An overview on the approximation of boundary Riemann problems through physical viscosity

AN OVERVIEW ON THE APPROXIMATION OF BOUNDARY RIEMANN PROBLEMS THROUGH PHYSICAL VISCOSITY

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Abstract. This note aims at providing an overview of some recent results concerning the viscous approximation of so-called boundary Riemann problems for nonlinear systems of conservation laws in small total variation regimes.

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1. Introduction

We consider nonlinear systems of conservation laws in one space dimension, namely partial differential equations in the form

$$\partial_t U + \partial_x \big[F(U) \big] = 0. \tag{1.1}$$

In the previous expression, the unknown $U(t, x)$ attains values in $\mathbb{R}^N$ and depends on the real variables $t \in [0, +\infty[$ and x. We specify in the following the domain where x varies and we assume that the system is strictly hyperbolic, namely the Jacobian matrix of the flux $DF(U)$ admits N real and distinct eigenvalues

$$\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_N(U) \tag{1.2}$$

for every $U \in \mathbb{R}^N$. We are concerned with the viscous approximation

$$\partial_t U_\varepsilon + \partial_x \big[F(U_\varepsilon) \big] = \varepsilon\, \partial_x \Big[B(U_\varepsilon) \partial_x U_\varepsilon \Big], \tag{1.3}$$

where $\varepsilon > 0$ is a parameter, the function F is as in (1.1) and B is a $(N \times N)$ matrix which depends on the underlying physical model. We refer to § 6 for a more extended discussion on the precise hypotheses we impose on B, here we just mention that loosely speaking these hypotheses imply that system (1.3) is (weakly) dissipative. Also, they are satisfied in physically relevant examples like the Navier-Stokes equations in one space variable, written in either Eulerian or Lagrangian coordinates.

This note aims at reviewing some results concerning the structure of the solutions of (1.1) that can be recovered as limits $\varepsilon \to 0^+$ of solutions of the viscous approximation (1.3). In particular, we will focus on initial-boundary value problems and on Riemann-type data. Note that, as we will see in the following, in the case of initial-boundary value problems the limit of the viscous approximation (1.3) *depends* on the viscosity matrix B.

The style of this note is fairly informal, and to keep the exposition as simple as possible we only give very few technical details. We refer to the original research papers for the rigorous proof of the results we discuss.

The exposition is organized as follows. In § 2 we introduce the Riemann and the so-called boundary Riemann problem and we provide some motivations for their analysis. In § 3 we recall Lax's Theorem on the solution of the Riemann problem in small total variation regimes. In § 4 we consider the viscous approximation of a boundary Riemann problem and we show that, even in the linear case, the limit *depends* on the underlying viscous mechanism. In § 5 we state our main result concerning the viscous approximation of a boundary Riemann problem in general nonlinear cases. Loosely speaking, this result can be viewed as an extension of Lax's Theorem to the case of initial-boundary value problems. Finally, in § 6 we discuss the hypotheses we impose on the viscosity matrix B.

2. The Riemann problem and the boundary Riemann problem

The Riemann problem is a particular Cauchy problem posed by coupling the systems of conservation laws (1.1) with the initial condition

$$U(0,x) = \begin{cases} U^- & x < 0, \\ U^+ & x > 0. \end{cases} \tag{2.1}$$

In the previous expression, U^-, $U^+ \in \mathbb{R}^N$ are given constant states.

The Riemann problem (1.1),(2.1) is the basis for the construction of approximation schemes like the Glimm scheme [14] and the wave front-tracking algorithm [8, 16]. These schemes have been used to establish global in time existence and uniqueness for general Cauchy problems: we refer to the books by Dafermos [11] and Bressan [8] for an extended overview. Note, in particular, that the very definition of the class of functions where one can prove uniqueness results for general Cauchy problems involve the analysis of Riemann problems, see the definition of Standard Riemann Semigroup in [8, p. 171]. Also, the Riemann problem describes the local in space and time structure of the admissible solution of a general Cauchy problem: for this, we refer to [8, p. 185] and to the references therein. The analysis of the Riemann problem provides insights on the asymptotic behavior of solutions to general Cauchy problems, see Liu [22]. Finally, we mention the importance of the Riemann problem in view of numerical applications and we refer to the book by LeVeque [20] for that.

