tool is AdS/CFT. For $\mathcal{N}=4$ instead, one could evaluate the current anomalous dimension both at weak and strong coupling, since the free theory belongs to the conformal manifold in this case. However, for $\mathcal{N}=4$ well-known non-renormalization theorems ensure that, as far as eq. (5.11) is concerned, the weak and strong coupling results are the same: the two-point function one has to compute involves 1/2 BPS operators, and this is known not to renormalize [123](recall we have to evaluate at $\lambda_\beta = 0$). Therefore, in what follows we will treat all three cases holographically.

The gravity dual of β-deformed $\mathcal{N}=4$ SYM and more general toric singularities, including the conifold and the $Y^{p,q}$ series, was found in [124] (see also [125]). This will allow us to treat the three different cases somewhat together.

Broken currents anomalous dimensions

In an $\mathcal{N}=1$ SCFT with chiral superfields Φ_i the coefficient C_J appearing in (5.6), can be computed using the R-charges and flavor quantum numbers of fermions in the theory via the following t'Hooft anomaly [117]

$$C_J = 36 \sum_i (\dim R_i)\,(1-r_i) \operatorname{Tr}_i\left(T^a T^b\right) \ . \tag{5.29}$$

Here r_i are the R-charges of the chiral superfields and R_i the representation they transform under gauge symmetry transformations (R-charges of chiral superfields are reported in appendix E.1). The values of C_J for the various theories is presented in table 5.1. The non-abelian flavor symmetry generators are in the fundamental representation and are normalized as $\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$. Note that, consistently, the result for $Y^{p,q}$ theories is positive definite, hence satisfying unitarity, for $p \geq q \geq 0$, which is the range for which $Y^{p,q}$ manifolds are defined.

Theory	Flavor group	Current central charge: C_J
$\mathcal{N}=4$ SYM	$SU(3)$	$6(N^2-1)$
Conifold theory	$SU(2)\times SU(2)$	$9N^2$
$Y^{p,q}$ theories	$SU(2)$	$6N^2\left(5pq^2-4p^3+\left(2p^2-q^2\right)\sqrt{4p^2-3q^2}\right)/q^2$
	$U(1)$	$48N^2p^2\left(2p-\sqrt{4p^2-3q^2}\right)/q^2$

Table 5.1: Central charges for the non-anomalous global currents

Next we turn to the calculation of C_O. Since $\mathcal{O}$ and $\mathcal{W}_\beta$ lie in the same representation of the flavor group, up to a group theory factor (which for all cases we consider turns out to be 1) they have the same normalization. Therefore, the value of C_O is the same as the value of the corresponding C_{W_β} (which is nothing but the component of the Zamolodchikov metric along the corresponding modulus).[9] The two-point function for W_β can be extracted from the bulk effective action for the dual massless scalars β which is known to be [124]

$$S = -\frac{N^2}{16\pi^2 R_E^3}\int d^5x\sqrt{g}\left[C\frac{\partial_\mu\beta\,\partial^\mu\overline{\beta}}{\mathrm{Im}\,\tau}\right], \qquad \beta = \gamma - \tau\sigma\ , \tag{5.30}$$

where τ is the axio-dilaton, R_E is the radius of AdS and the normalization C depends on the geometry of the compact manifold X_5 and reads

$$C = \langle g_{0,E}\rangle \frac{\mathrm{Vol}(S^5)}{\mathrm{Vol}(X_5)}\ . \tag{5.31}$$

In the above expression $\langle g_{0,E}\rangle$ is the average value of the determinant of the metric on the internal 2-torus that geometrically realizes the $U(1)\times U(1)$ symmetry in the dual field theory. The values of $\langle g_{0,E}\rangle$ in the three cases is presented in appendix E.2. The two-point function for the marginal operator W_β that one derives from (5.30) is

$$\langle W_\beta(x) W_\beta^\dagger(0)\rangle = \frac{N^2}{(2\pi)^4}\frac{C}{\mathrm{Im}\,\tau}\frac{4!}{|x|^8}\ , \tag{5.32}$$

[9]Here W_β is the F-component of the superpotential $\mathcal{W}_\beta$.

assuming a bulk/boundary coupling with unit normalization, $\int d^4x \; \beta W_\beta + h.c.$.[10] This gives the value of C_{W_β} and in turn C_O

$$C_O = 24 g_s N^2 C \; . \tag{5.33}$$

In Table 5.2 we list the value of C_O for various theories under consideration.

Theory	C_O
$\mathcal{N}=4$ SYM	$24\pi g_s N^3$
Conifold theory	$45\pi g_s N^3/2$
$Y^{p,q}$ theories	$24\pi g_s N^3 p\left(7pq^2 - 8p^3 + \left(4p^2-2q^2\right)\sqrt{4p^2-3q^2}\right)/q^4$

Table 5.2: Normalization of two-point functions of the marginally irrelevant operators.

Plugging the value of C_O and C_J in eq. (5.11) (and remembering that these deformations are complex) we obtain the values of γ. Table 5.3 contains our results (to express these anomalous dimensions in terms of the field theory parameter λ_β, one should take into account that, following the conventions of [124], there is a $(g_s)^{1/2}$ difference between λ_β and β; therefore, the resulting anomalous dimensions scale just with N).

Theory	Broken Flavor group	Current anomalous dimension: γ
$\mathcal{N}=4$ SYM	$SU(3)$	$\pi g_s N|\beta|^2$
Conifold theory	$SU(2)\times SU(2)$	$5\pi g_s N|\beta|^2/8$
$Y^{p,q}$ theories	$SU(2)$	$\pi g_s N|\beta|^2 \frac{p}{q^2}\left(\frac{2q^4-4p^4+p^2q^2+\sqrt{4p^2-3q^2}\left(2p^3-pq^2\right)}{q^4-4p^4+3p^2q^2}\right)$

Table 5.3: Anomalous dimensions for the broken currents belonging to the non-Cartan elements of the flavor group.

The g_s and N dependence of current anomalous dimensions can be equivalently obtained from the mass of the dual bulk gauge field. To see this, it is sufficient to look at the $\mu-\alpha$ component of the Einstein's equation. Schematically, we have

$$R_{\mu\alpha} \supset -\frac{1}{48}\frac{|G_3|^2}{\mathrm{Im}\,\tau} g_{\mu\alpha} \; , \tag{5.34}$$

[10]This is suggested by the fact that both parameters are periodic with the same period [124].

where the holomorphic three-form flux in the $\beta-$deformed geometry takes the form [124]

$$G_3 = -(\gamma - \tau\sigma)\, R_E^4\; d\left(12\omega_1 \wedge d\psi + iG\omega_2\right)\ , \tag{5.35}$$

where $R_E^4 = 4\pi N$ with $\alpha' = 1$. The 2-forms $\omega_1 \wedge d\psi, \omega_2$ and the function G are different for different cases but the form (5.35) for G_3 is the same for $S^5, T^{1,1}$ and $Y^{p,q}$. This implies that $|G_3|^2 \propto |\gamma - \tau\sigma|^2 R_E^8 R_E^{-6} = |\gamma - \tau\sigma|^2 R_E^2$. The extra R_E^{-6} comes from the metric used for contracting the indices in $|G_3|^2$. The Maxwell operator is normalized with an additional factor of R_E^{-2}. Therefore, after canonically normalizing the Maxwell operator we see that the mass term is proportional to

$$m^2 \propto \frac{|\gamma - \tau\sigma|^2 R_E^4}{\operatorname{Im}\tau} = |\gamma - \tau\sigma|^2 4\pi N g_s \equiv 4\pi g_s N |\beta|^2\ . \tag{5.36}$$

It would be interesting to reproduce the exact coefficient by analyzing the fluctuation equations for the gauge fields in detail and see whether the result matches with those in table 5.3. In the undeformed background the (massless) gauge fields dual to conserved currents are degenerate and lie in adjoint representation of the isometry group of X_5. When the β-deformation is turned on the degeneracy is partially lifted, making some of the gauge fields (those that belong to the non-cartan elements) massive. The β-deformation turns on modes that have dependence on the X_5 coordinates. If the explicit form of vector spherical harmonics on X_5 were known, it would become possible to perform degenerate state perturbation theory and obtain the mass splitting to leading order in the deformation. In this sense, our results for the anomalous dimensions in Table 5.3 give a prediction for bulk gauge field masses in the deformed background, to leading order in β.

5.5 Other examples

Current multiplet recombination put severe constraints on CFT data. For example, we have seen that for marginal deformations anomalous dimensions of weakly broken currents are fixed, to leading order, by the Zamolodchikov metric on the conformal manifold and by a global current central charge in the undeformed CFT.

In this chapter, we have considered deformations triggered by marginal deformations and shown that in all cases one can compute the anomalous dimension of broken currents. The techniques we have used can be applied to several other examples. Besides the β-deformation, the conformal manifold of $\mathcal{N} = 4$ SYM admits another symmetry breaking deformation, which breaks the flavor symmetry group fully, and which can be investigated field theoretically using perturbation theory. Note, also, that at generic points of the conformal manifold of $\mathcal{N} = 4$ SYM supersymmetry is broken from $\mathcal{N} = 4$ to $\mathcal{N} = 1$. The corresponding supersymmetry current operators acquire anomalous dimensions which one could also compute. Also the conifold theory, besides the β-deformation, admits two other exactly marginal deformations with different symmetry-breaking patterns.

We focused our attention on four-dimensional theories but there exist marginal deformations for SCFTs in three dimensions. An example is the β-deformation of the $\mathcal{N} = 6$ ABJM theory [126] which breaks the $SU(2) \times SU(2)$ flavor symmetry down to $U(1)^2$ [127] (and here, too, supersymmetry is partially broken).

Chapter 6

Summary

In this thesis, we have explored deformations of AdS/CFT in various concrete setups. We have studied various aspects of conformal vs non-conformal, SUSY preserving vs SUSY breaking, single-trace and double-trace deformations of conformal field theories that admit a holographic description. The original work of this thesis are documented in chapters 2-5. We highlight the major contributions below.

- In chapter 2, we used holography to study spontaneous breaking of supersymmetry in a strongly coupled $\mathcal{N} = 1$ theory. We presented a simpler derivation of supersymmetry Ward identities (originally derived in [31]) involving the supercurrent which encode the presence of the Goldstino, the massless mode associated to the breaking of supersymmetry.

- In chapter 3, we used holography to study non-supersymmetric vacua of the Klebanov-Strassler gauge theory which does not have a UV fixed point and therefore does not admit an asymptotically AdS gravity dual. We took the consistent truncation of the ten dimensional Type IIB supergravity on $T^{1,1}$ and restricted to a subsector involving only the metric and four scalar fields (along with their fermionic partners) that suffices for capturing the asymptotics in the gravity dual. One of the main results of this chapter is the holographic renormalisation of this subsector which required sorting out subtleties associated to the sources for dual operators. In particular, we calculate the renormalized one-point functions, which allows to derive Ward identities for both supercurrent and trace of the energy-momentum tensor. To derive such identities in this particular setup, it is imperative that we work at finite radial cut-off and identify sources for the dual operators with induced fields at cut-off surface. The fact that we were able to successfully use usual tools of AdS/CFT correspondence in this particular case, where the QFT does not admit a UV fixed point, is somewhat remarkable.

- In chapter 4, we used double trace deformation of a generic large-N CFT to study the phenomenon of multiplet recombination for scalar operators. We considered this phenomenon from the bulk perspective, by showing that a singleton in the UV theory can merge with a bulk scalar in the IR theory under double trace perturbations, in accordance with field theory expectations. This setting involves a very special limit of AdS/CFT correspondence, namely, where one of the operators saturate the unitarity bound. The computation is first done for a generic double-trace deformation (where both operators are above the unitarity bound), with the singleton limit taken at the end with appropriate rescalings. Multiplet recombination, in the context of holography, has mostly been discussed in the context of breaking of higher-spin currents. Our finding sheds light on the spin zero case as well.

- In the final chapter 5, we used the multiplet recombination technique to calculate leading order anomalous dimensions of broken currents due to symmetry breaking exactly marginal deformations in a wide variety of strongly coupled superconformal field theories that admit a holographic dual. Some original findings, in the context of symmetry breaking deformations of $O(N)$ vector models, are also presented in appendix D and has later been further studied in [128].

Appendix A

Details of Chapter 2

A.1 Equations of motion and leading asymptotics

In this section we give the equations of motion of both the bosonic and the fermionic fields that follow from the action (2.10) as well as the leading form of the their asymptotic solutions subjected to the boundary conditions of the background presented in the main text.

Bosonic sector

Equations of motion

$$\Box\phi = \frac{1}{12}\left(\sinh(4\phi) - 20\sinh(2\phi)\right) \ , \tag{A.1}$$

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + \frac{2}{3}g_{\mu\nu}\mathcal{V} \ . \tag{A.2}$$

Asymptotic solutions

In order to obtain the asymptotic solutions of the equations of motion it is necessary to pick a specific gauge for the metric. In the following gauge

$$ds_5^2 = dr^2 + \gamma_{\mu\nu}(r,x)dx^\mu dx^\nu \ , \tag{A.3}$$

where the canonical radial coordinate r is related to the coordinate z in (2.17) through $dr = -dz/z$. The leading asymptotics of the bosonic fields for any solution that asymptotes to the background (2.23) take the form

$$\begin{aligned}\phi(z,x) &\sim \phi_0(x)z \ , \\ \gamma_{\mu\nu}(z,x) &\sim \frac{1}{z^2}\left(\eta_{\mu\nu} + \gamma_{(0)\mu\nu}(x)\right) \ .\end{aligned} \tag{A.4a}$$

Fermionic sector

Equations of motion

$$\Gamma^M D_M \zeta - 2\Gamma^M \mathcal{F}_- \psi_M + \mathcal{M}\zeta = 0 \ , \tag{A.5}$$

$$\Gamma^{MNP} D_N \Psi_P - 4\mathcal{F}_+ \Gamma^M \zeta - m(\phi)\Gamma^{MN}\Psi_N = 0 \ . \tag{A.6}$$

Asymptotic solutions

In the gauge

$$\Psi_r = 0 \ , \tag{A.7}$$

the leading asymptotics of the fermions, for any bosonic solution that asymptotes to the background (2.23), take the form

$$\begin{aligned} \zeta^-(z,x) &\sim z^{3/2}\zeta_0^-(x) \ , \\ \Psi_\mu^+(z,x) &\sim z^{-1/2}\Psi_{(0)\mu}^+(x) \ . \end{aligned} \tag{A.8a}$$

Identification of the sources with dual operators

From the asymptotic solution we can read off the sources of the dual operators.

$$\phi_0 \leftrightarrow \mathcal{O}_F \ , \qquad \gamma_{(0)\mu\nu} \leftrightarrow T^{\mu\nu} \ , \tag{A.9}$$

$$\zeta_0^- \leftrightarrow \mathcal{O}_\psi \ , \qquad \Psi_{(0)\mu}^+ \leftrightarrow S^{-\mu} \ . \tag{A.10}$$

A.2 Local symmetries and transformation of the sources

The gauge-fixing conditions (A.3),(A.7) are preserved by Weyl rescalings and supersymmetry transformations. The transformation of the covariant sources under these gauge-preserving local transformations gives rise to the holographic Ward identities.

Weyl transformations

The transformation of the sources under Weyl transformations is

$$\begin{aligned} \delta_\sigma \gamma_{(0)\mu\nu} &= 2\sigma\gamma_{(0)\mu\nu} \ , & \delta_\sigma \Psi_{(0)\mu}^+ &= \frac{1}{2}\sigma\Psi_{(0)\mu}^+ \ , \\ \delta_\sigma \phi_0 &= -\sigma\phi_0 \ , & \delta_\sigma \zeta_0^- &= -\frac{3}{2}\sigma\zeta_0^- \ , \end{aligned} \tag{A.11}$$

Local supersymmetry transformations

The gauge fixing condition (A.7) on the gravitino leads to a differential equation for the supersymmetry parameter ϵ via eq. (2.16a), namely

$$\left(D_r + \frac{1}{3}m(\phi)\Gamma_r\right)\epsilon = 0\ , \tag{A.12}$$

In the gauged fixed metric (A.3) and projecting out the two chiralities, we have

$$\dot{\epsilon}^{\pm} \mp \frac{1}{3}m(\phi)\epsilon^{\pm} = 0\ . \tag{A.13}$$

The asymptotic solutions to these equations are

$$\begin{aligned} \epsilon^{+}(z,x) &\sim z^{-1/2}\epsilon_0^{+}(x)\ , \\ \epsilon^{-}(z,x) &\sim z^{1/2}\epsilon_0^{-}(x)\ , \end{aligned} \tag{A.14}$$

where the arbitrary spinors $\epsilon_0^{\pm}(x)$ parameterize respectively supersymmetry and superWeyl transformations on the boundary. The transformation of the covariant sources under these transformations is as follows.

