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AN EFFECTIVE MODEL FOR NEMATIC LIQUID CRYSTAL COMPOSITES WITH FERROMAGNETIC INCLUSIONS*

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Abstract. Molecules of a nematic liquid crystal respond to an applied magnetic field by reorienting themselves in the direction of the field. Since the dielectric anisotropy of a nematic is small, it takes relatively large fields to elicit a significant liquid crystal response. The interaction may be enhanced in colloidal suspensions of ferromagnetic particles in a liquid crystalline matrix—ferronematics—as proposed by Brochard and de Gennes in 1970. The ability of these particles to align with the field and simultaneously cause reorientation of the nematic molecules greatly increases the magnetic response of the mixture. Essentially the particles provide an easy axis of magnetization that interacts with the liquid crystal via surface anchoring. We derive an expression for the effective energy of ferronematic in the dilute limit, that is, when the number of particles tends to infinity while their total volume fraction tends to zero. The total energy of the mixture is assumed to be the sum of the bulk elastic liquid crystal contribution, the anchoring energy of the liquid crystal on the surfaces of the particles, and the magnetic energy of interaction between the particles and the applied magnetic field. The homogenized limiting ferronematic energy is obtained rigorously using a variational approach. It generalizes formal expressions previously reported in the physical literature.

Key words. ferronematics, liquid crystal, homogenization

AMS subject classifications. 35B27, 35Q56

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1. Introduction.

The study of magnetic particle suspensions in a liquid crystalline matrix was initiated with the theoretical article by Brochard and de Gennes [1] (July, 1970) and the experimental work carried out by Rault, Cladis, and Burger[1] [2] (June, 1970). The underlying mechanism behind a ferronematic system is a mechanical coupling between the nematic molecules and the magnetic particles, mostly realized by the surface anchoring energy.

Molecules of nematic liquid crystals have positive magnetic susceptibility, so they tend to align themselves in the direction of an applied magnetic field. However, since this magnetic susceptibility is small—of order 10^{-7}—it takes large fields of about 10^4 Oe to elicit a significant response. Brochard and de Gennes argued that the addition of paramagnetic ions to the system is not an efficient way to increase the magnetic susceptibility constant, since it would require a concentration of paramagnetic ions above $n = 10^{20}$ ions per cm^3. The latter is the limiting value that cannot be exceeded in order for the composite to remain a liquid crystal.

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[1]Both groups acknowledge an ongoing scientific communication while their works were underway.

The focus of subsequent research turned to suspensions of large ferromagnetic particles in the nematic matrix. Brochard and de Gennes identified the two key properties of such systems: strength of mechanical coupling and stability of suspension. The former guarantees that the magnetic field, acting on the liquid crystal through the magnetic particles, influences the nematic texture more than it would have in the absence of the particles. The latter property sets a limit on the size and concentration of particles to prevent clustering. The numbers arrived at from theoretical considerations set the particle length $l > 0.5 \times 10^{-2}\mu$m and a ratio $\frac{l}{d} \approx 10$, where d denotes the diameter of the particle. The theoretical prediction on particle volume fraction was not to exceed the value of $f = 10^{-3}$.

In their experiments, Rault, Cladis, and Burger [2] chose monodomain particles (grains) of $\gamma Fe_2O_{\frac{2}{3}}$ that were 0.35μm long (l) by 0.04 μm in diameter (d). The saturation magnetization was equal to 384 gauss with the easy axis being parallel to the long axis of the grain. Grains of these dimensions satisfy the criterion for mechanical coupling to the nematic liquid and also for mechanical rotation (as opposed to rotation of magnetization inside the grain in a reversed field). Typical grain concentrations were of the order of 2×10^{11} grains/cm^3, which corresponds to $f \approx 1.4 \times 10^{-4}$—well within the theoretical prediction by Brochard and de Gennes. For these physical parameters, Rault, Cladis, and Burger state the following [2]: *The ferronematic appeared to be very stable in the nematic-isotropic phases showing very little tendency to agglomerate. However, if a high field (*1 *kG) is applied to the sample in the isotropic phase, upon returning it to the nematic phase, we have observed long chains of grains about* 50μm. Both works assert that distortions of the nematic pattern in magnetic suspensions occur at very low fields: magnetizations range in the order of 0.1 to 1 gauss, instead of the values 10^{-4} to 10^{-3} for pure nematic liquid crystals, with a typical coupling gain of order 10^3.

Central to an understanding of the nematic-magnetic coupling is the question of how the grains align in the nematic. Brochard and de Gennes postulated strong anchoring of nematic molecules on surfaces of the particles along the magnetic moment assumed to coincide with the direction of the particle axis. Then the magnetic field around each grain is anisotropic, even when the grain has a spherical shape. This anisotropy, in turn, imposes a preferred orientation on the magnetic moment in the nematic phase. This effect turns out to be small for small grains, with the magnetic moment causing only a local disruption of the nematic alignment.

In their experimental work [3], Chen and Amer used particle coating that yields homeotropic anchoring of the liquid crystal on the magnetic grain to synthesize stable ferronematic systems. Although the length and aspect ratios of the particles, 0.5 μm and 7 : 1, respectively, are compatible with those considered in the previous works, the earlier theory assuming rigid parallel anchoring was found to be not applicable to the homeotropic case. Generally, the influence of surface anchoring on the effective properties of ferronematic composites stimulated intense experimental and theoretical research activity spanning three decades. In [4], [5], [6], the authors showed that the rigid anchoring approximation might be used only if the condition $\frac{Zd}{K} >> 1$ holds, where Z represents the surface energy density and K denotes a typical Frank constant. A calculation for MBBA data, with $K = 5 \times 10^{-7} \frac{\text{dyn}}{cm^2}$, and $10^{-3} < Z < 10^{-2}$, and $d = 0.07\mu$m gives $10^{-2} < \frac{Zd}{K} < 10^{-1}$, showing a finite surface energy of the system.

Assuming soft liquid crystal surface anchoring, Burylov and Raikher [7] proposed a macroscopic free energy density of the form

$$
\begin{aligned}
F =&\frac{1}{2}\{K_1(\operatorname{div}\mathbf{n})^2+K_2(\operatorname{curl}\mathbf{n}\cdot\mathbf{n})^2+K_3(\mathbf{n}\times\operatorname{curl}\mathbf{n})^2\}-\frac{1}{2}\chi_a(\mathbf{n}\cdot\mathbf{H})^2 \\
&-M_s f\,(\mathbf{m}\cdot\mathbf{H})+\left(\frac{fK_bT}{\nu}\right)\ln f+\left(\frac{AZf}{d}\right)(\mathbf{n}\cdot\mathbf{m})^2. \qquad (1.1)
\end{aligned}
$$

Here $K_i,\ i=1,2,3$, are elastic constants, f represents the volume fraction of the particles, χ_a is the anisotropic part of the diamagnetic susceptibility of nematic, and the positive constants ν and M_s denote the particle volume and the saturation magnetization, respectively. In the last term, $A=1-3\cos^2\alpha$ characterizes the type of anchoring, with α denoting the easy-angle orientation of the nematic on the particle surface.

The macroscopic free energy (1.1) has been investigated in theoretical and experimental works involving orientational transitions in ferronematic states [6], [8], [9]. In particular, [9] presents a nonlinear modification of the Rapini–Papoular energy that predicts a first order Fredericks transition. In [10] and [11], Kopcansky et al. use the modified theory to determine threshold fields in ferronematic transitions under combined electric and magnetic fields. In [12], the authors report on experimental studies of structural transitions in ferronematics subject to electric and magnetic fields, with the matrix consisting of 8CB and 6CHBT liquid crystals, respectively. While in both cases the anchoring was determined to be soft, it was found that $\mathbf{n}\perp\mathbf{m}$ in the first case, and $\mathbf{n}\|\mathbf{m}$ in the second. So, it was established then that both parallel and perpendicular anchoring may occur depending on the properties of the matrix (which, in turn, reflects the properties of the particle coating). Zadorozhnii et al. [13] provide a comprehensive analysis of the director—a unit vector in the direction of the preferred molecular alignment—switching for small and large values of the applied field in a nematic liquid crystal cell subject to homeotropic boundary conditions at the cell and particle walls. They show that the threshold field depends on the anchoring strength of the director on the particle surface.

Note that a closely related set of models [14], [15] exists for suspensions of ferroelectric nanoparticles in a nematic liquid crystalline matrix. The mechanical coupling between the particles and the nematic is still governed by the surface anchoring, but the particles interact with an electric and not a magnetic field.

In this work, we rigorously derive an expression for the effective ferronematic energy that reduces to the models described above under appropriate limits. We consider a collection of spheroidal particles with fixed randomly distributed locations in the matrix, and with magnetic moment pointing in the direction of an easy axis. The particles are taken as rotations and translations of the same spheroidal particle, located at the origin. We model the liquid crystalline matrix according to Ericksen's theory of nematics with variable degree of orientation [16]. Using a standard assumption of the mathematical literature on nematics that all elastic constants are equal, the state of a liquid crystal is described within the Ericksen theory by a vector $\mathbf{u}(\mathbf{x})$ pointing in the direction parallel to the "average" molecular orientation near the point $\mathbf{x}$. The magnitude of $\mathbf{u}(\mathbf{x})$—called the degree of orientation—describes the quality of the alignment. Here the nematic is in the isotropic state near $\mathbf{x}$ when $|\mathbf{u}(\mathbf{x})|=0$, while all nematic molecules are aligned in the direction parallel to $\mathbf{u}(\mathbf{x})$ when $|\mathbf{u}(\mathbf{x})|=1$.

Since the energy expression (1.1) of Burylov and Raikher is based on the Oseen–Frank theory for the nematic director $\mathbf{n}=\mathbf{u}/|\mathbf{u}|$ and the Ericksen theory reduces to that of Oseen–Frank when $|\mathbf{u}|\equiv\text{const}$, our model is more general than that of [7]. Under the assumption that the Frank elastic constants are equal (corresponding to

$K_1 = K_2 = K_3 =: K$ and the derivative terms in (1.1) reduced to $1/2\, K|\nabla \mathbf{n}|^2$), the bulk liquid crystal energy has the form of the Ginzburg–Landau energy for $\mathbf{u}$. The potential term in this energy penalizes for deviations of $|\mathbf{u}|$ from some constant value and replaces the hard length constraint of the Oseen–Frank theory.

We assume soft anchoring of the liquid crystal molecules on the surfaces of ferromagnetic particles as represented by the Rapini–Papoular energy term. This term has the same form within both the Ericksen and the Oseen–Frank theories, with $\mathbf{u}$ being replaced by $\mathbf{n}$ in the latter case. The surface energy contribution can be either positive or negative depending on whether parallel or perpendicular alignment of nematic molecules is preferred on particle surfaces. It turns out that the case when the surface energy is negative is the most challenging to analyze.

Mathematically, we consider a family of energy functionals, $\mathcal{F}_\epsilon$, parametrized by a quantity $\epsilon > 0$ that characterizes the geometry of the system—specifically, the size of the particles and the interparticle distance. The system is assumed to be dilute; that is, the volume fraction of the particles tends to 0 in the limit $\epsilon \to 0$. The parameter scalings in the model that are responsible for relative contributions of the different energy components are also formulated in terms of ϵ. Here we consider the choice of scalings that guarantees that the limiting contributions of the bulk and surface energies, as well as the energy of interaction between the particles and the applied magnetic field, are all of order $O(1)$. We show that for the same parametric regime the contribution from the energy of magnetic interaction between the particles is $o(1)$ in ϵ. This is consistent with the experimental observations that characterize dilute small particle systems in the absence of clustering.

We study the variational limit of the family of energies $\{\mathcal{F}_\epsilon\}$ as $\epsilon \to 0$. The limiting functional $\{\mathcal{F}_0\}$ represents the effective, or homogenized, energy of the system. Here the convergence is understood in the sense that the sequence of minimizers $\{u_\epsilon\}$ of $\{\mathcal{F}_\epsilon\}$ converges to a minimizer u of $\{\mathcal{F}_0\}$ in an appropriate functional space. The effective energy provides a benchmark for comparison with the formal expression for the ferronematic energy functional [7] given in (1.1).

The homogenized magnetic and surface energy contributions in (3.8) generalize those in (1.1) as the limit in this work is obtained under less restrictive assumptions on the geometry of the particles. The interaction between the liquid crystal and the particles is due to surface anchoring and is represented by the matrix A in (3.7) that encodes the information on the shape and size of the particles, their locations, and their orientation with respect to a fixed frame. Likewise, the effective magnetic moment $\mathbf{M}$ in (3.7) that couples the particles to the external magnetic field depends on the spatial and orientational distributions of the particles. For the high-aspect-ratio needle-like particles, the terms coupling the nematic director and the magnetic field to the particles reduce to their counterparts in (1.1).

