Algebraic contraction rate for distance between entropy solutions of scalar conservation laws

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Abstract

We establish an algebraic contraction rate in a modified Wasserstein distance for solutions of scalar conservation laws with uniformly convex flux. We also show that our estimate is optimal w.r.t. scaling in time and discuss why it gives non-trivial information in relation to the stability of the rarefaction wave.

Contents

1 Introduction

The aim of this paper is to establish a contraction result in a modified transport distance for entropy solutions to a scalar conservation law

$$\partial_t \theta + \partial_x f(\theta) = 0, \tag{1.1}$$

with uniformly convex flux f. This estimate gives non-trivial information in relation to the stability of the rarefaction wave solution of (1.1) and its contraction rate turns out to be optimal in terms of scaling in time.

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The intuition behind the arguments to derive this contraction result are geometric in nature. As noted already in [8], the Burgers' equation

$$\partial_t \theta + \partial_x(\theta(\theta - 1)) = 0, \tag{1.2}$$

a special case of equation (1.1), can be written formally as a gradient flow of the energy

$$F(\theta) = \int x\theta(x)dx,$$

with respect to the two-phase Wasserstein space. In [8] these insights are derived in the physical context of a relaxed version of a model of the flow of two immiscible fluids of different density and mobility in a porous medium. One well-known benefit of formally writing partial differential equations as a gradient flow in Wasserstein space is deriving contraction results, provided the energy is semi-convex.

Unfortunately, this is not the case when writing the Burgers' equation as a gradient flow as above, i.e. the energy F is not semi-convex. This can formally be seen by Lemma 3.2, which tells us that the Hessian of F is given by

$$\text{Hess } F(\theta) = \frac{-\partial_x f'(\theta)}{2} \,\text{id}\,.$$

Since $\eta(t, x) =: \eta_t(x) = H(x)$, where H is the Heaviside function, is a solution to (1.2) with formally

$$\partial_x f'(\eta_t) = 2\partial_x \eta_t = +\infty,$$

we see that in general the Hessian of F is not bounded from below. This is not surprising, since intuitively speaking this corresponds to the non-uniqueness of solutions to the initial value problem related to (1.2), see the discussion in [5]. To reestablish uniqueness for this initial value problem, a well-known selection principle is introduced: the notion of entropy solution. That these entropy solutions also play a special role in the above context of the gradient flow interpretation is the content of [5], namely the fact that the time-discretized gradient flow for F with respect to the two-phase Wasserstein metric converges to the entropy solution.

The idea of our work here is to use special distinguishing features of the entropy solution to obtain a contraction-like estimate for these solutions. Indeed, a careful analysis of the Hessian of F along entropy solutions shows us that the well-known Oleinik condition ensures a kind of semi-convexity which improves over time, namely

$$\text{Hess } F(\theta_t) \geq -\frac{1}{2t} \,\text{id}\,.$$

Geometrically speaking we establish a semi-convexity of the energy landscape along certain trajectories, despite the fact that the global energy landscape of F is highly non-convex. Altogether this provides an interesting example of a gradient flow which satisfies a certain contractivity in spite of the fact that it lives in an energy landscape that is not globally semi-convex.

2 Preliminaries

2.1 Preliminaries on scalar conservation laws

In this section, we quickly recall some facts about the entropy solution to a scalar conservation law, which can be for example found in [11].

We shall assume that the flux $f : \mathbb{R} \to \mathbb{R}$ is C^∞ and uniformly convex. The scalar conservation law associated to f is the initial value problem

$$\partial_t \theta + \partial_x f(\theta) = 0 \quad \text{in } \mathbb{R}^t_+ \times \mathbb{R}^x, \tag{2.1}$$

where $\theta(0, \cdot)$ is given. As it is well-known, due to the crossing of characteristics, in general after finite time classical solutions cease to exist, even in the case of smooth initial data. On the other hand, distributional solutions are non-unique in general. Hence there is the need to find a notion of solution which grants basic existence and uniqueness properties. It turns out that the correct notion of solution is that of *entropy* solution. One of the equivalent ways to introduce it is by the Oleinik principle, first observed in [7]: $\theta : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is an entropy solution for (2.1) provided it is a distributional solution and furthermore for any $t > 0$ satisfies

$$\partial_x f'(\theta_t) \le \frac{1}{t} \qquad \text{in the sense of distributions.} \tag{2.2}$$

A typical assumption about the initial value $\bar\theta$ for solutions of (2.1) is $\bar\theta \in L^1 \cap L^\infty(\mathbb{R})$. For more details on entropy solutions, see [11]. In this paper we will deal with a slightly modified version of these hypotheses. Specifically, denote by $H : \mathbb{R} \to \mathbb{R}$ the Heaviside function, i.e.

$$H(x) := \left\{ \begin{array}{ll} 1, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0, \end{array} \right\}.$$

We then consider initial data $\bar\theta$ satisfying $\bar\theta(x) \in [0, 1]$ for every $x \in \mathbb{R}$ (in particular $\bar\theta \in L^\infty(\mathbb{R})$) and $\bar\theta - H \in L^1(\mathbb{R})$. Existence, uniqueness, stability and convergence of the viscous approximation for this kind of initial data can be achieved with minor modifications as in the standard case. We refer to [3] or [11] for an overview of the theory and state without proof those basic properties we shall use later on. The first is the following existence, uniqueness and stability result, see e.g. [3, Theorem 6.2.2 & 6.2.3].

Theorem 2.1 (Existence, uniqueness and stability of entropy solutions)**.** *Let $f \in C^\infty(\mathbb{R})$ be uniformly convex and $\bar\theta : \mathbb{R} \to [0, 1]$ such that $\int \bar\theta(x) - H(x)\,\mathrm{d}x < \infty$. Then for the scalar conservation law* (2.1) *there exists a unique entropy solution* $(0, \infty) \times \mathbb{R} \ni (t, x) \mapsto \theta_t(x) \in [0, 1]$ *such that θ_t converges to $\bar\theta$ as $t \downarrow 0$ weakly in duality with $C_c(\mathbb{R})$.*

Furthermore, if $(\bar\theta^n)$ is a sequence of initial data as above converging to some $\bar\theta^\infty$ as $n \to \infty$ weakly in duality with $C_c(\mathbb{R})$ and θ^n, θ^∞ the corresponding entropy solutions, then the sequence $n \mapsto \theta^n_t$ converges to θ^∞_t weakly in duality with $C_c(\mathbb{R})$ for every $t \ge 0$.

A different way to introduce the notion of entropy solution is to perturb (2.1) by adding a second order regularizing term like

$$\partial_t \theta^\varepsilon + \partial_x f(\theta^\varepsilon) - \varepsilon \partial_x (f''(\theta^\varepsilon) \partial_x \theta^\varepsilon) = 0, \tag{2.3}$$

and let ε go to zero. The uniform ellipticity of (2.3) grants that for any initial datum $\bar\theta \in L^\infty(\mathbb{R})$ a solution $\theta : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ exists, is unique in the class of locally bounded functions and smooth in $(0, \infty) \times \mathbb{R}$, where the initial value $\bar\theta$ is assumed in the sense that θ_t converges to $\bar\theta$ weakly in duality with $C_c(\mathbb{R})$. Then one can show that as $\varepsilon \downarrow 0$ such solutions weakly converge to a specific distributional solution of (2.1), which turns out to be the same solution singled out by Oleinik's principle (see e.g. [3, Theorem 16.4.2]):

Theorem 2.2 (Convergence of the viscous approximation)**.** *Let $f \in C^\infty(\mathbb{R})$ be uniformly convex and $\bar\theta : \mathbb{R} \to [0,1]$ such that $\int \bar\theta(x) - H(x)\,\mathrm{d}x < \infty$. Denote by $\theta : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ the entropy solution of* (2.1) *starting from $\bar\theta$ and by $\theta^\varepsilon : (0,\infty) \times \mathbb{R} \to [0,1]$ the solution of* (2.3) *starting from $\bar\theta$.*

Then for every $t \geq 0$ the family (θ^ε_t) weakly converges to θ_t as $\varepsilon \downarrow 0$ in duality with $C_c(\mathbb{R})$.

A remarkable property of the solutions of the particular viscous approximation (2.3) is that they also satisfy Oleinik's condition. This is due to the specific choice of the second order term, as we illustrate in the following simple proposition.

Proposition 2.3. *Let $f \in C^\infty(\mathbb{R})$ be uniformly convex, $\bar\theta \in C^\infty(\mathbb{R})$ with $\bar\theta(x) \in [0,1]$ for every $x \in \mathbb{R}$, $T \geq 0$ and let $\theta \in C^\infty([T,\infty) \times \mathbb{R})$ be the solution of* (2.3) *such that $\theta_T = \bar\theta$. Assume that*

$$\partial_x\big(f'(\theta_T)\big)(x) \leq \frac{1}{T}, \qquad \forall x \in \mathbb{R}, \tag{2.4}$$

this condition being vacuous if $T = 0$.

Then for all $x \in \mathbb{R}$, $t > T$ we have

$$\partial_x\big(f'(\theta_t)\big)(x) \leq \frac{1}{t}. \tag{2.5}$$

Proof. Multiplying equation (2.3) by $f''(\theta)$ and differentiating with respect to x we find that the resulting equation can be rewritten as an equation in $w := \partial_x f'(\theta)$, which reads as

$$\partial_t w + w^2 + f'(\theta)\partial_x w - \varepsilon\partial_x(f''(\theta)\partial_x w) = 0. \tag{2.6}$$

Assume for a moment that $T > 0$ and notice that $\tilde w_t(x) := \frac{1}{t}$ also solves (2.6) and fulfills $\tilde w_T(x) \geq w_T(x)$ for every $x \in \mathbb{R}$. Since f is uniformly convex, the second-order term is uniformly elliptic and thus for (2.6) the comparison principle holds. Hence in this case the thesis follows. The general case can now be handled with an approximation argument based on the stability of the solutions of (2.3) w.r.t. convergence of the initial datum. □

By the convergence of the viscous approximation to the entropy solution we thus also see that Oleinik's principle can be formulated in the following seemingly stronger - but in fact equivalent - way: if $\theta : [T,\infty) \times \mathbb{R} \to [0,1]$ is an entropy solution of (2.1) starting from a function $\bar\theta$ fulfilling (2.4) in the sense of distributions, then θ_t fulfills (2.5) in the sense of distributions for every $t > T$.

2.2 The spaces $\mathcal{M}, \mathcal{M}_{t,m}$

We introduce here the space of functions $\mathcal{M}$ as follows. Recall that $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function.

Definition 2.4. *Denote by $\mathcal{M}$ the set of functions $\theta : \mathbb{R} \to [0,1]$ such that*

$$\left.\begin{aligned} L(\theta) &:= \sup\{x : \int_{-\infty}^x \theta = 0\} > -\infty, \\ R(\theta) &:= \inf\{x : \int_x^{+\infty} 1 - \theta = 0\} < +\infty, \end{aligned}\right\} \qquad \textit{bounded "mixing zone"}, \tag{2.7}$$

and

$$\int_{\mathbb{R}} \theta - H = 0, \qquad \textit{volume constraint.} \tag{2.8}$$

Notice that if θ is an entropy solution of (2.1) with admissible flux f (see Definition 2.13) such that θ_0 satisfies (2.7), then by the general principle of finite speed of propagation for solutions of (2.1) we deduce that θ_t satisfies (2.7) for every $t \geq 0$. Also, by the translation invariance of solutions of (2.1) it is not restrictive to assume that $\int \theta - H = 0$, which easily yields $\int \theta_t - H = 0$ for every $t \geq 0$. In summary, we have

$$\theta_0 \in \mathcal{M} \qquad \Rightarrow \qquad \theta_t \in \mathcal{M},\ \forall t \geq 0. \tag{2.9}$$

In order to state and prove our contraction estimate, we introduce the subspaces $\mathcal{M}_{t,m}$ depending on a time parameter $t \geq 0$ and a 'mobility function' m which will be linked to the equation.

