Model Order Reduction in Fluid Dynamics: Challenges and Perspectives

Model order reduction in fluid dynamics: challenges and perspectives

Toni Lassila, Andrea Manzoni, Alfio Quarteroni and Gianluigi Rozza

Abstract This chapter reviews techniques of model reduction of fluid dynamics systems. Fluid systems are known to be difficult to reduce efficiently due to several reasons. First of all, they exhibit strong nonlinearities – which are mainly related either to nonlinear convection terms and/or some geometric variability – that often cannot be treated by simple linearization. Additional difficulties arise when attempting model reduction of unsteady flows, especially when long-term transient behavior needs to be accurately predicted using reduced order models and more complex features, such as turbulence or multiphysics phenomena, have to be taken into consideration. We first discuss some general principles that apply to many parametric model order reduction problems, then we apply them on steady and unsteady viscous flows modelled by the incompressible Navier-Stokes equations. We address questions of *inf-sup* stability, certification through error estimation, computational issues and – in the unsteady case – long-time stability of the reduced model. Moreover, we provide an extensive list of literature references.

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1 Introduction

Numerical methods for Computational Fluid Dynamics (CFD) are by now essential in engineering applications dealing with flow simulation and control, such as the ones arising in aerodynamics, hydrodynamics and, more recently, in physiological flows. In despite of a constant increase in available computational power, numerical simulations of turbulent flows, multiscale and multiphysics phenomena, flows separation and/or bifurcation phenomena are still very demanding, possibly requiring millions or tens of millions of degrees of freedom and several days of CPU time on powerful parallel hardware architectures. This effort is even more substantial whenever we are interested in the repeated solution of the fluid equations for different values of model parameters, such as in flow control or optimal design problems (*many-query* contexts), or in *real time* flow visualization and output evaluation.

These problems represent a remarkable challenge to classical numerical approximations techniques, such as Finite Elements (FE), Finite Volumes or spectral methods. In fact, these methods require huge computational efforts (and also data/memory management) if we are interested to provide accurate response, thus making both *real-time* and *many-query* simulations unaffordable. For this reason, we need to rely on suitable *Reduced-Order Models* (ROMs) – that can reduce both the amount of CPU time and storage capacity – in order to enhance the computational efficiency in these contexts.

This chapter reviews the current state-of-the art for the model reduction of parametrized fluid dynamics equations. In particular, we focus on the incompressible Navier-Stokes equations, because of their ubiquitous presence in fluid flow applications and the fact that they involve the most important features and challenges relevant to *nonlinear model reduction*. These equations are usually written in primitive variables as follows: find the velocity field $\mathbf{u} : \Omega \times [0,T) \to \mathbb{R}^d$ and pressure field $p : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \frac{1}{\mathrm{Re}}\triangle \mathbf{u} &= 0, && \text{in } \Omega \times (0,T) \\ \nabla \cdot \mathbf{u} &= 0, && \text{in } \Omega \times (0,T) \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ denotes the fluid domain, $\mathrm{Re} = |\mathbf{u}_{\max}|L/\nu$ is the nondimensional Reynolds number, L is a characteristic length, ν is the fluid kinematic viscosity and $|\mathbf{u}|_{\max} = \max_{\mathbf{x}\in\Omega} |\mathbf{u}|$. In addition, suitable boundary conditions need to be prescribed in order to solve problem (1), see e.g. [51, 80, 118].

The Navier-Stokes equations are the most accurate continuum-based approximation for viscous flows where both convective and diffusive effects contribute, and they are known to accurately reproduce many interesting physical phenomena observed in fluids, such as the onset of turbulence. Concerning the functional setting required to frame the analysis of problem (1), let us denote $(H^1_0(\Omega))^d \subset V \subset (H^1(\Omega))^d$ and $Q \subset L^2(\Omega)$. The solution of (1) is such that $(\mathbf{u}, p) \in L^2(0,T;V) \times C^0(0,T;Q)$; see e.g. [102, 118] for the definition of Sobolev spaces and more details

about this functional setting. Moreover, let us introduce a further functional space $W \subseteq V \times Q$, denote $\langle \cdot, \cdot \rangle_X$ the scalar product over a generic space X and $\| \cdot \|_X$ its induced norm. When the subscript is omitted, $\langle \cdot, \cdot \rangle$ denotes in the following the L^2-scalar product and $\| \cdot \|$ the induced norm, respectively.

In many applications, the fluid problem can depend in addition on a number of parameters. In this case we deal with a parametric model reduction problem. We denote $\mu \in \mathscr{P} \subset \mathbb{R}^P$ a vector of P parameters of interest for a given fluid dynamics problem, as in the case of the Reynolds number appearing in (1). Other typical examples deal with different physical parametrizations (e.g. by considering Grashof number, Prandtl number, inflow velocity peaks, etc. as parameters) or geometrical parametrization, i.e. when the fluid domain $\Omega = \Omega(\mu)$ depends on a set of parameters allowing to describe/modify its shape. For the sake of simplicity, in this chapter we will focus on physical parameters, whereas several details about flexible but efficient geometrical parametrizations can be found e.g. in [87].

Model reduction of the Navier-Stokes equations is a challenging task because their solutions tend to exhibit complex phenomena at multiple temporal and spatial scales, which means they are difficult to reduce to low-dimensional models without losing at least some of the scales. In the case of unsteady flows, application of the standard "method of lines" to the time-discretization of the unsteady Navier-Stokes equations leads in three dimensions to the lack of sharp long-time stability estimates. It is well known [68] that application of the discrete Grönwall lemma leads to excessive growth of error bounds in time, because standard linear stability analysis of the unsteady Navier-Stokes equations results in stability constants that can be of the order $C_s \sim \exp(\mathrm{Re}\, T)$. While turbulence has sometimes been offered as an explanation to this difficulty, the underlying situation is more delicate. The same type of problem is exhibited by the one-dimensional Burgers' equation, which does not possess turbulent solutions. This also makes hard to provide meaningful error bounds for the solutions of ROMs for the unsteady Navier-Stokes equations.

During the last three decades, several efforts in theoretical foundations, numerical investigations and methodological improvements have made possible to develop general ideas in reduced order modelling and to tackle several problems arising in fluid dynamics. Among a number of early contributions, we want to highlight the most important – in our opinion – that date back to the late 1980s (see e.g. [39, 98, 114]). These were mainly based on *ad hoc* selection of the basis functions, without the benefit of a formal algorithm. Indeed, model reduction has come into play as a truly invaluable tool in CFD applications only once systematic strategies for constructing quasi-optimal bases were made available.

For the sake of exposition, we limit ourselves to describe two main algorithms for choosing the basis on which to build ROMs, namely the *Proper Orthogonal Decomposition* (POD) and the *(greedy) Reduced basis* (RB) methods. They share several features but have been historically introduced and developed to address different types of problems – POD is typically applied to build bases for *time*-dependent problems, while the greedy RB method is usually applied to build bases for *parameter*-

dependent problems. Moreover, we provide detailed remarks and references about extensions of these techniques and alternative strategies. We do not address in this review the case of *combined time and parameter*-dependent problems; the interested reader can refer to some recent works concerning error estimates for ROMs in the case of acoustic Helmholtz and incompressible Navier-Stokes equations [63], the Boussinesq equations [70], and the viscous Burgers' equation using the method of lines [93] or in the space-time formulation [131].

1.1 Proper Orthogonal Decomposition

POD is the leading model reduction tool for the unsteady Navier-Stokes equations. It was first introduced in [83] in the context of fluid dynamics as a method for discerning and analyzing coherent structures in experimental turbulent flows, and more recently in direct numerical simulations of turbulent flows in [53, 126], where also the concept of space-time windowing of POD has been introduced, to identify turbulent effects in transitional flow that are highly localized both in space and time.

POD techniques reduce the dimensionality of a system by transforming the original unknowns onto a new set of N_r variables (called POD modes, or principal components) such that the first few modes retain most of the energy present in all of the original unknowns. This allows to obtain a reduced, modal representation through a spectral decomposition which requires basic matrix computations (a singular value decomposition) also for nonlinear equations. For a deeper review on POD we recall here also the contribution of Bergman et al. and Grinberg et al. in MS&A Vol.8, A.Quarteroni-G.Rozza (Eds.), Springer, 2013.

For the reader's convenience, we recall briefly the POD based on the *method of snapshots*, as presented in [114]. An approximation $\mathbf{u}_r(\mathbf{x},t)$ to the solution $\mathbf{u}(\mathbf{x},t)$ of (1) is sought as the sum of a base flow $\bar{\mathbf{u}}$ and a linear combination of some spatial modes $\Psi_i(\mathbf{x})$ through a set of temporal coefficients, as follows:

$$\mathbf{u}(\mathbf{x},t) \approx \mathbf{u}_r(\mathbf{x},t) := \bar{\mathbf{u}}(\mathbf{x}) + \sum_{i=1}^{N_r} a_i(t)\Psi_i(\mathbf{x}), \tag{2}$$

for a suitable $N_r \geq 1$, where $\bar{\mathbf{u}}(\mathbf{x}) := \int_0^T \mathbf{u}(\mathbf{x},\tau)\, d\tau$ is the time-averaged base flow. This *ansatz* is reasonable assuming that the flow field can be approximated by a stochastic process that is stationary in time and ergodic [60]. In Section 4.3 we will discuss some extensions in situations where such assumptions do not hold.

The spatial modes are assumed to satisfy the orthogonality relation $\langle \Psi_i, \Psi_j \rangle = 0$ if $i \neq j$, for $\langle \cdot\,, \cdot \rangle$ denoting a convenient scalar product, whereas the coefficients $a_i(t)$ satisfy the following system of ODEs

$$\begin{aligned} \frac{da_i(t)}{dt} &= F_i + \sum_{j=1}^{N_r} A_{ij} a_j(t) + \sum_{j=1}^{N_r}\sum_{k=1}^{N_r} C_{ijk} a_j(t) a_k(t), \quad t \geq 0 \\ a_i(0) &= \langle \Psi_i, \mathbf{u}_0 \rangle, \end{aligned} \tag{3}$$

for $i = 1, \ldots, N_r$, where the functional forms of the reduced system coefficient tensors

$$\begin{aligned} F_i &:= -\frac{1}{\mathrm{Re}} \langle \nabla \Psi_i, \nabla \bar{\mathbf{u}} \rangle - \langle \Psi_i, (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} \rangle \\ A_{ij} &:= -\langle \Psi_i, (\bar{\mathbf{u}} \cdot \nabla) \Psi_j \rangle - \langle \Psi_i, (\Psi_j \cdot \nabla) \bar{\mathbf{u}} \rangle - \frac{1}{\mathrm{Re}} \langle \nabla \Psi_i, \nabla \Psi_j \rangle \\ C_{ijk} &:= -\langle \Psi_i, (\Psi_j \cdot \nabla) \Psi_k \rangle \end{aligned} \tag{4}$$

are obtained by Galerkin projection of the original system (1) on the spatial modes $\Psi_1, \ldots, \Psi_{N_r}$ in (2). The resulting ROM is referred to as a *Galerkin ROM*.

We point out that the pressure terms do not appear in these equations, and our space is defined as $X \equiv V$. In fact, by construction the POD modes $\{\Psi_i\}_{i=1}^{N_r}$ are discretely divergence-free. However, for some flows we could be interested either in evaluating the pressure field through the ROM, or to explicitly enforce the divergence-free constraint in the ROM; we will go back to this point in Section 3.1.

From the structure of (2) we note immediately that trajectories of the reduced solution $\mathbf{u}_r$ live in an N_r-dimensional submanifold of the full space. Thus the accuracy of the ROM is implicitly dependent on the assumption that the trajectories of the full-order system (1) can reasonably be approximated on a much lower-dimensional submanifold. As we will see in the following sections, these two ingredients, namely *(i)* the expression of the approximate solution in a reduced-order model as a linear combination of properly selected snapshots and *(ii)* a projection onto the subspace spanned by the snapshot solutions in order to find the weights in the linear combination, are peculiar also to the *(greedy) reduced basis* methods.

We now focus on computation of the spatial modes $\{\Psi_i\}_{i=1}^{N_r}$. We start from a set of snapshot solutions $\mathbf{U}_n(\mathbf{x}) := \mathbf{u}(\mathbf{x}, t^n)$ of the trajectory $\mathbf{u}(\mathbf{x}, t)$ at some selected times t^n, for $n = 1, \ldots, N_s$. These solutions can be either obtained through accurate numerical simulations of the discretized Navier-Stokes equations (1), or by experimental measurements of the physical system. In the former case, a POD approach is premised upon a "truth approximation" space $X_h \subset X$ of (typically very large) dimension, for which the snapshot solutions $\mathbf{U}_n(\mathbf{x}) := \mathbf{u}_h(\mathbf{x}, t^n)$ of the (truth approximation of the) trajectory $\mathbf{u}_h(\mathbf{x}, t)$ at some selected times t^n, for $n = 1, \ldots, N_s$. Nonetheless, we omit the subscript h wherever possible. The snapshots are typically equispaced in time along the entire period T and obtained after discarding the initial transient of the flow until a stable regime is reached and the flow can be modelled as a stochastic process that is stationary in time[1].

The POD space $X_{N_r}^{\mathrm{POD}} := \mathrm{span}\{\Psi_i : i = 1, \ldots, N_r\}$ of dimension $1 \leq N_r \leq N_s$, for a suitable N_s, is defined as the subspace which minimizes the least-squares discrep-

[1] In practice, N_r POD modes are required to resolve the first $N_r/2$ temporal harmonics, and these can be computed from $N_s = 2N_r$ snapshots [96].

ancy between the snapshots $\{\mathbf{U}_i(\mathbf{x})\}_{i=1}^{N_r}$ and their best approximation in the X-norm:

$$X_{N_r}^{\mathrm{POD}} := \underset{X_{N_r}\subset X_{N_s},\,\dim(X_{N_r})=N_r}{\arg\inf} \frac{1}{N_r}\sum_{i=1}^{N_s} \left\|\mathbf{U}_i(\cdot) - \Pi_{X_{N_r}}(\mathbf{U}_i(\cdot))\right\|^2_{(L^2(\Omega))^d}, \tag{5}$$

where $\Pi_{X_{N_r}}$ denotes the $(L^2(\Omega))^d$ projection onto the subspace X_{N_s}; for incompressible fluid problems this means that the POD basis is the best approximation basis in the sense of capturing the kinetic energy contained in the snapshots.

From a practical point of view, we form the correlation matrix $\mathbb{C}\in\mathbb{R}^{N_s\times N_s}$, whose components are

$$\mathbb{C}_{nm} := \frac{1}{T}\int_\Omega \left[\mathbf{U}_n(\mathbf{x}) - \bar{\mathbf{U}}(\mathbf{x})\right]\cdot\left[\mathbf{U}_m(\mathbf{x}) - \bar{\mathbf{U}}(\mathbf{x})\right]\, d\mathbf{x}, \tag{6}$$

where $\bar{\mathbf{U}}(\mathbf{x}) := \frac{1}{N}\sum_{n=1}^{N}\mathbf{U}_n(\mathbf{x})$ is the ensemble average that approximates the base flow $\bar{\mathbf{u}}$. Then, we compute the eigenpairs (λ_k,ψ_k), $k=1,\dots,N_s$ (with positive eigenvalues ordered by decreasing size) of $\mathbb{C}$. The central result of POD states that the *optimal* subspace $X_{N_r}^{\mathrm{POD}}$ of dimension N_r satisfying (5) is such that

$$\Psi_i = \tilde{\Psi}_i/\|\tilde{\Psi}_i\|_W, \qquad \tilde{\Psi}_i = \sum_{n=1}^{N_s}\psi_{i,n}(\mathbf{U}_n(\mathbf{x}) - \bar{\mathbf{U}}(\mathbf{x})), \qquad 1\le k\le N_s, \tag{7}$$

being $\psi_{in} = (\psi_i)_n$ the n-th component of the i-th eigenvector. In this way, the basis functions $\{\Psi_i\}_{i=1}^{N_s}$ are L^2-orthonormal[2].