2. Background.

Given the domain $\Omega \subset \mathbf{R}^3$, let $P_i \subset \Omega$ be an arbitrary collection of subsets of Ω such that $P_i \cap P_j = \emptyset$ for every $i \neq j$ where $i, j = 1, \ldots, n$. Suppose that the region $\Omega \setminus \cup_i P_i$ is occupied by a nematic liquid crystal and that for each $i = 1, \ldots, n$ the region P_i corresponds to a hard ferromagnetic particle embedded in the nematic matrix.

We will consider the liquid crystal configurations that can be described by the Ericksen theory for nematics with variable degree of orientation; we will neglect all flow effects and assume that all elastic constants are equal. Further, we will use the phenomenological Rapini–Papoular term in order to approximate the liquid crystal/ferromagnetic surface energy. Then the elastic energy of the liquid crystal is given by

$$\mathcal{F}^{el}_{lc} := \int_{\Omega\setminus\cup_i P_i} \left(K\,|\nabla \mathbf{u}|^2 + W(|\mathbf{u}|)\right)\,dV + q\int_{\cup_i \partial P_i} (\mathbf{u},\nu)^2\,d\sigma\,,$$

where $K > 0$ is the elastic constant, $q \in \mathbf{R}$ is the strength of the surface term, W is the bulk free energy of the undistorted state, and ν is the outward unit normal vector to ∂P_i.

Suppose that ferromagnetic particles are sufficiently small so that for every $i = 1,\ldots,N$ an ith particle can be characterized by a magnetization vector $\mathbf{m}_i$ pointing in the direction of an easy axis of the particle. In order to derive the expression for the magnetostatic contribution f^m to the free energy density of what is effectively a diamagnetic matrix interspersed with the ferromagnetic particles, we follow [17]. We have that

$$\left(\frac{\partial f^m}{\partial \mathbf{H}}\right)_T = -\mathbf{B}\,, \tag{2.1}$$

where $\mathbf{H}$ and $\mathbf{B}$ are the magnetic field and the magnetic induction, respectively (cf. equation (39.1) in [17]), and the derivative is taken holding the temperature T fixed. Assuming that $\mathbf{M}$ denotes the magnetic moment, the induction is given by

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})\,, \tag{2.2}$$

where μ_0 is the magnetic permeability of vacuum.

Suppose that the magnetic moment of the material can be written as

$$\mathbf{M} = \mathbf{m} + \chi\mathbf{H}\,, \tag{2.3}$$

and the material can exhibit both the spontaneous magnetization $\mathbf{m}$ (an independent thermodynamic variable) and the magnetization induced by the field (we assume it to be proportional to the field). The tensor χ is the magnetic susceptibility; it is generally small in diamagnetics, but it can be large in soft ferromagnetic bodies. In what follows, we will set $\mathbf{m} = \mathbf{0}$ in the liquid crystal, while we will set $\chi = 0$ in hard ferromagnetics.

Substituting (2.2) and (2.3) into (2.1) and integrating with respect to the field, we obtain

$$f^m(\mathbf{m},\mathbf{H}) = f^m(\mathbf{m},\mathbf{0}) - \mu_0(\mathbf{m},\mathbf{H}) - \frac{\mu(\mathbf{H},\mathbf{H})}{2}. \tag{2.4}$$

Here $\mu = \mu_0(\mathbf{I}+\chi)$ is the magnetic permeability tensor. Note that the energy $f^m(\mathbf{m},\mathbf{0})$ accounts for both the exchange and the anisotropy energies for a ferromagnetic body. We will ignore this splitting since we consider single-domain particles.

The expression (2.4) can be adjusted further by excluding the energy of the external field that would otherwise be created by the same sources in vacuum.

Let the fields $\mathbf{H}$ and $\mathbf{h}$ solve the (different) sets of Maxwell's equations under the same boundary conditions at infinity in the presence and in the absence of the material, respectively. Then $\mathbf{h}$ is the magnetic field in vacuum when there is no magnetizing body (cf. equation (32.1) in [17]).

Since the free energy of the field $\mathbf{h}$ is

$$\mathcal{F}^m_{\mathbf{h}} := -\int_{\mathbf{R}^3} \frac{\mu_0|\mathbf{h}|^2}{2}\,dV\,,$$

the adjusted free energy can be written as

$$\tilde{\mathcal{F}}^m := \int_{\mathbf{R}^3} f^m \, dV - \mathcal{F}^m_{\mathbf{h}} = \int_{\mathbf{R}^3} \left(f^m + \frac{\mu_0 |\mathbf{h}|^2}{2} \right) dV \,. \tag{2.5}$$

By rearranging terms, using Maxwell's equations, and integrating, one can show [17] that

$$\tilde{\mathcal{F}}^m = \int_{\mathbf{R}^3} \left(f^m + \frac{1}{2}(\mathbf{H}, \mathbf{B}) - \frac{\mu_0}{2}(\mathbf{M}, \mathbf{h}) \right) dV \,. \tag{2.6}$$

This equation can be simplified by taking (2.4) into account to obtain

$$\mathcal{F}^m = -\frac{\mu_0}{2} \int_{\mathbf{R}^3} \left((\mathbf{m}, \mathbf{H}) + (\mathbf{m}, \mathbf{h}) + \chi(\mathbf{H}, \mathbf{h}) \right) dV \,, \tag{2.7}$$

where we drop the tilde for convenience.

In a hard ferromagnetic material, the magnetic susceptibility $\chi = 0$. By denoting the demagnetizing field by $\mathbf{H}_i = \mathbf{H} - \mathbf{h}$ (2.7) reduces to

$$\mathcal{F}^m = -\frac{\mu_0}{2} \int_{\mathbf{R}^3} \left((\mathbf{m}, \mathbf{H}_i) + 2(\mathbf{m}, \mathbf{h}) \right) dV \,;$$

this is the sum of the magnetostatic and the Zeeman energies. Further, $\mathbf{H}_i$ vanishes as $x \to \infty$, and it satisfies the same set of Maxwell's equations as $\mathbf{H}$.

If the material is diamagnetic, then $\mathbf{m} = \mathbf{0}$ and χ is small enough so that the magnetic field is essentially unperturbed by the presence of magnetizing body. We conclude that

$$\mathcal{F}^m = -\frac{\mu_0}{2} \int_{\mathbf{R}^3} \chi(\mathbf{h}, \mathbf{h}) \, dV \,,$$

which is the standard form of the free energy for the diamagnetic bodies.

Now we establish the expressions for the magnetic free energy in various components of the composite. Suppose for now that the external field $\mathbf{h}$ is constant.

Using the same notation as above, the energy of interaction between the magnetic field and the (diamagnetic) liquid crystal (cf. [18], [19]) is given by

$$\mathcal{F}^m_{lc} := -\frac{\mu_0}{2} \int_{\Omega \setminus \cup_i P_i} \chi_{lc}(\mathbf{H}, \mathbf{h}) \, dV \,.$$

The magnetic susceptibility tensor χ_{lc} can be approximated as

$$\chi_{lc} = \frac{\chi_a |\mathbf{u}|}{s_{exp}} \left(\frac{\mathbf{u}}{|\mathbf{u}|} \otimes \frac{\mathbf{u}}{|\mathbf{u}|} - \frac{1}{3}\mathbf{I} \right) + \bar{\chi}\, \mathbf{I} \,.$$

Here $\chi_a = \chi_\parallel - \chi_\perp$ is the rescaled diamagnetic anisotropy, and

$$\bar{\chi} = \left(\chi_\parallel + 2\chi_\perp \right)/3$$

is the average susceptibility. The scaling factor s_{exp} is the value of the uniaxial order parameter $|\mathbf{u}|$ when the measurements of the susceptibility were taken, and it reflects the hysteresis behavior of the magnetic loading experiments. We point out that in a nematic $\chi_\parallel, \chi_\perp < 0$ and $0 < \chi_a \ll |\bar{\chi}|$ [20]. The smallness of $\chi_a/\bar{\chi}$ is the basis for assuming that the effect of the liquid crystal on the magnetic field is weak [18].

By setting $\chi = 0$ in (2.7), the free energy of the hard ferromagnetic particles is

$$\mathcal{F}_f^m := -\frac{\mu_0}{2} \sum_{i=1}^{N} \int_{P_i} \{(\mathbf{m}_i, \mathbf{H}) + (\mathbf{m}_i, \mathbf{h})\} \, dV \, .$$

By solving the Maxwell's equations of magnetostatics, we find that the total field $\mathbf{H}$ is given by

$$\mathbf{H} = -\nabla \phi \, ,$$

where the magnetic potential satisfies the equations

$$\begin{aligned} \Delta \phi = 0 \quad & \text{in } \cup P_i, \\ \operatorname{div}(\mu_{lc} \nabla \phi) = 0 \quad & \text{in } \{\cup P_i\}^c. \end{aligned}$$

The boundary conditions are

$$\left[-\mu \frac{\partial \phi}{\partial \nu} + (\mathbf{m}_i, \nu) \right] \bigg|_{\partial P_i} = 0$$

for every $i = 1, \ldots, N$ and

$$\nabla \phi = \mathbf{h}$$

at infinity. Here the magnetic permeability tensor $\mu = \mu_{lc} = \mu_0 (\mathbf{I} + \chi_{lc})$ in the liquid crystal and $\mu = \mu_0 \mathbf{I}$ in the ferromagnetic particles.

The equilibrium configuration of the composite can be found by minimizing the functional

$$\mathcal{F} := \mathcal{F}_{lc}^{el} + \mathcal{F}_{lc}^{m} + \mathcal{F}_f^m$$

with respect to $\mathbf{u}$ and $\mathbf{m}_i$.

3. Formulation of the problem. Suppose that the positions and orientations of prolate spheroidal particles are fixed and distributed randomly in the matrix, the spontaneous magnetic moments of the ferromagnetic particles are parallel to their long axes, and $\chi_a = 0$.

Consider the family of energy functionals $\mathcal{F}_\varepsilon$

$$\begin{aligned} \mathcal{F}_\varepsilon[\mathbf{u}] = & \int_{\Omega \setminus \cup \mathcal{P}_i^\varepsilon} \left\{ |\nabla \mathbf{u}|^2 + W(|\mathbf{u}|) \right\} dV + g_\varepsilon \int_{\cup \partial \mathcal{P}_i^\varepsilon} (\mathbf{u}, \nu)^2 \, d\sigma \\ & - \int_{\mathbf{R}^3} \{(\mathbf{m}_\varepsilon, \mathbf{H}_\varepsilon) + 2(\mathbf{m}_\varepsilon, \mathbf{h}_\varepsilon)\} \, dV, \end{aligned} \tag{3.1}$$

where $\varepsilon > 0$ is a small parameter related to the geometry of the system. Here

$$\begin{cases} \mathbf{m}_\varepsilon = \mathbf{m}_i^\varepsilon, & \mathbf{x} \in \mathcal{P}_i^\varepsilon, \\ 0, & \mathbf{x} \in \Omega \setminus \cup \mathcal{P}_i^\varepsilon, \end{cases} \tag{3.2}$$

and for simplicity, we set $W(t) = \left(1 - t^2\right)^2$. The magnetic field is given by

$$h_\varepsilon = |\mathbf{h}_\varepsilon| = \text{const}, \quad \mathbf{H}_\varepsilon = -\nabla \varphi, \tag{3.3}$$

with

$$\begin{cases} \triangle\varphi = 0, & x \in \mathbf{R}^3, \\ \left[-\mu_0 \frac{\partial\varphi}{\partial\nu} + (\mathbf{m}_i^\varepsilon, \nu)\right]\Big|_{\partial\mathcal{P}_i} = 0, & x \in \partial\mathcal{P}_i^\varepsilon. \end{cases} \tag{3.4}$$

We assume that for a prescribed $\mathbf{U} \in C^1(\Omega, \mathbf{R}^3)$,

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega. \tag{3.5}$$

Note that in (3.1) we ignore the energy $\mathcal{F}_{lc}^m$ of the weak interaction between the nematic and the magnetic fields.

For each $\varepsilon > 0$, we denote by $\mathbf{u}_\varepsilon$ a minimizer of (3.1). We study the limiting energy and the behavior of minimizers of $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$.