In the following, we always assume that the states U^- and U^+ in (2.1) are sufficiently close. This is not restrictive in view of most of the applications mentioned before because those applications are concerned with data with small total variation.

Distributional solutions of the Riemann problem (1.1),(2.1) are not unique. In 1957 Lax [19] introduced the celebrated criterion which is now named after him and established the existence of a distributional solution of (1.1),(2.1) satisfying this criterion, see Theorem 1 below. The analysis in [19] required some technical assumptions on the structure of the Jacobian matrix $DF(U)$ besides strict hyperbolicity. Those additional assumptions were relaxed and eventually removed in a series of papers by various authors including Liu [21], Tzavars [26], Dafermos [10] and Bianchini [4]. In particular, Liu [21] introduced an admissibility condition that is now named after him and that can be regarded as an extension of Lax's admissibility condition. Also, Bianchini [4] constructed solutions of (1.1),(2.1) that can be recovered as limits of the viscous approximation (1.3). Note, in particular, that the analysis in [4] implies that the limit of (1.3),(2.1) does *not* depend on the viscous mechanism, namely does not depend on the choice of B.

In the following, we focus on the so-called *boundary Riemann problem*, namely we assume that x varies in the domain $[0, +\infty[$ and we couple (1.1) with constant initial and boundary conditions. Note that the initial-boundary value problem obtained by coupling (1.1) with the data

$$U(t, 0) = \bar{U} \qquad U(0, x) \equiv U_0$$

is, in general, overdetermined and hence ill-posed, see the paper by Dubois and LeFloch [12] for a related discussion.

In the present note we focus on the solutions of the boundary Riemann problem that can be obtained as limits of a viscous dissipation. In other words, we couple the family of systems (1.3) with initial and boundary data

$$U_\varepsilon(t, 0) = U_b, \qquad U_\varepsilon(0, x) = U_0. \tag{2.2}$$

Here U_b, $U_0 \in \mathbb{R}^N$ are given constant states and as in the case of the Riemann problem we always assume that U_b and U_0 are sufficiently close. In the following, we will consider physical cases when the viscosity matrix B is not invertible. In this case, it may happen that the initial-boundary value problem (1.3),(2.2) is overdetermined and hence it is not well-posed, see [6, §2.2.1] for a more extended discussion. To simplify the exposition, in the following we always assume that (1.3),(2.2) is well-posed, but as a matter of fact the analysis can be extended to the general case (at the price of higher technicalities) by arguing as in [6]. Also, note that a rigorous proof of the the convergence of the viscous approximation (1.3),(2.2) is presently not available in the case of a general viscosity matrix B. There are, however, results under more restrictive assumptions, see for instance [2, 13, 17, 25].

The same considerations that motivate the analysis of the Riemann problem apply to the boundary Riemann problem. In particular, the analysis of the boundary Riemann problem is the basis for the construction of Glimm schemes and wave front-tracking algorithms for initial-boundary value problems, see the works by Goodman [15], Amadori [1] and the references in [11].

Also, as pointed out before, in the case when the domain has a boundary there is a further motivation for discussing the limit of the viscous approximation (1.3): the limit *depends* on the viscous mechanism, i.e. on the matrix B. In other words, the limit varies when B vary. Remarkably, this happens even in the linear case when F is linear and B is constant, see § 4. See also the work by Gisclon [13] and the book by Serre [24, §15.2] for an extended discussion of the nonlinear case. Finally, note that the fact that, for initial-boundary value problems, the limit depends on B has also relevant consequences from the numerical viewpoint, see the work by Mishra and Spinolo [23].

3. The solution of the Riemann problem

We now provide an heuristic formulation of the main result concerning the solvability of the Riemann problem (1.1),(2.1) in small total variation regimes.

Theorem 1. *Assume that*

- i) *system* (1.1) *is strictly hyperbolic, namely* (1.2) *holds.*
- ii) *The distance* $|U^+ - U^-|$ *is sufficiently small.*

Then we can exhibit a distributional solution U of the Riemann problem (1.1),(2.1) *such that*

1) *U is obtained by patching together at most countably many shocks, contact discontinuities and rarefaction waves.*
2) *Each shock and each contact discontinuity is Liu admissible.*

Some remarks are in order. First, the above result was first proved by Lax [19] under the additional assumption that every vector field is either genuinely nonlinear or linearly degenerate. These assumptions were relaxed in various papers, see in particular [10, 21, 26], and the final result was established in [4].