The assumption that we will make on m are the following.

Definition 2.5. *A function $m : [0,1] \to \mathbb{R}$ is called an admissible mobility, if*

- *$m \in C^\infty([0,1])$,*
- *$m(0) = m(1) = 0$,*
- *m is uniformly concave, i.e. there exists $\alpha > 0$ s.t. $m'' \leq -\alpha < 0$.*

Given an admissible mobility m, the spaces $\mathcal{M}_{t,m}$ are defined as follows.

Definition 2.6. *For $t = 0$ we put $\mathcal{M}_{0,m} := \mathcal{M}$ and for $t > 0$ the set $\mathcal{M}_{t,m} \subset \mathcal{M}$ denotes the set of functions $\theta \in \mathcal{M}$, s.t. additionally*

$$-\partial_x m'(\theta) \leq \frac{1}{t} \text{ in the distributional sense.} \tag{2.10}$$

Remark 2.7. We will always take $m = -f$, f being the flux in (2.1). Thus condition (2.10) is the Oleinik condition and entropy solutions θ of (2.1) are characterized among all distributional solutions by the requirement $\theta_t \in \mathcal{M}_{t,m}$.

Also, Oleinik's principle as stated after Proposition 2.3 grants that for an entropy solution θ of (2.1) it holds

$$\theta_T \in \mathcal{M}_{T,m} \qquad \Rightarrow \qquad \theta_t \in \mathcal{M}_{t,m}, \quad \forall t \geq T. \tag{2.11}$$

■

On the set $\mathcal{M}_{t,m}$ we now define a modified Wasserstein distance $\mathsf{d}_{t,m}$, which will turn out to be the correct distance for which to state our main contraction result.

First of all notice that $\mathcal{M}$ can be naturally equipped with the quadratic transportation distance W_2 by putting

$$W_2^2(\theta_1, \theta_2) := \inf_{T : T_\sharp \theta_1 = \theta_2} \int_{\mathbb{R}} |T(x) - x|^2 \theta_1(x)\, \mathrm{d}x.$$

As for the standard transport problem between measures of finite mass, it is easy to check that there exists a unique non decreasing map T_{opt} such that $(T_{opt})_\sharp \theta_1 = \theta_2$, and that this map is the unique minimizer of the problem above. Also thanks to the bounded mixing zone and volume constraint, for all $\theta_1, \theta_2 \in \mathcal{M}$ there exist $L < R$ s.t.

$$W_2^2(\theta_1, \theta_2) = W_2^2\left(\theta_1|_{[L,R]}, \theta_2|_{[L,R]}\right) < \infty.$$

Alternatively, it can be checked directly that the well-known Benamou-Brenier formula holds also in the case of measures with infinite mass and reads as

$$W_2^2(\theta_1,\theta_2) = \inf \int_0^1 \int_{\mathbb{R}} v_s(x)^2 \theta_s(x)\,\mathrm{d}x\,\mathrm{d}s, \tag{2.12}$$

the infimum being taken over all distributional solutions (θ, v) of the continuity equation

$$\partial_s \theta_s + \partial_x(v_s \theta_s) = 0,$$

such that $s \mapsto \theta_s$ is weakly continuous with $\theta_0 = \theta_1$ and $\theta_1 = \theta_2$.

Observe that the right-hand side of (2.12) can formally be viewed as a Riemannian distance, via the continuity equation, with respect to the Riemannian metric

$$g_\theta(\delta\theta_1, \delta\theta_2) := \int_{\mathbb{R}} v_1 v_2 \theta\,\mathrm{d}x,$$

where

$$\delta\theta_i + \partial_x(v_i\theta) = 0, \qquad \text{for } i = 1, 2.$$

Equivalently, putting $j := v\theta$ the continuity equation becomes

$$\partial_s \theta_s + \partial_x j_s = 0,$$

and the Benamou-Brenier formula reads as

$$W_2^2(\theta_1,\theta_2) = \inf \int_0^1 \int_{\mathbb{R}} \frac{j_s^2}{\theta_s}\,\mathrm{d}x\,\mathrm{d}s. \tag{2.13}$$

We are now formally changing the Riemannian metric to obtain a different distance, namely replace g by g_m defined by

$$g_{m,\theta}(\delta\theta_1, \delta\theta_2) := \int_{\mathbb{R}} v_1 v_2 m(\theta)\,\mathrm{d}x,$$

where

$$\delta\theta_i + \partial_x(v_i m(\theta)) = 0, \qquad \text{for } i = 1, 2.$$

This metric and the corresponding distance was already discussed in [2] and [4]. In our case we additionally we force all admissible curves between $\theta_1, \theta_2 \in \mathcal{M}_{t,m}$ to completely lie in $\mathcal{M}_{t,m}$.

Definition 2.8. *Let $\theta_1, \theta_2 \in \mathcal{M}_{t,m}$. Then*

$$\mathsf{d}_{t,m}^2(\theta_1,\theta_2) := \inf \int_0^1 \int v_s^2 m(\theta_s)\,\mathrm{d}x\,\mathrm{d}s, \tag{2.14}$$

the infimum being taken over all distributional solutions (θ, v) of

$$\partial_s \theta + \partial_x(v_s m(\theta_s)) = 0, \tag{2.15}$$

such that $s \mapsto \theta_s$ is weakly continuous with $\theta_s \in \mathcal{M}_{t,m}$ for every $s \in [0,1]$, $\theta(0,\cdot) = \theta_1$ and $\theta(1,\cdot) = \theta_2$.

Notice that putting $j := vm(\theta)$, equation (2.15) becomes

$$\partial_s \theta_s + \partial_x j_s = 0. \tag{2.16}$$

In the following it will be technically more convenient to work with the continuity equation written in this form. We collect in the definition below the basic properties of solutions of (2.16) which we will always demand from now on:

Definition 2.9 (Solutions of the continuity equation). *We say that $[0,1] \ni s \mapsto (\theta_s, j_s)$ is a solution of (2.16) provided it is a distributional solution and furthermore the map $s \mapsto \theta_s$ is weakly continuous in duality with $C_c(\mathbb{R})$.*

Thus an equivalent expression for $\mathsf{d}_{t,m}$ is

$$\mathsf{d}^2_{t,m}(\theta_1, \theta_2) := \inf \int_0^1 \int \frac{j_s^2}{m(\theta_s)} \, \mathrm{d}x \, \mathrm{d}s, \tag{2.17}$$

the infimum being taken over all solutions (θ, j) of (2.16) such that $\theta_0 = \theta_1$, $\theta_1 = \theta_2$, and $\theta_s \in \mathcal{M}_{t,m}$ for every $s \in [0,1]$. We remark that here and in the following the value of the ratio $\frac{|j|^2(x)}{m(\theta)(x)}$ is defined to be 0 if $m(\theta)(x) = j(x) = 0$ and $+\infty$ if $m(\theta)(x) = 0 \neq j(x)$.

It is obvious from the definition that $\mathsf{d}_{t,m}$ is symmetric, satisfies the triangle inequality and that $\mathsf{d}_{t,m}(\overline{\theta}, \overline{\theta}) = 0$. The fact that $\mathcal{M}_{t_2,m} \subset \mathcal{M}_{t_1,m} \subset \mathcal{M}$ for $0 \leq t_1 \leq t_2$ yields

$$\mathsf{d}_{0,m} \leq \mathsf{d}_{t_1,m} \leq \mathsf{d}_{t_2,m}.$$

Also, comparing (2.13) with (2.17) and using the trivial inequality $\frac{1}{z} \leq \frac{m'(0)}{m(z)}$ valid for any $z \in [0,1]$ we obtain

$$W_2 = \mathsf{d}_{0,id} \leq \sqrt{|m'(0)|}\mathsf{d}_{0,m} \leq \sqrt{|m'(0)|}\mathsf{d}_{t,m},$$

which in particular shows that $\mathsf{d}_{t,m}(\theta_1, \theta_2) = 0$ implies $\theta_1 = \theta_2$. As mentioned before, for a more in depth discussion of the distance $\mathsf{d}_{0,m}$, see [2] and [4]. A fact which a priori is not obvious - due to the constraint $\theta_t \in \mathcal{M}_{t,m}$ - is the finiteness of the distances $\mathsf{d}_{t,m}$. We will prove this in Proposition 3.1.

Proposition 2.10 (Regularization). *Let $t \geq 0$ and $s \mapsto (\theta_s, j_s)$ a solution of the continuity equation (2.16) such that $\theta_s \in \mathcal{M}_{t,m}$ for every $s \in [0,1]$ with $\sup_{s\in[0,1]} R(\theta_s) < \infty$ and $\inf_{s\in[0,1]} L(\theta_s) > -\infty$. Then there exists a sequence $t_n \uparrow t$ and a sequence of solutions $s \mapsto (\theta_{n,s}, j_{n,s})$ of the continuity equation (2.16) such that:*

i) for every $n \in \mathbb{N}$ the maps $(s,x) \mapsto \theta_{n,s}(x), j_{n,s}(x) \in \mathbb{R}$ are in $C^\infty([0,1] \times \mathbb{R})$,

ii) $\theta_{n,s} \in \mathcal{M}_{t_n,m}$ for every $n \in \mathbb{N}$, $s \in [0,1]$, with $\sup_{n,s} R(\theta_{n,s}) < \infty$ and $\inf_{n,s} L(\theta_{n,s}) > -\infty$

iii) $\lim_{n\to\infty} \mathsf{d}_{t_n,m}(\theta_{n,i}, \theta_i) = 0$, $i = 0, 1$,

iv) it holds

$$\lim_{n\to\infty} \iint_0^1 \frac{|j_{n,s}|^2(x)}{m(\theta_{n,s}(x))} \, \mathrm{d}s \, \mathrm{d}x = \iint_0^1 \frac{|j_s|^2(x)}{m(\theta_s(x))} \, \mathrm{d}s \, \mathrm{d}x.$$

Proposition 2.11 (Compactness)**.** *For every $n \in \mathbb{N}$ let $(\theta_{n,s}, j_{n,s})$ be a solution of the continuity equation* (2.16) *with $\theta_{n,s} \in \mathcal{M}$ for every $n \in \mathbb{N}$, $s \in [0,1]$. Assume that the sequence $n \mapsto \theta_{n,s}$ converges to some θ_s in duality with $C_c(\mathbb{R})$ for every $s \in [0,1]$ and that*

$$\sup_n \iint_0^1 \frac{|j_{n,s}|^2(x)}{\theta_{n,s}(x)} \,\mathrm{d}s\,\mathrm{d}x < \infty.$$

Then for some subsequence $n_k \uparrow +\infty$ the following is true:

i) $k \mapsto j_{n_k,s}$ converges to some j_s in duality with C_c for a.e. $s \in [0,1]$,

ii) $s \mapsto (\theta_s, j_s)$ solves the continuity equation (2.16)*,*

iii) $s \mapsto \theta_s$ is continuous in duality with $C_c(\mathbb{R})$.

