Hamiltonian perturbations of hyperbolic PDEs: from classification results to the properties of solutions

Hamiltonian perturbations of hyperbolic PDEs: from classification results to the properties of solutions

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Abstract

We begin with presentation of classification results in the theory of Hamiltonian PDEs with one spatial dimension depending on a small parameter. Special attention is paid to the deformation theory of integrable hierarchies, including an important subclass of the so-called integrable hierarchies of the topological type associated with semisimple Frobenius manifolds. Many well known equations of mathematical physics, such as KdV, NLS, Toda, Boussinesq etc., belong to this subclass, but there are many new integrable PDEs, some of them being of interest for applications. Connections with the theory of Gromov–Witten invariants and random matrices are outlined. We then address the problem of comparative study of singularities of solutions to the systems of first order quasilinear PDEs and their Hamiltonian perturbations containing higher derivatives. We formulate Universality Conjectures describing different types of critical behavior of perturbed solutions near the point of gradient catastrophe of the unperturbed one.

1 Introduction

The main subject of our research is the study of Hamiltonian perturbations of systems of hyperbolic[1] PDEs

$$u^i_t + A^i_j(u)u^j_x + \text{higher order derivatives} = 0, \quad i = 1, \dots, n.$$

(Here and below the summation over repeated indices will be assumed.) They can be obtained, in particular, by applying the procedure of *weak dispersion expansion*: starting from a system of PDEs

$$u^i_t + F^i(u, u_x, u_{xx}, \dots) = 0, \quad i = 1, \dots, n$$

[1]Always the strong hyperbolicity will be assumed, i.e., the eigenvalues of the $n \times n$ matrix $\left(A^i_j(u)\right)$ are all real and pairwise distinct for all $u = (u^1, \dots, u^n)$ in the domain under consideration.

with the analytic right hand side let us introduce *slow variables*

$$x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t.$$

Expanding in ϵ one obtains, after dividing by ϵ a system of the above form

$$\frac{1}{\epsilon}\, F^i(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \dots) = u^i_t + A^i_j(u) u^j_x + \epsilon\, \left(B^i_j(u) u^j_{xx} + \frac{1}{2} C^i_{jk}(u) u^j_x u^k_x \right) + \dots$$

assuming all the dependent variables are *slow*, i.e., the terms of the order $1/\epsilon$ vanish:

$$F^i(u, 0, 0, \dots) \equiv 0, \quad i = 1, \dots, n.$$

E.g., the celebrated Korteweg - de Vries (KdV) equation

$$u_t + u\, u_x + \frac{\epsilon^2}{12} u_{xxx} = 0 \tag{1.1}$$

is one of the most well known examples of such a weakly dispersive Hamiltonian PDE. Another class of examples comes from interpolated discrete systems. Let us consider the simplest example of Toda lattice: an infinite system of particles on the line with exponential interaction of neighbors. The equations of motion

$$\left.\begin{array}{rcl} \dot q_n & = & \frac{\partial H}{\partial p_n} \\ & & \\ \dot p_n & = & -\frac{\partial H}{\partial q_n} \end{array}\right\}, \quad n \in \mathbb{Z}$$

$$H = \sum_{n \in \mathbb{Z}} \frac{p_n^2}{2} + e^{q_n - q_{n-1}} \tag{1.2}$$

after the interpolation

$$\begin{array}{rcl} q_{n+1} - q_n & = & u(n\, \epsilon) \\ p_n & = & v(n\, \epsilon) \end{array}$$

can be recast into the form (1.6) via the (formal) Taylor expansion

$$\begin{array}{rclcl} u_t & = & \frac{1}{\epsilon} \left[v(x+\epsilon) - v(x) \right] & = & v_x + \frac{1}{2} \epsilon\, v_{xx} + \dots \\ & & & & \\ v_t & = & \frac{1}{\epsilon} \left[e^{u(x)} - e^{u(x-\epsilon)} \right] & = & e^u u_x - \frac{1}{2} \epsilon\, \left(e^u\right)_{xx} + \dots \end{array} \tag{1.3}$$

Another class of infinite expansions comes from a nonlocal evolution. An example of Camassa - Holm equation

$$u_t - u_{txx} = \frac{3}{2} u\, u_x - \left[u_x u_{xx} + \frac{1}{2} u\, u_{xxx} \right] \tag{1.4}$$

illustrates the procedure. After the introduction of the slow variables $x \mapsto \epsilon x, \quad t \mapsto \epsilon t$ we use the geometric series

$$\left(1 - \epsilon^2 \partial_x^2\right)^{-1} = 1 + \epsilon^2 \partial_x^2 + \epsilon^4 \partial_x^4 + \dots$$

in order to rewrite the Camassa - Holm equation in the form (1.6):

$$u_t = \frac{3}{2} u\, u_x + \varepsilon^2 \left(u\, u_{xxx} + \frac{7}{2} u_x u_{xx} \right) + O(\varepsilon^4) \tag{1.5}$$

Let us return back to the general setting. Loosely speaking the system of PDEs

$$u_t^i + A_j^i(u) u_x^j + \epsilon \left(B_j^i(u) u_{xx}^j + \frac{1}{2} C_{jk}^i(u) u_x^j u_x^k \right) + \cdots = 0, \quad i = 1, \ldots \tag{1.6}$$

depending on a small parameter ϵ will be considered as a Hamiltonian vector field on the "infinite dimensional manifold"

$$\mathcal{L}(M^n) \otimes \mathbb{R}[[\epsilon]] \tag{1.7}$$

where M^n is a n-dimensional manifold (in all our examples it will have the topology of a ball) and

$$\mathcal{L}(M^n) = \left\{ S^1 \to M^n \right\}$$

is the space of loops on M^n. The dependent variables

$$u = (u^1, \ldots, u^n) \in M^n$$

are local coordinates on M^n. In the expansion (1.6) the terms of order ϵ^k are *polynomials* of degree $k+1$ in the derivatives u_x, u_{xx}, ... where

$$\deg u^{(m)} = m, \quad m = 1, 2, \ldots$$

The coefficients of these polynomials are smooth functions defined in every coordinate chart on M^n. Clearly the above gradation on the ring of polynomial functions on the jet bundle $J^\infty(M^n)$ does not depend on the choice of local coordinates. The systems of the form (1.6) will be assumed to be Hamiltonian flows

$$u_t^i = \{u^i(x), H\} = \sum_{k \geq 0} \epsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij} \left(u; u_x, \ldots, u^{(m)} \right) \partial_x^{k-m+1} \frac{\delta H}{\delta u^j(x)} \tag{1.8}$$

with respect to *local Poisson brackets*

$$\begin{aligned} &\{u^i(x), u^j(y)\} = \sum_{k \geq 0} \epsilon^k \sum_{m=0}^{k+1} A_{k,m}^{ij} \left(u(x); u_x(x), \ldots, u^{(m)}(x) \right) \delta^{(k-m+1)}(x-y) \\ &\deg A_{k,m}^{ij} \left(u; u_x, \ldots, u^{(m)} \right) = m \end{aligned} \tag{1.9}$$

with *local Hamiltonians*

$$\begin{aligned} H &= \sum_{k \geq 0} \epsilon^k \int h_k \left(u; u_x, \ldots, u^{(k)} \right) dx \\ \deg h_k &\left(u; u_x, \ldots, u^{(k)} \right) = k. \end{aligned} \tag{1.10}$$

Here $\delta(x)$ is the Dirac delta-function. The meaning of the delta-function and of its derivatives is clear from the explicit expression (1.8). The integral in (1.10) is understood in the sense of formal variational calculus, i.e., for a differential polynomial $h = h(u; u_x, \dots, u^{(m)})$ the integral

$$\int h(u; u_x, \dots, u^{(m)})\, dx$$

is the class of equivalence of h modulo the total x-derivative:

$$h(u; u_x, \dots, u^{(m)}) \sim h(u; u_x, \dots, u^{(m)}) + \partial_x \left(f(u; u_x, \dots, u^{(m-1)}\right)$$

$$\partial_x = \sum_{k\geq 0} u^{i\,(k+1)} \frac{\partial}{\partial u^{i\,(k)}}.$$

$\delta H / \delta u^j(x)$ is the Euler – Lagrange operator

$$\frac{\delta H}{\delta u^j(x)} = \frac{\partial h}{\partial u^j} - \partial_x \frac{\partial h}{\partial u^j_x} + \partial_x^2 \frac{\partial h}{\partial u^j_{xx}} - \dots \quad \text{for} \quad H = \int h\, dx.$$

The coefficients of the Poisson bracket and Hamiltonian densities will always be assumed to be differential polynomials. The antisymmetry and Jacobi identity for the Poisson bracket (1.9) are understood as identities for formal power series in ϵ. The Poisson bracket (1.9) defines a structure of a Lie algebra $\mathcal{G}_{\rm loc}$ on the space of all local functionals

$$\{F, G\} = \int \frac{\delta F}{\delta u^i(x)} A^{ij} \frac{\delta G}{\delta u^j(x)}\, dx \tag{1.11}$$
$$A^{ij} := \sum_{k\geq 0} \epsilon^k \sum_{m=0}^{k+1} A^{ij}_{k,m} \left(u; u_x, \dots, u^{(m)} \right) \partial_x^{k-m+1}$$
$$F = \sum_{k\geq 0} \epsilon^k \int f_k(u; u_x, \dots, u^{(k)})\, dx, \quad G = \sum_{l\geq 0} \epsilon^l \int g_l(u; u_x, \dots, u^{(l)})\, dx$$
$$\deg f_k(u; u_x, \dots, u^{(k)}) = k, \qquad \deg g_l(u; u_x, \dots, u^{(l)}) = l.$$

The full ring of functions on the infinite dimensional manifold $\mathcal{L}(M^n) \otimes \mathbb{R}[[\epsilon]]$ is obtained by taking the suitably completed symmetric tensor algebra of $\mathcal{G}_{\rm loc}$.

Let us now introduce the class of "coordinate changes" on the infinite dimensional manifold $\mathcal{L}(M^n) \otimes \mathbb{R}[[\epsilon]]$. Define a *generalized Miura transformation*

$$u^i \mapsto \tilde{u}^i = \sum_{k\geq 0} \epsilon^k F^i_k \left(u; u_x, \dots, u^{(k)} \right) \tag{1.12}$$
$$\deg F^i_k \left(u; u_x, \dots, u^{(k)} \right) = k$$

$$\det \left(\frac{\partial F^i_0(u)}{\partial u^j} \right) \neq 0.$$

The coefficients $F_k^i\left(u; u_x, \dots, u^{(k)}\right)$ are differential polynomials. It is easy to see that the transformations of the form (1.12) form a group[2]. The classes of evolution PDEs (1.6), local Poisson brackets (1.9) and local Hamiltonians (1.10) are invariant with respect to the action of the group of generalized Miura transformations. We say that two objects of our theory (i.e., two evolutionary vector fields of the form (1.6), two local Poisson brackets of the form (1.9), or two local Hamiltonians of the form (1.10)) are *equivalent* if they are related by a generalized Miura transformation.

Our main goal is the *classification* of Hamiltonian PDEs (1.6), (1.8) with respect to the above equivalence relation. We will also address the problem of selection of *integrable* Hamiltonian PDEs. Last but not least, we will study the general properties of *solutions* to Hamiltonian PDEs of the form (1.6), (1.8).

2 Towards classification of Hamiltonian PDEs

The first step is the classification of local Poisson brackets (1.9) with respect to the action of the group of Miura-type transformations.

Theorem 2.1 *Under assumption*

$$\det\left(A^{ij}_{0,0}(u)\right) \neq 0 \tag{2.1}$$

any bracket of the form (1.9) *is equivalent to*

$$\left\{\tilde u^i(x), \tilde u^j(y)\right\} = \eta^{ij}\,\delta'(x-y), \quad \eta^{ij} = \eta^{ji} = \text{const}, \quad \det\left(\eta^{ij}\right) \neq 0. \tag{2.2}$$

The *proof* of this theorem consists of two parts. The first part deals with the analysis of the *leading term* of the Poisson bracket. Setting $\epsilon \to 0$ one obtains again a Poisson brackets of a simpler form

$$\left\{u^i(x), u^j(y)\right\}^{[0]} = A^{ij}_{0,0}(u(x))\delta'(x-y) + A^{ij}_{0,1}(u(x); u_x(x))\delta(x-y). \tag{2.3}$$

Here the coefficient $A^{ij}_{0,1}(u; u_x)$ depends linearly on u_x. This is a so-called *Poisson bracket of hydrodynamic type* introduced in 1983 by B.Dubrovin and S.P.Novikov [24]. One of the main results of [24] says that, under the assumption (2.1) the leading term

$$g^{ij}(u) := A^{ij}_{0,0}(u) \tag{2.4}$$

is a (contravariant) Riemannian or pseudo-Riemannian metric of the curvature zero on the underlying manifold M^n; the second coefficient must have the form

$$A^{ij}_{0,1}(u; u_x) = \Gamma^{ij}_k(u) u^k_x, \quad \Gamma^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u) \tag{2.5}$$

[2]To invert the transformation one has to solve the differential equation (1.12) for $u^1, \dots, u^n$. The solution in question is written as the formal WKB expansion.

where $\Gamma^j_{sk}(u)$ are the Christoffel coefficients for the Levi-Civita connection for the metric $g^{ij}(u)$. Due to triviality of the topology of M^n one can choose a global system of flat coordinates for the metric

$$\tilde{u}^i = \tilde{u}^i(u), \quad \frac{\partial \tilde{u}^i}{\partial u^k}\frac{\partial \tilde{u}^j}{\partial u^l} g^{kl}(u) = \eta^{ij} = \text{const}.$$

In these coordinates the Poisson bracket (2.3) takes the form (2.2).

The second part of the proof is based on the deformation theory of the Poisson bracket (2.2). We may assume that the original Poisson bracket (1.9) has the form

$$\left\{u^i(x), u^j(y)\right\} = \eta^{ij}\delta'(x-y) + O(\epsilon).$$

The first correction is a 2-cocycle in the Poisson cohomology of the "manifold" $\mathcal{L}(M^n) \otimes \mathbb{R}[[\epsilon]]$ equipped with the Poisson bracket (2.2). This first correction can be eliminated by a "change of coordinates", i.e., by a generalized Miura transformation, *iff* this 2-cocycle is trivial. To complete the proof of Theorem 2.1 one has to use triviality of the Poisson cohomology in positive degrees in ϵ proved in [38] (see also [12]).

Corollary 2.2 *Any system of Hamiltonian PDEs for slow dependent variables satisfying the nondegeneracy assumption* (2.1) *can be reduced to the following standard form*

$$u^i_t = \eta^{ij}\partial_x \frac{\delta H}{\delta u^j(x)}, \quad i = 1, \dots, n \tag{2.6}$$

with the Hamiltonian of the form (1.10). *Two such systems are equivalent* iff *the Hamiltonians are related by a canonical transformation*

$$\begin{aligned} &H \mapsto H + \epsilon\{F, H\} + \frac{\epsilon^2}{2}\{F, \{F, H\}\} + \dots, \\ &F = \sum_{k\geq 0} \epsilon^k \int f_k\left(u; u_x, \dots, u^{(k)}\right) dx, \quad \deg f_k\left(u; u_x, \dots, u^{(k)}\right) = k. \end{aligned} \tag{2.7}$$

In order to prove the second part of the corollary one has to use triviality in the positive degrees in ϵ of the first Poisson cohomology of the bracket (2.2). This implies that any canonical transformation close to identity must have the form

$$u^i(x) \mapsto \tilde{u}^i(x) = u^i(x) + \{F, u^i(x)\} + \frac{\epsilon^2}{2}\{F, \{F, u^i(x)\}\} + \dots \tag{2.8}$$

$$\{\tilde{u}^i(x), \tilde{u}^j(y)\} = \{u^i(x), u^j(y)\} = \eta^{ij}\delta'(x-y)$$

with the generating Hamiltonian F of the above form polynomial in jets in every order in ϵ.