The POD can equally be applied to the reduction of parametric fluid flow problems (see e.g. the parametric studies in [33] for rotating transitional flow, in [54] for modeling the airflow in a large public building, and in [67] for the analysis of turbulent plane channel flow). In fact, if the system (1) depends in addition on a vector $\mu\in\mathscr{P}\subset\mathbb{R}^P$ of P parameters of interest, we can follow the same procedure, except that the snapshots are now sampled also in the parameter space. It should be noted, however, that even if the POD procedure is the same in both the time interval and the parameter space, the practical results will differ considerably, due to the *causal* nature of time as opposed to other types of physical parameters.

So far we have not mentioned the treatment of boundary conditions that need to be imposed on (1). In the case of homogeneous boundary conditions, the snapshots as well as their linear combinations will naturally satisfy the same boundary conditions so that nothing special needs to be done. If we have non-homogeneous Dirichlet boundary conditions, the linear combinations of snapshots will not in general satisfy them, and neither will the ROM solution. To remedy this problem we can either subtract the non-homogeneous boundary values from the snapshots be-

[2] For numerical stability reasons the POD eigenvalues are usually not computed from the correlation matrix itself, but rather as the squares of the singular values of the snapshot matrix obtained by collecting all the snapshots as column vectors.

fore constructing the POD basis, or add an additional constraint equation to the ROM that enforces the boundary condition. These two methods can also be applied to parameter-dependent problems with multiple parameters in the boundary data. For a comparison of the two approaches we refer to [54], where both methods were found to produce similar results.

More difficulties arise when the non-homogeneous boundary conditions depend on time. This is a very typical case when POD-based ROMs are used for boundary control applications on unsteady flows. In [112] the time-dependent velocity boundary condition was handled by augmenting the Galerkin system (3) with a penalty term, so that (3) can be written as

$$\begin{aligned} &\frac{da_i(t)}{dt} = F_i + \sum_{j=1}^{N_r} A_{ij} a_j(t) + \sum_{j=1}^{N_r}\sum_{k=1}^{N_r} C_{ijk} a_j(t) a_k(t) + \tau \left[U_i^{\text{in}}(t) - \sum_{j=1}^{N_r} M_{ij} a_j(t) \right], \\ &a_i(0) = (\Psi_i, \mathbf{u}_0), \end{aligned} \tag{8}$$

where the boundary tensors are U^{in} and M are defined as

$$U_i^{\text{in}}(t) := \int_{\Gamma_{\text{in}}} \Psi_i(\mathbf{x}) \cdot \mathbf{u}_{\text{in}}(\mathbf{x})\, ds, \quad M_{ij} := \int_{\Gamma_{\text{in}}} \Psi_i(\mathbf{x}) \cdot \Psi_j(\mathbf{x})\, ds \tag{9}$$

with the assumption that the time-averaged base flow is zero on the inflow section Γ_{in}, i.e. $\bar{\mathbf{u}}|_{\Gamma_{\text{in}}} \equiv 0$. The penalty term $\tau > 0$ was chosen such that the correct asymptotically stable solution was obtained. This can be understood as a weak imposition of the Dirichlet condition that approaches strong imposition as $\tau \to \infty$.

1.2 Reduced basis construction by greedy algorithms

A popular strategy for constructing ROMs in the case of parameter-dependent problems is that of using *greedy* algorithms, based on the idea of selecting at each step the locally optimal element. This option can be seen as an alternative to POD strategy of previous section, yet preferable in the context of parametrized problems for reasons that will be sketched later on.

Before describing the greedy algorithm, let us formulate a steady version of problem (1), depending on a set of parameters $\mu \in \mathscr{P} \subset \mathbb{R}^P$, in a convenient way also for the following. Here we introduce the weak form, which was not the case in Sec. 1.1 to go from (2) to (3). The weak form of parametrized steady Navier-Stokes equations reads as follows: find $(\mathbf{u}, p) = (\mathbf{u}(\mu), p(\mu)) \in V \times Q$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{w}; \mu) + b(p, \mathbf{w}; \mu) + c(\mathbf{u}, \mathbf{u}, \mathbf{w}; \mu) = F(\mathbf{w}; \mu), & \forall \mathbf{w} \in V \\ b(q, \mathbf{u}; \mu) = G(q; \mu), & \forall q \in Q, \end{cases} \tag{10}$$

where the parametrized bilinear and trilinear forms are defined as follows:

$$a(\mathbf{v},\mathbf{w};\mu)=\int_\Omega \frac{\partial \mathbf{v}}{\partial x_i}\nu_{ij}(\cdot;\mu)\frac{\partial \mathbf{w}}{\partial x_j}d\Omega,\quad b(q,\mathbf{w};\mu)=-\int_\Omega q\chi_{ij}(\cdot;\mu)\frac{\partial w_j}{\partial x_i}d\Omega, \tag{11}$$

$$c(\mathbf{v},\mathbf{w},\mathbf{z};\mu)=\int_\Omega v_i\,\chi_{ji}(\cdot;\mu)\,\frac{\partial w_m}{\partial x_j}z_m\,d\Omega. \tag{12}$$

In what follows, we consider the more general case including the pressure field, so that $X=V\times Q$. Here μ may denote both physical and geometrical parameters, whose action on the problem is encoded by parametrized tensors $\nu(\cdot;\mu),\chi(\cdot;\mu)$. We point out that tensors components might depend *a priori* on both parameter components and spatial coordinates; see e.g. [87, 100, 109] for their complete derivation. Furthermore, $F(\cdot;\mu)$ and $G(\cdot;\mu)$ are linear forms accounting for non-homogeneous boundary data and source terms. Until stated otherwise, summation over repeated indices is understood.

The goal of the Reduced Basis (RB) method is to compute a low-dimensional approximation $(\mathbf{u}_r(\mu),p_r(\mu))$ of the solution to problem (10) by seeking a linear combination of well-chosen solutions[3] $(\Psi_i,\xi_i)=(\mathbf{u}(\mu^i),p(\mu^i))$ of problem (10), corresponding to specific choices of the parameter values:

$$\mathbf{u}_r(x;\mu):=\sum_{i=1}^{N_r}u_i(\mu)\Psi_i(x),\qquad\qquad p_r(x;\mu):=\sum_{i=1}^{N_r}p_i(\mu)\xi_i(x), \tag{13}$$

where the coefficients $u_i(\mu)$, $p_i(\mu)$ are computed by solving the following nonlinear algebraic system:

$$\begin{cases}\displaystyle\sum_{j=1}^{N_r}A_{ij}(\mu)u_j(\mu)+\sum_{j=1}^{N_r}B_{il}(\mu)p_l(\mu)+\sum_{j=1}^{N_r}C_{ijk}(\mu)u_j(\mu)u_k(\mu)=F_i(\mu),\\ \displaystyle\sum_{j=1}^{N_r}B^T u_j(\mu)=G_l(\mu),\end{cases} \tag{14}$$

with $i=1,\dots,N_r$. Reduced spaces for pressure and velocity fields (denoted respectively $Q_{N_r}^{RB}$ and $V_{N_r}^{RB}$) have the same dimension in the case of physical parametrizations, whereas geometrical parametrizations require modifying the velocity space in order to manage the divergence-free constraint; see Section 3.1. As in the case of problem (1), the functional forms appearing in (14) are obtained by Galerkin projection of the original problem (10) onto the RB space $X_{N_r}^{RB}=V_{N_r}^{RB}\times Q_{N_r}^{RB}$, spanned by the solutions (Ψ_i,ξ_i), so that, for $1\le i,j,k\le N_r$,

$$\begin{aligned}&A_{ij}(\mu):=a(\Psi_i,\Psi_j;\mu),\quad B_{kj}(\mu)=b(\xi_k,\Psi_j;\mu),\quad C_{ijk}(\mu)=c(\Psi_i,\Psi_j,\Psi_k;\mu)\\ &F_i(\mu):=F(\Psi_i;\mu),\quad G_l=G(\xi_l;\mu),\end{aligned} \tag{15}$$

[3] Gram-Schmidt orthonormalization is required in order to ensure the algebraic stability of the reduced basis approximation. Furthermore, in case of parameter-dependent geometries, the velocity space has to be enriched, as detailed in Section 3.

resulting again in a Galerkin ROM. In the parametrized setting the goal is to approximate uniformly well all the elements of the parametric manifold of solutions

$$M(\mu) = \{\mathbf{U}(\mu) := (\mathbf{u}(\mu), p(\mu)) \in X, \quad \mu \in \mathscr{P}\}$$

using finite dimensional subspaces $X_{N_r}^{RB}$ generated from elements of $M(\mu)$.

From a practical point of view, this approach is premised upon a classical Finite Element (FE) method "truth approximation" space $X_h \subset X$ of (typically very large) dimension. The RB method thus consists in a low-order approximation of the "truth" manifold $M_h = \{\mathbf{U}_h(\mu) := (\mathbf{u}_h(\mu), p_h(\mu)) \in X_h : \ \mu \in \mathscr{D}\}$. Nonetheless, we omit the subscript h wherever possible.

Next we address the construction of these subspaces. The so-called *greedy* algorithm, first proposed in [125], provides a quasi-optimal procedure for sampling the parameter space $\mathscr{P}$ – and so the manifold $M(\mu)$.

Thus, we seek a set of snapshot functions $\{\mathbf{U}(\mu^1), \mathbf{U}(\mu^2), \ldots, \mathbf{U}(\mu^{N_r})\}$ such that each $\mathbf{U}(\mu) \in M(\mu)$ is well approximated by the elements of the subspace $X_{N_r} = \text{span}\{\mathbf{U}(\mu^n), 1 \le n \le N_r\}$, according to the following algorithm:

$S_1 = \{\mu^1\}$; `compute` $\mathbf{U}(\mu^1)$; $X_1^{RB} = \text{span}\{\mathbf{U}(\mu^1)\}$;
`for` $n = 2 : N_r$
  `compute` $\mathbf{U}(\mu^n) = \arg\max_{\mathbf{W} \in M(\mu)} \|\mathbf{W} - \Pi_{X_{n-1}} \mathbf{W}\|_X$;
  `set` $S_n = S_{n-1} \cup \{\mu^n\}$;
  `set` $X_n^{RB} = X_{n-1}^{RB} \cup \text{span}\{\mathbf{U}(\mu^n)\}$;
  `if` $\max_{\mathbf{W} \in M(\mu)} \|\mathbf{W} - \Pi_{X_n^{RB}} \mathbf{W}\|_X \le \varepsilon_{\text{tol}}^*$
    `set` $N_r = n - 1$;
  `end`;
`end`.

where Π_{X_n} is the orthogonal projection w.r.t. the scalar product induced by $\|\cdot\|_X$ onto X_n^{RB}. Thus, at each step $n = 1, \ldots, N_r$, $\mathbf{U}(\mu^n)$ is the *worst case* element, which maximizes the error in approximating the subspace $M(\mu)$ using the elements of X_n^{RB}. However, this procedure (sometimes called *strong greedy* algorithm) is computationally infeasible: finding the maximum of the error of best approximation $\|\mathbf{W} - \Pi_{X_n} \mathbf{W}\|_X$ in X_n^{RB} would require a suitable maximization algorithm, which would also involve a large number of solutions of the full-order system (10). In a more feasible variant of this algorithm – sometimes called *weak greedy algorithm* – we replace the max over $\mathbf{W} \in M(\mu)$ with a max over a very fine sample $\Xi_{train} \subset \mathscr{D}$ of cardinality $|\Xi_{train}| = n_{train}$, and the true error $\|\mathbf{W} - \Pi_{X_n} \mathbf{W}\|_X$ with a suitable error estimate $\Delta_n(\mu)$, satisfying

$$c_\Delta \Delta_n(\mu) \le \|\mathbf{W} - \Pi_{X_n}^{RB} \mathbf{W}\|_X \le C_\Delta \Delta_n(\mu), \qquad \forall \mathbf{W} \in M(\mu) \tag{16}$$

for some constants $C_\Delta > c_\Delta > 0$. In this way, $\mathbf{U}(\mu^n) = \arg\max_{\mathbf{W}\in M(\mu)} \Delta_n(\mu)$ can be computed more effectively, under the assumption that the surrogate error $\Delta_n(\mu)$ is cheap to evaluate. In Sec. 3.2 we recall some a posteriori error estimates for reduced basis approximations for steady Navier-Stokes equations, and refer to [101] for their practical numerical implementation.

We point out that greedy-RB sampling methods are similar in objective to, but substantially different in approach from, the POD methods, which are more expensive from a computational standpoint. In fact, in the former we only need to compute the N_r retained snapshots (or *winning candidates*), which are typically very few. Only the error estimate has to be evaluated over the whole train set Ξ_{train}, which is very large – this is the reason why we require that the surrogate error must be cheap to evaluate. Instead, in the latter we must compute all the N_s candidate snapshots as well as compute the SVD of a large matrix.

We conclude this section by mentioning also some additional techniques quite close to POD for generating efficiently reduced spaces, the Centroidal Voronoi Tessellation (CVT) [25, 26, 46] and the Proper Generalized Decomposition (PGD) method [92], which has been recently applied to the solution of Navier-Stokes equations [47, 117]. Recent contributions are also contained in MS&A Vol.8, A.Quarteroni-G.Rozza (Eds.), Springer, 2013, see the chapter by Farhat and Amsallem dealing both with POD and Galerkin projection, and by Urban et al. A comparison on reduced representation approximations is provided instead by Bebendorf et al. in the same volume.

The rest of the chapter is structured as follows: In Section 2 we lay out some general guidelines that should be considered before attempting to build a ROM for any specific fluid problem. In Section 3 we address some issues related to approximation stability and error estimation which occur in the *reduced basis* approximation of steady-state solutions of parametrized Navier-Stokes equations. Moreover, in Section 4 we discuss specific issues related to the POD/Galerkin -based ROMs, such as the need for stabilizing the ROM, and how to ensure that the proper long-term behavior is recovered by the ROM. Some final remarks and a quick glance on current developments in the field are given in Section 5.

2 Some principles of model reduction of fluid systems

In this section we try to condense some fundamental principles to take into account when building ROMs that are known to most practitioners in the reduced-order modelling community but not always clearly communicated or established in literature. They are based both on our personal experience as well as on the general impression conveyed by state-of-the art literature on this subject. We have included motivating examples and several references to literature. Moreover, together with the

description of these fundamental principles, we also sketch the basic ingredients of reduced-order models for the computational reduction of PDEs.

2.1 "Never try to reduce the irreducible"

Once a full-order computational model for the fluid dynamics problem has been constructed, e.g. by means of finite elements or finite volumes discretizations, we may begin the process of constructing a suitable reduced-order model (ROM). The first step is to verify the assumption that the trajectories of the system live on a low-dimensional submanifold of the full space. From a practical point of view such a check is straightforward: it is sufficient to compute several trajectories of the full-order dynamical system, to collect snapshots into one matrix, and to perform a POD by using the singular value decomposition of this matrix. If the decay of the singular values is sufficiently rapid, then a limited number of modes will potentially suffice to represent the solution trajectories and an attempt at building a ROM can be performed.