Gravitino:

The transformation of the induced gravitino Ψ_μ under supersymmetry is

$$\delta_\epsilon \Psi_\mu = \left(\nabla_\mu + \frac{1}{3}m(\phi)\Gamma_\mu\right)\epsilon\ . \tag{A.15}$$

Projecting this equation on the positive chirality, which is the leading one asymptotically as follows from eq. (A.8) and which corresponds to the source of the supercurrent, we get

$$\delta_\epsilon \Psi_\mu^{+} = \partial_\mu \epsilon^{+} + \frac{2}{3}m(\phi)\Gamma_\mu \epsilon^{-}\ , \tag{A.16}$$

where we have used (2.20) in order to drop a term proportional to the VEV of the stress tensor (which is subleading asymptotically).

Metric:

The supersymmetry transformation of the vielbein e_μ^a is given by

$$\delta_\epsilon e_\mu^a = \frac{1}{2}\bar{\epsilon}\,\gamma^a\,\Psi_\mu + \text{h.c.}\,. \tag{A.17}$$

From this it follows that the corresponding variation of the induced metric is

$$\delta_\epsilon \gamma_{\mu\nu} = \bar{\epsilon}^{+}\Gamma_{(\mu}\Psi_{\nu)}^{+} + \bar{\epsilon}^{-}\Gamma_{(\mu}\Psi_{\nu)}^{-} + \text{h.c.}\ , \tag{A.18}$$

where the symmetrization is done with a factor of 1/2. From the leading terms, we obtain

$$\delta_\epsilon \gamma_{\mu\nu} = \bar{\epsilon}^+ \Gamma_{(\mu} \Psi^+_{\nu)} + \text{h.c.} \; . \tag{A.19}$$

Hypermultiplet sector:

The transformation of the fields in the hypermultiplet is

$$\delta_\epsilon \phi = -\frac{i}{2} \bar{\epsilon}^+ \zeta^- + \text{h.c.} \; , \qquad \delta_\epsilon \zeta^- = \frac{i}{2} \left(z\phi' + \frac{1}{2} \sinh(2\phi) \right) \epsilon^- \sim i\phi\epsilon^- \; . \tag{A.20}$$

Combining these results, we deduce that the covariant sources transform under ϵ^+ as

$$\begin{aligned} \delta_{\epsilon^+} \gamma_{\mu\nu} &= \bar{\epsilon}^+ \Gamma_{(\mu} \Psi^+_{\nu)} + \text{h.c.} \; , & \delta_{\epsilon^+} \Psi^+_\mu &= \partial_\mu \epsilon^+ \; , \\ \delta_{\epsilon^+} \phi &= -\frac{i}{2} \bar{\epsilon}^+ \zeta^- + \text{h.c.} \; , & \delta_{\epsilon^+} \zeta^- &= 0 \; , \end{aligned} \tag{A.21}$$

and under ϵ^- as

$$\begin{aligned} \delta_{\epsilon^-} \gamma_{\mu\nu} &= 0 \; , & \delta_{\epsilon^-} \Psi^+_\mu &= \Gamma_\mu \epsilon^- \; , \\ \delta_{\epsilon^-} \phi &= 0 \; , & \delta_{\epsilon^-} \zeta^- &= i\phi\epsilon^- \; . \end{aligned} \tag{A.22}$$

Appendix B

Consistent Truncation of Type IIB supergravity on $T^{1,1}$

In this appendix we give a brief summary of the consistent truncation of Type IIB supergravity on the conifold and the resulting 5-dimensional $\mathcal{N} = 2$ gauged supergravity theory. The theory we consider was obtained in [52, 53, 129, 130]. We then present the full 10-dimensional Klebanov-Strassler solution in terms of the $5D$ supergravity fields. We conclude by identifying the modes which are turned on in the solution and its interpretation in the dual field theory.

B.1 Consistent truncation

The $\mathcal{N} = 2$, type IIB supergravity contains the following bosonic degrees of freedom propagating in ten dimensional spacetime

$$\begin{array}{cc} \text{NS-NS sector} & \text{R-R Sector} \\ G_{\mu\nu}, B_{\mu\nu}, \phi & C_0, C_2, C_4 \end{array} \tag{B.1}$$

Each of these fields are functions of the ten dimensions coordinates. Consistent truncation proceed by expanding these field onto in a set of differential forms characterizing the geometric structure of $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$ coset space whose geometric properties is described in the next section. We adopt an ansatz for these 10D fields which retains all and only those modes of type IIB supergravity that are invariant under the action of $SU(2) \times SU(2)$. This automatically guarantees a consistent truncation because the truncated modes close under the action of the isometry group. In what follows, we present a basis of globally defined left-invariant differential forms on $T^{1,1}$ on which the Type IIB supergravity fields are expanded. We write down explicitly the truncation ansatz for the type IIB fields. Non-trivial cycles of the internal manifold can allow for additional terms in the expansion for the fluxes.

B.1.1 Geometry of the Conifold

The conifold Y_6 is a Ricci-flat Calabi-Yau space which can be viewed as a complex 3-dimensional surface embedded in $\mathbb{C}^4$ with the embedding given by:

$$\sum_i^4 z_i^2 = 0 \; . \tag{B.2}$$

It can can also be visualized as a cone over $T^{1,1}$ which is a homogenous space and is defined as the following coset

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} \; , \tag{B.3}$$

where the $U(1)$ is embedded diagonally in the two $SU(2)$'s; i.e; if σ_1^3 and σ_2^3 are the third generators of the two $SU(2)$ respectively then the $U(1)$ in (B.3) is generated by $\frac{\sigma_1^3+\Sigma_2^3}{2}$. Topologically this space is $S^2 \times S^3$ and can also be seen as a $U(1)$ fibration over $S^2 \times S^2$. This space admits an Einstein metric:

$$ds^2_{T^{1,1}} = a\left(d\psi + \sum_{i=1}^2 \cos\theta_i d\phi_i\right)^2 + b\sum_{i=1}^2\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) \tag{B.4}$$

that satisfies $R_{\alpha\beta} = 3g_{\alpha\beta}$ for $a = \frac{1}{9}$ and $b = \frac{1}{6}$. Here (θ_i, ϕ_i) parametrise the two 2-sphere as usual and ψ parametrizes the $U(1)$ fiber and ranges from $(0, 4\pi)$. In the following we provide the set of left-invariant differential forms on this space. These will be the building blocks of our truncation ansatz, presented in the next subsection.

We begin by defining the following one-forms on $T^{1,1}$

$$\begin{aligned}
e^1 &= -\sin\theta_1 d\phi_1, \\
e^2 &= d\theta_1, \\
e^3 &= -\sin\psi d\theta_2 + \cos\psi\sin\theta_2 d\phi_2 \\
e^4 &= \cos\psi d\theta_2 + \sin\psi\sin\theta_2 d\phi_2 \\
e^5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2
\end{aligned} \tag{B.5}$$

There is another useful basis obtained by an orthogonal transformation of (B.5):

$$\begin{pmatrix} g^1 \\ g^2 \\ g^3 \\ g^4 \\ g^5 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \\ e^5 \end{pmatrix} . \tag{B.6}$$

In these basis, the metric takes a particularly simple form:

$$ds^2_{T^{1,1}} = \frac{1}{9}\left(g^5\right)^2 + \frac{1}{6}\sum_{i=1}^{4}\left(g^i\right)^2 = \frac{1}{9}\left(e^5\right)^2 + \frac{1}{6}\sum_{i=1}^{4}\left(e^i\right)^2 \tag{B.7}$$

The volume enclosed by (a unit) $T^{1,1}$ is given by:

$$\int e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 = \frac{16\pi^3}{27}. \tag{B.8}$$

The left-invariant forms on the $T^{1,1}$ coset are spanned by the 1-form e^5 and the four 2-forms $e^{12}, e^{34}, e^{13}+e^{24}, e^{14}-e^{23}$, together with all their possible wedgings (here we use the short-hand notation e^{ij} for $e^i \wedge e^j$). We combine them into the following equivalent basis:

$$\begin{aligned}
&\eta = -\frac{1}{3}g^5 = -\frac{1}{3}e^5\ , && \Phi && =\frac{1}{6}\left(e^{12}+e^{34}\right) = \frac{1}{6}\left(g^{12}+g^{34}\right)\ ,\\
&J = \frac{1}{6}\left(e^{12}-e^{34}\right) = \frac{1}{6}\left(g^{14}-g^{23}\right)\ , && \Omega && =\frac{1}{6}\left(e^{13}+e^{24}-i\left(e^{14}-e^{23}\right)\right)\\
& && && =\frac{1}{6}\left(g^{13}+g^{24}+i\left(g^{12}-g^{34}\right)\right)\ .
\end{aligned} \tag{B.9}$$

These satisfy the algebraic conditions

$$\begin{aligned}
&\eta_\mu J^{\mu\nu} = \eta_\mu \Phi^{\mu\nu} = \eta_\mu \Omega^{\mu\nu} = 0\ ,\\
&d\eta = 2J,\quad d\,J = 0\ ,\qquad d\Omega = 3i\eta\wedge\Omega\ ,\qquad d\Phi = 0\ ,\\
&\Omega\wedge\Omega = \Omega\wedge J = \Omega\wedge\Phi = J\wedge\Phi = 0\ ,\qquad 2J\wedge J = \Omega\wedge\overline{\Omega} = -2\Phi\wedge\Phi\ .
\end{aligned} \tag{B.10}$$

It is useful to note some relations that these forms satisfy

$$\begin{aligned}
&\omega_2 = \frac{1}{2}\left(g^1\wedge g^2 + g^3\wedge g^4\right) = \frac{1}{2}\left(\sin\theta_1 d\theta_1 d\phi_1 - \sin\theta_2 d\theta_2 d\phi_2\right)\ ,\\
&\omega_3 = g^5\wedge\omega_2\ ,\qquad \mathrm{vol}(T^{1,1}) = \frac{1}{108}g^1\wedge g^2\wedge g^3\wedge g^4\wedge g^5\ ,\\
&\omega_2\wedge\omega_3 = 54\ \mathrm{vol}(T^{1,1})\ ,\qquad \int_{S^2}\omega_2 = 4\pi\ ,\qquad \int_{S^3}\omega_3 = 8\pi^2\ ,\\
&d(g^1\wedge g^3 + g^2\wedge g^4) = g^5\wedge\left(g^1\wedge g^2 - g^3\wedge g^4\right)\ ,\\
&d(g^1\wedge g^2 - g^3\wedge g^4) = -g^5\wedge\left(g^1\wedge g^3 + g^2\wedge g^4\right)\ ,\\
&dg^5\wedge dg^5 = -2g^1\wedge g^2\wedge g^3\wedge g^4\ ,\qquad \omega_2 = 3\Phi\ ,\qquad \omega_3 = -9\ \eta\wedge\Phi\ ,\\
&e^{1234} = g^{1234} = -18\ J\wedge J\ ,\qquad e^{12345} = g^{12345} = 54\ \eta\wedge J\wedge J\ .
\end{aligned} \tag{B.11a}$$

B.1.2 Reduction ansatz of Type IIB fields

We take the 10-dimensional spacetime to be a direct product space $M \times T^{1,1}$, where M is a 5-dimensional spacetime. The ansatz for the 10-dimensional metric (in Einstein frame) is

$$\begin{aligned} ds^2 &= e^{-\frac{2}{3}(4u+v)} g_{\mu\nu} dx^\mu dx^\nu + \frac{1}{6} e^{2u} \cosh t \left[e^{2w}(e^1e^1 + e^2e^2) + e^{-2w}(e^3e^3 + e^4e^4) \right] \\ &+ e^{2v}(\eta + A)^2 + \frac{1}{3} e^{2u} \sinh t \left[\cos\theta(e^1e^3 + e^2e^4) + \sin\theta(e^1e^4 - e^2e^3) \right] \end{aligned} \tag{B.12}$$

where x^μ are coordinates on the non-compact manifold M, whose metric is $g_{\mu\nu}(x)$. The dilaton and the RR axion are assumed independent of the internal coordinates: $\phi = \phi(x)$, $C_0 = C_0(x)$, and $F_1 = dC_0$. For the NSNS antisymmetric tensor B_2 and $H_3 = dB_2$ we have

$$B_2 = b_2 + b_1 \wedge (\eta + A) + b^J J + \mathrm{Re}(b^\Omega\, \Omega) + b^\Phi \Phi \ , \tag{B.13}$$

and for the corresponding field strength is

$$\begin{aligned} H_3 &= h_3 + h_2 \wedge (\eta + A) + h_1^J \wedge J + \mathrm{Re}\big[h_1^\Omega \wedge \Omega + h_0^\Omega\, \Omega \wedge (\eta + A) \big] \\ &\quad + h_1^\Phi \wedge \Phi + p\, \Phi \wedge (\eta + A) \ , \end{aligned} \tag{B.14}$$

where we have defined

$$\begin{aligned} h_3 &= db_2 - b_1 \wedge dA \,, & h_1^\Omega &= db^\Omega - 3iA\, b^\Omega \equiv Db^\Omega \ , \\ h_2 &= db_1 \,, & h_0^\Omega &= 3ib^\Omega \ , \\ h_1^J &= db^J - 2b_1 \equiv Db^J \,, & h_1^\Phi &= db^\Phi - p\, A \equiv Db^\Phi \, . \end{aligned} \tag{B.15}$$

The last term in (B.17) is due to the non-trivial three-cycle dual to ω_3. Similarly for the RR antisymmetric two-form C_2 and its gauge invariant field strength $F_3 = dC_2 - C_0 H_3$, we have:

$$C_2 = c_2 + c_1 \wedge (\eta + A) + c^J J + \mathrm{Re}(c^\Omega\, \Omega) + c^\Phi \Phi \ , \tag{B.16}$$

and for the corresponding field strength

$$\begin{aligned} F_3 &= g_3 + g_2 \wedge (\eta + A) + g_1^J \wedge J + \mathrm{Re}\big[g_1^\Omega \wedge \Omega + g_0^\Omega\, \Omega \wedge (\eta + A) \big] \\ &\quad + g_1^\Phi \wedge \Phi + q\, \Phi \wedge (\eta + A) \ , \end{aligned} \tag{B.17}$$

where

$$\begin{aligned} g_3 &= dc_2 - c_1 \wedge dA - C_0(db_2 - b_1 \wedge dA) \,, & g_1^\Omega &= Dc^\Omega - C_0 Db^\Omega \ , \\ g_2 &= dc_1 - C_0 db_1 \,, & g_0^\Omega &= 3i\left(c^\Omega - C_0 b^\Omega \right) \ , \\ g_1^J &= Dc^J - C_0 Db^J \,, & g_1^\Phi &= Dc^\Phi - C_0 Db^\Phi \ . \end{aligned} \tag{B.18}$$

Finally for the self dual 5-form field strength we have

$$
\begin{aligned}
F_5 \;=\;& f_5 + f_4 \wedge (\eta + A) + f_3^J \wedge J + f_2^J \wedge J \wedge (\eta + A) + \mathrm{Re}\left[f_3^\Omega \wedge \Omega + f_2^\Omega \wedge \Omega \wedge (\eta + A)\right] \\
&+ f_3^\Phi \wedge \Phi + f_2^\Phi \wedge \Phi \wedge (\eta + A) + f_1 \wedge J \wedge J + f_0\, J \wedge J \wedge (\eta + A)\,,
\end{aligned}
\tag{B.19}
$$

This ansatz contains redundant fields because the self duality of F_5 relates f_5, f_4, f_3^J, f_3^Ω, f_3^Φ to the lower forms f_0, f_1, f_2^J, f_2^Ω, f_2^Φ. These lower forms contain the independent degrees of freedom carried by the five form flux:

$$
\begin{aligned}
f_0 &= 3\,\mathrm{Im}(b^\Omega \overline{c^\Omega}) + p\, c^\Phi - q\, b^\Phi + k \\
f_1 &= Da + \frac{1}{2}\left(q\, b^\Phi - p\, c^\Phi\right) A + \frac{1}{2}\left[b^J\, Dc^J - b^\Phi\, Dc^\Phi + \mathrm{Re}(b^\Omega\, \overline{Dc^\Omega}) - \; b \leftrightarrow c\right] \\
f_2^J &= da_1^J + \frac{1}{2}\left[\, b^J dc_1 - b_1 \wedge Dc^J - \; b \leftrightarrow c\,\right], \\
f_2^\Omega &= Da_1^\Omega + 3ia_2^\Omega + \frac{1}{2}\left[b^\Omega dc_1 - b_1 \wedge Dc^\Omega + 3ic^\Omega b_2 - \; b \leftrightarrow c\right], \\
f_2^\Phi &= da_1^\Phi + \frac{1}{2}\left(q\, b_1 - p\, c_1\right) \wedge A + q\, b_2 - p\, c_2 + \frac{1}{2}\left[b^\Phi\, dc_1 - b_1 \wedge Dc^\Phi - \; b \leftrightarrow c\right]
\end{aligned}
\tag{B.20}
$$

