SBV regularity for genuinely nonlinear, strictly hyperbolic systems of conservation laws in one space dimension

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SBV REGULARITY FOR GENUINELY NONLINEAR, STRICTLY HYPERBOLIC SYSTEMS OF CONSERVATION LAWS IN ONE SPACE DIMENSION

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ABSTRACT. We prove that if $t \mapsto u(t) \in \mathrm{BV}(\mathbb{R})$ is the entropy solution to a $N \times N$ strictly hyperbolic system of conservation laws with genuinely nonlinear characteristic fields

$$u_t + f(u)_x = 0,$$

then up to a countable set of times $\{t_n\}_{n \in \mathbb{N}}$ the function $u(t)$ is in SBV, i.e. its distributional derivative u_x is a measure with no Cantorian part.

The proof is based on the decomposition of $u_x(t)$ into waves belonging to the characteristic families

$$u(t) = \sum_{i=1}^{N} v_i(t) \tilde{r}_i(t), \quad v_i(t) \in \mathcal{M}(\mathbb{R}), \ \tilde{r}_i(t) \in \mathbb{R}^N,$$

and the balance of the continuous/jump part of the measures v_i in regions bounded by characteristics. To this aim, a new interaction measure $\mu_{i,\mathrm{jump}}$ is introduced, controlling the creation of atoms in the measure $v_i(t)$.

The main argument of the proof is that for all t where the Cantorian part of v_i is not 0, either the Glimm functional has a downward jump, or there is a cancellation of waves or the measure $\mu_{i,\mathrm{jump}}$ is positive.

CONTENTS

1. INTRODUCTION

In this paper we consider the entropy solution to the hyperbolic system in one space dimension

$$\begin{cases} u_t + f(u)_x = 0, & u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N, \quad f \in C^2(\Omega, \mathbb{R}^N) \\ u(t = 0) = \bar{u} & \bar{u} \in \mathrm{BV}(\mathbb{R}; \Omega), \quad |\bar{u}|_{\mathrm{BV}} \ll 1. \end{cases} \tag{1.1}$$

We assume that each characteristic field is either genuinely nonlinear or linearly degenerate, and in what follows we will refer to the unique solution constructed by vanishing viscosity or wave-front tracking, see [Bre, Daf1].

While linearly degenerate families do not gain any regularity during the time evolution, the genuinely nonlinear families show a regularizing effect due to the non linearity of the eigenvalue. The most famous one is probably the decay of positive waves, which in the scalar case $N = 1$ takes the form

$$u_x^+ \leq \frac{1}{kt}, \tag{1.2}$$

where k is the genuinely nonlinearity constant,

$$k = \inf_u f''(u) > 0.$$

For a strictly hyperbolic system of conservation laws, even if the i-th family satisfies the genuinely nonlinearity condition

$$D\lambda_i(u)r_i(u) \geq k > 0,$$

where $\{\lambda_i\}_{i=1}^N$ and $\{r_i\}_{i=1}^N$ are the eigenvalues and eigenvectors of $A(u) := Df(u)$ with a suitable orientation, then it may happen that new positive waves are created at a later time. In this case, the estimate (1.2) takes the form

$$v_i^+(t)(B) \leq C\left\{\frac{\mathcal{L}^1(B)}{t-s} + Q(s) - Q(t)\right\}, \quad 0 \leq s < t, \tag{1.3}$$

where B is a Borel set, Q is the Glimm interaction potential and the constant C depends on k. The wave measures $v_i(t)$ are defined precisely in Section 2.1, and roughly speaking they are the part of $u_x(t)$ which has direction close to $r_i(u(t))$ and travels with a speed close to $\lambda_i(u(t))$.

Since $Q(t)$ is a decreasing function, an elementary argument yields that v_i^+ is absolutely continuous up to countably many times: in fact, if B is $\mathcal{L}^1$-negligible and $v_i^+(t)(B) > 0$, then by letting $s \nearrow t$ we obtain

$$Q(t) - \lim_{s \nearrow t} Q(s) \geq \frac{v_i^+(t)(B)}{C} > 0,$$

so that $t \mapsto Q(t)$ has a jump downward. Being Q decreasing, this can happen only countably many times.

A complementary estimate is the fact that also $v_i^-(t)$ has no Cantorian part. The first positive result has been given in [ADL], where it is shown that the solution $u(t)$ of a genuinely nonlinear scalar conservation law in one space dimension is SBV up to countably many times. In that paper, the authors consider the characteristic lines

$$\dot{x} = f'(u(t, x)), \quad u(0, x) = y,$$

and prove the following: every time a Cantorian part in $u_x(t)$ appears, then there is a set of positive measure A such that all the characteristics starting from $y \in A$ are defined in the interval $[0, t]$ but cannot be prolonged more than t. By the σ-finitness of $\mathcal{L}^1$, one can apply the same observation used to prove that the positive part of $u_x(t)$ is abolutely continuous up to countably many times, and deduce that up to countably many times the solution $u(t)$ is SBV.

The use of the measure of the set $A(t)$ of initial points for characteristics which can be prolonged up to time t has been applied to obtain extension of the above result: in [Rog] the SBV estimate is used for scalar balance laws, later extended to Temple systems in [AnN] and in [BDR] to the case of Hamilton-Jacobi equation in several space dimension with uniformly convex Hamiltonian. In the context of the Riemann problem for genuinely non-linear systems, the thesis has moreover been proved in [Daf].

The case of genuinely nonlinear systems of conservation laws is more complicate by the fact that centered rarefaction waves are created at $t > 0$, and thus the characteristics are not unique in the future. Thus, in estimating the $\mathcal{L}^1$-measure of the initial points, one has to take into account also that interaction points can generate centered rarefaction waves, so that the estimate should be something like

$$\mathcal{L}^1\Big\{\text{initial points of characteristics arriving at } t \text{ but not prolongable}\Big\}$$

$$\geq \Big\{\text{measure of the Cantorian part of } v_i(t)\Big\} - \Big\{\text{amount of interaction in } [0, t]\Big\}. \tag{1.4}$$

In this paper we use a different approach. Let $\tilde{\lambda}_i(t, x)$ be the Rankine-Hugoniot speed if u has a jump in the point (t, x) of the i-th family or the i-eigenvalue of $A(u)$ in the remaining cases. We first prove that not only the characteristic waves $v_i(t)$ satisfy a balance equation of the form

$$\partial_t v_i + \partial_x\big(\tilde{\lambda}_i v_i\big) = \mu_i,$$

with μ_i a measure bounded by the interaction-cancellation of waves, but also its atomic part $v_{i,\mathrm{jump}}(t)$ satisfy

$$\partial_t v_{i,\mathrm{jump}} + \partial_x\big(\tilde{\lambda}_i v_{i,\mathrm{jump}}\big) = \mu_{i,\mathrm{jump}},$$

with $\mu_{i,\mathrm{jump}}$ bounded measure. This measure $\mu_{i,\mathrm{jump}}$ differs from the interaction-cancellation measure because it is not 0 when an atomic part in v_i is created, and it describes the natural behavior of solutions to genuinely nonlinear conservation laws: it is easy to create a shock because of the nonlinearity, but you can remove it only by means of cancellation or strong interactions.

The second step is to use the two above balance equations to study the balance of v_i, $v_{i,\mathrm{jump}}$ and $v_{i,\mathrm{cont}} = v_i - v_{i,\mathrm{jump}}$ in regions bounded by characteristics. The key estimate we obtain is that

$$v_{i,\mathrm{cont}}(t)(B) \geq -C\bigg\{\frac{\mathcal{L}^1(B)}{\tau - t} + \mu^{ICJ}\big(\{[t, \tau] \times \mathbb{R}\}\big)\bigg\}, \quad 0 \leq t < \tau, \tag{1.5}$$

where the measure μ^{ICJ} is the interaction-cancellation measure μ^{IC} plus morally the measure $\sum_{i=1}^N |\mu_{i,\mathrm{jump}}|$. This is the companion estimate of (1.3), and using the same argument of the positive part we conclude with the main result of this paper (Corollary 3.3):

Theorem 1.1. *Let $u(t)$ be the entropy solution of the Cauchy problem*

$$\begin{cases} u_t + f(u)_x = 0, \\ u(t = 0) = \bar{u} \end{cases} \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^{\mathrm{N}}, \qquad f \in C^2(\Omega; \mathbb{R}^{\mathrm{N}})$$

for a strictly hyperbolic system of conservation laws where each characteristic field is genuinely non-linear, with initial datum $\bar{u}$ small in $\mathrm{BV}(\mathbb{R}; \Omega)$. Then $u(t) \in \mathrm{SBV}(\mathbb{R}; \Omega)$ out of at most countably many times.

1.1. **Structure of the paper.** The paper is organized as follows.

In Section 2 the main notation and assumptions are introduced: strict hyperbolicity and characteristic families, and the decomposition into wave measures. A few fundamental results concerning the hyperbolic systems of conservation laws are recalled: Lax's solution to the Riemann problem (Theorem 2.1) and Bressan's existence and uniqueness of a Lipschitz semigroup of solutions for small BV initial data (Theorem 2.2).

In Section 3 we prove Theorem 1.1 (Corollary 3.3 below). This result is a corollary of the fact that if the $\bar{\imath}$-th family is genuinely nonlinear, then the $\bar{\imath}$-th component $v_{\bar{\imath}}(t)$ of $u_x(t)$ is SBV up to countably many times (Corollary 3.2), and the latter is a consequence of the estimates (1.3) and (1.5) (Theorem 3.1). In this section the notion of interaction-cancellation is recalled, and the SBV estimates are derived assuming (1.5), whose proof is postponed to Section 5.

Since the proof depends on uniform estimates for the wave-front tracking approximations, in Section 4 we recall the basic properties of these approximated solutions. A key fact is the possibility to collect the jumps of the wave-front solution into two families: one is converging to the jump part of u_x, and the other to the continuous part of u_x. This is done by defining the maximal $(\varepsilon_0, \varepsilon_1)$-shocks (Definition 4.1): $0 < \varepsilon_0 < \varepsilon_1$ are two treshold parameters, fixing the minimal size of the jump (ε_0) and the lower bound for the maximal size of the jump (ε_1). This definition has already been used in [Bre] to study the structure of the semigroup solution u, which we recall in Theorem 4.2. From this result we obtain that the wave measure v_i^ν, $v_{i,\mathrm{jump}}^\nu$ for the wave-front tracking approximation u^ν and the products $\tilde{\lambda}_i^\nu v_i^\nu$, $\tilde{\lambda}_i^\nu v_{i,\mathrm{jump}}^\nu$ converge weakly (Corollaries 4.5, 4.6, 4.7).

In the last section, Section 5, we prove the decay estimate (1.5). First of all, we prove that the distributions

$$\partial_t v_i^\nu + \partial_x\big(\tilde{\lambda}_i^\nu v_i^\nu\big), \qquad\qquad \partial_t v_{i,\mathrm{jump}}^\nu + \partial_x\big(\tilde{\lambda}_i^\nu v_{i,\mathrm{jump}}^\nu\big)$$

are uniformly bounded measures: we denote them respectively by μ_i^ν and $\mu_{i,\mathrm{jump}}^\nu$ (Proposition 5.1). The latter measure is called the *jump balance measure*. Since the bounds do not depend on the aproximation parameter ν, it is possible to pass to the limit and to obtain the balance equations for the wave measures

v_i, $v_{i,\mathrm{jump}}$. The consequences of this fact however are not directly related with the SBV regularity, so we will address them in a forthcoming paper. Next, we study the balances of the measures v_i^ν, $v_{i,\mathrm{jump}}^\nu$ in regions bounded by minimal characteristics (Lemma 5.6). We then use an argument completely similar to the one used for the decay of positive waves in [Bre]: if $I = [a, b]$ is an interval and $v_{i,\mathrm{cont}}^-(I)$ is too negative, then either $v_{i,\mathrm{jump}}^-$ is cancelled or the characteristics $a(t)$, $b(t)$ starting from a, b collapse in a future time; in the last case the time t for which $a(t) = b(t)$ is of the order of the length of the interval divided by the amount of negative wave, or by the interaction measure, cancellation measure and jump wave balance measure in the region spanned in time by characteristics from the interval. In this way we give a precise meaning to the inequality (1.4). We thus obtain the estimate (1.5) first for the approximated wave-front tracking solution and for finitely many intervals, and then passing to the limit we recover the same estimate for the semigroup solution and for Borel sets (Lemmas 5.7, 5.9). Adding the already known decay estimate for positive waves, we obtain the desired result (Corollary 5.11).

2. General preliminary notions

Consider the Cauchy problem (1.1). The following assumptions are done:

(1) Strict hyperbolicity: we set $A(u) = \mathrm{D}f(u)$ and we assume that the eigenvalues $\{\lambda_i\}_i$ of A satisfy

$$\lambda_1(u) < \cdots < \lambda_\mathrm{N}(u), \qquad u \in \Omega.$$

We denote the unit right eigenvectors, and the left ones satisfying $r_i \cdot l_j = \delta_{ij}$, respectively by

$$r_1(u), \ldots, r_\mathrm{N}(u) \qquad l_1(u), \ldots, l_\mathrm{N}(u).$$

(2) Each $\bar\imath$-th characteristic field is either genuinely non-linear, i.e.

$$|\mathrm{D}\lambda_{\bar\imath}(u)r_{\bar\imath}(u)| \geq k > 0, \qquad u \in \Omega, \tag{2.1 : $\bar\imath$GN}$$

or linearly-degenerate, i.e.

$$\mathrm{D}\lambda_{\bar\imath}(u)r_{\bar\imath}(u) = 0, \qquad u \in \Omega. \tag{2.2 : $\bar\imath$LD}$$

By the general theory on hyperbolic systems of conservation laws, one has then the following theorem.