It is easy to construct examples where slow decay or even no decay of the singular values of empirical snapshots is obtained. Consider for instance the one-dimensional linear transport equation

$$\begin{aligned} \partial_t u(x,t) + c\partial_x u(x,t) = 0, \qquad & (x,t) \in \mathbb{R} \times (0,T) \\ u(x,0) = u_0(x), \quad & x \in \mathbb{R} \end{aligned} \tag{17}$$

with solution $u(x,t) = u_0(x-ct)$. Take N_s snapshots of this solution at times $t = 0, \Delta t, 2\Delta t, \ldots, (N_s-2)\Delta t, T$. Assume that $u_0 \in L^2(\mathbb{R})$ and localized so that the measure of its support $\lambda(\mathrm{spt}(u_0)) < |c|\Delta t/2$. Thus, it follows that

$$\int_{\mathbb{R}} u_j(\xi)\, u_k(\xi)\, d\xi = \int_{\mathbb{R}} u_0(\xi - cj\Delta t)\, u_0(\xi - ck\Delta t)\, d\xi = \|u_0\|_{L^2(\mathbb{R})}\, \delta_{jk} \tag{18}$$

so that the correlation matrix of the snapshots (6) is diagonal with all eigenvalues equal. The singular values of the snapshot matrix do not decay at all, so that snapshot-based POD is not successful at representing traveling waves.

Using the empirical singular values to measure the feasibility of model reduction can also be theoretically justified. As already mentioned, the subset where solutions of the dynamical system live has typically the structure of a compact manifold $M(\mu)$ belonging to some larger function space X. To quantify how well such a manifold can be approximated by *Galerkin projection* onto a low-dimensional subspace, one can rely on the concept of Kolmogorov n-width, defined as

$$d_n(M;X) := \inf_{X_n \subset X} \sup_{\mathbf{u} \in M} \inf_{\widetilde{\mathbf{u}} \in X_n} \|\mathbf{u} - \widetilde{\mathbf{u}}\|_X \tag{19}$$

where the first infimum is taken over all linear subspaces $X_n \subset X$ of dimension n. The decay of $d_n \to 0$ as $n \to \infty$ can then be used as a measure of how many (POD or greedy-RB) modes need to be considered for the ROM (2) – the faster the decay, the smaller need to be the dimension of the linear subspace.

In the case that one is able to obtain exponential convergence in the n-width, that is to say $d_n(M;X) \leq C\exp(-\alpha n^{\beta})$ for some constants $C, \alpha, \beta > 0$, exponential convergence is also inherited (albeit at a reduced rate) by reduced-order approximations, and the same equivalence holds also for algebraic convergence rates, as was recently proved in [18]. Results regarding the connection between n-width decay rates and greedy algorithm converges rates can be found in [18, 22] for parametric problems, in [55] for time-dependent problems, and results regarding the n-width decay rates for parameter-dependent elliptic PDEs in [77, 86]. We stress that such results rely on a suitable sampling algorithm (such as the greedy algorithm, that selects proper time instances t^n or parameter points μ^n) where to compute the snapshots $\mathbf{U}_n(\cdot,t^n)$ (respectively $\mathbf{U}_n(\cdot,\mu^n)$) according to a reliable estimate of the error between the ROM and the full-order model. This is in order to actually find a (quasi-)optimal approximation space. We will revisit this point in Section 2.2.

Exponentially fast convergence of numerical approximations is often linked to spectral approximations of smooth (analytic) functions. In the case of elliptic coercive PDEs with random coefficients it was shown (see e.g. [10], Lemma 3.2) that if an elliptic and uniformly coercive parametric bilinear form $a : X \times X \times \mathscr{P} \to \mathbb{R}$ (consider for instance the scalar equivalent of the one defined in (11)) is such that

$$a(w,w;\mu) \geq \nu_{\min}\|w\|_X^2 \quad \text{for all } w \in X, \mu \in \mathscr{P} \subset \mathbb{R} \tag{20}$$

and its dependence on μ is analytic, then also the solutions $u(\mu)$ of

$$a(u(\mu),w;\mu) = f(w) \quad \text{for all } w \in X \tag{21}$$

for any $f \in X'$ are analytic functions of μ, provided that the parameter range $\mathscr{P} = [\mu_{\min}, \mu_{\max}]$ is bounded. The analyticity is then sufficient to prove exponential convergence of certain approximations to the solutions by expanding the solution as a power series. For example, when an approximation $u_{h,p}$ is obtained by using the FE method in space (with mesh size h) and the spectral collocation method in parameter (with polynomial order p), an exponential convergence result was obtained in [10] (Theorem 4.1): for any $\mu \in \mathscr{D}$

$$\begin{aligned}\|u(\mu) - u_{h,p}(\mu)\|_{L_p^2(\mathscr{P})\otimes X} \leq \frac{1}{\sqrt{\nu_{\min}}} \inf_{w \in L_p^2(\mathscr{P})\otimes X} \left(\frac{1}{|\mathscr{P}|}\int_{\mathscr{P}\times\Omega} \nu|\nabla(u(\mu)-w)|^2\right)^{1/2} \\ + C\exp\left(-p\log\left[\frac{2\tau}{|\mathscr{P}|}\left(\sqrt{1+\frac{|\mathscr{P}|^2}{4\tau^2}}\right)\right]\right),\end{aligned} \tag{22}$$

where the (sub)exponential convergence rate in p depends on the distance $\tau > 0$ between $\mathscr{P}$ and the nearest singularity in the complex (parameter) plane. Unfortunately, theoretical results that give estimates on the regularity of Navier-Stokes solu-

tion with respect to parameters acting on boundary terms, external forces, or initial data require stringent assumptions of small data and small Reynolds number that are not usually fulfilled by realistic flows. Nevertheless, exponential convergence of ROM approximation is often recovered also in nonlinear fluid problems.

2.2 "If it is not in the snapshots, it is not in the ROM"

We now turn to the question of how to choose the dimension N_r of the reduced space, so that we can take advantage of a substantial computational reduction but dealing with a reliable reduced-order model. In the case of the greedy-RB algorithm, reliable error estimates $\Delta_n(\mu)$ satisfying (16) can be used to assess the quality of the ROM, so that the sampling procedure stops when the error between the full-order model and the ROM is (estimated) lower than a give threshold, say 10^{-m} with $m \geq 2$, *uniformly* over the parameter space.

In the POD case, we can rely on the Relative Information Content (RIC) of the POD basis, which is defined as the ratio between the sum of the retained POD modes vs. the sum of the whole set of eigenvalues of the correlation matrix:

$$\text{RIC} := \frac{\sum_{i=1}^{N_r} \lambda_i}{\sum_{i=1}^{N_s} \lambda_i}. \tag{23}$$

The RIC is usually chosen up to $100(1-\alpha)\%$ by retaining a limited number of the *most energetic* POD modes, being, say $\alpha \in [10^{-m}, 10^{-1}]$ for a suitable $m > 1$. Flow features that are not sufficiently energetic will be omitted in the POD and thus cannot be captured by the ROM. A possible way to check which features to retain is to use a spatially weighted L^2-norm in the computation of the POD that gives more weight to features located at particular sites of interest.

Individual snapshots in the ensemble can be weighted accordingly to their importance, as proposed in [33]. In a series of papers (see e.g. [31, 36, 37]) Navon et al proposed a dual-weighted POD method, where the weights assigned to each snapshot were derived from an adjoint related to the optimality system of a variational data assimilation problem in meteorology. It is also known that for compressible flows the choice of inner product and weighting of the different flow variables (velocity, pressure, speed of sound) in the snapshot matrix can have a large effect on the stability and accuracy of the ROM [14, 35]. Similarly, the H^1 inner product was recommended for the computation of POD modes for compressible Navier-Stokes equations in [66] for the purpose of enhancing stability.

On the other hand, in the case of parametrized problems, the approximation properties of the basis depend on the parameter points μ^k where the snapshots are computed. It is known that, in general, a POD basis computed at a single parameter point is not a good approximation for solutions computed at different parameter points. This is another reason, in addition to computational efficiency pointed out in Section 1.2, why a greedy algorithm should be chosen in order to manage with a careful

sampling of a parameter space. In summary, two typical improvements to the POD sampling process are adopted:

1. *Adaptivity*. In this case an initial POD basis is constructed and the resulting ROM used for simulations, but is later updated based on some problem-dependent criteria. This is a typical approach in ROM-based optimization and optimal control applications, such as those presented in [17, 103], where the snapshots and the POD are updated after every optimization step to improve the accuracy of the ROM near the optimal point. The price to be paid is that the cost of the optimization loop will increase due to the need of additional full-order simulations to update the ROM. The idea of a trust-region POD method was presented in [49]. In this case, the POD version of the optimality system is solved at each iteration within a trust-region radius $\Delta^{(k)}$ to obtain a quasi-optimal $c^{(k+1)}$ set of controls. Then the full-order Navier-Stokes equations are solved with the quasi-optimal controls to obtain $\mathbf{u}(c^{(k+1)})$. The discrepancy between the ROM prediction and the full-order solution is then measured, and if its too large the step is rejected and the trust region radius decreased, $\Delta^{(k+1)} < \Delta^{(k)}$. Otherwise, the step is accepted and the trust region possibly increased, $\Delta^{(k+1)} \geq \Delta^{(k)}$. The ROM is updated after each iteration step to incorporate the newly computed snapshots.

2. *Optimality* (or *near-optimality*). A priori error estimates for POD approximations were introduced in [72] and can be used to gauge the total number of POD modes to retain to achieve a given representation accuracy at one single parameter point. In this case, snapshots are typically chosen iteratively by measuring the error of the current ROM at different trial points of the parameter space, then computing snapshots at the parameter point where the maximum error (estimate) is obtained and adding them to the ROM, like in the greedy-RB algorithm, first proposed in [52], and now standard in the parametric model reduction community. For sampling in time POD-greedy strategies have been proposed for linear evolution equations in [56], the viscous nonlinear Burgers' equation in [93], and the Navier-Stokes equations in [127].
   In [73] the authors derived sensitivity equations to measure the effect of adding new snapshots in the POD basis and use them to find optimal locations for new snapshots that minimize the error between the POD-solution and the trajectory of the full-order system. This can avoid the expensive computation of full-order trial solutions typically needed in a POD-greedy approach. Furthermore, in [24] the POD procedure was extended to incorporate goal-oriented quantities related to specific outputs of interest over the entire range of parameters.

2.3 "Exploit the known structure of the solutions"

Both POD and greedy-RB strategies use a set of full-order solutions to build a global basis for the approximation of the solution of a PDE problem for any given time $t \in (0,T)$ or parameter value $\mu \in \mathscr{P}$. It is important to understand that the basis

functions of a ROM do not really tell us much about the dynamical structure of a time-dependent problem. In the case of parameter-dependent problems, a parameter value μ different from the snapshots $\mu^1, \ldots, \mu^{N_r}$ may result in a flow regime that is qualitatively very different (for instance when the flow is parametrized with respect to the Reynolds number) than those exhibited at the snapshot parameters. Also parametrized geometrical features can greatly affect the qualitative behavior of the solutions. Thus, in order to make the ROM capable to represent the physics of the full model correctly, we need to let the equations play a role also at the evaluation level, for any new problem instance to solve. This is the reason why, in equations (3) or (14), we follow a *projection* approach, rather than an *interpolation*-based strategy. This makes more reliable also the evaluation of outputs derived from the solution, such as energy, stresses, vorticity, etc.

We still have to explain how to pursue a strong computational reduction when solving the problem obtained by plugging the reduced solution into the equations. Thus, we need to equip ROMs of previous sections with an efficient implementation aiming at decoupling the generation and projection stages. Let us focus, for the sake of clarity, on the case of parametrized problems. In particular, two ingredients need to come into play, in order to obtain the so-called *Offline/Online* splitting:

1. *Affine parameter dependence*. In order to speed up the evaluation of a reduced approximation when the differential operators depend on some parameters, the key point is to isolate the contribute of parametrized quantities in the differential operators, so that expensive parameter-independent structures can be computed *Offline* and stored once, whereas inexpensive parameter-dependent quantities can be efficiently evaluated *Online* for each new value of the parameters.

   To make the *Online* evaluation step efficient, we need to take the parametrized quantities out of the integrals appearing in (11)-(12). The usual assumption required in the reduced-basis methods is the so-called *affine parameter dependence*, i.e. we require that parametrized forms (11)-(12) can be expressed as linear combinations of parameter-independent operators:

$$a(\mathbf{v},\mathbf{w};\mu) = \sum_{q=1}^{Q^a} \Theta_a^q(\mu) a^q(\mathbf{v},\mathbf{w}), \qquad b(q,\mathbf{w};\mu) = \sum_{q=1}^{Q^b} \Theta_b^q(\mu) b^q(q,\mathbf{w}); \quad (24)$$

$$c(\mathbf{v},\mathbf{w},\mathbf{z};\mu) = \sum_{q=1}^{Q^c} \Theta_c^q(\mu) c^q(\mathbf{v},\mathbf{w},\mathbf{z}) \quad (25)$$

   for some integers Q^a, Q^b, Q^c, where q is a condensed index of i, j quantities. This is straightforward when dealing with common physical parametrizations (e.g. by considering Reynolds number, Grashof number, Prandtl number, inflow velocity peaks, etc. as parameters [42, 44]) or simple affine geometrical parametrization – in all these cases, parametrized tensors entering in (11)-(12) depend only on parameter μ. Instead, when parametrized tensors depend also on x, affinity assumptions (24)-(25) can only be recovered by suitable approximations, such as

the ones based on the so-called Empirical Interpolation Method (EIM); see e.g. [15, 85].

2. *Reduced matrix structures*. Once the parameters have been taken out of the operators by requiring the affine parameter dependence (24)-(25), the reduced operators (15) can be expressed (e.g. for the diffusion term) as

$$A_{ij}(\mu) = a(\Psi_i, \Psi_j; \mu) = \sum_{q=1}^{Q^a} \Theta_a^q(\mu) a^q(\Psi_i, \Psi_j) = \sum_{q=1}^{Q^a} \Theta_a^q(\mu) A_{ij}^q.$$

In order to make the *Online* evaluation independent of the dimension of the full-order space, structures A^q and C^q corresponding to parameter-independent operators must be constructed properly and stored during the *Offline* stage.