B.1.3 The five-dimensional model

The 5-dimensional model resulting from the dimensional reduction of Type IIB supergravity on $T^{1,1}$ coset space gives rise to an $\mathcal{N} = 4$ gauged supergravity. By consistently turning off some of the $\mathcal{N} = 4$ multiplets, one finds a further truncation to an $\mathcal{N} = 2$ gauged supergravity which contains the Klebanov-Strassler solution. The resulting truncation contains apart from the gravity multiplet, a vector multiplet and three hyper multiplets. In total there are 13 scalar, 2 gauge fields and the metric in the bosonic part of the truncation. The action for the $\mathcal{N} = 2, D = 5$ gauged supergravity model takes the following form:

$$
S = S_{\mathrm{Kin}} + S_{\mathrm{Pot}} + S_{\mathrm{Top}}\ ,
$$

where the S_{Kin} part reads

$$
\begin{aligned}
S_{\mathrm{Kin}} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} \sqrt{-g_5}\ \Big(& R - \frac{28}{3} du^2 - \frac{4}{3} dv^2 - \frac{8}{3} du \lrcorner dv - dt^2 - \mathrm{sh}^2 t\ (d\theta - 3A)^2 \\
&- e^{-4u-\phi}\left[\mathrm{ch}\ 2t\, \left(h_1^\Phi\right)^2 + \mathrm{ch}^2 t\ |h_1^\Omega|^2 - \mathrm{sh}^2 t\ \mathrm{Re}\left(e^{-2i\theta}\left(h_1^\Omega\right)^2\right) + 2\mathrm{sh}\ 2t\ h_1^\Phi \lrcorner \mathrm{Re}\left(ie^{-i\theta} h_1^\Omega\right)\right] \\
&- e^{-4u+\phi}\left[\mathrm{ch}\ 2t\, \left(g_1^\Phi\right)^2 + \mathrm{ch}^2 t\ |g_1^\Omega|^2 - \mathrm{sh}^2 t\ \mathrm{Re}\left(e^{-2i\theta}\left(g_1^\Omega\right)^2\right) + 2\mathrm{sh}\ 2t\ g_1^\Phi \lrcorner \mathrm{Re}\left(ie^{-i\theta} g_1^\Omega\right)\right] \\
&- \frac{1}{2} d\phi^2 - \frac{1}{2} e^{2\phi} dC_0^2 - 2e^{-8u} f_1^2 - \frac{1}{2} e^{\frac{8}{3}(u+v)} (dA)^2 - e^{-\frac{4}{3}(u+v)} (da_1^J)^2 \Big)\ .
\end{aligned}
\tag{B.21}
$$

For clarity we mention that

$$du^2 = \partial_\mu u \partial^\mu u \quad , \qquad du \lrcorner\, dv = \partial_\mu u \partial^\mu v \ ,$$
$$|h_1^\Omega|^2 = h_\mu^\Omega \left(\overline{h^\Omega}\right)^\mu , \qquad \left(h_1^\Omega\right)^2 = h_\mu^\Omega \left(h^\Omega\right)^\mu \ . \tag{B.22}$$

The scalar manifold $\mathcal{M}$ describing the dynamics of the 13 scalars is

$$\mathcal{M} = SO(1,1) \times \frac{SO(3,4)}{SO(3)\times SO(4)} \ . \tag{B.23}$$

The potential terms that gives rise to masses and the cosmological constant reads

$$\begin{aligned}
S_{\text{Pot}} &= \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} (-2\mathcal{V}) \star 1, \\
&= \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \sqrt{-g_5} \ \Bigg(-4e^{-\frac{20}{3}u+\frac{4}{3}v} + 24 \text{ ch } t \ e^{-\frac{14}{3}u-\frac{2}{3}v} - 9e^{-\frac{8}{3}(u+v)}\text{sh}^2 t - 2e^{-\frac{8}{3}(4u+v)} f_0^2 \\
&- e^{-\frac{20}{3}u-\frac{8}{3}v-\phi}\left[\text{Re}\left(-e^{-2i\theta} \ \text{sh}^2 t \left(h_0^\Omega\right)^2\right) + 2i \ pe^{-i\theta}\text{sh}(2t) h_0^\Omega + \text{ch}^2 t \ |h_0^\Omega|^2 + p^2\left(1+2\text{sh}^2 t\right)\right] \\
&- e^{-\frac{20}{3}u-\frac{8}{3}v+\phi}\left[h\to g \ , \quad p\to (q-pC_0)\right]\Bigg) \ .
\end{aligned} \tag{B.24}$$

The potential gives rise to a masses for various 5D fields and is shown in Table 3.1 in chapter 3. Finally we have the topological interaction term is given by $S_{\text{Top}} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}} A \wedge da_1^J \wedge da_1^J$.

B.2 The deformed conifold and the 5D Klebanov-Strassler solution

B.2.1 The deformed conifold

The deformed conifold is a 3-dimensional complex space which is given by the following embedding in $\mathbb{C}^4$

$$\sum_i^4 z_i^2 = \epsilon^2 \ , \tag{B.25}$$

Here ϵ is the deformation parameter that prevents the three-cycle from shrinking to zero size at the tip of Y_6. This removes the conical singularity if the singular conifold presented in section B.1.1 and makes the geometry smooth. The metric of the deformed conifold is diagonal in the basis $\{g^1, g^2, g^3, g^4, g^5\}$ and is given by [28]:

$$\begin{aligned}
ds_6^2 =& \frac{1}{2}\epsilon^{4/3} K(\tau) \\
&\left(\frac{1}{3K^3(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2\left(\frac{\tau}{2}\right)\left((g^3)^2+(g^4)^2\right) + \sinh^2\left(\frac{\tau}{2}\right)\left((g^1)^2+(g^2)^2\right)\right) \ ,
\end{aligned} \tag{B.26}$$

where $K(\tau)$ is given by

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau} \underset{\tau\to 0}{\to} \left(\frac{2}{3}\right)^{1/3}$$
$$\underset{\tau\to\infty}{\sim} 2^{1/3}e^{-\tau/3}. \tag{B.27}$$

For large τ we can introduce another radial coordinate

$$r^2 = \frac{3}{2^{5/3}}\epsilon^{4/3}e^{2\tau/3}, \tag{B.28}$$

where the numerical factor in front is important to recover the metric of the singular conifold (eq. B.7) in the large τ limit. Later on it will be useful to have a $1 \oplus 5$ split of this metric:

$$ds_6^2 = \frac{\epsilon^{4/3}}{6K^2}d\tau^2 + ds_5^2 \tag{B.29}$$

where ds_5^2 in the basis (B.5) is:

$$ds_5^2 = \frac{1}{2}\epsilon^{4/3}K\left[\frac{1}{3K^3}\left(e^5\right)^2 + \frac{1}{2}\cosh\tau\left(\left(e^1\right)^2 + \left(e^2\right)^2 + \left(e^3\right)^2 + \left(e^4\right)^2\right) + \left(e^1e^3 + e^2e^4\right)\right]. \tag{B.30}$$

B.2.2 The 5D Klebanov-Strassler solution

We present the ansatz for finding the KS solution as written down in [28]. In the following F, f, k, l are function of the radial coordinate τ.

$$B_2 = g_sM(f\ g^{12} + k\ g^{34}) \tag{B.31a}$$
$$H_3 = dB_2 = g_sM\left(df\wedge g^{12} + dk\wedge g^{34} + \frac{1}{2}(k-f)\left(g^{513}+g^{524}\right)\right) \tag{B.31b}$$
$$F_3 = dC_2 - C_0H_3 = M\left((1-F)g^{345} + F\ g^{125} + dF\wedge\left(g^{13}+g^{24}\right)\right) \tag{B.31c}$$
$$F_5 = dC_4 + B_2\wedge F_3 = \left(N + g_sM^2l\right)\ g^{12345} \tag{B.31d}$$
$$ds_{10}^2 = h^{-1/2}(\tau)dx_\mu dx^\mu + h^{1/2}(\tau)ds_6^2. \tag{B.31e}$$

In the above, $l(\tau)$ is related to $F(\tau), k(\tau),$ and $f(\tau)$ by $l = f(1-F) + kF$. On this ansatz $H_{3\mu\nu\rho}F_3^{\mu\nu\rho} = 0$ holds identically. This inturn implies that the RR scalar C_0 is zero on this ansatz (this is because H_3 and dC_2 thread dual cycles thereby implying $H_3 \lrcorner dC_2 = 0$). Moreover we require $G_3 \equiv F_3 + \frac{i}{g_s}H_3$ to satisfy the imaginary self-dual (ISD) relation $\star_6 G_3 = i\ G_3$ which is required by supersymmetry. It then follows from the 10D equation of motion of the dilaton that ϕ is a constant.

Using the formulae in (B.11a) we new rewrite the above ansatz in terms of the differential forms presented in(B.9)

$$B_2 = 3g_sM\left((f+k)\Phi + (f-k)\mathrm{Im}\Omega\right) \tag{B.32a}$$

$$H_3 = 3g_sM\left(d(f+k)\wedge\Phi + d(f-k)\wedge\mathrm{Im}\Omega + 3(f-k)\eta\wedge\mathrm{Re}\Omega\right) \tag{B.32b}$$

$$F_3 = -9M\left(\eta\wedge\Phi + (2F-1)\ \eta\wedge\mathrm{Im}\Omega - \frac{2}{3}dF\wedge\mathrm{Re}\Omega\right) \tag{B.32c}$$

$$F_5 = \left(N + 54g_sM^2l(\tau)\right)\eta\wedge J\wedge J. \tag{B.32d}$$

From this form it is straightforward to read off the relation between the 5D scalars of $\mathcal{N}=2$ supergravity model and the functions $F(\tau), k(\tau), f(\tau)$ and $l(\tau)$ appearing in the Klebanov-Strassler ansatz (B.31).

- From the form of B_2 and H_3 we find that the non-zero fields are

$$b^\Phi = 3g_sM(f+k)\ ,\qquad b^\Omega = -3ig_sM(f-k)\ , \tag{B.33}$$

$$h_0^\Omega = 9g_sM(f-k)\ ,\qquad h_1^\Omega = -3ig_sM\ d(f-k)\ ,\qquad h_1^\Phi = 3g_sM\ d(f+k)\ . \tag{B.34}$$

- From F_3 we get:

$$g_0^\Omega = 18iM\left(F-\frac{1}{2}\right)\ ,\qquad g_1^\Omega = 6M\ dF\ ,\qquad q = -9M\ . \tag{B.35}$$

  which gives

$$c^\Omega = 6M\left(F-\frac{1}{2}\right) \tag{B.36}$$

  We can see that one can recover (B.35) from (B.36) by using equation (3.15) of [52] keeping in mind that $C_0 = 0$ on the KS ansatz.

- From F_5 we get that all the expansion coefficients are zero except for:

$$f_0 = N + 54g_sM^2l(\tau) \tag{B.37}$$

  which is indeed consistent with the first equation in (3.18) of [52] with the parameter k being identified with N (the D3 brane charge)[1].

- For the metric sector we make a convenient gauge choice in (B.12) for the non-compact five dimensional part where we separate the radial direction (that we will call τ) from the four spacetime dimensions (which we will denote by x^i) where the gauge theory

[1] Caution: the symbol k is used here in two different context. It appears as a function $k(\tau)$ of τ in the 10D KS ansatz (B.31) and as parameter in the expansion ansatz of F_5 (B.20). We should be alert to mark the difference.

lives

$$ds^2_{\mathcal{M}} = e^{2X} d\tau^2 + e^{2Y} G_{ij}(x,\tau) dx^i dx^j \tag{B.38}$$

The comparison of this metric ansatz with with (B.12) gives the following relations

$$\begin{aligned}
e^{2u} &= \frac{3}{2} h^{1/2} \epsilon^{4/3} K \ \sinh\tau \ , \qquad e^{2v} = \frac{3}{2}\frac{h^{1/2}\epsilon^{4/3}}{K^2} \ , \qquad e^{2w} = 1 \ , \\
e^{2X} &= \frac{1}{4} h^{4/3} \epsilon^{32/9} \left(\frac{3}{2}\right)^{2/3} K^{-4/3} \sinh^{4/3}\tau \ , \\
e^{2Y} &= h^{1/3} \epsilon^{20/9} \left(\frac{3}{2}\right)^{5/3} K^{2/3} \sinh^{4/3}\tau \ , \\
G_{ij}(x,\tau) &= \eta_{ij} \ , \qquad \tanh t = \text{sech}\ \tau \ , \qquad \theta = 0 \ .
\end{aligned} \tag{B.39}$$

Finally let us write down the solution for the functions $F(\tau), k(\tau), f(\tau)$ and $l(\tau)$ appearing in the KS ansatz.

$$\begin{aligned}
f(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1) \ , \qquad k(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1) \ , \\
l(\tau) &= \frac{\tau\coth\tau - 1}{4\sinh^2\tau}(\sinh 2\tau - 2\tau) \ , \qquad F(\tau) = \frac{\sinh\tau - \tau}{2\sinh\tau} \ , \\
h'(\tau) &= -\alpha \frac{f(1-F) + kF}{K^2(\tau)\sinh^2\tau} \ .
\end{aligned} \tag{B.40}$$

where $\alpha = 4(g_s M \alpha')^2 \epsilon^{-8/3}$. We have explicitly checked that this solution satisfies the equations of motion resulting from the $5D$ supergravity model.

B.3 Gauge / Gravity map

Before closing off this appendix we comment on the $5D$ fields that are turned on the KS solution and their dual field theory interpretation. The transverse-traceless part of the metric $g_{\mu\nu}$ as usual is dual to the energy momentum tensor. The graviphoton $A - 2a_1^J$ is dual to the $U(1)$ R-symmetry current which is anomalous in the KS gauge theory. Indeed on the KS background, the gauge field $A - 2a_1^J$ acquires a mass which is $\frac{2}{3} g_s q^2$. The massless dilaton $e^{-\phi}$ is dual to the sum of the inverse of the two gauge coupling squared (where the two gauge couplings g_1 and g_2 are associated to the two gauge groups in the dual field theory). On the KS solution the dilaton takes a constant value which translates into vanishing β-function for $\frac{1}{g_1^2} + \frac{1}{g_2^2}$. The scalar fields b^Φ is instead dual to the difference of the complexified gauge couplings [10]. Its asymptotic behavior on KS solution (B.33, B.40) is $b^\Phi \sim \tau \sim 3\log(r)$ which translates into the running of the difference of the gauge coupling squared. The RR scalar C_0 is dual to the sum of the two theta angles and is vanishing in the solution. On the other hand, the massless scalar c^Φ is dual to the difference of the two gauge theory theta angles which (through Higgs mechanism in the bulk) becomes the longitudinal mode of the massive graviphoton $A - 2a_1^J$. This spontaneous breakdown of $U(1)$ gauge symmetry in the bulk captures holographically the chiral anomaly associated to the $U(1)$ R-symmetry [131]. The scalars $(4u+v), (u-v)$ are dual to irrelevant operators of dimension 8 and 6 respectively. These scalars respectively parametrise the breathing and squashing mode of $T^{1,1}$. It is easy to see this if we look at the reduction ansatz for the metric (B.12). The breathing mode controls the volume of the internal manifold which is given by the determinant of the internal metric while the squashing mode controls the relative size of the $U(1)$ fiber with respect to the Kahler-Einstein base of the deformed conifold. Around the KS ansatz we easily see that the determinant is proportional to e^{4u+v} and the ratio of the relative size is given by e^{u-v}. The scalar t of mass $m^2 = -3$ controls the deformation of the conifold and is associated to dimension 3 gaugino condensate operator in the dual field theory.

Appendix C

Details of chapter 3

C.1 The 5d supergravity action

In this appendix we present a *further sub-truncation* of the $\mathcal{N} = 2$ 5D-dimensional gauge supergravity model discussed in appendix B. We also explicitly write down the fermionic section of this truncation which will be relevant for proving the supersymmetric Ward identities. As explained in the main text, this sub-truncation preserves an additional $U(1)$ symmetry [54].

