



Membrane duality revisited

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Received 24 September 2015; accepted 7 October 2015

Available online 22 October 2015

Editor: Herman Verlinde

Abstract

Just as string T-duality originates from transforming field equations into Bianchi identities on the string worldsheet, so it has been suggested that M-theory U-dualities originate from transforming field equations into Bianchi identities on the membrane worldvolume. However, this encounters a problem unless the target space has dimension $D = p + 1$. We identify the problem to be the nonintegrability of the U-duality transformation assigned to the pull-back map. Just as a double geometry renders manifest the $O(D, D)$ string T-duality, here we show in the case of the M2-brane in $D = 3$ that a generalized geometry renders manifest the $SL(3) \times SL(2)$ U-duality. In the case of M2-brane in $D = 4$, with and without extra target space coordinates, we show that only the $GL(4, R) \times R^4$ subgroup of the expected $SL(5, R)$ U-duality symmetry is realized.

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1. Introduction

1.1. The story so far

1.1.1. Strings, T-duality and double geometry

Some time ago [1], it was pointed out that strings moving in a D -dimensional space M^D with coordinates $X^\mu(\tau, \sigma)$, background metric $g_{\mu\nu}(X)$ and 2-form $B_{\mu\nu}(X)$ could usefully be described by a doubled geometry with $2D$ -dimensional coordinates

$$Z^M = (X^\mu, Y_\sigma) \quad (1.1)$$

and doubled metric¹

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu} & B_{\mu\rho} g^{\rho\sigma} \\ -g^{\mu\sigma} B_{\sigma\nu} & g^{\mu\nu} \end{pmatrix}. \quad (1.2)$$

The motivation was twofold; worldsheet and spacetime:

1. Worldsheet

In the case when M^D is the D -torus T^D , this renders manifest the $O(D, D)$ T-duality by combining worldsheet field equations and Bianchi identities via the constraint

$$\Omega_{MN} \epsilon^{ij} \partial_j Z^N = G_{MN} \sqrt{-\gamma} \gamma^{ij} \partial_j Z^N, \quad (1.3)$$

where

$$\Omega_{MN} = \begin{pmatrix} 0 & \delta_\mu^\beta \\ \delta^{\alpha\nu} & 0 \end{pmatrix}, \quad (1.4)$$

and γ_{ij} is the worldsheet metric.

2. Spacetime

In the case when M^D is a generic manifold, the $2D$ -dimensional diffeomorphisms with parameter $\xi^M = (\xi^\mu, \lambda_\alpha)$ suggest a way of unifying² D -dimensional diffeomorphisms

$$\delta g_{\mu\nu} = -\partial_\mu \xi^\rho g_{\rho\nu} - \partial_\nu \xi^\rho g_{\mu\rho} - \partial_\rho g_{\mu\nu} \xi^\rho, \quad (1.5)$$

and 2-form gauge invariance

$$\delta B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu. \quad (1.6)$$

After all, G_{MN} is just the Kaluza–Klein metric with spacetime metric $g_{\mu\nu}$, gauge field A_μ^a and internal metric g_{ab}

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu^a g_{ab} A_\nu^b & A_\mu^a g_{ab} \\ g_{ab} A_\nu^b & g_{ab} \end{pmatrix}, \quad (1.7)$$

where the “gauge field” is $B_{\mu\alpha}$ and the “internal” metric is $g^{\alpha\beta}$. If this programme were successful one would expect the $SL(D)/SO(D)$ coset of general relativity to be promoted to an $O(D, D)/(SO(D) \times SO(D))$, as conjectured in [3,5].

¹ G_{MN} had previously appeared in [2] with a different physical interpretation as a metric on phase space.

² An earlier alternative suggestion [3] was to use the non-symmetric metric $g_{\mu\nu} + b_{\mu\nu}$. The two alternatives are related by the two-vielbein approach [4].

In summary, the worldsheet goal of rendering manifest the string T-duality $O(D, D)$ by doubling the coordinates was achieved successfully in [1] and a T-dual worldsheet action using the doubled coordinates was then constructed in [6]. However, there were missing ingredients in the spacetime approach: the generalized diffeomorphisms were subsequently supplied in [4,7]

$$\delta G_{MN} = \xi^P \partial_P G_{MN} + (\partial_M \xi^P - \partial^P \xi_M) G_{PN} + (\partial_N \xi^P - \partial^P \xi_N) G_{MP}, \quad (1.8)$$

and the section condition subsequently supplied in [7]

$$\Omega^{MN} \partial_M \partial_N = 0. \quad (1.9)$$

(The need for the section condition has, however, been called into question [8,9].) Once these ingredients were included, it was possible also to build a generalized spacetime action for G_{MN} . This activity came to be known as “Double Field Theory.”

For further developments and variations on this doubled geometry theme, in addition to those already cited, including “Generalized geometry” and the E_{11} approach see, for example, [8, 10–42].

1.1.2. Branes, U-duality and M-theory

Following [1], it was pointed out [43–45] that membranes moving in a $(D \leq 4)$ -dimensional space M^D with coordinates $X^\mu(\tau, \sigma, \rho)$, background metric $g_{\mu\nu}(X)$ and 3-form $B_{\mu\nu\rho}(X)$ could usefully be described by a geometry with $[D + D(D - 1)/2]$ -dimensional coordinates

$$Z^M = (X^\mu, Y_{\rho\sigma}) \quad (1.10)$$

and generalized metric

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + B_{\mu\rho\sigma} g^{\rho\sigma\lambda\tau} B_{\lambda\tau\nu} & B_{\mu\rho\sigma} g^{\rho\sigma\lambda\tau} \\ g^{\mu\nu\rho\sigma} B_{\rho\sigma\nu} & g^{\mu\nu\rho\sigma} \end{pmatrix}, \quad (1.11)$$

where

$$g^{\alpha\beta\gamma\delta} = \frac{1}{2}(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}). \quad (1.12)$$

Once again, the motivation was twofold; worldvolume and spacetime:

1. Worldvolume

In the case when M^D is the D -torus T^D , the hope was to render manifest the M-theory U-dualities (using modern parlance) by combining worldvolume field equations and Bianchi identities. For example, the U-duality would be $SL(5, R)$ in the case $D = 4$. The restriction to $D \leq 4$ arises because, just as the usual coordinates X^μ correspond to momentum in the supersymmetry algebra, so the extra coordinates $Y_{\mu\nu}$ correspond to the M2 central charge. But for $D \geq 5$, this is not enough, as shown in Table 2 in [43]. There is also the M5 central charge with corresponding coordinates $Y_{\mu\nu\rho\sigma\tau}$, which first appears in $D = 5$. In Appendix A, we illustrate the emergence of extra coordinates from central charges in the M-theory algebra for general D . For example, in the $D = 7$ case $X^\mu, Y_{\mu\nu}, \tilde{Y}^{\mu\nu} \sim \epsilon^{\mu\nu\rho\sigma\tau\lambda\kappa} Y_{\rho\sigma\tau\lambda\kappa}$ and \tilde{X}^μ form a 56 of the U-duality symmetry $E_{7(7)}$.

2. Spacetime

If this programme were successful, one would expect the $SL(D)/SO(D)$ of general relativity to be promoted not merely to $O(D, D)/(SO(D) \times SO(D))$ but to $E_8/SO(16)$, with possible infinite-dimensional extensions involving E_9 , E_{10} and E_{11} as conjectured in [5,14,46]. Once again, however, the generalized diffeomorphisms, section conditions and U-invariant actions came later. This activity has become known as “Exceptional Field Theory”. For subsequent developments and variations on generalized geometry in M-theory and U-duality see, for example, [17,22–24,26,28–30,47–59], where the 5-brane and other extended objects were incorporated, as required for $D > 4$. The E_{11} approach [14] goes further with infinitely many coordinates of which those associated with the M-theory central charges are but a subset.