Theorem 2.1. *Let $\bar u = u^-\chi_{\{x<0\}} + u^+\chi_{x\geq 0}$. Then there exists a unique self-similar weak solution whose shocks satisfy the Lax compatibility condition:*

$$\lambda_i(\omega^-) \geq \lambda_i(\omega^-, \omega^+) \geq \lambda_i(\omega^+) \qquad \text{at each jump } [\omega^-, \omega^+] \text{ having speed } \lambda_i(\omega^-, \omega^+).$$

Proof. We recall just that one can define C^1-curves $\Psi_i(\sigma)(u_0)$ and the scalars $\lambda_i(u_0, \Psi_i(\sigma)u_0)$ by

$$\begin{cases} \dot\Psi_i(\sigma) = \tilde r_i(\Psi_i(\sigma)) & \sigma \geq 0 \\ f(\Psi_i(\sigma)) - f(u_0) = \lambda_i(u_0, \Psi_i(\sigma))\big(\Psi_i(\sigma) - u_0\big) & \sigma < 0 \end{cases}, \qquad \Psi_i(0)(u_0) = u_0,$$

where $\tilde r_i$ is a vector parallel to r_i and satisfying $\mathrm{D}\lambda_{\bar\imath}\tilde r_{\bar\imath} = 1$ if (2.1 : $\bar\imath$GN) holds, otherwise $|\tilde r_{\bar\imath}| = 1$ if instead (2.2 : $\bar\imath$LD) holds. The proof (see e.g. Th. 5.3, [Bre]) is then based on the inverse function theorem applied to the local C^1 homeomorphism $\Lambda : U(0) \ni \sigma \mapsto u^+ \in U(u^-)$

$$\Lambda(\sigma_1, \ldots, \sigma_\mathrm{N})(u^-) = \Psi_\mathrm{N}(\sigma_\mathrm{N}) \circ \cdots \circ \Psi_1(\sigma_1)(u^-).$$

It is applied in a domain where the Jacobian of the map is uniformly bounded away from 0. From the C^1-regularity of the curves Ψ_i, its Jacobian at $\sigma = 0$ is $\big[\tilde r_1|\ldots|\tilde r_\mathrm{N}\big](u^-) = \mathrm{D}\Lambda(\sigma = 0)(u^-)$. Under (2.1 : $\bar\imath$GN), if $u^+ = \Psi_{\bar\imath}(\sigma)(u^-)$ with $\sigma < 0$ we have a *compressive $\bar\imath$-shock*, otherwise a *centered $\bar\imath$-rarefaction wave* (which is continuous). In the case of linear degeneracy (2.2 : $\bar\imath$LD) we have an *$\bar\imath$-contact discontinuity*. $\quad\square$

Theorem 2.2. *There exists a closed domain $\mathcal{D} \subset L^1(\mathbb{R}; \Omega)$ and a unique distributional solution $u = u(t, x) = [u(t)](x)$ which is a Lipschitz semigroup $\mathcal{D} \times [0, +\infty) \to \mathcal{D}$ and which for piecewise constant initial data coincides, for a small time, with the solution of the Cauchy problem obtained piecing together the standard entropy solutions of the Riemann problems. Moreover, it lives in the space of BV functions.*

Below, we will refer to u as the semigroup solution, or equivalently as vanishing viscosity solution. When referring to pointwise values of u, we tacitly take its L^1-representative such that the restriction map $t \mapsto u(t) \in L^1(dx)$ is continuous from the right and $u(t)$ is pointwise continuous from the right in x.

2.1. **Decomposition of u_x into wave measures.** By the smallness assumption on the BV norm of the initial datum, one can assume that the eigenvalues of $A = \mathrm{D}f$ satisfy on Ω

$$\inf \lambda_1 \le \sup \lambda_1 \ < \ \ldots \ < \ \inf \lambda_\mathrm{N} \le \sup \lambda_\mathrm{N},$$

and the eigenvectors lie in different cones. We then decompose as in the literature (see e.g. pp. 93,201 of [Bre]) the vector-valued measures u_x, $f(u)_x$ along the right eigenvectors of $A = \mathrm{D}f$. We adopt the following notation.

(1) Under $(2.1 : \bar{\imath}\mathrm{GN})$ we normalize the $\bar{\imath}$-th right eigenvector of $A(u)$ so that $\mathrm{D}\lambda_{\bar{\imath}} r_{\bar{\imath}} = 1$, as in the proof of Theorem 2.1: denote it by $\tilde{r}_{\bar{\imath}}(u)$. Under $(2.2 : \bar{\imath}\mathrm{LD})$ just take $\tilde{r}_{\bar{\imath}}(u) = r_{\bar{\imath}}(u)$, so that $|\tilde{r}_{\bar{\imath}}| = 1$.

(2) We fix the left eigenvectors $\tilde{l}_1(u), \ldots, \tilde{l}_\mathrm{N}(u)$ so that $\tilde{l}_i \cdot \tilde{r}_j = \delta_{ij}$ still holds for $i, j \in \{1, \ldots, \mathrm{N}\}$.

(3) Given two values $u^{\pm} \in \Omega$, by the solution to the Riemann problem, briefly recalled in Theorem 2.1, there exists $\sigma = (\sigma_1, \ldots, \sigma_\mathrm{N}) \in \mathbb{R}^\mathrm{N}$ such that $u^+ = \Lambda(\sigma)(u^-)$. We introduce the values

$$\omega_0 = u^-, \qquad \omega_i = \Psi(\sigma_i)(\omega_{i-1}) \ i = 1, \ldots, \mathrm{N}. \tag{2.3}$$

We define $\lambda_i(u^+, u^-)$ as the i-th eigenvalue of the averaged matrix

$$A(\omega_i, \omega_{i-1}) = \int_0^1 A\big(\theta\omega_i + (1-\theta)\omega_{i-1}\big)d\theta$$

and $\tilde{l}_i(u^+, u^-)$, $\tilde{r}_i(u^+, u^-)$ vectors satisfying $\tilde{l}_i \cdot \tilde{r}_j = \delta_{ij}$ which are
  • if $\sigma_{\bar{\imath}} < 0$ and $(2.1 : \bar{\imath}\mathrm{GN})$ holds, left\right $\bar{\imath}$-eigenvectors of $A(\omega_{\bar{\imath}}, \omega_{\bar{\imath}-1})$ normalized so that

$$\lambda_{\bar{\imath}}(\omega_{\bar{\imath}}) - \lambda_{\bar{\imath}}(\omega_{\bar{\imath}-1}) = \tilde{l}_{\bar{\imath}} \cdot (\omega_{\bar{\imath}} - \omega_{\bar{\imath}-1}). \tag{2.4a}$$

  • otherwise, $\tilde{l}_i(u^+, u^-) = \int_0^1 \tilde{l}_i(\theta\omega_i + (1-\theta)\omega_{i-1})$. Then[1]

$$\sigma_i = \tilde{l}_i \cdot (\omega_i - \omega_{i-1}). \tag{2.4b}$$

We call *strength* of the i-th wave the value

$$\sigma_i = \tilde{l}_i \cdot (\omega_i - \omega_{i-1}).$$

In the genuinely non-linear case, by the parameterization choice it is equal to $\lambda_{\bar{\imath}}(\omega_{\bar{\imath}}) - \lambda_{\bar{\imath}}(\omega_{\bar{\imath}-1})$.

Remark 2.3. Notice that, as in the proof of Theorem 2.1, we defined $\lambda_i(u^+, u^-)$ as the Rankine-Hugoniot speed of the i-th wave of the Riemann problem $[u^-, u^+]$, in case it is a shock or a contact discontinuity: indeed since

$$f(\omega_i) - f(\omega_{i-1}) = \int_0^1 (\mathrm{D}f(\omega_{i-1} + \theta(\omega_i - \omega_{i-1}))(\omega_i - \omega_{i-1}))d\theta = A(\omega_i, \omega_{i-1})(\omega_i - \omega_{i-1}),$$

then, if the Rankine-Hugoniot condition $f(\omega_i) - f(\omega_{i-1}) = \lambda_i(\omega_i - \omega_{i-1})(\omega_i - \omega_{i-1})$ holds, $\omega_i - \omega_{i-1}$ is an i-eigenvector of $A(\omega_i, \omega_{i-1})$ with eigenvalue $\lambda_i(u^+, u^-)$. In case of rarefaction waves it is just an average speed, and we see below that in the wave front tracking construction one may choose this as the speed of the 'artificial' jump, up to small perturbations.

We recall now the wave decomposition of u_x into wave measures, and consequently of the flow $f(u)_x$. Let

$$\lambda_i(t, x) = \lambda_i\big(u(t, x^+), u(t, x^-)\big).$$

$$\tilde{l}_i(t, x) = \tilde{l}_i\big(u(t, x^+), u(t, x^-)\big), \qquad \tilde{r}_i(t, x) = \tilde{r}_i\big(u(t, x^+), u(t, x^-)\big).$$

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[1]Denote by $\sigma(u^+; u^-)$ the inverse function of $\sigma \mapsto u^+ = \Lambda(\sigma; u^-)$. By the inverse function theorem $J\sigma(u^+; u^-) = (D\Lambda)^{-1}(\sigma(u^+, u^-); u^-)$. If either the i-th component σ_i is nonnegative or the i-th characteristic field is linearly degenerate, by the ODE satisfied by Ψ_i one finds $\nabla\sigma_i(u^+; u^-) = \tilde{l}_i(u^+)$ at least when $u^+ = \Psi_i(\sigma_i)(u^-)$, being in that case $\dot{\Psi}_i(\sigma_i) = \tilde{r}_i(\Psi_i(\sigma_i))$. Then by the fundamental theorem of calculus and the definition of $\omega_0, \ldots, \omega_\mathrm{N}$

$$\sigma_i(u^+, u^-) = \sigma_i(\omega_i, \omega_{i-1}) = \int_0^1 \frac{d}{d\theta}\sigma_i\big(\omega_{i-1} + \theta(\omega_i - \omega_{i-1}), \omega_{i-1}\big)d\theta$$

$$= \int_0^1 \Big\{\tilde{l}_i\big(\omega_{i-1} + \theta(\omega_i - \omega_{i-1}), \omega_{i-1}\big) \cdot (\omega_i - \omega_{i-1})\Big\}d\theta = \tilde{l}_i \cdot (\omega_i - \omega_{i-1}).$$

Lemma 2.4. *Define the scalar measures $v_i := \tilde{l}_i \cdot u_x$. Then the following decomposition holds*

$$u_x = \sum_{i=1}^{N} v_i \tilde{r}_i(t,x), \tag{2.5a}$$

$$-u_t = f(u)_x = \sum_{i=1}^{N} \lambda_i(t,x) v_i \tilde{r}_i(t,x). \tag{2.5b}$$

Proof. The first equation is a direct consequence of the fact that $\tilde{r}_1, \ldots, \tilde{r}_N$ are linearly independent and $\tilde{l}_i \cdot \tilde{r}_j = \delta_{ij}$. For justifying the second one, denote by the structure of BV functions (Sect. 3.7 of [AFP])

$$u_x = \left[\mathrm{D}_x^{\mathrm{a}} u + \mathrm{D}_x^{\mathrm{c}} u \right] + \sum_{h \in \mathbb{N}} \left[u(t, \gamma_h(t)^+) - u(t, \gamma_h(t)^-) \right] \mathcal{L}^1(dt),$$

where $\mathrm{D}_x^{\mathrm{a}} u, \mathrm{D}_x^{\mathrm{c}} u$ are the absolutely continuous and Cantor part of u_x, while the sum is the jump part. The values $u(t, \gamma_h(t)^\pm)$ are indeed the values of the approximate jump at $(t, \gamma_h(t))$ by a fine property of semigroup solutions recalled e.g. in Theorem 4.2 below. Then by Volpert chain rule (Th. 3.99 of [AFP])

$$f(u)_x = \mathrm{D}f(u) \left[\mathrm{D}_x^{\mathrm{a}} u + \mathrm{D}_x^{\mathrm{c}} u \right] + \sum_{h \in \mathbb{N}} \left[f(u(t, \gamma_h(t)^+)) - f(u(t, \gamma_h(t)^-)) \right] \mathcal{L}^1(dt)_{\llcorner \mathrm{Dom}(\gamma_h)}$$

$$\overset{(2.3)}{=} A(u) \left[\mathrm{D}_x^{\mathrm{a}} u + \mathrm{D}_x^{\mathrm{c}} u \right] + \sum_{h \in \mathbb{N}} \sum_{i=1}^{N} \left[f(\omega_i(t, \gamma_h(t)^+)) - f(\omega_{i-1}(t, \gamma_h(t)^-)) \right] \mathcal{L}^1(dt)_{\llcorner \mathrm{Dom}(\gamma_h)}$$

$$\overset{(2.5a)}{=} \sum_{i=1}^{N} \lambda_i(t,x)(v_i)_{\mathrm{cont}} \tilde{r}_i(t,x) + \sum_{i=1}^{N} \lambda_i(t,x)(v_i)_{\mathrm{jump}} \tilde{r}_i(t,x) = \sum_{i=1}^{N} \lambda_i(t,x) v_i \tilde{r}_i(t,x),$$

where we also applied in the last step that each $\tilde{r}_i(u)$ is a right eigenvector of $A(u)$, and the Rankine-Hugoniot conditions of Remark 2.3 for the jump part. □

3. Main SBV regularity argument

Given a semigroup solution u, the wave decomposition (2.5) of u_x along $\tilde{r}_1, \ldots, \tilde{r}_N$ reduces the vectorial problem to scalar ones. In order to check that there is no Cantor part in u_x, by (2.5) we will indeed check that there is no Cantor part in each v_i.

The argument here is based on piecewise constant front-tracking approximations of the semigroup solution u, some recall is provided in Section 4. The magnitude of waves of each ν-front-tracking approximation may change in time only when discontinuity lines meet, and waves interact. In [Bre], Section 7.6, an approximate conservation principle for wave strengths is expressed by introducing two finite measures concentrated on interaction points of physical waves: the *interaction* and *interaction-cancellation* measures

$$\mu_\nu^I(\{P\}) = |\sigma'\sigma''| \qquad \mu_\nu^{IC}(\{P\}) = |\sigma'\sigma''| + \begin{cases} |\sigma'| + |\sigma''| - |\sigma' + \sigma''| & i = i' \\ 0 & i \neq i' \end{cases}. \tag{3.1}$$

Above σ', σ'' denote the incoming strengths of the two physical waves interacting at P.