The bosonic action, restricted to the fields relevant for our analysis in chapter 3, namely the metric $g_{\mu\nu}$ and the four scalars U, V, b^{Φ} and ϕ, can be written as a σ-model and reads

$$S_b = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(R - \mathcal{G}_{IJ}(\varphi)\partial_A\varphi^I\partial^A\varphi^J - \mathcal{V}(\varphi)\right) . \tag{C.1}$$

The fermionic action containing the gravitino Ψ_M and the four spinor fields ζ_U, ζ_V, ζ_{b^Φ} and ζ_ϕ can also be expressed in terms of sigma model language and, up to quadratic terms in the fermions takes the form

$$\begin{aligned} S_f = -\tfrac{1}{2\kappa^2}\int d^5x\sqrt{-g}\Big[\tfrac{1}{2}\Big(\overline{\Psi}_A\Gamma^{ABC}\mathcal{D}_B\Psi_C + i\mathcal{G}_{IJ}\overline{\zeta}^I\Gamma^A\left(\partial\!\!\!/\varphi^J - \mathcal{G}^{JK}\partial_K\mathcal{W}\right)\Psi_A + \text{h.c.}\Big) \\ +\tfrac{1}{2}\Big(\mathcal{G}_{IJ}\overline{\zeta}^I\left(\delta^J_K\nabla\!\!\!\!/ + \Gamma^J_{KL}[\mathcal{G}]\partial\!\!\!/\varphi^L\right)\zeta^K + h.c.\Big) + \mathcal{M}_{IJ}(\varphi)\overline{\zeta}^I\zeta^J\Big] . \end{aligned} \tag{C.2}$$

Here, $\kappa^2 = 8\pi G_5$ and the indices $A, B, ...$ are 5d space-time indices, while $I, J, ...$ are indices on the scalar manifold. In particular,

$$\varphi^I = \begin{pmatrix} U \\ V \\ b^{\Phi} \\ \phi \end{pmatrix}, \quad \zeta^I = \begin{pmatrix} \zeta_U \\ \zeta_V \\ \zeta_b \\ \zeta_\phi \end{pmatrix}, \quad \mathcal{G}_{IJ}(\varphi) = \begin{pmatrix} \frac{8}{15} & 0 & 0 & 0 \\ 0 & \frac{4}{5} & 0 & 0 \\ 0 & 0 & e^{-\frac{4}{5}(U+V)-\phi} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} . \tag{C.3}$$

The only non-zero components of the Christoffel symbol $\Gamma^K_{IJ}[\mathcal{G}]$ of the metric (C.3) on the scalar manifold are

$$\Gamma^U_{b^\Phi b^\Phi}[\mathcal{G}] = \frac{3}{4}e^{-\frac{4}{5}(U+V)-\phi}, \quad \Gamma^V_{b^\Phi b^\Phi}[\mathcal{G}] = \frac{1}{2}e^{-\frac{4}{5}(U+V)-\phi}, \quad \Gamma^\phi_{b^\Phi b^\Phi}[\mathcal{G}] = e^{-\frac{4}{5}(U+V)-\phi},$$
$$\Gamma^{b^\Phi}_{b^\Phi U}[\mathcal{G}] = -\frac{2}{5}, \quad \Gamma^{b^\Phi}_{b^\Phi V}[\mathcal{G}] = -\frac{2}{5}, \quad \Gamma^{b^\Phi}_{b^\Phi \phi}[\mathcal{G}] = -\frac{1}{2}\ . \tag{C.4}$$

The covariant derivative ∇_A and the supercovariant derivative $\mathcal{D}_A$ are defined as follows

$$\nabla_A = \partial_A + \frac{1}{4}\left(\omega_A\right)^{ab}\gamma_{ab}\ , \tag{C.5a}$$
$$\mathcal{D}_A = \nabla_A + \frac{1}{6}\Gamma_A\mathcal{W}\ , \tag{C.5b}$$

where $a, b, ...$ are indices on the tangent space and $(\omega_A)^{ab}$ is the spin connection of the 5d metric. The scalar potential takes the following form

$$\mathcal{V}(\varphi) = 2e^{-\frac{8}{3}U}(b^\Phi q - k)^2 + e^{-\frac{4}{15}(7U-3V)+\phi}q^2 - 24e^{-\frac{2}{15}(8U+3V)} + 4e^{-\frac{4}{15}(4U+9V)}\ , \tag{C.6}$$

where we used the following relations to connect to the notations adopted in the main text

$$q = \frac{9}{2}M\ , \quad k = -\frac{27\pi N}{2}\ , \tag{C.7}$$

with N and M being the number of regular and fractional D3 branes respectively. Both the scalar potential and the mass matrix $\mathcal{M}_{IJ}$ can be expressed in terms of the Papadopoulos-Tseytlin superpotential [132]

$$\mathcal{W}(\varphi) = (k - qb^\Phi)e^{-\frac{4}{3}U} + 3e^{-\frac{4}{15}(2U-3V)} + 2e^{-\frac{2}{15}(4U+9V)}\ , \tag{C.8}$$

through the relations

$$\mathcal{V}(\varphi) = \mathcal{G}^{IJ}\partial_I\mathcal{W}(\varphi)\partial_J\mathcal{W}(\varphi) - \frac{4}{3}\mathcal{W}(\varphi)^2\ , \tag{C.9a}$$
$$\mathcal{M}_{IJ}(\varphi) = \partial_I\partial_J\mathcal{W} - \Gamma^K_{IJ}[\mathcal{G}]\partial_K\mathcal{W} - \frac{1}{2}\mathcal{G}_{IJ}\mathcal{W}\ . \tag{C.9b}$$

The supersymmetry transformations to linear oder in ϵ are

$$\delta_\epsilon\zeta^I = -\frac{i}{2}\left(\partial\!\!\!/\varphi^I - \mathcal{G}^{IJ}\partial_J\mathcal{W}\right)\epsilon\ , \tag{C.10a}$$
$$\delta_\epsilon\Psi_A = \left(\nabla_A + \frac{1}{6}\mathcal{W}\Gamma_A\right)\epsilon\ , \tag{C.10b}$$
$$\delta_\epsilon\varphi^I = \frac{i}{2}\bar{\epsilon}\zeta^I + \text{h.c.}\ , \tag{C.10c}$$
$$\delta_\epsilon e^a_A = \frac{1}{2}\bar{\epsilon}\Gamma^a\Psi_A + \text{h.c.}\ . \tag{C.10d}$$

It follows that the BPS equations for Poincaré domain wall solutions of the form (3.5) are

$$e^{-X(z)} z \partial_z \varphi^I - \mathcal{G}^{IJ} \partial_J \mathcal{W} = 0 \ , \quad e^{-X(z)} z \partial_z \log\left(\frac{e^Y}{z}\right) + \frac{1}{3}\mathcal{W} = 0 \ . \tag{C.11}$$

C.2 Equations of motion and leading asymptotics

In this appendix we give the bosonic and fermionic equations of motion following from the action (C.1)+(C.2), as well as the leading form of the their asymptotic solutions, subject to KT boundary conditions.

C.2.1 Bosonic sector

In the bosonic sector the equations of motion are

$$\frac{1}{\sqrt{-g_5}} \partial_A \left(\sqrt{-g_5}\ g^{AB}\ \partial_B \phi\right) = e^{-\frac{4}{15}(7U-3V)+\phi} q^2 - e^{-\frac{4}{5}(U+V)-\phi} \left(db^\Phi\right)^2 \ , \tag{C.12a}$$

$$\frac{1}{\sqrt{-g_5}} \partial_A \left(\sqrt{-g_5}\ e^{-\frac{4}{5}(U+V)-\phi}\ g^{AB}\ \partial_B b^\Phi\right) = 2q\ e^{-\frac{8}{3}U} \left(b^\Phi q - k\right) \ , \tag{C.12b}$$

$$\begin{aligned} &\frac{16}{15}\frac{1}{\sqrt{-g_5}} \partial_A \left(\sqrt{-g_5}\ g^{AB}\ \partial_B U\right) + \frac{4}{5} e^{-\frac{4}{5}(U+V)-\phi} \left(db^\Phi\right)^2 + \frac{16}{3} e^{-\frac{8}{3}U} (b^\Phi q - k)^2 \\ &+ \frac{28}{15} e^{-\frac{4}{15}(7U-3V)+\phi} q^2 + \frac{64}{15} e^{-\frac{4}{15}(4U+9V)} - \frac{128}{5} e^{-\frac{2}{15}(8U+3V)} = 0 \ , \end{aligned} \tag{C.12c}$$

$$\begin{aligned} &\frac{8}{5}\frac{1}{\sqrt{-g_5}} \partial_A \left(\sqrt{-g_5}\ g^{AB}\ \partial_B V\right) + \frac{4}{5}\left(e^{-\frac{4}{5}(U+V)-\phi} \left(db^\Phi\right)^2 - q^2 e^{-\frac{4}{15}(7U-3V)+\phi}\right) \\ &+ \frac{48}{5}\left(e^{-\frac{4}{15}(4U+9V)} - e^{-\frac{2}{15}(8U+3V)}\right) = 0 \ , \end{aligned} \tag{C.12d}$$

$$\begin{aligned} R_{AB} = &\frac{8}{15}\partial_A U \partial_B U + \frac{4}{5}\partial_A V \partial_B V + e^{-\frac{4}{5}(U+V)-\phi} \partial_A b^\Phi \partial_B b^\Phi + \frac{1}{2}\partial_A \phi \partial_B \phi \\ &+ \frac{1}{3} g_{AB}\left(2e^{-\frac{8}{3}U}(b^\Phi q - k)^2 + e^{-\frac{4}{15}(7U-3V)+\phi} q^2 - 24 e^{-\frac{2}{15}(8U+3V)} + 4e^{-\frac{4}{15}(4U+9V)}\right) \ . \end{aligned} \tag{C.12e}$$

Asymptotic solutions

In order to obtain the asymptotic solutions of the equations of motion it is necessary to pick a specific gauge. In the gauge (C.25), the leading asymptotics of the bosonic fields for any solution that asymptotes to the KT solution take the form

$$
\begin{aligned}
&\gamma_{\mu\nu}(z,x) \sim \frac{h^{1/3}(z)}{z^2}\left(\left(1+\frac{1}{24}g_s q^2 h^{-1}(z)(1-4\log z)c(x)+\frac{1}{6}qh^{-1}(z)\ b(x)\right)\eta_{\mu\nu}+h_{\mu\nu}(x)\right),\\
&\phi(z,x) \sim \log g_s + c(x)\ ,\\
&b^{\Phi}(z,x) \sim b(x) - (1+c(x))\ g_s q \log z\ ,\\
&U(z,x) \sim \frac{5}{4}\log\left(h(z)+\frac{1}{8}g_s q^2(1-4\log z)c(x)+\frac{1}{2}q\ b(x)\right)\ ,\\
&V(z,x) = \mathcal{O}(z^4)\ ,
\end{aligned}
\tag{C.13}
$$

where the warp factor is given by

$$
h(z) = \frac{1}{8}\left(-4k + g_s q^2 - 4g_s q^2 \log z\right) + \mathcal{O}(z^4), \tag{C.14}
$$

and $h_{\mu\nu}(x), c(x)$ and $b(x)$ are infinitesimal sources.

C.2.2 Fermionic sector

The fermionic equations of motion take the form

$$
\not{\nabla}\zeta_\phi + \frac{i}{2}\Gamma^M \not{\partial}\phi\Psi_M - m_\phi\zeta_\phi + \mathcal{F}_-\mathcal{G}^{1/2}_{b^\Phi b^\Phi}\zeta_b = 0\ , \tag{C.15a}
$$

$$
\not{\nabla}\left(\mathcal{G}^{1/2}_{b^\Phi b^\Phi}\zeta_b\right) + \frac{i}{2}\Gamma^M\mathcal{F}_+\Psi_M + m_b\mathcal{G}^{1/2}_{b^\Phi b^\Phi}\zeta_b - \frac{1}{2}\mathcal{F}_+\zeta_\phi + \partial_U\mathcal{F}_-\zeta_U + \partial_V\mathcal{F}_-\zeta_V = 0\ , \tag{C.15b}
$$

$$
\not{\nabla}\zeta_U + \frac{i}{2}\Gamma^M\mathcal{B}^U_+\Psi_M + m_U\zeta_U + \frac{12}{5}\left(e^{-\frac{2}{15}(4U+9V)} - e^{-\frac{4}{15}(2U-3V)}\right)\zeta_V - \frac{15}{8}\partial_U\mathcal{F}_+\mathcal{G}^{1/2}_{b^\Phi b^\Phi}\zeta_b = 0\ , \tag{C.15c}
$$

$$
\not{\nabla}\zeta_V + \frac{i}{2}\Gamma^M\mathcal{B}^V_+\Psi_M + m_V\zeta_V + \frac{8}{5}\left(e^{-\frac{2}{15}(4U+9V)} - e^{-\frac{4}{15}(2U-3V)}\right)\zeta_U - \frac{5}{4}\partial_V\mathcal{F}_+\mathcal{G}^{1/2}_{b^\Phi b^\Phi}\zeta_b = 0\ , \tag{C.15d}
$$

$$
\Gamma^{ABC}\mathcal{D}_B\Psi_C - \frac{i}{2}\left(\frac{1}{2}\not{\partial}\phi\Gamma^A\zeta_\phi + \mathcal{F}_-\Gamma^A\mathcal{G}^{1/2}_{b^\Phi b^\Phi}\zeta_b + \frac{8}{15}\mathcal{B}^U_-\Gamma^A\zeta_U + \frac{4}{5}\mathcal{B}^V_-\Gamma^A\zeta_V\right) = 0\ , \tag{C.15e}
$$

where we have defined the following quantities

$$\mathcal{F}_\pm = \mathcal{G}^{1/2}_{b^\Phi b^\Phi}\left(\partial\!\!\!/ b^\Phi \pm e^{-\frac{4}{15}(2U-3V)+\phi}q\right) , \tag{C.16a}$$

$$\mathcal{B}^U_\pm = \partial\!\!\!/ U \pm \frac{1}{2}\left(5(k-b^\Phi q)e^{-\frac{4}{3}U} + 6e^{-\frac{4}{15}(2U-3V)} + 4e^{-\frac{2}{15}(4U+9V)}\right) , \tag{C.16b}$$

$$\mathcal{B}^V_\pm = \partial\!\!\!/ V \pm 3\left(e^{-\frac{2}{15}(4U+9V)} - e^{-\frac{4}{15}(2U-3V)}\right) \tag{C.16c}$$

$$m_\phi(\varphi) = \frac{1}{2}\mathcal{W} , \tag{C.16d}$$

$$m_b(\varphi) = \frac{1}{2}\mathcal{W} - 3e^{-\frac{4}{15}(2U-3V)} , \tag{C.16e}$$

$$m_U(\varphi) = \frac{1}{30}\left(\mathcal{W} + 84\left(k-b^\Phi q\right)e^{-\frac{4}{3}U}\right) , \tag{C.16f}$$

$$m_V(\varphi) = \frac{3}{10}\mathcal{W} - \frac{4}{5}(k-b^\Phi q)e^{-\frac{4}{3}U} + 2e^{-\frac{2}{15}(4U+9V)} . \tag{C.16g}$$

The fermion masses $m_\phi(\varphi)$, $m_b(\varphi)$, $m_U(\varphi)$, $m_V(\varphi)$ reproduce the masses shown in Table 3.1 in the AdS limit ($q \to 0$) with unit AdS radius ($k = -2$).

Asymptotic solutions

In the gauge (C.25) the leading asymptotics of the fermions, for any bosonic solution that asymptotes to the KT solution, take the form

$$\begin{aligned}
\Psi^+_\mu(z,x) &\sim z^{-1/2}h(z)^{1/12}\Psi^+_{(0)\mu}(x) \\
&\quad + iz^{-1/2}h(z)^{1/6}\gamma_\mu\left(-\frac{4}{5g_sq^2}h(z)^{11/12}\psi^-_1(x) + \frac{h(z)^{-7/4}}{12g_sq}\left(g_sq^2+12h(z)\right)\psi^-_2(x)\right) , \\
\zeta^-_\phi(z,x) &\sim z^{1/2}h(z)^{-1/12}\psi^-_1(x) , \\
\zeta^-_b(z,x) &\sim \frac{z^{1/2}h(z)^{-1/12}}{20q}\left(24h(z)-5g_sq^2\right)\psi^-_1(x) + z^{1/2}h(z)^{-3/4}\psi^-_2(x) , \\
\zeta^-_U(z,x) &\sim \frac{3}{4}z^{1/2}h(z)^{-1/12}\psi^-_1(x) + \frac{5}{8}q\ z^{1/2}h(z)^{-7/4}\psi^-_2(x) , \\
\zeta^+_V(z,x) &= \mathcal{O}(z^{3/2}) ,
\end{aligned} \tag{C.17}$$

where $h(z)$ is given in (C.14) and $\Psi^+_{(0)\mu}(x)$, $\psi^-_1(x)$, $\psi^-_2(x)$ are spinor sources of the indicated chirality. Notice that the limit $q \to 0$, corresponding to KW asymptotics, is a singular limit in these asymptotic solutions. In particular, the parameter q corresponds to a singular perturbation of the fermionic equations of motion (C.15).