In summary, in contrast with strings where both the worldsheet and spacetime approaches have been successful, the brane worldvolume approach seems problematical and, with the exception of [22,29], recent developments have tended to focus on the spacetime approach where the extra coordinates (1.10) and generalized metric (1.11) have proved valuable. In fact, the worldvolume approach has been questioned by Percacci and Sezgin [60], by Sen [61], and by Lukas and Ovrut [62]. They suggest that it works only for target space dimensions $D = p + 1$. In this case, the $D!/((D - p)!p!)$ wrapping modes on a D -torus ($D \geq p + 1$) and the D Kaluza–Klein modes are equal in number as in the case of a string. Sen argues that this equality is a requirement. If so, the $D = 3$ U-duality $SL(3) \times SL(2)$ might be expected, but the $D = 4$ U-duality $SL(5)$, would not.

In any event, the need to include coordinates corresponding to central charges in the M-theory algebra exposes a major difference between U-duality in M-theory and T-duality in string theory. In string theory, T-duality takes strings into strings, but in M-theory U-duality mixes up p -branes with different p . It seems unlikely, therefore, that the M2-brane worldvolume alone is sufficient. Somehow the totality of p -brane worldvolumes must conspire to give the full U-duality. This remains an unsolved problem.

Finally we note that the purpose of extra coordinates in both string and M-theory is to render the T and U dualities manifest. If one is content with non-manifest T-duality, one may invoke the Gaillard–Zumino (GZ) approach [63], as was done in [64]. The GZ approach to U duality is discussed below.

1.2. This paper

This paper is devoted purely to the worldvolume approach. We shall show:

- There is a problem with $SL(5, R)$, which manifests itself both in the GZ approach (which doesn't introduce extra coordinates), as well as in approaches in which extra coordinates are introduced [43,65]. In the GZ approach, as well as the approach of [43], we shall show that the obstacle to the realization of $SL(5, R)$ symmetry is the nonintegrability of the transformation rule for the pull-back map. In the approach of [65], we shall show that the proposed manifestly $SL(5, R)$ -invariant equation for a membrane in a target based on generalized geometry does not support linearized fluctuations about a Poincaré invariant vacuum solution.
- In the case of topological membranes, the $SL(2, R)$ symmetry is known but we shall formulate it in a double geometry setting.
- We shall rederive the result that the membrane in $d + 3$ dimensions has a Heisenberg subgroup of the $SL(2, R)$ symmetry [60], by making use of simple integrability considerations.

2. Topological membranes

Although p -branes in $p + 1$ dimensions carry no dynamical degrees of freedom [66], they are nevertheless of considerable interest. In the present context, they provide us with a setting in which we can get a handle on duality symmetries that transform field equations and Bianchi identities into each other. Everything we will do here applies to topological p -branes for general p , but for simplicity in notation as well as our special interest in $M2$ -branes, we shall focus on topological membranes.

The standard action for the closed membrane is

$$I = \int d^3\sigma \left[-\frac{1}{2}\sqrt{-\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + \frac{1}{6}\varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho B_{\mu\nu\rho} + \frac{1}{2}\sqrt{-\gamma} \right], \quad (2.1)$$

where in our conventions $\varepsilon^{012} = +1$. For the topological membrane we take $\mu = 0, 1, 2$.³ The $SL(3)$ is manifest. For simplicity, we shall take the metric tensor $g_{\mu\nu}$ and 3-form potential $B_{\mu\nu\rho}$ in the 3-dimensional target space to be constant.⁴

In this case, using the algebraic field equation $\gamma_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$, we have the identity

$$\sqrt{-\gamma} \gamma^{ij} g_{\mu\nu} \partial_j X^\nu = -\frac{1}{2}\sqrt{-g} \varepsilon_{\mu\nu\rho} \varepsilon^{ijk} \partial_j X^\nu \partial_k X^\rho. \quad (2.2)$$

Therefore, letting $B_{\mu\nu\rho} = \sqrt{-g} \varepsilon_{\mu\nu\rho} B$, the action can be written as

$$I = \frac{1}{3!} \int d^3\sigma \sqrt{-g} (1 + B) \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho \varepsilon_{\mu\nu\rho}. \quad (2.3)$$

We shall, however, use the form (2.1) of the action below, motivated by the fact that this form will make it easier to compare with what happens in the case of the non-topological membrane. The resulting equations are

$$\partial_i P_\mu^i = 0, \quad \gamma_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \quad (2.4)$$

where

$$P_\mu^i \equiv -\sqrt{-\gamma} \gamma^{ij} \partial_j X^\nu g_{\mu\nu} + \frac{1}{2} \varepsilon^{ijk} \partial_j X^\nu \partial_k X^\rho B_{\mu\nu\rho}. \quad (2.5)$$

There is also a conserved topological current:

$$\partial_i J^{i\mu\nu} = 0, \quad J^{i\mu\nu} = \varepsilon^{ijk} \partial_j X^\mu \partial_k X^\nu. \quad (2.6)$$

Because we are considering a target space that is three dimensional, we can make the definitions

$$J_\mu^i = \frac{1}{2} \varepsilon_{\mu\nu\rho} J^{i\nu\rho}, \quad B_{\mu\nu\rho} = B |g|^{1/2} \varepsilon_{\mu\nu\rho}. \quad (2.7)$$

³ This form of the action is in accordance with the terminology of ‘topological membrane’ we are using here. It should be noted, however, that the characterization of membranes as ‘topological’ also arises in the context of membranes that propagate in dimensions higher than three but with action that consists of Wess–Zumino term and no kinetic term. See, for example [67]. In general, branes with pure Wess–Zumino terms exhibit a huge symmetry enhancement; see, for example [68].

⁴ This is an important assumption. Otherwise the modification in equation (2.4) will obstruct the sought after duality symmetry. The spacetime background here can be viewed as being a subsector a time-like dimensional reduction of the bosonic sector of $D = 11$ supergravity down to 8 dimensions, where only the fields $(g_{\mu\nu}, B_{\mu\nu\rho})$ are kept, and the 8 dimensional Euclidean coordinates are taken to be constants. In a spacelike dimensional reduction, the signature of the metric $g_{\mu\nu}$ would be Euclidean, and we would take the 8 dimensional spacetime coordinates to be constant. All of our considerations apply for this case as well.

Noting that $\varepsilon^{ijk}\varepsilon_{\mu\nu\rho}\partial_j X^\nu\partial_k X^\rho = -2(\det\partial X)\gamma^{ij}g_{\mu\nu}\partial_j X^\nu$ and that $\det\partial X = |\gamma|^{1/2}|g|^{-1/2}$, we can write P_μ^i and J_μ^i as follows

$$P_\mu^i = -\sqrt{-\gamma}\gamma^{ij}g_{\mu\nu}(1+B)\partial_j X^\nu, \quad J_\mu^i = \sqrt{-\gamma}\gamma^{ij}g_{\mu\nu}|g|^{-1/2}\partial_j X^\nu. \quad (2.8)$$

From these equations we find

$$P_\mu^i = -(1+B)|g|^{1/2}J_\mu^i. \quad (2.9)$$

Note that this equation readily follows from the form of the action given in (2.3). We may now consider linear $GL(2, R) = SL(2, R) \times R$ transformations of the form⁵

$$\delta \begin{pmatrix} P_\mu^i \\ J_\mu^i \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} P_\mu^i \\ J_\mu^i \end{pmatrix} + \lambda \begin{pmatrix} P_\mu^i \\ J_\mu^i \end{pmatrix}. \quad (2.10)$$

We see that (2.9) is left invariant provided that $|g|$ and B transform such that

$$C \equiv -(1+B)|g|^{1/2} \quad (2.11)$$

transforms as

$$\delta C = b + 2aC - cC^2. \quad (2.12)$$

This can be seen from (2.9), noting that it implies $C\mathbb{1} = PJ^{-1}$. It represents the infinitesimal form of a fractional linear transformation of the real variable C , and gives a representation of the algebra $SL(2, R) \times R$. The fact that this symmetry acts on a combination of g and B is a consequence of the fact that the target spacetime is Lorentzian, as noted in [60].