We make the following assumptions:

1. The ferromagnetic particles consist of a family of N_ε prolate spheroids $\mathcal{P}_i^\varepsilon = \mathbf{x}_i^\varepsilon + \varepsilon^\alpha R_i^\varepsilon \mathcal{P}$, $i = 1, \ldots, N_\varepsilon$, where $\mathbf{x}_i^\varepsilon \in \mathbf{R}^3$ denotes a particle center and $\mathcal{P}$ is a reference spheroid with the long axis parallel to the z-coordinate axis, and R_i^ε is a rotation.
2. Given positive numbers $0 < d < D$, the distance between adjacent particles $|\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon| \in [d\varepsilon, D\varepsilon]$ and the distance between each particle and the boundary of Ω, $\operatorname{dist}(\mathbf{x}_i^\varepsilon, \partial\Omega) \in [d\varepsilon, D\varepsilon]$ for all $0 < i, j, \leq N_\varepsilon$. We further assume that the number of particles $N_\varepsilon < N\varepsilon^{-3}$ for some $N > 0$ uniformly in ε. These assumptions, in particular, rule out clustering of particles.
3. $|\mathbf{m}_i^\varepsilon| = m_\varepsilon = \operatorname{Vol}(\mathcal{P}_i^\varepsilon) m \varepsilon^{\beta_1}$, $\mathbf{h}_\varepsilon = \mathbf{h}\varepsilon^{\beta_2}$, and $g_\varepsilon = g\varepsilon^\gamma$, where m, $\mathbf{h}$, and g are given constants.
4. The parameters α, β_1, β_2, and γ satisfy

$$\begin{gathered} 1 < \alpha < 2, \quad 6\alpha + 2\beta_1 > 9, \\ \beta_2 + \beta_1 = 3 - 6\alpha, \quad \gamma = 3 - 2\alpha. \end{gathered} \tag{3.6}$$

5. There exist functions $A \in L^\infty\left(\Omega; M^{3\times 3}\right)$ and $\mathbf{M} \in L^\infty\left(\Omega; \mathbb{R}^3\right)$ such that

$$\begin{gathered} A^\varepsilon(\mathbf{x}) = \varepsilon^3 g \sum_i \delta(\mathbf{x} - \mathbf{x}_i^\varepsilon) R_i^\varepsilon \left(\int_{\partial\mathcal{P}} \nu \otimes \nu \, d\sigma\right) R_i^{\varepsilon T}, \\ \mathbf{M}^\varepsilon(\mathbf{x}) = \varepsilon^3 m \operatorname{Vol}^2(\mathcal{P}) \sum_i \delta(\mathbf{x} - \mathbf{x}_i^\varepsilon) R_i^\varepsilon \hat{\mathbf{z}} \end{gathered} \tag{3.7}$$

   converge to A and $\mathbf{M}$, respectively, in the sense of distributions.

Note that the total volume of the particles satisfies $\operatorname{Vol}(\cup\mathcal{P}_i^\varepsilon) = O\left(\varepsilon^{3\alpha-3}\right)$, so that the homogenization problem for (3.1) corresponds to a dilute limit when $\lim \operatorname{Vol}(\cup\mathcal{P}_i^\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$. The scalings on $\mathbf{h}_\varepsilon$ and g_ε ensure that the magnetic interaction between the applied field and the particles, the Ginzburg–Landau energy, and the surface energy are all $O(1)$, while the magnetic interactions between the particles are of order $o(1)$ and, therefore, can be neglected.

Remark 1. In order for assumption 5 to hold, it is sufficient to postulate weak-$\star$ convergence of the Radon measures A^ε and $\mathbf{M}^\varepsilon$ to Radon measures μ_A and $\mu_{\mathbf{M}}$, respectively. Indeed, assumption 2 and the uniform boundedness in i and ε of the rotation matrices R_i^ε would then guarantee that μ_A and $\mu_{\mathbf{M}}$ are both absolutely continuous and have the derivatives $A \in L^\infty\left(\Omega; M^{3\times 3}\right)$ and $\mathbf{M} \in L^\infty\left(\Omega; \mathbb{R}^3\right)$ with respect to Lebesgue measure.

Our principal goal is to prove the following theorem.

THEOREM 3.1. *Suppose that assumptions* 1–5 *hold. Then the sequence of minimizers* $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ *of the functionals* $\mathcal{F}_\epsilon[\mathbf{u}_\epsilon]$ *converges in the sense of* (4.12) *to a minimizer of the functional*

$$\mathcal{F}_0[\mathbf{u}] = \int_\Omega \left[|\nabla \mathbf{u}|^2 + \left(1 - |\mathbf{u}|^2\right)^2 + (A\mathbf{u}, \mathbf{u}) - 2(\mathbf{h}, \mathbf{M})\right] dV, \tag{3.8}$$

where A *and* $\mathbf{M}$ *are as defined in assumption* 5*.*

The matrix A and the vector $\mathbf{M}$ that appear in the statement of Theorem 3.1 describe the homogenized liquid crystal/ferromagnetic particle interaction and the effective magnetization density, respectively.

4. Main results.

We prove Theorem 3.1 in several steps as outlined below.

4.1. Liquid crystal energy.

First, we consider the energy (3.1) without the magnetic terms, that is,

$$\mathcal{E}_\varepsilon[\mathbf{u}] = \int_{\Omega \setminus \cup \mathcal{P}_i^\varepsilon} \left\{|\nabla \mathbf{u}|^2 + W(|\mathbf{u}|)\right\} dV + g_\varepsilon \int_{\cup \partial \mathcal{P}_i^\varepsilon} (\mathbf{u}, \nu)^2 \, d\sigma. \tag{4.1}$$

For each small $\varepsilon > 0$, we let $\mathbf{u}_\varepsilon$ be a minimizer of (4.1) subject to the Dirichlet boundary condition $\mathbf{u}_\varepsilon = \mathbf{U}$ on $\partial\Omega$.

We want to find the limiting functional of the family $\mathcal{E}_\varepsilon$ as $\epsilon \to 0$. Although our approach is developed for the prolate spheroidal particles, it can be easily extended to particles of arbitrary convex shapes. The method is based on the procedure developed in [21] for the case of spheres.

4.1.1. Compactness.

We first observe that the restriction of $\mathbf{U}$ to the domain $\Omega_\varepsilon = \Omega \setminus \cup \mathcal{P}_i^\varepsilon$ is an admissible function. Indeed,

$$\begin{aligned} \mathcal{E}_\varepsilon[\mathbf{U}] &= \int_{\Omega \setminus \cup \mathcal{P}_i^\varepsilon} \left\{|\nabla \mathbf{U}|^2 + W(|\mathbf{U}|)\right\} dV + g_\varepsilon \int_{\cup \partial \mathcal{P}_i^\varepsilon} (\mathbf{U}, \nu) \, d\sigma \\ &\le \int_\Omega \left\{|\nabla \mathbf{U}|^2 + W(|\mathbf{U}|)\right\} dV + g_\varepsilon \int_{\cup \partial \mathcal{P}_i^\varepsilon} (\mathbf{U}, \nu) \, d\sigma \\ &\le C\left(1 + g_\varepsilon N_\varepsilon |\partial \mathcal{P}| \varepsilon^{2\alpha}\right) \le C\left(1 + gN|\partial \mathcal{P}|\right) \le C, \end{aligned} \tag{4.2}$$

where C is a generic positive constant. Consequently,

$$\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] \le \mathcal{E}_\varepsilon[\mathbf{U}] \le C. \tag{4.3}$$

That is, $\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon]$ is uniformly bounded in ε.

The following lemma is needed for the proof of compactness of the sequence $\{\mathbf{u}_\varepsilon\}$ of energy minimizers of (4.1).

LEMMA 4.1. *Let* $\mathcal{P}$ *denote a prolate spheroid in* $\mathbf{R}^3$ *with minor and major axes* A *and* B*, respectively. Let* $\hat{\mathcal{P}} \supset \mathcal{P}$ *represent the prolate spheroid homothetic to* $\mathcal{P}$ *with axes* $\frac{\hat{A}}{A} = \frac{\hat{B}}{B} > 2$*. Then*

$$\begin{aligned} \int_{\partial \mathcal{P}} |\mathbf{u}|^2 \, d\sigma \le \frac{3B^2(1+\lambda)}{A} &\int_{\hat{\mathcal{P}} \setminus \mathcal{P}} |\nabla \mathbf{u}|^2 \, dV \\ &+ \left(1 + \frac{1}{\lambda}\right) \frac{24A^2}{7\hat{A}^3} \int_{\hat{\mathcal{P}} \setminus \mathcal{P}} |\mathbf{u}|^2 \, dV. \end{aligned} \tag{4.4}$$

Proof. Suppose that the center of the spheroid $\mathcal{P}$ is at the origin and its long axis is oriented along the z-axis. We introduce the coordinates

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = A^{-1} B \rho \cos\phi.$$

Then the volume element is given by $dV = A^{-1} B \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$, and $\rho = C$ defines a prolate spheroid with axes C and BC/A with the surface area element $d\sigma = A^{-1} C^2 \sin\phi \sqrt{B^2 \sin^2\phi + A^2 \cos^2\phi}\, d\theta \, d\phi$. We start with the relation

$$\mathbf{u}(A,\phi,\theta) = \mathbf{u}(t,\phi,\theta) - \int_A^t \mathbf{u}_\rho d\,\rho, \quad \text{where } t \in [A, \hat{A}]. \tag{4.5}$$

Let $\lambda > 0$ be fixed. Taking the square of (4.5) and applying Young's inequality gives

$$\begin{aligned} |\mathbf{u}|^2(A,\phi,\theta) &= |\mathbf{u}|^2(t,\phi,\theta) - 2\mathbf{u}(t,\phi,\theta) \cdot \int_A^t \mathbf{u}_\rho d\rho + \left| \int_A^t \mathbf{u}_\rho d\rho \right|^2 \\ &\le (1+\lambda) \left| \int_A^t \mathbf{u}_\rho d\rho \right|^2 + \left(1 + \frac{1}{\lambda}\right) |\mathbf{u}|^2(t,\phi,\theta). \end{aligned} \tag{4.6}$$

Further, by Hölder's inequality

$$\begin{aligned} \left| \int_A^t \mathbf{u}_\rho d\,\rho \right|^2 &\le \int_A^t |\mathbf{u}_\rho|^2 \rho^2 d\rho \int_A^t \rho^{-2} d\rho \\ &\le \frac{1}{A} \int_A^{\hat{A}} |\mathbf{u}|_\rho^2 \rho^2 d\rho. \end{aligned}$$

We multiply both sides of the inequality (4.6) by the determinant of the Jacobian, integrate in $\hat{\mathcal{P}} \setminus \mathcal{P}$, and use the fact that $|\mathbf{u}_\rho|^2 \le 3B^2 A^{-2} |\nabla \mathbf{u}|^2$:

$$\begin{aligned} &\frac{B}{A} \int_0^\pi \int_0^{2\pi} \int_A^{\hat{A}} |\mathbf{u}|^2(A,\phi,\theta) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &\quad \le \frac{1+\lambda}{A} \frac{\hat{A}^3 - A^3}{3} \int_0^\pi \int_0^{2\pi} \int_A^{\hat{A}} |\mathbf{u}_\rho|^2 A^{-1} B \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &\quad + \left(1 + \frac{1}{\lambda}\right) \int_0^\pi \int_0^{2\pi} \int_A^{\hat{A}} |\mathbf{u}|^2 A^{-1} B \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &\quad \le (1+\lambda) \frac{B^2 \left(\hat{A}^3 - A^3\right)}{A^3} \int_{\hat{\mathcal{P}} \setminus \mathcal{P}} |\nabla \mathbf{u}|^2 \, dV + \left(1 + \frac{1}{\lambda}\right) \int_{\hat{\mathcal{P}} \setminus \mathcal{P}} |\mathbf{u}|^2 \, dV. \end{aligned} \tag{4.7}$$

At the same time

$$\begin{aligned} &\frac{B}{A} \int_0^\pi \int_0^{2\pi} \int_A^{\hat{A}} |\mathbf{u}|^2(A,\phi,\theta) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &\quad = \frac{B(\hat{A}^3 - A^3)}{3A} \int_0^\pi \int_0^{2\pi} |\mathbf{u}|^2(A,\phi,\theta) \sin\phi \, d\theta \, d\phi \\ &\quad \ge \frac{\hat{A}^3 - A^3}{3A^2} \int_0^\pi \int_0^{2\pi} |\mathbf{u}|^2(A,\phi,\theta) a \sin\phi \sqrt{B^2 \sin^2\phi + A^2 \cos^2\phi} \, d\theta \, d\phi \\ &\quad = \frac{\hat{A}^3 - A^3}{3A^2} \int_{\partial \mathcal{P}} |\mathbf{u}|^2 \, d\sigma\,. \end{aligned} \tag{4.8}$$

Combining (4.7) with (4.8) and using the fact that $\hat{A} > 2A$, we obtain

$$\begin{aligned}\int_{\partial\mathcal{P}} |\mathbf{u}|^2\, d\sigma &\le \frac{3B^2(1+\lambda)}{A}\int_{\hat{\mathcal{P}}\setminus\mathcal{P}} |\nabla\mathbf{u}|^2\, dV + \left(1+\frac{1}{\lambda}\right)\frac{3A^2}{\hat{A}^3 - A^3}\int_{\hat{\mathcal{P}}\setminus\mathcal{P}} |\mathbf{u}|^2\, dV \\ &\le \frac{3B^2(1+\lambda)}{A}\int_{\hat{\mathcal{P}}\setminus\mathcal{P}} |\nabla\mathbf{u}|^2\, dV + \left(1+\frac{1}{\lambda}\right)\frac{24A^2}{7\hat{A}^3}\int_{\hat{\mathcal{P}}\setminus\mathcal{P}} |\mathbf{u}|^2\, dV\,. \qquad \square\end{aligned}$$

Next, we use the previous lemma to estimate the surface energy contribution in (4.1) in terms of the L^2-norms of $\mathbf{u}$ and $\nabla\mathbf{u}$.