C.3 Covariant sources for gauge-invariant operators

As was mentioned in section 3.3, the covariant sources of certain operators in the KS theory are composite in terms of bulk fields. In particular, the covariant source of the difference of the inverse gauge couplings square corresponds to the composite field $\widetilde{b}^\Phi = e^{-\phi} b^\Phi$. Inserting the asymptotic expansions (C.13) we find that $\widetilde{b}^\Phi$ asymptotes to

$$\widetilde{b}^\Phi \sim g_s^{-1} b(x) - q \log z \ , \tag{C.18}$$

and it is therefore sourced only by the $b(x)$ mode. Similarly, the composite field

$$\widetilde{U} = \frac{1}{8}\left(4qb^\Phi - 4k + q^2 e^\phi\right) e^{-\frac{4}{5}U} \ , \tag{C.19}$$

has the property that the modes $b(x)$ and $c(x)$ drop out of its asymptotic expansion so that $\widetilde{U} = 1$, up to normalizable modes. Moreover, the BPS equations (C.11) imply that $\widetilde{U}$ is a constant, up to a mode that has the right scaling to be identified with the source of a dimension 8 operator, which therefore corresponds to a supersymmetric irrelevant deformation [57,133]. These two properties allow us to identify $\widetilde{U}$ with the covariant source of the dimension 8 operator, which can therefore be consistently switched off by setting $\widetilde{U} = 1$. Finally, although this is not necessary, it is natural to define the stress tensor as the operator that couples only to the fluctuation $h_{\mu\nu}$ in (C.13), which can be achieved by defining the covariant source of the stress tensor as

$$\widetilde{\gamma}_{\mu\nu} = e^{-\frac{4}{15}U} \gamma_{\mu\nu} \ . \tag{C.20}$$

The covariant sources for the fermionic partners of these operators follow by supersymmetry and are given respectively by[1]

$$\begin{aligned}
\widetilde{\zeta}_b^- &= e^{-\phi}\left(\zeta_b^- - b^\Phi \zeta_\phi^-\right) \ , \\
\widetilde{\zeta}_U^- &= -\frac{4}{5}\widetilde{U}\zeta_U^- + \frac{1}{8}e^{-\frac{4}{5}U}\left(4q\zeta_b^- + q^2 e^\phi \zeta_\phi^-\right) \ , \\
\widetilde{\Psi}_\mu^+ &= e^{-\frac{2}{15}U}\left(\Psi_\mu^+ - \frac{2i}{15}\Gamma_\mu \zeta_U^-\right) \ .
\end{aligned} \tag{C.21}$$

The leading asymptotic behavior of these fields, following from (C.13) and (C.17), is

$$\begin{aligned}
&\widetilde{\zeta}_b^- \sim -\frac{1}{5g_s q} z^{1/2} h(z)^{-1/12} \left(4h(z) + 5k\right) \psi_1^-(x) + g_s^{-1} z^{1/2} h(z)^{-3/4} \psi_2^-(x) \ , \\
&\widetilde{\zeta}_U^- \sim 0 \ , \\
&\widetilde{\Psi}_\mu^+ \sim z^{-1/2}\left(h(z)^{-1/12}\left(\Psi_{(0)\mu}^+ - \frac{i}{10}\gamma_\mu \psi_1^-\right) - \frac{4i}{5g_s q^2} h(z)^{11/12} \gamma_\mu \psi_1^- + \frac{i}{g_s q} h(z)^{-3/4} \gamma_\mu \psi_2^-\right) \ ,
\end{aligned} \tag{C.22}$$

[1] In fact, the fully covariant with respect to $\widetilde{\gamma}_{\mu\nu}$ fermionic sources contain an additional factor of $e^{U/15} = e^{X/8}$, which comes from the covariantization of the spinor ϵ^+ with respect to $\widetilde{\gamma}_{\mu\nu}$. This extra factor would remove the factors of $e^{-X/8}$ from the definition of the fermionic one-point functions in (3.13), as well as an overall factor of $h^{-1/12}$ from the expansions (C.22). However, since we are working to linear order in the sources this factor does not play a crucial role and we have chosen not to include it in the definition of the fermion sources.

where $\widetilde{\zeta}_U^-$ is only sourced by a mode corresponding to an irrelevant operator of dimension 15/2, which can therefore be put to zero consistently.

The fact that the covariant sources $\widetilde{\gamma}_{\mu\nu}$, $\widetilde{b}^\Phi$ and $\widetilde{U}$, as well as their supersymmetric partners, are composite in terms of supergravity fields implies that some care is required when evaluating the partial derivatives in the definition of the one-point functions (3.11), where composite fields are held constant. In particular, expressing the supergravity fields in terms of the composite fields,

$$\begin{aligned}
b^\Phi &= e^\phi \widetilde{b}^\Phi \ , \\
e^{\frac{4}{5}U} &= \frac{1}{8}\widetilde{U}^{-1} e^\phi \left(4q\widetilde{b}^\Phi - 4ke^{-\phi} + q^2\right) \ , \\
\gamma_{\mu\nu} &= \frac{1}{2}\widetilde{U}^{-1/3} e^{\phi/3} \left(4q\widetilde{b}^\Phi - 4ke^{-\phi} + q^2\right)^{1/3} \widetilde{\gamma}_{\mu\nu} \ , \\
\zeta_b^- &= e^\phi \left(\widetilde{\zeta}_b^- + \widetilde{b}^\Phi \zeta_\phi^-\right) \ , \\
\zeta_U^- &= -\frac{5}{4}\widetilde{U}^{-1}\left(\widetilde{\zeta}_U^- - \widetilde{U}\left(4q\widetilde{b}^\Phi - 4ke^{-\phi} + q^2\right)^{-1}\left(4q\widetilde{\zeta}_b^- + (4q\widetilde{b}^\Phi + q^2)\zeta_\phi^-\right)\right) \ , \\
\Psi_\mu^+ &= \frac{1}{\sqrt{2}}\widetilde{U}^{-1/6} e^{\phi/6}\left(4q\widetilde{b}^\Phi - 4ke^{-\phi} + q^2\right)^{1/6}\left(\widetilde{\Psi}_\mu^+ + \frac{2i}{15}\widetilde{\Gamma}_\mu \zeta_U^-\right) \ ,
\end{aligned} \tag{C.23}$$

one obtains the following expressions for the partial derivatives of a generic function F with respect to the covariant bosonic sources

$$\begin{aligned}
\left.\frac{\partial F}{\partial \widetilde{\gamma}_{\mu\nu}}\right|_{\phi,\widetilde{b}^\Phi,\widetilde{U},\widetilde{\Psi}^+,\zeta_\phi^-,\widetilde{\zeta}_b^-,\widetilde{\zeta}_U^-} &= e^{\frac{4}{15}U}\frac{\partial F}{\partial \gamma_{\mu\nu}} \ , \\
\left.\frac{\partial F}{\partial \phi}\right|_{\widetilde{\gamma},\widetilde{b}^\Phi,\widetilde{U},\widetilde{\Psi}^+,\zeta_\phi^-,\widetilde{\zeta}_b^-,\widetilde{\zeta}_U^-} &= \frac{\partial F}{\partial \phi} + b^\Phi \frac{\partial F}{\partial b^\Phi} + \left(1 + \frac{k}{2}e^{-\frac{4}{5}U}\right)\left(\frac{5}{4}\frac{\partial F}{\partial U} + \frac{1}{3}\gamma_{\mu\nu}\frac{\partial F}{\partial \gamma_{\mu\nu}}\right) + \text{fermions} \ , \\
\left.\frac{\partial F}{\partial \widetilde{b}^\Phi}\right|_{\widetilde{\gamma},\phi,\widetilde{U},\widetilde{\Psi}^+,\zeta_\phi^-,\widetilde{\zeta}_b^-,\widetilde{\zeta}_U^-} &= e^\phi \frac{\partial F}{\partial b^\Phi} + \frac{q}{2}e^{-\frac{4}{5}U+\phi}\left(\frac{5}{4}\frac{\partial F}{\partial U} + \frac{1}{3}\gamma_{\mu\nu}\frac{\partial F}{\partial \gamma_{\mu\nu}}\right) + \text{fermions} \ .
\end{aligned} \tag{C.24}$$

These expressions are required in order to correctly evaluate the one-point functions (3.11).

C.4 Local symmetries and transformation of the sources

The bulk equations of motion dictate that certain components of the metric and of the gravitino are non-dynamical. In particular, the radial-radial and radial-transverse components of the metric (or, more precisely, the shift and lapse functions of the metric with respect to the radial coordinate), as well a the radial component of the gravitino, are non-dynamical and can be gauge-fixed to a convenient choice. We choose the gauge

$$ds_5^2 = dr^2 + \gamma_{\mu\nu}(r,x)dx^\mu dx^\nu, \quad \Psi_r = 0 \ , \tag{C.25}$$

where the canonical radial coordinate r is related to the coordinate z in (3.5) through $dr = -e^{X(z)}dz/z$. Moreover, for the domain wall ansatz in (3.5) we have

$$\gamma_{\mu\nu} = \frac{e^{2Y}}{z^2}\eta_{\mu\nu} \; . \tag{C.26}$$

The gauge-fixing conditions (C.25) are preserved by a subset of bulk diffeomorphisms and supersymmetry transformations. The transformation of the covariant sources under these gauge-preserving local transformations gives rise to the holographic Ward identities.[2]

C.4.1 Bulk diffeomorphisms

Infinitesimal bulk diffeomorphisms that preserve the gauge (C.25) are parameterized by a vector field satisfying the differential equations

$$\dot{\xi}^r = 0, \quad \dot{\xi}^\mu + \gamma^{\mu\nu}\partial_\nu \xi^r = 0 \; . \tag{C.27}$$

The general solution of these equations is

$$\xi^r = \sigma(x) \; , \tag{C.28}$$

$$\xi^\mu = \xi_o^\mu(x) - \int^r dr' \gamma^{\mu\nu}(r', x)\partial_\nu \sigma(x) \; , \tag{C.29}$$

where the arbitrary functions $\sigma(x)$ parameterizes Weyl transformations on the boundary [59], while $\xi_o^\mu(x)$ corresponds to boundary diffeomorphisms. The transformation of the supergravity fields under Weyl transformations (C.28) is

$$\begin{aligned}
&\delta_\sigma \gamma_{\mu\nu} = \sigma\dot{\gamma}_{\mu\nu} \sim 2e^{-\frac{8}{15}U}\sigma\gamma_{\mu\nu} \; , && \delta_\sigma \Psi_\mu^+ = \sigma\dot{\Psi}_\mu^+ \sim \frac{1}{2}e^{-\frac{8}{15}U}\Psi_\mu^+\sigma \; , \\
&\delta_\sigma \phi = \sigma\dot{\phi} \sim 0 \; , && \delta_\sigma \zeta_\phi^- = \sigma\dot{\zeta}_\phi^- \sim -\frac{1}{2}e^{-\frac{8}{15}U}\zeta_\phi^-\sigma \; , \\
&\delta_\sigma b^\Phi = \sigma\dot{b}^\Phi \sim qe^{-\frac{8}{15}U+\phi}\sigma \; , && \delta_\sigma \zeta_b^- = \sigma\dot{\zeta}_b^- \sim -e^{-\frac{8}{15}U}\left(\frac{1}{2}\zeta_b^- - qe^{\phi}\left(\zeta_\phi^- + \frac{8}{15}\zeta_U^-\right)\right)\sigma \; , \\
&\delta_\sigma U = \sigma\dot{U} \sim \frac{5}{8}q^2 e^{-\frac{4}{3}U+\phi}\sigma \; , && \delta_\sigma \zeta_U^- = \sigma\dot{\zeta}_U^- \sim -\frac{3q}{16}e^{-\frac{4}{3}U}\left(\zeta_b^- - \frac{7q}{4}e^{\phi}\zeta_\phi^-\right)\sigma \; .
\end{aligned} \tag{C.30}$$

[2]In fact, gauge-preserving bulk diffeomorphisms and local supersymmetry transformations cannot be considered separately since they mix. However, this mixing occurs only at asymptotically subleading orders and involves transverse derivatives on the transformation parameters. This implies that the mixing between gauge-preserving bulk diffeomorphisms and local supersymmetry transformations does not affect our results here, and so for simplicity we will treat them separately.

These imply that the covariant sources transform as

$$
\begin{aligned}
&\delta_\sigma \widetilde{\gamma}_{\mu\nu} \sim 2e^{-\frac{8}{15}U}\sigma\widetilde{\gamma}_{\mu\nu} \ , && \delta_\sigma \widetilde{\Psi}^+_\mu \sim \frac{1}{2}e^{-\frac{8}{15}U}\widetilde{\Psi}^+_\mu\sigma \ , \\
&\delta_\sigma \phi \sim 0 \ , && \delta_\sigma \zeta^-_\phi \sim -\frac{1}{2}e^{-\frac{8}{15}U}\zeta^-_\phi\sigma \ , \\
&\delta_\sigma \widetilde{b}^\Phi \sim qe^{-\frac{8}{15}U}\sigma \ , && \delta_\sigma \widetilde{\zeta}^-_b \sim -\frac{1}{2}e^{-\frac{8}{15}U}\left(\widetilde{\zeta}^-_b - \frac{16}{15}qe^\phi\zeta^-_U\right)\sigma \ , \\
&\delta_\sigma \widetilde{U} \sim 0 \ , && \delta_\sigma \widetilde{\zeta}^-_U \sim 0 \ .
\end{aligned} \tag{C.31}
$$

C.4.2 Local supersymmetry transformations

The gauge fixing condition (C.25) on the gravitino leads to a differential equation for the supersymmetry parameter ϵ via eq. (C.10b), namely

$$\left(\nabla_r + \frac{1}{6}\mathcal{W}\Gamma_r\right)\epsilon = 0 \ , \tag{C.32}$$

or, in gauge-fixed form and projecting out the two chiralities,

$$\dot{\epsilon}^\pm \mp \frac{1}{6}\mathcal{W}\epsilon^\pm = 0 \ . \tag{C.33}$$

The asymptotic solutions to these equations are

$$
\begin{aligned}
\epsilon^+(z,x) &= z^{-1/2}h(z)^{1/12}\epsilon^+_0(x) + \mathcal{O}(z^4) \ , \\
\epsilon^-(z,x) &= z^{1/2}h(z)^{-1/12}\epsilon^-_0(x) + \mathcal{O}(z^4) \ ,
\end{aligned} \tag{C.34}
$$

where the arbitrary spinors $\epsilon^\pm_0(x)$ parameterize respectively supersymmetry and superWeyl transformations on the boundary. The transformation of the covariant sources under these transformations is as follows.

Gravitino:

The transformation of the induced gravitino Ψ_μ under supersymmetry is

$$\delta_\epsilon\Psi_\mu = \left(\nabla_\mu + \frac{1}{6}\mathcal{W}\Gamma_\mu\right)\epsilon \ . \tag{C.35}$$

Projecting this equation on the positive chirality, which is the leading one asymptotically as follows from eq. (C.17) and which corresponds to the covariant source of the supercurrent, we get

$$\delta_\epsilon\Psi^+_\mu = \partial_\mu\epsilon^+ + \frac{1}{3}\Gamma_\mu\mathcal{W}\,\epsilon^- \ , \tag{C.36}$$

where we have used (C.11) in order to drop a term proportional to the VEV of the stress tensor (which is subleading asymptotically).