To make the $SL(2, R)$ symmetry manifest, we introduce a doubled system of coordinates $Z^{a\mu}$, with $a = 1, 2$, such that $X^\mu = -Z^{1\mu}$ and

$$P_\mu^i = \sqrt{-\gamma}\gamma^{ij}|g|^{-1/2}g_{\mu\nu}\partial_j Z^{2\nu}, \quad J_\mu^i = -\sqrt{-\gamma}\gamma^{ij}|g|^{-1/2}g_{\mu\nu}\partial_j Z^{1\nu}. \quad (2.13)$$

This doubling of coordinates is in accordance with the generalized target space geometry recently studied in [69] for maximal supergravity in eight dimensions. Using these definitions, it follows that (2.9) can be written as

$$\partial_i Z^{a\mu} = G^{ab}\epsilon_{bc}\partial_i Z^{c\mu}, \quad (2.14)$$

where

$$G^{ab} = \begin{pmatrix} |g|^{-1/2} & B \\ B & |g|^{1/2}(B^2 - 1) \end{pmatrix}, \quad (2.15)$$

transforming by conjugation under $SL(2, R)$. Note that $\det G_{ab} = -1$, such that the product $(\det G_{ab})(\det g_{\mu\nu}) = 1$. Denoting the inverse of this metric by G_{ab} , it transforms under infinitesimal $SL(2, R)$ transformations as $\delta G_{ab} = \Lambda_a^c G_{cb} + \Lambda_b^c G_{ac}$, with Λ as given in (2.10). Written out, this gives

$$\delta g^{1/2} = 2c|g|B - 2d|g|^{1/2}, \quad \delta B = -b|g|^{-1/2} - c|g|^{1/2}(B^2 - 1). \quad (2.16)$$

⁵ It is understood that there also exists the trivial $SL(3, R) \times R^3$ symmetry realized as $\delta P_\mu^i = (R_\mu^{\nu} + S^{(\mu)}\delta_\mu^\nu)P_\mu^i$ and $\delta J_\mu^i = (R_\mu^{\nu} + S^{(\mu)}\delta_\mu^\nu)J_\mu^i$, where R_μ^{ν} are real traceless matrices and $S^{(\mu)}$ are the real scaling parameters.

Using these rules, the transformation of $-|g|^{1/2}(1+B)$ indeed gives the result (2.12). Demanding manifest $SL(2, R)$ invariance thus fixes the separate variation of g and B under $SL(2, R)$, not just its combination (2.11).

It is important to note that the $SL(2, R)$ transformation under which $(\partial_i Z_{1\mu}, \partial_i Z_{2\mu})$ forms a doublet is embedded into (2.10) with a field-dependent scale transformation with parameter

$$\lambda = -a + \frac{1}{3}c(3+B), \tag{2.17}$$

as can be determined from (2.13).

Turning to the key equation (2.14), it can be written as $P_{ab}\partial_i Z^{b\mu} = 0$, where $P_{ab} = \frac{1}{2}(G_{ab} - \epsilon_{ab})$ is a projector, and it amounts to

$$\partial_i Z^{2\mu} = |g|^{1/2}(1+B)\partial_i Z^{1\mu}. \tag{2.18}$$

Acting with ∇_i does not yield the field equation $\partial_i(\sqrt{-\gamma}\gamma^{ij}g_{\mu\nu}\partial_j X^v) = 0$. However, the latter is identically satisfied for the topological membrane, since

$$\partial_i(\sqrt{-\gamma}\gamma^{ij}g_{\mu\nu}\partial_j X^v) \equiv -\frac{1}{2}\partial_i(\sqrt{-g}\epsilon_{\mu\nu\rho}\epsilon^{ijk}\partial_j X^v\partial_k X^\rho) = 0, \tag{2.19}$$

recalling that the target space metric is constant.

It is instructive to perform a double dimensional reduction [70] of (2.14). To this end, we let

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \phi g_{\mu\nu} & 0 \\ 0 & \phi^{-2} \end{pmatrix}, \quad \hat{B}_{\mu\nu 2} = |g|^{1/2}\epsilon_{\mu\nu} B, \tag{2.20}$$

where $g_{\mu\nu}, \phi$ and B are constants. Note that $\sqrt{-\hat{g}} = \sqrt{-g}$. Choosing the gauge $X^2 = \sigma^2$ and letting $\partial_2 Z^{a\mu} = 0$ gives

$$\hat{\gamma}_{\hat{i}\hat{j}} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & \phi^{-2} \end{pmatrix}, \quad \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ij}\gamma^{kl}\partial_k X^\mu \partial_\ell X^\nu g_{\mu\nu} = 0. \tag{2.21}$$

It follows that (2.14), or equivalently (2.18), holds for the topological string, where the indices now run over two values, namely, $i = 0, 1$ and $\mu = 0, 1$. Setting $\hat{i} = 2$ and $\hat{\mu} = 2$ in (2.18) fixes Z^{22} , giving it a linear dependence on the coordinate σ^2 . Comparing (2.14) for the topological string with (1.3), they are in fact, contrary to appearance, the same equation, with the identification $x^\mu = -Z^{1\mu}$ and $y_\mu = -\epsilon_{\mu\nu}Z^{2\nu}$. This can be seen by writing (1.3) as

$$G^{ab}\epsilon_{bc}\epsilon_{\mu\nu}\epsilon^{ij}\partial_j Z^{c\nu} = \bar{g}_{\mu\nu}\sqrt{-\gamma}\gamma^{ij}\partial_j Z^{a\nu}, \tag{2.22}$$

where $\bar{g}_{\mu\nu} = |g|^{-1/2}g_{\mu\nu}$. Thus

$$G^{1b}\epsilon_{bc}\epsilon_{\mu\nu}\epsilon^{ij}\partial_j Z^{c\nu} = \bar{g}_{\mu\nu}\sqrt{-\gamma}\gamma^{ij}\partial_j Z^{1\nu} = -\epsilon^{ij}\epsilon_{\mu\nu}\partial_j Z^{1\nu}, \tag{2.23}$$

where we have used the formula for the determinant of $\partial_j Z^{1\nu}$ in the second equation. From this we conclude

$$G^{1b}\epsilon_{bc}\partial_j Z^{c\nu} = -\partial_j Z^{1\nu}, \tag{2.24}$$

showing the equivalence of (2.14) and (1.3), up to a relative sign which can be attributed to convention choices.

3. Membrane in $D = 8$

The M2-brane action in $D = 8$ can be obtained from the M2-brane action in $D = 11$ by dimensional reduction on 3-torus. The bosonic sector of such a reduction has been studied in [62] where $SL(3, R) \times SL(2, R)$ symmetry could not be established. In a different approach aiming at a direct construction of an M2-brane action in $D = 8$ which couples to all the bosonic fields of the maximal supergravity theory in which the $SL(3, R) \times SL(2, R)$ symmetry is built in manifestly has been proposed [71]. However, the condition of $SL(2, R)$ symmetry puts nonlinear constraints on the field which have been solved only in a fashion that exhibits a two parameter subgroup of $SL(2, R)$ as a symmetry. More specifically, the bosonic sector of maximal $D = 8$ supergravity has the fields

$$(g_{\mu\nu}, B_{\mu\nu\rho}, C_{\mu\nu m}, A_{\mu}^{mr}, 7\phi), \quad m = 1, 2, 3, \quad r = 1, 2 \quad (3.1)$$

where the seven scalars parametrize the coset $(SL(3, R)/SO(3)) \times (SL(2, R)/SO(2))$, the vector fields transform as $(3, 2)$ of $SL(3, R) \times SL(2, R)$. The field strength of the 3-form field is combined with the dual field strength for a doublet of $SL(2, R)$. The gauge invariance of the pullbacks of the field strengths of the 1, 2, 3-form potentials requires the introduction of worldvolume fundamental potentials, resulting in the field strengths $h^r = db^r - \underline{B}^r + \dots$, $g_m = dc_m - \underline{C}_m + \dots$ and $f^{mr} = d\phi^{mr} - \underline{A}^{mr}$, where the underlined fields are the pullback of the target space forms. The action proposed in [71] then takes the form $I = \int \sqrt{-\gamma} \lambda (1 + \Phi(f, g) + \star h^r \star h^s G_{rs})$ where λ is a Lagrangian multiplier field, G_{rs} is $SL(2, R)$ matrix parametrized in terms of the $SL(2, R)/SO(2)$ coset scalars and Φ is a function of the field strengths (f^{mr}, g_m) . Duality relations for these field strengths are imposed by hand in addition to the field equations that follow from the action to ensure the correct number of propagation degrees of freedom, namely the 5 scalars coming from X^μ and 3 scalars ϕ^{m1} . The resulting field equations have not lent themselves to a solution in general, however, and a special solution discussed in [71] breaks $SL(2, R)$ symmetry.