If the same assumptions as in [19] hold, then U admits at most N shocks, contact discontinuities and rarefaction waves. Also, in part 2) of the statement we can replace the Liu admissibility criterion with the Lax admissibility criterion.

Finally, in the statement of the above theorem the expression "we can exhibit" means that we have an algorithm for determining the solution, see in particular [4, 19] for the explicit construction.

4. The linear case

To highlight some of the main features of the analysis, in this paragraph we go over the construction of the solution of the boundary Riemann problem in the simplest possible case. More precisely, we consider the family of linear systems

$$\partial_t U_\varepsilon + A\,\partial_x U_\varepsilon = \varepsilon B\,\partial^2_{xx} U_\varepsilon. \tag{4.1}$$

We assume that

hyp1) A is a $N \times N$ symmetric, nonsingular matrix. We term

$$0 < \lambda_p < \cdots < \lambda_N$$

the *positive* eigenvalues of A and $\vec{r}_p, \dots, \vec{r}_N$ the associated eigenvectors.

hyp2) B is a $N \times N$ symmetric, positive definite matrix.

We now couple (4.1) with the data (2.2) and we describe the limit $\varepsilon \to 0^+$. First, the limit is a distributional solution of the linear transport equation

$$\partial_t U + A\,\partial_x U = 0$$

and therefore it has the following structure. There is $(s_p, \dots, s_N) \in \mathbb{R}^{N-p+1}$ such that

$$U(t,x) = \begin{cases} s_p \vec{r}_p + \cdots + s_N \vec{r}_N + U_0 & 0 < x < \lambda_p t \\ s_{p+1} \vec{r}_{p+1} + \cdots + s_N \vec{r}_N + U_0 & \lambda_p t < x < \lambda_{p+1} t \\ \dots \\ U_0 & x > \lambda_N t \end{cases} \tag{4.2}$$

We set

$$\bar{U} := s_p \vec{r}_p + \cdots + s_N \vec{r}_N + U_0 \tag{4.3}$$

and we point out that, in general, $\bar{U} \neq U_b$, namely the *parabolic* boundary condition is not the same as the boundary condition in the *hyperbolic* limit.

By relying on the analysis in [24, §15.2] we infer that the relation between $\bar{U}$ and U_b is the following: there is a *boundary layer* $W : [0, +\infty[\to \mathbb{R}^N$ such that

$$\begin{cases} BW' = A(W - \bar{U}) \\ W(0) = U_b, \quad \lim_{y \to +\infty} W(y) = \bar{U}. \end{cases} \tag{4.4}$$

In the above expression, W' denotes the first derivate of W. We now make some remarks concerning (4.4). First, the function $y \mapsto W(y/\varepsilon)$ is a steady solution of the linear system (4.1). Second, whether or not the system (4.4) admits a solution *depends* on B. This is the reason why, even in the linear case, the limit of the viscous approximation (4.1) *depends* on the viscosity B. Note that in the case of the Riemann problem (1.3),(2.1) the variable x varies in the whole real line and we do not need to take into account boundary layers: this is the reason why, under suitable assumptions, one can prove that the limit of (1.3), (2.1) does not depend on the viscosity mechanism, see [4].

We now provide some more details on the limit solution of (2.2),(4.1). We focus on (4.4) and we make the relation between U_0 and U_b more explicit. We first point out that, by combining assumptions h1) and h2) with [3, Lemma 7.1], we infer that the number of *negative* eigenvalues of the matrix $B^{-1}A$ is exactly $p-1$. Also, if we denote by $\vec{\xi}_1, \dots, \vec{\xi}_{p-1}$ the associated eigenvectors we can conclude that the vectors $\xi_1, \dots, \vec{\xi}_{p-1}, \vec{r}_p, \dots, \vec{r}_N$ are linearly independent.