Proposition 2.12 (Lower semicontinuity of the action)**.** *Let m be an admissible mobility, and for $n \in \mathbb{N} \cup \{\infty\}$ let $\overline{\theta}_n : \mathbb{R} \to [0,1]$ and $\overline{j}_n \in L^1_{loc}(\mathbb{R})$ be given. Assume that*

$$\overline{\theta}_n \to \overline{\theta}_\infty \quad \text{and} \quad \overline{j}_n \to \overline{j}_\infty \qquad \text{as } n \to \infty \text{ weakly in duality with } C_c(\mathbb{R}).$$

Then

$$\varliminf_{n\to\infty} \int_{\mathbb{R}} \frac{|\overline{j}_n|^2(x)}{m(\overline{\theta}_n)(x)} \,\mathrm{d}x \geq \int_{\mathbb{R}} \frac{|\overline{j}_\infty|^2(x)}{m(\overline{\theta}_\infty)(x)} \,\mathrm{d}x.$$

2.3 Heuristic discussion of gradient flow structure

In [8] it has been noticed that the gradient flow of the functional

$$F(\theta) := \int y(\theta(y) - H(y))\,\mathrm{d}y,$$

on the space w.r.t. the metric g_{-f} produces solutions of the scalar conservation law

$$\partial_t \theta + \partial_x (f(\theta)) = 0. \tag{2.18}$$

In this section we first recall this result, then show that also the viscous approximation of (2.18) has a natural gradient flow interpretation w.r.t. the metric g_{-f} and then proceed with the formal computation of the Hessian of the energy functional which drives our contraction result. The content of this part is purely formal, the rigorous statements and proofs being given in the next sections.

Let us first restrict ourselves to a smaller subclass of fluxes f.

Definition 2.13. *A flux f is called an admissible flux if $(-f)$ is an admissible mobility.*

We will see in Proposition 3.6 that the choice to work with admissible fluxes only is not really restrictive.

In the following let f always be an admissible flux, and $m := -f$ the corresponding admissible mobility.

Fix $\theta \in \mathcal{M}$ and let $\delta\theta_1, \delta\theta_2 : \mathbb{R} \to \mathbb{R}$ be perturbations of it in $\mathcal{M}$. Notice that the volume constraint (2.8) forces $\int \delta\theta_i = 0$, $i = 1, 2$. According to formula (2.15) we introduce the functions ϕ_1, ϕ_2 as solutions of

$$\delta\theta_i + \partial_x(\partial_x \phi_i m(\theta)) = 0, \tag{2.19}$$

and then define $g_{m,\theta}(\delta\theta_1, \delta\theta_2)$ as

$$g_{m,\theta}(\delta\theta_1, \delta\theta_2) := \int \partial_x\phi_1\partial_x\phi_2 m(\theta)\,\mathrm{d}x = \int \phi_2\delta\theta_1\,\mathrm{d}x. \tag{2.20}$$

The first variation of F at θ in the direction $\delta\theta_1$ is given by

$$DF_\theta(\delta\theta_1) = \int y\delta\theta_1(y)\,\mathrm{d}y.$$

Hence $\delta\theta_2 = \nabla_{g_m}F(\theta)$ if and only if

$$\int y\delta\theta_1(y)\,\mathrm{d}y = g_{m,\theta}(\delta\theta_1, \delta\theta_2) \overset{(2.20)}{=} \int \phi_2(y)\delta\theta_1(y)\,\mathrm{d}y, \qquad \forall\delta\theta_1,$$

which gives $\phi_2 = y + const.$ so that (2.19) yields $\nabla_{g_m}F(\theta) = -\partial_x(\partial_x\phi_2 m(\theta)) = \partial_x(f(\theta))$.

Therefore a curve $t \mapsto \theta_t$ solves $\partial_t\theta_t = -\nabla_{g_m}F(\theta_t)$ if and only if the map $(t,x) \mapsto \theta_t(x)$ solves (2.18), as claimed.

We now turn to the viscous approximation. Let $u : [0,1] \to \mathbb{R}$ be such that $u'' = \frac{f''}{(-f)}$ and define the functional $U : \mathcal{M} \to \mathbb{R}$ by

$$U(\theta) := \int u(\theta(y))\,\mathrm{d}y.$$

In the case of the Burgers' equation, i.e. $f(\theta) = \theta(\theta - 1)$, we find that this is nothing but the entropy of mixing

$$U(\theta) := 2\int \big(\theta(y)\ln\theta(y) + (1-\theta(y))\ln(1-\theta(y))\big)\,\mathrm{d}y.$$

Its first variation at θ in the direction $\delta\theta_1$ is given by

$$DU_\theta(\delta\theta_1) = \int u'(\theta(y))\delta\theta_1(y)\,\mathrm{d}y.$$

Thus, as before, $\delta\theta_2 = \nabla_{g_m}U(\theta)$ if and only if

$$\int u'(\theta(y))\delta\theta_1(y)\,\mathrm{d}y = g_{m,\theta}(\delta\theta_1, \delta\theta_2) \overset{(2.20)}{=} \int \phi_2(y)\delta\theta_1(y)\,\mathrm{d}y, \qquad \forall\delta\theta_1,$$

i.e. $\phi_2 = u'(\theta) + const.$. Then by (2.19) we obtain $\nabla_{g_m}U(\theta) = -\partial_x(\partial_x\phi_2 m(\theta)) = \partial_x(f''(\theta)\partial_x\theta)$, having used the defining property of u. Therefore a curve $t \mapsto \theta_t$ solves $\partial_t\theta_t = -\nabla_{g_m}U(\theta_t)$ if and only if the map $(t,x) \mapsto \theta_t(x)$ solves

$$\partial_t\theta - \partial_x(f''(\theta)\partial_x\theta) = 0. \tag{2.21}$$

Collecting these two properties we deduce that the gradient flow of the functional

$$F_\varepsilon := F + \varepsilon U,$$

produces solutions of

$$\partial_t\theta + \partial_x(f(\theta)) - \varepsilon\partial_x(f''(\theta)\partial_x\theta) = 0. \tag{2.22}$$

We now informally discuss the properties of the Hessian of F_ε and show how this leads to an algebraic contraction rate for the entropy solutions of (2.18). In order to derive a formula for the Hessian of F_ε we follow the idea in [10]: rather than differentiating the functional along geodesics, we compute the rate of change of the metric tensor along gradient flows.

More precisely, let $E : \mathcal{M} \to \mathbb{R}$ be a functional and θ be solving

$$\partial_t \theta_t = -\nabla_{g_m} E(\theta_t).$$

Let $\delta\theta_0 \in T_{\theta_0}\mathcal{M}$ be a perturbation of θ_0 and observe that it evolves in time according to

$$\frac{\mathrm{D}}{\mathrm{d}t}\delta\theta_t = -\mathrm{Hess}_{g_m} E(\theta_t)(\delta\theta_t), \tag{2.23}$$

where we are writing $\frac{\mathrm{D}}{\mathrm{d}t}$ for the - formal - covariant derivative in $\mathcal{M}$. On the other hand, basic Riemannian calculus gives

$$\partial_t \tfrac{1}{2} g_{m,\theta_t}(\delta\theta_t, \delta\theta_t) = g_{m,\theta_t}\Big(\frac{\mathrm{D}}{\mathrm{d}t}\delta\theta_t, \delta\theta_t\Big).$$

Coupling this equality with (2.23) we get

$$\partial_t \tfrac{1}{2} g_{m,\theta_t}(\delta\theta_t, \delta\theta_t) = -g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m} E(\theta_t)(\delta\theta_t), \delta\theta_t\Big). \tag{2.24}$$

The right hand side of this equality contains the unknown $\mathrm{Hess}_{g_m} E(\theta_t)$, but the left hand side can be computed, hence we can use this equality to define the Hessian of E.

This computation was at the basis of [10], now we see how it translates in our context. Pick $E := F$. Then given that θ is a gradient flow of F, by the above discussion it solves (2.18). Hence the perturbation $\delta\theta$ solves

$$\partial_t \delta\theta + \partial_x(f'(\theta)\delta\theta) = 0.$$

We can use this equation to explicitly evaluate the left hand side of (2.24) (we postpone it to Lemma 3.2), the result is

$$g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m} F(\theta_t)(\delta\theta_t), \delta\theta_t\Big) = -\int \frac{1}{2}|\partial_x \phi_t|^2 m(\theta_t)\partial_x(f'(\theta_t))\,\mathrm{d}x. \tag{2.25}$$

This suggests that F is not semiconvex, because $\partial_x(f'(\theta))$ is not bounded from above independently on θ and thus we cannot expect an inequality like

$$g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m} F(\theta_t)(\delta\theta_t), \delta\theta_t\Big) \geq -C g_{m,\theta_t}(\delta\theta_t, \delta\theta_t) = -C\int |\partial_x\phi_t|^2 m(\theta_t)\,\mathrm{d}x$$

to hold. This lack of semiconvexity can be seen as the geometric counterpart of the fact that (2.18) has non-unique solutions in general. Yet, by the Oleinik principle we know that if θ_t is the *entropy* solution of (2.18), then it holds

$$g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m} F(\theta_t)(\delta\theta_t), \delta\theta_t\Big) \geq -\frac{1}{t}\int \frac{1}{2}|\partial_x\phi_t|^2 m(\theta_t)\,\mathrm{d}x,$$

or equivalently

$$\partial_t \tfrac{1}{2} g_{m,\theta_t}(\delta\theta_t, \delta\theta_t) \leq \frac{1}{t}\int \frac{1}{2}|\partial_x\phi_t|^2 m(\theta_t)\,\mathrm{d}x = \frac{1}{2t} g_{m,\theta_t}(\delta\theta_t, \delta\theta_t). \tag{2.26}$$

We can rewrite this inequality as

$$\partial_t\Big(\frac{1}{t}g_{m,\theta_t}(\delta\theta_t,\delta\theta_t)\Big)\le 0,$$

which integrated from 1 to t gives

$$g_{m,\theta_t}(\delta\theta_t,\delta\theta_t)\le t\, g_{m,\theta_1}(\delta\theta_1,\delta\theta_1).$$

This inequality can be seen as a contraction rate for the distance between two entropy solutions θ_t^1,θ_t^2. Once integrated it gives the bound

$$\mathsf{d}^2_{t,m}(\theta_t^1,\theta_t^2)\le t\,\mathsf{d}^2_{1,m}(\theta_1^1,\theta_1^2).$$

The rigorous justification of this inequality is the main content of this paper.

As a side remark, we notice that the choice of $m(z):=z$ in (2.25) provides a contraction rate similar to (2.26) for the standard Wasserstein distance W_2, even if the scalar conservation law (2.18) has not a gradient flow structure in the Wasserstein space (notice that in deriving (2.26) we didn't use the assumption $m=-f$, so that this argument is justified).