Our approach here is instead to consider a membrane propagating in $D = 8$ dimensions and coupled to the target space metric and 3-form potential only, and to study the duality symmetry of the standard membrane action. The background can be viewed as the truncated version of the maximal supergravity. In fact, all considerations below apply equally well to p -branes in $d + p$ dimensions propagating in the background of a metric and $p + 1$ potential. We thus consider an $8 + 3$ dimensional spacetime with coordinates

$$X^{\hat{\mu}} = (x^\mu, y^\alpha), \quad \mu = 0, \dots, 7, \quad \alpha = 1, 2, 3, \quad (3.2)$$

with x^0 being in the time direction. We take the background geometry to have the form

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{\alpha\beta}(x) \end{pmatrix} = \begin{pmatrix} g_{(3)}^\kappa \bar{g}_{\mu\nu} & 0 \\ 0 & g_{(3)}^{1/3} \bar{g}_{\alpha\beta} \end{pmatrix}, \quad (3.3)$$

where $g_{(3)} \equiv \det g_{\alpha\beta}$ and $\bar{g}_{\mu\nu}, \bar{g}_{\alpha\beta}$ are assumed to be $SL(2, R)$ invariant, and κ is an exponent to be determined. We take the only nonvanishing component of the 3-form to be $B_{\alpha\beta\gamma}$, and assume that all the target space background fields to depend on x^μ only. Thus, the action is

$$I = \int d^3\sigma \left[-\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} (\partial_i x^\mu \partial_j x^\nu g_{\mu\nu}(x) + \partial_i y^\alpha \partial_j y^\beta g_{\alpha\beta}(x)) + \frac{1}{2} \sqrt{-\gamma} + \frac{1}{3!} \varepsilon^{ijk} \partial_i y^\alpha \partial_j y^\beta \partial_k y^\gamma B_{\alpha\beta\gamma}(x) \right]. \quad (3.4)$$

In maximal supergravity theory in $D = 8$, the fields $(g_{\alpha\beta}, B_{\alpha\beta\gamma})$ contain five scalars that parametrize the coset $SL(3, R)/SO(3)$ and two scalar parameterizing the coset $SL(2, R)/SO(2)$. In addition to these fields and the metric $g_{\mu\nu}$ there are two triplet of vectors, $g_{\mu\alpha}$ and $B_{\mu\alpha\beta}$, that transform as $(3, 2)$ under three 2-forms $B_{\mu\nu\alpha}$ that transform as $(3, 1)$ under the U-duality group $SL(3, R) \times SL(2, R)$. We are neglecting the latter fields below, with the expectation that they would not effect the realization of $SL(3, R) \times SL(2, R)$ U-duality symmetry at the level of duality rotations on the worldvolume of the membrane in $D = 8$, should such symmetry exist at all.⁶

Turning the action (3.4), it implies that the induced worldvolume metric is given by

$$\gamma_{ij} = g_{(3)}^{\kappa} \bar{g}_{\mu\nu} \partial_i x^\mu \partial_j x^\nu + g_{(3)}^{1/3} \bar{g}_{\alpha\beta} \partial_i y^\alpha \partial_j y^\beta, \quad (3.5)$$

and the field equations are

$$\partial_i P_\mu^i = S_\mu, \quad \partial_i P_\alpha^i = 0, \quad (3.6)$$

where

$$\begin{aligned} P_\alpha^i &\equiv -\sqrt{-\gamma} \gamma^{ij} \partial_j y^\beta g_{\alpha\beta} + \frac{1}{2} \varepsilon^{ijk} \partial_j y^\beta \partial_k y^\gamma B_{\alpha\beta\gamma}, \\ P_\mu^i &\equiv -\sqrt{-\gamma} \gamma^{ij} \partial_j x^\nu g_{\mu\nu}, \\ S_\mu &\equiv -\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i x^\nu \partial_j x^\rho \partial_\mu g_{\nu\rho} - \frac{1}{6} \varepsilon^{ijk} \partial_i y^\alpha \partial_j y^\beta \partial_k y^\gamma \partial_\mu B_{\alpha\beta\gamma}. \end{aligned} \quad (3.7)$$

There is also a conserved topological current:

$$\partial_i J^{i\alpha\beta} = 0, \quad J^{i\alpha\beta} \equiv \varepsilon^{ijk} \partial_j y^\alpha \partial_k y^\beta, \quad (3.8)$$

such that, defining

$$B_{\alpha\beta\gamma}(x) = \varepsilon_{\alpha\beta\gamma} g_{(3)}^{1/2} B, \quad (3.9)$$

we have

$$\begin{aligned} P_\alpha^i &= -\left(\sqrt{-\gamma} \gamma^{ij} + \sqrt{V} V^{ij} B\right) g_{\alpha\beta} \partial_j y^\beta, \\ J_\alpha^i &\equiv \frac{1}{2} \varepsilon_{\alpha\beta\gamma} J^{i\beta\gamma} = \sqrt{V} V^{ij} g_{(3)}^{-1/2} g_{\alpha\beta} \partial_j y^\beta, \end{aligned} \quad (3.10)$$

where V^{ij} is the inverse of

$$V_{ij} \equiv \partial_i y^\alpha \partial_j y^\beta g_{\alpha\beta}. \quad (3.11)$$

We can combine these equations as

$$P_\alpha^i = -|g|^{1/2} (\delta_j^i + B X^i_j) J_\alpha^j, \quad (3.12)$$

where

$$X^i_j = \frac{(\gamma^{-1} V)^i_j}{\sqrt{-\det(\gamma^{-1} V)}}. \quad (3.13)$$

⁶ All fields have been kept in [62] where $SL(3, R) \times SL(2, R)$ duality as duality rotation symmetry on the membrane worldvolume is sought but not found.

In [60], the self-consistency of (3.12) was studied in detail, and it was shown that a two-parameter subgroup of $SL(2, R)$ can be realized. In doing so, the complicated transformation rule for the induced metric, which is no longer a scaling transformation we saw in the case of topological membrane, was taken into account in [60].⁷

Here, we shall avoid this complication and show that this same result can be derived more simply from the integrability of the transformation rule for $\partial_i y^\alpha$. Assuming that P_α^i and J_α^i transform as in (2.10), we can compute the transformation of $\partial_i y^\alpha$ by using (3.8) and (2.10), finding

$$\delta \partial_i y^\alpha = \left[\left(-a + c \left(1 + \frac{1}{3} B + \frac{1}{2} \text{tr} X \right) \right) \delta_i^j - c X^j_i \right] \partial_j y^\alpha. \tag{3.14}$$

As discussed above, the integrability of this variation is in general not guaranteed, even on-shell. Clearly the a transformation is always integrable, and so the remaining question is whether the c transformation in (3.14) is integrable. In order to test this on-shell, it will suffice to consider a particular membrane solution [72], for which $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ and we take

$$\begin{aligned} x^0 &= \sigma^0, & x^\mu &= \text{constant for } 1 \leq \mu \leq 7, \\ y^1 &= \alpha \sigma_1 \cos(\omega \sigma^0), & y^2 &= \alpha \sigma_1 \sin(\omega \sigma^0), & y^3 &= \beta \sigma_2. \end{aligned} \tag{3.15}$$