Example 2.3 *The Riemann equation*

$$v_t + v\, v_x = 0 \tag{2.9}$$

is a Hamiltonian system

$$v_t + \partial_x \frac{\delta H_0}{\delta v(x)} = 0$$

with the Hamiltonian

$$H_0 = \int \frac{v^3}{6}\, dx \tag{2.10}$$

and the Poisson bracket of the form (2.2)*:*

$$\{v(x), v(y)\} = \delta'(x-y). \tag{2.11}$$

Any Hamiltonian perturbation of this equation of order ϵ^4 can be reduced to the following normal form parametrized by two arbitrary functions of one variable $c = c(u)$, $p = p(u)$:

$$\begin{gathered} u_t + u\, u_x + \frac{\epsilon^2}{24}\left[2c\, u_{xxx} + 4c' u_x u_{xx} + c'' u_x^3\right] + \epsilon^4 \left[2p\, u_{xxxxx}\right. \\ \left. +2p'(5u_{xx}u_{xxx} + 3u_x u_{xxxx}) + p''(7u_x u_{xx}^2 + 6u_x^2 u_{xxx}) + 2p''' u_x^3 u_{xx}\right] = 0. \end{gathered} \tag{2.12}$$

The Hamiltonian has the form

$$H = \int \left[\frac{u^3}{6} - \epsilon^2 \frac{c(u)}{24} u_x^2 + \epsilon^4 p(u) u_{xx}^2\right] dx \tag{2.13}$$

Two such perturbations are equivalent iff *the associated functional parameters $c(u)$, $p(u)$ coincide [18].*

3 Deformation theory of integrable hierarchies

We will now concentrate on the study of integrable hyperbolic systems

$$v_t^i + A_j^i(v) v_x^j = 0, \quad i = 1, \dots, n$$

and their Hamiltonian perturbations. The word 'hyperbolic' will stand for *strong hyperbolicity*, i.e., all eigenvalues of the matrix $\left(A_j^i(v)\right)$, $v \in M^n$, will be assumed real and pairwise distinct. We will also consider the complex analytic situation where the eigenvalues of the matrix will be assumed to be distinct.

Let us first recall the main points of the theory of integrable hyperbolic PDEs.

Definition 3.1 *A hyperbolic system*

$$v_t^i + A_j^i(v)v_x^j = v_t^i + \eta^{ij}\partial_x \frac{\delta H_0}{\delta v^j(x)} = 0 \tag{3.1}$$

$$H_0 = \int h(v)\,dx, \quad A_j^i(v) = \eta^{is}\frac{\partial^2 h(v)}{\partial v^s \partial v^j}$$

is called integrable *if the Lie algebra of first integrals* F_0 *of the form*

$$F_0 = \int f(v)\,dx, \quad \{H_0, F_0\} = 0 \tag{3.2}$$

possesses the following property of maximality*: solutions* $f = f(v)$ *to the overdetermined system of equations*

$$\frac{\partial^2 f}{\partial v^i \partial v^l}\eta^{ij}\frac{\partial^2 h}{\partial v^j \partial v^k} = \frac{\partial^2 f}{\partial v^i \partial v^k}\eta^{ij}\frac{\partial^2 h}{\partial v^j \partial v^l}, \quad k, l = 1, \ldots, n \tag{3.3}$$

equivalent to (3.2) *depend on the maximal number (*$= n$*) of arbitrary functions of one variable.*

First integrals of an integrable system of hyperbolic PDEs form a maximal Abelian subalgebra in the Lie algebra $\mathcal{G}_{\rm loc}$ of local Hamiltonians [26]. The Hamiltonian flow

$$v_s^i + B_j^i(v)v_x^j = v_s^i + \eta^{ij}\partial_x \frac{\delta F_0}{\delta v^j(x)}, \quad B_j^i(v) = \eta^{is}\frac{\partial^2 f(v)}{\partial v^s \partial v^j} \tag{3.4}$$

generated by any solution to (3.2) is an infinitesimal symmetry of the hyperbolic system (3.1):

$$\frac{\partial}{\partial s}\frac{\partial v^i}{\partial t} = \frac{\partial}{\partial t}\frac{\partial v^i}{\partial s}, \quad i = 1, \ldots, n. \tag{3.5}$$

All these symmetries commute pairwise due to commutativity of the Lie algebra of conservation laws. Integrability of the Hamiltonian hyperbolic system (3.1) is equivalent to its diagonalizability (i.e., existence of *Riemann invariants*) [66]. Recall that the necessary and sufficient condition for diagonalizability is vanishing of the Haantjes tensor [43] in the case under consideration written in the form

$$\begin{aligned} \mathrm{H}_{ijk} &= (h_{ipq}h_{jr}h_{ks} + h_{jpq}h_{kr}h_{is} + h_{kpq}h_{ir}h_{js})\,h_{ab}\eta^{pa}\delta^{qbsr} \\ \delta^{ijkl} &:= \det\begin{pmatrix} \eta^{ik} & \eta^{il} \\ \eta^{jk} & \eta^{jl} \end{pmatrix} \end{aligned} \tag{3.6}$$

where we use short notations for the derivatives of the Hamiltonian density

$$h_{ij} := \frac{\partial^2 h}{\partial v^i \partial v^j}, \quad h_{ijk} := \frac{\partial^3 h}{\partial v^i \partial v^j \partial v^k}.$$

The tensor (3.6) is totally antisymmetric. For $n = 1$ and $n = 2$ any hyperbolic system is integrable. For $n \geq 3$ there are $n(n-1)(n-2)/6$ integrability constraints $\mathrm{H}_{ijk} = 0$, $i < j < k$.

Given a symmetry (3.4) the functions $v^1(x,t), \ldots, v^n(x,t)$ implicitly defined by the system of n equations written in the form

$$\det\left[(\lambda - x)\cdot \mathrm{id} + t\, A(v) - B(v)\right] \equiv \lambda^n, \quad A(v) = \left(A^i_j(v)\right), \quad B(v) = \left(B^i_j(v)\right) \tag{3.7}$$

give a solution to the original hyperbolic system. Any solution to this system satisfying certain genericity conditions can be obtained in this way [66].

Let us now consider Hamiltonian perturbations

$$u^i_t + \eta^{ij}\partial_x \frac{\delta H}{\delta u^j(x)} = 0, \quad H = H_0 + O(\epsilon), \quad H_0 = \int h(u)\, dx \tag{3.8}$$

of an integrable hyperbolic system

$$v^i_t + \eta^{ij}\partial_x \frac{\delta H_0}{\delta v^j(x)} = 0, \quad i = 1, \ldots, n. \tag{3.9}$$

(We use different notations $v = v(x,t)$ and $u = u(x,t)$ for the dependent variables of the unperturbed/perturbed systems resp. for a convenience later on.)

Definition 3.2 *The perturbed system* (3.8) *is called* N-integrable *if there exists a linear differential operator*

$$D_N = D^{[0]} + \epsilon\, D^{[1]} + \epsilon^2 D^{[2]} + \cdots + \epsilon^N D^{[N]}$$

$$D^{[0]} = \mathrm{id}, \quad D^{[k]} = \sum b^{[k]}_{i_1 \ldots i_{m(k)}}(u; u_x, \ldots, u^{(k)}) \frac{\partial^{m(k)}}{\partial u^{i_1} \ldots \partial u^{i_{m(k)}}} \tag{3.10}$$

$$\deg b^{[k]}_{i_1 \ldots i_{m(k)}}(u; u_x, \ldots, u^{(k)}) = k, \quad k \geq 1$$

acting on the commuting Hamiltonians (3.3) - (3.5) *such that, for any two solutions* $f(u)$, $g(u)$ *to the equations* (3.3) *the Hamiltonians*

$$H^f_N := \int D_N f\, dx, \quad H^g_N := \int D_N g\, dx \tag{3.11}$$

satisfy

$$\{H^f_N, H^g_N\} = O\left(\epsilon^{N+1}\right). \tag{3.12}$$

Moreover, we require that

$$H = D_N h + O\left(\epsilon^{N+1}\right), \tag{3.13}$$

so the Hamiltonians (3.11) *satisfy also*

$$\{H, H^f_N\} = O\left(\epsilon^{N+1}\right)$$

for any solution $f = f(u)$ *to the equations* (3.3).

The perturbed system (3.8) *is called* integrable *if it is* N*-integrable for any* $N \geq 0$.

In the formula (3.10) $m(k)$ is some positive integer depending on k. The summation is taken over all indices $i_1, \ldots, i_{m(k)}$ from 1 to n. As usual the coefficients $b^{[k]}_{i_1 \ldots i_{m(k)}}(u; u_x, \ldots, u^{(k)})$ are graded homogeneous differential polynomials of degree k. It is easy to see that

$$m(k) = \left[\frac{3k}{2}\right]. \tag{3.14}$$

As the D-operator makes sense only acting on the common kernel of the linear operators

$$h_{jk}\,\eta^{ij}\frac{\partial^2}{\partial v^i \partial v^l} - h_{jl}\,\eta^{ij}\frac{\partial^2}{\partial v^i \partial v^k}, \quad k,\, l = 1, \ldots, n, \tag{3.15}$$

the coefficients are not determined uniquely.

For a N-integrable system any symmetry (3.3) - (3.5) can be extended to a Hamiltonian flow

$$\begin{gathered} u^i_s + \eta^{ij}\partial_x \frac{\delta F}{\delta v^j(x)} = 0 \\ F = F_0 + O(\epsilon), \quad F_0 = \int f(u)\, dx, \quad F = \int D_N f\, dx \end{gathered} \tag{3.16}$$

satisfying

$$\frac{\partial}{\partial s}\frac{\partial u^i}{\partial t} - \frac{\partial}{\partial t}\frac{\partial u^i}{\partial s} = O\left(\epsilon^{N+1}\right), \quad i = 1, \ldots, n. \tag{3.17}$$

All these symmetries commute pairwise modulo terms of order ϵ^{N+1}.

The linear differential operator D_N giving an extension of the symmetries of the dispersionless system will be called *D-operator* for an N-integrable system. For an integrable system such an operator exists for any N; in this case we will omit the label N.

Example 3.3 *The D-operator for the KdV hierarchy has the form*

$$f \mapsto Df = \frac{1}{\sqrt{2}} \operatorname{res}\left(\partial^{1/2} f\right)(L), \quad L = \frac{\epsilon^2}{2}\partial_x^2 + u(x). \tag{3.18}$$

Here $f = f(u)$ is an arbitrary function. In particular choosing

$$f(u) = \frac{u^{k+2}}{(k+2)!}$$

one obtains the Hamiltonian densities of the KdV hierarchy

$$Df = \frac{1}{\sqrt{2}}\frac{2^{k+2}}{(2k+3)!!} \operatorname{res} L^{\frac{2k+3}{2}}.$$

Starting from the above definition we develop a "perturbative" approach to the study of integrability that can be used for

• finding obstructions to integrability;

• classification of integrable PDEs.

Example 3.4 *One-dimensional system of particles with neighboring interaction*

$$H = \sum \frac{1}{2} p_n^2 + P(q_n - q_{n-1}) \tag{3.19}$$

with the potential $P(u)$ (generalized Fermi - Pasta - Ulam system) after interpolation

$$\begin{aligned} q_n(t) - q_{n-1}(t) &= w(\epsilon\, n, \epsilon\, t) \\ p_n(t) &= v(\epsilon\, n, \epsilon\, t) \end{aligned} \tag{3.20}$$

and substitution

$$u = \frac{\epsilon\, \partial_x}{1 - e^{-\epsilon\, \partial_x}}\, w \tag{3.21}$$

the following system

$$\begin{aligned} u_t &= v_x \\ v_t &= \epsilon^{-1} \left[P'\left(\frac{e^{\epsilon\, \partial_x} - 1}{\epsilon\, \partial_x} u \right) - P'\left(\frac{1 - e^{-\epsilon\, \partial_x}}{\epsilon\, \partial_x} u \right) \right] \\ &= \partial_x P'(u) + \frac{\epsilon^2}{24} \left[2\, P''(u)\, u_{xxx} + 4\, P'''(u)\, u_x u_{xx} + P^{IV}(u)\, u_x^3 \right] + \mathcal{O}(\epsilon^4). \end{aligned} \tag{3.22}$$

The above formulae are understood as formal power series in ϵ:

$$w = \frac{e^{\epsilon\, \partial_x} - 1}{\epsilon\, \partial_x}\, u = \frac{1}{\epsilon} \int_x^{x+\epsilon} u(s)\, ds = u + \sum_{k \geq 1} \frac{\epsilon^k}{(k+1)!}\, u^{(k)},$$

$$u = w + \frac{1}{2}\, \epsilon\, w' + \sum_{k>1} \frac{B_k}{k!} \epsilon^k w^{(k)}, \tag{3.23}$$

B_k are the Bernoulli numbers. The equations (3.22) is a Hamiltonian system

$$\begin{aligned} u_t &= \partial_x \frac{\delta H}{\delta v(x)} \\ v_t &= \partial_x \frac{\delta H}{\delta u(x)} \end{aligned}$$

$$H = \int h\,dx = \int \left[\frac{1}{2}\,v^2(x) + P\left(w(x) - w(x-\epsilon)\right)\right]\,dx$$

$$h = \frac{1}{2}\,v^2 + P(u) - \frac{\epsilon^2}{24}\,P''(u)\,u_x^2 + \frac{\epsilon^4}{5760}\,\left[8\,P''(u)\,u_{xx}^2 - P^{IV}(u)\,u_x^4\right] + \mathcal{O}(\epsilon^6) \tag{3.24}$$

(modulo inessential total derivatives).

In the dispersionless limit $\epsilon \to 0$ (3.22) *reduces to the nonlinear wave equation written as a system*

$$\begin{aligned} u_t &= v_x \\ v_t &= \partial_x P'(u). \end{aligned} \tag{3.25}$$

So, the dispersionless system (3.25) *is integrable for an arbitrary potential* $P(u)$*. The perturbed system* (3.22) *is 2-integrable* iff *the potential* $P(u)$ *satisfies*

$$P''P^{IV} = (P''')^2$$

that is, only for

$$P(u) = k\,e^{c\,u} + a\,u + b$$

for some constants a, b, c, k [20]. So, the generalized FPU system (3.22) *is integrable only when it coincides with Toda lattice.*

Example 3.5 *The perturbed Riemann wave equation* (2.12) *is 5-integrable for an arbitrary choice of the functional parameters* $c(u)$*,* $p(u)$ *[19]. Indeed, the first integrals of the unperturbed system have the form*

$$F_0 = \int f(v)\,dx$$

for an arbitrary function $f(v)$*. Define deformed functionals by the formula*

$$F = \int D_{c,p} f\,dx$$

where the D-operator $D_5 \equiv D_{c,p}$ *(see [20] for details) transforming the first integrals of the unperturbed system to the first integrals (modulo* $O(\epsilon^6)$*) of the perturbed one*

$$\begin{aligned} D_{c,p} f = f - \frac{\epsilon^2}{24} c\,f''' u_x^2 + \epsilon^4 \Bigg[&\left(p\,f''' + \frac{c^2\,f^{(4)}}{480}\right)\,u_{xx}^2 \\ - &\left(\frac{c\,c''\,f^{(4)}}{1152} + \frac{c\,c'\,f^{(5)}}{1152} + \frac{c^2\,f^{(6)}}{3456} + \frac{p'\,f^{(4)}}{6} + \frac{p\,f^{(5)}}{6}\right)\,u_x^4 \Bigg] \end{aligned} \tag{3.26}$$

It is an interesting open problem to prove existence and uniqueness, for a given pair of the functional parameters $c(u)$, $p(u)$, of an extension to all orders in ϵ of the perturbed system (2.12) in order to obtain an integrable PDE. So far existence of such an extension is known only for some particular cases including

• KdV: $c(u) = \text{const}$, $p(u) = 0$.

• Volterra lattice

$$\dot{c}_n = c_n(c_{n+1} - c_{n-1}), \quad n \in \mathbb{Z}.$$

Here $c(u) = 2$, $p(u) = -\frac{1}{240}$.

• Camassa–Holm equation (1.4) has $c(u) = 8u$, $p(u) = \frac{1}{3}u$.

We will now consider a particular class of systems of *bihamiltonian* PDEs. They admit a Hamiltonian description with respect to two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ of the form (1.8) - (1.9) with two different Hamiltonians of the form (1.10):

$$u^i_t = \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2, \quad i = 1, \ldots, n. \tag{3.27}$$

The Poisson brackets must satisfy the *compatibility condition*: the linear combination

$$a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2$$

must be a Poisson bracket for arbitrary constant coefficients a_1, $a_2 \in \mathbb{R}$. We will now formulate additional assumptions that ensure integrability of the bihamiltonian system (3.27). Denote $g_1^{ij}(u)$ and $g_2^{ij}(u)$ the contravariant metrics of the form (2.4) associated with the Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ respectively.

Definition 3.6 *We say that the bihamiltonian structure* (3.27) *of the form* (1.9) is strongly nondegenerate *if none of the roots of the characteristic equation*

$$\det\left(g_2^{ij}(u) - \lambda\, g_1^{ij}(u)\right) = 0 \tag{3.28}$$

is a locally constant function on $M^n \ni u$. It is called semisimple *if these roots are pairwise distinct.*

Theorem 3.7 *Any system of PDEs admitting a bihamiltonian description with respect to a strongly nondegenerate semisimple bihamiltonian structure is integrable.*

Sketch of the proof. Denote $\lambda = w^1(u)$, ..., $\lambda = w^n(u)$ the roots of the characteristic equation (3.28). Under assumptions of the theorem these roots give a system of local coordinates on M^n [22, 23]. In these coordinates any bihamiltonan dispersionless system (3.9) becomes diagonal. This proves integrability of the dispersionless system.