In performing the necessary computations, we will take advantage of passing to the viscous approximation of (2.18) in order to gain regularity of the objects involved. It turns out that also such viscous approximation has a natural geometric counterpart, as we shall now explain. If we take $E:=U$ in the Hessian computation and we recall that a gradient flow of U solves (2.21), we deduce that a perturbation $\delta\theta$ solves

$$\partial_t\delta\theta-\partial_x\big(f'''(\theta)\delta\theta\partial_x\theta+f''(\theta)\partial_x\delta\theta\big)=0.$$

Starting from this and using the arguments above we can compute (the actual computation is quite lengthy, we postpone it to Lemma 3.2) the Hessian of U as

$$g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m}U(\theta_t)(\delta\theta_t),\delta\theta_t\Big)=\int -|\partial_{xx}\phi_t|^2f''(\theta_t)f(\theta_t)+\frac{1}{2}|\partial_x\phi_t|^2\big(\partial_x(f'(\theta_t))\big)^2\,\mathrm{d}x\ge 0. \tag{2.27}$$

Hence U is geodesically convex in $\mathcal{M}$.

Interestingly enough, the augmented functional F_ε turns out to be $-\frac{|f|_\infty}{8\varepsilon}$-convex. Indeed from (2.25) and (2.27) we get

$$\begin{aligned}g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m}F_\varepsilon(\theta_t)(\delta\theta_t),\delta\theta_t\Big)&=\int\frac{1}{2}|\partial_x\phi_t|^2f(\theta_t)\partial_x(f'(\theta_t))\,\mathrm{d}x\\&\quad+\varepsilon\int -|\partial_{xx}\phi_t|^2f''(\theta_t)f(\theta_t)+\frac{1}{2}|\partial_x\phi_t|^2\big(\partial_x(f'(\theta_t))\big)^2\,\mathrm{d}x\\&\ge\frac{1}{2}\int|\partial_x\phi_t|^2f(\theta_t)\partial_x(f'(\theta_t))\,\mathrm{d}x+\varepsilon\frac{1}{2}\int|\partial_x\phi_t|^2\big(\partial_x(f'(\theta_t))\big)^2\,\mathrm{d}x\end{aligned}$$

and the Young inequality gives

$$\begin{aligned}\frac{1}{2}\int|\partial_x\phi_t|^2f(\theta_t)\partial_x(f'(\theta_t))\,\mathrm{d}x&=-\frac{1}{2}\int|\partial_x\phi_t|^2(-f(\theta_t))\partial_x(f'(\theta_t))\,\mathrm{d}x\\&\ge-\frac{|f|_\infty}{8\varepsilon}\int|\partial_x\phi_t|^2(-f(\theta_t))\,\mathrm{d}x-\varepsilon\frac{1}{2}\int|\partial_x\phi_t|^2(\partial_x(f'(\theta_t)))^2\,\mathrm{d}x,\end{aligned}$$

and therefore

$$g_{m,\theta_t}\Big(\mathrm{Hess}_{g_m} U(\theta_t)(\delta\theta_t), \delta\theta_t\Big) \geq -\frac{|f|_\infty}{8\varepsilon}\int |\partial_x \phi_t|^2(-f(\theta_t))\,\mathrm{d}x.$$

This semiconvexity can be interpreted as the geometric counterpart of the fact that solutions of (2.22) are unique.

3 Rigorous statements and proofs

3.1 Finiteness of the new distance

The aim of this section is to prove that $\mathsf{d}_{t,m}$ is finite on $\mathcal{M}_{t,m}$.

Proposition 3.1. *For any $t \geq 0$, any admissible mobility m and any $\theta^0, \theta^1 \in \mathcal{M}_{t,m}$ it holds*

$$\mathsf{d}_{t,m}(\theta^0, \theta^1) < \infty.$$

Furthermore, there exists a solution $(\theta_s(x), j_s(x))$ of the continuity equation (2.16) *such that $\theta_0 = \theta^0$, $\theta_1 = \theta^1$ and*

$$\iint_0^1 \frac{|j_s|^2(x)}{m(\theta_s)(x)}\,\mathrm{d}s\,\mathrm{d}x = \mathsf{d}^2_{t,m}(\theta^0, \theta^1). \tag{3.1}$$

Proof. The second part of the statement follows by a standard compactness-lower semicontinuity argument based on Propositions 2.11 and 2.12, thus we focus on the proof of the finiteness of the distance.
Setting up. Clearly we can assume $t > 0$. Recall that $\exists M > 0$ s.t.

$$\begin{aligned} \theta^0(x) = \theta^1(x) &= 0, \qquad \text{for } x < -M,\\ \theta^0(x) = \theta^1(x) &= 1, \qquad \text{for } x > M, \end{aligned}$$

and observe that to get the thesis it is sufficient to prove that

$$\mathsf{d}_{t,m}(\theta^0, \bar\theta) < \infty,$$

where $\bar\theta \in \mathcal{M}_{t,m}$ is some specific density depending only on θ^0, θ^1. We choose $\bar\theta$ to be the rarefaction wave at time T, i.e.

$$\bar\theta(x) := \left\{ \begin{array}{cc} 0, & x \leq -Tm'(0),\\ (-m')^{-1}(\frac{x}{T}), & -Tm'(0) < x < -Tm'(1),\\ 1, & -Tm'(1) \leq x, \end{array} \right\},$$

where $T \geq t$ is so large that

$$\begin{aligned} T_1 &:= -Tm'(0) \leq -(M+1),\\ T_2 &:= -Tm'(1) \geq (M+1). \end{aligned}$$

It is obvious that $-\partial_x m'(\bar\theta) \le \frac{1}{t}$, so that $\bar\theta \in \mathcal{M}_{t,m}$. Notice also that the smoothness of m grants that the function $(-m')^{-1}$ is C^1 and thus for some constants $c_1, c_2 > 0$ it holds

$$c_1 \le \big((-m')^{-1}\big)'(z) \le c_2, \qquad \forall z \in [-m'(0), -m'(1)], \tag{3.2}$$

hence up to increasing T we can also assume that

$$\frac{1}{T} \le c_2. \tag{3.3}$$

Observe that by a standard reparametrization argument one gets

$$\inf\left\{\int_0^1 \int \frac{j_s^2}{m(\theta_s)}\,\mathrm{d}x\,\mathrm{d}s\right\} = \left(\inf\left\{\int_0^1 \sqrt{\int \frac{j_s^2}{m(\theta_s)}\,\mathrm{d}x}\,\mathrm{d}s\right\}\right)^2,$$

(see for instance the proof of Thm. 5.4 in [4] for the details), hence to conclude it is sufficient to find a weakly continuous curve $s \to \theta_s \in \mathcal{M}_{t,m}$ such that for some j the continuity equation (2.16) holds and

$$\int_0^1 \sqrt{\int \frac{j_s^2}{m(\theta_s)}\,\mathrm{d}x}\,\mathrm{d}s < \infty. \tag{3.4}$$

Definition of the interpolating curve. Notice that since $-m'' > \alpha > 0$, $-m'$ is invertible. Then put

$$\eta_s(x) := (-m')^{-1}\Big(-(1-s)m'\big(\theta^0(x)\big) - sm'\big(\bar\theta(x)\big)\Big),$$

(observe that for the case $m(\theta) = \theta(1-\theta)$, the above is just the linear interpolation between θ^0 and $\bar\theta$). A direct consequence of the definition is that

$$\begin{aligned} &0 \le \eta_s \le 1 \\ &\eta_s(x) = 0 \qquad \text{for } x < T_1, \\ &\eta_s(x) = 1 \qquad \text{for } x > T_2. \end{aligned} \tag{3.5}$$

Also, the equality $-m'\big(\eta_s(x)\big) = -(1-s)m'\big(\theta^0(x)\big) - sm'\big(\bar\theta(x)\big)$ and the fact that $\theta^0, \bar\theta \in \mathcal{M}_{t,m}$ yield

$$-\partial_x(m'(\eta_s)) \le \frac{1}{t}, \qquad \text{in the sense of distributions.} \tag{3.6}$$

Furthermore, the inequality $-m'' \ge \alpha$ grants that $(-m')^{-1}$ is Lipschitz and therefore $[0,1] \ni s \mapsto \eta_s(x) \in [0,1]$ is Lipschitz uniformly on $x \in \mathbb{R}$. Hence defining $h_s := \int(\eta_s - H)dy \in \mathbb{R}$, we have that the map $s \mapsto h_s$ is Lipschitz as well.

Now put

$$\theta_s(x) := \eta_s(x - h_s).$$

By construction it holds $\int(\theta_s - H)dy = 0$ and the map $[0,1] \times \mathbb{R} \mapsto \theta_s(x)$ is continuous. Taking into account (3.5) we deduce $\theta_s \in \mathcal{M}$ for every $s \in [0,1]$ while the property (3.6) gives $\theta_s \in \mathcal{M}_{t,m}$ for every $s \in [0,1]$.

Define

$$j_s(x) := h_s'\eta_s(x - h_s) - \int_{-\infty}^x \partial_s\eta_s(y - h_s)\,\mathrm{d}y,$$

so that for a.e. $s \in [0,1]$, $j_s(x)$ is well defined for every $x \in \mathbb{R}$. By direct computation one can check that (θ, j) solve the continuity equation (2.16) in the sense of distributions. It is also easy to see that it holds

$$\begin{aligned} j_s(x) &= 0 \text{ for } x \le K_1 + h_s, \\ j_s(x) &= 0 \text{ for } x \ge K_2 + h_s, \end{aligned} \tag{3.7}$$

and

$$\sup_{s,x} |j_s(x)| < \infty. \tag{3.8}$$

Energy estimate. Our goal is to prove the bound (3.4). Start observing that $-m'' > \alpha > 0$ and $m(0) = m(1) = 0$ yield that $\frac{1}{m(\theta)} \le C \frac{1}{\theta(1-\theta)}$ for $C = \frac{2}{\alpha}$. Hence, letting $I_1 := [T_1, T_1 + \frac{1}{2}]$, $I_2 := [T_1 + \frac{1}{2}, T_2 - \frac{1}{2}]$ and $I_3 := [T_2 - \frac{1}{2}, T_2]$ and recalling (3.5), to conclude it is sufficient to show that

$$\int_{I_i} \frac{\big|j_s(y + h_s)\big|^2}{\eta_s(y)(1 - \eta_s(y))}\, \mathrm{d}y \le \frac{C}{s}, \qquad \forall s \in [0,1], \quad i = 1,2,3, \tag{3.9}$$

for some constant $C > 0$.