Thus we have the induced metric

$$\gamma_{ij} = \partial_i y^\alpha \partial_j y^\beta \delta_{\alpha\beta} + \partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}, \tag{3.16}$$

giving

$$\gamma_{00} = -1 + \alpha^2 \omega^2 \sigma_1^2, \quad \gamma_{11} = \alpha^2, \quad \gamma_{22} = \beta^2. \tag{3.17}$$

It is straightforward to verify that this is a solution of the equations of motion. We see that in this background

$$\begin{aligned} X - \frac{1}{2} \text{tr} X &= \text{diag}(f, h, h), \\ f &= -\frac{1}{2} \alpha \omega \sigma_1 (1 - \alpha^2 \omega^2 \sigma_1^2)^{-1/2} - (\alpha \omega \sigma_1)^{-1} (1 - \alpha^2 \omega^2 \sigma_1^2)^{1/2}, \\ h &= \frac{1}{2} \alpha \omega \sigma_1 (1 - \alpha^2 \omega^2 \sigma_1^2)^{-1/2}. \end{aligned} \tag{3.18}$$

Let us consider the transformations $\delta \partial_1 y^2$ and $\delta \partial_2 y^2$ under c , which from (3.15), (3.14) and (3.18) will therefore give

$$\delta \partial_1 y^2 = 0, \quad \delta \partial_2 y^2 = c \beta h. \tag{3.19}$$

Checking the integrability, we see from the first equation that $\partial_2 \delta \partial_1 y^2 = 0$, whereas from the second equation $\partial_1 \delta \partial_2 y^2 = c \beta \frac{\partial h}{\partial \sigma_1} \neq 0$. This example is therefore sufficient to show that the proposed transformation for $\delta \partial_i y^\alpha$ under the c transformation, subject only to the use of the membrane equations of motion, is not integrable.

Choosing $\kappa = -\frac{1}{6}$, equations (3.12) transform properly under the remaining $SL(2, R)$ transformations

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \tag{3.20}$$

⁷ In [43] only the second term is kept, and therefore it effectively deals with membrane in D dimensional target where $\alpha = 1, \dots, D$.

provided the background parameters g and B transform according to (2.16). As a consequence, the transformation of the induced metric takes the form

$$\delta\gamma_{ij} = -\frac{2a}{3}\gamma_{ij}, \quad \delta V_{ij} = -\frac{2a}{3}V_{ij} \tag{3.21}$$

Equations (2.9) can be combined as

$$X^j{}_i \partial_j Z^{a\alpha} = G^{ab} \epsilon_{bc} \partial_i Z^{c\alpha} . \tag{3.22}$$

The presence of the matrix $X^i{}_j$ shows that the duality symmetry of this equation is the Heisenberg group with the underlying algebra parametrized as in (3.20).

4. Membrane in $D = 4$

In this section we shall study the action (2.1) for $D = 4$ target spacetime with Lorentzian signature. In this action (2.1), the target space fields can be interpreted as the bosonic sector of $N = 1, D = 4$ supergravity in which the cosmological constant is dualized to a 3-form potential [73–75]. Supermembrane action in this setting exists [76,77] and it has been studied in detail in [75]. The spacetime background can also be viewed as being a subsector a time-like dimensional reduction of the bosonic sector of $D = 11$ supergravity down to 7 dimensions, where only the fields $(g_{\mu\nu}, B_{\mu\nu\rho})$ are kept, and the 7 dimensional Euclidean coordinates are taken to be constants. In a spacelike dimensional reduction, the signature of the metric $g_{\mu\nu}$ would be Euclidean, and again, we would take the 7 dimensional spacetime coordinates to be constant. Our considerations apply to this case as well but we shall adhere to the Lorentzian signature for concreteness.

We shall consider two approaches to the problem of duality rotations in this theory. In the first approach, due to Gaillard and Zumino [63], there is no need to introduce any extra coordinates. Rather, one examines the consistency of the duality rotations, since the definition of the conjugate momentum field associated with the worldvolume scalar fields involves the topological current whose conservation is the Bianchi identity. In a second approach considered in [43–45], one introduces extra coordinates in order to try to achieve a manifest realization of the duality symmetry.

4.1. Gaillard–Zumino approach

The equations of motion for a membrane in a four-dimensional target space with coordinates X^μ ($\mu = 0, \dots, 3$) can be written as

$$\partial_i P_\mu^i = 0, \quad \gamma_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \tag{4.1}$$

where

$$P_\mu^i \equiv -g_{\mu\nu} \mathcal{F}^{i\nu} + \frac{1}{2} B_{\mu\nu\rho} J^{i\nu\rho} \tag{4.2}$$

and

$$\mathcal{F}^{i\mu} \equiv \sqrt{-\gamma} \gamma^{ij} \partial_j X^\mu \quad J^{i\mu\nu} \equiv \epsilon^{ijk} \partial_j X^\mu \partial_k X^\nu . \tag{4.3}$$

It is important to note that (4.3) implies the relation⁸

$$J^{i\mu\nu} = \gamma^{i\ell} \epsilon_{\ell jk} \mathcal{F}^{j\mu} \mathcal{F}^{k\nu} . \tag{4.4}$$

⁸ We use the convention $\epsilon^{012} = -\epsilon_{012} = 1$.

It will also be useful to note the relation

$$-\gamma \gamma^{ij} = \mathcal{F}^{i\mu} \mathcal{F}^{j\nu} g_{\mu\nu} . \quad (4.5)$$

The question then is what is the largest set of transformations that transforms P_μ^i and $J^{i\mu\nu}$ into each other in a consistent manner, such that the system of field equations $\partial_i P_\mu^i = 0$ and the Bianchi identity $\partial_i J^{i\mu\nu} = 0$ remain invariant.

Our task is to check whether the equations of motion are invariant under $SL(5, R)$ transformations which can be parametrized as

$$\Lambda^M_N = \begin{pmatrix} a^\mu{}_\nu + \frac{1}{4}a \delta_\nu^\mu & -\frac{1}{6}\varepsilon^{\mu\nu\rho\sigma} b_{\nu\rho\sigma} \\ \frac{1}{6}\varepsilon_{\nu\mu\rho\sigma} c^{\mu\rho\sigma} & -a \end{pmatrix} , \quad (4.6)$$

where $a^\mu{}_\nu$ is traceless. It acts on a 5-plet of $SL(5, R)$ as $\delta V_M = -\Lambda^P{}_M V_P$. Thus, defining the components of a triplet of second-rank antisymmetric tensor of K^i_{MN} as $K^i_{\mu 5} := P_\mu^i$ and $K^i_{\mu\nu} := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{i\rho\sigma}$, it follows from $\delta K^i_{MN} = 2\Lambda^P{}_{[M} K^i_{N]P}$ that

$$\delta \begin{pmatrix} P_\mu^i \\ J^{i\mu\nu} \end{pmatrix} = \begin{pmatrix} -a^\rho{}_\mu + \frac{3}{4}a \delta_\mu^\rho & \frac{1}{2}b_{\mu\rho\sigma} \\ c^{\mu\nu\rho} & 2a^{[\mu}{}_{[\rho} \delta_{\sigma]}^\nu] - \frac{1}{2}a \delta_{\rho\sigma}^{\mu\nu} \end{pmatrix} \begin{pmatrix} P_\rho^i \\ J^{i\rho\sigma} \end{pmatrix} \quad (4.7)$$

where $\delta_{\rho\sigma}^{\mu\nu} = \frac{1}{2}(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu)$. Next, assembling $(g_{\mu\nu}, B_{\mu\nu\rho})$ into a symmetric $SL(5, R)$ matrix G_{MN} , where $M, N = 1, \dots, 5$, with identifications

$$\begin{aligned} G_{\mu\nu} &= g^{-2/5} g_{\mu\nu} , & G_{\mu 5} &= G_{5\mu} = \frac{1}{3!} g^{-2/5} g_{\mu\alpha} \varepsilon^{\alpha\beta\gamma\delta} B_{\beta\gamma\delta} , \\ G_{55} &= g^{3/5} \left(1 + \frac{1}{3!} B^2 \right) , & B^2 &\equiv B^{\mu\nu\rho} B_{\mu\nu\rho} , \end{aligned} \quad (4.8)$$