Classical results on dynamical systems ensure that (4.4) admits a solution if and only if there are $(s_1, \dots, s_{p-1})$ such that

$$U_b = s_1 \vec{\xi}_1 + \cdots + s_{p-1} \vec{\xi}_{p-1} + \bar{U}.$$

By recalling (4.3) we arrive at

$$U_b = s_1 \vec{\xi}_1 + \cdots + s_{p-1} \vec{\xi}_{p-1} + s_p \vec{r}_p + \cdots + s_N \vec{r}_N + U_0,$$

which can be solved for $(s_1, \dots, s_N)$ because the vectors $\xi_1, \dots, \vec{\xi}_{p-1}, \vec{r}_p, \dots, \vec{r}_N$ are linearly independent. Note again that the solution $(s_1, \dots, s_N)$, and hence the function U in (4.2), depends on the choice of B since the vectors $\vec{\xi}_1, \dots, \vec{\xi}_{p-1}$ do.

5. The solution of the boundary Riemann problem

We now state state our main result concerning the viscous approximation (1.3),(2.2) of a boundary Riemann problem in small total variation regimes, see [5, 6] for the proof.

Theorem 2. *Assume that*

i) *system* (1.1) *is strictly hyperbolic, namely* (1.2) *holds.*
ii) *The distance* $|U_b - U_0|$ *is sufficiently small.*
iii) *The Kawashima-Shizuta conditions KS1,...,KS5 in § 6.1 hold.*
iv) *Either condition C1 or condition C2 in § 6.2 holds.*

Then we can exhibit a distributional solution U of the conservation law (1.1) *such that*

1. *the function U attains the Cauchy datum $U(0, \cdot) \equiv U_0$.*
2. *U is obtained by patching together at most countably many shocks, contact discontinuities and rarefaction waves.*
3. *Each shock and each contact discontinuity is Liu admissible.*
4. *The trace on the t axis is well-defined, namely there is $\bar{U} \in \mathbb{R}^N$ such that*

$$\lim_{x \to 0^+} U(t, x) = \bar{U} \quad \text{for almost every } t \in [0, +\infty[.$$

5. *The* parabolic *trace U_b and the* hyperbolic *trace $\bar{U}$ satisfy the following relation: there is $\underline{U} \in \mathbb{R}^N$ such that*
   i) *$\underline{U}$ is close to $\bar{U}$ and $F(\underline{U}) = F(\bar{U})$.*
   ii) *There is a boundary layer $W : [0, +\infty[\to \mathbb{R}^N$ such that*

$$\begin{cases} B(W)W' = F(W) - F(\underline{U}) \\ W(0) = U_b, \quad \lim_{y \to +\infty} W(y) = \underline{U}. \end{cases} \tag{5.1}$$

Some remarks are in order. First, we refer to § 6.3 for some more detailed comment on assumptions iii) and iv) in the statement of the above theorem. Here we only point out that hypotheses i),... iv) are motivated by physical examples, in particular they are satisfied if system (1.3) are the Navier-Stokes equations of fluid dynamics.

Second, the above result applies to both the *boundary characteristic* case and to the *non characteristic boundary* case. The boundary is characteristic if one of the eigenvalues of the Jacobian matrix $DF(U)$ can attain the value 0, and it is non characteristic otherwise. Handling the boundary characteristic case is definitely more challenging from the technical viewpoint. As pointed out by Joseph and LeFloch [17], in the non characteristic boundary case conditions 5i) and 5ii) in the statement of Theorem 2 imply that $\bar{U} = \underline{U}$ owing to the Local Invertibility Theorem. Hence, in the non characteristic boundary case the relation between the parabolic trace U_b and the hyperbolic trace $\bar{U}$ is that there is a boundary layer $W : [0, +\infty[\to \mathbb{R}^N$ such that

$$\begin{cases} B(W)W' = F(W) - F(\bar{U}) \\ W(0) = U_b, \quad \lim_{y \to +\infty} W(y) = \bar{U}. \end{cases}$$

Note that in the case when B is constant and $F(W) = AW$ the above system boils down to (4.4).

Third, in the statement of the above theorem the sentence "we can exhibit" should be interpreted in the same sense as in Lax's Theorem 1: it means that we have an explicit algorithm to determine the solution. Forth, the uniqueness of the solution satisfying conditions $1, \dots, 5$ above has been established under some additional conditions on B, see the work by Christoforou and Spinolo [9]. Finally, we refer to Joseph and LeFloch [17] for the analysis of self-similar viscous approximations of boundary Riemann problems.

6. Hypotheses

We now introduce the precise hypotheses we impose in the statement of Theorem 2. More precisely, in § 6.1 we define a set of hypotheses that were introduced in the celebrated work by Kawashima and Shizuta [18]. In § 6.2 we recall two new conditions introduced in [5, 6]. Finally, in § 6.3 we comment on the conditions defined in § 6.1 and § 6.2.