We remark that the basis functions are given by full-order approximations of (10) for selected values of the parameters, under the form

$$\Psi_i(x) = \sum_{m=1}^{N^{\mathbf{u}}} \Psi_i^m \phi_m^{\mathbf{u}}(x), \quad \xi_i(x) = \sum_{m=1}^{N^p} \xi_i^m \phi_m^p(x),$$

where $\{\phi_m^{\mathbf{u}}(x)\}_{m=1}^{N^{\mathbf{u}}}$, $\{\phi_m^p(x)\}_{m=1}^{N^p}$ are two bases of the full-order (velocity, resp. pressure) approximation spaces, of dimension $N^{\mathbf{u}}$, N^p, respectively. Thus, the assembling of reduced-order algebraic structures (15) can be efficiently performed by combining the matrices collecting the basis functions, given by

$$\mathbb{Z}_{\mathbf{u}} = [\Psi_1 \,|\ldots|\, \Psi_{N_r}] \in \mathbb{R}^{N^{\mathbf{u}} \times N_r}, \qquad \mathbb{Z}_p = [\xi_1 \,|\ldots|\, \xi_{N_r}] \in \mathbb{R}^{N^p \times N_r},$$

and the full-order algebraic structures. It is straightforward to check that, e.g.,

$$a^q(\Psi_i, \Psi_j) = \sum_{m=1}^{N^{\mathbf{u}}} \sum_{n=1}^{N^{\mathbf{u}}} \Psi_i^m a^q(\phi_m^{\mathbf{u}}, \phi_n^{\mathbf{u}}) \Psi_j^n, \qquad \text{i.e.} \qquad A^q(\mu) = \mathbb{Z}_{\mathbf{u}}^T \, \tilde{A}^q(\mu) \, \mathbb{Z}_{\mathbf{u}},$$

where $\tilde{A}_{mn}^q(\mu) = a^q(\phi_m^{\mathbf{u}}, \phi_n^{\mathbf{u}})$ is the full-order stiffness matrix corresponding to the bilinear form $a^q(\cdot,\cdot)$. The same procedure can be applied to pressure and nonlinear terms as well; see e.g. [87, 100] for a detailed explanation.

Both these ingredients, together with the snapshot selection procedures and (wherever available) rigorous error estimates allow to successfully apply Galerkin ROMs to incompressible flows. However, some caveats should be mentioned.

For instance, the evaluation of the trilinear convective term – given by $C_{ijk} a_j(t) a_k(t)$ in (3) or $C_{ijk}(\mu) u_j(\mu) u_k(\mu)$ in (14) – even in the reduced-order formulation requires evaluating tensorial terms of relatively large sizes. This is even more involved when the size Q^c of the affine expansion (25) is large. For more general nonpolynomial nonlinearities of the form

$$\left\langle \mathbf{f}\left(\bar{\mathbf{u}}(x)+\sum_{i=1}^{N_r} a_i(t)\Psi_i(x)\right), \bar{\mathbf{u}}(x)+\sum_{j=1}^{N_r} a_j(t)\Psi_j(x)\right\rangle \tag{26}$$

deflating the nonlinear terms to their full-order representations may be necessary in order to evaluate the nonlinear terms, negating many advantages of using a ROM in the first place. In order to reduce the online cost of evaluating the nonlinear term(s), several "hyper-reduction" techniques have been proposed, such as DEIM [30] (Discrete Empirical Interpolation Method), DBPIM [12] (Discrete Best Points Interpolation Method), MPE (Missing Point Estimation) [8] and GNAT [5] (Gauss-Newton with Approximated Tensor quantities). In general, most of these methods attempt to approximate the nonlinearity using linear combinations of the POD basis functions

$$\mathbf{f}\left(\bar{\mathbf{u}}(x)+\sum_{i=1}^{N_r} a_i(t)\Psi_i(x)\right) \approx f_0\,\bar{\mathbf{u}}(x)+\sum_{i=1}^{N_r} \widetilde{f}_i(t)\,\Psi_i(x) \tag{27}$$

and differ mainly on the strategy of choosing the approximation coefficients $\widetilde{f}_i(t)$. When the nonlinearity is treated using a Newton algorithm, a similar approximation can be applied to the Jacobian J_f, see e.g. [5, 30]. A contribution on discrete EIM (DEIM) is the chapter by Antil et al. in MS&A Vol.8, A.Quarteroni-G.Rozza (Eds.), Springer, 2013.

2.4 "Divide and conquer whenever possible"

The *rationale* behind the efficacy of the ROMs we have discussed so far is the regularity of the parameter dependence in the case of parametrized problems like (10) – respectively, time continuity in the case of time-dependent problems like (1). In other words, solutions to these problems lie on a low-dimensional *manifold*, as already pointed out in Section 1.2. The more regular the manifold (and the parametric dependence), the more conveniently the solution can be approximated by a suitable combination of snapshots.

However, even laminar flow can experience strong qualitative changes (bifurcations) when critical parameters such as the Reynolds number is varied. For example, the flow behind a cylinder experiences first a transition from steady flow to a time-periodic flow, then a loss of periodicity in the vortex shedding, and finally transition to a chaotic turbulent regime as the Reynolds number is gradually increased. In order to make sure that a ROM approximates correctly the fluid flow in some range of the parameter(s), we require that the parameter space (or the time interval) are chosen such that the manifold is locally a branch of nonsingular solutions.

Although quite restrictive, this is a standard assumption also in the case of full-order approximations, based e.g. on the FE method (see e.g. [21, 27]). Nevertheless, bases constructed using the greedy algorithm provide reliable approximations also

in the case of bifurcation points included in the parameter space; for instance, ROMs have been used to track particular solution branches past the bifurcation point, see e.g. [97, 119]. In case of parametrized flows, in order to minimize the required number of basis functions, a good ROM should be tailored so that different flow regimes can be captured in a reliable way. Since POD-based ROMs provide poor approximations away from the parameter values for which the snapshot solutions were computed, it rarely makes sense to try and develop one global approximation basis for the entire parameter space. Many works have been focused in these last years on possible strategies to rectify this aspect.

One possibility is to combine ROMs computed for different physical flow regimes. In [3, 4] the ROMs computed at different parameter points were interpolated to obtain a new ROM that was valid also in the intermediate zone between the original parameter points. In [58, 59] the parametric sensitivities of the POD modes were computed and added to the snapshot set, which improved the validity of the reduced solutions away from the parametric snapshots. However, in a more involved geometrical parametrization case the ROM failed completely, as it did not converge to the exact solution even when the number of POD modes was increased.

A "compact POD" approach based on goal-oriented Petrov-Galerkin projection was proposed in [28], in order to minimize the approximation error subject to a chosen output criteria, also including sensitivity information (with proper weighting coming from the Taylor-expansion) and including "mollification" of basis functions far away from the snapshot parameter. A further option, described in [5], exploits a *k-means* clustering procedure to construct local ROMs by grouping together nearby snapshots. In this way, once the snapshots have been computed, the reduced space is partitioned in subregions and a local reduced basis is assigned to each subregion. This can be seen as an adaptive version of a former strategy based on the so-called *Centroidal Voronoi Tessellation*, introduced in [46] and extended in [25, 26].

Finally, we mention that local ROMs can be properly combined also in view of a further computational reduction for instance in the solution of parametrized problems featuring a repetitive geometrical structure – such as networks, or multidomain configurations. The Reduced Basis Element (RBE) method combines domain decomposition with parametric ROMs, by exploiting nonconforming approaches – such as mortar methods or discontinuous Galerkin methods – between the subdomains and the greedy RB method within each subdomain. Recent application of the RBE method to fluid flows can be found e.g. in [43, 65, 81]. A more advanced variant exploits static condensation at the interdomain level [62] by connecting (at some interfaces, or ports, during the online stage) a library of reference, interchangeable components.

3 Model reduction of steady viscous flows

In this section we summarize those features which are peculiar to ROMs for parametrized steady viscous flows, such as *inf-sup* stability, correct treatment of pressure, suitable a posteriori error estimates. We also point out the analogies with the case of linear viscous flows modelled by Stokes equations. In particular, we exploit a greedy algorithm for the construction of the reduced space: at each step the basis of snapshots is augmented by the solution corresponding to the largest error estimate. The downside is that the method is completely reliant on the existence of computable a posteriori error bounds, which are not really available for the unsteady Navier-Stokes equations, as we mentioned. This is the main reason why, so far, this method has largely been limited to steady Navier-Stokes equations.

3.1 A question of stability: inf-sup constants and supremizers

A feature of the standard POD-Galerkin ROM (3) is that the pressure term $-\nabla p$ has been completely eliminated. In fact, assuming that the POD modes Ψ_i satisfy the strong incompressibility constraint by construction, $\nabla \cdot \Psi_i = 0$ pointwise, integration by parts of the pressure-gradient term evaluated on the POD modes gives

$$(\nabla p, \Psi_i) = \int_\Omega \nabla p \cdot \Psi_i \, d\mathbf{x} = -\int_\Omega p(\nabla \cdot \Psi_i) \, d\mathbf{x} + \int_{\partial\Omega} p(\Psi_i \cdot n) \, ds, \tag{28}$$

which demonstrates that the pressure only enters the ROM on the boundary and for enclosed flows ($\Psi_i \cdot n \equiv 0$ on $\partial\Omega$) it vanishes completely from the equations. For instance, this is the case of a standard driven cavity problem. It should be noted, however, that the situation also depends on the choice of the adopted spatial discretization. For standard FE discretizations the incompressibility of solutions applies only elementwise, i.e.

$$\int_{K \in \mathscr{T}_h} \nabla \cdot \Psi_i \, d\mathbf{x} = 0 \quad \text{for all mesh elements } K \in \mathscr{T}_h \tag{29}$$

so that unless piecewise constant functions in each mesh element K are used for the pressure, the term $-\int_\Omega p(\nabla \cdot \Psi_i) \, d\mathbf{x}$ does not vanish identically. Nevertheless, this term is neglected for many flows as small and unnecessary to enforce the incompressibility of the ROM solutions. It is known that neglecting the pressure term for convectively unstable shear layers, especially ones with two-dimensional mixing layers, can result in large errors as was demonstrated in [96]. Pressure-extended ROMs include also the pressure in the equations, either by deriving the necessary terms in the expansion (3) to account for the pressure [96], or by performing a separate POD to construct another basis $\{\Phi_j\}_{j=1}^{N_r}$ for the pressure field [16, 79]. The benefit of the latter approach is that the pressure field is immediately recovered with-

out any post-processing steps necessary.

We focus our analysis on pressure-extended ROMs, using a greedy algorithm to also build a basis for the pressure. In this way, for each selected parameter value, we compute both the (truth FE approximation of the) velocity and the pressure fields. Reduced velocity and pressure spaces result as follows:

$$V_{N_r} := \text{span}(\Psi_j \,:\, j = 1,\dots,N_r), \quad Q_{N_r} := \text{span}(\Phi_j \,:\, j = 1,\dots,N_r),$$

(we omit the superscript RB for the sake of brevity) where $\Psi_j \in V_h$ and $\Phi_j \in Q_h$ for any $j = 1,\dots,N_r$, being V_h and Q_h the truth velocity and pressure approximation spaces. Mathematically, a necessary and sufficient condition ensuring the ROM stability is the *reduced (Brezzi) inf-sup condition*

$$\beta_r(\mu) := \inf_{q\in Q_{N_r}} \sup_{\mathbf{v}\in V_{N_r}} \frac{b(q,\mathbf{v};\mu)}{\|q\|_Q\|\mathbf{v}\|_V} > 0, \tag{30}$$

which is obviously related to, but not implied by, the *full-order (Brezzi) inf-sup condition*

$$\beta_h(\mu) := \inf_{q\in Q_h} \sup_{\mathbf{v}\in V_h} \frac{b(q,\mathbf{v};\mu)}{\|q\|_Q\|\mathbf{v}\|_V} > 0, \tag{31}$$

for velocity and pressure spaces $V_h \supset V_{N_r}$ and $Q_h \supset Q_{N_r}$. We recall that $b(q,\mathbf{v};\mu)$ denotes the pressure/divergence bilinear form, defined in (11). We also point out that now the stability factors such as $\beta_r(\mu)$, $\beta_h(\mu)$ are functions of the parameter vector μ, rather than *constants*, as in usual discretization techniques. We remind that (31) is ensured e.g. by choosing as $V_h \times Q_h$ the space of Taylor-Hood $\mathbb{P}_2 - \mathbb{P}_1$ elements (see [19, 20]); however, this choice is not restrictive, the whole construction keeps holding for other spaces combinations as well (e.g. [99]).

Instead, in order to fulfill the reduced inf-sup condition (30), we define for each pressure basis function Φ_j the corresponding inner supremizer velocity function [106, 109]

$$T^\mu \Phi_j := \underset{\mathbf{v}\in V_h}{\text{argsup}} \frac{b(\Phi_j,\mathbf{v};\mu)}{\|\mathbf{v}\|_V}, \tag{32}$$

which can be obtained by solving the discrete elliptic problem

$$(T^\mu \Phi_j, \mathbf{v})_V = b(\Phi_j,\mathbf{v};\mu), \quad \text{for all } \mathbf{v} \in V_h. \tag{33}$$

By applying (32) and enriching the RB velocity space V_h to include the inner supremizers, we define a new extended velocity space as

$$V_{N_r}^* := V_{N_r} \oplus \text{span}(T^\mu \Phi_j \,:\, j = 1,\dots,N_r),$$

such that

$$\begin{aligned} 0 < \beta_h(\mu) &= \inf_{q\in Q_h} \sup_{\mathbf{v}\in V_h} \frac{b(q,\mathbf{v};\mu)}{\|q\|_Q\|\mathbf{v}\|_V} \leq \inf_{q\in Q_r} \sup_{\mathbf{v}\in V_h} \frac{b(q,\mathbf{v};\mu)}{\|q\|_Q\|\mathbf{v}\|_V} \\ &= \inf_{q\in Q_r} \frac{b(q,T^\mu q;\mu)}{\|q\|_Q\|T^\mu q\|_V} \leq \inf_{q\in Q_r} \sup_{\mathbf{v}\in V_{N_r}^*} \frac{b(q,\mathbf{v};\mu)}{\|q\|_Q\|\mathbf{v}\|_V} = \beta_r(\mu). \end{aligned} \tag{34}$$

Thus, the inf-sup stability of the full-order space now implies the stability of the supremizer-enchanced reduced space, provided that this latter is enriched with the solutions of the *supremizer* equation (33).

We remark that, by enriching V_{N_r} with the supremizers $\{T^\mu \Phi_j\}_{j=1}^{N_r}$, the new RB velocity space $V_{N_r}^*$ has dimension $2N_r$, the double of the dimension N_r of the RB pressure space.

The treatment of the (Brezzi) inf-sup stability through the supremizer operator is common to Stokes and Navier-Stokes equations, and more in general to any problem written under a *saddle-point* form. Further details about the efficient construction of the supremizer solutions and the Gram-Schmidt orthonormalization of the RB basis functions can be found for instance in [87, 106, 109], whereas a general context drawn for saddle-point problems has been developed in [50].