it follows from $\delta G_{MN} = -2\Lambda^P{}_{(M} G_{N)P}$ that

$$\delta g_{\mu\nu} = -2a^\sigma{}_{(\mu} g_{\nu)\sigma} + \frac{5}{6} \left(a + \frac{2}{15} c \cdot B \right) g_{\mu\nu} - c^{\alpha\beta}{}_{(\mu} B_{\nu)\alpha\beta} , \quad (4.9)$$

$$\delta B_{\mu\nu\rho} = -3a^\sigma{}_{[\mu} B_{\nu\rho]\sigma} + \frac{5}{4} \left(a - \frac{2}{15} c \cdot B \right) B_{\mu\nu\rho} + b_{\mu\nu\rho} + c_{\mu\nu\rho} , \quad (4.10)$$

where indices are lowered on the parameters $c^{\mu\nu\rho}$ using the metric $g_{\mu\nu}$, and we have defined $c \cdot B \equiv c^{\alpha\beta\gamma} B_{\alpha\beta\gamma}$. Next, the variation of $\mathcal{F}^{i\mu}$ can be found from (4.2) by using the variations (4.7), (4.9) and (4.10). The result is

$$\delta \mathcal{F}^{i\mu} = a^\mu{}_\nu \mathcal{F}^{i\nu} - \frac{1}{12} \left(a + \frac{4}{3} c \cdot B \right) \mathcal{F}^{i\mu} + \frac{1}{2} c^\mu{}_{\nu\rho} J^{i\nu\rho} + \frac{1}{2} c^{\mu\nu\rho} B_{\nu\rho\sigma} \mathcal{F}^{i\sigma} . \quad (4.11)$$

In deriving this variation, one makes use of the identity $B^\lambda{}_{\alpha\beta} c^{\alpha\beta\sigma} B_{\sigma\nu\rho} = \frac{1}{3} c \cdot B B^\lambda{}_{\nu\rho}$, which can be proven by writing $c^{\mu\nu\rho}$ and $B_{\mu\nu\rho}$ in terms of dual vector fields. Finally, from the variations above, one can also determine the variation of γ_{ij} by using (4.5), finding

$$\delta \gamma_{ij} = \frac{1}{3} \left(a - \frac{1}{3!} c \cdot B + \frac{1}{2} \varepsilon \right) \gamma_{ij} , \quad (4.12)$$

where

$$\varepsilon \equiv \frac{1}{\sqrt{-\gamma}} \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho c_{\mu\nu\rho} . \quad (4.13)$$

Now we turn to a key test for the above transformation rules, which is the requirement that the transformations of $J^{i\mu\nu}$, $\mathcal{F}^{i\mu}$, and γ_{ij} , must be compatible with equation (4.4). The question of whether this nontrivial condition holds was raised in [60]. In fact, as we shall show here, it does actually hold. Firstly, the invariance of (4.4) under the a - and b -dependent transformations is manifest. The nontrivial check is the invariance under the c -dependent transformations, which requires that

$$0 = \left(J_i^{\sigma[\mu} c^{\nu]\alpha\beta} + \frac{1}{2} J_i^{\alpha\beta} c^{\mu\nu\sigma} + \frac{1}{6} J_i^{\mu\nu} c^{\alpha\beta\sigma} \right) B_{\alpha\beta\sigma} - \varepsilon_{ijk} J^{j\alpha\beta} \mathcal{F}^{k[\mu} c^{\nu]}_{\alpha\beta} + \frac{1}{6} \varepsilon J_i^{\mu\nu} - c^{\mu\nu}{}_{\sigma} \mathcal{F}_i^{\sigma} . \quad (4.14)$$

The first three terms sum up to zero since $J_i^{[\sigma\mu} c^{\nu\alpha\beta]} = 0$ identically. The remaining three terms also sum up to zero, upon using the fact that $\gamma^{ij} \partial_i X^\mu \partial_j X^\nu = g^{\mu\nu} - n^\mu n^\nu$, where n^μ is normal to the brane in the target space, i.e. $\partial_i X^\mu g_{\mu\nu} n^\nu = 0$. There remains, however, the condition that the variation of $\partial_i X^\mu$, which follows from the first equation in (4.3), must be curl-free. Using (4.11) and (4.12) in the first equation in (4.3) we find that

$$\delta \partial_i X^\mu = \left[a^\mu{}_\sigma - \frac{1}{4} \left(a + \frac{1}{3} c \cdot B + \frac{1}{3} \varepsilon \right) \delta^\mu_\sigma + \frac{1}{2} c^{\mu\alpha\beta} B_{\alpha\beta\sigma} \right] \partial_i X^\sigma + \frac{1}{2} c^\mu{}_{\nu\rho} \frac{\gamma_{ij} J^{j\nu\rho}}{\sqrt{-\gamma}} . \quad (4.15)$$

Thus the integrability condition amounts to

$$0 = \varepsilon^{ijk} \partial_j \delta \partial_k X^\mu = -\frac{1}{12} \varepsilon^{ijk} \partial_j \varepsilon \partial_k X^\mu + \frac{1}{2} \varepsilon^{ijk} c^\mu{}_{\rho\sigma} \partial_j \left(\frac{\gamma_{kl} J^{\ell\rho\sigma}}{\sqrt{-\gamma}} \right) . \quad (4.16)$$

Using the field equation $\nabla_i \partial^i X^\mu = 0$, this equation can be simplified to read

$$0 = V_k^{\rho\sigma} \left(\gamma^{ki} g^{\mu\nu} + 2\gamma^{k[m} \gamma^{i]n} \partial_m X^\mu \partial_n X^\nu \right) c_{\nu\rho\sigma} , \quad (4.17)$$

where we have defined

$$V_i^{\mu\nu} \equiv \nabla_i \partial^j X^{[\mu} \partial_j X^{\nu]} . \quad (4.18)$$

As we did earlier, we can most conveniently check this equation by considering a particular membrane solution [72], namely

$$X^\mu = (\sigma^0, \alpha\sigma_1 \cos(\omega\sigma^0), \alpha\sigma_1 \sin(\omega\sigma^0), \beta\sigma_2) , \quad (4.19)$$

which solves the equations of motion (4.1). It suffices to consider the integrability condition (4.16) for $i = 0$ and $\mu = 0$. It is then immediately evident that the first term on the right-hand side of (4.16) gives zero, whereas the second term gives a non-vanishing result that is proportional to the parameter $c^0{}_{12}$. Thus the integrability condition is not satisfied, and so the proposed $SL(5, R)$ transformation of $\partial_i X^\mu$ is not valid.

The subgroup that is consistent with the curl-free condition is therefore the semi-direct product $GL(4, R) \ltimes R^4$, generated by a , $a^\mu{}_\rho$, and $b_{\mu\nu\rho}$.

4.2. Introduction of extra coordinates

In seeking a manifestly realized $SL(5, R)$ symmetry, six extra coordinates were introduced in [43], such that together with the four coordinates X^μ of spacetime they form a 10-plet of $SL(5, R)$. The extra coordinates are antisymmetric tensorial, and are denoted by $Y^{\mu\nu}$. The field equations and Bianchi identities of the membrane were cast into a manifestly $SL(5, R)$ -covariant form. However, the equations satisfied by the extended system are problematic, and this can be seen as follows. Setting $B_{\mu\nu\rho} = 0$ and taking $g_{\mu\nu} = \eta_{\mu\nu}$ for simplicity, these equations take the form [43]

$$\begin{aligned}\sqrt{-\gamma}\gamma^{ij}\partial_j X^\mu &= 2\varepsilon^{ijk}\partial_j Y^{\mu\nu}\partial_k X_\nu, \\ \sqrt{-\gamma}\gamma^{ij}\partial_j Y^{\mu\nu} &= \varepsilon^{ijk}\partial_j X^\mu\partial_k X^\nu.\end{aligned}\quad (4.20)$$

Taking the curl of the second equation gives the integrability condition

$$V_i^{\mu\nu} = 0, \quad (4.21)$$

where we have used the field equation $\nabla_i\partial^i X^\mu = 0$ to simplify the result. This integrability condition, which implies second-order differential constraints over and above the field equations, therefore poses a problem with the desired $SL(5, R)$ duality-symmetric system of membrane equations. Note also in the GZ approach as well as the approach in which extra coordinates are introduced, the obstacle to the sought-after $SL(5, R)$ symmetry is the nonvanishing of the expression $V_i^{\mu\nu}$ defined in (4.18).