Since $-m'$ is increasing, the fact that $\theta^0(x) \in [0,1]$ for every $x \in \mathbb{R}$ and that $-m'(\bar\theta(x)) = \frac{x}{T}$ for $x \in \cup_{i=1}^3 I_i$ we get

$$-(1-s)m'(\theta^0(x)) - sm'(\bar\theta(x)) \ge -(1-s)m'(0) + s\frac{x}{T} = -m'(0) + s\frac{x - T_1}{T},$$

having recalled that $T_1 = -m'(0)T$. Hence from (3.2) we get

$$\eta_s(x) \ge sc_1 \frac{x - T_1}{T}, \qquad \forall x \in \cup_{i=1}^3 I_i, \quad \forall s \in [0,1]. \tag{3.10}$$

Similarly, from the inequality

$$-(1-s)m'(\theta^0(x)) - sm'(\bar\theta(x)) \le -(1-s)m'(1) + s\frac{x}{T} = -m'(1) - s\frac{T_2 - x}{T},$$

and (3.2) we deduce

$$\eta_s(x) \le 1 - sc_1 \frac{T_2 - x}{T}. \tag{3.11}$$

Estimate over I_1. For $x \in I_1$ we have $\theta^0(x) = 0$ and $-m'(\bar\theta(x)) = \frac{x}{T}$ and thus

$$\eta_s(x) = (-m')^{-1}\Big(- (1-s)m'(0) + s\frac{x}{T}\Big) = (-m')^{-1}\Big(- m'(0) + s\frac{x - T_1}{T}\Big). \tag{3.12}$$

From the second in (3.2) we therefore get

$$\eta_s(x) \le s\, c_2 \,\frac{x - T_1}{T} \le \frac{1}{2}, \qquad \forall s \in [0,1],\ x \in I_1, \tag{3.13}$$

having used (3.3). Coupling this bound with (3.10) we get

$$\eta_s(x)(1 - \eta_s(x)) \ge sc_1 \frac{x - T_1}{2T}, \qquad \forall s \in [0,1],\ x \in I_1. \tag{3.14}$$

From (3.12) and (3.2) we deduce $|\partial_s \eta_s(x)| \le c_2 \frac{x-T_1}{T}$, which together with the first inequality in (3.13) and the definition of $j_s(x)$ yields

$$|j_s(x+h_s)| \le C(x-T_1), \qquad \forall s\in[0,1],\ x\in I_1. \tag{3.15}$$

Hence

$$\int_{I_1} \frac{|j_s(x+h_s)|^2}{\eta_s(x)\big(1-\eta_s(x)\big)}\,\mathrm{d}x \le C\int_{I_1}\frac{(x-T_1)^2}{s(x-T_1)}\,\mathrm{d}x = \frac{C}{s}.$$

Estimate over I_2. From (3.10) and (3.11) we get

$$\frac{sc_1}{2T} \le \eta_s(x) \le 1-\frac{sc_1}{2T}, \qquad \forall s\in[0,1],\ x\in I_2.$$

Recalling from (3.8) that $j_s(x)$ is uniformly bounded in s,x we thus obtain

$$\begin{aligned}\int_{I_2} \frac{|j_s(x+h_s)|^2}{\eta_s(x)(1-\eta_s(x))}\,\mathrm{d}x &\le C\int_{I_2}\frac{1}{\eta_s(x)(1-\eta_s(x))}\,\mathrm{d}x\\ &= C\int_{I_2}\frac{1}{\eta_s(x)}\,\mathrm{d}x + C\int_{I_2}\frac{1}{1-\eta_s(x)}\,\mathrm{d}x \le \frac{C}{s}.\end{aligned}$$

Estimate over I_3. The argument is similar to the one used in I_1. For $x\in I_3$ we have $\theta^0(x)=1$ and $-m'(\bar\theta(x)) = \frac{x}{T}$ and thus

$$\eta_s(x) = (-m')^{-1}\Big(-(1-s)m'(1)+s\frac{x}{T}\Big) = (-m')^{-1}\Big(-m'(1)-s\frac{T_2-x}{T}\Big). \tag{3.16}$$

Hence from the second in (3.2) we get

$$\eta_s(x) \ge 1 - sc_2\frac{T_2-x}{T} \ge \frac{1}{2}, \qquad \forall s\in[0,1],\ x\in I_3, \tag{3.17}$$

which together with (3.11) gives

$$\eta_s(x)(1-\eta_s(x)) \ge sc_1\frac{T_2-x}{2T}, \qquad \forall s\in[0,1],\ x\in I_3.$$

From (3.16) we obtain $|\partial_s\eta_s(x)| \le c_2\frac{T_2-x}{T}$. Using this bound with the first inequality in (3.17), the definition of $j_s(x)$ and the fact that $j_s(T_2+h_s)=0$ (see the second in (3.7)) we get

$$|j_s(x+h_s)| = |j_s(T_2+h_s)-j_s(x+h_s)| \le h_s'|1-\eta_s(x)| + \int_x^{T_2}|\partial_s\eta_s|(y)\,\mathrm{d}y \le C(T_2-x).$$

Therefore we get

$$\int_{I_3} \frac{|j_s(x+h_s)|^2}{\eta_s(x)(1-\eta_s(x))}\,\mathrm{d}x \le C\int_{I_3}\frac{(T_2-x)^2}{s(T_2-x)}\,\mathrm{d}x \le \frac{C}{s}.$$

□

3.2 Contraction result

Lemma 3.2 (Hessian of F_ε)**.** *Let f, m be respectively an admissible flux and an admissible mobility, $M > 0$ and $a \in (0, \frac{1}{2})$. Let $\bar\theta \in C^\infty(\mathbb{R})$ be $2M$-periodic, i.e. $\bar\theta(x) = \bar\theta(x + 2M)$ for every $x \in \mathbb{R}$, and such that $\bar\theta(x) \in [a, 1-a]$ for every $x \in \mathbb{R}$. Let $T > 0$ and $\theta \in C^\infty([T,\infty)\times\mathbb{R})$ be the solution of*

$$\begin{cases} \partial_t\theta + \partial_x f(\theta) - \varepsilon\partial_x(f''(\theta)\partial_x\theta) &= 0, & \text{on } [T,\infty)\times[-M,M], \\ \theta_T &= \bar\theta, & \text{on } [-M,M], \\ \theta_t(x) &= \theta_t(x+2M) & \forall t \geq T,\ x \in \mathbb{R}. \end{cases} \tag{3.18}$$

Also, let $\overline{\delta\theta} \in C^\infty(\mathbb{R})$ be $2M$-periodic, with $\int_{-M}^{M} \overline{\delta\theta} = 0$ and $\delta\theta \in C^\infty([T,\infty)\times\mathbb{R})$ the solution of

$$\begin{cases} \partial_t\delta\theta + \partial_x(f'(\theta)\delta\theta) - \varepsilon\partial_x\big(f'''(\theta)\delta\theta\partial_x\theta + f''(\theta)\partial_x\delta\theta\big) &= 0, & \text{on } [T,\infty)\times[-M,M], \\ \delta\theta_T &= \overline{\delta\theta}, & \text{on } [-M,M], \\ \delta\theta_t(x) &= \delta\theta_t(x+2M), & \forall t \geq T,\ x \in \mathbb{R}. \end{cases} \tag{3.19}$$

Finally, define $\phi \in C^\infty([T,\infty)\times\mathbb{R})$ by

$$\begin{aligned} \partial_x\phi_t(x) &:= -\frac{1}{m(\theta_t(x))}\int_{-M}^{x} \delta\theta_t(y)\,\mathrm{d}y, \qquad \text{for } x \in [-M,M] \\ \phi_t(0) &:= 0. \end{aligned} \tag{3.20}$$

Then the map $t \mapsto \int_{-M}^{M} |\partial_x\phi_t|^2(x) m(\theta_t(x))\,\mathrm{d}x$ is C^∞ and it holds

$$\begin{aligned} \partial_t\frac{1}{2}\int_{-M}^{M} |\partial_x\phi|^2 m(\theta)\,\mathrm{d}x = &\int_{-M}^{M} \frac{1}{2}|\partial_x\phi|^2 m(\theta)\partial_x(f'(\theta))\,\mathrm{d}x \\ &+ \varepsilon\int_{-M}^{M} -|\partial_{xx}\phi|^2 f''(\theta)m(\theta) + \frac{1}{2}|\partial_x\phi|^2\partial_x(f'(\theta))\partial_x(m'(\theta))\,\mathrm{d}x \end{aligned} \tag{3.21}$$

for every $t \geq T$.

In particular, if $m = -f$ and $\partial_x(f'(\bar\theta)) \leq \frac{1}{T}$ on $\mathbb{R}$ it holds

$$\int_{-M}^{M} |\partial_x\phi_t|^2 m(\theta_t)\,\mathrm{d}x \leq \frac{t}{T}\int_{-M}^{M} |\partial_x\phi_T|^2 m(\theta_T)\,\mathrm{d}x, \qquad \forall t \geq T. \tag{3.22}$$

Proof. By the maximum principle we know that $\theta_t(x) \in [a, 1-a]$ for every $t \geq T$ and $x \in [-M,M]$, hence $\inf_{t,x} m(\theta_t(x)) > 0$, furthermore, the assumption $\int_{-M}^{M} \overline{\delta\theta} = 0$ yields $\int_{-M}^{M} \delta\theta_t = 0$ for every $t \geq T$ and therefore we have

$$\partial_x\phi_t(-M) = \partial_x\phi_t(M) = 0, \qquad \forall t \geq T.$$

This shows that the definition (3.20) of ϕ is well posed and ϕ is indeed C^∞, which further implies that $t \mapsto \int_{-M}^{M} |\partial_x\phi_t|^2(x)m(\theta_t(x))\,\mathrm{d}x$ is C^∞ as well.

Now notice that by the definition of ϕ we know that the equation

$$\delta\theta + \partial_x(m(\theta)\partial_x\phi) = 0 \tag{3.23}$$

holds. Differentiating it in t we get

$$\partial_t\delta\theta + \partial_x\big(\partial_t m(\theta)\partial_x\phi + m(\theta)\partial_{tx}\phi\big) = 0. \tag{3.24}$$

We now proceed with the computations. For notational simplicity we are going to omit the explicit dependence of the functions on t, x and the interval of integration, which will always be $[-M, M]$. Notice that due to the periodicity of the functions we will never have boundary terms when performing integration by parts.

We have

$$\begin{aligned}
\partial_t \frac{1}{2}\int (\partial_x\phi)^2 m(\theta)\,\mathrm{d}x &= \int \partial_x\phi\partial_{tx}\phi m(\theta) + \frac{1}{2}|\partial_x\phi|^2\partial_t m(\theta)\,\mathrm{d}x \\
&= \int -\phi\partial_x(\partial_{tx}\phi m(\theta)) + \frac{1}{2}|\partial_x\phi|^2\partial_t m(\theta)\,\mathrm{d}x \\
\text{by (3.24)} \qquad &= \int \phi\partial_t\delta\theta + \phi\partial_x(\partial_t m(\theta)\partial_x\phi) + \frac{1}{2}|\partial_x\phi|^2\partial_t m(\theta)\,\mathrm{d}x \\
&= \int \phi\partial_t\delta\theta - \frac{1}{2}|\partial_x\phi|^2 m'(\theta)\partial_t\theta\,\mathrm{d}x.
\end{aligned}$$

Using (3.18) and (3.19) we then obtain

$$\begin{aligned}
&\partial_t \frac{1}{2}\int |\partial_x\phi|^2 m(\theta)\,\mathrm{d}x \\
&\qquad = \int \frac{1}{2}|\partial_x\phi|^2 m'(\theta)\big(\partial_x f(\theta) - \varepsilon\partial_x(f''(\theta)\partial_x\theta)\big)\,\mathrm{d}x \\
&\qquad\quad - \int \phi\big(\partial_x(f'(\theta)\delta\theta) - \varepsilon\partial_x(f'''(\theta)\delta\theta\partial_x\theta + f''(\theta)\partial_x\delta\theta)\big)\,\mathrm{d}x \\
&\qquad = \underbrace{\int \frac{1}{2}|\partial_x\phi|^2 m'(\theta)\partial_x f(\theta) - \phi\partial_x(f'(\theta)\delta\theta)\,\mathrm{d}x}_{=:A} \\
&\qquad\quad + \varepsilon\underbrace{\int -\frac{1}{2}|\partial_x\phi|^2 m'(\theta)\partial_x(f''(\theta)\partial_x\theta) + \phi\partial_x\big(f'''(\theta)\delta\theta\partial_x\theta + f''(\theta)\partial_x\delta\theta\big)\,\mathrm{d}x}_{=:B}.
\end{aligned}$$