3.2 Certification of ROMs for the steady Navier-Stokes equations

We now introduce the main aspects related with *a posteriori* error estimation in the RB context for parametrized steady Navier-Stokes equations. This approach is in analogy with the so-called *(Babuška) inf-sup stability theory* [11], which can be seen as a generalization to the Petrov-Galerkin case of the Lax-Milgram result for the Galerkin-type formulation. Its application to the Stokes problem is just a possible use, as shown in [106], where a general framework to compute error bounds for noncoercive problems solved by the RB method has been introduced. Within this framework, a joint residual-based estimation for velocity and pressure fields in the Stokes case can be easily obtained under the form

$$\|\mathbf{U}_h(\mu) - \mathbf{U}_r(\mu)\|_X \leq \frac{\|r(\mu)\|_{X'}}{\beta_{S,h}^{LB}(\mu)} =: \Delta_{N_r}(\mu), \qquad \forall\, \mu \in \mathscr{D}, \tag{35}$$

where:

- $\mathbf{U}_h(\mu) = (\mathbf{u}_h(\mu), p_h(\mu)) \in X_h = V_h \times Q_h$ and $\mathbf{U}_r(\mu) = (\mathbf{u}_r(\mu), p_r(\mu)) \in X_{N_r} = V_{N_r}^* \times Q_{N_r}$ denote the truth and the RB approximations of velocity and pressure;
- $\|r(\mu)\|_{X'} = \sup_{\mathbf{V}\in X_h} r(\mathbf{V};\mu)/\|\mathbf{V}\|_X$ is the dual norm of the global residual

$$r(\mathbf{V};\mu) := r_{\mathbf{u}}^S(\mathbf{v};\mu) + r_p(q;\mu),$$

being

$$\begin{aligned} r_{\mathbf{u}}^S(\mathbf{v};\mu) &:= F(\mathbf{v};\mu) - a(\mathbf{u}_r(\mu),\mathbf{v};\mu) - b(p_r(\mu),\mathbf{v};\mu), \\ r_p(q;\mu) &:= G(q;\mu) - b(q,\mathbf{u}_r(\mu);\mu); \end{aligned} \tag{36}$$

- the bilinear form $A_S(\cdot,\cdot;\mu) : X \times X \to \mathbb{R}$ denotes the global Stokes operator

$$A_S(\mathbf{U},\mathbf{V};\mu) := a(\mathbf{u},\mathbf{v};\mu) + b(p,\mathbf{v};\mu) + b(q,\mathbf{u};\mu); \tag{37}$$

- $\beta^{LB}_{S,h}(\mu)$ is a computable lower bound for the Babuška inf-sup stability factor $\beta_{S,h}(\mu)$, involving the global Stokes operator:

$$\exists\, \beta^{LB}_{S,h}(\mu) > 0 :\ \beta_{S,h}(\mu) = \inf_{\mathbf{U}\in X_h} \sup_{\mathbf{V}\in X_h} \frac{A_S(\mathbf{U},\mathbf{V};\mu)}{\|\mathbf{U}\|_X \|\mathbf{V}\|_X} \geq \beta^{LB}_{S,h}(\mu), \quad \forall\, \mu \in \mathscr{D}. \tag{38}$$

In this way, the stability of the reduced basis approximation is based on Brezzi's saddle point theory (and the introduction of a supremizer operator on the pressure terms), whereas a rigorous *a posteriori* error estimation procedure for velocity and pressure fields is based on Babuška's inf-sup constant.

Alternatively, we could rely on the Brezzi's theory also for the sake of error estimation, by deriving two distinct error bounds for velocity and pressure, as shown in [50]. However, despite their similar effectivity, these latter require the approximation of two stability factors (for stiffness and pressure/divergence terms) and of a continuity constant (for the stiffness term), which entail larger computational costs in a parametrized context.

In the Navier-Stokes case we can instead obtain a rigorous *a posteriori* error estimation by relying on the so-called Brezzi-Rappaz-Raviart (BRR) theory [21, 27], which is useful for the analysis of a wider class of nonlinear equations. We require some slight modifications with respect to the linear preliminaries, even if also for the Navier-Stokes problem the *a posteriori* error estimation takes advantage of the dual norm of residuals and of an effective lower bound of a suitable (parametric) stability factor, given in this case by the Babuška inf-sup constant referred not to the global Navier-Stokes operator

$$A(\mathbf{U},\mathbf{V};\mu) = A_S(\mathbf{U},\mathbf{V};\mu) + C(\mathbf{U},\mathbf{U},\mathbf{V};\mu), \tag{39}$$

but to its *Fréchet* derivative (with respect to the first variable), defined as

$$dA(\mathbf{W};\mu)(\mathbf{U},\mathbf{V}) = A_S(\mathbf{U},\mathbf{V};\mu) + C(\mathbf{W},\mathbf{U},\mathbf{V};\mu) + C(\mathbf{U},\mathbf{W},\mathbf{V};\mu), \tag{40}$$

when evaluated at $\mathbf{W} \in X$. In both cases, we denote by $C(\mathbf{U},\mathbf{U},\mathbf{V};\mu) = c(\mathbf{u},\mathbf{u},\mathbf{v};\mu)$. In this framework, a joint residual-based estimation for velocity and pressure fields in the Navier-Stokes case takes the following form: for any $N_r \geq N^*(\mu)$,

$$\|\mathbf{U}_h(\mu) - \mathbf{U}_r(\mu)\|_X \leq \frac{\beta^{LB}_{NS,h}(\mu)}{2\gamma(\rho;\mu)} \left(1 - \sqrt{1 - \tau_{N_r}(\mu)}\right) =: \Delta_{N_r}(\mu), \quad \forall\, \mu \in \mathscr{D} \tag{41}$$

provided that $\tau_{N_r}(\mu) < 1$. In particular:

- $\tau_{N_r}(\mu)$ is a *non-dimensional* measure of the residual, defined as

$$\tau_{N_r}(\mu) = \frac{4\gamma(\rho;\mu)\|r(\mu)\|_{X'}}{(\beta^{LB}_{NS,h}(\mu))^2};$$

moreover, we denote $N^*(\mu)$ the smallest N_r such that $\tau_{N_r}(\mu) < 1$, for all $N_r \geq N^*(\mu)$. Since $\|r(\mu)\|_{X'}$ – and thus $\tau_{N_r}(\mu)$ – undergoes a fast decrease when N_r increases, usually $N^*(\mu) < 10$, so that (41) holds for reasonable dimensions N_r;

- $\gamma(\rho_h;\mu)$ is the (discrete) continuity constant of the trilinear form $c(\cdot,\cdot,\cdot;\mu)$, depending on the Sobolev embedding constant ρ_h defined as

$$\rho_h^2 = \sup_{v \in V_h} \frac{\|v\|^2_{L^4(\Omega)}}{(v,v)_H};$$

- the dual norm $\|r(\mu)\|_{X'}$ of the global residual, which is given in this case by

$$\begin{aligned} r(\mathbf{V};\mu) &:= r_{\mathbf{u}}(\mathbf{v};\mu) + r_p(q;\mu), \\ r_{\mathbf{u}}(\mathbf{v};\mu) &:= r^S_{\mathbf{u}}(\mathbf{v};\mu) - c(\mathbf{u}_r(\mu),\mathbf{u}_r(\mu),\mathbf{v};\mu); \end{aligned} \tag{42}$$

- $\beta^{LB}_{NS,h}(\mu)$ is a computable lower bound for the Babuška inf-sup stability factor $\beta_{NS,h}(\mu)$, involving the *Fréchet* derivative of the global Navier-Stokes operator:

$$\exists\, \beta^{LB}_{NS,h}(\mu) > 0 :\ \beta_{NS,h}(\mu) = \inf_{\mathbf{V} \in X_h} \sup_{\mathbf{W} \in X_h} \frac{dA(\mathbf{U}_h(\mu);\mu)(\mathbf{V},\mathbf{W})}{\|\mathbf{V}\|_X \|\mathbf{W}\|_X} \geq \beta^{LB}_{NS,h}(\mu). \tag{43}$$

We remark that the framework described above is essential the nonlinear extension of the much simpler linear *a posteriori* error estimation (35), to which the nonlinear error estimation (41) reduces in the limit that $\|r(\mu)\|_{X'} \to 0$.

A posteriori error estimation for the Navier-Stokes problem poses, from a computational standpoint, more severe challenges than for Stokes problem. We do not provide any detail about the evaluation of these quantities; the interested reader can refer, for instance, to [76, 87, 94, 124].

3.3 Relevant computational issues

Finally we point out the most relevant computational difficulties encountered in developing/applying the methodology presented in this section. We focus, in particular, on the evaluation of the *a posteriori* error bounds, a crucial aspect when attempting to build a reduced space with the greedy RB method.

With respect to linear problems, where the computational speedup between a reduced basis method and a truth approximation is usually about 10^2, reduction may be even larger (sometimes up to one order of magnitude) in nonlinear problems. In this case, nonlinear solvers might require several iterations to converge to the solution. Each iteration entails a large linear system to solve in the case of the truth approximation. Instead, a reduced-order model requires at each iteration of the nonlinear solver the solution of a small linear system, which can be assembled by exploiting the precomputed structures (14).

Nevertheless, we need to rely on a suitable Offline/Online splitting to speed up our computation. Such a strategy is also required to evaluate in a very small amount of time the error estimates (35) or (41), so that all the parametric-dependent quantities appearing in these formulas can exploit the affine parametric dependence.

Moreover, error estimates should to be uniformly effective across entire parameter range, to avoid the greedy algorithm skew towards particular locations in parameter space. In this case, the basis resulting from the selected snapshots could be inadequate to uniformly approximate the whole manifold of solutions, or result larger than required. Essentially, we pursue the following strategy:

1. *Stability factors*. If the (Navier)-Stokes operator is parameter-dependent, so is the lower bound of the stability factor (38) or (43). In this case, computing its lower bound according to a suitable Offline/Online splitting is not easy. We face it by using the so-called *Successive Constraint Method* (SCM)[4] which converts the eigenproblem corresponding to the computation of (38) or (43) on the successive solution of suitable linear optimization problems.
   This algorithm has been applied for the first time to saddle point Stokes problems in [106], while a first extension to the nonlinear Navier-Stokes case has been considered in [87]. In case of physical parametrizations (for instance, involving the Reynolds number) and large parametric variations, stability factors might undergo large variations and the SCM algorithm is able to capture this behavior. Instead, according to our own experience, in case of geometrical parametrizations arising from local shape changes or simple scaling (or affine) transformations, piecewise constant approximations of the stability factors can provide good result at a very lower cost. In more involved cases, alternative heuristic strategies to derive lower bounds of stability factors might take advantage of suitable interpolation techniques (see e.g. [87]).
2. *Residuals*. A suitable Offline/Online splitting can be used to evaluate the dual norms of residuals (36)-(42). Indeed, these quantities can be expressed as the sum of products of μ-dependent known functions and μ-independent inner products, formed of more complicated but precomputable quantities, involving the Riesz representations of $r_{\mathbf{u}}(\mu)$ and $r_p(\mu)$.
   As already remarked in Section 2.3, in the case of nonlinear convective terms tensorial terms of relatively large sizes are generated; they depend on both the dimension N_r of the reduced spaces and the parametric complexity Q^c of the trilinear convective term. Unfortunately, evaluating and storing these structures might become computationally infeasible, so that an Offline/Online splitting for evaluating the dual norms of residuals is not always practicable.

A Galerkin projection is well suited for symmetric and coercive PDEs, as in this case it provides the optimal approximation in the corresponding energy norm. In the

[4] This algorithm has been first introduced in [64] for both coercive and noncoercive problems, analyzed in [107] in the coercive case and afterwards improved in [32]. A general version using the so-called "natural norm" [110] has been analyzed in [61], where it has been applied to noncoercive problems such as Helmholtz equations – the simpler coercive case can be seen as a particular instance where the stability factor is just the coercivity constant.

case of convection-dominated flows, symmetry is broken and no a priori optimality can be ascertained. Indeed, a large gap between the magnitude of the observed nonlinear residuals and the true error between full and reduced solution may exist.

A remedy consists in using Petrov-Galerkin methods, with different spaces of test and trial functions. They are usually presented in the guise of stabilization methods, such as in the case of the Streamline-Upwind Petrov-Galerkin (SUPG) method. However, one is then left with the question of how to choose the test space. Recent works on optimal or near-optimal choice of Petrov-Galerkin test spaces were presented in [40, 41] and [38]. These options are "optimal" in the sense that they give the best possible ratio of continuity constant to stability constant in the energy norm estimates. In the finite element or discontinuous Galerkin context, optimal test spaces are usually avoided, as this would lead to using test functions with global support. However, in the ROM setting one does not care too much if the reduced order system is full, as it is typically small enough to be solved with direct solvers (the reduced dimension N_r is typically in the range $10 - 10^2$).

In fact, the optimal test spaces are precisely equivalent to the method of supremizers used in [109, 106] to stabilize ROMs for the Stokes equations. Unfortunately, in the parametrized setting one has to face the fact that the optimal test spaces (and also the supremizers) usually depend explicitly on the parameters and thus suitable strategies to recover the Offline/Online splitting must be devised.

4 Model reduction of unsteady viscous flows

In this section we provide an overview of some reduction techniques available for unsteady viscous flows. We do not restrict ourselves to parametrized problems and RB methods; rather, we provide a quick survey of more general ROM techniques based on the study of the stability of the underlying dynamical system – arising for instance from *model order reduction for ODEs* – addressed in the following chapters of the book. We start by recalling that current approaches for constructing reduced basis approximations of time-dependent parametrized PDEs exploit a combined POD-greedy procedure – POD in time to capture the causality associated with the evolution equation, greedy procedure for sampling the parameter space and treat more efficiently extensive ranges of parameter variation [93].

Certified RB methods have been applied to parametrized (moderate Reynolds) unsteady viscous flows in [70], where a nonisothermal viscous flow is modelled by Boussinesq equations describing natural convection. Parameters are the Grashof number and the gravity direction. In [48] an improved *h-p* adaptive certified method is introduced to address the same natural convection problem, which has also been applied to a multiscale Stokes Fokker-Planck system modelling liquid crystals in [71]. More recent contributions in the field adopt a *space-time* Petrov-Galerkin variational approach to improve the control of the exponentially growing energy estimates in the linear case [123] dealing with convection-conduction problems, for Burgers' equations [131], Boussinesq equations for moderate Grashof number flows

exhibiting steady periodic responses [129] and even addressing interesting hydrodynamic stability problems for moderate Reynolds number flows in an eddy-promoted channel [130].

4.1 Model reduction of linearized time-invariant systems

The POD modes discussed in Section 2.2 only represent the statistical information content of the set of snapshots without taking into account the underlying dynamical system. Many examples of fluid dynamics where a POD-Galerkin ROM described exactly the limit cycle of the system exist, however they completely miss the long-time dynamical behavior of its trajectories.

An example of a dynamical system whose POD-modes are able to exactly represent the stable limit cycle, but for which a Galerkin ROM gives incorrect dynamics was described in [95]. The quadratically nonlinear ODE system

$$\begin{cases} \dot{u}_1(t) = \mu u_1(t) - u_2(t) - u_1(t)u_3(t) \\ \dot{u}_2(t) = \mu u_2(t) + u_1(t) - u_2(t)u_3(t) \\ \dot{u}_3(t) = -u_3(t) + u_1^2(t) + u_2^2(t) \end{cases},$$

has one fixed point at $\mathbf{u} = (0,0,0)$, which is unstable, and an asymptotically stable limit cycle $\mathbf{u}_{\mathrm{LS}}(t) = (\sqrt{\mu}\cos(t), \sqrt{\mu}\sin(t), \mu)$. All trajectories tend towards the limit cycle. Since

$$\frac{1}{t-t_0}\int_{t_0}^{t} \mathbf{u}(\tau)\, d\tau \quad \underset{t\to\infty}{\longrightarrow} \quad (0,0,\mu),$$

the POD basis of dimension 2 is given by

$$\bar{\mathbf{u}} := (0,0,\mu), \quad \Psi_1 = (1,0,0), \quad \Psi_2 = (0,1,0)$$

and is able to exactly represent the stable limit cycle:

$$\mathbf{u}_{\mathrm{LS}}(t) = \bar{\mathbf{u}} + \sqrt{\mu}\cos(t)\Psi_1 + \sqrt{\mu}\sin(t)\Psi_2.$$

However, Galerkin projection on the POD basis of dimension 2 using the Euclidean inner product leads to

$$\mathbf{u}_r(t) := \bar{\mathbf{u}} + a_1(t)\Psi_1 + a_2(t)\Psi_2.$$

The coefficients of the ROM are given by the dynamical system

$$\begin{cases} \dot{a}_1(t) = -a_2(t) \\ \dot{a}_2(t) = a_1(t) \end{cases},$$

which is only marginally stable and whose trajectories remain on a circle of radius $r = (a_1^2(t_0) + a_2^2(t_0))^{1/2}$ for all time without converging asymptotically towards the correct limit cycle.