A different proposal has been made in [65], where the original coordinates are embedded into an $SL(5, R)$ 10-plet Z^{MN} via

$$Z^{\mu 5} \equiv X^\mu, \quad Z^{\mu\nu} \equiv Y^{\mu\nu}. \quad (4.22)$$

The following $SL(5)$ covariant equation was proposed in [65]:

$$\sqrt{-\Gamma}\Gamma^{ij}\partial_j Z^{MN} = c\varepsilon^{ijk}\partial_j Z^{MP}\partial_k Z^{NQ}G_{PQ}. \quad (4.23)$$

Here c is an arbitrary constant, and the induced $SL(5)$ -invariant metric is given by

$$\Gamma_{ij} = -\frac{1}{2}\partial_i Z^{MN}\partial_j Z_{MN}, \quad (4.24)$$

where $SL(5, R)$ indices are raised and lowered with the metric G_{MN} . Note the scale invariance of the equation (4.23) under the rescaling $Z \rightarrow \lambda Z$. The duality equation (4.23) induces the equations of motion

$$\nabla^i\partial_i Z^{MN} = 0, \quad (4.25)$$

which resemble the original membrane equations of motion, except that the worldvolume metric γ_{ij} is now replaced by the $SL(5)$ -invariant metric Γ_{ij} . The curl of (4.23) also implies the integrability equation

$$\nabla_i\partial^j Z^{P[M}\partial_j Z^{N]Q}G_{PQ} = 0. \quad (4.26)$$

Unlike the previous proposal discussed above, this does not yield an immediate contradiction, since it involves the original X^μ as well as the new $Y^{\mu\nu}$ coordinates. Indeed, taking $B_{\mu\nu\rho} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$, we see that the equation (4.23) gives

$$\sqrt{-\Gamma}\Gamma^{ij}\partial_j X^\mu = -c\varepsilon^{ijk}\partial_j Y^{\mu\nu}\partial_k X_\nu, \tag{4.27}$$

$$\sqrt{-\Gamma}\Gamma^{ij}\partial_j Y^{\mu\nu} = c\varepsilon^{ijk}\partial_j X^\mu\partial_k X^\nu + c\varepsilon^{ijk}\partial_j Y^{\mu\sigma}\partial_k Y^\nu{}_\sigma. \tag{4.28}$$

Apart from the fact that the induced worldvolume metrics are different, we see that the second equation above contains an extra term, in comparison to that given in (4.20). Consequently, its integrability condition will indeed mix X^μ and $Y^{\mu\nu}$, thereby avoiding an immediate conflict. However, to test whether the system described by (4.23) makes sense, we should also examine the spectrum of small fluctuations around a vacuum solution that respects the worldvolume Poincaré symmetry.

Such a background can be taken to be given by

$$G_{MN} = \eta_{MN}, \quad \partial_i \bar{Z}^{MN} = \lambda \sigma_i^{MN}, \tag{4.29}$$

where λ is an arbitrary constant, η_{MN} is the $SO(p, q) \subset SL(5, R)$ invariant tensor, and σ_i^{MN} specifies the embedding of $SO(2, 1)$ into $SO(p, q) \subset SL(5, R)$, for which we choose the canonical normalization

$$[\sigma_i, \sigma_j] = \varepsilon_{ijk} \sigma^k. \tag{4.30}$$

Owing to the scale invariance of (4.23), the parameter λ drops out, and for convenience we choose it so that we may identify the worldvolume metric (4.24) with $\eta_{ij} = \text{diag}(-, +, +)$:

$$\bar{\Gamma}_{ij} = \frac{1}{2} \lambda^2 \text{Tr}(\sigma_i \sigma_j) = \eta_{ij}. \tag{4.31}$$

Equation (4.23) then yields

$$[\sigma_i, \sigma_j] = \frac{1}{\lambda c} \varepsilon_{ijk} \sigma^k \implies c\lambda = 1. \tag{4.32}$$

Denoting the spin of the representation σ_i by j (assuming the representation carries a single spin), we have the relation

$$\text{Tr}(\sigma_i \sigma_j) = \frac{1}{3} j(j+1)(2j+1) \eta_{ij}, \tag{4.33}$$

which together with (4.31) determines

$$c^2 = \frac{1}{6} j(j+1)(2j+1). \tag{4.34}$$

Defining the fluctuations around this background as

$$Z^{MN} = \bar{Z}^{MN} + \phi^{MN}, \tag{4.35}$$

we fix the gauge freedom of worldvolume diffeomorphisms by imposing

$$\sigma_i^{MN} \phi_{MN} = 0. \tag{4.36}$$

The expansion of equation (4.23) to linear order in the fluctuations then gives

$$\eta^{ij} \partial_j \phi_{MN} + \varepsilon^{ijk} \left(\sigma_{jM}{}^P \partial_k \phi_{PN} - \sigma_{jN}{}^P \partial_k \phi_{PM} \right) = 0. \tag{4.37}$$

The final analysis depends on the particular choice of generators σ_i embedding $SO(2, 1)$ into $SL(5, R)$. Three inequivalent choices correspond to the decompositions⁹

⁹ We do not consider the decompositions $5 \rightarrow 2+3$ and $5 \rightarrow 1+4$ because they do not allow for a symmetric invariant η_{MN} .

$$\begin{aligned}
A) : & \quad 5 \rightarrow 5, \\
B) : & \quad 5 \rightarrow 3 + 1 + 1, \\
C) : & \quad 5 \rightarrow 2 + 2 + 1.
\end{aligned} \tag{4.38}$$

For case A), using an explicit spin-2 representation for the generators σ_i implies that the invariant tensor η_{MN} is of signature (2, 3), and one may verify that equation (4.23) reduces to

$$\partial_{(i}\phi_{jkl)} = 0, \tag{4.39}$$

for the components $\phi_{ijk} \equiv (\sigma_{(i}\sigma_j\sigma_k)^{MN}\phi_{MN}$ surviving the gauge condition (4.36). This shows that around this background the fluctuations do not admit any non-trivial dynamics.

Similarly, in case B) equations (4.23) restrict the fluctuations to

$$\partial_{(i}\phi_{j)4} = 0, \quad \partial_{(i}\phi_{j)5} = 0, \quad \partial_i\phi_{45} = 0, \tag{4.40}$$

which again kills all dynamics for the fluctuation components surviving the gauge condition (4.36).

Finally, in case C) the background is most conveniently given in terms of the 't Hooft symbols

$$(\sigma_i)_{MN} = -\frac{1}{2}\delta_M^m\eta^{nN}\varepsilon_{imn} + \frac{1}{2}\eta_{iM}\delta_4^N - \frac{1}{2}\delta_i^N\eta_{M4} \tag{4.41}$$

(with $\eta_{44} = -1$). In this case, the degeneracy of the spin 1/2 representations introduces an additional factor of 2 into (4.34), such that $c = \frac{1}{\sqrt{2}}$. The fluctuation equations from (4.23), together with the gauge condition (4.36), imply that

$$\partial_i\phi_{45} = -\frac{1}{2}\epsilon_i{}^{jk}\partial_j\phi_k, \quad \partial_{(i}\phi_{j)} = 0, \tag{4.42}$$

which again kills all dynamics for the fluctuation components surviving the gauge condition.