Use (3.23) to obtain

$$\begin{aligned}
A &= \int \frac{1}{2}|\partial_x\phi|^2 m'(\theta)f'(\theta)\partial_x\theta + \phi\partial_x\big(f'(\theta)\partial_x(m(\theta)\partial_x\phi)\big)\,\mathrm{d}x \\
&= \int \frac{1}{2}|\partial_x\phi|^2 m'(\theta)f'(\theta)\partial_x\theta - \partial_x\phi f'(\theta)\partial_x(m(\theta)\partial_x\phi)\,\mathrm{d}x \\
&= \int -\frac{1}{2}|\partial_x\phi|^2 m'(\theta)f'(\theta)\partial_x\theta - \frac{1}{2}\partial_x|\partial_x\phi|^2 m(\theta)f'(\theta)\,\mathrm{d}x = \int \frac{1}{2}|\partial_x\phi|^2 m(\theta)\partial_x(f'(\theta))\,\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
B &= \int -\frac{1}{2}|\partial_x\phi|^2 m'(\theta)\partial_{xx}(f'(\theta)) + \partial_x\phi\partial_x(f''(\theta))\partial_x(m(\theta)\partial_x\phi) + \partial_x\phi f''(\theta)\partial_{xx}(m(\theta)\partial_x\phi)\,\mathrm{d}x \\
&= \int -\frac{1}{2}|\partial_x\phi|^2 m'(\theta)\partial_{xx}(f'(\theta)) + |\partial_x\phi|^2\partial_x(f''(\theta))\partial_x(m(\theta)) + \frac{1}{2}\partial_x|\partial_x\phi|^2\partial_x(f''(\theta))m(\theta) \\
&\quad + |\partial_x\phi|^2 f''(\theta)\partial_{xx}(m(\theta)) + \partial_x|\partial_x\phi|^2 f''(\theta)\partial_x(m(\theta)) + \partial_{xxx}\phi\partial_x\phi f''(\theta)m(\theta)\,\mathrm{d}x.
\end{aligned}$$

Conclude noticing that

$$\begin{aligned}
&\int \frac{1}{2}\partial_x|\partial_x\phi|^2\partial_x(f''(\theta))(m(\theta)) + \partial_x|\partial_x\phi|^2 f''(\theta)\partial_x(m(\theta)) + \partial_{xxx}\phi\partial_x\phi f''(\theta)m(\theta)\,\mathrm{d}x \\
&\quad = \int \frac{1}{2}\partial_x|\partial_x\phi|^2\partial_x(f''(\theta)m(\theta)) + \partial_{xxx}\phi\partial_x\phi f''(\theta)m(\theta) + \frac{1}{2}\partial_x|\partial_x\phi|^2 f''(\theta)\partial_x(m(\theta))\,\mathrm{d}x \\
&\quad = \int -|\partial_{xx}\phi|^2 f''(\theta)m(\theta) - \frac{1}{2}|\partial_x\phi|^2\Big(\partial_x(f''(\theta))\partial_x(m(\theta)) + f''(\theta)\partial_{xx}(m(\theta))\Big)\,\mathrm{d}x,
\end{aligned}$$

and therefore

$$\begin{aligned}
B = &\int -|\partial_{xx}\phi|^2 f''(\theta)m(\theta) \\
&\qquad + \frac{1}{2}|\partial_x\phi|^2\Big(- m'(\theta)\partial_{xx}(f'(\theta)) + \partial_x(f''(\theta))\partial_x(m(\theta)) + f''(\theta)\partial_{xx}(m(\theta))\Big)\,\mathrm{d}x \\
= &\int -|\partial_{xx}\phi|^2 f''(\theta)m(\theta) + \frac{1}{2}|\partial_x\phi|^2\partial_x(f'(\theta))\partial_x(m'(\theta))\,\mathrm{d}x.
\end{aligned}$$

This yields (3.21). For proving (3.22) observe that in the case $m = -f$ from (3.21) it obviously follows that

$$\partial_t\frac{1}{2}\int_{-M}^{M}|\partial_x\phi_t|^2(-f(\theta_t))\,\mathrm{d}x \le \int_{-M}^{M}\frac{1}{2}|\partial_x\phi_t|^2(-f(\theta_t))\partial_x(f'(\theta_t))\,\mathrm{d}x,$$

and that since we are on a periodic domain and $\partial_x(f'(\bar\theta)) \le \frac{1}{T}$, Oleinik holds (cp. Proposition 2.3), i.e.

$$\partial_x(f'(\theta_t)) \le \frac{1}{t}.$$

This implies that

$$\partial_t\frac{1}{2}\int_{-M}^{M}|\partial_x\phi_t|^2(-f(\theta_t))\,\mathrm{d}x \le t^{-1}\frac{1}{2}\int_{-M}^{M}|\partial_x\phi_t|^2(-f(\theta_t))\,\mathrm{d}x,$$

which can be rewritten as

$$\partial_t\left(t^{-1}\frac{1}{2}\int_{-M}^{M}|\partial_x\phi_t|^2(-f(\theta_t))\,\mathrm{d}x\right) \le 0.$$

Integrating yields (3.22). □

Theorem 3.3. *Let f be an admissible flux and $\overline{\theta}^0, \overline{\theta}^1 \in \mathcal{M}_{1,-f}$. Let $[1,\infty)\times\mathbb{R} \ni (t,x) \mapsto \theta_t^i(x)$, $i = 0,1$ be the entropy solutions of the scalar conservation law*

$$\partial_t\theta + \partial_x(f(\theta)) = 0, \tag{3.25}$$

subject to the initial condition

$$\theta_1^i = \overline{\theta}^i.$$

Then

$$\mathsf{d}^2_{(t,-f)}(\theta_t^0, \theta_t^1) \le t\,\mathsf{d}^2_{(1,-f)}(\bar\theta^0, \bar\theta^1). \tag{3.26}$$

Proof. Start by applying Proposition 3.1 to $\bar{\theta}^0, \bar{\theta}^1$ and $t = 1$ to find an optimal curve for $\mathsf{d}_{(1,-f)}$ connecting $\bar{\theta}^0$ and $\bar{\theta}^1$, i.e. a solution $(\overline{\theta}_s(x), \overline{j}_s(x))$ of the continuity equation (2.16) such that $\bar{\theta}_i = \bar{\theta}^i$ for $i = 0, 1$ and fulfilling (3.1). Then we regularize this curve of initial data by applying Proposition 2.10 with $t := 1$ to find a sequence $t_n \uparrow 1$ and maps $\bar{\theta}_n, \bar{j}_n \in C^\infty([0,1]\times\mathbb{R})$ fulfilling $(i), (ii), (iii), (iv)$ of the statement.

Setting up of the approximation scheme. We now make the curve of initial data periodic in x. For this we pick $M > 0$ so large that

$$M \geq 2\max\{|L(\theta_{n,s})|, |R(\theta_{n,s})|\}, \qquad \forall s \in [0,1],\ n \in \mathbb{N},$$

and define

$$\bar{\theta}^M_{n,s}(x) := \begin{Bmatrix} \bar{\theta}_{n,s}(x), & x \in [-M, \frac{M}{2}), \\ 2(1 - \frac{x}{M}), & x \in [\frac{M}{2}, M]. \end{Bmatrix} = \min\Big\{\bar{\theta}_{n,s}(x), 2(1 - \frac{x}{M})\Big\}, \quad \forall x \in [-M, M],$$

smooth it out a bit at $\frac{M}{2}$ and M in order for its $2M$-periodic extension to $\mathbb{R}$ (still denoted by $\bar{\theta}^M_{n,s}$) to be smooth. Notice that such smoothing can be done in a way that

$$\partial_x(f'(\bar{\theta}^M_{n,s})) \leq \frac{1}{t_n}, \qquad \forall x \in \mathbb{R},\ s \in [0,1],\ n \in \mathbb{N}. \tag{3.27}$$

Next we change the curve of initial data such that their values lie in $(0,1)$. Choose $a \in (0, \frac{1}{2})$ and define $\overline{\theta}^{a,M}_{n,s}, \overline{\delta\theta}^{a,M}_{n,s}, \overline{j}^{a,M}_{n,s} \in C^\infty(\mathbb{R})$ as

$$\begin{aligned} \bar{\theta}^{a,M}_{n,s}(x) &:= (1-2a)\bar{\theta}^M_{n,s}(x) + a, \\ \overline{\delta\theta}^{a,M}_{n,s}(x) &:= (1-2a)\partial_s\overline{\theta}^M_{n,s}(x), \\ \bar{j}^{a,M}_{n,s}(x) &:= (1-2a)\bar{j}^M_{n,s}(x) = \int_{-\infty}^x \overline{\delta\theta}^{a,M}_{n,s}(y)\,\mathrm{d}y, \end{aligned}$$

for every $x \in \mathbb{R}$. The smoothness of f and $\bar{\theta}^M_{n,s}$ and (3.27) give that

$$\partial_x(f'(\bar{\theta}^{a,M}_{n,s}))(x) \leq \frac{1}{T_{a,n}}, \qquad \forall x \in \mathbb{R}, \tag{3.28}$$

for some $T_{a,n}, T_a$ satisfying $T_{a,n} \to T_a$ as $n \to \infty$ and $T_a \uparrow 1$ as $a \downarrow 0$. Then let $\theta^{a,M,\varepsilon}_{n,s,t} \in C^\infty([T_{a,n}, \infty) \times \mathbb{R})$ be the solution of (3.18) with $T_{a,n}, \bar{\theta}^{a,M}_{n,s}$ in place of $T, \overline{\theta}$. Similarly, let $\delta\theta^{a,M,\varepsilon}_{n,s,t} \in C^\infty([T_{a,n}, \infty) \times \mathbb{R})$ be the solution of (3.19) with $T_{a,n}, \overline{\delta\theta}^{a,M}_{n,s} = \overline{\delta\theta}^{a}_{n,s}$ in place of $T, \overline{\delta\theta}$ and denote by $\partial_x\phi^{a,M,\varepsilon}_{n,s,t}$ the quantity defined by (3.20) accordingly. Similarly, define $j^{a,M,\varepsilon}_{n,s,t} \in C^\infty([T_{a,n}, \infty) \times \mathbb{R})$ as the $2M$-periodic function such that

$$j^{a,M,\varepsilon}_{n,s,t}(x) = \int_{-M}^x \delta\theta^{a,M,\varepsilon}_{n,s,t}(y)\,\mathrm{d}y, \qquad \forall x \in [-M, M],$$

and notice that

$$m(\theta^{a,M,\varepsilon}_{n,s,t})\partial_x\phi^{a,M,\varepsilon}_{n,s,t} = j^{a,M,\varepsilon}_{n,s,t}.$$

Now (3.22) reads

$$\int_{-M}^{M} \frac{|j^{a,M,\varepsilon}_{n,s,t}|^2(x)}{m(\theta^{a,M,\varepsilon}_{n,s,t}(x))}\,\mathrm{d}x \le \frac{t}{T_{a,n}} \int_{-M}^{M} \frac{|\bar{j}^{a,M}_{n,s}|^2(x)}{m(\bar{\theta}^{a,M}_{n,s}(x))}\,\mathrm{d}x,$$

which integrated yields

$$\int_0^1 \int_{-M}^{M} \frac{|j^{a,M,\varepsilon}_{n,s,t}|^2(x)}{m(\theta^{a,M,\varepsilon}_{n,s,t}(x))}\,\mathrm{d}x\,\mathrm{d}s \le \frac{t}{T_{a,n}} \int_0^1 \int_{-M}^{M} \frac{|\bar{j}^{a,M}_{n,s}|^2(x)}{m(\bar{\theta}^{a,M}_{n,s}(x))}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.29}$$