Let us now construct a complete set of commuting bihamiltonian flows. Without loss of generality we may assume the metric g_1^{ij} to be constant in the coordinates u^1, ..., u^n. For a given $\lambda \in \mathbb{R}$ consider the generalized Miura transformation

$$u^i \mapsto \tilde{u}^i = F^i(u; u_x, \ldots; \epsilon; \lambda)$$

reducing the Poisson pencil

$$\{\ ,\ \}_2 - \lambda\,\{\ ,\ \}_1$$

to the constant form (2.2):

$$\{\tilde u^i(x), \tilde u^j(y)\}_2 - \lambda\,\{\tilde u^i(x), \tilde u^j(y)\}_1 = -\lambda g_1^{ij}\delta'(x-y).$$

Using triviality of the Poisson cohomology of this bracket (see above) one can prove [26] that the reducing transformation exists for sufficiently large $|\lambda|$. Moreover, it admits an expansion

$$\tilde u^i = u^i + \sum_{p\geq 1} \frac{F_p^i(u; u_x, u_{xx}, \dots; \epsilon)}{\lambda^p}, \quad i = 1, \dots, n$$

$$F_p^i(u; u_x, u_{xx}, \dots; \epsilon) = \sum_{k\geq 0} \epsilon^k F^i_{p,[k]}(u; u_x, \dots, u^{(k)}), \quad \deg F^i_{p,[k]}(u; u_x, \dots, u^{(k)}) = k.$$

The Hamiltonians

$$H_p^i = \int F_p^i(u; u_x, u_{xx}, \dots; \epsilon)\, dx, \quad i = 1, \dots, n, \quad p = 0,\, 1,\, 2, \dots$$

give a complete family of commuting bihamiltonian flows,

$$\{H_p^i, H_q^j\}_{1,2} = 0, \quad i,\, j = 1, \dots, n, \quad p,\, q = 0,\, 1,\, 2, \dots$$

Taking into account the previous theorems, we will now focus on the deformation theory of bihamiltonian PDEs. A compatible pair of Poisson brackets defines a pair of anticommuting differentials

$$\partial_1^2 = \partial_2^2 = \partial_1\partial_2 + \partial_2\partial_1 = 0$$

on the local multivectors on the loop space $\mathcal{L}(M^n)\otimes\mathbb{R}[[\epsilon]]$. Cohomologies of any of these differentials vanish in positive degrees in ϵ (see above). Define *bihamiltonian cohomology* by

$$H^k(\partial_1, \partial_2) = \frac{\mathrm{Ker}\,\partial_1\partial_2|_{\Lambda^{k-1}}}{(\mathrm{Im}\partial_1 + \mathrm{Im}\partial_2)_{\Lambda^{k-2}}}, \quad k \geq 2.$$

For $k = 1$ the denominator vanishes; for $k = 0$ the bihamiltonian cohomology is defined by

$$H^0(\partial_1, \partial_2) = \mathrm{Ker}\,\partial_1 \cap \mathrm{Ker}\,\partial_2.$$

In particular, the second bihamiltonian cohomology describes the infinitesimal deformation space of a given dispersionless bihamiltonian structure.

Let us first make a digression about dispersionless bihamiltonian structures (also called bihamiltonian structures of the hydrodynamic type)

$$\begin{aligned}\{v^i(x), v^j(y)\}_2 - \lambda\,\{v^i(x), v^j(y)\}_1 = \left(g_2^{ij}(v(x)) - \lambda\, g_1^{ij}(v(x))\right)\delta'(x-y)\\ + \left(\Gamma^{ij}_{k\,1}(v) - \lambda\,\Gamma^{ij}_{k\,2}(v)\right) v_x^k\,\delta(x-y).\end{aligned} \tag{3.29}$$

The metrics $g_1^{ij}(v)$ and $g_2^{ij}(v)$ form a so-called *flat pencil* [16], i.e., the contravariant Christoffel coefficients for the metric $g_2^{ij}(v) - \lambda\, g_1^{ij}(v)$ are equal to

$$\Gamma^{ij}_{k\,1}(v) - \lambda\, \Gamma^{ij}_{k\,2}(v)$$

where

$$\Gamma^{ij}_{k\,1}(v) = -g_1^{is}(v)\Gamma^j_{sk1}(v) \quad \text{and} \quad \Gamma^{ij}_{k\,2}(v) = -g_2^{is}(v)\Gamma^j_{sk2}(v)$$

are the contravariant Christoffel coefficients for the metrics $g_1^{ij}(v)$ and $g_2^{ij}(v)$ resp. Moreover the curvature of the metric $g_2^{ij}(v) - \lambda\, g_1^{ij}(v)$ must vanish identically in λ. Assuming the bihamiltonian structure (3.29) to be strongly nondegenerate and semisimple one can reduce [22, 33, 55] the theory of flat pencils of metrics to the study of compatibility conditions

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i,\, j,\, k \text{ distinct}$$

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i,\, j} \gamma_{ki}\gamma_{kj} = 0$$

$$w^i \partial_i \gamma_{ij} + w^j \partial_j \gamma_{ji} + \sum_{k \neq i,\, j} w^k \gamma_{ki}\gamma_{kj} + \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) = 0$$

of the following overdetermined system of linear differential equations with rational coefficients for an auxiliary vector function $\psi = (\psi_1(w), \dots, \psi_n(w))$

$$\begin{aligned} &\partial_i \psi_j = \gamma_{ji}\psi_i, \quad i \neq j \\ &\partial_i \psi_i + \sum_{k \neq i} \gamma_{ki} \frac{w^k - \lambda}{w^i - \lambda}\psi_k + \frac{1}{2\,(w^i - \lambda)}\psi_i = 0. \end{aligned} \tag{3.30}$$

Here $w^1, \dots, w^n$ are the roots of the characteristic equation (3.28); in these coordinates both the metrics become diagonal

$$\frac{\partial w^i}{\partial v^k}\frac{\partial w^j}{\partial v^l} g_1^{kl}(v) = g_1^{ii}(w)\delta^{ij}, \quad \frac{\partial w^i}{\partial v^k}\frac{\partial w^j}{\partial v^l} g_2^{kl}(v) = w^i g_1^{ii}(w)\delta^{ij}.$$

The coefficients $\gamma_{ij} = \gamma_{ij}(w)$ in (3.30) are the rotation coefficients of the first metric

$$\gamma_{ij}(w) := H_i^{-1}\partial_i H_j, \quad i \neq j, \quad H_i = \left(g_1^{ii}(w)\right)^{-1/2}. \tag{3.31}$$

To the best of our knowledge all nontrivial examples of flat pencils of metrics come from Frobenius manifolds (see below).

Let us now consider an ϵ-deformation of the Poisson pencil (3.29)

$$\begin{aligned} &\{u^i(x), u^j(y)\}_2 - \lambda\, \{u^i(x), u^j(y)\}_1 = \\ &= \left(g_2^{ij}(u(x)) - \lambda\, g_1^{ij}(u(x))\right)\delta'(x-y) + \left(\Gamma^{ij}_{k\,1}(u) - \lambda\, \Gamma^{ij}_{k\,2}(u)\right) u^k_x\, \delta(x-y) \\ &+ \sum_{k \geq 1} \epsilon^k \sum_{l=0}^{k+1} \left[A^{ij}_{k,l;2}(u(x); u_x, \dots, u^{(l)}) - \lambda\, A^{ij}_{k,l;1}(u(x); u_x, \dots, u^{(l)})\right]\, \delta^{(k-l+1)}(x-y) \\ &\deg A^{ij}_{k,l;a}(u; u_x, \dots, u^{(l)}) = l, \quad a = 1,\, 2. \end{aligned} \tag{3.32}$$

We begin with formulating an important *quasitriviality theorem* [22] saying that the bihamiltonian cohomology becomes trivial as soon as we extend the class of generalized Miura transformations allowing rational dependence on the jet coordinates.

Definition 3.8 *The bihamiltonian structure* (3.32) *is said to be* trivial *if it can be obtained from the leading term* (3.29) *by a λ-independent Miura-type transformation*

$$u^i = v^i + \sum_{k\geq 1} \varepsilon^k F_k^i(v; v_x, \dots, v^{(k)}), \tag{3.33}$$
$$\deg F_k^i(v; v_x, \dots, v^{(k)}) = k, \quad i = 1, \dots, n$$

whith the coefficients $F_k^i(v; v_x, \dots, v^{(k)})$ being graded homogeneous polynomials in the derivatives. It is called quasitrivial *if it is not trivial and there exists a transformation*

$$u^i = v^i + \sum_{k\geq 1} \varepsilon^k F_k^i(v; v_x, \dots, v^{(m_k)}) \tag{3.34}$$

reducing (3.32) *to* (3.29) *but the functions F_k^i depend* rationally *on the jet coordinates $v^{i,m}$, $m \geq 1$ with*

$$\deg F_k^i(v; v_x, \dots, v^{(m_k)}) = k, \quad i = 1, \dots, n, \quad k \geq 1 \tag{3.35}$$

and m_k are some positive integers. If such a transformation (3.33) *or* (3.34) *exists, it is called a* reducing transformation *of the bihamiltonian structure* (3.32).

Example 3.9 *For the Poisson pencil known in the theory of KdV hierarchy*

$$\{u(x), u(y)\}_2 - \lambda \{u(x), u(y)\}_1 = (u(x) - \lambda)\, \delta'(x-y) + \frac{1}{2}\, u_x \delta(x-y) + \frac{1}{8}\epsilon^2 \delta'''(x-y) \tag{3.36}$$

the reducing transformation reads

$$u = v - \frac{\epsilon^2}{12} (\log v_x)_{xx} + \epsilon^4 \left[\frac{v_{xxxx}}{288 v_x^2} - \frac{7 v_{xx} v_{xxx}}{480 v_x^3} + \frac{v_{xx}^3}{90 v_x^4}\right]_{xx} + O\left(\epsilon^6\right). \tag{3.37}$$

It is a canonical transformation

$$v \mapsto u = v + \epsilon\{v(x), K\} + \frac{\epsilon^2}{2}\{\{v(x), K\}, K\} + \dots$$

generated by the Hamiltonian

$$K = -\int \left[\frac{\epsilon}{24}\, v_x\, \log v_x + \frac{\epsilon^3}{5760} \frac{v_{xx}^3}{v_x^3} + O\left(\epsilon^5\right)\right] dx. \tag{3.38}$$

Theorem 3.10 (see [22][3]). *Any strongly nondegenerate semisimple bihamiltonian structure (3.32) is quasitrivial. The coefficients F_k^i of the reducing transformation (3.34) have the form*

$$F_k^i(v; v_x, \dots, v^{(m_k)}) \in C^\infty(M^n)\left[v_x, \dots, v^{(m_k)}\right]\left[\left(w_x^1 w_x^2 \dots w_x^n\right)^{-1}\right] \qquad (3.39)$$
$$m_k \le \left[\frac{3\,k}{2}\right].$$

Here $w^i = w^i(v)$ are the roots of the characteristic equation (3.28)*.*

The reducing transformation for the bihamiltonian structure (3.32) establishes a correspondence between solutions of any bihamiltonian system (3.27) admitting regular expansion in ϵ

$$u^i(x,t;\epsilon) = u_0^i(x,t) + \sum_{k\ge 1}\epsilon^k u_k^i(x,t), \quad i = 1, \dots, n$$

and monotone solutions to the dispersionless limit

$$\begin{aligned} &v_t^i = \{v^i(x), H_1^0\}_1^0 = \{v^i(x), H_2^0\}_2^0, \quad i = 1, \dots, n \\ &\{\ ,\ \}_{1,2}^0 := \{\ ,\ \}_{1,2}\big|_{\epsilon=0}, \quad H_{1,2}^0 := H_{1,2}\big|_{\epsilon=0}. \end{aligned} \qquad (3.40)$$

By definition the solution $v = v(x,t)$ is called monotone if

$$\partial_x w^i(v(x,t)) \neq 0, \quad i = 1, \dots, n$$

for all real x, t.

Therefore the problem of solving of any system of bihamiltonian PDEs of the above form can be reduced to solving *linear PDEs* (3.3) (see details in [22]).

We will now address the problem of classification of bihamiltonian structures (3.32) with a *given* dispersionless limit (3.29). First we will associate with any such a perturbation a collection of n functions of one variable called central invariants [50, 22, 23]. With any Poisson bracket of the form (1.9) we associate a matrix valued series in an auxiliary variable p:

$$\pi^{ij}(u;p) = \sum_{k\ge 0} A_{k,0}^{ij}(u) p^k. \qquad (3.41)$$

Recall that the coefficients $A_{k,0}^{ij}$ have degree 0 in the jet variables, so they may depend only on u. In this way for a bihamiltonian structure one obtains two matrix valued series $\pi_1^{ij}(u;p)$ and $\pi_2^{ij}(u;p)$. Recall that the leading terms of these series are $A_{0,0_1}^{ij}(u) = g_1^{ij}(u)$, $A_{0,0_2}^{ij}(u) = g_2^{ij}(u)$. Consider the characteristic equation

$$\det\left(\pi_2^{ij}(u;p) - \lambda\,\pi_1^{ij}(u;p)\right) = 0. \qquad (3.42)$$

[3] For $n = 1$ the proof of quasitriviality theorem was also obtained in [1]. Apparently the author of [1] was not aware about results of the paper [22]

The roots $\lambda^i(u;p)$, ..., $\lambda^n(u;p)$ have the form

$$\lambda^i(u;p) = \sum_{k\geq 0} \lambda^i_k(u) p^k, \quad \lambda^i_0(u) = w^i(u), \quad \lambda^i_k(u) = 0 \quad \text{for} \quad k = \text{odd}.$$

Put

$$c_i = \frac{1}{3} \frac{\lambda^i_2(u)}{g^{ii}_1(w)}, \quad i = 1, \ldots, n. \tag{3.43}$$

Definition 3.11 *The functions $c_i \in \mathcal{C}^\infty(M^n)$ are called* central invariants *of the bihamiltonian structure* (3.32).

Theorem 3.12 (see [50, 22, 23]). *1) The central invariant c_i is a function of one variable w^i, $i = 1, \ldots, n$.*

2). Two strongly nondegenerate semisimple bihamiltonian structures are equivalent iff *they have the same central invariants. In particular, the bihamiltonian structure is trivial* iff *it has all central invariants equal to zero.*

Example 3.13 *For the bihamiltonian structure* (3.36) *the central invariant is constant $c_1 = \frac{1}{24}$. For the bihamiltonian structure of the Camassa–Holm hierarchy*

$$\{u(x), u(y)\}_2 - \lambda \{u(x), u(y)\}_1 = (u(x) - \lambda)\, \delta'(x-y) + \frac{1}{2} u_x \delta(x-y) + \lambda \frac{\epsilon^2}{8} \delta'''(x-y)$$

the central invariant equals

$$c_1 = \frac{1}{24}\, w, \quad w = u.$$

So the KdV and Camassa–Holm hierarchies are not equivalent.

The theory of central invariants gives a description of the space of infinitesimal deformations of bihamiltonian systems of hydrodynamic type. It remains an open problem to prove vanishing of higher obstructions and establish existence of an integrable hierarchy with a given dispersionless limit and given central invariants. In the next section we will formulate some partial existence results for bihamiltonian PDEs associated with semisimple Frobenius manifolds.

4 Frobenius manifolds and integrable hierarchies of the topological type

Frobenius structures on M^n yield a particular subclass of bihamiltonian structure of hydrodynamic type on the loop space $\mathcal{L}(M^n)$. Recall [18] that a Frobenius structure

$(\ \cdot\ , e, <\ ,\ >, E, d)$ on M^n consists of a structure of a Frobenius algebra[4] $(x, y) \mapsto x \cdot y \in T_v M^n$ on the tangent spaces $T_v M^n = (A_v, <\ ,\ >_v)$ depending (smoothly, analytically etc.) on the point $v \in M^n$. It must satisfy the following axioms.

FM1. The curvature of the metric $<\ ,\ >_v$ on M^n vanishes. Denote ∇ the Levi-Civita connection for the metric. The unity vector field e must be flat, $\nabla e = 0$.

FM2. Let c be the 3-tensor $c(x, y, z) :=< x \cdot y, z >$, x, y, $z \in T_v M^n$. The 4-tensor $(\nabla_w c)(x, y, z)$ must be symmetric in x, y, z, $w \in T_v M^n$.

FM3. A vector field $E \in Vect(M^n)$ (called *Euler vector field*) must be fixed on M^n such that

$$Lie_E(x \cdot y) - Lie_E x \cdot y - x \cdot Lie_E y = x \cdot y$$

$$Lie_E <\ ,\ >= (2 - d) <\ ,\ >$$

for some number $d \in k$ called *charge*.