More precisely, given a polygonal region Γ with edges transversal to the waves it encounters (Figure 1), one considers the total amount $W_{\mathrm{in}}^{i\pm}$, $W_{\mathrm{out}}^{i\pm}$ of positive and negative i-waves entering and exiting Γ. Considering the 'fluxes' (the amount of positive\negative ith-waves entering\exiting Γ)

$$W_{\mathrm{in}}^i = W_{\mathrm{in}}^{i+} - W_{\mathrm{in}}^{i-} \qquad W_{\mathrm{out}}^i = W_{\mathrm{out}}^{i+} - W_{\mathrm{out}}^{i-}, \qquad i = 1, \ldots, N,$$

then one has the upper bounds

$$|W_{\mathrm{out}}^i - W_{\mathrm{in}}^i| \leq C\mu^I(\Gamma) \qquad |W_{\mathrm{out}}^{i\pm} - W_{\mathrm{in}}^{i\pm}| \leq C\mu^{IC}(\Gamma).$$

Since the above measures satisfy uniform estimates w.r.t. ν, by compactness one can define measures μ^I, μ^{IC} on the entropy solution by taking w^*-limits of the corresponding measures on a suitable sequence of ν-front-tracking approximations.

Assume that the i-th characteristic field is genuinely non-linear, $(2.1 : \bar{i}\mathrm{GN})$. A balance of the kind above will lead in Lemma 5.9 below to the following estimate on a semigroup solution u: the density w.r.t. time of the continuous part $(v_{\bar{i}})_{\mathrm{cont}}$ of $v_{\bar{i}} = \tilde{l}_{\bar{i}} \cdot u_x$ is controlled at time s by the Lebesgue measure
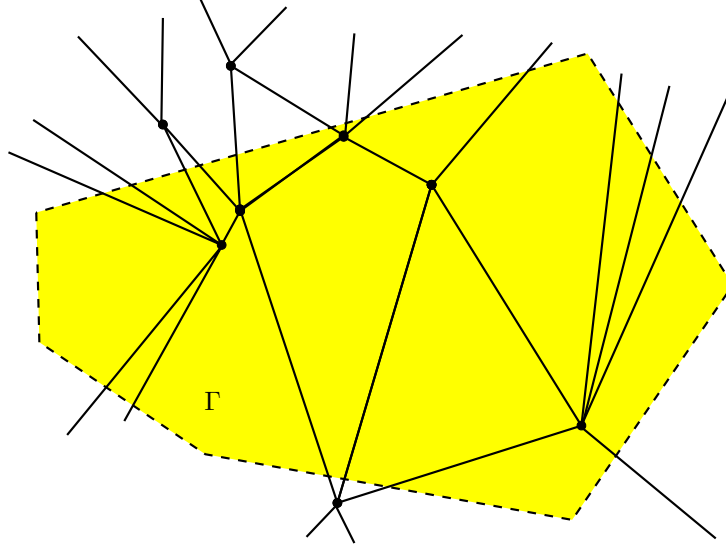
FIGURE 1. The yellow area represents the region Γ in the plane (x, t), waves enter and exit.

and by a finite measure on a horizontal strip around s. This estimate is analogous to the decay of positive waves proved in Theorem 10.3 of [Bre]. We state it here, postponing the proof to Section 5.

Theorem 3.1. *Suppose* $(2.1 : \bar{\imath}\mathrm{GN})$ *holds for the* $\bar{\imath}$*-th characteristic field. Denoting*

$$v_{\bar{\imath}} = \int \big[v_{\bar{\imath}}(t)\big] dt, \qquad v_{\bar{\imath}}(t) = (v_{\bar{\imath}}(t))_{\mathrm{cont}} + (v_{\bar{\imath}}(t))_{\mathrm{jump}} : \quad (v_{\bar{\imath}}(t))_{\mathrm{jump}} \text{ purely atomic,}$$

then there exists a finite, nonnegative Radon measure $\mu_{\bar{\imath}}^{ICJ}$ *on* $\mathbb{R}^+ \times \mathbb{R}$ *such that for* $s > \tau > 0$

$$\mathcal{O}(1)\Big\{\mathcal{L}^1(B)/\tau + \mu_{\bar{\imath}}^{ICJ}([s - \tau, s + \tau] \times \mathbb{R})\Big\} \geq |(v_{\bar{\imath}}(s))_{\mathrm{cont}}|(B) \qquad \forall B \text{ Borel subset of } \mathbb{R}. \qquad (3.2 : \bar{\imath})$$

The Radon measure $\mu_{\bar{\imath}}^{ICJ}$ is defined in the next section as a w^*-limit of measures (5.4) relative to front-tracking approximations. It takes into account both general interactions and cancellations and also balances for exclusively the jump part of $v_{\bar{\imath}}$. In the statement $v_{\bar{\imath}}(s)$ is the representative continuous from the right, coinciding with $\tilde{l}_i \cdot (u(s))_x$.

Following an argument already in [ADL], the estimate $(3.2 : \bar{\imath})$ allows to prove that $v_{\bar{\imath}}$ is made by merely a jump part concentrated on the graphs of countably many time-like Lipschitz curves and a part absolutely continuous w.r.t. the Lebesgue measure. Indeed, $(v_{\bar{\imath}})_{\mathrm{cont}}$ is the integral w.r.t. $\mathcal{L}^1(dt)$ of $(v_{\bar{\imath}}(t))_{\mathrm{cont}}$: if $|(v_{\bar{\imath}}(s))_{\mathrm{cont}}|(B) > 0$ at some time s for an $\mathcal{L}^1$-negligible set B, then by $(3.2 : \bar{\imath})$ the time marginal of $\mu_{\bar{\imath}}^{ICJ}$ has an atom at s, which may happen only countably many times.

Corollary 3.2. *Let* u *be a semigroup solution of the Cauchy problem for the strictly hyperbolic system* (1.1). *Consider the* $\bar{\imath}$*-wave measure* $v_{\bar{\imath}} = \tilde{l}_{\bar{\imath}} \cdot u_x$. *If* $(2.1 : \bar{\imath}\mathrm{GN})$ *holds, then* $v_{\bar{\imath}}$ *has no Cantor part.*

If all the characteristic fields are genuinely non-linear, by the wave decomposition (2.5) the above estimate yields then $\mathrm{SBV}([0, T] \times \mathbb{R}; \mathbb{R}^N)$ regularity of u for all $T > 0$.

Corollary 3.3. *Let* u *be the semigroup solution of the Cauchy problem*

$$\begin{cases} u_t + f(u)_x = 0, \\ u(t = 0) = \bar{u} \end{cases} \qquad u : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^N, \qquad f \in C^2(\Omega; \mathbb{R}^N)$$

for a strictly hyperbolic system of conservation laws where each characteristic field is genuinely non-linear, with initial datum $\bar{u}$ *small in* $\mathrm{BV}(\mathbb{R}; \Omega)$. *Then* $u(t) \in \mathrm{SBV}(\mathbb{R}; \Omega)$ *out of at most countably many times.*

Proof of Corollaries 3.2 and 3.3. By the general theory on 1-dimensional systems of conservation laws, u belongs to $\mathrm{BV}([0, T] \times \mathbb{R}; \Omega)$ for all $T > 0$. By the structure theorem of BV functions (see Sect. 3.7 of [AFP]), the derivative of u can be decomposed into a jump part, concentrated on a 1-countably

rectifiable set, a part which is absolutely continuous w.r.t. the Lebesgue measure, and a remaining part—the Cantor part—which is singular w.r.t. $\mathcal{L}^2$ and vanishes on sets having finite Hausdorff 1-dimensional measure:

$$\mathrm{D}u = \nabla u\, \mathcal{L}^2 + \mathrm{D}^{\mathrm{c}}\, u + \mathrm{D}^{\mathrm{j}}\, u.$$

We want to prove that the Cantor parts $\mathrm{D}^{\mathrm{c}}_t\, u$, $\mathrm{D}^{\mathrm{c}}_x\, u$ of the both components u_t, u_x of $\mathrm{D}u$ vanish.

Denote by $u(t)$ the 1-space-dimensional restriction of u at time t, which is a function of x. By the slicing theory of BV functions (Theorems 3.107-108 in [AFP]), not only

$$u_x = \int u(t)_x dt, \tag{3.3}$$

but moreover the Cantor part of u_x is the integral, w.r.t. $\mathcal{L}^1(dt)$, of the Cantor part of $u(t)$: one has also the disintegration

$$\mathrm{D}^{\mathrm{c}}_x\, u = \int \mathrm{D}^{\mathrm{c}}\big(u(t)\big)dt.$$

Since, moreover, by combining Volpert chain rule (Th. 3.99 of [AFP]) and the conservation law one has

$$\mathrm{D}^{\mathrm{c}}_t\, u = f'(u)\, \mathrm{D}^{\mathrm{c}}_x\, u,$$

it suffices to show that for $\mathcal{L}^1$-a.e. t the function $u(t)$ belongs to $\mathrm{SBV}(\mathbb{R};\mathbb{R}^{\mathrm{N}})$.

If one sets $v_i(t) = \tilde{l}_i \cdot u(t)_x$, then by the slicing (3.3) and as in the wave decomposition (2.5) one derives

$$v_i = \int v_i(t)dt \qquad i = 1,\ldots,\mathrm{N}, \qquad u(t)_x = \sum_{i=1}^{\mathrm{N}} \tilde{r}_i(t,x)v_i(t).$$

As a consequence, there is a Cantor part at time s in some $v_{\bar{\imath}}(s)$ precisely when $u(s)_x$ has a Cantor part.

It remains then to prove that $(2.1 : \bar{\imath}\mathrm{GN})$ implies that $v_{\bar{\imath}}(s)$ has no Cantor part for $\mathcal{L}^1$-a.e. s. By Theorem 3.1, the assumption of genuine nonlinearity $(2.1 : \bar{\imath}\mathrm{GN})$ implies that the estimate $(3.2 : \bar{\imath})$ holds for the $\bar{\imath}$-th characteristic field. The fact that $v_{\bar{\imath}}(s)$ has a Cantor part means that there exists an $\mathcal{L}^1$-negligible compact set K with $v_{\bar{\imath}}(s)(K) > 0$ and with no atom of $v_{\bar{\imath}}(s)$. Then for all $s > \tau > 0$

$$
\begin{aligned}
0 < |v_{\bar{\imath}}(s)|(K) &= |(v_{\bar{\imath}}(s))_{\mathrm{cont}}|(K) \\
&\overset{(3.2:\bar{\imath})}{\leq} \mathcal{O}(1)\Big\{\mathcal{L}^1(K)/\tau + \mu_{\bar{\imath}}^{ICJ}\big([s-\tau, s+\tau] \times \mathbb{R}\big)\Big\}.
\end{aligned}
$$

As we are taking $\mathcal{L}^1(K) = 0$, by outer regularity of Borel measures when $\tau \downarrow 0$ this means that

$$\mu_{\bar{\imath}}^{ICJ}\big(\{s\} \times \mathbb{R}\big) > 0.$$

It is indeed true that if $\mathcal{L}^1(K) = 0$ then $\mu_{\bar{\imath}}^{ICJ}(\{s\} \times K) \geq \mathcal{O}(1)|v_{\bar{\imath}}(s)_{\mathrm{cont}}|(K) > 0$. As the above measures are locally finite, this can thus happen at most countably many times: for all other times t the continuous part of $v_{\bar{\imath}}(t)$ is absolutely continuous.

As a consequence, if $(2.1 : \bar{\imath}\mathrm{GN})$ holds for all $\bar{\imath} = 1,\ldots,\mathrm{N}$ then $u(t) \in \mathrm{SBV}(\mathbb{R};\mathbb{R}^{\mathrm{N}})$ out of countably many times, the times when the time marginal of anyone of the various μ_i^{ICJ} has a jump, $i = 1,\ldots,\mathrm{N}$. This yields the membership of u in $\mathrm{SBV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R};\mathbb{R}^{\mathrm{N}})$. $\square$

4. Recalls on the approximation by front-tracking solutions

We recall in this section a result about the convergence of a suitable sequence of ν-approximate front-tracking solutions (Pages 219-220 in [Bre]).

The ν-approximate front-tracking solutions $\{u^\nu\}_\nu$ are, roughly speaking, piecewise-constant functions obtained approximating by a step function the initial data, (approximatively) solving Riemann problems at discontinuity points and piecing together these solutions until the time they interact and discontinuity lines cross each other: at that time the procedure starts again. By the construction, which allows small perturbations of the speed, only two discontinuity lines are allowed to cross at one time. Each outgoing i-rarefaction wave $[u^-, u^+ = \Psi_i(\sigma)(u^-)]$ in the approximate solution of the Riemann problems is decomposed into small jumps $[\omega_h, \omega_{h+1} = \Psi_i(\sigma_h)(\omega_h)]$ of strength σ_h at most ν; among the various possibilities, we let the h-th jump $[\omega_h, \omega_{h+1}]$ travel with the mean speed $\lambda_i(u^+, u^-)$. In order to control the number of discontinuity lines, if the interacting wave fronts are small enough a simplified Riemann solver is used, which leaves unchanged the size of the incoming waves introducing a non-physical wave

front traveling with fixed speed higher then λ_{N}; the total size of non-physical waves is controlled at each time by a constant ε_ν.

At each time t, the restriction $u^\nu(t)$ is a step function: its derivative consists of finitely many deltas. Below one can see that if one fixes suitable thresholds, it is possible to group these deltas in two families: up to subsequences, the largest of them converge in the ν-limit to the jumps of the entropy solution u at time t, for $\mathcal{L}^1$-a.e. time, while the others tail off up to the remaining continuous part of $u(t)_x$. We distinguish the jumps, excluding interaction times, depending on their characteristic family.

Looking at the (t, x)-variables instead of time-restrictions of u^ν, the derivative of u^ν is concentrated on polygonal lines and consists only of the jump part. Nevertheless, these broken lines can be grouped as follows in order to distinguish those converging to the jump set of u and those part of u_x^ν converging to the continuous part of u_x.

Definition 4.1 (Maximal $(\varepsilon_0, \varepsilon_1)$-shock front). A *maximal $(\varepsilon_0, \varepsilon_1)$-shock front of the i-th family of a ν-approximate front-tracking solution u^ν* is any maximal (w.r.t. inclusion) polygonal line $(t, \gamma(t))$ in the (t, x)-plane, with $t^- \leq t \leq t^+$,

- whose segments are i-shocks of u^ν with strength $|\sigma| \geq \varepsilon_0$, and at least once $|\sigma| \geq \varepsilon_1 > \varepsilon_0$;
- whose nodes are interaction points of u^ν;
- which is on the left of any other polygonal line it intersects and having the above properties.