In order to capture the correct temporal dynamics, the characteristics of the dynamical system (fixed points, periodic solutions, and their (in)stability) should be preserved by the ROM – such ROMs are built based on analyzing the stability of the underlying dynamical system. In this section we discuss some, namely, linearized time-invariant flows, which exhibit asymptotically stable periodic steady-states.

For *linear time-invariant systems* (LTIs), system-theoretical reduction methods such as *balanced truncation* [7, 91] are more effective, in the sense that they provide a ROM that has nearly the best possible approximation error. A linearized input-output system in state-space form is

$$\begin{cases} \dfrac{d\mathbf{U}}{dt}(t) = A\mathbf{U}(t) + B\mathbf{S}(t) & \text{for } t \in (t_0, t_f) \\ \mathbf{Y}(t) = C\mathbf{U}(t) & \text{for } t \in (t_0, t_f) \\ \mathbf{U}(t_0) = \mathbf{U}_0 \end{cases} \tag{44}$$

with inputs (controls) $\mathbf{S}$ and outputs (observations) $\mathbf{Y}$. If the system (44) is stable, the controllability and observability Gramians are the matrices defined respectively as

$$W_c = \int_{t_0}^{t_f} e^{A\tau} B B^* e^{A^*\tau}\, d\tau, \quad W_o = \int_{t_0}^{t_f} e^{A^*\tau} C^* C e^{A\tau}\, d\tau \tag{45}$$

which can be computed from the Lyapunov equations:

$$AW_c + W_c A^* + BB^* = 0, \qquad A^* W_o + W_o A + C^* C = 0;$$

see e.g. [105] for further details. The controllability Gramian W_c measures to what degree each state of the system (44) is excited by an input; in particular, W_c is positive definite if and only if all states are reachable with some input $\mathbf{S}(t)$. Instead, the observability Gramian W_o measures to what degree each state excites future outputs; in particular, W_o is positive definite if and only if any initial state $\mathbf{U}(t_0) = \mathbf{U}_0$ can be uniquely determined from $\mathbf{Y}(t)$ on (t_0, t_f).

A balancing transformation $\mathbb{T}$ is sought to transform the state variables of the LTI into equivalent "balanced state variables", $\hat{\mathbf{U}} = \mathbb{T}\mathbf{U}$, in a way that the transformed Gramians become equal and diagonal:

$$\mathbb{T}^{-1} W_c \mathbb{T}^{-*} = \mathbb{T}^* W_o \mathbb{T} = \Sigma = \mathrm{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_N). \tag{46}$$

In the balanced coordinates, the states that are least influenced by the input also have the least influence on the output, and such a balancing transformation exists as long

as the system is both controllable and observable (i.e., both W_c and W_o are positive definite). The $\{\hat{\sigma}_i\}$ are called the Hankel singular values; when sorted in descending order, we can split the balanced LTI system into two parts:

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix}\hat{\mathbf{U}}_1\\ \hat{\mathbf{U}}_2\end{bmatrix}(t) = \begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{U}}_1\\ \hat{\mathbf{U}}_2\end{bmatrix}(t) + \begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}\mathbf{S} & \text{for } t \in (t_0, t_f) \\ \mathbf{Y}(t) = \begin{bmatrix}\hat{C}_1 & \hat{C}_2\end{bmatrix}\begin{bmatrix}\hat{\mathbf{U}}_1\\ \hat{\mathbf{U}}_2\end{bmatrix}(t) & \text{for } t \in (t_0, t_f) \\ \hat{\mathbf{U}}(t_0) = \mathbb{T}\mathbf{U}_0, & \end{cases} \tag{47}$$

where $\dim(\hat{\mathbf{U}}_1) = r$ and $\dim(\hat{\mathbf{U}}_2) = N - r$. A balanced truncation ROM is then obtained by retaining only the balanced state variables related to the first r Hankel singular values:

$$\begin{cases} \dfrac{d}{dt}\hat{\mathbf{U}}_1(t) = \hat{A}_{11}\hat{\mathbf{U}}_1 + \hat{B}_1\mathbf{S} & \text{for } t \in (t_0, t_f) \\ \tilde{\mathbf{Y}}(t) = \hat{C}_1\hat{\mathbf{U}}_1(t) & \text{for } t \in (t_0, t_f)\,. \\ \hat{\mathbf{U}}_1(t_0) = \mathbb{T}_1\mathbf{U}_0 & \end{cases} \tag{48}$$

In other words, balanced truncation involves first changing the coordinates according to (46), and then truncating the least controllable/observable states, which have little effect on the input-output behavior.

When the exact transfer function $G(s) = C(sI - A)^{-1}B$ of the LTI system is compared with the one obtained after balanced truncation, $\hat{G}(s) = \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1$, we have the following results [7]:

1. Any ROM with r states and transfer function $\tilde{G}_r(s)$ has operator norm error at least $\|G - \tilde{G}_r\|_\infty > \hat{\sigma}_{r+1}$, where $\hat{\sigma}_{r+1}$ is the $(r+1)$st Hankel singular value.
2. The balanced truncation ROM with r states and transfer function $\hat{G}_r(s)$ has operator norm error bounded by $\|G - \hat{G}_r\|_\infty < 2\sum_{i=r+1}^{N}\hat{\sigma}_i$.
3. If the full-order system (44) is stable, so is the balanced truncation ROM (48).

The Hankel singular values $\{\hat{\sigma}_i\}$ characterize also the Kolmogorov n-width discussed in Section 2.1 of the range space of the Hankel operator, see [45]. As already discussed in Sect. 2.1, the main requirement for constructing efficient ROMs is that the associated singular values decay reasonably fast. Previously, we used the decay of the empirical POD singular values to measure this, whereas in the balanced truncation method one looks at the Hankel singular values. In fact, there exists an interesting connection between the Hankel singular values and the empirical POD singular values – it was pointed out in [105] that the POD modes are *equivalent* to the modes obtained by balanced truncation provided that the snapshots $\mathbf{U}_i$ are taken as the impulse responses of the system and the inner product equal to the one induced by the observability Gramian.

Balanced truncation methods based on explicitly computing the Gramians in (45) by solving Lyapunov equations are generally too expensive to apply to large linear systems with millions of state variables. A possible remedy is the *balanced truncation POD* method [74, 128], in which the exact Gramians (45) are approximated using a method of snapshots:

$$\begin{aligned} W_c^e &= \frac{1}{K}\sum_{k=1}^{K}\frac{1}{w_k^2}\int_{t_0}^{t_f}\left(\xi_k(\tau)-\bar{\xi}_k\right)\left(\xi_k(\tau)-\bar{\xi}_k\right)^*\,d\tau, \\ W_o^e &= \frac{1}{K}\sum_{k=1}^{K}\frac{1}{w_k^2}\int_{t_0}^{t_f}Q_k\left(\zeta_k(\tau)-\bar{\zeta}_k\right)^*\left(\zeta_k(\tau)-\bar{\zeta}_k\right)Q_k^*\,d\tau. \end{aligned} \tag{49}$$

Here the empirical trajectories $\xi_k(t)$ and empirical outputs $\zeta_k(t)$ are computed by solving the system (44) using generalized impulse controls $\mathbf{S}_k(t) = w_k Q_k \mathbf{e}_k \delta(t)$, where $w_k > 0$ are positive weights, $Q_k \in \mathbb{R}^{P\times P}$ are orthogonal matrices, $\mathbf{e}_k \in \mathbb{R}^P$ are Euclidean unit vectors, and $\delta(t)$ is the one-dimensional Dirac delta distribution:

$$\begin{cases} \dfrac{d\xi_k}{dt}(t) = A\xi_k(t) + B\mathbf{S}_k(t) & \text{for } t \in (t_0,t_f) \\ \zeta_k(t) = C\xi_k(t) & \text{for } t \in (t_0,t_f) \\ \xi(t_0) = \mathbf{U}_0 \end{cases} \tag{50}$$

for each $k = 1,\ldots,K$. In the case of LTI systems, the empirical Gramians (49) coincide with the exact Gramians (45) provided that $K \geq P$ empirical impulse responses are computed. It was proposed in [75] to use the same balanced truncation POD method for dealing also with nonlinear flows. In this case the empirical Gramians (49) – which are (approximate) finite Gramians – are obtained by solving the nonlinear system and taking snapshots of the trajectory. By using these finite Gramians to perform the balancing we obtain the following, empirical balancing transformation $\mathbb{T}_e = [\,\mathbb{T}_{e,1}\,\mathbb{T}_{e,2}\,]$:

$$\mathbb{T}_e^{-1} W_c^e \mathbb{T}_e^{-*} = \mathbb{T}_e^* W_o^e \mathbb{T} = \Sigma = \operatorname{diag}(\hat{\sigma}_1,\ldots,\hat{\sigma}_N). \tag{51}$$

The balanced POD modes were applied in [105] to a linearized flow in a plane channel and a comparison was made between POD, balanced truncation, and balanced POD methods. The conclusion was that the balanced POD modes produced nearly identical results with the balanced truncation modes, and both methods significantly outperformed the standard POD modes. Another comparison on a problem of designing closed-loop controllers for flow over a cavity was done in [13], where again the balanced POD modes achieved a stable closed-loop controller with fewer ROM degrees-of-freedom. A difficulty related to balanced truncation is that the linearized system must be stable. An extension to unstable linear systems was proposed in [132] by decoupling the dynamics on the stable and unstable subspaces, and then truncating the relatively uncontrollable and unobservable modal representations on each subspace (see e.g. [132] for further details). This strategy was used

to propose reduced-order controllers around linearly unstable steady states for flow around a cylinder in [122] and for flow past a flat plate in [1].

4.2 Stabilization of ROMs for unsteady Navier-Stokes equations

As mentioned before, usually a standard Galerkin projection-based ROM does not produce satisfactory results when applied to nonlinear unsteady Navier-Stokes equations. There do exist exceptions – for nonautonomous problems with strong external sources, such as periodically driven inflow, long-time drifting from asymptotically stable states was not observed in [84, 113]. The drifting of ROM trajectories in the general case is however a well-known problem and many attempts have been made to remedy it.

First works on stabilization experimented in adding artificial viscosity [9] to the reduced equations. The idea was further developed by extending the spectral vanishing viscosity method of Tadmor [115] to the Navier-Stokes equations in [111]. In long-time simulation of convection dominated flows some type of closure model that takes into account the energy transfer between the ROM modes is needed. In [29] a driven cavity problem at Re = 20,000 was successfully stabilized by adding a linear damping term in the Galerkin ROM. The computation of correct limit cycles was done in [2] by applying a shooting method. For a review of various stabilization methods for Galerkin ROMS we refer to [16].

4.3 Dynamic mean-field representations and shift-modes

In many fluid dynamics systems, the Reynolds decomposition (2) together with Galerkin projection leads to unstable ROMs because the interaction between the time-averaged mean flow $\bar{\mathbf{u}}$ and the oscillating part of the flow field represented by the POD modes is neglected. In [95] this problem was analyzed and identified moving from the consideration that a Galerkin model without dynamic mean-field correction is unable to represent correctly the unstable fixed point of the dynamical system, which leads to structurally unstable ROMs (small perturbations in the model can cause divergent trajectories). This was found to occur even in problems where theoretically the POD-Galerkin ROM was able to capture the stable attractor exactly. As a result the periodic limit cycle was correctly captured, but transient dynamics of the ROM were off by orders of magnitude.

The simplest method proposed in [95] to correct the mean-field approximation error of POD is the inclusion of a *shift-mode* Ψ_Δ, which is added to the POD basis in order to represent the correct unstable fixed point of the full-order system, resulting in the extended POD *ansatz*

$$\mathbf{u}(x,t) \approx \mathbf{u}_r(x,t) := \bar{\mathbf{u}}(x) + a_0(t)\Psi_\Delta(x) + \sum_{i=1}^{N_r} a_i(t)\Psi_i(\mathbf{x}). \tag{52}$$

For instance, in the case of the unsteady cylinder wake flow, the unstable fixed point corresponds to the solution $\mathbf{u}_s$ of the steady Navier-Stokes flow. The shift-mode is obtained by applying a Gram-Schmidt process to the correction term $\mathbf{u}_\Delta := \bar{\mathbf{u}} - \mathbf{u}_s$:

$$\Psi_\Delta^* := \mathbf{u}_\Delta - \sum_{i=1}^{N_r} (\mathbf{u}_\Delta, \Psi_i)\Psi_i, \qquad \Psi_\Delta := \frac{\Psi_\Delta^*}{\|\Psi_\Delta^*\|_\Omega} \tag{53}$$

and applying the Galerkin ROM to the expanded POD basis of dimension $N_r + 1$. This allows the ROM to represent exactly the unstable fixed point of the system. A comprehensive discussion of the various other types of mean-field corrections and their effects on the ROM predictions can be found in [116].

4.4 Model reduction of periodic steady-state solutions

In Section 4.3 we have discussed the difficulties of building ROMs that are capable of accurately representing the transient dynamics of unsteady flows. In many applications of fluid dynamics, for example in turbomachinery or in large "straight" arteries in the human circulatory system, the behavior of the flow is such that all trajectories approach a single stable periodic solution. One option is then to disregard the simulation of the transient, and concentrate only on approximating the periodic steady-state solution.

For linearized flows the frequency-domain POD technique was introduced in [69]. It replaces the time-domain representation of the Galerkin-projected equations with a Fourier-domain representation for each individual harmonic. For fully nonlinear flows the individual harmonics are coupled by the nonlinear terms and no term-by-term analysis of the harmonics can be performed. To solve this problem, the *Harmonic Balance* (HB) method used for the study of harmonic ODEs was adapted for the efficient solution of time-periodic flows in [57, 89, 90]. After suitable spatial discretization of (1) the system

$$\begin{bmatrix} \dot{\mathbf{u}}_h \\ 0 \end{bmatrix} = -\begin{bmatrix} -(\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + \nu \triangle \mathbf{u}_h + \mathbf{f}(t) \\ -\nabla \cdot \mathbf{u}_h \end{bmatrix} = -\begin{bmatrix} \mathbf{S}_1(\mathbf{U}) \\ \mathbf{S}_2(\mathbf{U}) \end{bmatrix} = -\mathbf{S}(\mathbf{U}), \tag{54}$$

is obtained, where the spatial operator $\mathbf{S}(\mathbf{U})$ depends nonlinearly on the solution $\mathbf{U} := (\mathbf{u}_h, p_h) \in \mathbb{R}^{N_h^u + N_h^p}$, N_h^u and N_h^p being the number of degrees of freedom of the discrete velocity and pressure fields, respectively.