In the flat coordinates $v^1, \ldots, v^n$ for the metric $<\ ,\ >$ the structure constants of the algebra structure are locally given by triple derivatives of a function $F(v)$ called *potential* of the Frobenius manifold:

$$\left\langle \frac{\partial}{\partial v^i} \cdot \frac{\partial}{\partial v^j}, \frac{\partial}{\partial v^k} \right\rangle = \frac{\partial^3 F(v)}{\partial v^i \partial v^j \partial v^k}. \tag{4.1}$$

This function must satisfy WDVV associativity equations, including the quasihomogeneity condition

$$E\, F = (3 - d)F + \text{quadratic terms};$$

the Euler vector field E depends at most linearly on the flat coordinates, i.e. $\nabla\nabla E = 0$ (see details in [18]).

Using the metric $<\ ,\ >$ one also obtains an algebra structure on the cotangent planes $T_v^* M^n$. The two contravariant metrics (i.e., bilinear forms on M^n) are defined by the following formulae

$$\begin{aligned}(\omega_1, \omega_2)_1 &=< \omega_1, \omega_2 > \\ (\omega_1, \omega_2)_2 &= i_E(\omega_1 \cdot \omega_2).\end{aligned} \tag{4.2}$$

Remarkably this pair of metrics form a flat pencil. The commuting Hamiltonians of the associated bihamiltonian dispersionless hierarchy are expressed via the horizontal sections of the so-called deformed flat connection on M^n. Any choice of such a basis of horizontal sections gives a *calibration* of the Frobenius manifold.

One of equations of the integrable hierarchy on $\mathcal{L}(M^n)$ has a particularly simple form resembling the Riemann wave equation

$$\mathbf{v}_t + \mathbf{v} \cdot \mathbf{v}_x = 0, \quad \mathbf{v} = (v^1, \ldots, v^n) \in M^n \simeq T_{\mathbf{v}} M^n \tag{4.3}$$

[4]Recall that a commutative associative unital algebra A over a field k is called *Frobenius algebra* if it is equipped with a nondegenerate symmetric *invariant* k-bilinear form, i.e., $< x \cdot y, z >=< x, y \cdot z >$ for all x, y, $z \in A$.

where we identify the points of the manifold with the points in the tangent plane using the flat coordinates.

We arrive at the problem of *reconstruction* of the integrable hierarchy with the given dispersionless limit (4.3). This can be done for the case of *semisimple* Frobenius manifolds. By definition the Frobenius structure is called semisimple if the algebra structure on the tangent planes $T_v M^n$ is semisimple for all[5] $v \in M^n$. The bihamiltonian structure associated with the flat pencil of metrics (4.2) will be strongly nondegenerate semisimple *iff* the Frobenius manifold is semisimple.

Theorem 4.1 *For any calibrated semisimple Frobenius manifold structure on M^n there exists a unique integrable hierarchy*

$$\begin{aligned} &\frac{\partial u^i}{\partial t^{j,p}} = \partial_x \sum_{g\geq 0} \epsilon^{2g} K^i_{j,p;g}(u; u_x, \dots, u^{(2g)}), \quad i,\, j = 1, \dots, n, \quad p \geq 0 \\ &\deg K^i_{j,p;g}(u; u_x, \dots, u^{(2g)}) = 2g \end{aligned} \tag{4.4}$$

with the right hand sides polynomial in jets in every order in ϵ with the dispersionless term

$$\frac{\partial v^i}{\partial t^{j,p}} = \partial_x K^i_{j,p;0}(v) \tag{4.5}$$

defined as above by the Frobenius manifold and all central invariants equal to $1/24$.

The clue to the proof of this theorem [29] is in invariance of the equations of the hierarchy with respect to the Virasoro symmetries

$$\tau \mapsto \tau + \alpha\, L_m \tau + O(\alpha^2), \quad m \geq -1 \tag{4.6}$$

acting linearly onto the tau-function of the hierarchy. The tau-function

$$\tau = \tau(\mathbf{t}; \epsilon) = \exp \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t}), \tag{4.7}$$

$\mathbf{t} = (t^{i,p})_{1\leq i \leq n,\ p\geq 0}$ of any solution[6]

$$u^i(x, \mathbf{t}; \epsilon) = \sum_{g\geq 0} \epsilon^{2g} u^i_g(x, \mathbf{t}) \tag{4.8}$$

to the hierarchy satisfies

$$u^i(x, \mathbf{t}; \epsilon) = \epsilon^2 \eta^{ij} \frac{\partial^2}{\partial x \partial t^{j,0}} \log \tau(\mathbf{t}; \epsilon), \quad i = 1, \dots, n. \tag{4.9}$$

[5]Here we are considering only a small ball in the Frobenius manifold. Globally the Frobenius manifolds under consideration are only generically semisimple.

[6]More general solutions admitting regular expansions in ϵ are obtained from (4.8) by ϵ-dependent shifts $\mathbf{t} \mapsto \mathbf{t} - \mathbf{t}_0(\epsilon)$.

Existence of such a tau-function is the main reason for appearance of Frobenius manifolds in the theory of integrable hierarchies [26, 22, 29]. The Virasoro operators have the form [28]

$$\begin{aligned} L_m &= L_m(\epsilon^{-1}\mathbf{t}, \epsilon\partial/\partial\mathbf{t}) \\ &= \sum \epsilon^2 a_m^{i,p;j,q} \frac{\partial^2}{\partial t^{i,p}\partial t^{j,q}} + \sum b_{m\,j,q}^{\;\;i,p}\, t^{j,q} \frac{\partial}{\partial t^{i,p}} + \epsilon^{-2} c^m_{i,p;j,q}\, t^{i,p}\, t^{j,q} + d_0\delta_{m,0} \end{aligned} \tag{4.10}$$

where the constant d_0 and the constant coefficients $a_m^{i,p;j,q}$, $b_{m\,j,q}^{\;\;i,p}$, $c^m_{i,p;j,q}$ for every $m \geq -1$ depend on the Frobenius manifold. The hierarchy (4.4) is obtained from the known dispersionless limit (4.5) by a quasitriviality transformation of the form

$$v^i \mapsto u^i = v^i + \eta^{ij} \frac{\partial^2}{\partial x \partial t^{j,0}} \sum_{g\geq 0} \epsilon^{2g} F_g(v; v_x, \dots, v^{(3g-2)}) \tag{4.11}$$

where the terms of expansion are rational functions of jet variables of the degree

$$\deg F_g(v; v_x, \dots, v^{(3g-2)}) = 2g-2, \quad g \geq 2.$$

These terms are determined from the system of *Virasoro constraints* [26]. In particular, for $g = 1$ from this procedure one obtains the formula previously derived [28] from the universal identities [37] for the genus 1 Gromov–Witten invariants

$$F_1 = \frac{1}{24}\sum_{i=1}^n \log w_x^i - \log\left[\tau_I(w) J^{1/24}(w)\right], \quad J(w) = \det\left(\frac{\partial v^i}{\partial w^j}\right). \tag{4.12}$$

Here $\tau_I(w)$ is the so-called isomonodromy tau-function[7] of the Frobenius manifold.

Reducing the system of Virasoro constraints to the so-called *universal loop equation* [26, 18] one proves existence [29] and uniqueness [26] of the solution. Moreover, using Virasoro invariance one proves that the resulting hierarchy (4.4) is polynomial in jet variables in every order in ϵ.

Remark 4.2 *One can also prove that the conserved quantities of* (4.4) *obtained by applying the quasitriviality transformation* (4.11) *to the Hamiltonians of the dispersionless hierarchy* (4.5) *depend polynomially on the jet variables, in every order in* ϵ*. It remains to prove that also the coefficients of the resulting bihamiltonian structure depend polynomially on the jet variables.*

Definition 4.3 *The integrable hierarchy associated by the above construction with a given calibrated semisimple Frobenius manifold* M^n *is called* integrable hierarchy of the topological type.

[7]We have changed the sign in the definition [28] of the isomonodromy tau-function.

Let us describe the structure of solutions of an integrable hierarchy of the topological type. The *vacuum solution* $\tau_{\rm vac}(\mathbf{t};\epsilon)$ is defined by the system of Virasoro constraints

$$L_m \tau_{\rm vac}(\mathbf{t};\epsilon) = 0, \quad m \geq -1. \tag{4.13}$$

Any other solution to the hierarchy adimitting a regular expansion in ϵ is obtained by an ϵ-dependent shift $\mathbf{t} \mapsto \mathbf{t} - \mathbf{t}_0(\epsilon)$. In particular, the *topological solution* is specified by the so-called *dilaton shift*

$$\tau_{\rm top}(\mathbf{t};\epsilon) = \tau_{\rm vac}(\mathbf{t} - \mathbf{t}_{\rm dilaton};\epsilon), \quad t^{i,p}_{\rm dilaton} = \delta^i_1 \delta^p_1. \tag{4.14}$$

The corresponding topological solution (4.9) to the integrable hierarchy of the topological type admits an expansion

$$u^i_{\rm top}(x,\mathbf{t};\epsilon) = \sum_{g\geq 0} \epsilon^{2g} \sum_{m\geq 0} \sum_{\mathbf{j},\mathbf{p}} a^i_{\mathbf{j},\mathbf{p},g}(t^{1,0}+x, t^{2,0}, \dots, t^{n,0}) t^{j_1,p_1} \dots t^{j_m,p_m} \tag{4.15}$$

Here the summation over multiindices $\mathbf{j},\mathbf{p} = (j_1,\dots,j_m,p_1,\dots,p_m)$ extends over all values

$$1 \leq j_1,\dots,j_m \leq n, \quad 1 \leq p_1,\dots,p_m.$$

The coefficients of the expansion are given in terms of certain functions $a^i_{\mathbf{j},\mathbf{p},g}(v^1,\dots,v^n)$ smooth on the semisimple part $M_{ss} \subset M$ of the Frobenius manifold. Here M_{ss} is defined as the subset of the semisimple Frobenius manifold on which the operator of multiplication by the Euler vector field is regular semisimple. Note that for a generic semisimple Frobenius manifold the functions $a^i_{\mathbf{j},\mathbf{p},g}(v^1,\dots,v^n)$ have a complicated singularity at the origin.

Example 4.4 *For $n = 1$ one has only one Frobenius manifold with the potential $F(v) = \frac{1}{6}v^3$. The associated integrable hierarchy of the topological type coincides with the KdV hierarchy*

$$\begin{aligned} u_{t_0} &= u_x \\ u_{t_1} &= u\,u_x + \frac{\epsilon^2}{12} u_{xxx} \\ u_{t_2} &= \frac{1}{2}u^2 u_x + \frac{\epsilon^2}{12}(2u_x u_{xx} + u\,u_{xxx}) + \frac{\epsilon^4}{240} u^V, \quad \dots \end{aligned}$$

represented in the Lax form as follows:

$$\epsilon \frac{\partial L}{\partial t_k} = [A_k, L], \quad L = \frac{\epsilon^2}{2}\frac{d^2}{dx^2} + u, \quad A_k = \frac{2^{\frac{2k+1}{2}}}{(2k+1)!!}\left(L^{\frac{2k+1}{2}}\right)_+$$

The vacuum solution to the KdV hierarchy reads

$$\tau^{\rm KdV}_{\rm vac} = \frac{1}{(-t_1)^{1/24}} \exp\left\{ \frac{1}{\epsilon^2}\left[-\frac{t_0^3}{6\,t_1} - \frac{t_0^4 t_2}{24\,t_1^3} + O(t_0^5)\right] + \left[\frac{t_0 t_2}{24\,t_1^2} - \frac{t_0^2 t_3}{48\,t_1^3} + \frac{t_0^2 t_2^2}{24\,t_1^4} + O(t_0^3)\right] \right.$$
$$\left. +\epsilon^2\left[-\frac{t_4}{1152\,t_1^3} + \frac{29\,t_2 t_3}{5760\,t_1^4} - \frac{7\,t_2^3}{1440\,t_1^5} + O(t_0)\right] + O(\epsilon^4)\right\}. \tag{4.16}$$

After the dilaton shift $t_1 \mapsto t_1 - 1$ one obtains [69, 46] the generating function of the intersection numbers of the tautological classes $\psi_i = c_1(\mathcal{L}_i) \in H^2(\bar{\mathcal{M}}_{g,n})$ on the moduli spaces $\bar{\mathcal{M}}_{g,n}$ of stable algebraic curves

$$\log \tau_{\rm top}^{\rm KdV} = \frac{1}{\epsilon^2}\left(\frac{t_0^3}{6} + \frac{t_0^3\, t_1}{6} + \frac{t_0^3\, t_1^2}{6} + \frac{t_0^3\, t_1^3}{6} + \frac{t_0^3\, t_1^4}{6} + \frac{t_0^4\, t_2}{24} + \frac{t_0^4\, t_1\, t_2}{8}\right.$$
$$\left. + \frac{t_0^4\, t_1^2\, t_2}{4} + \frac{t_0^5\, t_2^2}{40} + \frac{t_0^5\, t_3}{120} + \frac{t_0^5\, t_1\, t_3}{30} + \frac{t_0^6\, t_4}{720} + \dots\right)$$
$$+\left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0\, t_2}{24} + \frac{t_0\, t_1\, t_2}{12} + \frac{t_0\, t_1^2\, t_2}{8} + \frac{t_0^2\, t_2^2}{24}\right.$$
$$\left. + \frac{t_0^2\, t_3}{48} + \frac{t_0^2\, t_1\, t_3}{16} + \frac{t_0^3\, t_4}{144} + \dots\right)$$
$$+\epsilon^2\left(\frac{7\, t_2^3}{1440} + \frac{7\, t_1\, t_2^3}{288} + \frac{29\, t_2\, t_3}{5760} + \frac{29\, t_1\, t_2\, t_3}{1440} + \frac{29\, t_1^2\, t_2\, t_3}{576} + \frac{5\, t_0 t_2^2 t_3}{144}\right.$$
$$+ \frac{29\, t_0\, t_3^2}{5760} + \frac{29\, t_0\, t_1\, t_3^2}{1152} + \frac{t_4}{1152} + \frac{t_1\, t_4}{384} + \frac{t_1^2\, t_4}{192} + \frac{t_1^3\, t_4}{96} + \frac{11\, t_0\, t_2\, t_4}{1440}$$
$$\left. + \frac{11\, t_0\, t_1\, t_2\, t_4}{288} + \frac{17\, t_0^2\, t_3\, t_4}{1920} + \dots\right) + O(\epsilon^4)$$
$$= \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g(\mathbf{t}), \quad \mathcal{F}_g(\mathbf{t}) = \sum \frac{1}{n!} t_{p_1}\dots t_{p_n} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{p_1} \wedge \dots \wedge \psi_n^{p_n}$$

The definition (4.9) of tau-function yields a familiar formula

$$u = \epsilon^2 \partial_x^2 \log \tau, \quad x = t_0$$

for solutions to the KdV hierarchy. The topological solution $u(x, \mathbf{t})$ can be also characterized by the initial data

$$u(x, 0; \epsilon^2) = x.$$

The Virasoro symmetries of the KdV hierarchy are generated by the operators

$$L_m = \frac{\epsilon^2}{2} \sum_{k+l=m-1} \frac{(2k+1)!!\,(2l+1)!!}{2^{m+1}} \frac{\partial^2}{\partial t_k \partial t_l} \tag{4.17}$$
$$+ \sum_{k\geq 0} \frac{(2k+2m+1)!!}{2^{m+1}(2k-1)!!}\, t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{16}\delta_{m,0}, \quad m \geq 0,$$
$$L_{-1} = \sum_{k\geq 1} t^k \frac{\partial}{\partial t_{k-1}} + \frac{1}{2\epsilon^2} t_0^2.$$

Example 4.5 *Choosing the shift vector in the form*

$$t_k^0 = \frac{(-1)^{k+1}}{k!\,(k-1)!}, \quad k \geq 1, \quad t_0 = 0$$

one obtains the generating function of the Weil–Petersson volumes of the moduli spaces

$$\log \tau_{\rm top}^{\rm KdV}(x, -1, \frac{1}{1!\,2!}, -\frac{1}{2!\,3!}, \dots; \epsilon^2) = \sum_{g=0}^{\infty} \left(\frac{\epsilon}{\pi^3}\right)^{2g-2} \sum_n Vol(\mathcal{M}_{g,n}) \left(\frac{x}{\pi^2}\right)^n. \qquad (4.18)$$

This is a reformulation of the result of P.Zograf and Yu.I.Manin [73, 51].