LEMMA 4.2. *Let $\varepsilon > 0$, $\lambda > 0$ be as in Lemma* 4.1. *Then*

$$\begin{aligned}|g_\varepsilon| \int_{\cup\partial\mathcal{P}_i^\varepsilon} (\mathbf{u}\cdot\nu)^2\, d\sigma \le C(1+\lambda)\Bigg[\varepsilon \int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV \\ + \lambda^{-1}\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon} |\mathbf{u}|^2\, dV\Bigg]\end{aligned} \tag{4.9}$$

for any admissible function $\mathbf{u}$, *where the constant C is independent of ε.*

Proof. Let C denote a generic constant independent of ε. Setting $A = \varepsilon^\alpha a$, $B = \varepsilon^\alpha b$, and $\hat{A} = d\varepsilon/2$, we apply Lemma 4.1 to the surface integral term

$$\begin{aligned}\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{u}\cdot\nu)^2\, d\sigma &\le \int_{\partial\mathcal{P}_i^\varepsilon} |\mathbf{u}|^2\, d\sigma \\ &\le \varepsilon^\alpha \frac{3b^2(1+\lambda)}{a}\int_{\hat{\mathcal{P}}_i^\varepsilon\setminus\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV + \varepsilon^{2\alpha-3}\left(1+\frac{1}{\lambda}\right)\frac{192a^2}{7d^3}\int_{\hat{\mathcal{P}}_i^\varepsilon\setminus\mathcal{P}_i^\varepsilon} |\mathbf{u}|^2\, dV \\ &\le C(1+\lambda)\left[\varepsilon^\alpha \int_{\hat{\mathcal{P}}_i^\varepsilon\setminus\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV + \varepsilon^{2\alpha-3}\lambda^{-1}\int_{\hat{\mathcal{P}}_i^\varepsilon\setminus\mathcal{P}_i^\varepsilon} |\mathbf{u}|^2\, dV\right].\end{aligned}$$

Then, since $g_\varepsilon = g\varepsilon^{3-2\alpha}$ and $1 < \alpha < 2$, we have

$$\begin{aligned}|g_\varepsilon| \int_{\cup\partial P_i^\varepsilon} (\mathbf{u}\cdot\nu)^2\, d\sigma &= |g_\varepsilon| \sum_{i=1}^{N_\varepsilon}\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{u}\cdot\nu)^2\, d\sigma \\ &\le C(1+\lambda)\sum_{i=1}^{N_\varepsilon}\left[\varepsilon\int_{\hat{\mathcal{P}}_i^\varepsilon\setminus\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV + \lambda^{-1}\int_{\hat{\mathcal{P}}_i^\varepsilon\setminus\mathcal{P}_i^\varepsilon} |\mathbf{u}|^2\, dV\right] \\ &\le C(1+\lambda)\left[\varepsilon\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV + \lambda^{-1}\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon} |\mathbf{u}|^2\, dV\right]. \qquad \square\end{aligned} \tag{4.10}$$

We are now in the position to prove the following theorem.

THEOREM 4.3. *If a sequence of admissible functions $\{\mathbf{u}_\varepsilon\}$ satisfies $\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] < M$ for some constant $M > 0$ uniformly in ε, then there exists a constant $\tilde{M} > 0$ such that $\|\mathbf{u}_\varepsilon\|_{H^1(\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon)} < \tilde{M}$ uniformly in ε.*

Proof. Suppose that $\{\mathbf{u}_\varepsilon\}$ satisfies $\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] < M$ for some constant $M > 0$ uniformly in ε. Using Lemma 4.2 with $\lambda = 1$, the assumption on $W(t)$, and Hölder's inequality, we have

$$\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}\left\{|\nabla\mathbf{u}|^2+|\mathbf{u}|^4\right\}dV \le M+2\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\mathbf{u}|^2\,dV+|g_\varepsilon|\int_{\cup_i\partial\mathcal{P}_i^\varepsilon}(\mathbf{u}\cdot\nu)^2\,d\sigma$$
$$-|\Omega\setminus\cup\mathcal{P}_i^\varepsilon| \le C\epsilon\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\nabla\mathbf{u}|^2\,dV$$
$$+C\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\mathbf{u}|^2\,dV+M_1$$
$$\le C\epsilon\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\nabla\mathbf{u}|^2\,dV$$
$$+C|\Omega|^{1/2}\left(\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\mathbf{u}|^4\,dV\right)^{\frac{1}{2}}+M_1\,,$$

where $M_1>0$ is a constant independent of ε. Let ε be small enough so that $C\epsilon<\frac{1}{2}$. Then

$$\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}\left(\frac{1}{2}|\nabla\mathbf{u}|^2+|\mathbf{u}|^4\right)dV \le M_2\left[1+\left(\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\mathbf{u}|^4\,dV\right)^{\frac{1}{2}}\right]$$

uniformly in ε for some constant $M_2>0$. Using the same arguments as in [21], we conclude that there exists a constant $\tilde{M}>0$ such that $\|\mathbf{u}_\varepsilon\|_{H^1\left(\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon\right)}<\tilde{M}$ uniformly in ε. ☐

Note that the proof of Theorem 4.3 is trivial if $g>0$ when the boundary term is nonnegative.

Due to our assumptions on the distributions and the sizes of the spheroids $\mathcal{P}_i^\varepsilon$, the domains in the sequence $\Omega\setminus\cup\mathcal{P}_i^\varepsilon$ are strongly connected [22]; that is, for every function $\mathbf{u}\in H^1(\Omega\setminus\cup\mathcal{P}_i^\varepsilon)$, there exists an extension $\tilde{\mathbf{u}}\in H^1(\Omega)$ such that

$$\|\tilde{\mathbf{u}}\|_{H^1(\Omega)}\le C\,\|\mathbf{u}\|_{H^1\left(\Omega\setminus\cup\mathcal{P}_i^\varepsilon\right)}\,, \tag{4.11}$$

where $C>0$ is independent of ε. Note that a sufficient condition for (4.11) is the existence of a "security layer" around each particle having thickness comparable with the diameter of the particle as $\varepsilon\to 0$ [23]. It follows that there exists a sequence $\{\tilde{\mathbf{u}}_\varepsilon\}$ of extended minimizers that is uniformly bounded in $H^1(\Omega)$ and, up to a subsequence, converges to some $\mathbf{u}_0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Thus

$$\int_{\Omega\setminus\cup\mathcal{P}_i^\varepsilon}|\mathbf{u}_\varepsilon-\mathbf{u}_0|^2\,dV\to 0 \tag{4.12}$$

as $\varepsilon\to 0$. Further, by the trace theorem,

$$(\mathbf{u}_0-\mathbf{U})|_{\partial\Omega}=\mathbf{0}.$$

In order to identify the limiting functional and to demonstrate that $\mathbf{u}_0$ is its minimizer, we now prove the following theorem.

THEOREM 4.4. *Suppose that*

$$\mathcal{E}[\mathbf{u}]:=\int_\Omega\left[|\nabla\mathbf{u}|^2+\left(1-|\mathbf{u}|^2\right)^2+(A\mathbf{u},\mathbf{u})\right]dV \tag{4.13}$$

for every $\mathbf{u} \in H^1(\Omega)$*. Given* $\mathbf{w} \in C^\infty(\bar{\Omega})$*, there exists a sequence* $\{\mathbf{w}^\varepsilon\} \subset H^1(\Omega)$ *such that*

$$\mathcal{E}_\varepsilon\left[\mathbf{w}^\varepsilon\right] \to \mathcal{E}[\mathbf{w}] \tag{4.14}$$

when $\varepsilon \to 0$*.*

Proof. We begin by constructing a test function. Let $\mathbf{w} \in C^\infty(\bar{\Omega})$, and set

$$\mathbf{w}^\varepsilon := \mathbf{w} + \mathbf{z}^\varepsilon = \mathbf{w} + \sum_i \left(\mathbf{u}_i^\varepsilon - \mathbf{w}\right)\phi\left(\frac{|\mathbf{x}-\mathbf{x}_i^\varepsilon|}{\varepsilon^\kappa}\right), \tag{4.15}$$

where $\kappa \in (1,\alpha)$ and the function $\phi \in C^\infty(\mathbb{R}^+)$ satisfies

$$\phi(t) = \begin{cases} 1 & \text{if } t < \frac{1}{2}, \\ 0 & \text{if } t > 1. \end{cases}$$

For every $i = 1, \ldots, N_\varepsilon$, the function $\mathbf{u}_i^\varepsilon$ is a solution of the following problem:

$$\begin{cases} \Delta \mathbf{u}_i^\varepsilon - \frac{1}{\varepsilon^{2\alpha}}\left(\mathbf{u}_i^\varepsilon - \mathbf{w}_i\right) = \mathbf{0} & \text{in } B_{\varepsilon^\kappa}(\mathbf{x}_i^\varepsilon)\backslash\mathcal{P}_i^\varepsilon, \\ \frac{\partial \mathbf{u}_i^\varepsilon}{\partial \nu} + g_\varepsilon\left(\mathbf{u}_i^\varepsilon, \nu\right)\nu = \mathbf{0} & \text{on } \partial\mathcal{P}_i^\varepsilon, \\ \mathbf{u}_i^\varepsilon = \mathbf{w}_i & \text{when } |\mathbf{x}| = \varepsilon^\kappa, \end{cases} \tag{4.16}$$

where $\mathbf{w}_i = \mathbf{w}(\mathbf{x}_i^\varepsilon)$.

To understand the behavior of a solution to (4.16), for a fixed $i \in \{1, \ldots, N_\varepsilon\}$, we rescale the lengths by the characteristic size of the particle, $\mathbf{y} = \varepsilon^{-\alpha}(\mathbf{x} - \mathbf{x}_i^\varepsilon)$, and set $\hat{\mathbf{u}}_i^\varepsilon(\mathbf{y}) := \mathbf{u}_i^\varepsilon\left(\mathbf{x}_i^\varepsilon + \varepsilon^\alpha \mathbf{y}\right) - \mathbf{w}_i$. Then

$$\begin{cases} \Delta \hat{\mathbf{u}}_i^\varepsilon - \hat{\mathbf{u}}_i^\varepsilon = \mathbf{0} & \text{in } B_{\varepsilon^{\kappa-\alpha}}(\mathbf{0})\backslash\mathcal{P}_i, \\ \frac{\partial \hat{\mathbf{u}}_i^\varepsilon}{\partial \nu} + g\varepsilon^{3-\alpha}\left(\hat{\mathbf{u}}_i^\varepsilon + \mathbf{w}_i, \nu\right)\nu = \mathbf{0} & \text{on } \partial\mathcal{P}_i, \\ \hat{\mathbf{u}}_i^\varepsilon = \mathbf{0} & \text{when } |\mathbf{y}| = \varepsilon^{\kappa-\alpha}, \end{cases} \tag{4.17}$$

where the spheroid $\mathcal{P}_i = \varepsilon^{-\alpha}\mathcal{P}_\varepsilon^i$ is centered at the origin. Note that $\hat{\mathbf{u}}_i^\varepsilon$ is a critical point of the functional

$$\hat{E}_\varepsilon^i[\mathbf{u}] := \int_{B_R(\mathbf{0})\backslash\mathcal{P}_i}\left[|\nabla \mathbf{u}|^2 + |\mathbf{u}|^2\right]dV + g\varepsilon^{3-\alpha}\int_{\partial\mathcal{P}_i}\left(\mathbf{u} + \mathbf{w}_i, \nu\right)^2 d\sigma, \tag{4.18}$$

where $\mathbf{u} \in H_0^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$ and $R = \varepsilon^{\kappa-\alpha}$. We can assume that $\hat{\mathbf{u}}_i^\varepsilon$ is a global minimizer of $\hat{E}_\varepsilon^i$ over $H_0^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$ once we prove the following lemma.

LEMMA 4.5. *The* $\min_{H_0^1(B_R(\mathbf{0})\backslash\mathcal{P}_i)} \hat{E}_\varepsilon^i$ *is attained, and the minimizer satisfies*

$$\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV \le C\varepsilon^{6-2\alpha}, \tag{4.19}$$

$$\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} |\hat{\mathbf{u}}_i^\varepsilon|^2\, dV \le C\varepsilon^{6-2\alpha}, \tag{4.20}$$

$$\int_{\partial\mathcal{P}_i} |\hat{\mathbf{u}}_i^\varepsilon|^2\, d\sigma \le C\varepsilon^{6-2\alpha}. \tag{4.21}$$

Proof. 1. *Boundedness from above.* Since $\mathbf{u} \equiv \mathbf{0}$ is in $H_0^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$,

$$\min_{H_0^1(B_R(\mathbf{0})\backslash\mathcal{P}_i)} \hat{E}_\varepsilon^i \le \hat{E}_\varepsilon^i[\mathbf{0}] = g\varepsilon^{3-\alpha}\int_{\partial\mathcal{P}_i}\left(\mathbf{w}(\mathbf{x}_i^\varepsilon), \nu\right)^2 d\sigma < 1 \tag{4.22}$$

when ε is sufficiently small.