6.1. **Kawashima-Shizuta conditions.**

We assume that there is an invertible change of variables $U \mapsto V$ such that, if U_ε satisfies (1.3), then the function V is a solution of the quasi-linear system

$$E(V)\partial_t V + A(V)\partial_x V = \varepsilon D(V)\partial^2_{xx} V + \varepsilon\, \mathcal{G}(V, \partial_x V). \tag{6.1}$$

The $N \times N$ matrices E, A and D and the function $\mathcal{G} \in \mathbb{R}^N$ satisfy for every $V \in \mathbb{R}^N$ the following requirements:

KS1) the matrix E is symmetric and positive definite.

KS2) The matrix A is symmetric.

KS3) The matrix D admits the following block decomposition:

$$D(V) = \begin{pmatrix} 0 & 0 \\ 0 & D_{22}(V) \end{pmatrix}.$$

The $(N-1) \times (N-1)$ matrix D_{22} is symmetric and positive definite.

KS4) The so-called *Kawashima-Shizuta* condition holds, namely

$$\operatorname{Ker} D \cap \big\{\text{eigenvalues of } E^{-1}A\big\} = \emptyset.$$

In the above formula, $\operatorname{Ker} D$ denotes the kernel of D and $\emptyset$ the empty set.

KS5) The term $\mathcal{G}(V, \partial_x V)$ is basically of lower order with respect to the other ones in (6.1), see [18] for the precise requirements satisfied by $\mathcal{G}$.

6.2. **Conditions on a_{11}.** We first introduce some notation: we block-decompose the matrix A and the dependent variable V as

$$A(V) = \begin{pmatrix} a_{11}(V) & A_{21}^t(V) \\ A_{21}(V) & A_{22}(V) \end{pmatrix} \qquad V = \begin{pmatrix} v_1 \\ V_2 \end{pmatrix}$$

In the previous expression, the function a_{11} and A_{21} attain values in $\mathbb{R}$ and $\mathbb{R}^{N-1}$, respectively, A_{21}^t denotes the transpose of A_{21} and $A_{22}(V)$ is a $(N-1) \times (N-1)$ matrix. Also, $v_1 \in \mathbb{R}$ and $V_2 \in \mathbb{R}^{N-1}$.

We now define the conditions C1 and C2 as follows:

C1) The function a_{11} does not change sign. Namely, either $a_{11}(V) > 0$ for every V or $a_{11}(V) = 0$ for every V or $a_{11}(V) < 0$ for every V.

C2) The following implication holds: if $a_{11}(V) = 0$ then

$$\partial_{v_1} a_{11} = 0$$

and

$$\nabla_{V_2} a_{11} = \alpha A_{21}^t \text{ for some } \alpha \neq 0 \text{ (possibly depending on } V).$$

In the above expression, ∂_{v_1} denotes the partial derivative with respect to v_1 and ∇_{V_2} the gradient with respect to V_2.

6.3. **Comments.** As mentioned in § 6.1, conditions KS1),. . . ,KS5) were introduced in the fundamental paper by Kawashima and Shizuta [18] and they are modeled on the Navier-Stokes and MHD equations of fluid dynamics and magneto-fluid dynamics, respectively.

We now comment on conditions C1 and C2, which are new and are introduced in [6] and [5], respectively. First, we point out that Theorem 2 is established in [6] under condition C1. Note furthermore that condition C1 is satisfied by the Navier-Stokes equations written in Lagrangian coordinates, but it is not satisfied by the same equations written in Eulerian coordinates [1]. For this reason, in the later work [5] we have replaced condition C1 with condition C2, which is satisfies by the Navier-Stokes equations written in Eulerian coordinates.

From the technical viewpoint, the basic idea underpinning the introduction of either condition C1 or C2 is the following: the key point in the proof of Theorem 2 is the analysis of the boundary layers. Note that, if we do not impose any further conditions besides KS1,. . . ,KS5, the boundary layers can exhibit "weird behaviors", see the examples in [6, § 2.2.2]. Loosely speaking, both condition C1 and condition C2 imply that the boundary layers are "nicely behaved". Note, however, that establishing Theorem 2 under condition C2 is definitely more challenging from the technical viewpoint as in this case the boundary layers satisfy an ordinary differential equation with singularity. The proof is based on results concerning the analysis of singular differential equations established in [7].