Metric:

The supersymmetry transformation of the vielbein e^a_μ is given by

$$\delta_\epsilon e^a_\mu = \frac{1}{2}\,\bar\epsilon\,\gamma^a\,\Psi_\mu + \text{h.c.}\,. \tag{C.37}$$

From this it follows that the corresponding variation of the induced metric is

$$\delta_\epsilon \gamma_{\mu\nu} = \bar\epsilon^+ \Gamma_{(\mu}\Psi^+_{\nu)} + \bar\epsilon^- \Gamma_{(\mu}\Psi^-_{\nu)} + \text{h.c.}\ , \tag{C.38}$$

where the symmetrization is done with a factor of 1/2. Dropping the term proportional to Ψ^-_μ that is related to the one-point function of the supercurrent and is asymptotically subleading, we obtain

$$\delta_\epsilon \gamma_{\mu\nu} = \bar\epsilon^+ \Gamma_{(\mu}\Psi^+_{\nu)} + \text{h.c.}\ . \tag{C.39}$$

Hypermultiplet sector:

The transformation of the fields in the hypermultiplet is

$$\begin{aligned}
&\delta_\epsilon \phi = \frac{i}{2}\bar\epsilon^+ \zeta^-_\phi + \text{h.c.}\ , \quad \delta_\epsilon \zeta^-_\phi = -\frac{i}{2}\Gamma^z \partial_z \phi\ \epsilon^- \sim 0\ , \\
&\delta_\epsilon b^\Phi = \frac{i}{2}\bar\epsilon^+ \zeta^-_b + \text{h.c.}\ , \quad \delta_\epsilon \zeta^-_b = -\frac{i}{2}\left(\Gamma^z\partial_z b^\Phi + e^{-\frac{4}{15}(2U-3V)+\phi} q\right)\epsilon^- \sim -iqe^{-\frac{8}{15}U+\phi}\epsilon^-\ , \\
&\delta_\epsilon U = \frac{i}{2}\bar\epsilon^+ \zeta^-_U + \text{h.c.}\ , \quad \delta_\epsilon \zeta^-_U = -\frac{i}{2}\left(\Gamma^z\partial_z U - \partial_U \mathcal{W}\right)\epsilon^- \sim i\partial_U\mathcal{W}\epsilon^- \sim -i\frac{q^2}{3}e^{-\frac{4}{3}U+\phi}\epsilon^-\ .
\end{aligned} \tag{C.40}$$

Combining these results, we deduce that the covariant sources transform under $\epsilon^\pm$ as

$$\begin{aligned}
&\delta_{\epsilon^+}\widetilde\gamma_{\mu\nu} \sim \bar\epsilon^+\widetilde\Gamma_{(\mu}\widetilde\Psi^+_{\nu)} + \text{h.c.}\ , && \delta_{\epsilon^+}\widetilde\Psi^+_\mu \sim e^{-\frac{2}{15}U}\partial_\mu\epsilon^+\ , \\
&\delta_{\epsilon^+}\phi \sim \frac{i}{2}\bar\epsilon^+\zeta^-_\phi + \text{h.c.}\ , && \delta_{\epsilon^+}\zeta^-_\phi \sim 0\ , \\
&\delta_{\epsilon^+}\widetilde b^\Phi \sim \frac{i}{2}\bar\epsilon^+\widetilde\zeta^-_b + \text{h.c.}\ , && \delta_{\epsilon^+}\widetilde\zeta^-_b \sim 0\ , \\
&\delta_{\epsilon^+}\widetilde U \sim \frac{i}{2}\bar\epsilon^+\widetilde\zeta^-_U + \text{h.c.}\ , && \delta_{\epsilon^+}\widetilde\zeta^-_U \sim 0\ .
\end{aligned} \tag{C.41}$$

where $\widetilde\Gamma_\mu = \widetilde e^a_\mu \gamma_a = e^{-\frac{2}{15}U} e^a_\mu\gamma_a$, and

$$\begin{aligned}
&\delta_{\epsilon^-}\widetilde\gamma_{\mu\nu} \sim 0\ , && \delta_{\epsilon^-}\widetilde\Psi^+_\mu \sim e^{-\frac{8}{15}U}\widetilde\Gamma_\mu\epsilon^-\ , \\
&\delta_{\epsilon^-}\phi \sim 0\ , && \delta_{\epsilon^-}\zeta^-_\phi \sim 0\ , \\
&\delta_{\epsilon^-}\widetilde b^\Phi \sim 0\ , && \delta_{\epsilon^-}\widetilde\zeta^-_b \sim -iqe^{-\frac{8}{15}U}\epsilon^-\ , \\
&\delta_{\epsilon^-}\widetilde U \sim 0\ , && \delta_{\epsilon^-}\widetilde\zeta^-_U \sim -i\frac{7}{30}q^2 e^{-\frac{4}{3}U+\phi}\epsilon^-\ .
\end{aligned} \tag{C.42}$$

Appendix D

Recombination along RG flows and anomalous dimensions

In chapter 5 we gave examples of multiplet recombination triggered by exactly marginal deformations. In this appendix we would like to consider the phenomenon of multiplet recombination triggered by relevant deformations. In section D.1 we demonstrate how one can make use multiplet recombination to calculate leading anomalous dimensions. We consider two examples. The first one is the free scalar field theory deformed by $\lambda\phi^4$ interaction and the second one is the free $O(N)$ vector model deformed by a symmetry-breaking interaction term that breaks the $O(N)$ symmetry in various ways. We consider these models in $d = 4-\epsilon$ dimensions where the deformation is relevant. In section D.3, we consider holographic RG flows that give rise to order one anomalous dimensions. In such situations it is difficult to map conformal families of the two CFTs and obtain concrete realization of multiplet recombination. Still, it remains true that in the deformed theory there is (at least) a spin-1 short operator less, and a spin-1 long operator more (with all its descendants)[1]. Therefore, one can prescribe the term 'current multiplet recombination' even for these more intricate situations.

D.1 $\lambda\phi^4$ theory

Consider a scalar field theory with quartic coupling

$$S = \int d^dx \quad \left(\frac{1}{2}\partial\phi\cdot\partial\phi + \frac{1}{4!}\lambda\phi^4\right) \ , \tag{D.1}$$

In $d = 4-\epsilon$ the β-function of the coupling λ has a non-trivial zero

$$\lambda_* = \frac{16\pi^2}{3}\epsilon \ . \tag{D.2}$$

[1]This is true as long as there are no emergent symmetries in the IR. The latter, however, would not affect the multiplet recombination we are discussing, and hence we do not consider such a possibility.

known as the Wilson-Fisher fixed point. At this fixed point, the theory is a weakly interacting conformal field theory. The interaction induces small anomalous dimensions for all local operators $\mathcal{O}_\Delta$ in the theory: $\Delta \to \Delta + \gamma$. In particular, the field ϕ acquires a small anomalous dimension γ_ϕ. We want to calculate γ_ϕ to the leading order in the interaction. The traditional method to calculate the leading order correction to γ_ϕ involves a two loop Feynman diagram Fig. D.1. This is problem 13.2 in Peskin & Schroeder. However, we can

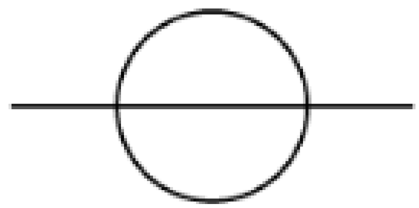

Figure D.1: Leading order contribution to γ_ϕ comes from this two-loop diagram.

completely avoid the computation of this diagram by making use of multiplet recombination.

At the UV point, ϕ is a primary operator that satisfies the shortening condition $\Box\phi = 0$ which means that the scalar descendant $\Box\phi$ is absent. However, at the Wilson-Fisher fixed point (D.1), the following operator relation holds (it essentially follows from the equations of motion)

$$\Box\phi = \frac{\lambda_*}{3!}\phi^3 \ , \tag{D.3}$$

This relation implies that the operator ϕ^3, which was a primary operator in the free CFT, becomes a descendant of ϕ at $\lambda = \lambda_*$ Since this is an operator relation, it follows that

$$\langle \Box\phi(x)\Box\phi(0)\rangle = \left(\frac{\lambda_*}{3!}\right)^2 \langle \phi^3(x)\phi^3(0)\rangle \ . \tag{D.4}$$

The left hand side is equal to

$$\langle \Box\phi(x)\Box\phi(0)\rangle = \frac{h}{(2\pi)^{d/2}|x|^{2\Delta+4}} \ , \tag{D.5}$$

where $h = 16\Delta(\Delta+1)(\Delta - \frac{d-2}{2})\left(\Delta - \frac{d-4}{2}\right)$, and the right hand side is equal to

$$\left(\frac{\lambda_*}{3!}\right)^2 \langle \phi^3(x)\phi^3(0)\rangle = \left(\frac{\lambda_*}{3!}\right)^2 \frac{3!}{(2\pi)^{3d/2}|x|^{2\Delta+4}} \ , \tag{D.6}$$

Equating them and solving for Δ, gives

$$\Delta = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108} + \mathcal{O}\left(\epsilon^3\right) \ , \tag{D.7}$$

from which we see that, to leading order $\gamma_\phi = \frac{1}{108}\epsilon^2$.

D.2 The $O(N)$ model

The action of the free $O(N)$ model in d dimensions is

$$S = \frac{1}{2}\sum_{i=1}^{N}\int d^d x \ \ (\partial\phi_i)^2 \ \ , \tag{D.8}$$

where ϕ_i are N real scalar fields. This theory possesses a global $O(N)$ symmetry, and the set of corresponding currents reads

$$J^a_\mu = -\partial_\mu\phi_i (T^a)^{ij}\phi_j, \qquad\quad a = 1,..,\frac{N(N-1)}{2} \ , \tag{D.9}$$

where T^a are generators of $O(N)$ (normalized here as Tr $\left(T^a T^b\right) = -2\delta^{ab}$). Using the scalar two-point function

$$\langle\phi_i(x)\phi_j(0)\rangle = \frac{\delta_{ij}}{(2\pi)^{d/2}|x|^{d-2}} \ , \tag{D.10}$$

we get the following two-point function for the currents

$$\langle J^a_\mu(x) J^b_\nu(0)\rangle = 2(d-2)\delta^{ab}\frac{I_{\mu\nu}}{(2\pi)^d|x|^{2d-2}} \ , \tag{D.11}$$

where $I_{\mu\nu}$ is defined in eq. (5.6). In $d = 4-\epsilon$ dimensions we see that $C_J = 4-2\epsilon$. We would like to deform this theory by a relevant deformation such that the resulting theory has a fixed point with (partially) broken global symmetry. To this end, let us consider the following deformation which breaks $O(N)$ to $O(N-1)$

$$S_{\text{def}} = \int d^d x \left(\frac{g_1}{4!}\phi_1^4 + \frac{g_2}{4}\phi_1^2\sum_{j=2}^{N}\phi_j^2 + \frac{g_3}{4!}\left(\sum_{j=2}^{N}\phi_j^2\right)^2\right) \ . \tag{D.12}$$

Let us first choose $g_2 = 0$. In this case, we have two decoupled sectors, a ϕ^4 theory and an interacting $O(N-1)$ model (which implies that g_2 will not be generated quantum mechanically either).

The RG flow resulting from this deformation ends up in a weakly interacting IR fixed point of the Wilson-Fisher type where the values of the couplings (g_{1*}, g_{3*}) are

$$g_{1*} = \frac{16\pi^2}{3}\epsilon + \mathcal{O}(\epsilon^2), \qquad g_{3*} = \frac{48\pi^2}{N+7}\epsilon + \mathcal{O}(\epsilon^2) \ , \tag{D.13}$$

and are obtained as the zeros of the following beta-function

$$\beta_{g_1} = -g_1\epsilon + \frac{3}{16\pi^2}g_1^2 \; , \qquad \beta_{g_3} = -g_3\epsilon + \frac{1}{48\pi^2}g_3^2(N+7) \; . \tag{D.14}$$

The deformation gives rise to the following anomalous currents which were otherwise conserved

$$\partial^\mu J^a_\mu = -\frac{g_{1*}}{3!}(T^a)^{1j}\phi_j\phi_1^3 + \frac{g_{3*}}{3!}(T^a)^{1j}\sum_{k=2}^{N}\phi_1\phi_j\phi_k\phi_k \; . \tag{D.15}$$

In total there are $N-1$ broken currents. Computing the two-point functions of operators on the right-hand side, which provides the values of C_O, one finally gets from eq. (5.11)

$$\gamma_J = \left(\frac{1}{108} + \frac{N+1}{4(N+7)^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \; . \tag{D.16}$$

This is nothing but the sum of the anomalous dimensions of constituent fields, ϕ_1 and ϕ_j ($j \neq 1$). This is expected because for $g_2 = 0$ the broken currents are composed of fields belonging to decoupled sectors.

The symmetry-breaking pattern we discussed here is an instance of the more general one $O(N) \to O(N-M) \times O(M)$, which is a straightforward generalization of the action (D.12)

$$\int d^dx \left[\frac{g_1}{4!}\left(\sum_{i=1}^{M}\phi_i^2\right)^2 + \frac{g_2}{4}\sum_{i=1}^{M}\phi_i^2\sum_{j=M+1}^{N}\phi_j^2 + \frac{g_3}{4!}\left(\sum_{j=M+1}^{N}\phi_j^2\right)^2\right] . \tag{D.17}$$

Assuming (g_1, g_2, g_3) correspond to the fixed-point values, one can use the trick of multiplet recombination to calculate a general formula for the anomalous dimension of broken currents:

$$\gamma_J = \frac{1}{(4\pi)^4}\left((M+2)\left(\frac{g_1}{3!} - \frac{g_2}{2}\right)^2 + (N-M+2)\left(\frac{g_3}{3!} - \frac{g_2}{2}\right)^2\right) \; , \tag{D.18}$$

and of elementary fields:

$$\gamma_{\phi_i} = \frac{1}{(4\pi)^4}\left((M+2)\left(\frac{g_1}{3!}\right)^2 + (N-M)\left(\frac{g_2}{2}\right)^2\right), \quad i = 1, \; ..., \; M \tag{D.19}$$

$$\gamma_{\phi_i} = \frac{1}{(4\pi)^4}\left((N-M+2)\left(\frac{g_3}{3!}\right)^2 + M\left(\frac{g_2}{2}\right)^2\right), \quad i = M+1, \; ..., \; N \; . \tag{D.20}$$

In agreement with general expectations, from above equations and eq. (D.9), it follows that whenever $g_2 = 0$ the anomalous dimension of broken currents equals the sum of anomalous dimensions of constituents elementary fields, but it does not otherwise.

To obtain the fixed-points we need the β-function of the couplings g_i. The one-loop β functions of the couplings g_i in $d = 4 - \epsilon$ dimensions read

$$
\begin{aligned}
\beta_{g_1} &= -g_1\epsilon + \frac{1}{16\pi^2}\left(\frac{g_1^2}{3}(M+8) + 3g_2^2(N-M)\right), \\
\beta_{g_2} &= -g_2\epsilon + \frac{g_2}{48\pi^2}\left(g_1(M+2) + g_3(N-M+2) + 12g_2\right), \\
\beta_{g_3} &= -g_3\epsilon + \frac{1}{16\pi^2}\left(\frac{g_3^2}{3}(N-M+8) + 3g_2^2 M\right) .
\end{aligned}
\tag{D.21}
$$

One can calculate the matrix of first derivatives of the β functions at its zeros and analyse the stability of these fixed points. It turns out that some of them are unstable, meaning that they could be reached only by fine-tuning of the UV couplings.

Besides the decoupled theory, $g_2 = 0$, for specific values of M and N there exist fixed points also when $g_2 \neq 0$. Here, we present few of the many possible fixed points and specify their nature, i.e., whether they are stable or unstable.

1. $\mathbf{M = 1}$: This case was discussed in (D.12) with the coupling g_2 switched off. For certain values of N one can also find fixed points with $g_2 \neq 0$, and with broken $O(N)$ symmetry. For example, this kind of (unstable in this case) fixed point exists for $N = 3$, but does not exist for $N = 4$.

2. $\mathbf{N = 2M}$: In this case, for any M there exist the following zeros of the β function equations (apart from the fixed points with two decoupled sectors, when $g_2 = 0$):

$$
g_{1*} = \frac{24\pi^2\epsilon}{M+4}\ , \quad g_{2*} = \frac{8\pi^2\epsilon}{M+4}\ , \quad g_{3*} = \frac{24\pi^2\epsilon}{M+4} \qquad \text{(Stable for } M < 2\text{)}
\tag{D.22}
$$

$$
g_{1*} = \frac{24\pi^2 M\epsilon}{M^2+8}\ , \quad g_{2*} = \frac{8\pi^2(4-M)\epsilon}{M^2+8}\ , \quad g_{3*} = \frac{24\pi^2\epsilon M}{M^2+8} \qquad \text{(Stable for } M = 3\text{)}
\tag{D.23}
$$

Note, however, that, regardless of their nature, current multiplet recombination occurs at any of *those* fixed points where the $O(N)$ symmetry is broken. In the case of $N = 2M$, the first fixed point corresponds to a preserved $O(N)$ symmetry. Therefore there is no multiplet recombination here. In the second case, which preserves only the $O(M) \times O(M)$ subgroup, the current multiplet recombination occurs and anomalous dimension (computed through the techniques of section 5.1) reads

$$
\gamma_J = \frac{\epsilon^2(M+2)(M-2)^2}{2(M^2+8)^2} \underset{M\to\infty}{\sim} \frac{\epsilon^2}{2M}\ .
\tag{D.24}
$$

D.3 AdS-to-AdS domain walls

We would like now to consider symmetry-breaking relevant deformations connecting $\mathcal{N}=1$ SCFT at strong coupling. This is outside the realm of (perturbative) QFT, and hence we will rely on holography. Flows of this kind are described by BPS solutions of five-dimensional $\mathcal{N}=2$ supergravity with an AdS-to-AdS domain-wall metric and one or more scalars having non-trivial profiles.

Note that in five-dimensional $\mathcal{N}=2$ supergravity scalars belong either to hypermultiplets or to vector multiplets. The former are dual to chiral operators, the latter to real linear multiplets (which contain the spin-one currents). Therefore, flows triggered by superpotential deformations imply that hypermultiplet scalars in general run. If the chiral operators are charged under a given symmetry, the corresponding bulk gauge fields undergo a Higgs mechanism and so, by supersymmetry, also the vector multiplet scalars are expected to run.

As an illustrative example, we consider below one such scenario. This corresponds to a SCFT with $U(1)_{\tilde{R}} \times U(1)$ symmetry (the always-present superconformal R symmetry and an abelian flavor symmetry) perturbed by a charged, relevant deformation $\mathcal{O}$ triggering a RG flow towards an IR fixed point. If there are no emergent symmetries in the IR, at such a fixed point only a $U(1)_R$ superconformal R-symmetry is preserved.[2] The current associated to the $U(1)$ symmetry recombines and acquires an anomalous dimension.