Let $\varepsilon \downarrow 0$ and $n \uparrow +\infty$. The convergence of the solutions $\theta^{a,M,\varepsilon}_{n,s,t}$ of the viscous approximation (3.18) to entropy solutions and the stability of the latter with respect to convergence of the initial datum yields that

$\theta^{a,M,\varepsilon}_{n,s,t} \to \theta^{a,M}_{s,t}$ weakly in duality with $C_c(\mathbb{R})$ as $n \uparrow \infty$ and $\varepsilon \downarrow 0$ for any $s \in [0,1]$ and $t \ge T_a$,

where $\theta^{a,M}_{s,t}$ denotes the entropy solution of (3.18) with $\varepsilon = 0$ and $T_a, \bar{\theta}^{a,M}_s$ in place of $T, \bar{\theta}$. The uniform bound (3.29) and Proposition 2.11 grant that there are vector fields $[T_a,\infty) \times \mathbb{R} \ni (t,x) \mapsto j^{a,M}_{s,t}(x)$ such that the continuity equation (2.16) is fulfilled. Then Proposition 2.12 yields

$$\int_0^1 \int_{-M}^{M} \frac{|j^{a,M}_{s,t}|^2(x)}{m(\theta^{a,M}_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s \le \varliminf_{\substack{\varepsilon\downarrow 0\\ n\uparrow\infty}} \int_0^1 \int_{-M}^{M} \frac{|j^{a,M,\varepsilon}_{n,s,t}|^2(x)}{m(\theta^{a,M,\varepsilon}_{n,s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s, \qquad \forall t \ge 1. \tag{3.30}$$

Notice that for M large enough depending on t, the left hand side can be rewritten as

$$\int_0^1 \int_{-M}^{M} \frac{|j^{a,M}_{s,t}|^2(x)}{m(\theta^{a,M}_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s = \int_0^1 \int_{-M}^{M} \frac{|j^{a}_{s,t}|^2(x)}{m(\theta^{a}_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s = \int_0^1 \int_{\mathbb{R}} \frac{|j^{a}_{s,t}|^2(x)}{m(\theta^{a}_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s, \tag{3.31}$$

where $\theta^a_{s,t}$ denotes the entropy solution in the whole $\mathbb{R}$ with initial datum $\bar{\theta}^a_s(x) := (1-2a)\bar{\theta}_s(x) + a$ and $j_{s,t}$ are the corresponding momentum vector fields defined in analogy with the formulas above. These equalities are due to the finite speed of propagation of the equation. More precisely, the cone of propagation $C(t)(x_0)$ at a point x_0 up to time t is given by

$$C(t)(x_0) = \left\{(x+x_0,t) | 0 \le t \le 1, f'(0)t \le x \le f'(1)t\right\}.$$

This guarantees that for M sufficiently large, neither the periodization nor the affine term in the definition of $\theta^{a,M}_{s,t}$ affect the evolution of the solution in $[-M,M]$ up to our fixed time t. Also observe that $j^a_{s,t}(x) = 0$ for $x > \frac{M}{2}$ since then $\partial_s \theta^a_{s,t}(x) = 0$. By the stability of solutions and the lower semicontinuity of the action (Proposition 2.12) we find

$$\int_0^1 \int_{\mathbb{R}} \frac{|j_{s,t}|^2(x)}{m(\theta_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s \le \varliminf_{a\downarrow 0} \int_0^1 \int_{\mathbb{R}} \frac{|j^{a}_{s,t}|^2(x)}{m(\theta^{a}_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s,$$

where $\theta_{s,t} = \theta^a_{s,t}$ for $a = 0$, so that from this bound and (3.29), (3.30), (3.31) we get

$$\int_0^1 \int_{\mathbb{R}} \frac{|j_{s,t}|^2(x)}{m(\theta_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s \le t \varliminf_{a\downarrow 0} \varliminf_{n\uparrow+\infty} \int_0^1 \int_{-M}^{M} \frac{|\bar{j}^{a,M}_{n,s}|^2(x)}{m(\bar{\theta}^{a,M}_{n,s}(x))}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.32}$$

Now notice that arguing as for (3.31), we have

$$\int_{-M}^{M} \frac{|\bar{j}_{n,s}^{a,M}|^2(x)}{m(\bar{\theta}_{n,s}^{a,M}(x))}\,\mathrm{d}x = \int_{\mathbb{R}} \frac{|\bar{j}_{n,s}^{a}|^2(x)}{m(\bar{\theta}_{n,s}^{a}(x))}\,\mathrm{d}x, \tag{3.33}$$

where $\bar{\theta}_{n,s}^a := (1-2a)\bar{\theta}_{n,s} + a$ and $\bar{j}_{n,s}^a := (1-2a)\bar{j}_{n,s}$. By the convexity of $\mathbb{R}\times\mathbb{R}^+ \ni (\alpha,\beta) \mapsto \frac{|\alpha|^2}{m(\beta)}$ and the definition of $\bar{j}_{n,s}^a, \bar{\theta}_{n,s}^a$ we obtain

$$\int_{\mathbb{R}} \frac{|\bar{j}_{n,s}^{a}|^2(x)}{m(\bar{\theta}_{n,s}^{a}(x))}\,\mathrm{d}x \le \int_{\mathbb{R}} \frac{|\bar{j}_{n,s}|^2(x)}{m(\bar{\theta}_{n,s}(x))}\,\mathrm{d}x. \tag{3.34}$$

Thus recalling property (iv) of Proposition 2.10, from (3.32), (3.33) and (3.34) we deduce

$$\int_0^1 \int_{\mathbb{R}} \frac{|j_{s,t}|^2(x)}{m(\theta_{s,t})(x)}\,\mathrm{d}x\,\mathrm{d}s \le t \int_0^1 \int_{\mathbb{R}} \frac{|\bar{j}_s|^2(x)}{m(\bar{\theta}_s(x))}\,\mathrm{d}x\,\mathrm{d}s = t\,\mathsf{d}_{1,-f}^2(\overline{\theta}^0, \overline{\theta}^1).$$

Recalling Proposition 2.3 and the fact that $\overline{\theta}_s \in \mathcal{M}_{1,-f}$ for every $s \in [0,1]$ we deduce that $\theta_{s,t} \in \mathcal{M}_{t,-f}$ for every $s \in [0,1]$ and $t \ge 1$. Hence for given $t \ge 1$ the curve $s \mapsto (\theta_{s,t}, j_{s,t})$ is admissible in the definition of $\mathsf{d}_{t,-f}$ and the claim is proven. □

We now interpret this contraction rate in terms of rescaled solutions: this will show that every entropy solution of (3.25) with initial data in $\mathcal{M}$ converges - after rescaling - to the rarefaction wave. More precisely, for every entropy solution θ of (3.25) define the rescaled solution $\hat{\theta}$ by

$$\hat{\theta}_t(x) := \theta_t(tx). \tag{3.35}$$

Observe that the rarefaction wave θ^{rar} is self similar, in the sense that

$$\hat{\theta}_t^{rar}(x) = \theta_t^{rar}(tx) = \left\{ \begin{array}{cc} 0, & x \le f'(0), \\ (f')^{-1}(x), & f'(0) < x < f'(1), \\ 1, & f'(1) \le x, \end{array} \right\},$$

so that the rescaled solution does not depend on time. The convergence of rescaled solutions to the rescaled rarefaction wave is then a consequence of the following simple corollary:

Corollary 3.4. *Let f be an admissible flux and $\overline{\theta}^0, \overline{\theta}^1 \in \mathcal{M}_{1,-f}$. Let $[1,\infty)\times\mathbb{R} \ni (t,x) \mapsto \theta_t^i(x)$, $i = 0,1$ be the entropy solutions of the scalar conservation law*

$$\partial_t \theta^i + \partial_x(f(\theta^i)) = 0,$$

subject to the initial condition

$$\theta_1^i = \overline{\theta}^i.$$

Define the rescaled solutions $\hat{\theta}^i$, $i = 0,1$ according to formula (3.35).

Then

$$\mathsf{d}_{1,-f}^2(\hat{\theta}_t^0, \hat{\theta}_t^1) \le \frac{1}{t^2}\,\mathsf{d}_{1,-f}^2(\theta_1^0, \theta_1^1), \qquad \forall t \ge 1.$$

Proof. Fix $t \geq 1$ and let $s \mapsto (\theta_s, j_s)$ be a solution of the continuity equation (2.16) with $\theta_s \in \mathcal{M}_{t,-f}$ for every $s \in [0,1]$ and $\theta_i = \theta_t^i$, $i = 0, 1$. Define $(\hat{\theta}_s, \hat{j}_s)$ by

$$\hat{\theta}_s(x) := \theta_s(tx),$$
$$\hat{j}_s(x) := \tfrac{1}{t} j_s(tx),$$

so that $s \mapsto (\hat{\theta}_s, \hat{j}_s)$ still solves the continuity equation, $\hat{\theta}_s \in \mathcal{M}_{1,-f}$ for every $s \in [0,1]$ and $\hat{\theta}_i = \hat{\theta}_t^i$ for $i = 0, 1$.

Hence according to the definition of $\mathsf{d}_{1,-f}$ we have

$$d^2_{1,-f}(\hat{\theta}_t^0, \hat{\theta}_t^1) \leq \iint_0^1 \frac{|\hat{j}_s|^2(x)}{m(\hat{\theta}_s)(x)} \,\mathrm{d}s\,\mathrm{d}x = \frac{1}{t^3} \iint_0^1 \frac{|j_s|^2(x)}{m(\theta_s)(x)} \,\mathrm{d}s\,\mathrm{d}x.$$

Taking the infimum over all admissible curves in the definition of $\mathsf{d}_{t,-f}(\theta_t^0, \theta_t^1)$ we deduce

$$d^2_{1,-f}(\hat{\theta}_t^0, \hat{\theta}_t^1) \leq \frac{1}{t^3}\, \mathsf{d}^2_{t,-f}(\theta_t^0, \theta_t^1).$$

Thus Theorem 3.3 implies the claim. □

3.3 Sharpness on the contraction result

We now show that our contraction result is both non-trival and sharp. For simplicity, we restrict to the case

$$f(z) := z(z-1),$$

but all the discussion can be carried out for general fluxes.