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References

[1] Debora Amadori. Initial-boundary value problems for nonlinear systems of conservation laws. *NoDEA Nonlinear Differential Equations Appl.*, 4(1):1–42, 1997.

[2] Fabio Ancona and Stefano Bianchini. Vanishing viscosity solutions of hyperbolic systems of conservation laws with boundary. In *"WASCOM 2005"—13th Conference on Waves and Stability in Continuous Media*, pages 13–21. World Sci. Publ., Hackensack, NJ, 2006.

[3] Sylvie Benzoni-Gavage, Denis Serre, and Kevin Zumbrun. Alternate Evans functions and viscous shock waves. *SIAM J. Math. Anal.*, 32(5):929–962, 2001.

[4] Stefano Bianchini. On the Riemann problem for non-conservative hyperbolic systems. *Arch. Ration. Mech. Anal.*, 166(1):1–26, 2003.

[5] Stefano Bianchini and Laura V. Spinolo. The boundary Riemann problem limit of mixed hyperbolic-parabolic systems. *In preparation.*

[6] Stefano Bianchini and Laura V. Spinolo. The boundary Riemann solver coming from the real vanishing viscosity approximation. *Arch. Ration. Mech. Anal.*, 191(1):1–96, 2009.

[7] Stefano Bianchini and Laura V. Spinolo. Invariant manifolds for a singular ordinary differential equation. *J. Differential Equations*, 250(4):1788–1827, 2011.

[8] Alberto Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[9] Cleopatra Christoforou and Laura V. Spinolo. A uniqueness criterion for viscous limits of boundary Riemann problems. *J. Hyperbolic Differ. Equ.*, 8(3):507–544, 2011.

[10] Constantine M. Dafermos. Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method. *Arch. Rational Mech. Anal.*, 52:1–9, 1973.

[11] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[12] François Dubois and Philippe LeFloch. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J. Differential Equations*, 71(1):93–122, 1988.

[13] Marguerite Gisclon. Étude des conditions aux limites pour un système strictement hyperbolique, via l'approximation parabolique. *J. Math. Pures Appl. (9)*, 75(5):485–508, 1996.

[14] James Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, 18:697–715, 1965.

[15] Jonathan Goodman. *Initial Boundary Value Problems for Hyperbolic Systems of Conservation Laws.* PhD Thesis. California University. 1982.

[16] Helge Holden and Nils Henrik Risebro. *Front tracking for hyperbolic conservation laws*, volume 152 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.

[17] K. T. Joseph and Philippe G. LeFloch. Boundary layers in weak solutions of hyperbolic conservation laws. II. Self-similar vanishing diffusion limits. *Commun. Pure Appl. Anal.*, 1(1):51–76, 2002.

[18] Shuichi Kawashima and Yasushi Shizuta. On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. *Tohoku Math. J. (2)*, 40(3):449–464, 1988.

[19] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[20] Randall J. LeVeque. *Numerical methods for conservation laws.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1990.

[21] Tai Ping Liu. The Riemann problem for general 2×2 conservation laws. *Trans. Amer. Math. Soc.*, 199:89–112, 1974.

[22] Tai Ping Liu. Linear and nonlinear large-time behavior of solutions of general systems of hyperbolic conservation laws. *Comm. Pure Appl. Math.*, 30(6):767–796, 1977.

[23] Siddhartha Mishra and Laura V. Spinolo. Accurate numerical schemes for approximating initial-boundary value problems for systems of conservation laws. *J. Hyperbolic Differ. Equ.*, 12:61–86, 2015.

[24] Denis Serre. *Systems of conservation laws. 2.* Cambridge University Press, Cambridge, 2000. Geometric structures, oscillations, and initial-boundary value problems, Translated from the 1996 French original by I. N. Sneddon.

[25] Laura V. Spinolo. Vanishing viscosity solutions of a 2×2 triangular hyperbolic system with Dirichlet conditions on two boundaries. *Indiana Univ. Math. J.*, 56(1):279–364, 2007.

[26] Athanasios E. Tzavaras. Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws. *Arch. Rational Mech. Anal.*, 135(1):1–60, 1996.

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