A two-parameter family of $\mathcal{N}=2$ supergravity theories describing flows of this kind was derived long ago [33]. This is $\mathcal{N}=2$ supergravity coupled to a vector multiplet and a hypermultiplet, with scalar manifold

$$\mathcal{M} = O(1,1) \times \frac{SU(2,1)}{SU(2)\times U(1)} \ . \tag{D.25}$$

The first factor is parametrized by the vector multiplet real scalar ρ, while the second factor by the four scalars belonging to the hypermultiplet, $q^X = (V, \sigma, \theta, \tau)$. The two gauge fields, the graviphoton A_M and the one sitting in the vector multiplet, B_M, gauge a $U(1)\times U(1)$ subgroup of the isometry group of the hyperscalar manifold. The graviphoton is dual to the R symmetry, and the gauge field B_M to the $U(1)$ flavor symmetry.

This theory admits different classes of solutions, depending on the gauging. For instance, there exist (a) domain-wall solutions which provide a holographic version [134] of the so-called τ_U conjecture, originally proposed in Ref. [135], (b) non-supersymmetric solutions which have been used to construct models of (holographic) gauge mediation [136]. We will focus, instead, on supersymmetric AdS-to-AdS solutions.

This model has been widely studied and we refer to Ref. [33] for any technical detail. In what follows we just summarize the results we need for our analysis.

[2]The IR R symmetry is different from the UV one; i.e., it is a combination of the original R symmetry and the (broken) flavor symmetry. Indeed, a relevant deformation breaks explicitly conformal invariance and in turn the superconformal UV R symmetry $U(1)_{\tilde{R}}$.

What we are interested in are supersymmetric solutions admitting a critical point (i.e. an AdS stationary point of the gravity superpotential) which preserves a $U(1)\times U(1)$ symmetry, and a second critical point preserving a $U(1)$ symmetry. As discussed in Ref. [33] (see also Ref. [134]), the existence of such fixed points selects a subclass of gaugings, parametrized by two real parameters, β and γ, subject to the condition

$$(\beta-1)(1-2\zeta)>0 \quad \cap \quad \zeta>0 \quad \text{where} \quad \zeta=\frac{1-\beta}{2\gamma-1}\ . \tag{D.26}$$

The UV and IR fixed points sit at

$$P_{UV} \quad : \quad q^X=(1,0,0,0)\ ,\ \rho=1 \tag{D.27}$$

$$P_{IR} \quad : \quad q^X=(1-\xi^2,0,\xi\cos\varphi,\xi\sin\varphi)\ ,\ \rho=(2\zeta)^{1/6}\ , \tag{D.28}$$

in field space, with

$$\xi=\sqrt{\frac{2-4\zeta}{3\beta-1-4\zeta}}\quad ,\quad \varphi\in[0,\,2\pi]\ . \tag{D.29}$$

Note that P_{IR} is in fact a circle of stationary points, parametrized by φ. This is an exactly marginal deformation of the IR SCFT, which does not play any role for what we want to do next.

For any value of β and γ satisfying the constraint (D.26), there exists a smooth domain-wall (numerical) solution interpolating between P_{UV} and P_{IR} [33,134]. Since P_{UV} and P_{IR} preserve different symmetries, these domain walls describe, holographically, RG flows along which current multiplets recombine. Note that, as advertised, both the hyperscalars and the real scalar ρ run (they have different values at P_{UV} and P_{IR}).

To read the gauge field masses, the relevant part of the $\mathcal{N}=2$ Lagrangian is

$$-\frac{1}{4}a_{IJ}F^I_{\mu\nu}F^{J\mu\nu}-\frac{1}{2}\left(g^2 g_{XY}K^X_I K^Y_J\right)A^I_\mu A^{\mu I}\ , \tag{D.30}$$

where a_{IJ} is a function of the vector scalar multiplet ρ, g controls the value of the cosmological constant and g_{XY} is the metric on the hyperscalar manifold. The Killing vectors are functions of the scalar fields, hence the gauge symmetry can be Higgsed or exactly realized depending on the scalar profiles. All flows interpolating between P_{UV} and P_{IR} admit a vanishing Killing vector [33], hence a massless gauge field and, correspondingly, a preserved $U(1)$ symmetry (which can be shown to be an R symmetry [33,134]). This reduces to the superconformal R symmetry $U(1)_{\tilde{R}}$ in the UV and to the superconformal R-symmetry $U(1)_R$ in the IR. The second Killing vector, associated to the gauge field B_M, instead, vanishes at P_{UV}, only. This implies that B_M is massless at the UV fixed point, and massive elsewhere. Evaluating (D.30) on the IR endpoint of the flow, one finds, in units of the IR AdS radius

$L_{IR} = (g\, W_{IR})^{-1}$ (where W_{IR} is the value of the supergravity superpotential at P_{IR})

$$m_A^2 = 0 \quad , \quad m_B^2 = \frac{3}{4}\left(\frac{(2\beta+2\gamma-3)(6\beta\gamma+\beta-2\gamma-3)}{(2\gamma-1)^{4/3}\,(1-\beta)^{2/3}}\right) \ . \tag{D.31}$$

Plugging the above formula into the mass/dimension relation (5.12) one gets the holographic prediction for the $U(1)$ flavor current anomalous dimension.

As a consistency check, one can evaluate (D.31) for $\beta = -1, \gamma = \frac{3}{2}$, which, as shown in Ref. [33], corresponds to the FGPW flow [137]. This is known to describe, holographically, the $\mathcal{N} = 1^*$ mass deformation of $\mathcal{N} = 4$ theory. One gets $m_B^2 = 6$ and in turn $\Delta = 2 + \sqrt{7}$, in agreement with expectations [33, 137].

The supergravity model we have considered is a prototype of more general ones. It is amusing to see how holography lets one have control on how multiplets recombine even in RG flows which might be extremely intricate from a field theory perspective, and how it makes so the description of in principle very complicated UV/IR operator maps so transparent.

Appendix E

β-deformed SCFTs and the dual geometry

E.1 β-deformations: matter fields quantum numbers

In this section we report the quantum numbers of matter fields of the $\mathcal{N}=4$ SYM, conifold theory and $Y^{p,q}$ theories.

1. $\mathcal{N}=4$ SYM: When written in the $\mathcal{N}=1$ language, $\mathcal{N}=4$ SYM theory contains 3 chiral superfields Φ^i that transform in the fundamental representation of $SU(3)$. The R-charges of each of these is 2/3 as is evident from the $\mathcal{N}=4$ superpotential.

2. Conifold theory: The theory contains two kind of bifundamental matter fields $A_\alpha, B_{\dot\alpha}$. They share the same R-charge $R=1/2$, and, correspondingly, the same scaling dimension $\Delta = 3/4$. The fields A_α transform in the $(\frac{1}{2},0)$ of the flavor symmetry group $SU(2)\times SU(2)$. The $B_{\dot\alpha}$ transform instead in the $(0,\frac{1}{2})$.

3. $\mathcal{N}=1$ $Y^{p,q}$ theories: The theory contains four different kinds of bifundamental matter fields which are either singlets or doublets under the $SU(2)$ flavor symmetry. There are p doublets labelled U_α, q doublets labelled V_α, $p-q$ singlets labelled Z and $p+q$ singlets labelled Y. Under the $U(1)$ flavor (non-R) symmetry these fields have charges $0,1,-1,1$ respectively whereas under the $U(1)$ R symmetry they have the following charges

$$
\begin{aligned}
r_U &= \frac{2}{3}pq^{-2}\left(2p-\sqrt{4p^2-3q^2}\right), \\
r_V &= \frac{1}{3}q^{-1}\left(3q-2p+\sqrt{4p^2-3q^2}\right), \\
r_Z &= \frac{1}{3}q^{-2}\left(-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2}\right), \\
r_Y &= \frac{1}{3}q^{-2}\left(-4p^2+3q^2-2pq+(2p-q)\sqrt{4p^2-3q^2}\right) .
\end{aligned}
\tag{E.1}
$$

E.2 Volumes of X_5 and the 2-torus

In this section we give the expressions for $\mathrm{Vol}(S^5)/\mathrm{Vol}(X_5)$ and $\langle g_{0,E}\rangle$ which were needed to derive C_O in section 5.4. The ratio of the volumes defined in eq. (5.31), are (see [124] and references therein for details)

$$\frac{\mathrm{Vol}(S^5)}{\mathrm{Vol}(T^{1,1})} = \frac{27}{16}\ , \qquad \frac{\mathrm{Vol}(S^5)}{\mathrm{Vol}(Y^{p,q})} = \frac{3p^2\left(3q^2 - 2p^2 + p\sqrt{4p^2-3q^2}\right)}{q^2\left(2p + \sqrt{4p^2-3q^2}\right)}\ . \tag{E.2}$$

The average value of the determinant of the internal two-torus $\langle g_{0,E}\rangle$ can be computed from the corresponding metrics given in [124]. We summarize them below.

1. $\mathbf{S^5}$: The 2-torus in (3.12) of [124] is parametrized by the coordinates (φ_1, φ_2). The average volume is $\langle g_{0,E}\rangle = \pi N$

2. $\mathbf{T^{1,1}}$: This case is slightly subtle. The 2-torus in this case is parametrized by the coordinates $\varphi_{1,2} = \frac{\phi_1 \pm \phi_2}{2}$, where $\phi_{1,2}$ are the coordinates appearing in the standard line element ((A.18) [124]) of $T^{1,1}$. Taking this into account one finds[1] $\langle g_{0,E}\rangle = \frac{5\pi}{9}N$.

3. $\mathbf{Y^{p,q}}$: Here the two-torus in eq (A.24) of [124] is parametrized by (α, ϕ). We have $\langle g_{0,E}\rangle = \langle g_0\rangle R_E^4$, where the determinant g_0 and R_E have been defined in appendix A.2 of [124]. Upon computing the average we find

$$\langle g_{0,E}\rangle = \frac{7p^2 - 6q^2 - p\sqrt{4p^2-3q^2}}{9p(p^2-q^2)}\pi N\ . \tag{E.3}$$

   In computing $\langle g_0\rangle$ we have used the following relation for a

$$a = \frac{1}{2} - \frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}\ , \tag{E.4}$$

   and the integration over the y coordinate is in the range (y_1, y_2)

$$y_1 = \frac{1}{4p}\left(2p - 3q - \sqrt{4p^2-3q^2}\right), \quad y_2 = \frac{1}{4p}\left(2p + 3q - \sqrt{4p^2-3q^2}\right)\ . \tag{E.5}$$

[1]The last equality of (4.6) in [124] has a typo. We thank O. Lunin for a discussion on this point.

Bibliography

[1] M. Bertolini, D. Musso, I. Papadimitriou, and H. Raj, *A goldstino at the bottom of the cascade*, JHEP **11** (2015) 184, [arXiv:1509.0359].

[2] V. Bashmakov, M. Bertolini, L. Di Pietro, and H. Raj, *Scalar Multiplet Recombination at Large N and Holography*, JHEP **05** (2016) 183, [arXiv:1603.0038].

[3] V. Bashmakov, M. Bertolini, and H. Raj, *Broken current anomalous dimensions, conformal manifolds, and renormalization group flows*, Phys. Rev. **D95** (2017), no. 6 066011, [arXiv:1609.0982].

[4] H. Raj, *A note on sphere free energy of p-form gauge theory and Hodge duality*, arXiv:1611.0250.

[5] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, [hep-th/9510017].

[6] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[7] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. **323** (2000) 183–386, [hep-th/9905111].

[8] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS / CFT correspondence*, in Strings, Branes and Extra Dimensions: TASI 2001: Proceedings, pp. 3–158, 2002. hep-th/0201253.

[9] S. Kachru and E. Silverstein, *4-D conformal theories and strings on orbifolds*, Phys. Rev. Lett. **80** (1998) 4855–4858, [hep-th/9802183].

[10] I. R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, Nucl. Phys. **B536** (1998) 199–218, [hep-th/9807080].

[11] C. P. Herzog, I. R. Klebanov, and P. Ouyang, *Remarks on the warped deformed conifold*, in Modern Trends in String Theory: 2nd Lisbon School on g Theory Superstrings Lisbon, Portugal, July 13-17, 2001, 2001. hep-th/0108101.

[12] M. K. Benna and I. R. Klebanov, *Gauge-String Dualities and Some Applications*, in String theory and the real world: From particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics, 87th Session, Les Houches, France, July 2-27, 2007, pp. 611–662, 2008. arXiv:0803.1315.

[13] M. J. Strassler, *The Duality cascade*, in Progress in string theory. Proceedings, Summer School, TASI 2003, Boulder, USA, June 2-27, 2003, pp. 419–510, 2005. hep-th/0505153.

[14] I. R. Klebanov and A. M. Polyakov, *AdS dual of the critical O(N) vector model*, Phys. Lett. **B550** (2002) 213–219, [hep-th/0210114].

[15] S. Giombi and X. Yin, *The Higher Spin/Vector Model Duality*, J. Phys. **A46** (2013) 214003, [arXiv:1208.4036].

[16] S. Giombi and I. R. Klebanov, *One Loop Tests of Higher Spin AdS/CFT*, JHEP **12** (2013) 068, [arXiv:1308.2337].

[17] S. Giombi, I. R. Klebanov, and A. A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin AdS_{d+1}/CFT_d*, Phys. Rev. **D90** (2014), no. 2 024048, [arXiv:1402.5396].

[18] S. Giombi, I. R. Klebanov, and B. R. Safdi, *Higher Spin AdS_{d+1}/CFT_d at One Loop*, Phys. Rev. **D89** (2014), no. 8 084004, [arXiv:1401.0825].

[19] S. Giombi, I. R. Klebanov, and Z. M. Tan, *The ABC of Higher-Spin AdS/CFT*, arXiv:1608.0761.

[20] E. Witten, *Multitrace operators, boundary conditions, and AdS / CFT correspondence*, hep-th/0112258.

[21] W. Mueck, *An Improved correspondence formula for AdS / CFT with multitrace operators*, Phys. Lett. **B531** (2002) 301–304, [hep-th/0201100].

[22] I. Papadimitriou, *Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT*, JHEP **05** (2007) 075, [hep-th/0703152].

[23] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, Adv. Theor. Math. Phys. **2** (1998) 781–846, [hep-th/9712074].

[24] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, *Correlation functions in the CFT(d) / AdS(d+1) correspondence*, Nucl. Phys. **B546** (1999) 96–118, [hep-th/9804058].

[25] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, JHEP **07** (1998) 023, [hep-th/9806087].

[26] M. Bianchi, D. Z. Freedman, and K. Skenderis, *Holographic renormalization*, Nucl. Phys. **B631** (2002) 159–194, [hep-th/0112119].

[27] M. Bianchi, D. Z. Freedman, and K. Skenderis, *How to go with an RG flow*, JHEP **08** (2001) 041, [hep-th/0105276].

[28] I. R. Klebanov and M. J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities*, JHEP **08** (2000) 052, [hep-th/0007191].

[29] A. Dymarsky, I. R. Klebanov, and N. Seiberg, *On the moduli space of the cascading SU(M+p) x SU(p) gauge theory*, JHEP **01** (2006) 155, [hep-th/0511254].

[30] S. Kachru, J. Pearson, and H. L. Verlinde, *Brane / flux annihilation and the string dual of a nonsupersymmetric field theory*, JHEP **06** (2002) 021, [hep-th/0112197].

[31] R. Argurio, M. Bertolini, D. Musso, F. Porri, and D. Redigolo, *Holographic Goldstino*, Phys. Rev. **D91** (2015), no. 12 126016, [arXiv:1412.6499].

[32] A. Ceresole and G. Dall'Agata, *General matter coupled N=2, D = 5 gauged supergravity*, Nucl. Phys. **B585** (2000) 143–170, [hep-th/0004111].

[33] A. Ceresole, G. Dall'Agata, R. Kallosh, and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, Phys. Rev. **D64** (2001) 104006, [hep-th/0104056].

[34] J. A. Strathdee, *EXTENDED POINCARE SUPERSYMMETRY*, Int. J. Mod. Phys. **A2** (1987) 273.

[35] S. S. Gubser, *Curvature singularities: The Good, the bad, and the naked*, Adv. Theor. Math. Phys. **4** (2000) 679–745, [hep-th/0002160].

[36] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, JHEP **10** (2004) 075, [hep-th/0407071].

[37] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253–291, [hep-th/9802150].

[38] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett. **B428** (1998) 105–114, [hep-th/9802109].

[39] I. R. Klebanov and N. A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, Nucl. Phys. **B574** (2000) 263–274, [hep-th/9911096].

[40] I. R. Klebanov and A. A. Tseytlin, *Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories*, Nucl. Phys. **B578** (2000) 123–138, [hep-th/0002159].

[41] R. Argurio, M. Bertolini, S. Franco, and S. Kachru, *Gauge/gravity duality and meta-stable dynamical supersymmetry breaking*, JHEP **01** (2007) 083, [hep-th/0610212].

[42] R. Argurio, M. Bertolini, S. Franco, and S. Kachru, *Meta-stable vacua and D-branes at the conifold*, JHEP **06** (2007) 017, [hep-th/0703236].

[43] O. DeWolfe, S. Kachru, and M. Mulligan, *A Gravity Dual of Metastable Dynamical Supersymmetry Breaking*, Phys. Rev. **D77** (2008) 065011, [arXiv:0801.1520].