Example 4.6 *The hierarchy of the topological type associated with the two-dimensional Frobenius manifold with the potential*

$$F(u,v) = \frac{1}{2} u\, v^2 + e^u$$

coincides with the extended Toda hierarchy [8] associated with the difference Lax operator

$$L = \Lambda + v + e^u \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}.$$

The hierarchy contains two infinite sequences of time variables

$$\epsilon\, \frac{\partial L}{\partial t_k} = \frac{1}{(k+1)!} \left[(L^{k+1})_+, L \right], \quad \epsilon\, \frac{\partial L}{\partial s_k} = \frac{2}{k!} \left[\left(L^k (\log L - c_k) \right)_+, L \right]$$

$$c_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$$

In particular, for $k=0$ one obtains the standard Toda lattice equations (1.2) *written in the form* (1.3)*, $t = t_0$. The tau-function of a solution $u = u(\mathbf{s}, \mathbf{t}; \epsilon)$, $v = v(\mathbf{s}, \mathbf{t}; \epsilon)$ to the hierarchy is defined by*

$$u = \log \frac{\tau(s_0 + \epsilon) \tau(s_0 - \epsilon)}{\tau^2(s_0)}$$

$$v = \epsilon \frac{\partial}{\partial t_0} \log \frac{\tau(s_0+\epsilon)}{\tau(s_0)}$$

$x = s_0$ (see details in [8]). The Virasoro symmetries of the hierarchy are generated by the operators [27]

$$\begin{aligned} L_m &= \epsilon^2 \sum_{k=1}^{m-1} k!\,(m-k)! \frac{\partial^2}{\partial t_{k-1} \partial t_{m-k-1}} \\ &+ \sum_{k \geq 1} \frac{(m+k)!}{(k-1)!} \left(s_k \frac{\partial}{\partial s_{m+k}} + t_{k-1} \frac{\partial}{\partial t_{m+k-1}} \right) + 2 \sum_{k \geq 0} \alpha_m(k) s_k \frac{\partial}{\partial t_{m+k-1}}, \quad m > 0 \\ L_0 &= \sum_{k \geq 1} k \left(s_k \frac{\partial}{\partial s_k} + t_{k-1} \frac{\partial}{\partial s_{k-1}} \right) + \sum_{k \geq 1} 2\, s_k \frac{\partial}{\partial t_{k-1}} + \frac{1}{\epsilon^2} s_0^2 \\ L_{-1} &= \sum_{k \geq 1} \left(t_k \frac{\partial}{\partial t_{k-1}} + s_k \frac{\partial}{\partial s_{k-1}} \right) + \frac{1}{\epsilon^2} s_0\, t_0 \end{aligned}$$

$$\alpha_m(0) = m!, \quad \alpha_m(k) = \frac{(m+k)!}{(k-1)!} \sum_{j=k}^{m+k} \frac{1}{j}, \quad k > 0.$$

According to [36, 58, 27] for the topological solution to the extended Toda hierarchy the tau-function generates the Gromov–Witten invariants of $\mathbf{P}^1$ *and their descendents*

$$\log \tau_{\rm top}^{\rm Toda}(s_0, t_0, s_1, t_1, \dots; \epsilon^2) = \log \tau_{\rm vac}^{\rm Toda}(s_0, t_0, s_1 - 1, t_1, \dots; \epsilon^2) = \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{F}_g$$

$$\mathcal{F}_g = \sum \frac{1}{n!} t_{\alpha_1, p_1} \dots t_{\alpha_n, p_n} \int_{\left[\bar{\mathcal{M}}_{g,n}(\mathbf{P}^1, \beta)\right]} {\rm ev}_1^* \phi_{\alpha_1} \wedge \psi_1^{p_1} \wedge \dots \wedge {\rm ev}_n^* \phi_{\alpha_n} \wedge \psi_n^{p_n}.$$

Here $\phi_1 = 1 \in H^0(\mathbf{P}^1)$, $\phi_2 \in H^2(\mathbf{P}^1)$ *is a basis in the cohomology,* $\int_{\mathbf{P}^2} \phi_2 = 1$,

$$t_{1,p} = s_p, \quad t_{2,p} = t_p,$$

$$\bar{\mathcal{M}}_{g,n}(\mathbf{P}^1, \beta) = \left\{ f : (C_g, x_1, \dots, x_n) \to \mathbf{P}^1, \ \beta = \textit{degree of the map } f \right\}$$

are the moduli spaces of stable maps with values in the complex projective line.

Example 4.7 *Toda hierarchy and enumeration of ribbon graphs/triangulations of Riemann surfaces. A different choice of a shift*[8] *in the vacuum tau-function of the extended Toda hierarchy gives*

$$\begin{aligned} &\log \tau_{\rm vac}^{\rm Toda}(s_0, t_0, s_1, t_1 - 1, s_2, t_2, \dots; \epsilon)|_{t_0 = t_1 = 0, \quad t_k = (k+1)! \lambda_{k+1}; \quad s_0 = x, \ s_k = 0} \\ &= \frac{x^2}{2\epsilon^2} \left(\log x - \frac{3}{2} \right) - \frac{1}{12} \log x + \zeta'(-1) \\ &+ \sum_{g\geq 2} \left(\frac{\epsilon}{x} \right)^{2g-2} \frac{B_{2g}}{2g(2g-2)} + \sum_{g\geq 0} \epsilon^{2g-2} F_g(x; \lambda_3, \lambda_4, \dots) \end{aligned} \tag{4.19}$$

where B_{2g} *are Bernoulli numbers,* $\zeta(s)$ *the Riemann zeta-function,*

$$F_g(x; \lambda_3, \lambda_4, \dots) = \sum_n \sum_{k_1, \dots, k_n} a_g(k_1, \dots, k_n) \lambda_{k_1} \dots \lambda_{k_n} x^h,$$

$$h = 2 - 2g - \left(n - \frac{|k|}{2} \right), \quad |k| = k_1 + \dots + k_n,$$

generate the numbers of fat graphs

$$a_g(k_1, \dots, k_n) = \sum_\Gamma \frac{1}{\#\, {\rm Sym}\, \Gamma}$$

where

Γ = a connected fat graph of genus g with n vertices of the valencies $k_1, \dots, k_n$,

[8]One can show that the new shift corresponds to the topological tau-function of the extended nonlinear Schrödinger hierarchy [27]. The tau-function of the latter is obtained from the tau-function of the extended Toda hierarchy by a permutation of times $t_p \leftrightarrow s_p$, $p \geq 0$.

Sym Γ *is the symmetry group of the graph. E.g., for genus 1, one vertex of valency 4 the unique graph is shown on the picture (borrowed from [3])*

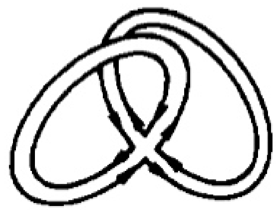

Fig. 1. Fat graph of genus 1 with one vertex of valency 4

So,

$$
\begin{aligned}
F = \epsilon^{-2} \Big[& \frac{1}{2}x^2\left(\log x - \frac{3}{2}\right) + 6x^3{\lambda_3}^2 + 2x^3\lambda_4 + 216x^4{\lambda_3}^2\lambda_4 + 18x^4{\lambda_4}^2 \\
&+288x^5{\lambda_4}^3 + 45x^4\lambda_3\lambda_5 + 2160x^5\lambda_3\lambda_4\lambda_5 + 90x^5{\lambda_5}^2 + 5400x^6\lambda_4{\lambda_5}^2 + 5x^4\lambda_6 \\
&+1080x^5{\lambda_3}^2\lambda_6 + 144x^5\lambda_4\lambda_6 + 4320x^6{\lambda_4}^2\lambda_6 + 10800x^6\lambda_3\lambda_5\lambda_6 + 27000x^7{\lambda_5}^2\lambda_6 \\
&+300x^6{\lambda_6}^2 + 21600x^7\lambda_4{\lambda_6}^2 + 36000x^8{\lambda_6}^3\Big] - \frac{1}{12}\zeta'(-1) - \frac{1}{12}\log x \\
&+\frac{3}{2}x{\lambda_3}^2 + x\lambda_4 + 234x^2{\lambda_3}^2\lambda_4 + 30x^2{\lambda_4}^2 + 1056x^3{\lambda_4}^3 + 60x^2\lambda_3\lambda_5 \\
&+6480x^3\lambda_3\lambda_4\lambda_5 + 300x^3{\lambda_5}^2 + 32400x^4\lambda_4{\lambda_5}^2 + 10x^2\lambda_6 + 3330x^3{\lambda_3}^2\lambda_6 \\
&+600x^3\lambda_4\lambda_6 + 31680x^4{\lambda_4}^2\lambda_6 + 66600x^4\lambda_3\lambda_5\lambda_6 + 283500x^5{\lambda_5}^2\lambda_6 \\
&+2400x^4{\lambda_6}^2 + 270000x^5\lambda_4{\lambda_6}^2 + 696000x^6{\lambda_6}^3 \\
&+\epsilon^2\Big[-\frac{1}{240x^2} + 240x{\lambda_4}^3 + 1440x\lambda_3\lambda_4\lambda_5 + \frac{1}{2}165x{\lambda_5}^2 + 28350x^2\lambda_4{\lambda_5}^2 \\
&+675x{\lambda_3}^2\lambda_6 + 156x\lambda_4\lambda_6 + 28080x^2{\lambda_4}^2\lambda_6 + 56160x^2\lambda_3\lambda_5\lambda_6 + 580950x^3{\lambda_5}^2\lambda_6 \\
&+2385x^2{\lambda_6}^2 + 580680x^3\lambda_4{\lambda_6}^2 + 2881800x^4{\lambda_6}^3\Big] + \dots
\end{aligned}
$$

The proof uses Toda equations [35, 34] for the Hermitean matrix integral [3, 52]

$$
\begin{aligned}
Z_N(\lambda;\epsilon) &= \frac{1}{\mathrm{Vol}(N)}\int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr}\,V(A)}\,dA \\
&= \tau_{\mathrm{vac}}(s_0, t_0, s_1, t_1 - 1, s_2, t_2, \dots; \epsilon)|_{t_0=t_1=0,\quad t_k=(k+1)!\lambda_{k+1};\quad s_0=x,\ s_k=0}
\end{aligned}
\tag{4.20}
$$

$$
V(A) = \frac{1}{2}A^2 - \sum_{k\geq 3}\lambda_k A^k, \quad dA = \prod_{i=1}^N dA_{ii}\prod_{i<j} d\mathrm{Re}A_{ij}d\mathrm{Im}A_{ij}
$$

understood as a formal saddle point expansion near the Gaussian point $\lambda_3 = \lambda_4 = \cdots = 0$ where one has to replace [65]

$$N \mapsto \frac{x}{\epsilon}$$

expanding the normalizing factor

$$\mathrm{Vol}(N) = \frac{2^{N/2}\pi^{\frac{N^2}{2}}\epsilon^{-\frac{N^2}{2}+\frac{1}{12}}}{\prod_{k=0}^{N-1} k!}$$

(related to the volume of the quotient of the unitary group $U(N)$ over the maximal torus $[U(1)]^N$) in the asymptotic series with the help of the asymptotic expansion of Barnes G-function [2]. Observe that the solution $u = u(x, \mathbf{t}, \mathbf{s}; \epsilon)$, $v = v(x, \mathbf{t}, \mathbf{s}; \epsilon)$ associated with the tau-function (4.19) can be characterized by the initial data

$$e^{u(x,0,0;\epsilon)} = x, \quad v(x, 0, 0; \epsilon) = 0$$

in agreement with the three term recursion relation

$$2zH_n(z) = H_{n+1}(z) + 2nH_{n-1}(z)$$

for Hermite polynomials.

For a convergent matrix integral the formal expansion (4.19) coincides with the asymptotic expansion of the integral in the so-called one-cut case, i.e., under the assumption that the large N distribution of the eigenvalues of the Hermitean random matrix A consists of a single interval [30, 31, 4]. The phase transitions from the one-cut to multi-cut behavior can be considered in the general setting of Universality Conjectures of the theory of Hamiltonian PDEs (see below).

Example 4.8 *The Drinfeld–Sokolov construction [14] associates a hierarchy of bihamiltonian integrable systems of the form* (1.8) - (1.10), (2.1), (3.27) *with every untwisted Lie algebra* $\hat{\mathfrak{g}}$. *The associated Frobenius manifold is isomorphic [23] to the one obtained in [15] (for the more general case of orbit spaces of a finite Coxeter group) as the natural polynomial Frobenius structure on the orbit space*

$$M^n = \mathfrak{h}/W(\mathfrak{g})$$

of the Weyl group. Here n *is the rank of the simple Lie algebra* $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ *is the Cartan subalgebra. The suitably ordered central invariants of the Drinfeld – Sokolov bihamiltonian structure for an untwisted affine Lie algebra* $\hat{\mathfrak{g}}$ *are given by the formula [23]*

$$c_i = \frac{1}{48}\langle \alpha_i^\vee, \alpha_i^\vee \rangle_{\mathfrak{g}}, \quad i = 1, \dots, n, \tag{4.21}$$

where $\alpha_i^\vee \in \mathfrak{h}$ *are the coroots of the simple Lie algebra* $\mathfrak{g}$. *Here* $\langle\ ,\ \rangle_{\mathfrak{g}}$ *is the normalized Killing form,*

$$\langle a, b \rangle_{\mathfrak{g}} := \frac{1}{2h^\vee}\mathrm{tr}(\mathrm{ad}\, a \cdot \mathrm{ad}\, b), \tag{4.22}$$

where $h^\vee$ is the dual Coxeter number. Thus, the Drinfeld–Sokolov hierarchy is equivalent to an integrable hierarchy of the topological type only for simply laced simple Lie algebras $\mathfrak{g}$.

We will not discuss here other examples of integrable hierarchies of the topological type. For convenience of the reader we give a list of some important examples[9] of Frobenius manifolds and the associated known, but not only, integrable hierarchies of the topological type[10]. Some of the applications to Gromov–Witten invariants and oscillatory asymptotics mentioned in this table still exist only as conjectures.

n	Potential F of Frobenius manifold	Hierarchy
1	$\frac{1}{6}v^3$	KdV
2	$\frac{1}{2}u\,v^2 + u^4$	Boussinesq
2	$\frac{1}{2}u\,v^2 + e^u$	Toda
2	$\frac{1}{2}u\,v^2 + \frac{1}{2}u^2\left(\log u - \frac{3}{2}\right)$	NLS
2	$\frac{1}{2}u\,v^2 - Li_3\left(e^{-u}\right)$	Ablowitz–Ladik
3	$\frac{1}{2}(u\,w^2 + u^2 v) + \frac{1}{6}v^2 w^2 + \frac{1}{60}w^5$	Drinfeld–Sokolov hierarchy of A_3 type, intersection theory on the moduli spaces of “spin 3 curves” [70, 32]
3	$\frac{1}{2}(u\,v^2 + v\,w^2) - \frac{1}{24}w^4 + 4w\,e^u$	A generalization of Toda hierarchy [7] for a difference Lax operator of bidegree (2,1); orbifold Gromov–Witten invariants of an orbicurve with one point of order 2 [54]
3	$\frac{1}{2}(\tau\,v^2 + v\,u^2) - \frac{i\pi}{48}u^4 E_2(\tau)$	Higher corrections to elliptic Whitham asymptotics, the KdV case
4	$\frac{i}{4\pi}\tau\,v^2 - 2\,u\,v\,w + u^2\log\left[\frac{\pi}{u}\frac{\theta_1'(0\|\tau)}{\theta_1(2w\|\tau)}\right]$	Higher corrections to elliptic Whitham asymptotics, the Toda/NLS case

Table 1. List of some Frobenius manifolds and the associated integrable hierarchies of the topological type

Recall that the potential of a generic semisimple Frobenius manifold of dimension $n \geq 3$ is expressed via solutions of certain monodromy preserving deformation equations (for $n = 3$ reducing to Painlevé-VI transcendents). So, the coefficients of a generic integrable hierarchy of the topological type will be expressed via these transcendents. The hierarchies shown in Table 1 correspond to particular solutions to the monodromy preserving deformation equations reducing to classical functions.

[9]We do not consider here an interesting example of the hierarchy, obtained by a nonstandard reduction [39] of the 2D Toda involved in the description of the equivariant GW invariants [59] of $\mathbf{P}^1$. It remains to better understand the place of this hierarchy in our general framework.

[10]Strictly speaking the example of Ablowitz–Ladik hierarchy does not fit into the general scheme as the function F does not satisfy the quasihomogeneity condition. Nevertheless, the Ablowitz–Ladik hierarchy possesses many properties of integrable hierarchies of the topological type. We will consider this example in a separate publication.

Remark 4.9 *Very recently the theory of Gromov–Witten invariants of orbicurves with polynomial quantum cohomology has been addressed by P.Rossi [60]. (Previously the theory of Gromov–Witten invariants of the same orbicurves has been analyzed by A.Takahashi from the point of view of homological mirror symmetry [62].) Rossi proved that for all these orbicurves*[11] *the associated Frobenius manifold coincides with the one defined by Y.Zhang and the author in [25] on the orbit spaces of simply laced extended affine Weyl groups. It would be interesting to obtain a realization of the associated integrable hierarchies of the topological type and relate it with the higher genus orbifold Gromov–Witten invariants and their descendents.*

At the end of this section we will explain a connection [29] of the theory of integrable hierarchies of the topological type with A.Givental's theory of the so-called total descendent potential [40] associated with an arbitrary semisimple Frobenius manifold.