The method starts from the assumption that this system admits a periodic steady-state solution $\mathbf{U}_\infty(t)$ with known period T, so that $\mathbf{U}_\infty(t) = \mathbf{U}_\infty(t+T)$ for all t. If in addition the spatial operator is time periodic with the same period T, they can both be represented using Fourier series expansions as

$$\mathbf{U}_\infty(t) = \sum_{k=-\infty}^{\infty} \widehat{\mathbf{U}}_k \exp\left(\frac{2\pi i k t}{T}\right), \qquad \mathbf{S}(\mathbf{U}_\infty) = \sum_{k=-\infty}^{\infty} \widehat{\mathbf{S}}_k(\mathbf{U}_\infty) \exp\left(\frac{2\pi i k t}{T}\right), \quad (55)$$

where each $\widehat{\mathbf{U}}_k$ and $\widehat{\mathbf{S}}_k(\mathbf{U}_\infty)$ is a (discrete representation of a) complex-valued vector field over Ω; by expressing $\widehat{\mathbf{S}}_k = \widehat{\mathbf{S}}_k(\mathbf{U}_\infty)$ we mean that each coefficient in the expansion of $\mathbf{S} = \mathbf{S}(\mathbf{U}_\infty)$ depends on (potentially all of) the spatial coefficients $\{\widehat{\mathbf{U}}_k\}_k$ of $\mathbf{U}_\infty(t)$. Since the periodic steady-state solution satisfies equation (54), its complex Fourier coefficients $\widehat{\mathbf{U}}_k \in \mathbb{C}^{N_h^u + N_h^p}$ must satisfy

$$\frac{2\pi i k}{T}\widehat{\mathbf{U}}_k + \widehat{\mathbf{S}}_k(\mathbf{U}_\infty) = 0, \quad \text{for all } k \in \mathbb{Z}. \quad (56)$$

The *harmonic balance* (HB) method starts by truncating the Fourier series to $2N+1$ terms and matching only those terms in (56), i.e.

$$\frac{2\pi i k}{T}\widehat{\mathbf{U}}_k + \widehat{\mathbf{S}}_k(\mathbf{U}_\infty) = 0, \quad \text{for all } k = -N, \ldots, N. \quad (57)$$

For real-valued fields $\widehat{\mathbf{U}}_{-k} = \overline{\widehat{\mathbf{U}}_k}$, so that only $N+1$ equations need to be solved.

If the flow is linear, all the harmonics decouple and we only need to solve $N+1$ uncoupled steady equations. For nonlinear flows, each $\widehat{\mathbf{S}}_k(\mathbf{U}_\infty)$ depends on all the $\widehat{\mathbf{U}}_k$ for $k = -N, \ldots, N$ and thus the system (57) is a fully coupled nonlinear system of $(N+1) \times (N_h^u + N_h^p)$ complex-valued equations. Due to the nonlinearity of the spatial operator its Fourier series coefficients cannot be computed directly. This problem is solved either using the alternating frequency/time domain method, as was done in [57, 90], or by the asymptotic numerical method, as was done in [34]. Once the Fourier coefficients are known, the periodic steady state solution can be reconstructed with arbitrary temporary precision.

An advantage of HB compared to POD is that no full-order transient simulations need to be performed until the periodic steady-state is reached, nor is the ROM dependent on the initial condition of these simulations. For a comparison between POD and HB we refer to [82]. We remark that the HB method is very efficient in reducing the temporal complexity, as typically only $N < 10$ terms are needed to accurately represent the solution. However, it has no effect on the spatial complexity of the problem. Like many space-time formulations it requires the solution of a system that is several times larger than the one solved when using the more standard method of lines. So far the HB method has been applied mainly to industrial problems, such as the design and simulation of turbomachinery [57] and problems in aeroelasticity [120].

5 Conclusions

In this chapter we have presented an overview of model reduction methods for incompressible fluid dynamics, both in the steady and unsteady flow cases. The main focus was on Galerkin-projection based ROMs, and the main strategies for constructing the low-order projection basis have been discussed. Theoretical properties of ROMs for fluid problems are related to, e.g.: the possibility to reduce the dynamics of a fluid system to a low-dimensional submanifold, measured for instance by the very fast exponential convergence of empirical POD singular values or of the Kolmogorov n-width; the lack of long-time stability of Galerkin ROMs and the need for stabilization; the error estimation of the ROM in the case of steady flow problems; the gain of computational efficiency thanks to the online/offline paradigm that allows fast real-time ROM simulations as well as to the use of hyperreduction for treating the nonlinear terms in an efficient way.

Ad hoc reduced order modelling techniques have recently been proposed for optimal flow control problems [104, 108, 121], optimal shape design of devices related with fluid flows [6, 58, 23, 88], and the treatment of fluid-structure interaction problems [76, 78].

Far from having covered the subject exhaustively, we hope nonetheless that this chapter could offer the reader a contribution for understanding which type of ROM may be the best for his or her particular fluid dynamics application, having made extensive reference to available results in the literature.

References

1. Ahuja, S., Rowley, C.: Feedback control of unstable steady states of flow past a flat plate using reduced-order estimators. J. Fluid Mech **645**, 447–478 (2010)
2. Akhtar, I., Nayfeh, A., Ribbens, C.: On the stability and extension of reduced-order Galerkin models in incompressible flows. Theor. Comp. Fluid Dyn. **23**(3), 213–237 (2009)
3. Amsallem, D., Cortial, J., Carlberg, K., Farhat, C.: A method for interpolating on manifolds structural dynamics reduced-order models. Int. J. Numer. Methods Engr. **80**(9), 1241–1258 (2009)
4. Amsallem, D., Farhat, C.: An online method for interpolating linear parametric reduced-order models. SIAM J. Sci. Comput. **33**(5), 2169–2198 (2011)
5. Amsallem, D., Zahr, M., Farhat, C.: Nonlinear model order reduction based on local reduced-order bases. Int. J. Numer. Methods Engr. **92**(10), 891–916 (2012)
6. Antil, H., Heinkenschloss, M., Hoppe, R.: Domain decomposition and balanced truncation model reduction for shape optimization of the Stokes system. Optim. Methods Softw. **26**(4–5), 643–669 (2011)
7. Antoulas, A.: Approximation of Large-Scale Dynamical Systems. SIAM (2005)
8. Astrid, P., Weiland, S., Willcox, K., Backx, T.: Missing point estimation in models described by proper orthogonal decomposition. IEEE. T. Automat. Contr. **53**(10), 2237–2251 (2008)
9. Aubry, N., Holmes, P., Lumley, J., Stone, E.: The dynamics of coherent structures in the wall region of a turbulent boundary layer. J. Fluid Mech. **192**(115), 173,355 (1988)
10. Babuška, I., Nobile, F., Tempone, R.: A stochastic collocation method for elliptic partial differential equations with random input data. SIAM J. Numer. Anal. **45**(3), 1005–1034 (2007)

11. Babuška, I.: Error-bounds for finite element method. Numer. Math. **16**, 322–333 (1971)
12. Baiges, J., Codina, R., Idelsohn, S.: Explicit reduced order models for the stabilized finite element approximation of the incompressible Navier-Stokes equations. Int. J. Numer. Methods Fluids (2013). DOI: 10.1002/fld.3777
13. Barbagallo, A., Sipp, D., Schmid, P.J.: Closed-loop control of an open cavity flow using reduced-order models. J. Fluid Mech. **641**(1), 1–50 (2009)
14. Barone, M.F., Kalashnikova, I., Segalman, D.J., Thornquist, H.K.: Stable Galerkin reduced order models for linearized compressible flow. J. Comp. Phys. **228**(6), 1932–1946 (2009)
15. Barrault, M., Maday, Y., Nguyen, N., Patera, A.: An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. C. R. Acad. Sci. Paris. Sér. I Math. **339**(9), 667–672 (2004)
16. Bergmann, M., Bruneau, C., Iollo, A.: Enablers for robust POD models. J. Comp. Phys. **228**(2), 516–538 (2009)
17. Bergmann, M., Cordier, L.: Optimal control of the cylinder wake in the laminar regime by trust-region methods and POD reduced-order models. J. Comp. Phys. **227**(16), 7813–7840 (2008)
18. Binev, P., Cohen, A., Dahmen, W., DeVore, R., Petrova, G., Wojtaszczyk, P.: Convergence rates for greedy algorithms in reduced basis methods. SIAM J. Math. Anal. **43**, 1457–1472 (2011)
19. Brezzi, F.: On the existence, uniqueness, and approximation of saddle point problems arising from Lagrangian multipliers. R.A.I.R.O., Anal. Numér. **2**, 129–151 (1974)
20. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods, *Springer Series in Computational Mathematics*, vol. 15. Springer Verlag (1991)
21. Brezzi, F., Rappaz, J., Raviart, P.: Finite dimensional approximation of nonlinear problems. Part I: Branches of nonsingular solutions. Numer. Math. **36**, 1–25 (1980)
22. Buffa, A., Maday, Y., Patera, A., Prud'homme, C., Turinici, G.: A priori convergence of the greedy algorithm for the parametrized reduced basis. ESAIM Math. Modelling Numer. Anal. **46**(3), 595–603 (2012)
23. Bui-Thanh, T., Willcox, K., Ghattas, O.: Parametric reduced-order models for probabilistic analysis of unsteady aerodynamics applications. AIAA J. **46**(10) (2008)
24. Bui-Thanh, T., Willcox, K., Ghattas, O., van Bloemen Waanders, B.: Goal-oriented, model-constrained optimization for reduction of large-scale systems. J. Comp. Phys. **224**(2), 880–896 (2007)
25. Burkardt, J., Gunzburger, M., Lee, H.: Centroidal Voronoi tessellation-based reduced-order modeling of complex systems. SIAM J. Sci. Comput. **28**(2), 459–484 (2006)
26. Burkardt, J., Gunzburger, M., Lee, H.: POD and CVT-based reduced-order modeling of Navier-Stokes flows. Comput. Meth. Appl. Mech. Engrg. **196**(1-3), 337–355 (2006)
27. Caloz, G., Rappaz, J.: Numerical analysis for nonlinear and bifurcation problems. In: P. Ciarlet, J. Lions (eds.) Handbook of Numerical Analysis, Vol. V, Techniques of Scientific Computing (Part 2), pp. 487–637. Elsevier Science B.V. (1997)
28. Carlberg, K., Farhat, C.: A low-cost, goal-oriented 'compact proper orthogonal decomposition' basis for model reduction of static systems. Int. J. Numer. Methods Engr. **86**(3), 381–402 (2011)
29. Cazemier, W., Verstappen, R., Veldman, A.: Proper orthogonal decomposition and low-dimensional models for driven cavity flows. Phys. Fluids **10**, 1685 (1998)
30. Chaturantabut, S., Sorensen, D.: Nonlinear model reduction via discrete empirical interpolation. SIAM J. Sci. Comput. **32**(5), 2737–2764 (2010)
31. Chen, X., Akella, S., Navon, I.: A dual-weighted trust-region adaptive POD 4-D Var applied to a finite-volume shallow water equations model on the sphere. Int. J. Numer. Methods Fluids **68**(3), 377–402 (2012)
32. Chen, Y., Hesthaven, J., Maday, Y., Rodriguez, J.: A monotonic evaluation of lower bounds for inf-sup stability constants in the frame of reduced basis approximations. C. R. Acad. Sci. Paris. Sér. I Math. **346**, 1295–1300 (2008)

33. Christensen, E., Brøns, M., Sørensen, J.: Evaluation of proper orthogonal decomposition–based decomposition techniques applied to parameter-dependent nonturbulent flows. SIAM J. Sci. Comput. **21**, 1419–1434 (2000)
34. Cochelin, B., Vergez, C.: A high order purely frequency-based harmonic balance formulation for continuation of periodic solutions. J. Sound Vibration **324**(1), 243–262 (2009)
35. Colonius, T., Rowley, C., Freund, J., Murray, R.: On the choice of norm for modeling compressible flow dynamics at reduced-order using the POD. In: Proc. 41st IEEE Conf. on Decision and Control, vol. 3, pp. 3273–3278. IEEE (2002)
36. Daescu, D., Navon, I.: Efficiency of a POD-based reduced second-order adjoint model in 4D-Var data assimilation. Int. J. Numer. Methods Fluids **53**(6), 985–1004 (2007)
37. Daescu, D., Navon, I.: A dual-weighted approach to order reduction in 4DVAR data assimilation. Monthly Weather Review **136**(3), 1026–1041 (2008)
38. Dahmen, W., Huang, C., Schwab, C., Welper, G.: Adaptive Petrov-Galerkin methods for first order transport equations. SIAM J. Numer. Anal. **50(5)** (2012)
39. Deane, A., Kevrekidis, I., Karniadakis, G., Orszag, S.: Low-dimensional models for complex geometry flows: Application to grooved channels and circular cylinders. Phys. Fluids **3**(10), 2337–2354 (1991)
40. Demkowicz, L., Gopalakrishnan, J.: A class of discontinuous Petrov-Galerkin methods. part I: The transport equation. Comput. Methods Appl. Mech. Engr. **199**(23-24), 1558–1572 (2010)
41. Demkowicz, L., Gopalakrishnan, J.: A class of discontinuous Petrov–Galerkin methods. II. optimal test functions. Numer. Methods Partial Differential Equations **27**(1), 70–105 (2011)
42. Deparis, S.: Reduced basis error bound computation of parameter-dependent Navier-Stokes equations by the natural norm approach. SIAM J. Num. Anal. **46**(4), 2039–2067 (2008)
43. Deparis, S., Løvgren, A.: Stabilized reduced basis approximation of incompressible three-dimensional Navier-Stokes equations in parametrized deformed domains. J. Sci. Comput. **50(1)**, 198–212 (2012)
44. Deparis, S., Rozza, G.: Reduced basis method for multi-parameter-dependent steady Navier-Stokes equations: Applications to natural convection in a cavity. J. Comp. Phys. **228**(12), 4359–4378 (2009)
45. Djouadi, S.: On the connection between balanced proper orthogonal decomposition, balanced truncation, and metric complexity theory for infinite dimensional systems. In: Proc. Am. Control Conf., June 30–July 2, Baltimore, MD, 2010 pp. 4911–4916 (2010)
46. Du, Q., Gunzburger, M.: Model reduction by proper orthogonal decomposition coupled with centroidal Voronoi tessellation. In: Proc. Fluids Engineering Division Summer Meeting, FEDSM2002-31051, ASME (2002)
47. Dumon, A., Allery, C., Ammar, A.: Proper general decomposition (PGD) for the resolution of Navier-Stokes equations. J. Comp. Phys. **230**, 1387–1407 (2011)
48. Eftang, J., Knezevic, D., Patera, A.: An "hp" certified reduced basis method for parametrized parabolic partial differential equations. Math. Comput. Model. Dynam. Syst. **17**(4), 395–422 (2011)
49. Fahl, M.: Trust-region methods for flow control based on reduced order modelling. Ph.D. thesis, Universität Trier (2001)
50. Gerner, A., Veroy, K.: Certified reduced basis methods for parametrized saddle point problems. SIAM J. Sci. Comp. **34**(5), A2812–A2836 (2012)
51. Girault, V., Raviart, P.A.: Finite element methods for Navier-Stokes equations: Theory and algorithms. Springer-Verlag, Berlin and New York (1986)
52. Grepl, M., Patera, A.: A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. ESAIM Math. Modelling Numer. Anal. **39(1)**, 157–181 (2005)
53. Grinberg, L., Yakhot, A., Karniadakis, G.: Analyzing transient turbulence in a stenosed carotid artery by proper orthogonal decomposition. Ann. Biomed. Eng. **37**(11), 2200–2217 (2009)