The family of i-maximal $(\varepsilon_0, \varepsilon_1)$-shock fronts of u^ν is totally ordered by the displacements of the polygonal lines on $\mathbb{R} \times \mathbb{R}^+$, and, up to extracting a subsequence, we are allowed to assume that its cardinality is a constant $M^i_{(\varepsilon_0,\varepsilon_1)}$ independent of ν. Let $\gamma^{\nu,i}_{(\varepsilon_0,\varepsilon_1),m} : [t^{\nu,i,-}_{(\varepsilon_0,\varepsilon_1),m}, t^{\nu,i,+}_{(\varepsilon_0,\varepsilon_1),m}] \to \mathbb{R}$ denote the uniformly Lipschitz paths of i-maximal $(\varepsilon_0, \varepsilon_1)$-shock fronts in u_ν and consider their graphs

$$\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)} = \bigcup_{m=1}^{M^i_{(\varepsilon_0,\varepsilon_1)}} \mathrm{Graph}\left(\gamma^{\nu,i}_{(\varepsilon_0,\varepsilon_1),m}\right).$$

Notice that the set $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$ enlarges as $(\varepsilon_0, \varepsilon_1)$ goes to 0.

Consider sequences $0 < 2^k \varepsilon_0^k \leq \varepsilon_1^k \downarrow 0$, where ε_0^k is not necessarily a power but has just an apex k. Up to subsequences and a diagonal argument, by a suitable labeling of the curves one can assume that for each i, k, m fixed the Lipschitz curves $\gamma^{\nu,i}_{(\varepsilon_0^k,\varepsilon_1^k),m}$ converge uniformly

$$t^{\nu,i,\pm}_{(\varepsilon_0^k,\varepsilon_1^k),m} \to t^{i,\pm}_{(\varepsilon_0^k,\varepsilon_1^k),m}, \qquad \gamma^{\nu,i}_{(\varepsilon_0^k,\varepsilon_1^k),m} \to \gamma^i_{(\varepsilon_0^k,\varepsilon_1^k),m} \qquad \text{as } \nu \to \infty \tag{4.1a}$$

to *distinct* Lipschitz curves which cover the jump set of u: out of countably many points in the (x, t)-plane, either u is continuous, and equal to the pointwise limit of u^ν at that point, or has jump. In the case of a jump, the jump point belongs to the graph of some $\gamma^i_{(\varepsilon_0^k,\varepsilon_1^k),m}$ for a suitable triple i, k, m; moreover, the left\right limits of the front-tracking approximation at the jump curve $\gamma^{\nu,i}_{(\varepsilon_0^k,\varepsilon_1^k),m}$ converge, for m, k fixed and $\nu \to \infty$, to the left\right limits of u at that jump. Below one finds the precise statement.

Theorem 4.2 (Th. 10.4 in [Bre]). *The jump part of u is concentrated on the graphs of $\gamma^i_{(\varepsilon_0^k,\varepsilon_1^k),m}$,*

$$\mathcal{J} = \bigcup_{i=1}^{\mathrm{N}} \bigcup_{m,k} \mathrm{Graph}\left(\gamma^i_{(\varepsilon_0^k,\varepsilon_1^k),m}\right).$$

Moreover, u is continuous and equal to the pointwise limit of u^ν out of $\mathcal{J}$. Define the countable set

$$\begin{aligned} \Theta = &\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_3 \\ = &\left\{ \textit{jump points of the initial datum } \bar{u} \right\} \bigcup \left\{ \textit{atoms of } \mu^{IC} \right\} \\ &\bigcup \left\{ \textit{intersection of any } \gamma^i_{(\varepsilon_0^k,\varepsilon_1^k),m} \textit{ and } \gamma^j_{(\varepsilon_0^k,\varepsilon_1^k),m} \textit{ with } i \neq j \right\} \bigcup \left\{ \textit{endpoints of } \gamma^i_{(\varepsilon_0^k,\varepsilon_1^k),m} \right\}. \end{aligned}$$

Then[2], *at each point* $(t, \gamma(t)) = \lim_\nu (t_\nu, \gamma^\nu(t_\nu))$ *of* $\mathcal{J} \setminus \Theta$ *one has a jump between the distinct values*

$$u(t, \gamma(t)^-) = \lim_\nu u^\nu(t_\nu, \gamma^\nu(t_\nu)^-) = \lim_{\substack{(s,y) \to (t, \gamma(t)) \\ y < \gamma(s)}} u(s, y), \tag{4.1b}$$

$$u(t, \gamma(t)^+) = \lim_\nu u^\nu(t_\nu, \gamma^\nu(t_\nu)^+) = \lim_{\substack{(s,y) \to (t, \gamma(t)) \\ y > \gamma(s)}} u(s, y). \tag{4.1c}$$

We now stress some consequences of the above analysis. On one hand, the fine approximation of u yields in turn piecewise-constant approximations of any function depending on time-space through sufficiently smooth composition with u, and depending on $u^\pm$ at jump points. The countably many points of Θ are not considered in these approximations statement, and not relevant. In particular, Theorem 4.2 yields the following corollary relative to the functions $\tilde{l}_i$, $\tilde{r}_i$, λ_i introduced at Page 5.

Notation 4.3. When adding an apex ν we refer to the corresponding quantities relative to the ν-front tracking approximation u^ν: at each point $\lambda_i(t, x) := \lambda_i(u(t, x^+), u(t, x^-))$ and so on, while $\lambda_i^\nu(t, x) := \lambda_i(u^\nu(t, x^+), u^\nu(t, x^-))$ and similarly

$$\tilde{l}_i^\nu(t, x) := \tilde{l}_i\big(u^\nu(t, x^+), u^\nu(t, x^-)\big), \qquad \tilde{r}_i^\nu(t, x) := \tilde{r}_i\big(u^\nu(t, x^+), u^\nu(t, x^-)\big). \tag{4.2a}$$

Remark 4.4. Front-tracking approximations do not satisfy in general the precise Rankine-Hugoniot conditions, but they may have a small error. Then we define the functions, for $i = 1, \ldots, N$,

$$\bar{\lambda}_i^\nu(t, x) = \begin{cases} \lambda_i(u^\nu(t, x)) & \text{at points which are not on } i\text{-fronts of } u^\nu \\ \dot{\gamma}^\nu(t) & \text{at discontinuity points of the } i\text{-th characteristic family} \end{cases} \tag{4.2b}$$

which are the actual speed of propagation of the outgoing jump of u^ν if belonging to the corresponding characteristic family. In general $\lambda_i^\nu(t, \gamma^\nu(t)) \neq \dot{\gamma}^\nu(t)$ and the difference vanishes as $\nu \to \infty$.

Corollary 4.5. *At each point out of* $\Theta \cup \mathcal{J}$ *the functions* $\tilde{l}_i$, $\tilde{r}_i$, λ_i *are the pointwise limits of their piecewise constant front-tracking approximations* λ_i^ν, $\tilde{l}_i^\nu$, $\tilde{r}_i^\nu$ *introduced above.*

Moreover, at each point $(t, \gamma(t)) = \lim_\nu (t_\nu, \gamma^\nu(t_\nu))$ *of* $\mathcal{J} \setminus \Theta$ *one has*

$$\tilde{l}_i(t, \gamma(t)) = \lim_\nu \tilde{l}_i^\nu(t_\nu, \gamma^\nu(t_\nu)), \qquad \tilde{r}_i(t, \gamma(t)) = \lim_\nu \tilde{r}_i^\nu(t_\nu, \gamma^\nu(t_\nu)),$$

$$\lambda_i(t, \gamma(t)) = \lim_\nu \lambda_i^\nu(t_\nu, \gamma^\nu(t_\nu)) = \lim_\nu \dot{\gamma}^\nu(t).$$

Proof. Out of $\Theta \cup \mathcal{J}$ the statement follows just by the pointwise convergence of u^ν to u provided by Theorem 4.2, because being $f \in C^1(\Omega; \mathbb{R}^N)$ the eigenvalues\eigenvectors of $A(u)$ depend continuously on the variable u. Also $\tilde{l}_i$, $\tilde{r}_i$, λ_i depend continuously on the values $u^\pm$, being suitable means of continuous functions on the segment $[u^-, u^+]$, and they depend on the time-space variables through the composition with the left and right limits $u \setminus u^\nu$ at the jump: the claim at a jump point $(t, \gamma(t)) = \lim_\nu (t_\nu, \gamma^\nu(t_\nu))$ out of Θ follows then by the convergence of $u^\nu(t_\nu, \gamma^\nu(t_\nu)^\pm)$ to $u^\pm(t, \gamma(t))$. $\qquad \square$

Corollary 4.6. *Each i-maximal $(\varepsilon_0^k, \varepsilon_1^k)$-shock front in u_ν converges (locally) as $\nu \to \infty$ to the corresponding graph of $\gamma_{(\varepsilon_0^k, \varepsilon_1^k), m}^i$ in the Hausdorff distance:* $d_H\big(\text{Graph } \gamma_{(\varepsilon_0^k, \varepsilon_1^k), m}^{\nu, i}, \text{Graph } \gamma_{(\varepsilon_0^k, \varepsilon_1^k), m}^i\big) \to 0$. *Moreover, there is a pointwise convergence also for the slopes* $\dot{\gamma}_{(\varepsilon_0^k, \varepsilon_1^k), m}^{\nu, i}(t)$ *out of a countable set of t.*

Proof. This follows directly by (4.1) and Corollary 4.5 taking into account that the ν-front tracking approximate solutions (approximatively) satisfy on i-shocks the Rankine-Hugoniot conditions

$$f(u^+) - f(u^-) = \lambda_i(u^+, u^-)(u^+ - u^-). \qquad \square$$

On the other hand, Theorem 4.2 allows handy approximations of the jump part of u_x, and therefore of the jump part of the wave measures v_i. More precisely, consider for fixed $(\varepsilon_0, \varepsilon_1)$ the measures

$$w_{\text{jump}}^{\nu, i, (\varepsilon_0, \varepsilon_1)} = u_x^\nu \llcorner_{\mathcal{J}_{(\varepsilon_0, \varepsilon_1)}^{\nu, i}}, \qquad v_{\text{jump}}^{\nu, i, (\varepsilon_0, \varepsilon_1)} = \tilde{l}_i \cdot w_{\text{jump}}^{\nu, i, (\varepsilon_0, \varepsilon_1)},$$

---

[2]For obtaining the central term below we refer more precisely to formulas (10.78-79) in [Bre] for the genuinely non-linear case, and to an adaptation of the proof after the absurd (10.83) in [Bre] for the linearly-degenerate case. These formula show also that the curves in (4.1a) may intersect only at endpoints for k fixed, as we claimed before the theorem. For the purpose of this paper only the genuinely non-linear statement will be relevant, thus we have not specified that the curves γ^ν are actually defined in a different way for linearly degenerate characteristics.

which are concentrated on finitely many segments, up to any finite time.

Corollary 4.7. *There exists a sequence ν_k such that*

$$\mathrm{D}^{\mathrm{j}}_x u = \underset{k}{\mathrm{weak}^*\mathrm{lim}} \sum_{i=1}^{\mathrm{N}} w_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}, \qquad (v_i)_{\mathrm{jump}} = \underset{k}{\mathrm{weak}^*\mathrm{lim}}\, v_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}.$$

Denoting by $\mathrm{D}^{\mathrm{a}}_x u_x = \nabla u \mathcal{L}^2_{\llcorner \mathbb{R}^+ \times \mathbb{R}}$ the absolutely continuous part, $\mathrm{D}^{\mathrm{c}}_x u$ the Cantor part of u_x, one has

$$\mathrm{D}^{\mathrm{a}}_x u + \mathrm{D}^{\mathrm{c}}_x u = \underset{k}{\mathrm{weak}^*\mathrm{lim}}\, w_{\mathrm{cont}}^{\nu_k,(\varepsilon_0^k,\varepsilon_1^k)}, \qquad w_{\mathrm{cont}}^{\nu_k,(\varepsilon_0^k,\varepsilon_1^k)} := \left(u^{\nu_k}\right)_x - \sum_{i=1}^{\mathrm{N}} w_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}.$$

Similarly each $\lambda_i(v_i)_{\mathrm{jump}}$ is the limit of $\bar\lambda_i^{\nu_k} v_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}$, while $\lambda_i(v_i)_{\mathrm{cont}}$ the one of $\bar\lambda_i^{\nu_k}\left[v_i^{\nu_k} - v_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}\right]$.

Remark 4.8. By inspection of the proof below, one can see that the convergence in the statement of Corollary 4.7 holds in the same way for the jump and continuous part of $\left(u(t)\right)_x$, for each time t except at most the ones corresponding to the points in Θ of Theorem 4.2.

Proof. The BV structure of the semigroup solution u (Sect. 3.7 of [AFP]) gives the formula for the jump part of $\mathrm{D}u$: for $M_k^i = M_{(\varepsilon_0^k,\varepsilon_1^k)}^i$, it is the matrix-valued measure

$$\mathrm{D}^{\mathrm{j}} u = \left(u^+ - u^-\right) \otimes n\, \mathcal{H}^1_{\llcorner \mathcal{J}} = \lim_{k\to\infty} \sum_{i=1}^{\mathrm{N}} \sum_{m=1}^{M_k^i} \left(u^+ - u^-\right) \otimes n\, \mathcal{H}^1_{\llcorner \mathrm{Graph}\left(\gamma_{(\varepsilon_0^k,\varepsilon_1^k),m}^i\right)}.$$

Moreover, each addend of the sum is the narrow limit of the analogous one relative to u^ν on the corresponding $(\varepsilon_0^k,\varepsilon_1^k)$-shock front: by (4.1b), (4.1c), Corollary 4.6 and the dominated convergence theorem

$$\sum_{m=1}^{M_k^i} \left(u^+ - u^-\right) \otimes n\, \mathcal{H}^1_{\llcorner \mathrm{Graph}\left(\gamma_{(\varepsilon_0^k,\varepsilon_1^k),m}^i\right)} = \underset{\nu}{\mathrm{weak}^*\mathrm{lim}} \sum_{m=1}^{M_k^i} \left(u^{\nu+} - u^{\nu-}\right) \otimes n^\nu\, \mathcal{H}^1_{\llcorner \mathrm{Graph}\left(\gamma_{(\varepsilon_0^k,\varepsilon_1^k),m}^{\nu,i}\right)}$$

$$= \underset{\nu}{\mathrm{weak}^*\mathrm{lim}}\left(u^{\nu+} - u^{\nu-}\right) \otimes n^\nu\, \mathcal{H}^1_{\llcorner \mathcal{J}_{(\varepsilon_0^k,\varepsilon_1^k)}^{\nu,i}}.$$

Collecting the two limits, one has the expression of the x-component

$$\mathrm{D}^{\mathrm{j}}_x u = \lim_k \left(\underset{\nu}{\mathrm{weak}^*\mathrm{lim}} \sum_{i=1}^{\mathrm{N}} w_{\mathrm{jump}}^{\nu,i,(\varepsilon_0^k,\varepsilon_1^k)}\right).$$

In particular, one can find a sequence ν_k such that

$$\mathrm{D}^{\mathrm{j}}_x u = \underset{k}{\mathrm{weak}^*\mathrm{lim}} \sum_{i=1}^{\mathrm{N}} w_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}.$$

Since u^{ν_k} converges to u in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$, then the x-derivative converges weakly*. One then takes the difference between the two terms $(u^{\nu_k})_x$ and $w_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}$, and takes their weak*-limit, in order to see that each $w_{\mathrm{cont}}^{\nu_k,(\varepsilon_0^k,\varepsilon_1^k)}$ weakly*-converge to the continuous part of u_x, which is the difference between u_x itself and its jump part.