Let H be a n-dimensional linear space equipped with a symmetric nondegenerate bilinear form $< \ , \ >$. Denote $\mathcal{H}$ the *Givental symplectic space* of the H-valued functions on the unit circle $|z| = 1$ that can be extended to an analytic function in an annulus. A symplectic structure on $\mathcal{H}$ is defined by the formula

$$\omega(f, g) := \frac{1}{2\pi i} \oint_{|z|=1} < f(-z), g(z) > dz. \tag{4.23}$$

A natural polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{4.24}$$

is given by the subspaces $\mathcal{H}_+/\mathcal{H}_-$ of functions that can be continued analytically inside/outside the unit circle (the functions in $\mathcal{H}_-$ must also vanish at $z = \infty$). Explicitly the canonical coordinates are given by the components $q^{\alpha,k}$, $p_{\alpha,k}$ of the coefficients of the Laurent expansion

$$f(z) = \cdots - p_2^* z^2 + p_1^* z - p_0^* + \frac{q^0}{z} + \frac{q^1}{z^2} + \ldots \tag{4.25}$$

Here we consider $q^k \in H$, $p_k \in H^*$,

$$q^k = q^{\alpha,k} e_\alpha, \quad p_k = p_{\alpha,k} e^\alpha, \tag{4.26}$$

$e_1, \ldots, e_n$ is a basis in H, $e^1, \ldots, e^n$ is the dual basis in H^*.

Any matrix valued function $G(z)$ on the unit circle $|z| = 1$ with values in $Aut(H)$ satisfying

$$G^*(-z)G(z) = 1 \tag{4.27}$$

defines a symplectomorphism

$$G : \mathcal{H} \to \mathcal{H}, \quad f(z) \mapsto G(z)f(z), \quad \omega(Gf, Gg) = \omega(f, g). \tag{4.28}$$

[11] The class of orbicurves considered in [60] includes the orbicurves with two singularities. For this subclass the relationship with the extended affine Weyl groups of A type has been established by T.Milanov and H.-H.Tseng [54].

Quantising the symplectomorphism (4.28) one obtains a *quantum canonical transformation* $\hat{G}$ acting on the Fock space $S^\bullet\mathcal{H}_-$ of functionals on the space $\mathcal{H}_-$ of vector-valued functions

$$q(z) = \frac{q_0}{z} + \frac{q_1}{z^2} + \dots, \quad q_k \in H, \quad |z| > 1 \tag{4.29}$$

analytic on the exterior part of the unit circle. The Fock space can be realized by polynomials in an infinite sequence of variables $t^{i,k}$, $i = 1, \dots, n$, $k \geq 0$. The operators $\hat{q}^{i,k}$ act on the Fock space by multiplication by $\epsilon^{-1}t^{i,k}$ and the operators $\hat{p}_{i,k}$ act by differentiation

$$\hat{q}^{i,k} f(\mathbf{t}) = \epsilon^{-1} t^{i,k} f(\mathbf{t}), \quad \hat{p}_{i,k} = \epsilon \frac{\partial}{\partial t^{i,k}} f(\mathbf{t}). \tag{4.30}$$

The quantization of $\hat{G}$ can be easily achieved in case the logarithm $g(z) = \log G(z)$ is well defined. Indeed, let us consider the quadratic Hamiltonian

$$H_g = \frac{1}{4\pi i} \oint_{|z|=1} \langle f(-z), g(z)\, f(z) \rangle \, dz = \frac{1}{2} p\, A\, p^* + q^* B\, p^* + \frac{1}{2} q^* C\, q \tag{4.31}$$

for some semiinfinite matrices A, B, C. The symplectomorphism G is the time 1 shift generated by the Hamiltonian H_g. Put

$$\hat{G} := e^{\hat{H}_g} \tag{4.32}$$

where

$$\hat{H}_g =: H_g(\hat{p}, \hat{q}) := \frac{1}{2} \epsilon^2 \frac{\partial}{\partial t} A \left(\frac{\partial}{\partial t} \right)^* + t^* B \left(\frac{\partial}{\partial t} \right)^* + \frac{1}{2\epsilon^2} t^* C\, t \tag{4.33}$$

is the standard normal ordering quantization of the quadratic Hamiltonian.

A more general situation occurs when the function $G(z)$ admits a Riemann–Hilbert factorization

$$G(z) = G_0^{-1}(z) G_\infty(z), \quad |z| = 1 \tag{4.34}$$

where the matrix valued functions $G_0(z)$ and $G_\infty(z)$ are analytic and invertible for $|z| < 1$ and $1 < |z| \leq \infty$ resp. The solution, if exists, is uniquely determined by the normalization condition $G_\infty(\infty) = 1$. The logarithms $g_0(z) = \log G_0(z)$ and $g_\infty(z) = \log G_\infty(z)$ are obviously well defined. Therefore one obtains the quantized operators $\hat{G}_0$ and $\hat{G}_\infty$ by applying the formula (4.32). Put

$$\hat{G} := \gamma_G \hat{G}_0^{-1} \hat{G}_\infty \tag{4.35}$$

for a suitable multiplier γ_G.

The Givental's formula expresses the so-called total descendent potential as the result of action of certain quantum canonical transformation onto a particular element of the (completed) Fock space. The latter is chosen in the form of a product of n copies of vacuum tau-functions (4.16) of the KdV hierarchy

$$\tau_{\rm KdV}^{\rm vac}(\mathbf{t}^1; \epsilon) \dots \tau_{\rm KdV}^{\rm vac}(\mathbf{t}^n; \epsilon). \tag{4.36}$$

Here

$$\mathbf{t}^i = (t^{i,0}, t^{i,1}, t^{i,2}, \dots), \quad i = 1, \dots, n.$$

The last step of the Givental's construction is in the choice of the symplectomorphism $G(z)$. At this point one has to use the parametrization of semisimple Frobenius manifolds by the data of certain Riemann–Hilbert problem [17, 18]. Reducing the Riemann–Hilbert problem to the standard form (4.34) one obtains a matrix valued function $G_w(z)$ on the unit circle satisfying (4.27) depending on the point $w \in M^n$ of the Frobenius manifold, and also depending on $n(n-1)/2$ *monodromy data* (the moduli of semisimple Frobenius manifolds; see details in [18]). Givental proves that the result of action of the quantized canonical transformation $\hat{G}_w = \gamma_G \hat{G}_0^{-1}(w)\hat{G}_\infty(w)$ on the vector (4.36) is well defined in every order in ϵ. Moreover, he proves that the function

$$\hat{G}_w \tau_{\text{KdV}}^{\text{vac}}(\mathbf{t}^1; \epsilon) \dots \tau_{\text{KdV}}^{\text{vac}}(\mathbf{t}^n; \epsilon) \tag{4.37}$$

does not depend on the choice of the semisimple point w when choosing

$$\gamma_G = \tau_I^{-1}(w)$$

the multiplier in the quantization formula (4.35).

Theorem 4.10 *For an arbitrary semisimple Frobenius manifold the function* (4.37) *is the vacuum tau-function for the integrable hierarchy of the topological type associated with the Frobenius manifold. The Givental's total descendent potential is the tau-function of the topological solution to the hierarchy obtained by the dilaton shift* (4.14)

Proof is based on the representation of the Givental's formula in the form

$$\hat{G}_w \tau_{\text{KdV}}^{\text{vac}}(\mathbf{t}^1; \epsilon) \dots \tau_{\text{KdV}}^{\text{vac}}(\mathbf{t}^n; \epsilon) = \exp\left[\frac{1}{\epsilon^2}\mathcal{F}_0 + \sum_{g \ge 1} \epsilon^{2g-2} F_g(v; v_x, \dots, v^{(3g-2)})\right]$$

where

$$v^i = \eta^{ij} \frac{\partial^2 \mathcal{F}_0}{\partial x \partial t^{j,0}}, \quad i = 1, \dots, n$$

using validity of the Virasoro constraints for this function proven in [40]. The theorem then follows from the uniqueness of the solution to the system of Virasoro constraints [26].

Corollary 4.11 *Consider the semisimple Frobenius manifold $M^n_{\mathbf{P}^N} = QH^*(\mathbf{P}^N)$, $n = N+1$, given by the quantum cohomology of the N-dimensional complex projective space. Then the total Gromov–Witten potential $\mathcal{F}^{\mathbf{P}^N}(\mathbf{t}; \epsilon)$ (see the formula* (4.38) *below) is equal to the logarithm of the topological tau-function of the integrable hierarchy of the topological type associated with $M^n_{\mathbf{P}^N}$.*

Recall (see above) that for $N = 0$ the hierarchy in question coincides with KdV, for $N = 1$ this is the extended Toda hierarchy; starting from $N = 2$ the integrable hierarchy is a new gadget of the theory of integrable systems.

Remark 4.12 *The action of the Givental operator* $\hat{G}_\infty(w)$ *on the product of* topological *tau-functions*

$$\hat{G}_\infty(w)\tau^{\rm top}_{\rm KdV}(\mathbf{t}^1;\epsilon)\dots\tau^{\rm top}_{\rm KdV}(\mathbf{t}^n;\epsilon)$$

is also well defined. Moreover, the result is a power series with respect to the new time variables. The coefficients of these power series depend on the point $w \in M$ *of the Frobenius manifold. These series play an important role in the classification of semisimple cohomological field theories obtained by C.Teleman [63].*

The above considerations suggest that the intrinsic structure of integrable hierarchies of the topological type is closely related to the topology of the Deligne–Mumford spaces. Let us formulate a more precise conjecture about such a relation. Among all differential equations for the total Gromov –Witten potential

$$\mathcal{F}^X(\mathbf{t};\epsilon) = \sum_{g\geq 0}\epsilon^{2g-2}\mathcal{F}^X_g(\mathbf{t}) \tag{4.38}$$

$$\mathcal{F}^X_g(\mathbf{t}) = \sum_m \sum_{\beta\in H_2(X;\mathbf{Z})}\frac{1}{m!}t^{\alpha_1,p_1}\dots t^{\alpha_m,p_m} < \tau_{p_1}(\phi_{\alpha_1})\dots\tau_{p_m}(\phi_{\alpha_m}) >_{g,\beta}$$

$$< \tau_{p_1}(\phi_{\alpha_1})\dots\tau_{p_m}(\phi_{\alpha_m}) >_{g,\beta} := \int_{[X_{g,m,\beta}]^{\rm virt}} {\rm ev}_1^*(\phi_{\alpha_1})\wedge c_1^{p_1}(\mathcal{L}_1)\wedge\dots\wedge{\rm ev}_m^*(\phi_{\alpha_m})\wedge c_1^{p_m}(\mathcal{L}_m)$$

$$X_{g,m,\beta} := \{f\ :\ (C_g, x_1,\dots,x_m)\to X,\quad f_*[C_g] = \beta\in H_2(X;\mathbb{Z})\}$$

$${\rm ev}_i : X_{g,m,\beta}\to X,\quad {\rm ev}_i(f) = f(x_i)$$

of a smooth projective variety X the *universal* identities are of particular interest. By definition they are those relations between Gromov–Witten invariants and their descendents

$$< \tau_{p_1}(\phi_{\alpha_1})\dots\tau_{p_m}(\phi_{\alpha_m}) >_{g,\beta}$$

that do not depend on X. According to the ideas of Y.-P.Lee [48, 49] the above universal relations are determined by the tautological ring of Deligne–Mumford spaces $\bar{M}_{g,n}$. For example, in genus 0 one has the already familiar WDVV equations. Also the topological recursion relations for the descendents, and also the Getzler's universal identities [37] for genus 1 GW invariants etc. An example of non-universal differential equations for $\mathcal{F}^X$ is given by the Virasoro constraints. The coefficients of the Virasoro operators depend on the classical cohomology ring of X together with the first Chern class $c_1(X)$.

Let us proceed to formulation of our main conjecture that relates the theory of integrable PDEs with the theory of Gromov–Witten invariants and their descendents.

For a smooth projective X denote $H^{\rm alg}(X) \subset H^*(X)$ the subspace generated by (k,k) forms (we do not impose the restriction $H^{\rm odd}(X)=0$). Introduce the differential ideal $I_{GW}^{\rm alg}(n)$ generated by polynomial identities for the derivatives of the form

$$\langle\!\langle \tau_{p_1}(\phi_{\alpha_1})\dots\tau_{p_m}(\phi_{\alpha_m})\rangle\!\rangle_g = \frac{\partial^m}{\partial t^{\alpha_1,p_1}\dots\partial t^{\alpha_m,p_m}}\mathcal{F}^X$$

with

$$\phi_{\alpha_i} \in H^{\rm alg}(X), \quad i=1,\dots,m$$

for all $m\geq 0$ valid for *all* X with $\dim H^{\rm alg}(X)=n$.

Another differential ideal $I_{KdV}(n)$ is generated by polynomial differential equations for the logarithmic derivatives of the tau-function $\tau_{\rm top}$ valid for *an arbitrary* n-dimensional semisimple Frobenius manifold M^n.

Conjecture 4.13 *For any $n\geq 0$*

$$I_{KdV}(n) = I_{GW}^{\rm alg}(n).$$

5 Critical behaviour in Hamiltonian PDEs, the dispersionless case

The new integrable hierarchies described in the previous section are written as infinite formal expansions in ϵ. For practical applications of PDEs of this type one has to truncate them at some order in ϵ. The natural question arises: how do the properties of solutions depend on the truncation order? What part of these properties is *universal*, i.e., independent of the choice of a generic solution and, possibly, of the choice of a Hamiltonian PDE?

The idea suggested by the author in [19] is to classify the types of *critical behavior* of solutions to Hamiltonian PDEs. By definition this is the behavior of a solution to the Hamiltonian PDE near the points of weak singularities (also called gradient catastrophes) of the dispersionless limit of the PDE. The idea of universality suggests that, up to simple transformations there exists only finite number of types of critical behavior.

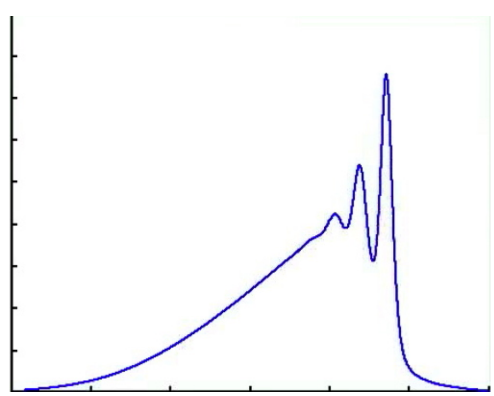

Fig. 2. Critical behavior in the KdV equation, cf. [72, 42]

In the present section we will briefly describe the local structure of gradient catastrophes for the systems of first order Hamiltonian PDEs. In the next section we will formulate and discuss the universality conjectured for Hamiltonian perturbations of these PDEs.

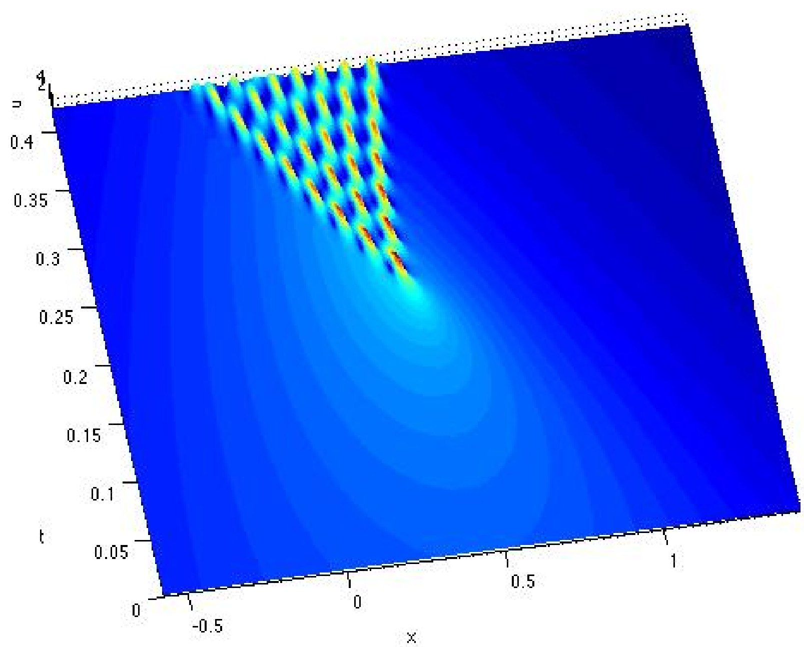

Fig. 3. Critical behavior in the focusing NLS equation; the graph of $u = |\psi|^2$ is shown

Solutions to hyperbolic systems typically have a finite life time. Let us begin with considering the simplest situation of the scalar nonlinear transport equation

$$v_t + a(v)v_x = 0. \tag{5.1}$$

As in Example 2.3 the equation (5.1) can be considered as an integrable Hamiltonian system with the Hamiltonian and Poisson bracket of the form

$$H_f^0 = \int f(v)\, dx, \quad f''(v) = a(v), \quad \{v(x), v(y)\} = \delta'(x-y). \tag{5.2}$$

The solution $v = v(x,t)$ to the Cauchy problem $v(x,0) = v_0(x)$ for (5.1) exists till the time $t = t_0$ of *gradient catastrophe*. At this point $x = x_0$, $t = t_0$, $v = v_0$,

$$v(x,t) \to v_0, \quad v_x(x,t) \to \infty \quad \text{for } (x,t) \to (x_0, t_0), \quad t < t_0.$$

The following statement is well known.