[44] R. Argurio, M. Bertolini, L. Pietro, F. Porri, and D. Redigolo, *Supercurrent multiplet correlators at weak and strong coupling*, JHEP **04** (2014) 123, [arXiv:1310.6897].

[45] R. Argurio, D. Musso, and D. Redigolo, *Anatomy of new SUSY breaking holographic RG flows*, JHEP **03** (2015) 086, [arXiv:1411.2658].

[46] I. Bena, M. Grana, and N. Halmagyi, *On the Existence of Meta-stable Vacua in Klebanov-Strassler*, JHEP **09** (2010) 087, [arXiv:0912.3519].

[47] B. Michel, E. Mintun, J. Polchinski, A. Puhm, and P. Saad, *Remarks on brane and antibrane dynamics*, JHEP **09** (2015) 021, [arXiv:1412.5702].

[48] G. S. Hartnett, *Localised Anti-Branes in Flux Backgrounds*, JHEP **06** (2015) 007, [arXiv:1501.0656].

[49] D. Cohen-Maldonado, J. Diaz, T. van Riet, and B. Vercnocke, *Observations on fluxes near anti-branes*, JHEP **01** (2016) 126, [arXiv:1507.0102].

[50] S. Kuperstein, B. Truijen, and T. Van Riet, *Non-SUSY fractional branes*, JHEP **03** (2015) 161, [arXiv:1411.3358].

[51] I. Papadimitriou and K. Skenderis, *AdS / CFT correspondence and geometry*, IRMA Lect. Math. Theor. Phys. **8** (2005) 73–101, [hep-th/0404176].

[52] D. Cassani and A. F. Faedo, *A Supersymmetric consistent truncation for conifold solutions*, Nucl. Phys. **B843** (2011) 455–484, [arXiv:1008.0883].

[53] I. Bena, G. Giecold, M. Grana, N. Halmagyi, and F. Orsi, *Supersymmetric Consistent Truncations of IIB on $T^{1,1}$*, JHEP **04** (2011) 021, [arXiv:1008.0983].

[54] A. Buchel, *Effective Action of the Baryonic Branch in String Theory Flux Throats*, JHEP **09** (2014) 117, [arXiv:1405.1518].

[55] I. Bena, G. Giecold, M. Grana, N. Halmagyi, and S. Massai, *The backreaction of anti-D3 branes on the Klebanov-Strassler geometry*, JHEP **06** (2013) 060, [arXiv:1106.6165].

[56] O. Aharony, A. Buchel, and A. Yarom, *Holographic renormalization of cascading gauge theories*, Phys. Rev. **D72** (2005) 066003, [hep-th/0506002].

[57] S. S. Gubser, *Dilaton driven confinement*, hep-th/9902155.

[58] A. Kehagias and K. Sfetsos, *On Running couplings in gauge theories from type IIB supergravity*, Phys. Lett. **B454** (1999) 270–276, [hep-th/9902125].

[59] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz, *Diffeomorphisms and holographic anomalies*, Class. Quant. Grav. **17** (2000) 1129–1138, [hep-th/9910267].

[60] S. Frolov, I. R. Klebanov, and A. A. Tseytlin, *String corrections to the holographic RG flow of supersymmetric SU(N) x SU(N + M) gauge theory*, Nucl. Phys. **B620** (2002) 84–108, [hep-th/0108106].

[61] R. Kallosh, F. Quevedo, and A. M. Uranga, *String Theory Realizations of the Nilpotent Goldstino*, JHEP **12** (2015) 039, [arXiv:1507.0755].

[62] E. A. Bergshoeff, K. Dasgupta, R. Kallosh, A. Van Proeyen, and T. Wrase, $\overline{\text{D3}}$ *and dS*, JHEP **05** (2015) 058, [arXiv:1502.0762].

[63] S. Ferrara, R. Kallosh, and A. Linde, *Cosmology with Nilpotent Superfields*, JHEP **10** (2014) 143, [arXiv:1408.4096].

[64] A. Buchel, *Transport properties of cascading gauge theories*, Phys. Rev. **D72** (2005) 106002, [hep-th/0509083].

[65] O. Aharony, A. Buchel, and A. Yarom, *Short distance properties of cascading gauge theories*, JHEP **11** (2006) 069, [hep-th/0608209].

[66] M. Berg, M. Haack, and W. Mueck, *Bulk dynamics in confining gauge theories*, Nucl. Phys. **B736** (2006) 82–132, [hep-th/0507285].

[67] M. Berg, M. Haack, and W. Mueck, *Glueballs vs. Gluinoballs: Fluctuation Spectra in Non-AdS/Non-CFT*, Nucl. Phys. **B789** (2008) 1–44, [hep-th/0612224].

[68] M. Haack and W. Muck, *Holographic renormalization of two-point functions in non-AdS/non-CFT*, Adv. High Energy Phys. **2010** (2010) 103630.

[69] S. Rychkov and Z. M. Tan, *The ϵ-expansion from conformal field theory*, J. Phys. **A48** (2015), no. 29 29FT01, [arXiv:1505.0096].

[70] S. Ghosh, R. K. Gupta, K. Jaswin, and A. A. Nizami, *ϵ-Expansion in the Gross-Neveu model from conformal field theory*, JHEP **03** (2016) 174, [arXiv:1510.0488].

[71] A. Raju, *ϵ-Expansion in the Gross-Neveu CFT*, JHEP **10** (2016) 097, [arXiv:1510.0528].

[72] M. Berkooz, A. Sever, and A. Shomer, *'Double trace' deformations, boundary conditions and space-time singularities*, JHEP **05** (2002) 034, [hep-th/0112264].

[73] S. S. Gubser and I. Mitra, *Double trace operators and one loop vacuum energy in AdS / CFT*, Phys. Rev. **D67** (2003) 064018, [hep-th/0210093].

[74] S. S. Gubser and I. R. Klebanov, *A Universal result on central charges in the presence of double trace deformations*, Nucl. Phys. **B656** (2003) 23–36, [hep-th/0212138].

[75] T. Hartman and L. Rastelli, *Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT*, JHEP **01** (2008) 019, [hep-th/0602106].

[76] M. Flato and C. Fronsdal, *Quantum Field Theory of Singletons: The Rac*, J. Math. Phys. **22** (1981) 1100.

[77] C. Fronsdal, *The Dirac Supermultiplet*, Phys. Rev. **D26** (1982) 1988.

[78] M. J. Duff, *Anti-de Sitter space, branes, singletons, superconformal field theories and all that*, Int. J. Mod. Phys. **A14** (1999) 815–844, [hep-th/9808100].

[79] A. Starinets, *Singleton field theory and Flato-Fronsdal dipole equation*, Lett. Math. Phys. **50** (1999) 283–300, [math-ph/9809014].

[80] T. Ohl and C. F. Uhlemann, *Saturating the unitarity bound in $AdS/CFT_(AdS)$*, JHEP **05** (2012) 161, [arXiv:1204.2054].

[81] S. Giombi and I. R. Klebanov, *Interpolating between a and F*, JHEP **03** (2015) 117, [arXiv:1409.1937].

[82] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, *Generalized F-Theorem and the ϵ Expansion*, JHEP **12** (2015) 155, [arXiv:1507.0196].

[83] O. Aharony, Z. Komargodski, and S. Yankielowicz, *Disorder in Large-N Theories*, JHEP **04** (2016) 013, [arXiv:1509.0254].

[84] I. R. Klebanov and E. Witten, *AdS / CFT correspondence and symmetry breaking*, Nucl. Phys. **B556** (1999) 89–114, [hep-th/9905104].

[85] M. Porrati and C. C. Y. Yu, *Notes on Relevant, Irrelevant, Marginal and Extremal Double Trace Perturbations*, JHEP **11** (2016) 040, [arXiv:1609.0035].

[86] A. B. Zamolodchikov, *Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory*, JETP Lett. **43** (1986) 730–732. [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)].

[87] Z. Komargodski and A. Schwimmer, *On Renormalization Group Flows in Four Dimensions*, JHEP **12** (2011) 099, [arXiv:1107.3987].

[88] Z. Komargodski, *The Constraints of Conformal Symmetry on RG Flows*, JHEP **07** (2012) 069, [arXiv:1112.4538].

[89] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safdi, *Towards the F-Theorem: N=2 Field Theories on the Three-Sphere*, JHEP **06** (2011) 102, [arXiv:1103.1181].

[90] H. Casini and M. Huerta, *On the RG running of the entanglement entropy of a circle*, Phys. Rev. **D85** (2012) 125016, [arXiv:1202.5650].

[91] D. E. Diaz and H. Dorn, *Partition functions and double-trace deformations in AdS/CFT*, JHEP **05** (2007) 046, [hep-th/0702163].

[92] M. Bianchi, J. F. Morales, and H. Samtleben, *On stringy AdS(5) x S**5 and higher spin holography*, JHEP **07** (2003) 062, [hep-th/0305052].

[93] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, *Higher spin symmetry and N=4 SYM*, JHEP **07** (2004) 058, [hep-th/0405057].

[94] M. Bianchi, *Higher spin symmetry (breaking) in N=4 sym theory and holography*, Comptes Rendus Physique **5** (2004) 1091–1099, [hep-th/0409292].

[95] M. Bianchi, P. J. Heslop, and F. Riccioni, *More on La Grande Bouffe*, JHEP **08** (2005) 088, [hep-th/0504156].

[96] L. Girardello, M. Porrati, and A. Zaffaroni, *3-D interacting CFTs and generalized Higgs phenomenon in higher spin theories on AdS*, Phys. Lett. **B561** (2003) 289–293, [hep-th/0212181].

[97] R. G. Leigh and A. C. Petkou, *Singleton deformation of higher-spin theory and the phase structure of the three-dimensional O(N) vector model*, Phys. Rev. **D88** (2013), no. 4 046006, [arXiv:1212.4421].

[98] P. J. Heslop and F. Riccioni, *On the fermionic Grande Bouffe: More on higher spin symmetry breaking in AdS/CFT*, JHEP **10** (2005) 060, [hep-th/0508086].

[99] J. N. Laia and D. Tong, *Flowing Between Fermionic Fixed Points*, JHEP **11** (2011) 131, [arXiv:1108.2216].

[100] D. Anselmi, *The N=4 quantum conformal algebra*, Nucl. Phys. **B541** (1999) 369–385, [hep-th/9809192].

[101] A. V. Belitsky, J. Henn, C. Jarczak, D. Mueller, and E. Sokatchev, *Anomalous dimensions of leading twist conformal operators*, Phys. Rev. **D77** (2008) 045029, [arXiv:0707.2936].

[102] P. Basu and C. Krishnan, *ϵ-expansions near three dimensions from conformal field theory*, JHEP **11** (2015) 040, [arXiv:1506.0661].

[103] E. D. Skvortsov, *On (Un)Broken Higher-Spin Symmetry in Vector Models*, in Proceedings, International Workshop on Higher Spin Gauge Theories: Singapore, Singapore, November 4-6, 2015, pp. 103–137, 2017. arXiv:1512.0599.

[104] S. Giombi and V. Kirilin, *Anomalous dimensions in CFT with weakly broken higher spin symmetry*, JHEP **11** (2016) 068, [arXiv:1601.0131].

[105] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, Class. Quant. Grav. **30** (2013) 104003, [arXiv:1204.3882].

[106] D. Green, Z. Komargodski, N. Seiberg, Y. Tachikawa, and B. Wecht, *Exactly Marginal Deformations and Global Symmetries*, JHEP **06** (2010) 106, [arXiv:1005.3546].

[107] L. Di Pietro, Z. Komargodski, I. Shamir, and E. Stamou, *Quantum Electrodynamics in d=3 from the Expansion*, Phys. Rev. Lett. **116** (2016), no. 13 131601, [arXiv:1508.0627].

[108] S. Giombi, I. R. Klebanov, and G. Tarnopolsky, *Conformal QED_d, F-Theorem and the ϵ Expansion*, J. Phys. **A49** (2016), no. 13 135403, [arXiv:1508.0635].

[109] R. G. Leigh and M. J. Strassler, *Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory*, Nucl. Phys. **B447** (1995) 95–136, [hep-th/9503121].

[110] S. Benvenuti and A. Hanany, *Conformal manifolds for the conifold and other toric field theories*, JHEP **08** (2005) 024, [hep-th/0502043].

[111] B. Kol, *On conformal deformations*, JHEP **09** (2002) 046, [hep-th/0205141].

[112] B. Kol, *On Conformal Deformations II*, arXiv:1005.4408.

[113] Y. Tachikawa, *Five-dimensional supergravity dual of a-maximization*, Nucl. Phys. **B733** (2006) 188–203, [hep-th/0507057].

[114] J. Louis and C. Muranaka, *Moduli spaces of AdS_5 vacua in $\mathcal{N} = 2$ supergravity*, JHEP **04** (2016) 178, [arXiv:1601.0048].

[115] O. Aharony, B. Kol, and S. Yankielowicz, *On exactly marginal deformations of N=4 SYM and type IIB supergravity on AdS(5) x S**5*, JHEP **06** (2002) 039, [hep-th/0205090].

[116] A. Ashmore, M. Gabella, M. Graña, M. Petrini, and D. Waldram, *Exactly marginal deformations from exceptional generalised geometry*, JHEP **01** (2017) 124, [arXiv:1605.0573].

[117] D. Anselmi, J. Erlich, D. Z. Freedman, and A. A. Johansen, *Positivity constraints on anomalies in supersymmetric gauge theories*, Phys. Rev. **D57** (1998) 7570–7588, [hep-th/9711035].

[118] E. Barnes, E. Gorbatov, K. A. Intriligator, and J. Wright, *Current correlators and AdS/CFT geometry*, Nucl. Phys. **B732** (2006) 89–117, [hep-th/0507146].

[119] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, *Supersymmetric AdS(5) solutions of M theory*, Class. Quant. Grav. **21** (2004) 4335–4366, [hep-th/0402153].

[120] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, Commun. Math. Phys. **262** (2006) 51–89, [hep-th/0411238].

[121] M. Bertolini, F. Bigazzi, and A. L. Cotrone, *New checks and subtleties for AdS/CFT and a-maximization*, JHEP **12** (2004) 024, [hep-th/0411249].

[122] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, and J. Sparks, *An Infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, JHEP **06** (2005) 064, [hep-th/0411264].

[123] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, *Three point functions of chiral operators in $D = 4$, N=4 SYM at large N*, Adv. Theor. Math. Phys. **2** (1998) 697–718, [hep-th/9806074].

[124] O. Lunin and J. M. Maldacena, *Deforming field theories with U(1) x U(1) global symmetry and their gravity duals*, JHEP **05** (2005) 033, [hep-th/0502086].

[125] A. Butti, D. Forcella, L. Martucci, R. Minasian, M. Petrini, and A. Zaffaroni, *On the geometry and the moduli space of beta-deformed quiver gauge theories*, JHEP **07** (2008) 053, [arXiv:0712.1215].

[126] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **10** (2008) 091, [arXiv:0806.1218].

[127] M. S. Bianchi and S. Penati, *The Conformal Manifold of Chern-Simons Matter Theories*, JHEP **01** (2011) 047, [arXiv:1009.6223].

[128] H. Osborn and A. Stergiou, *Seeking Fixed Points in Multiple Coupling Scalar Theories in the ε Expansion*, arXiv:1707.0616.

[129] J. T. Liu and P. Szepietowski, *Supersymmetry of consistent massive truncations of IIB supergravity*, Phys. Rev. **D85** (2012) 126010, [arXiv:1103.0029].

[130] N. Halmagyi, J. T. Liu, and P. Szepietowski, *On $N = 2$ Truncations of IIB on $T^{1,1}$*, JHEP **07** (2012) 098, [arXiv:1111.6567].

[131] I. R. Klebanov, P. Ouyang, and E. Witten, *A Gravity dual of the chiral anomaly*, Phys. Rev. **D65** (2002) 105007, [hep-th/0202056].

[132] G. Papadopoulos and A. A. Tseytlin, *Complex geometry of conifolds and five-brane wrapped on two sphere*, Class. Quant. Grav. **18** (2001) 1333–1354, [hep-th/0012034].

[133] K. A. Intriligator, *Maximally supersymmetric RG flows and AdS duality*, Nucl. Phys. **B580** (2000) 99–120, [hep-th/9909082].

[134] M. Bertolini, L. Di Pietro, and F. Porri, *Holographic R-symmetric flows and the τ_U conjecture*, JHEP **08** (2013) 071, [arXiv:1304.1481].

[135] M. Buican, *A Conjectured Bound on Accidental Symmetries*, Phys. Rev. **D85** (2012) 025020, [arXiv:1109.3279].

[136] R. Argurio, M. Bertolini, L. Di Pietro, F. Porri, and D. Redigolo, *Holographic Correlators for General Gauge Mediation*, JHEP **08** (2012) 086, [arXiv:1205.4709].

[137] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, *Renormalization group flows from holography supersymmetry and a c theorem*, Adv. Theor. Math. Phys. **3** (1999) 363–417, [hep-th/9904017].