The convergence of $v_{\mathrm{jump}}^{\nu,i,(\varepsilon_0,\varepsilon_1)} = \tilde l_i^{\nu_k} \cdot w_{\mathrm{jump}}^{\nu_k,i,(\varepsilon_0^k,\varepsilon_1^k)}$ to $(v_i)_{\mathrm{jump}}$ and of $\bar\lambda_i^\nu v_{\mathrm{jump}}^{\nu,i,(\varepsilon_0,\varepsilon_1)}$ to $\lambda_i(v_i)_{\mathrm{jump}}$ follow as above, taking into account also Corollary 4.5. $\square$

5. Main estimate

As explained in Section 3, the SBV regularity of u amounts to the estimate

$$\exists C > 0 \qquad C\left\{\mathcal{L}^1(B)/\tau + \mu_i^{ICJ}([s - \tau, s + \tau] \times \mathbb{R})\right\} \geq |(v_i(s))_{\mathrm{cont}}|(B) \qquad \forall\, s > \tau > 0,$$

which shows that in the case of genuine non-linearity the non-atomic part $(v_i(s))_{\mathrm{cont}}$ of $v_i(s)$ is controlled by the Lebesgue 1-dimensional measure and the interaction-cancellation-jump wave balance measure of a strip around the time s.

In proving it we consider the *wave balance* and *jump wave balance* measures

$$\mu_i = \partial_t v_i + \partial_x(\lambda_i v_i) \qquad \mu_{i,\text{jump}} = \partial_t(v_i)_{\text{jump}} + \partial_x(\lambda_i v_i)_{\text{jump}}, \qquad i = 1, \dots, N. \qquad (5.1)$$

In Lemmas 5.3, 5.4 below we prove that they are indeed Radon measures: they are controlled by using the interaction and interaction-cancellation measures μ^I, μ^{IC}, in the ν-front-tracking approximation together with other terms vanishing in the limit. The negative part of $\mu_{i,\text{jump}}$ however may not be absolutely continuous w.r.t. μ^{IC}: this is why the statement holds with the measures μ_i^{ICJ}, which dominates $\mu^{IC} + |\mu_{i,\text{jump}}|$ for $i = 1, \dots, N$.

We will first prove estimates of the wave measures on piecewise constant ν-approximate front-tracking solutions $\{u^\nu\}$; we then obtain the claims by passing to the limit on a suitable subsequence.

5.1. **Wave and jump wave balance measures.** We consider the distributions

$$\partial_t v_i + \partial_x(\lambda_i v_i) =: \mu_i \qquad\qquad \mu_i^\nu := \partial_t v_i^\nu + \partial_x(\bar\lambda_i^\nu v_i^\nu),$$

$$\partial_t(v_i)_{\text{jump}} + \partial_x(\lambda_i v_i)_{\text{jump}} =: \mu_{i,\text{jump}} \qquad \mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)} := \partial_t\left(v_i^\nu \llcorner_{\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}}\right) + \partial_x\left(\bar\lambda_i^\nu v_i^\nu \llcorner_{\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}}\right).$$

Notice first that by Corollary 4.7 the distributions in the left column are limits of the ones in the right column along sequences ν_k, $(\varepsilon_0^k, \varepsilon_1^k)$, as $k \to \infty$. This is why it seems natural to us to define them, even though we will apply in the present paper only the 'discrete' ones and we are not going to take advantage of the ones on the l.h.s..

For the rest of the paper we will often omit the index k: limits will be tacitly taken on subsequences of the one of Corollary 4.7. For every $\nu, (\varepsilon_0, \varepsilon_1), i$ the following holds.

Proposition 5.1. *The distributions μ_i, $\mu_{i,\text{jump}}$ and μ_i^ν, $\mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)}$ are finite Radon measures.*

We prove in this section the proposition above, direct consequence of the more specific statements in Lemmas 5.3, 5.4. We call these measures the *i-th wave\jump wave balance measures* respectively of u and u^ν. We introduce them with the aim of managing finer balances for the variation of waves, distinguishing different families and the part of their variation only due to shocks. This is technically more difficult and it is postponed to a future work: in the next section we give only rough balances on strips, for ν-front-tracking approximations, that will be applied in order to derive $(3.2 : \bar\imath)$.

Before stating the lemmas, let us illustrate in the scalar case what the two measures reduce to.

Example 5.2. In the scalar case $N = 1$, the speed $\lambda(u)$ is just $f'(u)$, genuine nonlinearity reads like $f''(u) \geq k > 0$, the parameterization choice is

$$\tilde{l} = \begin{cases} f''(u) & \text{at continuity points,} \\ \dfrac{f'(u^+) - f'(u^-)}{u^+ - u^-} & \text{at jump points.} \end{cases}$$

The wave measure $v = \tilde{l} u_x$ is then

$$v = f''(u)(\mathrm{D}_x^a u + \mathrm{D}_x^c u) + \frac{f'(u^+) - f'(u^-)}{u^+ - u^-} \mathrm{D}_x^j u = \left(f'(u)\right)_x, \qquad (\circ)$$

where the last equality holds by Volpert chain rule (Th. 3.99 of [AFP]).

In the case of a smooth solution one can directly compute that the two measures vanish:

$$\begin{aligned} \mu = \partial_t v + \partial_x(\lambda v) &= \partial_t(f''(u)u_x) + \partial_x(f'(u)f''(u)u_x) \\ &= \partial_t(\partial_x f'(u)) + \partial_x(f'(u)f''(u)u_x) = \partial_x(\partial_t f'(u)) + \partial_x(f'(u)f''(u)u_x) \\ &= \partial_x\big[f''(u)\big(u_t + f'(u)u_x\big)\big] = 0. \end{aligned}$$

One can see that the measure μ, if defined on the solution, vanishes also in the BV case:

$$\begin{aligned} \int \varphi \mu &= -\int \varphi_t\big(\tilde{l}u_x\big) - \int \varphi_x \lambda\big(\tilde{l}u_x\big) \overset{(\star)}{=} -\int \varphi_t\big(\tilde{l}u_x\big) + \int \varphi_x\big(\tilde{l}u_t\big) \\ &\overset{(\circ)}{=} -\int \varphi_t\big(f'(u)\big)_x + \int \varphi_x\big(\tilde{l}u_t\big) = -\int \varphi_x\big(f'(u)\big)_t + \int \varphi_x\big(\tilde{l}u_t\big) \\ &= \int \varphi_x\big[-\big(f'(u)\big)_t + \big(\tilde{l}u_t\big)\big] \overset{(\bullet)}{=} 0, \end{aligned}$$

where we applied repeatedly Volpert chain rule (for example at ($\bullet$)) and at ($\star$) we applied the conservation law $-u_t = (f(u))_x = \lambda u_x$.

The measure μ_{jump} defined on the solution, in the case of BV regularity, can be similarly computed as

$$-\int \varphi \mu_{\text{jump}} \stackrel{(\odot)}{=} \int (\varphi_t + \lambda \varphi_x) \lambda(u^+, u^-) \, \mathrm{D}_x^{\mathrm{j}} u = \sum_{k \in \mathbb{N}} \int \frac{d}{dt} \Big[\varphi(t, \gamma_k(t)) \Big] \Big(\lambda^+ - \lambda^- \Big)(t, \gamma_k(t)) dt,$$

where we denoted by $\{\gamma_k\}_{k \in \mathbb{N}}$ Lipschitz curves covering the jump set of u_x. We now make some heuristics. Knowing that μ is a Radon measure, one obtains that the strength $\lambda^+(t, \gamma_k(t)) - \lambda^-(t, \gamma_k(t)) = \int_{u^-(t,\gamma_k(t))}^{u^+(t,\gamma_k(t))} f''(s) ds$ of the jump γ_k is a function of bounded variation on time intervals where γ_k is separated from the other curves. In that case, if one had some regularity of $u^{\pm}(t, \gamma_k(t))$ one could derive

$$\int \varphi \mu_{\text{jump}} = \sum_{k \in \mathbb{N}} \Bigg\{ \int \varphi(t, \gamma_k(t)) \big[(f''(u^+) u_t^+ - f''(u^-) u_t^-) - \lambda(f''(u^+) u_x^+ - f''(u^+) u_x^-)) \big](t, \gamma_k(t)) dt$$

$$+ \big\{ \varphi \big[f'(u^+) - f'(u^-) \big] \big\}|_{(t_k^0, \gamma_k(t_k^0))} - \big\{ \varphi \big[f'(u^+) - f'(u^-) \big] \big\}|_{(t_k^1, \gamma_k(t_k^1))} \Bigg\}.$$

We now show that μ_i^{ν} is a measure concentrated on interaction points, and mainly on interactions between physical waves, where it is controlled by the interaction measure μ^I. At the interaction point (t, x), remembering that only one interaction may take place at one time,

$$v_i^{\nu}(t)(\mathbb{R}) - \lim_{\epsilon \downarrow 0} v_i^{\nu}(t - \epsilon)(\mathbb{R}) = \mu_i^{\nu}(\{(t, x)\}).$$

Lemma 5.3. *The distribution μ_i^{ν} is a Radon measure satisfying $|\mu_i^{\nu}| \leq \mathcal{O}(1)\mu_{\nu}^I + \rho^{\nu}$ for a purely atomic measure $|\rho^{\nu}| \leq \mathcal{O}(1)\varepsilon_{\nu}$ concentrated on interaction points involving non-physical waves. In the w^*-limit in ν then one has $|\mu_i| \leq \mathcal{O}(1)\mu^I$.*

Proof. We proceed by direct computation of the distribution, that will be a measure concentrated at interaction points. Then we estimate the value at each point.

1) **Computing μ_i^{ν}.** Fix an index ν and let $\{\ell_m\}_{m=1}^{L_{\nu}}$ be time-parametrized curves whose graphs are respectively the discontinuity segments of u^{ν}. Given any test function $\varphi \in C_c^{\infty}((0, +\infty) \times \mathbb{R})$ one has

$$-\langle \varphi, \mu_i^{\nu} \rangle = \iint (\varphi_t + \bar{\lambda}_i \varphi_x) v_i^{\nu} = \sum_{m=1}^{L_{\nu}} \int_{\tau_m^-}^{\tau_m^+} \big[(\varphi_t + \bar{\lambda}_i \varphi_x) \tilde{l}_i \cdot (u^+ - u^-) \big]_{x=\ell_m(t)} dt$$

$$= \sum_{m=1}^{L_{\nu}} \int_{\tau_m^-}^{\tau_m^+} \frac{d}{dt} \Big(\varphi(t, \ell_m(t)) \Big) \tilde{l}_i \cdot (u^+ - u^-) dt$$

$$= \sum_{m=1}^{L_{\nu}} \Big(\varphi(\tau_m^+, \ell_m(\tau_m^+)) - \varphi(\tau_m^-, \ell_m(\tau_m^-)) \Big) \tilde{l}_i \cdot (u^+ - u^-).$$

As noticed in (2.4), if $u^+ = \Psi_i(\sigma_i) u^-$ by construction $\tilde{l}_j \cdot (u^+ - u^-) = 0$ for $j \neq i$, and for $j = i$ it is the strength of the wave, that we denote again by σ_i (with an abuse of notation in case of genuinely non-linear shocks). The non-vanishing terms in the summation above therefore are the ones of discontinuities of the i-th family and non-physical fronts. If $\{(\tau_k, z_k)\}_k$ is the collection of 'nodes' of i-th fronts of u^{ν} for $t > 0$, the computation above yields

$$\mu_i^{\nu} = \sum_k p_k \delta_{(\tau_k, z_k)} + \rho^{\nu}, \tag{5.2a}$$

where, denoting at each node by σ_i', σ_i'' the i-th component of the incoming strengths, σ_i the outgoing,

$$p_k = \sigma_i - \sigma_i' - \sigma_i'' \tag{5.2b}$$

while ρ^{ν} is concentrated on the nodes of non-physical waves, with

$$\rho^{\nu}(\{P\}) = \tilde{l}_i^+ \cdot \sigma - \tilde{l}_i^- \cdot \sigma',$$

$\sigma = u^+ - u^-, \sigma' = u'^+ - u'^-$ being the outgoing\incoming strengths of the non-physical wave at P, $\tilde{l}_i^{\pm}$ the outgoing\incoming values of $\tilde{l}_i$ on the non-physical front. Notice that σ' or σ'' in the above expression may also vanish, for example in the last $\sigma' = 0$ when a new non-physical wave originates.

2) `Estimates`. In order to bound μ_i^ν one considers the Glimm functional of [Bre]

$$\boldsymbol{\Upsilon}^\nu = \sum_\alpha |\sigma_\alpha| + C_0 \sum_{(\alpha,\beta)} |\sigma_\alpha \sigma_\beta| \qquad C_0 \gg 1,$$

where α, β index the discontinuities at time t, with corresponding strengths σ_α, σ_β, and the second summation ranges over couples (α,β) whose discontinuity points $x_\alpha < x_\beta$ either belong to families $k_\alpha > k_\beta \in \{1,\dots,\mathrm{N}+1\}$ or belong to a same family but at least one is a genuinely non-linear shock. Let Υ^ν be the same functional, but neglecting in both the summations all terms involving strengths of non-physical waves. Since the total strength of non-physical waves at fixed time is controlled by ε_ν by construction of ν-approximate front-tracking solution, then $|\boldsymbol{\Upsilon}^\nu - \Upsilon^\nu| \leq \mathcal{O}(1)\varepsilon_\nu$.