5. Conclusions

Using the p -brane worldvolume approach to U-dualities, we have confirmed that there is a problem when $D \neq p + 1$, focusing on $p = 2$ in $D = 4$ where the expected $SL(5, R)$ fails to materialize, and $p = 2$ in $D = 3$, which we refer to as the topological membrane, where the expected $SL(3) \times SL(2)$ does arise. In the case of the topological membrane, we have introduced extra coordinates to make the U-duality symmetry manifest. The features we have found for $D = 4$ are the same whether we use the approach where extra coordinates are introduced in order to make U-dualities manifest [43,65], or in the Gaillard–Zumino approach where the symmetries are not manifest. In the latter approach, as well as that of [43], we have shown that the $SL(5, R)$ U-duality fails due to the nonintegrability of the transformation rule of the pull-back map. In the approach of [65] where a manifestly $SL(5, R)$ -invariant equation is proposed for a membrane in a target based on generalized geometry, we have shown that these equations do not support linearized fluctuations about a Poincaré invariant vacuum solution.

These problems extend to the worldvolume treatment of Berman and Perry [22], and also to the approach in Hatsuda et al. [29], which reformulates the diffeomorphism constraints for an M2-brane coupled to a supergravity background in $D = 4$ in an $SL(5, R)$ -covariant form. In both cases the problem is that they use transformation rules that are not integrable, for the reasons we have explained above. More specifically, in [29], the $SL(5, R)$ transformations of the time components J^0 and P^0 are used to assert the $SL(5, R)$ invariance of the Hamiltonian constraint,

which is a quadratic form in these variables. By worldvolume Lorentz symmetry, however, also the space components must transform in the same way, which is equivalent to our (4.7). Then, our discussion leading to the nonintegrability of the resulting transformation rule for $\partial_i X^\mu$ continues to be an obstacle for $SL(5, R)$ invariance. We expect this will also appear in the case of $SO(5, 5)$ symmetry of M5 branes that has been proposed in [30].

Going beyond $D = 5$ only exacerbates the problem, since the U-duality multiplets involve the M5-brane charges as well the M2-brane charges and the momentum, as shown in Appendix A. One possible approach to this problem may be along the lines studied in [71] for the case of M2-brane in $D = 8$. In this approach, firstly one keeps all the target space fields arising in the dimensional reduction of $D = 11$ supergravity down to d dimensions. Next, one ensures the gauge invariance of the pull-backs of all the target space form fields by introducing appropriate worldvolume potentials, in the same way the worldvolume vector fields are introduced in D-brane actions. Then, one imposes suitable duality equations that exhibit the expected U-duality symmetry group manifestly, while maintaining the correct number of degrees of freedom. The challenge in this approach is to resolve the resulting highly nonlinear constraint equations in a way that maintains the U-duality symmetry. So far, these equations have been solved for a restricted class of supergravity background such that only the two parameter Heisenberg subgroup of $SL(2, R)$ we encountered in our treatment of membrane in $D = 8$ has been realized [71].

The $E_{n(n)}$ symmetry of $D = 11$ supergravity compactified on the torus T^n also arises if ten-dimensional type IIA or IIB supergravity is compactified on the torus T^{n-1} . An alternative approach to finding a worldvolume realization of these U-duality symmetries could be to look at string theories in $(n - 1)$ dimensions rather than the membrane in n dimensions. Ideas along these lines have been pursued in [78–81].

Acknowledgements

We are grateful to Alexandros Anastasiou, Eric Bergshoeff, Leron Borsten, Guillaume Bossard, Martin Cederwall, Mia Hughes, William Linch, Silvia Nagy, Yi Pang, Daniel Robbins, Andy Royston, Arkady Tseytlin and Peter West for discussions. JXL acknowledges support by a key grant from the NSF of China with Grant No. 11235010. HS thanks Mitchell Institute for Fundamental Physics and Astronomy at Texas A&M University, and ES thanks Lyon University, and CNP and ES thank the Mitchell Foundation for hospitality at the Great Brampton House Workshop on Cosmology and Strings, April 2015, where part of this work was carried out. CNP is supported in part by DOE grant DE-FG02-13ER42020, and ES is supported in part by NSF grant PHY-1214344.

Appendix A. Supergravity compactifications

Recall that the 11 = dimensional M-theory superalgebra is given by [82] :

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C)_{\alpha\beta} P_M (\Gamma^{MN} C)_{\alpha\beta} Z_{MN} + (\Gamma^{MNPQR} C)_{\alpha\beta} Z_{MNPQR}, \quad (\text{A.1})$$

where Q_α transforms as **32** of $SO(10, 1)$. The total number of components of all charges on the RHS is

$$\mathbf{11} + \mathbf{55} + \mathbf{462} = \mathbf{528}, \quad (\text{A.2})$$

which is, algebraically, the maximum possible number since the LHS is a symmetric 32×32 matrix. The spatial components of the momentum P_1 and the central charges Z_2, Z_5 are associ-

ated with the plane wave W1, the 2-brane M2 and the 5-brane M5; the temporal components are associated with their duals, the KK-monopole K6, and objects we can call K9 and W10.

After dimensional reduction to n dimensions, the 528 charges will form representations of $SO(n - 1, 1) \times SO(D)$ as in (A.3).

$n = 11 - D$	p	0	1	2	3	4	5	
11			11	55			462	
10		1	10 + 10	45		210	252	
9		(1, 3)	(9, 3)	(36, 1)	(84, 1)	(126, 3)		
8		(1, 6)	(8, 4)	(28, 2)	(56, 4)	(70, 3)		
7		(1, 10)	(7, 6)	(21, 6)	(35, 10)			(A.3)
6		(1, 16)	(6, 12)	(15, 16)	(20, 10)			
5		(1, 28)	(5, 28)	(10, 36)				
4		(1, 56)	(4, 64)	(6, 36)				
3		(1, 120)	(3, 136)					
2		(1, 256)	(2, 136)					
1		528						

However, the charges carried by the waves, branes and monopoles do not fall into representations of $SO(n - 1, 1) \times SO(D)$ because they discriminate between the temporal and spatial components. For example, writing $M = (0, I)$ with $I = 1, 2, \dots, 10$, the 45 $SO(10)$ M5 charges are given by Z_{IJ} and the 10 K9 charges by Z_{0J} or equivalently $\tilde{Z}_{IJKLMNOPQ}$. These $SO(n - 1) \times SO(D)$ reps are given in (A.4).

$n = 11 - D$	p	0	1	2	3	4	5	6	7	8	9	10
11			10	45			252	210			10	1
10		1	9 + 9	36		126	126 + 126	84		9	1 + 1	
9		(1, 3)	(8, 3)	(28, 1)	(56, 1)	(70, 3)	(56, 3)	(28, 1)	(8, 1)	(1, 3)		
8		(1, 6)	(7, 4)	(21, 2)	(35, 4)	(35, 6)	(21, 4)	(7, 2)	(1, 4)			
7		(1, 10)	(6, 6)	(15, 6)	(20, 10)	(15, 10)	(6, 6)	(1, 6)				
6		(1, 16)	(5, 12)	(10, 16)	(10, 20)	(5, 16)	(1, 12)					
5		(1, 28)	(4, 28)	(6, 36)	(4, 36)	(1, 28)						
4		(1, 56)	(3, 64)	(3, 72)	(1, 64)							
3		(1, 120)	(2, 136)	(1, 136)								
2		(1, 256)	(1, 272)									
1		(1, 528)										

(A.4)

Note that only the 0-brane charges can be assigned to a representation (fundamental) of the non-compact U-duality as opposed to its maximal compact subgroup. In $D = 3$, for example, we have the $(3, 2)$ of $SL(3) \times SL(2)$ with generalized coordinates

$$Z^M = (X^\mu, Y_{\rho\sigma}) \quad \mu = 1, 2, 3 \tag{A.5}$$

In $D = 4$ we have the 10 of $SL(5)$ with generalized coordinates

$$Z^M = (X^\mu, Y_{\rho\sigma}) \quad \mu = 1, 2, 3, 4 \tag{A.6}$$

In $D = 7$ we have the 56 of $E_{7(7)}$ with generalized coordinates

$$Z^M = (X^\mu, Y_{\rho\sigma}, \tilde{Y}_{\lambda\tau}, \tilde{X}^\nu) \quad \mu = 1, 2, 3, 4, 5, 6, 7 \tag{A.7}$$

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