Theorem 5.1 *Up to shifts, Galilean transformations and rescalings near the point of gradient catastrophe the generic solution approximately behaves as the root $v = v(x,t)$ of cubic equation*

$$x = v\,t - \frac{v^3}{6}$$

(bifurcation diagram of A_3 singularity).

Proof. The solution can be found by the *method of characteristics*:

$$x = a(v)\,t + b(v) \tag{5.3}$$

for an arbitrary smooth function $b(v)$. At the point of gradient catastrophe one has

$$\begin{aligned} x_0 &= a(v_0)t_0 + b(v_0) \\ 0 &= a'(v_0)t_0 + b'(v_0) \\ 0 &= a''(v_0)t_0 + b''(v_0) \end{aligned} \tag{5.4}$$

(an inflection point). We impose the *genericity assumption*

$$\kappa := -\left(a'''(v_0)t_0 + b'''(v_0)\right) \neq 0. \tag{5.5}$$

Introduce the new variables

$$\begin{aligned} \bar{x} &= x - a_0(t - t_0) - x_0 \\ \bar{t} &= t - t_0 \\ \bar{v} &= v - v_0. \end{aligned}$$

Here $a_0 = a(v_0)$, $a_0' := a'(v_0)$ etc. Rescaling

$$\bar{x} \mapsto \lambda\,\bar{x}, \quad \bar{t} \mapsto \lambda^{\frac{2}{3}}\,\bar{t}, \quad \bar{v} \mapsto \lambda^{\frac{1}{3}}\,\bar{v}, \tag{5.6}$$

substituting in $x = a(v)\,t + b(v)$ and expanding at $\lambda \to 0$ one obtains, after division by λ

$$\bar{x} = a_0'\bar{t}\,\bar{v} - \frac{1}{6}\kappa\,\bar{v}^3 + O\left(\lambda^{\frac{1}{3}}\right).$$

Similar arguments can be applied to the two-component systems. We will consider here only the case of the nonlinear wave equation [20]

$$u_{tt} - \partial_x^2 P'(u) = 0 \tag{5.7}$$

for a given smooth function $P(u)$. The equation (5.7) is linear for a quadratic function $P(u)$; we assume therefore that

$$P'''(u) \neq 0.$$

The system (3.25) can be written in the Hamiltonian form

$$u_t = \partial_x \frac{\delta H}{\delta v(x)}$$
$$v_t = \partial_x \frac{\delta H}{\delta u(x)} \tag{5.8}$$

with the Hamiltonian

$$H = \int \left[\frac{1}{2} v^2 + P(u)\right] dx. \tag{5.9}$$

The associated Poisson bracket is standard (see (2.2))

$$\{u(x), v(y)\} = \delta'(x - y) \tag{5.10}$$

The system is hyperbolic on the domain of convexity of $P(u)$,

$$(u, v) \in \mathbb{R}^2 \quad \text{such that} \quad P''(u) > 0 \tag{5.11}$$

and elliptic when $P(u)$ becomes concave. Denote $r_\pm$ the Riemann invariants of the system,

$$r_\pm = v \pm Q(u), \quad \text{where} \quad Q'(u) = \sqrt{P''(u)}. \tag{5.12}$$

The equations (3.3) for the conserved quantities of (5.8) reduce to

$$f_{uu} = P''(u) f_{vv}. \tag{5.13}$$

The generic solution $(u(x,t), v(x,t))$ can be locally determined from the implicit function equations

$$x = f_u(u, v)$$
$$t = f_v(u, v). \tag{5.14}$$

The points (x_0, t_0, u_0, v_0) of catastrophe are determined from the system

$$\left.\begin{array}{rcl} x_0 & = & f_u(u_0, v_0) \\ t_0 & = & f_v(u_0, v_0) \\ 0 & = & f_{uv}^2(u_0, v_0) - P''(u_0) f_{vv}^2(u_0, v_0) \end{array}\right\}. \tag{5.15}$$

Let us first consider the hyperbolic catastrophe, $P''(u_0) > 0$. Let (x_0, t_0, u_0, v_0) be the first catastrophe, i.e., the solution is smooth for $t < t_0$ for sufficiently small $|x - x_0|$. At a generic critical point only one of the Riemann invariants breaks up. Let it be r_-. Introduce the shifted characteristic variables

$$\bar{x}_\pm = (x - x_0) \pm \sqrt{P''(u_0)}(t - t_0) \tag{5.16}$$

and shifted Riemann invariants

$$\bar{r}_\pm = r_\pm - r_\pm(u_0, v_0).$$

Theorem 5.2 *Up to rescalings near the first point of hyperbolic gradient catastrophe the generic solution to the nonlinear wave equation approximately behaves as the solution to the system*

$$\begin{aligned} \bar{x}_+ &= \bar{r}_+ \\ \bar{x}_- &= \bar{r}_+ \bar{r}_- - \frac{1}{6}\bar{r}_-^3. \end{aligned} \tag{5.17}$$

Observe that (5.17) is one of the normal forms of singularities of smooth maps $\mathbb{R}^2 \to \mathbb{R}^2$ classified by H.Whitney in [68].

Let us now consider elliptic critical points (5.15), $P''(u_0) < 0$. In this case the Riemann invariants (5.12) are complex conjugate. So they have a simultaneous break up. Therefore the critical points are located at isolated points (x_0, t_0) of the (x, t) plane. In order to describe the local structure of the generic solution near the critical point let us introduce complex variables

$$z = (x - x_0) + i\, c_0(t - t_0), \quad w = (v - v_0) + i\, c_0(u - u_0) \tag{5.18}$$

where

$$c_0 = \sqrt{-P''(u_0)}.$$

Theorem 5.3 *Near the point of elliptic gradient catastrophe the generic solution to the nonlinear wave equation approximately behaves as the solution to the complex quadratic equation*

$$z = \frac{1}{2} a_0 w^2, \quad a_0 = f_{uvv}(u_0, v_0) + i\, c_0 f_{vvv}(u_0, v_0) \neq 0. \tag{5.19}$$

Separating the real and imaginary parts of (5.19) one obtains a description of the critical behavior (5.19) in terms of the so-called elliptic umbilic catastrophe [64].

Similar description can be obtained for the critical behavior of solutions to any of the commuting flows

$$\begin{aligned} u_s &= \partial_x f_v(u, v) \\ v_s &= \partial_x f_u(u, v) \end{aligned} \tag{5.20}$$

where $f = f(u, v)$ is an arbitrary solution to (5.13). The details can be found in [20].

Example 5.4 *Consider the (focusing) nonlinear Schrödinger equation*

$$i\,\psi_t + \frac{1}{2}\,\psi_{xx} + |\psi|^2\psi = 0 \tag{5.21}$$

written in the coordinates

$$u = |\psi|^2, \quad v = \frac{1}{2i}\left(\frac{\psi_x}{\psi} - \frac{\bar\psi_x}{\bar\psi}\right),$$

i.e.,

$$\begin{aligned} & u_t + (u\,v)_x = 0 \\ & v_t + v\,v_x - u_x = \frac{1}{4}\left(\frac{u_{xx}}{u} - \frac{1}{2}\frac{u_x^2}{u^2}\right)_x . \end{aligned}$$

The dispersionless limit

$$\begin{aligned} & u_t + (u\,v)_x = 0 \\ & v_t + v\,v_x - u_x = 0 \end{aligned} \tag{5.22}$$

is an infinitesimal symmetry of the nonlinear wave equation with

$$P(u) = -u\,(\log u - 1).$$

The system (5.22) *is of elliptic type due to obvious inequality $u > 0$. So its generic critical points have the form* (5.19).

For $n \geq 3$ it is not difficult to see that critical points of a generic solution to any *integrable* first order quasilinear system can be essentially described by the same singularities of the types (5.17) or (5.19). At the moment we do not have a classification of the singularity types for solutions to non integrable quasilinear systems.

6 Universality in Hamiltonian PDEs

In the previous section we classified the types of generic critical behavior of solutions to dispersionless Hamiltonian PDEs of low order. In the present section we will study the effects of higher order Hamiltonian perturbations. It turns out that, the above list of types of critical behavior given in terms of algebraic functions has to be replaced by another list given in terms of particular Painlevé transcendents and their higher order generalizations.

Let us begin with describing one of these special functions.

Consider the following fourth order ODE for the function $U = U(X)$ depending on T as on the parameter

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]. \tag{6.1}$$

The equation (6.1) is usually considered as the fourth order analogue of the classical Painlevé-I equation P_I (see below); it is denoted P_I^2. The following result was proved by T.Claeys and M.Vanlessen [10].

Theorem 6.1 *For any $T \in \mathbb{R}$ there exists a solution to* (6.1) *real and smooth for all real X. For large $|X|$ the solution has the asymptotic behaviour*

$$U \sim -(6X)^{1/3}, \quad |X| \to \infty. \tag{6.2}$$

Actually, the solution of interest has been constructed for all real X and T by solving certain Riemann–Hilbert problem depending on X and T as on the parameters. The main difficulty was to prove existence of a solution to the Riemann–Hilbert problem for all $(X, T) \in \mathbb{R}^2$. This solution will be denoted $U(X, T)$.

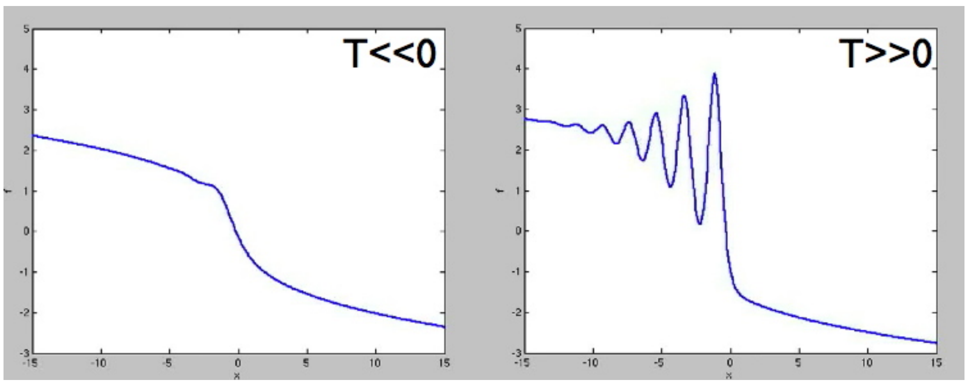

Fig. 4. The solution $U(X, T)$ to the ODE (6.1) for two instants of time T

The conjectural existence of the smooth solution to the ODE4 has been first discussed (for the particular value $T = 0$) by É.Brézin, G.Marinari, A.Parisi [6] and by G.Moore [56] in the setting of the theory of random matrices. Within the class (6.2) the uniqueness can be established using results of G.Moore [56] and A.Menikoff [53].

Importance of the smooth solution to the ODE (6.1) for the so-called *Gurevich - Pitaevsky* solution to KdV was discussed by B.Suleimanov [61] and V.Kudashev and B.Suleimanov [47].

Remark 6.2 *A somewhat stronger conjecture was formulated by the author in [19]. It says that for any real T there exists a* unique *real solution to* (6.1) *smooth for all real X. No assumptions about the asymptotic behavior are needed. This conjecture remains open.*

We are now ready to formulate, following [19], the Universality Conjecture about critical behavior of solutions to a generic Hamiltonian perturbations

$$u_t + \partial_x \frac{\delta H_f^\epsilon}{\delta u(x)} \equiv u_t + a(u)\, u_x + O(\epsilon^2) = 0, \quad H_f^\epsilon = H_f^0 + O(\epsilon^2) \tag{6.3}$$

of the scalar hyperbolic equation (5.1). Recall that all these perturbations have been classified in (2.12) up to the order $O(\epsilon^5)$. Consider the solution $u(x, t; \epsilon)$ to (6.3) that tends to a solution $v(x, t)$ as $\epsilon \to 0$ to the unperturbed equation (5.1) for sufficiently small $t < t_0$. Assume that $v(x, t)$ is smooth for $t < t_0$ for all x with sufficiently small $x - x_0$ having a point of gradient catastrophe at $(x = x_0, t = t_0, v = v_0)$.

Conjecture 6.3 *1) For sufficiently small $\epsilon > 0$ and $|x - x_0|$ there exists a positive δ such that the solution $u(x, t; \epsilon)$ can be locally smoothly extended for $t < t_0 + \delta$.*

2) Near the point (x_0, t_0) it behaves in the following way

$$u \simeq v_0 + \left(\frac{\epsilon^2 c_0}{\kappa^2}\right)^{1/7} U\left(\frac{x - a_0(t - t_0) - x_0}{(\kappa\, c_0^3 \epsilon^6)^{1/7}};\ \frac{a_0'(t - t_0)}{(\kappa^3 c_0^2 \epsilon^4)^{1/7}}\right) + O\left(\epsilon^{4/7}\right) \tag{6.4}$$

where

$$a_0 = a(v_0), \quad a_0' = a'(v_0),$$

c_0 and κ are some nonzero constants, $U(X, T)$ the solution to (6.1) described in Theorem 6.1.

We will not reproduce here the arguments of [19] supporting this conjecture. It was analyzed numerically by T.Grava and C.Klein [41]. A rigorous proof of the conjecture for solutions to the KdV equation with analytic rapidly decreasing initial data was recently obtained by T.Claeys and T.Grava [9] by using the so-called steepest descent method, due to P.Deift and X.Zhou (see in [13]).

Remarkably, the same special function $U(X, T)$ appears in the description of the critical behavior of solutions to second order Hamiltonian systems near a hyperbolic critical point. We will give a sketch of the following Universality Conjecture for Hamiltonian perturbations of the nonlinear wave equation (5.7) inspired by results of [20].

Conjecture 6.4 *Let $r_\pm$ and $x_\pm$ be as in (5.12), (5.16). Then for a solution to a generic Hamiltonian perturbation of (5.7) near the generic critical point of the form (5.17) one has*

$$\begin{aligned} r_+ &\simeq r_+^0 + c\, x_+ + \alpha_+ \epsilon^{4/7} U''\left(a\, \epsilon^{-6/7} x_-; b\, \epsilon^{-4/7} x_+\right) + \mathcal{O}\left(\epsilon^{6/7}\right) \\ r_- &\simeq r_-^0 + \alpha_-\, \epsilon^{2/7} U\left(a\, \epsilon^{-6/7} x_-; b\, \epsilon^{-4/7} x_+\right) + \mathcal{O}\left(\epsilon^{4/7}\right) \end{aligned} \tag{6.5}$$

where $U = U(X; T)$ is the same solution described in Theorem 6.1.

Proof of this conjecture remains an open problem. Observe recent result of [11] about asymptotics in Hermitean random matrices near singular edge points: for the recurrence coefficients

$$a_n(s, t) = a_n^0 + \tfrac{1}{2} c\, n^{-2/7} U(c_1 n^{6/7} s, c_2 n^{4/7} t) + O\left(n^{-3/7}\right)$$

$$b_n(s, t) = b_n^0 + c\, n^{-2/7} U(c_1 n^{6/7} s, c_2 n^{4/7} t) + O\left(n^{-3/7}\right)$$

This result support Conjecture 6.4 for the case of solutions to equations of Toda hierarchy with some particular initial data. Also numerical results obtained in the beginning of '90s in the theory of random matrices (see Fig. 5) qualitatively support Conjecture 6.4.