By choosing C_0 big enough and the smallness of the total variation, by those estimates in Lemma 7.2, Page 133, of [Bre] and Pages 137, 138, at each interaction at time τ

$$|\mu_i^\nu|(\{Q\}) \leq \mathcal{O}(1)|\sigma'\sigma''| \leq \mathcal{O}(1)\{\boldsymbol{\Upsilon}^\nu(\tau_k^-) - \boldsymbol{\Upsilon}^\nu(\tau_k^+)\}.$$

If $\mathfrak{T}^p$ is the set of interaction times involving only physical waves, by definition of the interaction measure

$$|\mu_i^\nu|_{\llcorner \mathfrak{T}^p \times \mathbb{R}} \leq \mathcal{O}(1)\mu_\nu^I{}_{\llcorner \mathfrak{T}^p \times \mathbb{R}}, \qquad \mu_\nu^I(\mathbb{R}^+ \times \mathbb{R}) \leq \boldsymbol{\Upsilon}^\nu(0).$$

For the set $\mathfrak{T}^n$ of interaction times involving some non-physical front, instead one has the bound

$$
\begin{aligned}
|\rho^\nu| &\leq \sum_{\tau \in \mathfrak{T}^n} \mathcal{O}(1)\{\boldsymbol{\Upsilon}^\nu(\tau^-) - \boldsymbol{\Upsilon}^\nu(\tau^+)\} \\
&\overset{(\circ)}{=} \sum_{\tau \in \mathfrak{T}^n} \mathcal{O}(1)\{\boldsymbol{\Upsilon}^\nu(\tau^-) - \boldsymbol{\Upsilon}^\nu(\tau^+) - [\Upsilon^\nu(\tau^-) - \Upsilon^\nu(\tau^+)]\} \\
&\overset{(\star)}{\leq} \sum_{\tau \in \mathfrak{T}^p \cup \mathfrak{T}^n} \mathcal{O}(1)\{\boldsymbol{\Upsilon}^\nu(\tau^-) - \boldsymbol{\Upsilon}^\nu(\tau^+) - [\Upsilon^\nu(\tau^-) - \Upsilon^\nu(\tau^+)]\} + \mathcal{O}(1)\varepsilon_\nu|\mu_\nu^I| \\
&= \mathcal{O}(1)\{[\boldsymbol{\Upsilon}^\nu(0) - \Upsilon^\nu(0)] - [\boldsymbol{\Upsilon}^\nu(\infty) - \Upsilon^\nu(\infty)]\} + \mathcal{O}(1)\varepsilon_\nu|\mu_\nu^I| \leq \mathcal{O}(1)\varepsilon_\nu,
\end{aligned}
$$

because at interactions involving also non-physical waves Υ^ν does not vary $(\circ)$, while at interactions of two physical waves σ', σ'' e.g. of the i-th family most of the terms cancel $(\star)$:

$$\boldsymbol{\Upsilon}^\nu(\tau^-) - \boldsymbol{\Upsilon}^\nu(\tau^+) - [\Upsilon^\nu(\tau^-) - \Upsilon^\nu(\tau^+)] \leq \mathcal{O}(1)\left(\sum_{k \neq i} |\sigma^k| + |\sigma' + \sigma'' - \sigma|\right) \sum_{\beta \text{ non physical}} |\sigma_\beta|.$$

One can gain the estimates on μ_i in the ν-limit because $\boldsymbol{\Upsilon}^\nu(0)$ can be bounded uniformly in ν. $\qquad \square$

The following lemma deals instead with the jump-wave balance measure, which is again concentrated on interactions. At the interaction point (t,x) one has

$$v_i^\nu(t)\big(\mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}\big) - \lim_{\epsilon \downarrow 0} v_i^\nu(t-\epsilon)\big(\mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}\big) = \mu_i^\nu\big(\{(t,x)\}\big).$$

The positive part is absolutely continuous w.r.t. the interaction-cancellation measure, up to a remainder. The bound on the negative part is obtained instead considering the history of the interacting waves, and involves the Glimm functional Υ.

Notations are illustrated in Figure 2. Denote by $\{(t_h, x_h)\}_h$ the finite set of terminal points of the i-th $(\varepsilon_0,\varepsilon_1)$-jump set $\mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}$ of u^ν and let $\{\gamma_h\}_h$, $\{\sigma_h\}_h$ denote the relative maximal $(\varepsilon_0,\varepsilon_1)$-shock front and the strength $\leq \varepsilon_0$ just after becoming below the threshold (if any, otherwise vanishes).

Lemma 5.4. *The positive part of $\mu_{i,\mathrm{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)}$ satisfies the following inequality*

$$\mu_{i,\mathrm{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)} \leq \mathcal{O}(1)\mu_\nu^{IC} - \sum_h \sigma_h \delta_{(t_h,x_h)} \qquad |\sigma_h| \leq \mathcal{O}(1)\frac{\varepsilon_0}{\varepsilon_1 - \varepsilon_0}\mu_\nu^{IC}(\gamma_h).$$

Also the negative part of $\mu_{i,\mathrm{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)}$ is concentrated on the set of nodes of $\mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}$ and its mass is uniformly bounded by the Glimm functional Υ. In the w^*-limit of Corollary 4.7 one gets

$$-\rho \leq \mu_{i,\mathrm{jump}} \leq \mathcal{O}(1)\mu^{IC} \qquad \rho \geq 0 \text{ finite}.$$

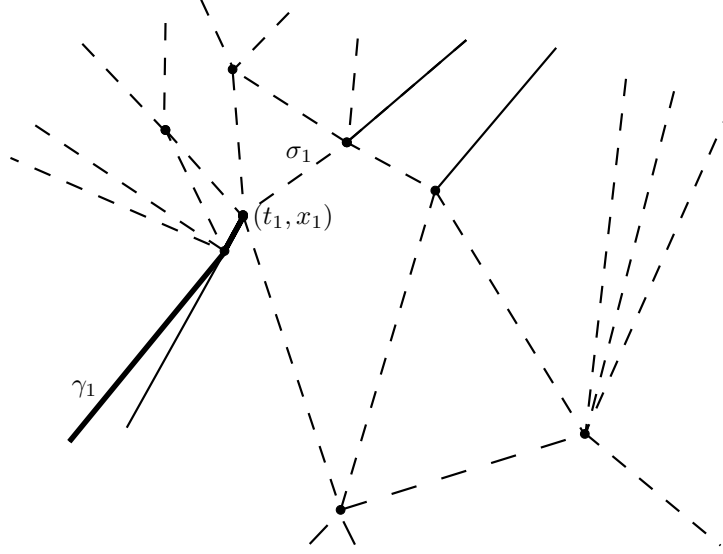*Proof.* As before, we proceed by direct computation of the measure.

FIGURE 2. Discontinuity lines of u^ν belonging to the i-th $(\varepsilon_0, \varepsilon_1)$-jump set $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$. Dashed lines either are below the threshold (e.g. σ_1) or belong to a different family $j \neq i$.

1) `Computing` $\mu^{\nu,(\varepsilon_0,\varepsilon_1)}_{i,\mathrm{jump}}$. One has to take into account that some of the shocks of u^ν are not present in the maximal $(\varepsilon_0, \varepsilon_1)$-shock fronts, and none of the rarefaction fronts and non-physical waves appears. Fix an index ν and let $\{\gamma_m\}^{L_\nu}_{m=1}$ be time-parametrized curves whose graphs are respectively the discontinuity *segments* of u^ν *present* in $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$. Given any test function $\varphi \in C^\infty_c((0, +\infty) \times \mathbb{R})$, and omitting for simplicity most of the indices $\nu, (\varepsilon_0, \varepsilon_1), i$ from the second line below up to the end of the proof, one has

$$-\langle \varphi, \mu^{\nu,(\varepsilon_0,\varepsilon_1)}_{i,\mathrm{jump}} \rangle = \iint \big(\varphi_t + \bar{\lambda}^\nu{}_i \varphi_x\big) v^\nu_i \llcorner_{\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}}$$

$$= \sum_m \int^{\tau^+_m}_{\tau^-_m} \frac{d}{dt}\Big(\varphi(t, \gamma_m(t))\Big) \tilde{l} \cdot (u^+ - u^-) dt$$

$$= \sum_m \Big(\varphi(\tau^+_m, \gamma_m(\tau^+_m)) - \varphi(\tau^-_m, \gamma(\tau^-_m))\Big) \sigma_i.$$

If $\{(\tau_k, z_k)\}_k$ are the nodes in $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$ for $t > 0$, then the computation above can be rewritten as

$$\mu^{\nu,(\varepsilon_0,\varepsilon_1)}_{i,\mathrm{jump}} = \sum_k q_k \delta_{(\tau_k, z_k)}, \tag{5.3a}$$

where, denoting at each node by σ', σ'' the i-incoming strengths, σ the outgoing, one has

$$q_k = \begin{cases} \sigma - \sigma' - \sigma'' & \text{at triple points of } \mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}, \\ \sigma & \text{at initial points of maximal fronts of } \mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}, \\ -\sigma' & \text{when a maximal front of } \mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)} \text{ ends without merging into another}, \\ \sigma - \sigma' & \text{otherwise.} \end{cases} \tag{5.3b}$$

2) `Upper bound`. At triple points of $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$ one has $|q_k| \leq \mathcal{O}(1)\mu^I_\nu$ by the interaction estimates in Lemma 7.2 of [Bre] and by definition of the interaction measure (3.1).

At initial points of $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$, by genuine nonlinearity $q_k \leq 0$.

At internal nodal points of a front where

- another shock of the same family, not belonging to any front in $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$, merges, again $q_k \leq 0$;
- a rarefaction wave front interacts, a cancellation occurs and $|q_k| = |\sigma - \sigma'| \leq \mu^{IC}_\nu(t_k, x_k)$ by definition of the interaction-cacellation measure in (3.1);
- any curve of different family interact, interaction takes place and $|q_k| = |\sigma - \sigma'| \leq \mathcal{O}(1)\mu^I_\nu(t_k, x_k)$.

At terminal points, since the shock gets cancelled from $\mathcal{J}^{\nu,i}_{(\varepsilon_0,\varepsilon_1)}$, there must be a cancellation by a rarefaction front or by a wave of different family. If γ_k is the maximal $(\varepsilon_0, \varepsilon_1)$-shock front through the point, by inspection one can see that the strength must increase from a value $\sigma_0 \leq -\varepsilon_1$ at some point up to a value $\leq -\varepsilon_0$ on the last segment, which colliding the other wave front becomes σ_1 and exceeds $-\varepsilon_0$. By the interaction estimates, at nodal points of each front the strength may increase at most of the amount of interaction-cancellation at that point. One can then conclude

$$\varepsilon_1 - \varepsilon_0 \leq |\sigma_0| - |\sigma_1| \leq \text{Tot.Var.}^+(u, \gamma_k) \leq \mathcal{O}(1)\mu_\nu^{IC}(\gamma_k).$$

This yields the final estimate, at terminal points:

$$q_k = -\sigma_1 + (\sigma_1 - q_k) \leq \frac{\varepsilon_0}{\varepsilon_1 - \varepsilon_0}(\varepsilon_1 - \varepsilon_0) + \mathcal{O}(1)\mu_\nu^I(t_k, x_k) \leq \frac{\varepsilon_0}{\varepsilon_1 - \varepsilon_0}\mathcal{O}(1)\mu_\nu^{IC}(\gamma_k) + \mathcal{O}(1)\mu_\nu^I(t_k, x_k).$$

Notice that σ_1 is one of the σ_k in the first statement.

By summing up the different contributions, being

$$\sum |\sigma_h| \leq \mathcal{O}(1)\frac{\varepsilon_0}{\varepsilon_1 - \varepsilon_0}\mu_\nu^{IC}(\mathbb{R}^+ \times \mathbb{R}),$$

one gains that $\mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)} - \mathcal{O}(1)\mu_\nu^{IC} + \sum \sigma_h \delta_{(t_h,x_h)}$ is a signed measure, and thus $\mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)}$ is a Radon measure. We got an upper bound, uniform w.r.t ν, of its positive mass, and we also got that the limit in ν in the sense of distributions, i.e. $\mu_{i,\text{jump}}$, is a Radon measure. Notice that $\sum |\sigma_h|$ by the estimate above vanishes in the limit of Corollary 4.7, as we have chosen $\varepsilon_1^k \geq 2^k \varepsilon_0^k$.

3) **Lower bound.** We derive now a lower bound uniform w.r.t ν of the negative mass of $\mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)}$, suitable to estimate that the negative part of the limit $\mu_{i,\text{jump}}$ is actually a finite measure.