2. *Boundedness from below.* When $g \geq 0$, the result is automatic as the functional $\hat{E}^i_\varepsilon$ is nonnegative. Suppose that $g < 0$. Let $\varepsilon > 0$ be small enough so that $\mathcal{P}_i \subset B_R(\mathbf{0})$, and choose $\mathbf{u} \in C_0^\infty(B_R(\mathbf{0}))$ such that the support of $\mathbf{u}$ is contained in $B_R(\mathbf{0})$. Following the same line of reasoning as in the proof of Lemma 4.1 and switching to spherical coordinates with the z-axis along the long axis of the spheroid $\mathcal{P}_i$, we have

$$\mathbf{u}(\rho(\phi), \theta, \phi) = -\int_{\rho(\phi)}^R \mathbf{u}_r(r, \theta, \phi)\, dr\,,$$

where

$$\rho(\phi) = \frac{ab}{\left(b^2 \sin^2\phi + a^2 \cos^2\phi\right)^{\frac{1}{2}}} \tag{4.23}$$

is the equation of the spheroid. By Hölder's inequality

$$\begin{aligned}\left(\int_{\rho(\phi)}^R \mathbf{u}_r(r, \theta, \phi)\, dr\right)^2 &\leq \int_{\rho(\phi)}^R |\mathbf{u}_r(r, \theta, \phi)|^2 r^2\, dr \int_{\rho(\phi)}^R r^{-2}\, dr \\ &\leq \frac{1}{\rho(\phi)} \int_{\rho(\phi)}^R |\mathbf{u}_r(r, \theta, \phi)|^2 r^2\, dr\,;\end{aligned}$$

then

$$|\mathbf{u}(\rho(\phi), \theta, \phi)|^2 \leq \frac{1}{\rho(\phi)} \int_{\rho(\phi)}^R |\mathbf{u}_r(r, \theta, \phi)|^2 r^2\, dr\,. \tag{4.24}$$

For the prolate spheroid with the long axis in the direction of the z-axis, the element of the surface area is given by

$$d\sigma = \left(\rho^2 + \rho_\phi^2\right)^{\frac{1}{2}} \rho \sin\phi\, d\theta\, d\phi\,. \tag{4.25}$$

Multiplying (4.24) by the Jacobian and integrating, we obtain

$$\begin{aligned}&\int_0^\pi \int_0^{2\pi} |\mathbf{u}(\rho(\phi), \theta, \phi)|^2 \left(\rho^2 + \rho_\phi^2\right)^{1/2} \rho \sin\phi\, d\theta\, d\phi \\ &\qquad \leq \int_0^\pi \int_0^{2\pi} \int_{\rho(\phi)}^R |\mathbf{u}_r(r, \theta, \phi)|^2 \left(\rho^2 + \rho_\phi^2\right)^{1/2} r^2 \sin\phi\, dr\, d\theta\, d\phi\,;\end{aligned}$$

then

$$\begin{aligned}\int_{\partial\mathcal{P}_i} |\mathbf{u}|^2\, d\sigma &\leq \max_{\phi \in [0,\pi]} \left(\rho^2 + \rho_\phi^2\right)^{1/2} \int_{B_R(\mathbf{0}) \setminus \mathcal{P}_i} |\nabla \mathbf{u}|^2\, dV \\ &\leq C \int_{B_R(\mathbf{0}) \setminus \mathcal{P}_i} |\nabla \mathbf{u}|^2\, dV\,,\end{aligned} \tag{4.26}$$

where the constant $C > 0$ depends only on $\mathcal{P}_i$.

Using (4.26), we obtain the following estimate:

$$
\begin{aligned}
\hat{E}_\varepsilon^i[\mathbf{u}] &= \int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} \left[|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2\right] dV + g\varepsilon^{3-\alpha}\int_{\partial\mathcal{P}_i} (\mathbf{u}+\mathbf{w}_i,\nu)^2\, d\sigma \\
&\geq \int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} \left[|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2\right] dV - 2|g|\varepsilon^{3-\alpha}\left[\int_{\partial\mathcal{P}_i} (\mathbf{u},\nu)^2\, d\sigma\right. \\
&\left. + \int_{\partial\mathcal{P}_i} (\mathbf{w}_i,\nu)^2\, d\sigma\right] = \left(1 - C\varepsilon^{3-\alpha}\right)\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} |\nabla\mathbf{u}|^2\, dV \\
&+ \int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} |\mathbf{u}|^2\, dV - 2|g|\varepsilon^{3-\alpha}\int_{\partial\mathcal{P}_i} (\mathbf{w}_i,\nu)^2\, d\sigma \\
&\geq \frac{1}{2}\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} \left[|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2\right] dV - 1
\end{aligned} \tag{4.27}
$$

when ε is sufficiently small uniformly in $\mathbf{u}$. It follows that

$$
\hat{E}_\varepsilon^i[\mathbf{u}] > -1
$$

for the same values of ε. Since $C_0^\infty\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$ is dense in $H_0^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$, the inequalities (4.26) and (4.27) hold for all $\mathbf{u} \in H^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$.

3. *Existence of a minimizer.* Suppose that $\{\mathbf{u}_k\} \subset H_0^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$ is a minimizing sequence for $\hat{E}_\varepsilon^i$. For a sufficiently small ε, from (4.22) and (4.27) we can assume that

$$
\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} \left[|\nabla\mathbf{u}|^2 + |\mathbf{u}|^2\right] dV < 2 \tag{4.28}
$$

uniformly in k. Then, up to a subsequence, $\{\mathbf{u}_k\}$ converges weakly in the space $H_0^1\left(B_R(\mathbf{0})\backslash\mathcal{P}_i\right)$ to a $\hat{\mathbf{u}}_i^\varepsilon$ that minimizes $\hat{E}_\varepsilon^i$ by the lower semicontinuity of (4.18) and the trace theorem.

4. *Properties of the minimizer.* In this part of the proof, C denotes various constants that depend on $\mathcal{P}_i$ and $\mathbf{w}_i$ only. Multiplying (4.17) by $\hat{\mathbf{u}}_i^\varepsilon$ and integrating by parts over $B_R(\mathbf{0})\backslash\mathcal{P}_i$, we have

$$
\begin{aligned}
&\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} \left[|\nabla\hat{\mathbf{u}}_i^\varepsilon|^2 + |\hat{\mathbf{u}}_i^\varepsilon|^2\right] dV \\
&\qquad = -g\varepsilon^{3-\alpha}\int_{\partial\mathcal{P}_i} (\hat{\mathbf{u}}_i^\varepsilon + \mathbf{w}_i,\nu)\,(\hat{\mathbf{u}}_i^\varepsilon,\nu)\, d\sigma\,.
\end{aligned} \tag{4.29}
$$

From (4.27) and Hölder's inequality it follows that

$$
\begin{aligned}
\int_{\partial\mathcal{P}_i} (\hat{\mathbf{u}}_i^\varepsilon + \mathbf{w}_i,\nu)\,(\hat{\mathbf{u}}_i^\varepsilon,\nu)\, d\sigma &\leq \int_{\partial\mathcal{P}_i} (\hat{\mathbf{u}}_i^\varepsilon,\nu)^2\, d\sigma + C\left(\int_{\partial\mathcal{P}_i} (\hat{\mathbf{u}}_i^\varepsilon,\nu)^2\, d\sigma\right)^{1/2} \\
&\leq C\left[\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} |\nabla\hat{\mathbf{u}}_i^\varepsilon|^2\, dV + \left(\int_{B_R(\mathbf{0})\backslash\mathcal{P}_i} |\nabla\hat{\mathbf{u}}_i^\varepsilon|^2\, dV\right)^{1/2}\right]
\end{aligned} \tag{4.30}
$$

when ε is small enough. Now, combining (4.29) and (4.30), we obtain that

$$\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV \le C\varepsilon^{3-\alpha}\left[\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV + \left(\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV\right)^{1/2}\right], \tag{4.31}$$

$$\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\hat{\mathbf{u}}_i^\varepsilon|^2\, dV \le C\varepsilon^{3-\alpha}\left[\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV + \left(\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV\right)^{1/2}\right]. \tag{4.32}$$

From (4.31), we find that

$$\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\nabla \hat{\mathbf{u}}_i^\varepsilon|^2\, dV \le C\varepsilon^{6-2\alpha},$$

and then, from (4.32),

$$\int_{B_R(\mathbf{0})\setminus\mathcal{P}_i} |\hat{\mathbf{u}}_i^\varepsilon|^2\, dV \le C\varepsilon^{6-2\alpha}$$

uniformly in $\varepsilon \ll 1$. Finally, (4.21) follows from (4.19) and (4.26). ◻

Recall that $R = \varepsilon^{\kappa-\alpha}$. Rewriting (4.19)–(4.21) in terms of $\mathbf{x}$ gives

$$\int_{B_{\varepsilon^\kappa}(\mathbf{x}_i^\varepsilon)\setminus\mathcal{P}_i^\varepsilon} |\nabla \mathbf{u}_i^\varepsilon|^2\, dV \le C\varepsilon^{6-\alpha}, \tag{4.33}$$

$$\int_{B_{\varepsilon^\kappa}(\mathbf{x}_i^\varepsilon)\setminus\mathcal{P}_i^\varepsilon} |\mathbf{u}_i^\varepsilon - \mathbf{w}_i|^2\, dV \le C\varepsilon^{6+\alpha}, \tag{4.34}$$

$$\int_{\partial\mathcal{P}_i^\varepsilon} |\mathbf{u}_i^\varepsilon - \mathbf{w}_i|^2\, d\sigma \le C\varepsilon^{6} \tag{4.35}$$

when ε is sufficiently small. Furthermore,

$$\begin{aligned} g_\varepsilon \int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{u}_i^\varepsilon, \nu)^2\, d\sigma &= g\varepsilon^{3-2\alpha}\int_{\partial\mathcal{P}_i^\varepsilon} \{(\mathbf{u}_i^\varepsilon - \mathbf{w}_i, \nu) + (\mathbf{w}_i,\nu)\}^2\, d\sigma \\ &= g\varepsilon^{3-2\alpha}\left[\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{w}_i,\nu)^2\, d\sigma + 2\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{w}_i,\nu)(\mathbf{u}_i^\varepsilon - \mathbf{w}_i,\nu)\, d\sigma \right. \\ &\quad \left. + \int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{u}_i^\varepsilon - \mathbf{w}_i,\nu)^2\, d\sigma\right]. \end{aligned} \tag{4.36}$$

By Hölder's inequality, (4.35), and the fact that $\mathbf{w} \in C^\infty(\bar{\Omega})$, we have

$$\begin{aligned} &\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{w}_i,\nu)(\mathbf{u}_i^\varepsilon - \mathbf{w}_i,\nu)\, d\sigma \\ &\quad \le \left(\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{w}_i,\nu)^2\, d\sigma\right)^{1/2}\left(\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{u}_i^\varepsilon - \mathbf{w}_i,\nu)^2\, d\sigma\right)^{1/2} \\ &\quad \le \left(\int_{\partial\mathcal{P}_i^\varepsilon} (\mathbf{w}_i,\nu)^2\, d\sigma\right)^{1/2}\left(\int_{\partial\mathcal{P}_i^\varepsilon} |\mathbf{u}_i^\varepsilon - \mathbf{w}_i|^2\, d\sigma\right)^{1/2} = O\left(\varepsilon^4\right). \end{aligned} \tag{4.37}$$

Since the last integral in (4.36) is $O\left(\varepsilon^6\right)$, we conclude that

$$g_\varepsilon \int_{\partial \mathcal{P}_i^\varepsilon} \left(\mathbf{u}_i^\varepsilon, \nu\right)^2 d\sigma = g \varepsilon^{3-2\alpha} \int_{\partial \mathcal{P}_i^\varepsilon} \left(\mathbf{w}_i, \nu\right)^2 d\sigma + O\left(\varepsilon^{7-2\alpha}\right) . \tag{4.38}$$

We now return to estimating $\mathcal{E}_\varepsilon\left[\mathbf{w}^\varepsilon\right]$. From (4.15) we have

$$\nabla \mathbf{w}^\varepsilon = \nabla \mathbf{w} + \nabla \mathbf{z}^\varepsilon ,$$

where

$$\begin{aligned} \nabla \mathbf{z}^\varepsilon = \sum_i \Big\{ &\phi\left(\varepsilon^{-\kappa}\left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|\right) \nabla\left(\mathbf{u}_i^\varepsilon - \mathbf{w}\right) \\ &+ \frac{1}{\varepsilon^\kappa}\phi'\left(\varepsilon^{-\kappa}\left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|\right) \frac{\mathbf{x}-\mathbf{x}_i^\varepsilon}{\left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|} \otimes \left(\mathbf{u}_i^\varepsilon - \mathbf{w}\right) \Big\} . \end{aligned} \tag{4.39}$$