54. Gunzburger, M., Peterson, J., Shadid, J.: Reducer-order modeling of time-dependent PDEs with multiple parameters in the boundary data. Comput. Methods Appl. Mech. Engrg. **196**, 1030–1047 (2007)
55. Haasdonk, B.: Convergence rates of the POD-greedy method. ESAIM Math. Modelling Numer. Anal. (2013). DOI:10.1051/m2an/2012045
56. Haasdonk, B., Ohlberger, M.: Reduced basis method for finite volume approximations of parametrized linear evolution equations. ESAIM Math. Modelling Numer. Anal. **42**(02), 277–302 (2008)
57. Hall, K., Thomas, J., Clark, W.: Computation of unsteady nonlinear flows in cascades using a harmonic balance technique. AIAA J. **40**(5), 879–886 (2002)
58. Hay, A., Borggaard, J., Akhtar, I., Pelletier, D.: Reduced-order models for parameter dependent geometries based on shape sensitivity analysis. J. Comp. Phys. **229**(4), 1327–1352 (2010)
59. Hay, A., Borggaard, J., Pelletier, D.: Local improvements to reduced-order models using sensitivity analysis of the proper orthogonal decomposition. J. Fluid Mech. **629**, 41–72 (2009)
60. Holmes, P., Lumley, J., Berkooz, G.: Turbulence, coherent structures, dynamical systems and symmetry. Cambridge Univ. Press (1998)
61. Huynh, D., Knezevic, D., Chen, Y., Hesthaven, J., Patera, A.: A natural-norm successive constraint method for inf-sup lower bounds. Comput. Meth. Appl. Mech. Engrg. **199**(29–32), 1963–1975 (2010)
62. Huynh, D., Knezevic, D., Patera, A.: A static condensation reduced basis element method: approximation and a posteriori error estimation. ESAIM Math. Modelling Numer. Anal. **47**(1), 213–251 (2013)
63. Huynh, D., Knezevic, D., Peterson, J., Patera, A.: High-fidelity real-time simulation on deployed platforms. Comp. Fluids **43**(1), 74–81 (2011)
64. Huynh, D., Rozza, G., Sen, S., Patera, A.: A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability costants. C. R. Acad. Sci. Paris. Sér. I Math. **345**, 473–478 (2007)
65. Iapichino, L., Quarteroni, A., Rozza, G.: A reduced basis hybrid method for the coupling of parametrized domains represented by fluidic networks. Comput. Methods Appl. Mech. Engrg. **221–222**, 63–82 (2012)
66. Iollo, A., Lanteri, S., Désidéri, J.: Stability properties of POD-Galerkin approximations for the compressible Navier-Stokes equations. Theor. Comp. Fluid Dyn. **13**(6), 377–396 (2000)
67. Johansson, P., Andersson, H., Rønquist, E.: Reduced-basis modeling of turbulent plane channel flow. Comput. Fluids **35**(2), 189–207 (2006)
68. Johnson, C., Rannacher, R., Boman, M.: Numerics and hydrodynamic stability: toward error control in computational fluid dynamics. SIAM J. Numer. Anal. **32(4)**, 1058–1079 (1995)
69. Kim, T.: Frequency-domain Karhunen-Loève method and its application to linear dynamic systems. AIAA J. **36**(11), 2117–2123 (1998)
70. Knezevic, D., Nguyen, N., Patera, A.: Reduced basis approximation and a posteriori error estimation for the parametrized unsteady Boussinesq equations. Math. Mod. and Meth. in Appl. Sc. **21(7)**, 1415–1442 (2011)
71. Knezevic, D.J.: Reduced basis approximation and a posteriori error estimates for a multiscale liquid crystal model. Math. Comput. Model. Dynam. Syst. **17**(4), 443–461 (2011)
72. Kunisch, K., Volkwein, S.: Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. SIAM J. Numer. Anal. **40**(2), 492–515 (2003)
73. Kunisch, K., Volkwein, S.: Optimal snapshot location for computing POD basis functions. ESAIM Math. Modelling Numer. Anal. **44**(3), 509 (2010)
74. Lall, S., Marsden, J., Glavaški, S.: Empirical model reduction of controlled nonlinear systems. In: Proc. IFAC World Congress Vol. F Int. Federation Automatic Control, Beijing, 1999 pp. 473–478 (1999)
75. Lall, S., Marsden, J., Glavaški, S.: A subspace approach to balanced truncation for model reduction of nonlinear control systems. Int. J. Robust Nonlinear Control **12**(6), 519–535 (2002)

76. Lassila, T., Manzoni, A., Quarteroni, A., Rozza, G.: A reduced computational and geometrical framework for inverse problems in haemodynamics. Tech. Rep. Nr. 12.2011, MATHICSE-EPFL (2011)
77. Lassila, T., Manzoni, A., Quarteroni, A., Rozza, G.: Generalized reduced basis methods and n-width estimates for the approximation of the solution manifold of parametric PDEs. In: F. Brezzi, P. Colli Franzone, U. Gianazza, G. Gilardi (eds.) Analysis and Numerics of Partial Differential Equations, *INdAM Series*, vol. 4. Springer (2013). Re-printed also on Bollettino Unione Matematica Italiana (UMI), under permission/agreement Springer-UMI
78. Lassila, T., Quarteroni, A., Rozza, G.: A reduced basis model with parametric coupling for fluid-structure interaction problem. SIAM J. Sci. Comput. **34(2)**, A1187–A1213 (2012)
79. Leblond, C., Allery, C., Inard, C.: An optimal projection method for the reduced-order modeling of incompressible flows. Comput. Methods Appl. Mech. Engrg. **200**, 2507–2527 (2011)
80. Lions, P.: Mathematical Topics in Fluid Mechanics. Oxford Lecture Series in Mathematics and Its Applications. Clarendon Press (1996)
81. Løvgren, A., Maday, Y., Rønquist, E.: A reduced basis element method for the steady Stokes problem. ESAIM Math. Modelling Numer. Anal. **40(3)**, 529–552 (2006)
82. Lucia, D., Beran, P., Silva, W.: Reduced-order modeling: new approaches for computational physics. Prog. Aerosp. Sci. **40**(1-2), 51–117 (2004)
83. Lumley, J.: The structure of inhomogeneous turbulent flows. In: A.M.Yaglom and V.I. Tatarski (Eds.). Atmospheric turbulence and radio wave propagation pp. 166–178 (1967)
84. Ma, X., Karniadakis, G., Park, H., Gharib, M.: DPIV-driven flow simulation: a new computational paradigm. Proc. R. Soc. A **459**(2031), 547–565 (2003)
85. Maday, Y., Nguyen, N., Patera, A., Pau, G.: A general multipurpose interpolation procedure: the magic points. Commun. Pure Appl. Anal. **8(1)** (2009)
86. Maday, Y., Patera, A., Turinici, G.: Global a priori convergence theory for reduced-basis approximation of single-parameter symmetric coercive elliptic partial differential equations. C. R. Acad. Sci. Paris. Sér. I Math. **335**, 1–6 (2002)
87. Manzoni, A.: Reduced models for optimal control, shape optimization and inverse problems in haemodynamics. Ph.D. thesis, École Polytechnique Fédérale de Lausanne (2012)
88. Manzoni, A., Quarteroni, A., Rozza, G.: Shape optimization of cardiovascular geometries by reduced basis methods and free-form deformation techniques. Int. J. Numer. Methods Fluids **70**(5), 646–670 (2012)
89. Maple, R., King, P., Orkwis, P., Wolff, J.: Adaptive harmonic balance method for nonlinear time-periodic flows. J. Comp. Phys. **193**(2), 620–641 (2004)
90. McMullen, M.: The application of non-linear frequency domain methods to the Euler and Navier-Stokes equations. Ph.D. thesis, Stanford University (2003)
91. Moore, B.: Principal component analysis in linear systems: Controllability, observability, and model reduction. IEEE Trans. Automat. Contr. **26(1)** (1981)
92. Néron, D., Ladevèze, P.: Proper generalized decomposition for multiscale and multiphysics problems. Arch. Comput. Methods Engrg. **17**(4), 351–372 (2010)
93. Nguyen, N., Rozza, G., Patera, A.: Reduced basis approximation and a posteriori error estimation for the time-dependent viscous Burgers equation. Calcolo **46**(3), 157–185 (2009)
94. Nguyen, N., Veroy, K., Patera, A.: Certified real-time solution of parametrized partial differential equations. In: Yip, S. (Ed.). Handbook of Materials Modeling pp. 1523–1558 (2005)
95. Noack, B., Afanasiev, K., Morzynski, M., Tadmor, G., Thiele, F.: A hierarchy of low-dimensional models for the transient and post-transient cylinder wake. J. Fluid Mech. **497**(1), 335–363 (2003)
96. Noack, B., Papas, P., Monkewitz, P.: The need for a pressure-term representation in empirical Galerkin models of incompressible shear flows. J. Fluid Mech. **523**(1), 339–365 (2005)
97. Noor, A., Peters, J.: Reduced basis technique for nonlinear analysis of structures. AIAA J. **18(4)**, 455–462 (1980)
98. Peterson, J.: The reduced basis method for incompressible viscous flow calculations. SIAM J. Sci. Stat. Comput. **10**, 777–786 (1989)
99. Quarteroni, A.: Numerical Models for Differential Problems, *Modeling, Simulation and Applications (MS&A)*, vol. 2. Springer-Verlag Italia, Milano (2009)

100. Quarteroni, A., Rozza, G.: Numerical solution of parametrized Navier-Stokes equations by reduced basis methods. Numer. Methods Partial Differential Equations **23(4)**, 923–948 (2007)
101. Quarteroni, A., Rozza, G., Manzoni, A.: Certified reduced basis approximation for parametrized partial differential equations in industrial applications. J. Math. Ind. **1**(3) (2011)
102. Quarteroni, A., Valli, A.: Numerical Approximation of Partial Differential Equations. Springer-Verlag, Berlin-Heidelberg (1994)
103. Ravindran, S.: Reduced-order adaptive controllers for fluid flows using POD. J. Sci. Comput. **15**(4), 457–478 (2000)
104. Ravindran, S.: A reduced-order approach for optimal control of fluids using proper orthogonal decomposition. Int. J. Numer. Meth. Fluids **34**, 425–448 (2000)
105. Rowley, C.: Model reduction for fluids, using balanced proper orthogonal decomposition. Int. J. Bifur. Chaos Appl. Sci. Engrg. **15**(3), 997–1014 (2005)
106. Rozza, G., Huynh, D., Manzoni, A.: Reduced basis approximation and a posteriori error estimation for Stokes flows in parametrized geometries: roles of the inf-sup stability constants. Numer. Math. (in press) (2013). DOI:10.1007/s00211-013-0534-8
107. Rozza, G., Huynh, D., Patera, A.: Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. Arch. Comput. Methods Engrg. **15**, 229–275 (2008)
108. Rozza, G., Manzoni, A., Negri, F.: Reduction strategies for PDE-constrained optimization problems in haemodynamics. In: J. Eberhardsteiner et.al. (ed.) Proceedings of the European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2012), Vienna, Austria, September 10-14, 2012 (2012)
109. Rozza, G., Veroy, K.: On the stability of reduced basis methods for Stokes equations in parametrized domains. Comput. Methods Appl. Mech. Engrg. **196**(7), 1244–1260 (2007)
110. Sen, S., Veroy, K., Huynh, P., Deparis, S., Nguyen, N., Patera, A.: "natural norm" a posteriori error estimators for reduced basis approximations. J. Comp. Phys. **217**(1), 37–62 (2006)
111. Sirisup, S., Karniadakis, G.: A spectral viscosity method for correcting the long-term behavior of POD models. J. Comp. Phys. **194**(1), 92–116 (2004)
112. Sirisup, S., Karniadakis, G.: Stability and accuracy of periodic flow solutions obtained by a POD-penalty method. J. Phys. D **202**(3), 218–237 (2005)
113. Sirisup, S., Karniadakis, G., Yang, Y., Rockwell, D.: Wave–structure interaction: simulation driven by quantitative imaging. Proc. R. Soc. A **460**(2043), 729–755 (2004)
114. Sirovich, L.: Turbulence and the dynamics of coherent structures. Part I: Coherent structures. Q. Appl. Math. **45**(3), 561–571 (1987)
115. Tadmor, E.: Convergence of spectral methods for nonlinear conservation laws. SIAM J. Numer. Anal. **26(1)**, 30–44 (1989)
116. Tadmor, G., Lehmann, O., Noack, B., Morzyński, M.: Mean field representation of the natural and actuated cylinder wake. Phys. Fluids **22**, 034,102 (2010)
117. Tamellini, L., Le Maître, O., Nouy, A.: Model reduction based on proper generalized decomposition for the stochastic steady incompressible Navier-Stokes equations. Tech. Rep. 26, MOX - Modellistica e calcolo scientifico, Politecnico di Milano (2012)
118. Temam, R.: Navier-Stokes Equations. AMS Chelsea, Providence, Rhode Island (2001)
119. Terragni, F., Vega, J.M.: On the use of POD-based ROMs to analyze bifurcations in some dissipative systems. Physica D (2012). DOI: 10.1016/j.physd.2012.04.009
120. Thomas, J., Hall, K., Dowell, E.: A harmonic balance approach for modeling nonlinear aeroelastic behavior of wings in transonic viscous flow. In: 44 th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference (2003)
121. Tonn, T., Urban, K., Volkwein, S.: Optimal control of parameter-dependent convection-diffusion problems around rigid bodies. SIAM J. Sci. Comput. **32**(3), 1237–1260 (2010)
122. Tu, J., Rowley, C.: An improved algorithm for balanced POD through an analytic treatment of impulse response tails. J. Comp. Phys. **231(16)** (2012)
123. Urban, K., Patera, A.T.: An improved error bound for reduced basis approximation of linear parabolic problems. Math. Comp. (2013). Accepted

124. Veroy, K., Patera, A.: Certified real-time solution of the parametrized steady incompressible Navier-Stokes equations: rigorous reduced-basis a posteriori error bounds. Int. J. Numer. Meth. Fluids **47(8-9)**, 773–788 (2005)
125. Veroy, K., Prud'homme, C., Rovas, D.V., Patera, A.T.: *A posteriori* error bounds for reduced basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations. In: Proceedings of the 16th AIAA Computational Fluid Dynamics Conference (2003). Paper 2003-3847
126. Wang, Z., Akhtar, I., Borggaard, J., Iliescu, T.: Proper orthogonal decomposition closure models for turbulent flows: A numerical comparison. Comput. Meth. Appl. Mech. Engrg. **237–240**, 10–26 (2012)
127. Weller, J., Lombardi, E., Bergmann, M., Iollo, A.: Numerical methods for low-order modeling of fluid flows based on POD. Int. J. Numer. Methods Fluids **63**(2), 249–268 (2010)
128. Willcox, K., Peraire, J.: Balanced model reduction via the proper orthogonal decomposition. AIAA J. **40**(11), 2323–2330 (2002)
129. Yano, M.: A space-time Petrov-Galerkin certified reduced basis method: application to the Boussinesq equations (2012). Preprint
130. Yano, M., Patera, A.T.: A space-time variational approach to hydrodynamic stability theory (2013). Preprint
131. Yano, M., Patera, A.T., Urban, K.: A space-time certified reduced basis method for Burgers' equation (2013). Preprint
132. Zhou, K., Salomon, G., Wu, E.: Balanced realization and model reduction for unstable systems. Int. J. Robust Nonlinear Control **9**(3), 183–198 (1999)