Consider the nonnegative measure $\bar{\mu} = -\mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)} + \mathcal{O}(1)\mu_\nu^{IC} + \sum |\sigma_h'|\delta_{(t_h,x_h)}$. Consider for $\alpha > 0$ the Lipschitz test function $\varphi_\alpha(t) = \chi_{[0,T+\alpha]}(t) - (t-T)/\alpha\chi_{[T,T+\alpha]}(t)$: being the time marginal $v_{i,\text{jump}}^\nu(t)$ of $v_{i,\text{jump}}^\nu$ absolutely continuous and locally finite one has

$$\bar{\mu}([0,T] \times \mathbb{R}) \leq \int \varphi_\alpha d\bar{\mu} = -\int \varphi_\alpha \mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)} + \mathcal{O}(1)\int \varphi_\alpha d\mu_\nu^{IC} + \sum_h |\sigma_h'|\varphi_\alpha(t_h)$$

$$\leq \iint ((\varphi_\alpha)_t + \bar{\lambda}_i^\nu(\phi_\alpha)_x)d[v_{i,\text{jump}}^\nu(t)]dt + [v_{i,\text{jump}}^\nu(0)](\mathbb{R}) + \mathcal{O}(1)\mu_\nu^{IC}([0,T+\alpha] \times \mathbb{R})$$

$$= -\frac{1}{\alpha}\int_T^{T+\alpha} [v_{i,\text{jump}}^\nu(t)](\mathbb{R})dt + [v_{i,\text{jump}}^\nu(0)](\mathbb{R}) + \mathcal{O}(1)\mu_\nu^{IC}([0,T+\alpha] \times \mathbb{R})$$

By the uniform bound on $|v_{i,\text{jump}}^\nu(t)|$, as $\alpha \downarrow 0$ one obtains

$$\bar{\mu}([0,T] \times \mathbb{R}) \leq \Delta_{0,t}\text{Tot.Var.}(v_{i,\text{jump}}^\nu) + \mathcal{O}(1)\mu_\nu^{IC}([0,T+\alpha] \times \mathbb{R}) \leq \mathcal{O}(1)\Upsilon^\nu(0). \qquad \square$$

5.2. Balances on characteristic regions. We already discussed the distinction between the part of u_x^ν which approximates the jump part of u_x and what remains: now we denote shortly

$$v_{i,\text{jump}}^\nu = v_i^\nu \llcorner_{\mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}} \qquad\qquad \mu_{i,\text{jump}}^\nu = \mu_{i,\text{jump}}^{\nu,(\varepsilon_0,\varepsilon_1)}$$

$$v_{i,\text{cont}}^\nu = v_i^\nu - v_{i,\text{jump}}^\nu \qquad\qquad \mu_{i,\text{cont}}^\nu = \mu_i^\nu - \mu_{i,\text{jump}}^\nu.$$

In the proof of Lemmas 5.3, 5.4 we derived the balances

$$v_i^\nu(t) - v_i^\nu(t^-) = \mu_i^\nu(\{t\} \times \mathbb{R}), \qquad v_{i,\text{jump}}^\nu(t) - v_{i,\text{jump}}^\nu(t^-) = \mu_{i,\text{jump}}^\nu(\{t\} \times \mathbb{R}).$$

We estimate in this section balances of these measures in regions bounded by generalized $\bar{\imath}$-th characteristics, based on the assumption of genuine non-linearity of the $\bar{\imath}$-th characteristic field. A relevant measure will be the approximated interaction-cancellation-jump balance measure

$$\mu_{\bar{\imath},\nu}^{ICJ} = \mu^{IC} + |\mu_{\bar{\imath},\text{jump}}^\nu|. \tag{5.4}$$

The measure $\mu_{\bar{\imath}}^{ICJ}$ is defined as a w^*-limit of the measures $\mu_{\bar{\imath},\nu}^{ICJ}$.

We recall that a *(generalized) i-th characteristic* is an absolutely continuous curve $x(t)$ satisfying for a.e. t the differential inclusion

$$\dot{x}(t) \in [\lambda_i(u(t,x(t)^+)), \lambda_i(u(t,x(t)^-))].$$

Due to the presence of discontinuities, if u is a semigroup solution of a system of conservation laws or a ν-approximate front-tracking solution there are several backward and forward characteristics, and they may collapse. They are polygonal lines whose direction changes at interaction points or hitting discontinuities, which are their nodes.

One can select the *minimal* i-th characteristics of the piecewise constant approximations starting at t_0 from any point $\bar{x}$, which means

$$x(t; t_0, x_0) = \min\left\{ \ x(t): \ x(t = t_0) = x_0, \ \dot{x}(t) \in [\lambda_i(u(t, x^+)), \lambda_i(u(t, x^-))] \ \right\}.$$

Analogously, one can select the *maximal* one.

For the rest of the paper we will select for simplicity minimal generalized characteristics. We stress however that the same statements hold if one selects any other family $y^\nu(t; t_0, x)$ of generalized characteristics of u^ν which, as the minimal and maximal ones, have the semigroup property

$$y^\nu(t_0; t_0, x_0) = x_0; \qquad y^\nu(t + \tau; t_0, x_0) = y^\nu(\tau; t, y^\nu(t, t_0, x_0)).$$

Notation 5.5. Given an interval $I = [a, b]$, we define the region $A_{[a,b]}^{t_0,\tau}$ bounded by the minimal i-characteristics $a(t), b(t)$ starting at t_0 respectively from a, b, and its time-section $I(t)$, as

$$A_{[a,b]}^{t_0,\tau} = \left\{ (t, x): \ t_0 < t \le t_0 + \tau, \ a(t) \le x \le b(t) \right\}, \qquad I(t) = [a(t), b(t)]. \tag{5.5}$$

Given $J = I_1 \cup \cdots \cup I_M$ the union of any disjoint closed intervals $I_1, \ldots, I_M$, $M \in \mathbb{N}$, we denote by $A_J^{t_0,\tau}$ the union of the regions $A_{I_h}^{t_0,\tau}$. Similarly, $J(t) = A_J^{t_0,\tau} \cap \{t\} \times \mathbb{R} = I_1(t) \cup \cdots \cup I_M(t)$.

Lemma 5.6 (Approximate wave balances)**.** *Assume* $(2.1 : \bar{\imath}\mathrm{GN})$ *and consider regions bounded by* $\bar{\imath}$-*characteristics as above. Then for any* $\tau, t_0 \ge 0$ *and* $t = t + \tau_0$

$$v_{\bar{\imath}}^\nu(t)\big(J(t)\big) - v_{\bar{\imath}}^\nu(t_0)\big(J\big) \le \mathcal{O}(1)\big(\mu_\nu^{IC}\big(A_J^{t_0,\tau}\big) + \varepsilon_\nu\big),$$

$$v_{\bar{\imath},\mathrm{jump}}^\nu(t)\big(J(t)\big) - v_{\bar{\imath},\mathrm{jump}}^\nu(t_0)\big(J\big) \le \mu_{\bar{\imath},\mathrm{jump}}^\nu\big(A_J^{t_0,\tau}\big),$$

$$v_{\bar{\imath},\mathrm{cont}}^\nu(t)\big(J(t)\big) - v_{\bar{\imath},\mathrm{cont}}^\nu(t_0)\big(J\big) \le \mathcal{O}(1)\big(\mu_{\bar{\imath},\nu}^{ICJ}\big(A_J^{t_0,\tau}\big) + \varepsilon_\nu\big).$$

The lemma shows that the variation of both the 'continuous' and 'jump' part of the wave measures on characteristic regions is controlled by finite, purely atomic measures, concentrated on interaction points. While for points in the interior of the region a perfect balance holds, with the suitable measures, in the proof one defines 'fluxes through the boundaries' in order to take into account the variation due to waves that enter\exit the region. Then the fluxes are controlled by the assumption of genuine non-linearity and the interaction-cancellation measure, by the choice of the speed of the rarefaction as a mean of the values $\lambda(u^\pm)$.

Proof. Fix for simplicity of notations $t_0 = 0$, $A_-^\tau = A_-^{0,\tau}$. Given a closed interval $[a, b]$ we briefly denote

$$v_{\bar{\imath}}^\nu(t) := v_{\bar{\imath}}^\nu(t)\big([a(t), b(t)]\big). \tag{5.6a}$$

We prove before the first balance, which is most of the work, and then we explain the others.

0) Contribution of non-physical waves. The total strength of non-physical fronts at each finite time is controlled by ε_ν, as well as the mass $|\rho^\nu|$ of μ^ν due to interactions involving non-physical waves (Lemma 5.3). Since thus non-physical interactions and waves are not relevant, they may be neglected by removing from both μ^ν, v^ν the relative terms: denote by $\hat{\mu}^\nu = \mu^\nu - \rho^\nu$, $\hat{v}^\nu = v^\nu \llcorner_{\{\text{physical waves}\}}$ this simplified measures. It suffices to obtain the claim for these measures, then the other follows.

1) Instantaneous estimate. The region $A_{[a,b]}^\tau$ is bounded by generalized $\bar{\imath}$-characteristics: then the function $v_{\bar{\imath}}^\nu(t)$ may vary only at times $\hat{t}$ where an interaction takes place in $[a(\hat{t}), b(\hat{t})]$, an i-rarefaction wave leaves the region $A_{[a,b]}^\tau$ or an i-shock enters. The following cases may occur: if physical waves in $A_{[a,b]}^\tau$ change at $(\hat{t}, \hat{x})$, then one can define $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$ by

$$v_{\bar{\imath}}^\nu(\hat{t}^+) - v_{\bar{\imath}}^\nu(\hat{t}^-) = \mu_{\bar{\imath}}^\nu\big(\{(\hat{t}, \hat{x})\}\big) + \Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}(\{(\hat{t}, \hat{x})\}).$$

Formula (5.2) and the inspection of all the possible cases, that we perform without taking into account that $a(t), b(t)$ are minimal, yield that $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}(\{(\hat{t}, \hat{x})\})$ is different from 0 only if $\hat{x}$ coincides with $a(\hat{t})$ or $b(\hat{t})$. The value can be computed exactly. The most relevant cases are described below, and they

correspond to interaction among physical waves of the same family. The others are analogous. For example, we do not consider when only part of a rarefaction exits the region: one can see that the value of $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$ is controlled by above and below by two cases we consider (the whole rarefaction exists, or no part of it exits). In these cases, that we illustrate in Figure 3, $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$ is precisely equal to
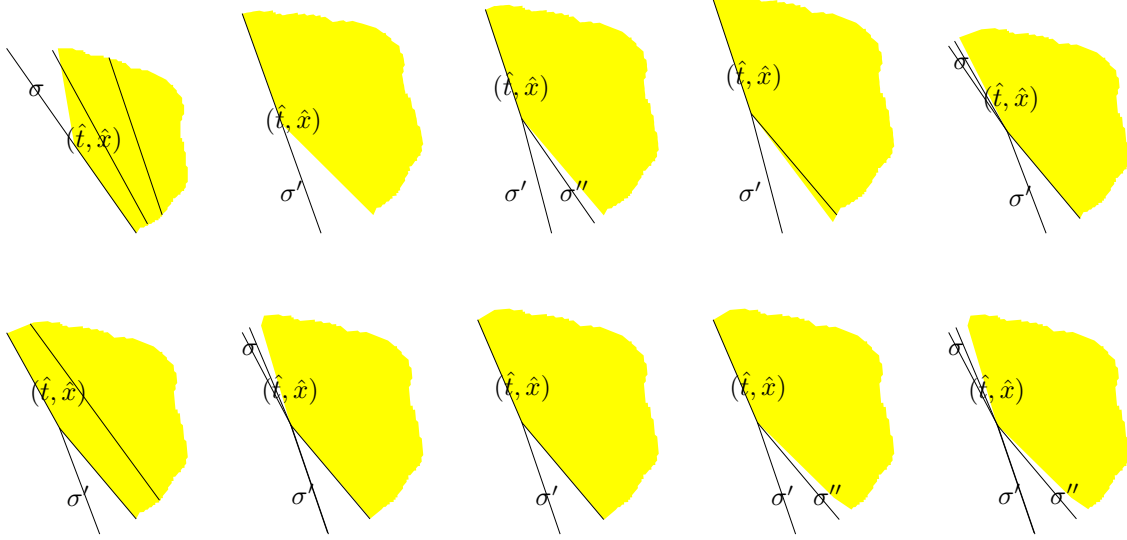

FIGURE 3. Illustration of the flux function $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}(\{(\hat{t},\hat{x})\})$ of Lemma 5.6

$-\sigma;\ \sigma \geq 0$	If an $\bar{\imath}$-rarefaction wave exits.
$\sigma';\ \sigma' \leq 0$	If an $\bar{\imath}$-shock enters.
$\sigma' + \sigma'';\ \sigma', \sigma'' \leq 0$	If two $\bar{\imath}$-shocks out of $A_{[a,b]}^{\tau}$ interact.
$\sigma';\ \sigma' \leq 0$	If two $\bar{\imath}$-shocks, one in one out of $A_{[a,b]}^{\tau}$, interact.
$-\sigma + \sigma';\ \sigma' \leq 0,\ \sigma \geq 0$	If an $\bar{\imath}$-shock out of $A_{[a,b]}^{\tau}$ interacts with an $\bar{\imath}$-rarefaction wave, and a $\bar{\imath}$-rarefaction wave exits the region $A_{[a,b]}^{\tau}$.
$\sigma';\ \sigma' \leq 0$	If an $\bar{\imath}$-shock out of $A_{[a,b]}^{\tau}$ interacts with an $\bar{\imath}$-rarefaction wave, and no $\bar{\imath}$-rarefaction wave exits the region $A_{[a,b]}^{\tau}$.
$-\sigma + \sigma';\ \sigma, \sigma' \geq 0$	If an $\bar{\imath}$-rarefaction wave out of $A_{[a,b]}^{\tau}$ interacts with a $\bar{\imath}$-shock in $A_{[a,b]}^{\tau}$, and a $\bar{\imath}$-rarefaction wave exits the region $A_{[a,b]}^{\tau}$.
$\sigma';\ \sigma' \geq 0$	If an $\bar{\imath}$-rarefaction wave out of $A_{[a,b]}^{\tau}$ interacts with a $\bar{\imath}$-shock in $A_{[a,b]}^{\tau}$, and no $\bar{\imath}$-rarefaction wave exits the region $A_{[a,b]}^{\tau}$.
$\sigma' + \sigma'';\ \sigma' \geq 0, \sigma'' \leq 0$	If an $\bar{\imath}$-rarefaction wave and $\bar{\imath}$-shock both out of $A_{[a,b]}^{\tau}$ interact, and no $\bar{\imath}$-rarefaction wave exits the region $A_{[a,b]}^{\tau}$.
$-\sigma + \sigma' + \sigma'';\ \sigma, \sigma' \geq 0, \sigma'' \leq 0$	If an $\bar{\imath}$-rarefaction wave and $\bar{\imath}$-shock both out of $A_{[a,b]}^{\tau}$ interact, and a $\bar{\imath}$-rarefaction wave exits the region $A_{[a,b]}^{\tau}$.