Then, since the supports of $\phi\left(\varepsilon^{-\kappa}\left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|\right)$ and $\phi\left(\varepsilon^{-\kappa}\left|\mathbf{x}-\mathbf{x}_j^\varepsilon\right|\right)$ are mutually nonintersecting for any $i \neq j \in 1, \ldots, N_\varepsilon$, using the definition of ϕ, we have

$$\begin{aligned} &\int_{\Omega \setminus \cup_i \mathcal{P}_i^\varepsilon} \left|\nabla \mathbf{z}_i^\varepsilon\right|^2 dV \le 2\sum_i \int_{\Omega\setminus\mathcal{P}_i^\varepsilon} \phi^2\left(\varepsilon^{-\kappa}\left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|\right) \left|\nabla\left(\mathbf{u}_i^\varepsilon - \mathbf{w}\right)\right|^2 dV \\ &+ \frac{2}{\varepsilon^{2\kappa}} \sum_i \int_{\Omega\setminus\mathcal{P}_i^\varepsilon} \left[\phi'\left(\varepsilon^{-\kappa}\left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|\right)\right]^2 \left|\mathbf{u}_i^\varepsilon - \mathbf{w}\right|^2 dV \\ &\le C \sum_i \int_{B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\setminus\mathcal{P}_i^\varepsilon} \left[\left|\nabla\left(\mathbf{u}_i^\varepsilon - \mathbf{w}\right)\right|^2 + \frac{1}{\varepsilon^{2\kappa}}\left|\mathbf{u}_i^\varepsilon - \mathbf{w}\right|^2\right] dV , \end{aligned} \tag{4.40}$$

where C depends on ϕ only. Since $\mathbf{w} \in C^\infty(\bar\Omega)$, the following estimates hold:

$$\begin{aligned} \left|\mathbf{u}_i^\varepsilon(\mathbf{x}) - \mathbf{w}(\mathbf{x})\right|^2 &\le 2\left|\mathbf{u}_i^\varepsilon(\mathbf{x}) - \mathbf{w}_i\right|^2 + 2\left|\mathbf{w}(\mathbf{x}) - \mathbf{w}_i\right|^2 \\ &\le C\left[\left|\mathbf{u}_i^\varepsilon(\mathbf{x}) - \mathbf{w}_i\right|^2 + \left|\mathbf{x}-\mathbf{x}_i^\varepsilon\right|^2\right] , \end{aligned} \tag{4.41}$$

$$\left|\nabla\left(\mathbf{u}_i^\varepsilon(\mathbf{x}) - \mathbf{w}(\mathbf{x})\right)\right|^2 \le 2\left|\nabla \mathbf{u}_i^\varepsilon(\mathbf{x})\right|^2 + C \tag{4.42}$$

for every $\mathbf{x} \in B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\setminus\mathcal{P}_i^\varepsilon$, where $C > 0$ is a constant that depends on $\mathbf{w}$ only. Therefore, by (4.33) and (4.34) we obtain

$$\begin{aligned} &\int_{B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\setminus\mathcal{P}_i^\varepsilon} \left|\nabla\left(\mathbf{u}_i^\varepsilon - \mathbf{w}\right)\right|^2 dV \\ &\le 2\int_{B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\setminus\mathcal{P}_i^\varepsilon} \left|\nabla \mathbf{u}_i^\varepsilon\right|^2 dV + C\left|B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\right| = O(\varepsilon^{\min\{6-\alpha, 3\kappa\}}) \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} &\frac{1}{\varepsilon^{2\kappa}} \int_{B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\setminus\mathcal{P}_i^\varepsilon} \left|\mathbf{u}_i^\varepsilon - \mathbf{w}\right|^2 dV \\ &\le C\left[\frac{1}{\varepsilon^{2\kappa}} \int_{B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\setminus\mathcal{P}_i^\varepsilon} \left|\mathbf{u}_i^\varepsilon - \mathbf{w}_i\right|^2 dV + \varepsilon^{3\kappa}\right] = O\left(\varepsilon^{\min\{6+\alpha-2\kappa, 3\kappa\}}\right) . \end{aligned} \tag{4.44}$$

Here $\left|B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)\right|$ is the volume of $B_{\varepsilon^\kappa}\left(\mathbf{x}_i^\varepsilon\right)$. It follows that

$$\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\nabla \mathbf{z}_i^\varepsilon\right|^2 dV = O\left(\varepsilon^{\min\{3-\alpha, 3(\kappa-1)\}}\right) = o(1) , \tag{4.45}$$

since $1 < \alpha < 2$, $1 < \kappa < \alpha$, and there are $O\left(\varepsilon^{-3}\right)$ spheroidal particles. In addition, by Hölder's inequality,

$$\begin{aligned}\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon}|\nabla\mathbf{w}^\varepsilon|^2\,dV &= \int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon}|\nabla\mathbf{z}_i^\varepsilon+\nabla\mathbf{w}|^2\,dV \\ &= \int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon}|\nabla\mathbf{w}|^2\,dV+o(1)=\int_{\Omega}|\nabla\mathbf{w}|^2\,dV+O\left(\varepsilon^{3(\alpha-1)}\right)+o(1) \\ &= \int_{\Omega}|\nabla\mathbf{w}|^2\,dV+o(1)\end{aligned}\tag{4.46}$$

when ε is small. This result extends to $\mathbf{w}\in H^1(\Omega)$ by a density argument.

Next, consider the asymptotic behavior of the nonlinear term. Extending continuously $\mathbf{w}^\varepsilon$ to $\tilde{\mathbf{w}}^\varepsilon\in H^1(\Omega)$ and using the uniform boundedness of $\mathbf{w}_\varepsilon$ in $H^1(\Omega)$ (e.g., from (4.46) and Poincaré's inequality), we conclude that there is a subsequence such that $\tilde{\mathbf{w}}^\varepsilon\rightharpoonup\mathbf{w}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ where $1<p<6$. Since the Lebesgue measure of the set $\cup_i\mathcal{P}_i^\varepsilon$ converges to zero when $\varepsilon\to0$ and $\mathbf{w}\in C^\infty(\bar{\Omega})$, we have that

$$\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon}\left(1-|\mathbf{w}^\varepsilon|^2\right)^2dV\to\int_\Omega\left(1-|\mathbf{w}|^2\right)^2dV\tag{4.47}$$

as $\varepsilon\to0$.

Finally, by (4.38), we determine that

$$\begin{aligned}g\varepsilon^{3-2\alpha}\sum_i\int_{\partial\mathcal{P}_i^\varepsilon}(\mathbf{u}_i^\varepsilon,\nu)^2\,d\sigma &= g\varepsilon^{3-2\alpha}\sum_i\int_{\partial\mathcal{P}_i^\varepsilon}(\mathbf{w}_i,\nu)^2\,d\sigma+O\left(\varepsilon^{4-2\alpha}\right) \\ &= g\varepsilon^{3-2\alpha}\sum_i\int_{\partial\mathcal{P}_i^\varepsilon}(\mathbf{w}_i,\nu)^2\,d\sigma+o(1) \\ &= g\varepsilon^3\sum_i\int_{\partial\mathcal{P}_i}(\mathbf{w}_i,\nu)^2\,d\sigma+o(1)\,,\end{aligned}\tag{4.48}$$

since $\alpha<2$. Thus

$$\begin{aligned}\mathcal{E}_\varepsilon[\mathbf{w}^\varepsilon] &= \int_\Omega\left[|\nabla\mathbf{w}|^2+\left(1-|\mathbf{w}|^2\right)^2\right]dV \\ &\quad+g\varepsilon^3\sum_i\int_{\partial\mathcal{P}_i}(\mathbf{w}_i,\nu)^2\,d\sigma+o(1)\end{aligned}\tag{4.49}$$

when ε is small. It remains to determine the asymptotic limit of the boundary term as $\varepsilon\to0$. The sum in this term can be rewritten as

$$\begin{aligned}g\varepsilon^3\sum_i\int_{\partial\mathcal{P}_i}\left(\mathbf{w}\left(\mathbf{x}_i^\varepsilon\right),\nu\right)^2d\sigma &= g\varepsilon^3\sum_i\int_{\partial\mathcal{P}_i}\left(w_k\left(\mathbf{x}_i^\varepsilon\right)\mathbf{e}_k,\nu\right)^2d\sigma \\ &= g\varepsilon^3\sum_i\left[\int_{\partial\mathcal{P}_i}(\mathbf{e}_k,\nu)(\mathbf{e}_j,\nu)\,d\sigma\right]w_j\left(\mathbf{x}_i^\varepsilon\right)w_k\left(\mathbf{x}_i^\varepsilon\right) \\ &= g\varepsilon^3\sum_i\left[\int_{\partial\mathcal{P}}(\mathbf{e}_k,R_i^\varepsilon\nu)(\mathbf{e}_j,R_i^\varepsilon\nu)\,d\sigma\right]w_j\left(\mathbf{x}_i^\varepsilon\right)w_k\left(\mathbf{x}_i^\varepsilon\right) \\ &= \sum_{jk}\left\langle A_{jk}^\varepsilon,w_jw_k\right\rangle\,,\end{aligned}\tag{4.50}$$

where $\mathbf{e}_k,\ k=1,2,3$, is an orthonormal basis in $\mathbb{R}^3$, the rotation matrices R_i^ε are such that $\mathcal{P}_i = R_i^\varepsilon \mathcal{P}$ for every $i=1,\dots,N_\varepsilon$, and the distributions

$$A^\varepsilon_{jk}\left(\mathbf{x}\right) = g\varepsilon^3 \sum_i \delta\left(\mathbf{x}-\mathbf{x}_i^\varepsilon\right) \int_{\partial\mathcal{P}} (\mathbf{e}_k, R_i^\varepsilon \nu)(\mathbf{e}_j, R_i^\varepsilon \nu)\, d\sigma$$

for every $j,k=1,2,3$. Further, $w_k = \left(\mathbf{w}\left(\mathbf{x}_i^\varepsilon\right), \mathbf{e}_k\right)$, where $k=1,2,3$ and we assume summation over the repeated indices. Thus, from our assumptions on the geometry of the domain

$$\mathcal{E}_\varepsilon\left[\mathbf{w}^\varepsilon\right] \to \int_\Omega \left[\left|\nabla \mathbf{w}\right|^2 + \left(1-\left|\mathbf{w}\right|^2\right)^2 + \left(A\mathbf{w},\mathbf{w}\right)\right] dV \tag{4.51}$$

for every $\mathbf{w} \in C^\infty(\bar{\Omega})$. ∎

THEOREM 4.6. *Let a sequence of minimizers $\{\mathbf{u}_\varepsilon\}$ of $\mathcal{E}_\varepsilon$ be such that the sequence $\{\tilde{\mathbf{u}}_\varepsilon\}$ of extensions of $\{\mathbf{u}_\varepsilon\}$ to Ω converges weakly in $H^1(\Omega)$ to some $\mathbf{u} \in H^1(\Omega)$. Then*

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] \geq \mathcal{E}[\mathbf{u}]\,, \tag{4.52}$$

where $\mathcal{E}$ is defined by (4.13).

Proof. Suppose that there is $\{\mathbf{u}_\delta\} \subset C^1(\Omega)$ such that $\mathbf{u}_\delta \to \mathbf{u}$ strongly in $H^1(\Omega)$ and the extensions to Ω of minimizers $\mathbf{u}_\varepsilon$ of $\mathcal{E}_\varepsilon$ converge $\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u}$ weakly in $H^1(\Omega)$. We construct $\mathbf{u}_\delta^\varepsilon = \mathbf{u}_\delta + \mathbf{z}_\delta^\varepsilon$ in the same way as in (4.15) so that their extensions $\tilde{\mathbf{u}}_\delta^\varepsilon \rightharpoonup \mathbf{u}_\delta$ converge weakly in $H^1(\Omega)$ along with $\mathcal{E}_\varepsilon\left[\mathbf{u}_\delta^\varepsilon\right] \to \mathcal{E}\left[\mathbf{u}_\delta\right]$ as $\varepsilon \to 0$. Let $\tilde{\zeta}_\delta^\varepsilon := \tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{u}}_\delta^\varepsilon$, and denote its restriction to $\Omega\backslash \cup_i \mathcal{P}_i^\varepsilon$ by ζ_ε^δ. Then $\tilde{\zeta}_\varepsilon^\delta \rightharpoonup \zeta_\delta := \mathbf{u} - \mathbf{u}_\delta$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for $p<6$ as $\varepsilon \to 0$.