Once again, the rarefaction wave θ^{rar} solution to

$$\partial_t \theta + \partial_x f(\theta) = 0, \tag{3.36}$$

will play a distinguished role in this discussion. With this choice of f it is given by the expression

$$\theta_t^{rar}(x) := \left\{ \begin{array}{cc} 0, & x \leq -t, \\ \frac{x+t}{2t}, & -t < x < t, \\ 1, & t \leq x \end{array} \right\}.$$

We also observe that the simple inequality

$$\frac{1}{z(1-z)} \geq \frac{1}{z}, \qquad \forall z \in [0,1],$$

yields for $t \geq 1$

$$\mathsf{d}_{t,-f}(\overline{\theta}^0, \overline{\theta}^1) \geq \mathsf{d}_{1,-f}(\overline{\theta}^0, \overline{\theta}^1) \geq W_2(\overline{\theta}^0, \overline{\theta}^1), \qquad \forall \overline{\theta}^0, \overline{\theta}^1 \in \mathcal{M}_{t,-f}. \tag{3.37}$$

By 'non-trivial' we mean that the contraction result provides an estimate which is better than the one obtained by triangle inequality. To see this, we start claiming that $W_2(\theta_t^{rar}, \theta_1^{rar}) \sim t^{\frac{3}{2}}$. Indeed we have

$$V_t^{rar}(x) := \int_{-\infty}^{x} \theta_t^{rar}(y)\,\mathrm{d}y = \left\{ \begin{array}{ll} 0, & x \leq -t, \\ t\left(\frac{x}{2t} + \frac{1}{2}\right)^2, & -t \leq x \leq t, \\ x, & t \leq x, \end{array} \right\}$$

thus for $z > 0$ it holds

$$(V_t^{rar})^{-1}(z) = \left\{ \begin{array}{ll} 2\sqrt{tz} - t, & z \leq t, \\ z, & z \geq t, \end{array} \right\}$$

and therefore, using the known fact that in one dimension the optimal transport distance between two measures can be expressed in terms of their cumulative distribution functions (see for example [12, Theorem 2.18])

$$\begin{aligned} W_2^2(\theta_t^{rar}, \theta_1^{rar}) &= \int_0^{\infty} |(V_t^{rar})^{-1}(z) - (V_1^{rar})^{-1}(z)|^2\,\mathrm{d}z \\ &= \int_0^1 |2\sqrt{z}(\sqrt{t} - 1) + 1 - t|^2\,\mathrm{d}z + \int_1^t |2\sqrt{tz} - t - z|^2\,\mathrm{d}z \sim t^3, \end{aligned}$$

which is our claim.

Now let θ be an entropy solution of (3.36) and notice that if we try to bound the $d_{1,-f}$-distance between θ_t and θ_t^{rar} via the triangle inequality

$$\mathsf{d}_{1,-f}(\theta_t, \theta_t^{rar}) \leq \mathsf{d}_{1,-f}(\theta_t, \theta_1) + \mathsf{d}_{1,-f}(\theta_1, \theta_1^{rar}) + \mathsf{d}_{1,-f}(\theta_1^{rar}, \theta_t^{rar}), \tag{3.38}$$

due to the fact that

$$\mathsf{d}_{1,-f}(\theta_1^{rar}, \theta_t^{rar}) \geq W_2(\theta_1^{rar}, \theta_t^{rar}) \geq t^{\frac{3}{2}},$$

we see that the right-hand side of (3.38) is of order at least $t^{\frac{3}{2}}$.

On the other hand, by Theorem 3.3 and (3.37) we obtain

$$\mathsf{d}_{1,-f}(\theta_t, \theta_t^{rar}) \leq \mathsf{d}_{t,-f}(\theta_t, \theta_t^{rar}) \leq \sqrt{t}\,\mathsf{d}_{1,-f}(\theta_1, \theta_1^{rar}),$$

and thus a scaling of order $\sqrt{t}$, which is certainly better than $t^{\frac{3}{2}}$ at the regime of large times.

Now we show that the contraction result is sharp: to this aim we will exhibit an entropy solution θ of (3.36) such that $\mathsf{d}_{t,-f}(\theta_t, \theta_t^{rar})$ is of order $\sqrt{t}$.

Proposition 3.5. *There exists an entropy solution θ of* (3.36) *such that*

$$ct \leq \mathsf{d}^2_{(t,-f)}(\theta_t, \theta_t^{rar}) \leq Ct, \qquad \forall t \geq 1,$$

for some constants c, C.

Proof. Theorem 3.3, combined with Proposition 3.1 to ensure $d^2_{(1,-f)}(\theta_1, \theta_1^{rar}) \lesssim 1$, provides the bound from above, so we only have to prove the bound from below. Recalling inequality (3.37) it is sufficient to show that

$$W_2^2(\theta_t, \theta_t^{rar}) \gtrsim t.$$

Let $\theta_t(x)$ be the entropy solution to (3.36) which is obtained via cutting θ_t^{rar} at 0 and inserting a piece of the constant function $\frac{1}{2}$. In formulas this reads

$$\theta_t(x) := \left\{ \begin{array}{cc} 0, & x \leq -(1+t), \\ \frac{x+1+t}{2t}, & -(1+t) < x \leq -1, \\ \frac{1}{2}, & -1 < x \leq 1, \\ \frac{x-1+t}{2t}, & 1 < x \leq 1+t, \\ 1, & 1+t < x, \end{array} \right\}.$$

By looking at the transport from θ_t to θ_t^{rar}, we see that heuristically we have to transport mass $\sim t$ by distance ~ 1, thus it should hold

$$W_2^2(\theta_0(t,\cdot), \theta_1(t,\cdot)) \sim t.$$

More formally, by defining

$$V_t(x) = \int_{-\infty}^{x} \theta_t(y)\,\mathrm{d}y,$$

we compute for $x \leq -1$

$$V_t(x) := \left\{ \begin{array}{cc} 0, & x \leq -(1+t), \\ t\left(\frac{x+1}{2t} + \frac{1}{2}\right)^2, & -(1+t) < x \leq -1, \end{array} \right\}.$$

This yields for $0 < z \leq \frac{t}{4}$

$$V_t^{-1}(z) = \left\{ \begin{array}{cc} -(t+1) + 2\sqrt{tz}, & z < \frac{t}{4}, \end{array} \right\}.$$

Thus we have

$$\begin{aligned} W_2^2(\theta_t, \theta_t^{rar}) &= \int_0^\infty |V_t^{-1}(z) - (V_t^{rar})^{-1}(z)|^2\,\mathrm{d}z \geq \int_0^{\frac{t}{4}} |V_t^{-1}(z) - (V_t^{rar})^{-1}(z)|^2\,\mathrm{d}z \\ &= \int_0^{\frac{t}{4}} |-(t+1) + 2\sqrt{tz} + t - 2\sqrt{tz}|^2\,\mathrm{d}z = \frac{t}{4} \sim t, \end{aligned}$$

and the proof is complete. □

3.4 Comments about admissible fluxes

In this final section we show that the notion of admissible fluxes in the sense of Definition 2.13 is not really restrictive: by a simple scaling argument we can always reduce to the case of admissible fluxes.

Proposition 3.6. *Let $f \in C^\infty(\mathbb{R})$ with $f'' \geq \alpha > 0$, and $\theta \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ a weak solution of*

$$\partial_t \theta + \partial_x f(\theta) = 0, \tag{3.39}$$

s.t. there exist $L(t), R(t)$ with

$$\begin{aligned} \theta_t(x) &= \inf_{x,t} \theta_t(x) =: \theta_-, \qquad &&\text{for } x < L(t), \\ \theta_t(x) &= \sup_{x,t} \theta_t(x) =: \theta_+, \qquad &&\text{for } x > R(t). \end{aligned} \tag{3.40}$$

Consider the function $\tilde{\theta}$ defined by

$$\tilde{\theta}_t(x) = \frac{1}{\theta_+ - \theta_-}\Big(\theta_{(\theta_+ - \theta_-)t}\big(x - (f(\theta_-) - f(\theta_+))t - a\big) - \theta_-\Big), \tag{3.41}$$

where a is an appropriately chosen constant depending on f and θ_0 only (see the proof).

Then $\tilde{\theta}$ is a weak solution of

$$\partial_t \tilde{\theta} + \partial_x \tilde{f}(\tilde{\theta}) = 0, \tag{3.42}$$

such that $\tilde{\theta}_t \in M$ for all $t \geq 0$, where $\tilde{f}$ is given by

$$\tilde{f}(z) := f\big((\theta_+ - \theta_-)z + \theta_-\big) + (f(\theta_-) - f(\theta_+)z - f(\theta_-),$$

and in particular is an admissible flux in the sense of Definition 2.13.

Furthermore, θ is an entropy solution of (3.39) *if and only if $\tilde{\theta}$ is an entropy solution of* (3.42).

Proof. The fact that $\tilde{\theta}$ is a weak solution of (3.42) comes from straightforward computation and simple algebraic manipulations also show that $\tilde{f}$ is an admissible flux.

We prove that $\tilde{\theta}_t \in M$ for every $t \geq 0$. Notice that by definition we have

$$0 \leq \tilde{\theta}_t(x) \leq 1, \forall t \geq 0,\ x \in \mathbb{R},$$

and

$$\begin{aligned} \tilde{\theta}_t(x) &= 0, \qquad &&\text{for } x << 0, \\ \tilde{\theta}_t(x) &= 1, \qquad &&\text{for } x >> 0. \end{aligned}$$

Since $\tilde{\theta}$ is a solution to a scalar conservation law with flux $\tilde{f}$ s.t. $\tilde{f}(0) = \tilde{f}(1)$ we observe that there exists $b \in \mathbb{R}$ s.t.

$$\int_{\mathbb{R}} \tilde{\theta}_t(y) - H(y)\,\mathrm{d}y = b, \text{ for a.e. } t \geq 0.$$

If $b \neq 0$, replacing a by $a + b$ in the definition of $\tilde{\theta}$ we find that for this new choice

$$\int_{\mathbb{R}} \tilde{\theta}_t(y) - H(y)\,\mathrm{d}y = 0,$$

which yields $\tilde{\theta}_t \in M$.

To see that θ is an entropy solution if and only if $\tilde{\theta}$ is, we use the Oleinik principle:

$$\theta_t \text{ entropy solution to conservation law with flux } f \qquad \Leftrightarrow \qquad \partial_x f'(\theta_t) \leq \frac{1}{t} \text{ for a.e. } t > 0.$$

The fact that Oleinik's condition holds for $(\tilde{\theta}, \tilde{f})$ if and only if it holds for (θ, f) is a simple calculation guided by the formal computation

$$\begin{aligned}\partial_x \tilde{f}'(\tilde{\theta}(t,x)) = \tilde{f}''(\tilde{\theta}(t,x))\partial_x\tilde{\theta}(t,x) &= (\theta_+ - \theta_-)^2 f''(\theta((\theta_+ - \theta_-)t, \tilde{x}))\frac{1}{(\theta_+ - \theta_-)}(\partial_x\theta)((\theta_+ - \theta_-)t, \tilde{x}) \\ &\leq (\theta_+ - \theta_-)^2 \frac{1}{(\theta_+ - \theta_-)}\frac{1}{(\theta_+ - \theta_-)t} = \frac{1}{t},\end{aligned}$$

where $\tilde{x} := x - (f(\theta_-) - f(\theta_+))t - a$. We omit the details. □

This proposition allows us to use our main Theorem 3.3 to compare the shape of two solutions θ^0, θ^1 of the same scalar conservation law, given that (3.40) holds for both solutions with the same value for i, s, since then their transformations $\tilde{\theta}^0, \tilde{\theta}^1$ via (3.41) both yield solutions to the same equation which fits in the framework of Theorem 3.3.

The only part of the transformation (3.41) that depends on the solution itself is then the translation parameter a which is needed in order to obtain the same mass for $\tilde{\theta}^0$ and $\tilde{\theta}^1$, a necessary condition to ensure that the Wasserstein distance between them is finite. Reversing the transformations of (3.41) we are e.g. able to obtain

$$W_2^2\big(\theta_t^0(\cdot - a_0), \theta_t^1(\cdot - a_1)\big) \lesssim t.$$

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