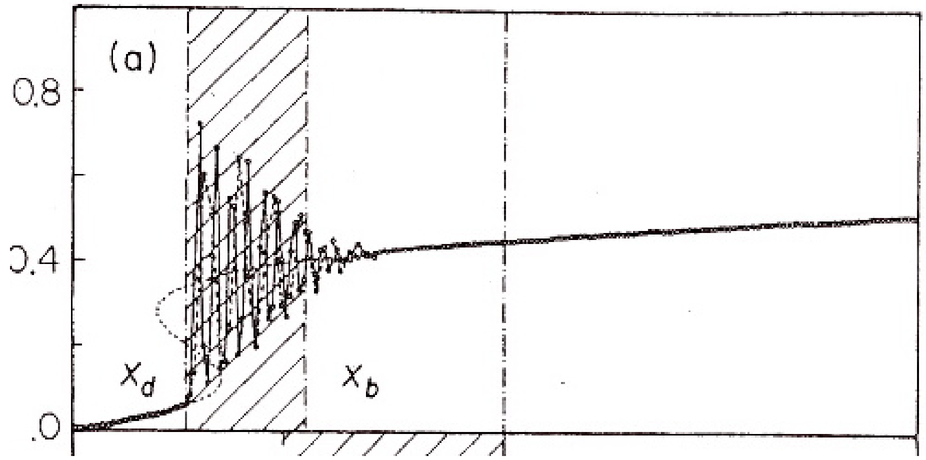

Fig. 5. Oscillatory behavior of correlation functions in the random matrix models, after [45]. The oscillatory zone corresponds to the two-cut region

We will now introduce another special function needed for the description of the critical behavior near *elliptic* critical points. The special function in question is defined as a particular solution to the classical Painlevé-I (P_I) equation for the function $W = W(Z)$, $Z \in \mathbb{C}$

$$W'' = 6W^2 - Z. \tag{6.6}$$

It is known that any solution to P_I is a meromorphic function on the complex plane. The following result was proved in 1913 by P.Boutroux [5].

Theorem 6.5 *1) Poles of a generic solution to P_I accumulate along five rays*

$$\arg Z = \frac{2\pi\, n}{5}, \quad n = 0, \pm 1, \pm 2. \tag{6.7}$$

2) For any three consecutive rays there exists a unique *so-called* tritronquée *solution such that the lines of poles truncate along these three rays for large $|Z|$.*

Let us consider the *tritronquée* solution $W_0(Z)$ associated with the triple of rays (6.7) with $n = 0$ and $n = \pm 1$. Due to Boutroux theorem this solution has at most finite number of poles in the sector

$$|\arg Z| < \frac{4\pi}{5} - \delta$$

for any small positive δ. In [21] arguments were found suggesting the following

Conjecture 6.6 *The* tritronquée *solution $W_0(Z)$ is holomorphic in the sector*

$$|\arg Z| < \frac{4\pi}{5}. \tag{6.8}$$

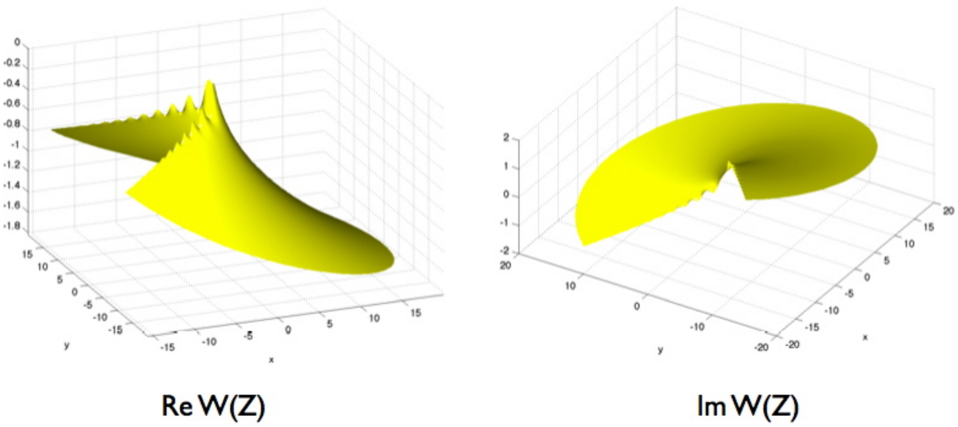

Fig. 6. The *tritronquée* solution $W_0(Z)$ to the P_I equation in the sector $|\arg Z| < \frac{4\pi}{5}$

We are now ready to formulate the Universality Conjecture for the critical behavior of solutions to Hamiltonian perturbations to the nonlinear wave equation (5.7) near a generic elliptic gradient catastrophe point.

Conjecture 6.7 *Let w and z be as in* (5.18). *Then for a solution to a generic Hamiltonian perturbation of* (5.7) *near the generic critical point of the form* (5.19) *one has*

$$w \simeq w^0 + \alpha\, \epsilon^{2/5} W_0\left(\epsilon^{-4/5}\, \beta\, z\right) + \mathcal{O}\left(\epsilon^{4/5}\right) \tag{6.9}$$

for some nonzero complex constants α, β depending on the choice of the solution.

The complex constant β is such that the argument of the *tritronquée* solution $W_0(Z)$ belongs to the sector $|\arg Z| < \frac{4\pi}{5}$ for any $x \in \mathbb{R}$ for sufficiently small $|t - t_0|$.

The conjecture first appeared in [21] in the description of the critical behaviour in the focusing NLS equation (5.22). It remains completely open, as well as the previous conjecture about poles of the *tritronquée* solution $W_0(Z)$ to P_I.

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References

[1] A.Barakat, On the moduli space of deformations of bihamiltonian hierarchies of hydrodynamic type, arXiv:math/0602596.

[2] Barnes G-function – from Wolfram MathWorld,

`http://mathworld.wolfram.com/BarnesG-Function.html`

[3] D.Bessis, C.Itzykson, J.-B.Zuber, Quantum field theory techniques in graphical enumeration, *Adv. Appl. Math.* **1** (1980) 109-157.

[4] P.M.Bleher, A.R.Its, Alexander, Asymptotics of the partition function of a random matrix model, *Ann. Inst. Fourier* **55** (2005) 1943–2000.

[5] P.Boutroux, Recherches sur les transcendants de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre. *Ann. École Norm* **30** (1913) 265 - 375.

[6] É.Brézin, E.Marinari, G.Parisi, A nonperturbative ambiguity free solution of a string model. *Phys. Lett.* **B 242** (1990) 35–38.

[7] G.Carlet, The extended bigraded Toda hierarchy. *J. Phys. A* **39** (2006) 9411–9435.

[8] G.Carlet, B.Dubrovin, Y.Zhang, The extended Toda hierarchy. *Mosc. Math. J.* **4** (2004) 313–332, 534.

[9] T.Claeys, T.Grava, Universality of the break-up profile for the KdV equation in the small dispersion limit using the Riemann-Hilbert approach, arXiv:0801.2326.

[10] T.Claeys, M.Vanlessen, The existence of a real pole-free solution of the fourth order analogue of the Painlevé I equation. *Nonlinearity* **20** (2007), no. 5, 1163–1184.

[11] T. Claeys, M. Vanlessen, Universality of a double scaling limit near singular edge points in random matrix models, *Comm. Math. Phys.* **273** (2007), no. 2, 499–532.

[12] L.Degiovanni, F.Magri, V.Sciacca, On deformation of Poisson manifolds of hydrodynamic type. *Comm. Math. Phys.* **253** (2005) 1–24.

[13] P.Deift, *Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert Approach*, Courant Lecture Notes **3**, New York University 1999.

[14] V. Drinfeld, V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, *Journal of Mathematical Sciences* **30** (1985), no. 2, 1975 - 2036. Translated from *Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki (Noveishie Dostizheniya)* **24** (1984) 81–180.

[15] B. Dubrovin, Topological conformal field theory from the point of view of integrable systems. In: "Integrable Quantum Field Theories", Eds. L.Bonora, G.Mussardo, A.Schwimmer, L.Girardello, and M.Martellini, 283–302. Plenum Press, 1993; Differential geometry of the space of orbits of a Coxeter group. Preprint SISSA 29/93/FM, February 1993, 30 pp. Published in: Surveys in Differential Geometry, Vol. IV (1999), 181–212.

[16] B.Dubrovin, Flat pencils of metrics and Frobenius manifolds, Proceedings of 1997 Taniguchi Symposium "*Integrable Systems and Algebraic Geometry*", Editors M.-H. Saito, Y.Shimizu and K.Ueno, 47-72. World Scientific, 1998.

[17] B.Dubrovin, Painlevé transcendents and topological field theory, In: "*The Painlevé Property: One Century Later*", R.Conte (Ed.), Springer Verlag, 1999, p.287-412.

[18] B.Dubrovin, WDVV and Frobenius manifolds, In: *Encyclopedia of Mathematical Physics*, eds. J.-P. Françoise, G.L. Naber and Tsou S.T., Oxford: Elsevier, 2006 (ISBN 978-0-1251-2666-3), Vol. 1, p. 438.

[19] B.Dubrovin, On Hamiltonian perturbations of hyperbolic systems of conservation laws, II: universality of critical behaviour, *Comm. Math. Phys.* **267** (2006) 117 - 139.

[20] B.Dubrovin, On universality of critical behaviour in Hamiltonian PDEs, *Amer. Math. Soc. Transl.* **224** (2008) 59-109.

[21] B.Dubrovin, T.Grava, C.Klein, On universality of critical behaviour in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the *tritronquée* solution to the Painlevé-I equation, arXiv:0704.0501, *J. Nonlinear Sci.*, Published online: 5 June 2008.

[22] B.Dubrovin, S.-Q.Liu, Y.Zhang, On Hamiltonian perturbations of hyperbolic systems of conservation laws I: quasitriviality of bihamiltonian perturbations. *Comm. Pure Appl. Math.* **59** (2006) 559-615.

[23] B.Dubrovin, S.-Q.Liu, Y.Zhang, Frobenius manifolds and central invariants for the Drinfeld–Sokolov bihamiltonian structures, *Adv. Math.* **219** (2008) 780-837.

[24] B.Dubrovin, S.Novikov, Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov–Whitham averaging method. *Dokl. Akad. Nauk SSSR* **270** (1983), no. 4, 781–785.

[25] B.Dubrovin, Y.Zhang, Extended Affine Weyl Groups and Frobenius Manifolds, *Compositio Math.* **111** (1998) 167-219.

[26] B.Dubrovin, Y.Zhang, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants. ArXiv:math.DG/0108160.

[27] B.Dubrovin, Y.Zhang, Virasoro symmetries of the extended Toda hierarchy. *Comm. Math. Phys.* **250** (2004) 161–193.

[28] B.Dubrovin, Y.Zhang, Frobenius manifolds and Virasoro constraints, *Selecta Mathematica* **5** (1999) 423-466.

[29] B.Dubrovin, Y.Zhang, Universal integrable hierarchy of the topological type, in preparation.

[30] N.M.Ercolani, K.D.T.-R. McLaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration, *Int. Math. Res. Not.* **14** (2003) 755–820.

[31] N.M.Ercolani, K.D.T.-R. McLaughlin, V.U.Pierce, Random matrices, graphical enumeration and the continuum limit of Toda lattices, *Comm. Math. Phys.* **278** (2008) 31–81.

[32] C.Faber, S.Shadrin, D.Zvonkine, Tautological relations and the r-spin Witten conjecture. ArXiv:math.AG/0612510.

[33] E.V. Ferapontov, Compatible Poisson brackets of hydrodynamic type, *J. Phys.* **A 34** (2001), no. 11, 2377-2388.

[34] A.S.Fokas, A.R.Its, A.V.Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* **147** (1992) 395-430.

[35] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov, A. Orlov, Matrix models of 2D gravity and Toda theory, *Nucl. Phys.* **B 357** (1991) 565-618.

[36] E. Getzler, The Toda conjecture. In: *Symplectic geometry and mirror symmetry* (Seoul, 2000), 51–79, World Sci. Publishing, River Edge, NJ, 2001.

[37] E. Getzler, Intersection theory on $\bar{M}_{1,4}$ and elliptic Gromov-Witten invariants, *J. Amer. Math. Soc.* **10** (1997), 973–998.

[38] E.Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.* **111** (2002) 535–560.

[39] E.Getzler, The equivariant Toda lattice. *Publ. Res. Inst. Math. Sci.* **40** (2004) 507–536.

[40] A. Givental, Gromov–Witten invariants and quantization of quadratic Hamiltonians, *Mosc. Math. J.* **1** (2001) 551–568, 645.

[41] T.Grava, C.Klein, Numerical study of a multiscale expansion of KdV and Camassa-Holm equation, *Comm. Pure Appl. Math.* **60** (2007) 1623-1664.

[42] A.Gurevich, L.Pitaevski, Nonstationary structure of a collisionless shock wave, *Sov. Phys. JETP Lett.* **38** (1974) 291-297.

[43] J.Haantjes, On X_m-forming sets of eigenvectors, Indag. *Math.* **17** (1955) 158-162.

[44] *Important Developments in Soliton Theory.* Edited by A. S. Fokas and V. E. Zakharov. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1993.

[45] J.Jurkiewicz, Chaotic behavior in one matrix model, *Phys. Lett.* **B 261** (1991) 260-268.

[46] M.Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.* **147** (1992) 1–23.

[47] V.Kudashev, B.Suleĭmanov, A soft mechanism for the generation of dissipationless shock waves, *Phys. Lett.* **A 221** (1996) 204–208.

[48] Y.-P.Lee, Invariance of tautological equations I: conjectures and applications, *J. Eur. Math. Soc.* **10** (2008) 399–413.

[49] Y.-P.Lee, Invariance of tautological equations II: Gromov–Witten theory, arXiv:math/0605708.

[50] S.-Q. Liu, Y. Zhang, Deformations of semisimple bihamiltonian structures of hydrodynamic type, *J. Geom. Phys.* **54** (2005) 427–453.

[51] Yu.I. Manin, P. Zograf, Invertible cohomological field theories and Weil-Petersson volumes, *Ann. Inst. Fourier* **50** (2000) 519–535.

[52] M.L.Mehta, *Random Matrices*, 2nd edition, Academic Press, New York, 1991.

[53] A.Menikoff, The existence of unbounded solutions of the Korteweg-de Vries equation. *Comm. Pure Appl. Math.* **25** (1972), 407–432.

[54] T.Milanov, H.-H.Tseng, The spaces of Laurent polynomials, $\mathbf{P}^1$-orbifolds, and integrable hierarchies. ArXiv:math.AG/0607012.

[55] O.I. Mokhov, Compatible and almost compatible pseudo-Riemannian metrics, *Funct. Anal. Appl.* **35** (2001), no. 2, 100–110.

[56] G.Moore, Matrix models of 2D gravity and isomonodromic deformation. In: *Common Trends in Mathematics and Quantum Field Theories* (Kyoto, 1990). *Progr. Theoret. Phys. Suppl.* **102** (1990), 255-285 (1991).

[57] S.Novikov, S.Manakov, L.Pitaevskiĭ, V.Zakharov, *Theory of Solitons. The Inverse Scattering Method.* Translated from the Russian. Contemporary Soviet Mathematics. Consultants Bureau [Plenum], New York, 1984.

[58] A.Okounkov, R.Pandharipande, Gromov-Witten theory, Hurwitz theory, and completed cycles. *Ann. of Math.* (2) **163** (2006) 517-560.

[59] A. Okounkov and R. Pandharipande, The equivariant Gromov-Witten theory of $\mathbf{P}^1$. *Ann. of Math.* (2) **163** (2006) 561-605.

[60] P.Rossi, Gromov–Witten theory of orbicurves, the space of tri-polynomials and symplectic field theory of Seifert fibrations, arXiv:0808.2626.

[61] B.Suleĭmanov, Onset of nondissipative shock waves and the "nonperturbative" quantum theory of gravitation. *J. Experiment. Theoret. Phys.* **78** (1994), 583–587; translated from *Zh. Èksper. Teoret. Fiz.* **105** (1994), no. 5, 1089–1097.

[62] A.Takahashi, Weighted projective lines associated to regular systems of weights of dual type, arXiv:0711.3907.

[63] C.Teleman, The structure of 2D semi-simple field theories, arXiv:0712.0160.

[64] R.Thom, *Structural Stability and Morphogenesis: An Outline of a General Theory of Models.* Reading, MA: Addison-Wesley, 1989.

[65] G.'t Hooft, A Planar Diagram Theory for Strong Interactions, *Nucl. Phys.* **B 72** (1974) 461.

[66] S. Tsarev, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method. *Math. USSR Izv.* **37** (1991), 397-419.

[67] *What is Integrability?* Edited by V. E. Zakharov. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.

[68] H.Whitney, On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane. *Ann. of Math.* (2) **62** (1955), 374-410.

[69] E.Witten, Two-dimensional gravity and intersection theory on moduli space. *Surveys in Differential Geometry* **1** (1991) 243-310.

[70] E. Witten, Algebraic geometry associated with matrix models of two-dimensional gravity. In: *Topological methods in modern mathematics* (Stony Brook, NY, 1991), 235-269, Publish or Perish, Houston, TX (1993).

[71] E.Witten, The N matrix model and gauged WZW models. *Nuclear Phys.* **B 371** (1992) 191-245.

[72] N.Zabusky, M.Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **15** (1965) 240-243.

[73] P. Zograf, Weil-Petersson volumes of moduli spaces of curves and the genus expansion in two dimensional gravity, arXiv:math/9811026.