We begin by observing that the expression for $\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon]$ can be rewritten as

$$\begin{aligned}
\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] = {} & \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon^\delta] + \int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left|\nabla\zeta_\varepsilon^\delta\right|^2 dV + 2\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left(\nabla\zeta_\varepsilon^\delta, \nabla\mathbf{u}_\varepsilon^\delta\right)\, dV \\
& + \int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^4 dV - 2\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 dV - 4\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV \\
& + 2\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 \left|\mathbf{u}_\varepsilon^\delta\right|^2 dV + 4\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)^2 dV \\
& + 4\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left|\mathbf{u}_\varepsilon^\delta\right|^2 \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV + 4\int_{\Omega\backslash\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV \\
& + 2\sum_i g_\varepsilon \int_{\partial\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \nu\right)^2 d\sigma + \sum_i g_\varepsilon \int_{\partial\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \nu\right)\left(\mathbf{u}_\varepsilon^\delta, \nu\right)\, d\sigma
\end{aligned} \tag{4.53}$$

so that the inequality

$$\mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] \geq \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon^\delta] + \Phi_{\varepsilon\delta}$$

holds, where

$$
\begin{aligned}
\Phi_{\varepsilon\delta} :=\ & 2\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left(\nabla\zeta_\varepsilon^\delta, \nabla\mathbf{u}_\varepsilon^\delta\right)\, dV \\
& - 2\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 dV - 4\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV \\
& + 4\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\mathbf{u}_\varepsilon^\delta\right|^2 \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV + 4\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV \\
& + 2\sum_i g_\varepsilon \int_{\partial\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \nu\right)^2 d\sigma + \sum_i g_\varepsilon \int_{\partial\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \nu\right)\left(\mathbf{u}_\varepsilon^\delta, \nu\right)\, d\sigma\,.
\end{aligned} \tag{4.54}
$$

We need to estimate each term on the right-hand side of (4.54). In the remainder of the proof, $C > 0$ denotes a constant independent of ε and δ.

(a) Beginning with the first term, we write

$$
\begin{aligned}
\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} (\nabla\mathbf{u}_\delta^\varepsilon, \nabla\zeta_\delta^\varepsilon)\, dV = & \int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} (\nabla\mathbf{u}_\delta, \nabla\zeta_\delta^\varepsilon)\, dV \\
& + \int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} (\nabla\mathbf{z}_\delta^\varepsilon, \nabla\zeta_\delta^\varepsilon)\, dV\,.
\end{aligned} \tag{4.55}
$$

We have

$$
\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} (\nabla\mathbf{u}_\delta, \nabla\zeta_\delta^\varepsilon)\, dV = \int_\Omega \left(\nabla\mathbf{u}_\delta, \nabla\tilde{\zeta}_\delta^\varepsilon\right) dV - \int_{\cup_i\mathcal{P}_i^\varepsilon} \left(\nabla\mathbf{u}_\delta, \nabla\tilde{\zeta}_\delta^\varepsilon\right) dV\,. \tag{4.56}
$$

The second integral in (4.56) can be estimated with the help of Minkowski's and Hölder's inequalities,

$$
\begin{aligned}
& \int_{\cup_i\mathcal{P}_i^\varepsilon} \left(\nabla\mathbf{u}_\delta, \nabla\tilde{\zeta}_\delta^\varepsilon\right) dV \le \left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}_\delta|^2\, dV\right)^{1/2} \left(\int_{\cup_i\mathcal{P}_i^\varepsilon} \left|\nabla\tilde{\zeta}_\delta^\varepsilon\right|^2 dV\right)^{1/2} \\
& \le \left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}_\delta|^2\, dV\right)^{1/2} \left(\int_\Omega \left[|\nabla\tilde{\mathbf{u}}_\delta^\varepsilon|^2 + |\nabla\tilde{\mathbf{u}}_\varepsilon|^2\right] dV\right)^{1/2} \\
& \le C\left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}_\delta|^2\, dV\right)^{1/2} \le C\left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}_\delta - \nabla\mathbf{u} + \nabla\mathbf{u}|^2\, dV\right)^{1/2} \\
& \le C\left[\left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV\right)^{1/2} + \left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}_\delta - \nabla\mathbf{u}|^2\, dV\right)^{1/2}\right] \\
& \le C\left[\left(\int_{\cup_i\mathcal{P}_i^\varepsilon} |\nabla\mathbf{u}|^2\, dV\right)^{1/2} + \left(\int_\Omega |\nabla\mathbf{u}_\delta - \nabla\mathbf{u}|^2\, dV\right)^{1/2}\right] \\
& \to C\left(\int_\Omega |\nabla\mathbf{u}_\delta - \nabla\mathbf{u}|^2\, dV\right)^{1/2}
\end{aligned}
$$

when $\varepsilon \to 0$ because $\mathbf{u} \in H^1(\Omega)$ and $|\cup_i\mathcal{P}_i^\varepsilon| \to 0$. Here $C > 0$ is independent of δ. Consider now the first integral in (4.56). By the weak convergence of $\tilde{\zeta}_\delta^\varepsilon$ to $\mathbf{u}_\delta - \mathbf{u}$

and Hölder's inequality, we have that

$$\begin{aligned}
&\int_\Omega \left(\nabla \mathbf{u}_\delta, \nabla\tilde{\zeta}_\delta^\varepsilon\right) dV \to \int_\Omega \left(\nabla \mathbf{u}_\delta, \nabla(\mathbf{u}_\delta - \mathbf{u})\right) dV \\
&\le \left(\int_\Omega |\nabla \mathbf{u}_\delta|^2\right)^{1/2} \left(\int_\Omega |\nabla \mathbf{u} - \nabla \mathbf{u}_\delta|^2\right)^{1/2} \\
&\le C\|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)}
\end{aligned}$$

when $\varepsilon \to 0$.

Now, for the second term in (4.55), we find using (4.45) and Hölder's inequality

$$\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} (\nabla \mathbf{z}_\delta^\varepsilon, \nabla\zeta_\delta^\varepsilon)\, dV \le \left(\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} |\nabla \mathbf{z}_\delta^\varepsilon|^2\right)^{1/2} \left(\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} |\nabla\zeta_\delta^\varepsilon|^2\right)^{1/2} \to 0$$

when $\varepsilon \to 0$.

(b) Consider the second term in (4.54). Since $\tilde{\zeta}_\varepsilon^\delta$ converges weakly to $\mathbf{u} - \mathbf{u}_\delta$ in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for $p < 6$ when $\varepsilon \to 0$, we have that

$$\begin{aligned}
\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 dV &\le \int_\Omega \left|\tilde{\zeta}_\varepsilon^\delta\right|^2 dV \to \int_\Omega |\mathbf{u} - \mathbf{u}_\delta|^2\, dV \\
&\le \|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)}
\end{aligned}$$

as $\varepsilon \to 0$.

(c) Using Hölder's inequality, we find that

$$\begin{aligned}
&\left|\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV\right| \le \left(\int_\Omega \left|\tilde{\mathbf{u}}_\varepsilon^\delta\right|^2 dV\right)^{1/2} \left(\int_\Omega \left|\tilde{\zeta}_\varepsilon^\delta\right|^2 dV\right)^{1/2} \\
&\to \left(\int_\Omega |\mathbf{u}_\delta|^2\, dV\right)^{1/2} \left(\int_\Omega |\mathbf{u} - \mathbf{u}_\delta|^2\, dV\right)^{1/2} \le C\|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)}
\end{aligned}$$

when $\varepsilon \to 0$.

(d) Estimating in the same way as in (c), we obtain

$$\begin{aligned}
&\left|\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\mathbf{u}_\varepsilon^\delta\right|^2 \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV\right| \le \left(\int_\Omega \left|\tilde{\mathbf{u}}_\varepsilon^\delta\right|^4 dV\right)^{3/4} \left(\int_\Omega \left|\tilde{\zeta}_\varepsilon^\delta\right|^4 dV\right)^{1/4} \\
&\to \left(\int_\Omega |\mathbf{u}_\delta|^4\, dV\right)^{3/4} \left(\int_\Omega |\mathbf{u} - \mathbf{u}_\delta|^4\, dV\right)^{1/4} \le C\|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)}
\end{aligned}$$

by Sobolev embedding.

(e) Estimating in the same way as in (c), we obtain

$$\begin{aligned}
&\left|\int_{\Omega\setminus\cup_i\mathcal{P}_i^\varepsilon} \left|\zeta_\varepsilon^\delta\right|^2 \left(\zeta_\varepsilon^\delta, \mathbf{u}_\varepsilon^\delta\right)\, dV\right| \le \left(\int_\Omega \left|\tilde{\zeta}_\varepsilon^\delta\right|^4 dV\right)^{3/4} \left(\int_\Omega \left|\tilde{\mathbf{u}}_\varepsilon^\delta\right|^4 dV\right)^{1/4} \\
&\to \left(\int_\Omega |\mathbf{u}_\delta|^4\, dV\right)^{1/4} \left(\int_\Omega |\mathbf{u} - \mathbf{u}_\delta|^4\, dV\right)^{3/4} \le C\|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)}^3
\end{aligned}$$

by Sobolev embedding.

(f) We use (4.10) with $\lambda = 1$ to obtain

$$\begin{aligned} g_\varepsilon \sum_i \int_{\partial \mathcal{P}_i^\varepsilon} (\zeta_\delta^\varepsilon, \nu)^2 \, d\sigma &\le C \left[\varepsilon \int_{\Omega \setminus \cup \mathcal{P}_i^\varepsilon} |\nabla \zeta_\delta^\varepsilon|^2 \, dV + \int_{\Omega \setminus \cup \mathcal{P}_i^\varepsilon} |\zeta_\delta^\varepsilon|^2 \, dV \right] \\ &\le C \left[\varepsilon \int_{\Omega \setminus \cup \mathcal{P}_i^\varepsilon} |\nabla \zeta_\delta^\varepsilon|^2 \, dV + \int_\Omega \left| \tilde{\zeta}_\delta^\varepsilon \right|^2 dV \right] \\ &\to C \int_\Omega |\mathbf{u} - \mathbf{u}_\delta|^2 \, dV \le C \|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)} \end{aligned}$$

by the strong convergence of $\tilde{\mathbf{u}}_\delta^\varepsilon$ and $\tilde{\mathbf{u}}_\varepsilon$ in $L^p(\Omega),\ 1 < p < 6$, to $\mathbf{u}_\delta$ and $\mathbf{u}$, respectively.

(g) Using Hölder's inequality, we get

$$\begin{aligned} &\left| \sum_i g_\varepsilon \int_{\partial \mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \nu\right) \left(\mathbf{u}_\varepsilon^\delta, \nu\right) \, d\sigma \right| \\ &\le \sum_i \left(\int_{\partial \mathcal{P}_i^\varepsilon} |g_\varepsilon| \left(\mathbf{u}_\varepsilon^\delta, \nu\right)^2 \, dV \right)^{1/2} \left(\int_{\partial \mathcal{P}_i^\varepsilon} |g_\varepsilon| \left(\zeta_\varepsilon^\delta, \nu\right)^2 \, dV \right)^{1/2} . \end{aligned}$$

As in (f), applying (4.10) with $\lambda = 1$, we have that

$$\limsup_{\varepsilon \to 0} \int_{\partial \mathcal{P}_i^\varepsilon} |g_\varepsilon| \left(\mathbf{u}_\varepsilon^\delta, \nu\right)^2 \, dV \le C \int_\Omega |\mathbf{u}_\delta|^2 \, dV \, ;$$

then, with the help of (f), we obtain the estimate

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \left| \sum_i g_\varepsilon \int_{\partial \mathcal{P}_i^\varepsilon} \left(\zeta_\varepsilon^\delta, \nu\right) \left(\mathbf{u}_\varepsilon^\delta, \nu\right) \, d\sigma \right| \\ &\quad \le C \left(\int_\Omega |\mathbf{u}_\delta|^2 \, dV \right)^{1/2} \left(\int_\Omega |\mathbf{u} - \mathbf{u}_\delta|^2 \, dV \right)^{1/2} \le C \|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)} \, . \end{aligned}$$

Now, from (a)–(g) and (4.54), the inequality

$$\limsup_{\varepsilon \to 0} |\Phi_{\varepsilon\delta}| \le C \|\mathbf{u} - \mathbf{u}_\delta\|_{H^1(\Omega)} \tag{4.57}$$

holds when δ is small. Thus, using (4.54), we conclude that

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] \ge \lim_{\delta \to 0} \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon^\delta] - \lim_{\delta \to 0} \limsup_{\varepsilon \to 0} |\Phi_{\varepsilon\delta}| \, ,$$

and

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon] \ge \mathcal{E}[\mathbf{u}] \, ,$$

because $\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon[\mathbf{u}_\varepsilon^\delta] = \mathcal{E}[\mathbf{u}_\delta]$ and $\mathcal{E}$ is continuous with respect to the strong convergence of $\mathbf{u}_\delta$ to $\mathbf{u}$ in $H^1(\Omega)$. ☐

From (4.51) and (4.52) it follows that $\mathcal{E}[\mathbf{u}] \le \mathcal{E}[\mathbf{w}]$ for every $\mathbf{w} \in H^1(\Omega)$; hence $\mathbf{u}$ minimizes $\mathcal{E}$ over $H^1(\Omega)$.

4.2. Magnetic energy. Having established the asymptotics of the liquid crystalline component of the energy, we now turn our attention to magnetic interactions. Consider (3.4) for the prolate spheroidal particle $\mathcal{P}$ with semiaxes $a > b$ and the long axis oriented in the direction of the z-axis. It is well known [17] that the solution to this problem in the exterior of $\mathcal{P}$ is given by

$$\phi = \frac{4\pi a\, b^2 m}{\left(a^2-b^2\right)^{3/2}}\left(\tanh^{-1}(t)-t\right) z \tag{4.58}$$

in cylindrical coordinates (ρ,θ,z), where $t = \xi^{-1/2}\left(a^2-b^2\right)^{1/2}$ and ξ is the largest root of

$$\frac{\rho^2}{\xi+b^2-a^2}+\frac{z^2}{\xi}=1\,.$$

Further, m is the density of the magnetic moment, so that $\mathbf{m} = \frac{4\pi a\, b^2 m}{3}\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}$ is a unit vector in the direction of the z-axis. Assuming that $a \ll \left(\rho^2+z^2\right)^{1/2}$ and expanding in $a/\left(\rho^2+z^2\right)^{1/2}$, we find that

$$\phi = \frac{32\pi a\, b^2 m}{3r^3} z + O\left(z(a/r)^5\right), \tag{4.59}$$

where $r = \sqrt{\rho^2+z^2} = |\mathbf{x}|$. Note that the leading term in (4.59) is identical to that for a sphere of the same volume as $\mathcal{P}$ and centered at the origin [24]. The leading order term in the expansion of the magnetic filed $\mathbf{H}$ generated by the ferromagnetic particle $\mathcal{P}$ is given by

$$\mathbf{H}(\mathbf{x}) = \frac{32\pi a\, b^2 m}{3r^3}\left(\frac{3z}{r^2}\mathbf{x}-\hat{\mathbf{z}}\right)+O\left((a/r)^5\right)\;;$$

then

$$|\mathbf{H}(\mathbf{x})| = O\left(m(a/r)^3\right) \tag{4.60}$$

when $a/r \ll 1$.

Now consider the term corresponding to the magnetic interaction between the particles $\mathcal{P}_i^\varepsilon$ and $\mathcal{P}_j^\varepsilon$ for some $i,j = 1,\dots,N_\varepsilon$. We have

$$\begin{aligned}\int_{\mathcal{P}_i^\varepsilon}(\mathbf{H}_j^\varepsilon,\mathbf{m}_i^\varepsilon)\,dV+\int_{\mathcal{P}_j^\varepsilon}(\mathbf{H}_i^\varepsilon,\mathbf{m}_j^\varepsilon)\,dV &= O\left(\frac{|m_i^\varepsilon|\left|m_j^\varepsilon\right|\mathrm{Vol}(\mathcal{P}_i^\varepsilon)\mathrm{Vol}(\mathcal{P}_j^\varepsilon)}{d^3}\right)\\ &= O\left(\varepsilon^{6\alpha+2\beta_1-3}\right)\;;\end{aligned} \tag{4.61}$$

then

$$\int_{\mathbb{R}^3}(\mathbf{m}_\varepsilon,\mathbf{H}_\varepsilon)\;dV = O\left(N_\varepsilon^2\varepsilon^{6\alpha+2\beta_1-3}\right) = O\left(\varepsilon^{6\alpha+2\beta_1-9}\right)\to 0 \tag{4.62}$$

when $\varepsilon\to 0$ by our assumptions α and β_1.

Finally, we consider the interaction between the external magnetic field and ferromagnetic particles. We have

$$\begin{aligned}\int_{\mathbb{R}^3}(\mathbf{m}_\varepsilon,\mathbf{h}_\varepsilon)\,dV &= \sum_i\int_{\mathcal{P}_i^\varepsilon}(\mathbf{m}_i^\varepsilon,\mathbf{h}_\varepsilon)\,dV\\ &= \int_\Omega(\mathbf{h},\mathbf{M}^\varepsilon)\;dV\to\int_\Omega(\mathbf{h},\mathbf{M})\;dV\end{aligned} \tag{4.63}$$

by (3.6) and (3.7), where $\mathbf{M}$ is the effective magnetic moment density.

Combining the results for the liquid crystal and magnetic energies, we conclude that the minimizers of the family of functionals $\mathcal{F}_\varepsilon$ converge to a minimizer of the functional

$$\mathcal{F}_0[\mathbf{u}] = \int_\Omega \left[|\nabla \mathbf{u}|^2 + \left(1 - |\mathbf{u}|^2\right)^2 + (A\mathbf{u}, \mathbf{u}) - 2(\mathbf{h}, \mathbf{M}) \right] dV, \tag{4.64}$$

concluding the proof of Theorem 3.1.

5. Example: Periodically distributed particles. Now suppose that the particles are distributed periodically in Ω with their centers of mass positioned at the vortices of a cubic lattice with the side ϵ. If we assume that there is a continuous function

$$R : \Omega \to \mathrm{Orth}^+ := \left\{ X \in M^{3\times 3} : XX^T = I,\ \det X = 1 \right\},$$

such that $R_i^\varepsilon = R\left(\mathbf{x}_i^\varepsilon\right)$ for every $i = 1, \ldots, N_\varepsilon$ and $\varepsilon > 0$, then

$$\mathbf{M}(\mathbf{x}) = m\, R(\mathbf{x})\hat{\mathbf{z}}, \tag{5.1}$$

and

$$\begin{aligned} A(\mathbf{x}) &= g\, R(\mathbf{x}) \left(\int_{\partial\mathcal{P}} \nu \otimes \nu d\sigma \right) R^T(\mathbf{x}) \\ &= g\, R(\mathbf{x})(\lambda_1(\hat{\mathbf{z}} \otimes \hat{\mathbf{z}}) + \lambda_2\, (I - \hat{\mathbf{z}} \otimes \hat{\mathbf{z}}))R^T(\mathbf{x}) \\ &= \frac{g}{m^2}(\lambda_1(\mathbf{M}(\mathbf{x}) \otimes \mathbf{M}(\mathbf{x})) + \lambda_2\, (I - \mathbf{M}(\mathbf{x}) \otimes \mathbf{M}(\mathbf{x}))), \end{aligned} \tag{5.2}$$

where λ_1 and λ_2 are the two distinct eigenvalues of $\int_{\partial\mathcal{P}} \nu \otimes \nu d\sigma$. The coupling terms in (4.64) then take the form

$$\frac{g(\lambda_1 - \lambda_2)}{m^2}\, (\mathbf{M}, \mathbf{u})^2 + \frac{g\lambda_2}{m^2}|\mathbf{u}|^2 - 2(\mathbf{h}, \mathbf{M}). \tag{5.3}$$

For a needle-like prolate spheroid with a high aspect ratio we have that $\lambda_1 \ll \lambda_2$ and the coefficient $\Lambda := \frac{g(\lambda_1 - \lambda_2)}{m^2}$ in front of $(\mathbf{M}, \mathbf{u})^2$ has a sign opposite that of g. Hence nematic molecules align perpendicular to $\mathbf{M}$ when $\Lambda > 0$ and parallel to $\mathbf{M}$ when $\Lambda < 0$. Since the model in [7] assumes that $|\mathbf{u}| = 1$, the middle term in (5.3) can be neglected, and the remaining interaction terms in (5.3) coincide with those in (1.1) up to a difference in notation.

6. Summary. We derive an expression for the effective energy of a dilute ferronematic composite consisting of identical spheroidal magnetic particles distributed in a nematic liquid crystalline matrix. The particles are assumed to be well separated from each other, and the boundary of the domain and the distributions of their positions and orientations are subject to certain convergence properties in the limit of decreasing particle size. We model the liquid crystal according to the Ericksen theory of nematics with variable degree of orientation and impose soft anchoring conditions on the surfaces of ferromagnetic particles as represented by the Rapini–Papoular energy term. Further, we consider a parametric regime in which the relative contributions of various components of the energy are of the same order of magnitude.

The homogenized energy derived in this work is more general than what can be found in the literature as it is obtained within a more general theory under less restrictive assumptions on the geometry of the composite. The effective interaction between the liquid crystal and the particles is due to surface anchoring and is represented by the matrix that encodes the information on the shape and size of the particles, their locations, and their orientation with respect to a fixed frame. Likewise, the effective magnetic moment that couples the particles to the external magnetic field depends on the spatial and orientational distributions of the particles. For the high-aspect-ratio needle-like particles, the coupling terms reduce to their counterparts derived in [7] on the basis of the Oseen–Frank theory for the nematic director.

REFERENCES

[1] F. Brochard and P. G. de Gennes, *Theory of magnetic suspensions in liquid crystals*, J. Phys. France, 31 (1970), pp. 691–708.

[2] J. Rault, P. E. Cladis, and J. P. Burger, *Ferronematics*, Phys. Lett., 32A (1970), pp. 199–200.

[3] S. H. Chen and N. M. Amer, *Observation of macroscopic collective behavior and new texture in magnetically doped liquid crystals*, Phys. Rev. Lett., 51 (1983), pp. 2298–2301.

[4] S. V. Burylov, L. Raikher, and A. N. Zakhlevnykh, *Orientational structure and magnetic properties of ferronematic in an external field*, Zh. Eksp. Teor. Fiz., 91 (1986), pp. 545–551.

[5] S. V. Burylov and L. Raikher, *Ferronematics: On the development of the continuum theory approach*, J. Magnetism Magnetic Materials, 85 (1990), pp. 74–76.

[6] S. V. Burylov and L. Raikher, *Orientation of a solid particle embedded in a monodomain nematic liquid crystal*, Phys. Rev. E, 50 (1994), pp. 358–367.

[7] S. V. Burylov and L. Raikher, *Macroscopic properties of ferronematics caused by orientational interactions on the particle surfaces.* I*, Extended continuum model*, Mol. Cryst. Liq. Cryst., 258 (1995), pp. 107–122.

[8] L. Raikher, *Orientational Frederiks-like transitions in ferronematics*, in Phase Transitions in Complex Fluids, P. Toledano and A. M. Figueiredo Neto, eds, World Scientific, Singapore, 1998, pp. 295–315.

[9] R. E. Bena and E. Petrescu, *Surface effects on magnetic Freedericksz transition in ferronematics with soft particle anchoring*, J. Magnetism Magnetic Materials, 263 (2003), pp. 353–359.

[10] P. Kopcansky, M. Koneracka, I. Potocova, M. Timko, J. Jadzyn, and G. Czechowski, *The anchoring of nematic molecules on magnetic particles in some types of ferronematics*, Czech J. Phys., 51 (2001), p. 59.

[11] I. Potocova, P. Kopcansky, M. Koneracka, M. Timko, J. Jadzyn, and G. Czechowski, *The influence of magnetic field on electric Fredericksz transition in* 8*CB-based ferronematic*, J. Magnetism Magnetic Materials, 196–197 (1999), pp. 578–580.

[12] P. Kopcansky, I. Potocova, M. Koneracka, M. Timko, A. Jansen, J. Jadzyn, and G. Czechowski, *The anchoring of nematic molecules on magnetic particles in some types of ferroelectrics*, J. Magnetism Magnetic Materials, 289 (2005), pp. 101–104.

[13] V. I Zadorozhnii, A. N. Vasilev, V. Yu. Reshetnyak, K. S. Thomas, and T. J. Slukin, *Nematic director response in ferronematic cells*, Europhys. Lett., 73 (2006), pp. 408–414.

[14] L. M. Lopatina and J. V. Selinger, *Theory of ferroelectric nanoparticles in nematic liquid crystals*, Phys. Rev. Lett., 102 (2009), 197802.

[15] L. M. Lopatina and J. V. Selinger, *Maier-Saupe-type theory of ferroelectric nanoparticles in nematic liquid crystals*, Phys. Rev. E, 84 (2011), 041703.

[16] J. L. Ericksen, *Liquid crystals with variable degree of orientation*, Arch. Ration. Mech. Anal., 113 (1991), pp. 97–120.

[17] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed., Course of Theoretical Physics 8, Butterworth–Heinemann, Oxford, UK, 1984.

[18] E. G. Virga, *Variational Theories for Liquid Crystals*, Chapman & Hall, London, 1994.

[19] N. Mottram and C. Newton, *Introduction to Q-Tensor Theory*, Technical report, University of Strathclyde, Glasgow, UK, 2004.

[20] P. G. De Gennes and J. Prost, *The Physics of Liquid Crystals*, Clarendon Press, Oxford, UK, 1993.

[21] L. Berlyand and E. Khruslov, *Competition between the surface and the boundary layer energies in a Ginzburg-Landau model of a liquid crystal composite*, Asymptot. Anal., 29 (2002), pp. 185–219.

[22] E. Khruslov, *Asymptotic behavior of the second boundary value problem under fragmentation of the domain's boundary*, Mat. Sb., 106 (1978), pp. 604–621.
[23] L. Berlyand, *Homogenization of the Ginzburg-Landau functional with surface energy term*, Asymptot. Anal., 21 (1999), pp. 379–399.
[24] L. Borcea and O. Bruno, *On the magneto-elastic properties of elastomer-ferromagnet composites*, J. Mech. Phys. Solids, 49 (2001), pp. 2877–2919.