The explanation is that when a physical interaction takes place at the boundary one has to balance the part of the waves that are out of the region we are looking at. *The assumption of genuine-nonlinearity of the $\bar{\imath}$-th characteristic field is what determines the sign of the big discontinuities, the $\bar{\imath}$-shocks.* Considering that, the contribution of $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$ may be positive only in the last four cases, when an $\bar{\imath}$-rarefaction wave comes from the outer region and hits a shock: in this case we now show that cancellation occurs and $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$ is controlled by the amount of cancellation in $\mu_{\bar{\imath}}^{IC}$ at that point. Indeed, suppose that an $\bar{\imath}$-rarefaction ℓ' hits an $\bar{\imath}$-shock ℓ'': the value u_m of u^{ν} between the two fronts should be the same, while the incidence condition implies $\dot{\ell}' > \dot{\ell}''$. Moreover the speeds $\dot{\ell}', \dot{\ell}''$ can be estimated by the intermediate value u_m and

the strength of the waves: one has

$$\lambda_{\bar{\imath}}(u_m) - \sigma'_{\bar{\imath}}/4 > \dot{\ell}' > \dot{\ell}'' > \lambda_{\bar{\imath}}(u_m) + 3\sigma''_{\bar{\imath}}/4 \qquad \Rightarrow \qquad -3\sigma''_{\bar{\imath}} \geq \sigma'_{\bar{\imath}}.$$

This says that if a cancellation occurs, then the shock should be of the same size as the rarefaction. Therefore the rarefaction even when not cancelled is controlled by the amount of cancellation. Then

$$\Phi^{\nu,\text{out}}_{\bar{\imath},[a,b]}(\{(\hat{t},\hat{x})\}) \leq 3\mu^{IC}_{\bar{\imath}}(\{(\hat{t},\hat{x})\}).$$

More formally, the 'flux' $\Phi^{\nu,\text{out}}_{\bar{\imath},[a,b]}$ could be defined at once considering the exterior trace of u^ν on $A^\tau_{[a,b]}$.

2) Single interval estimate. By the choice of $A^\tau_{[a,b]}$, which contains $I(\tau)$ but not $I(0)$, there is no flux concerning waves exiting at τ and no one concerning waves entering at 0—even if an interaction takes place at τ or at 0—because $v_{\bar{\imath}}(t)$ is continuous from the right. Indeed interactions at $t = 0$ are not taken into account by $\mu^\nu_{\bar{\imath}}$ on $A^\tau_{[a,b]}$. We recall that now we temporary neglect non-physical waves.

Piecing together the various physical interactions one obtains

$$\hat{v}^\nu_{\bar{\imath}}(\tau) - \hat{v}^\nu_{\bar{\imath}}(0) = \hat{\mu}^\nu_{\bar{\imath}}\big(A^\tau_{[a,b]}\big) + \Phi^{\nu,\text{out}}_{\bar{\imath},[a,b]}(\text{Graph}(a) \cup \text{Graph}(b)) \qquad \Phi^{\nu,\text{out}}_{\bar{\imath},[a,b]}(B) \leq \mathcal{O}(1)\mu^{IC}(B), \qquad (5.6b)$$

where $B \subset \text{Graph}(a) \cup \text{Graph}(b)$. This formula holds also in the case $a = b$. When removing the hat the same equation holds with an additional term uniformly controlled by ε_ν, however we prefer to take non-physical wave into account after having obtained the balance for countably many intervals.

3) Estimate on more intervals. Consider now two disjoint closed intervals $I_0 = [x_0, x_1]$, $I_1 = [x_2, x_3]$ and denote the relative selected $\bar{\imath}$-characteristics starting out from x_h by $x_h(t)$, $I_h(t) = [x_{2h}(t), x_{2h+1}(t)]$. Notice that the same equation as in (5.6b) holds immediately for $A^{t_0,\tau}_{I_1 \cup I_2}$ on time intervals where $x_1(t_0 + \tau) < x_2(t_0 + \tau)$. We consider thus the case $x_1(t) = x_2(t)$ at time $\hat{t} \leq t_0 + \tau$ and then, by the semigroup property, up to time $t_0 + \tau$.
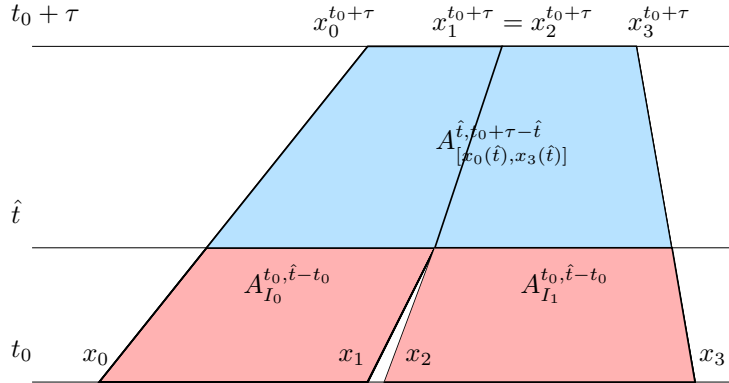

FIGURE 4. Illustration of Lemma 5.6, balance on two intervals.

As illustrated in Figure 4, decompose $A^{t_0,\tau}_{I_0} \cup A^{t_0,\tau}_{I_1}$ into the regions $A^{t_0,\tau'}_{I_0}, A^{t_0,\tau'}_{I_1}, A^{\hat{t},\tau-\tau'}_{[x_0(\hat{t}),x_3(\hat{t})]}$, with $\tau' = \hat{t} - t_0$, where one applies separately (5.6b). In order to shorten notations, fix again $t_0 = 0$:

$$\hat{v}^\nu_{\bar{\imath}}(\tau)\big([x_0(\tau), x_3(\tau)]\big) - \hat{v}^\nu_{\bar{\imath}}(0)\big(I_0 \cup I_1\big)$$
$$= \hat{v}^\nu_{\bar{\imath}}(\tau)\big([x_0(\tau), x_3(\tau)]\big) - \hat{v}^\nu_{\bar{\imath}}(\hat{t})\big([x_0(\hat{t}), x_3(\hat{t})]\big)$$
$$+ \hat{v}^\nu_{\bar{\imath}}(\hat{t})\big([x_0(\hat{t}), x_1(\hat{t})]\big) + \hat{v}^\nu_{\bar{\imath}}(\hat{t})\big([x_2(\hat{t}), x_3(\hat{t})]\big) - \hat{v}^\nu_{\bar{\imath}}(\hat{t})\big(\{x_1(\hat{t}) = x_2(\hat{t})\}\big)$$
$$- \hat{v}^\nu_{\bar{\imath}}(0)\big(I_0\big) - \hat{v}^\nu_{\bar{\imath}}(0)\big(I_1\big).$$

If one denotes briefly $\Phi^{\nu,\text{out}}_{\bar{\imath},I_0}((\hat{t},\tau]) = \Phi^{\nu,\text{out}}_{\bar{\imath},I_0}(A^{\hat{t},\tau}_{I_0})$ and so on, the balances yield

$$\hat{v}^\nu_{\bar{\imath}}(\tau)\big([x_0(\tau), x_3(\tau)]\big) - \hat{v}^\nu_{\bar{\imath}}(0)\big(I_0 \cup I_1\big)$$
$$= \hat{\mu}^\nu_{\bar{\imath}}\big(A^{\hat{t},\tau-\hat{t}}_{[x_0(\hat{t}),x_3(\hat{t})]}\big) + \hat{\mu}^\nu_{\bar{\imath}}\big(A^{\hat{t}}_{I_0}\big) + \hat{\mu}^\nu_{\bar{\imath}}\big(A^{\hat{t}}_{I_1}\big) - \hat{v}^\nu_{\bar{\imath}}(\hat{t})\big(\{x_1(\hat{t})\}\big)$$
$$+ \Phi^{\nu,\text{out}}_{\bar{\imath},I_0}((0,\hat{t}]) + \Phi^{\nu,\text{out}}_{\bar{\imath},I_1}((0,\hat{t}]) + \Phi^{\nu,\text{out}}_{\bar{\imath},[x_0(\hat{t}),x_3(\hat{t})]}((\hat{t},\tau]).$$

It remains then to notice that as the regions $A_{I_0}^{\hat{t}}$, $A_{I_1}^{\hat{t}}$ overlap at the interaction point then

$$\hat{\mu}_{\bar{\imath}}^{\nu}\big(A_{[x_0(\hat{t}),x_3(\hat{t})]}^{\hat{t},\tau-\hat{t}}\big) + \hat{\mu}_{\bar{\imath}}^{\nu}\big(A_{I_0}^{\hat{t}}\big) + \hat{\mu}_{\bar{\imath}}^{\nu}\big(A_{I_1}^{\hat{t}}\big) = \hat{\mu}_{\bar{\imath}}^{\nu}\big(A_{I_0\cup I_1}^{\hat{t}}\big) + \hat{\mu}_{\bar{\imath}}^{\nu}(\{(\hat{t},x_1(\hat{t}))\}).$$

Moreover, the waves entering at $(\hat{t},x_1(\hat{t}))$ from outside the region $A_{I_0\cup I_1}^{\tau} = A_{I_0}^{\tau} \cup A_{I_1}^{\tau}$ are counted both in $\Phi_{\bar{\imath},I_0}^{\nu,\mathrm{out}}((0,\hat{t}])$ and $\Phi_{\bar{\imath},I_1}^{\nu,\mathrm{out}}((0,\hat{t}])$, but only once in $\Phi_{\bar{\imath},I_0\cup I_1}^{\nu,\mathrm{out}}((0,\tau])$, while the ones interacting at $(\hat{t},x_1(\hat{t}))$ and coming from the interior of $A_{I_0\cup I_1}^{\tau}$ are counted precisely once in $\Phi_{\bar{\imath},I_0}^{\nu,\mathrm{out}}((0,\hat{t}]) + \Phi_{\bar{\imath},I_1}^{\nu,\mathrm{out}}((0,\hat{t}])$: since the amount of these waves from the region in between is precisely $\hat{v}_{\bar{\imath}}^{\nu}(\hat{t})\big(\{x_1(\hat{t})\}\big) - \hat{\mu}_{\bar{\imath}}^{\nu}(\{(\hat{t},x_1(\hat{t}))\})$, therefore

$$\Phi_{\bar{\imath},I_0}^{\nu,\mathrm{out}}((0,\hat{t}]) + \Phi_{\bar{\imath},I_1}^{\nu,\mathrm{out}}((0,\hat{t}]) + \Phi_{\bar{\imath},[x_0(\hat{t}),x_3(\hat{t})]}^{\nu,\mathrm{out}}((\hat{t},\tau])$$
$$= \hat{v}_{\bar{\imath}}^{\nu}(\hat{t})\big(\{x_1(\hat{t})\}\big) - \hat{\mu}_{\bar{\imath}}^{\nu}(\{(\hat{t},x_1(\hat{t}))\}) + \Phi_{\bar{\imath},I_0\cup I_1}^{\nu,\mathrm{out}}((0,\tau]).$$

Collecting the terms one obatins

$$\hat{v}_{\bar{\imath}}^{\nu}(\tau)\big([x_0(\tau),x_3(\tau)]\big) - \hat{v}_{\bar{\imath}}^{\nu}(0)\big(I_0 \cup I_1\big) = \hat{\mu}_{\bar{\imath}}^{\nu}\big(A_{I_0}^{\tau} \cup A_{I_1}^{\tau}\big) + \Phi_{\bar{\imath},I_0\cup I_1}^{\nu,\mathrm{out}}((0,\tau]).$$

The reasoning applied to two intervals I_0, I_1 can be applied as well to any finite number, obtaining the analogous equation. Since the flux can be estimated by the interaction-cancellation measure, one finds then that for all $M \in \mathbb{N}$ and real intervals $I_1,\ldots,I_M$

$$\hat{v}_{\bar{\imath}}^{\nu}(\tau)\big(I_1(\tau)\cup\cdots\cup I_M(\tau)\big) - \hat{v}_{\bar{\imath}}^{\nu}(0)\big(I_1\cup\cdots\cup I_M\big) \le (\hat{\mu}_{\bar{\imath}}^{\nu} + \mathcal{O}(1)\mu_{\nu}^{IC})\big(A_{I_1}^{\tau}\cup\cdots\cup A_{I_M}^{\tau}\big).$$

Since the total strength of non-physical fronts at each finite time is controlled by ε_{ν}, as well as the mass $|\rho^{\nu}|$ of $\mu_{\bar{\imath}}^{\nu}$ due to interactions involving non-physical waves and $\mu_{\bar{\imath}}^{\nu}$ (Lemma 5.3), from above

$$v_{\bar{\imath}}^{\nu}(\tau)\big(I_1(\tau)\cup\cdots\cup I_M(\tau)\big) - v_{\bar{\imath}}^{\nu}(0)\big(I_1\cup\cdots\cup I_M\big) \le (\mu_{\bar{\imath}}^{\nu} + \mathcal{O}(1)\mu^{IC})\big(A_{I_1\cup\cdots\cup I_M}^{\tau}\big) + \mathcal{O}(1)\varepsilon_{\nu} \qquad (5.7)$$
$$\le \mathcal{O}(1)\Big(\mu^{IC}\big(A_{I_1\cup\cdots\cup I_M}^{\tau}\big) + \varepsilon_{\nu}\Big).$$

4) `Balance for the jump part.` For the 'jump part' by (5.3) one can repeat the same argument as above with the relative measures, but the fluxes are non-positive because only i-shocks are involved, so that one does not need to estimate the positive part (which vanishes) of the fluxes by the interaction-cancellation measure: therefore

$$v_{\bar{\imath},\mathrm{jump}}^{\nu}(\tau)\big([a(\tau),b(\tau)]\big) - v_{\bar{\imath},\mathrm{jump}}^{\nu}(0)\big([a,b]\big) = \mu_{\bar{\imath},\mathrm{jump}}^{\nu}\big(A_{[a,b]}^{\tau}\big) + \Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}}\big(A_{[a,b]}^{\tau}\big) \qquad (5.8)$$

holds with new fluxes $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}}$ which take into account only the contribute in $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$ due to i-shocks in $\mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}$. In particular this flux is non-positive. Explicitly the new flux at the boundary $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}}$ is

$$\begin{cases} \sigma'; \ \sigma' \le 0 & \text{If an } \bar{\imath}\text{-shock of strength } \sigma' \text{ in } \mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i} \text{ enters, possibly interacting with a front} \\ & \text{wave either in } A_{[a,b]}^{\tau} \text{ or not belonging to } \mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i}. \\ \sigma' + \sigma''; \ \sigma',\sigma'' \le 0 & \text{If two } \bar{\imath}\text{-shocks out of } A_{[a,b]}^{\tau} \text{ and in } \mathcal{J}_{(\varepsilon_0,\varepsilon_1)}^{\nu,i} \text{ interact.} \\ 0 & \text{Otherwise.} \end{cases}$$

The generalization to countably many intervals holds precisely as before.
5) `Balance for the continuous part.` By subtracting (5.8) to (5.6b) one finds the analogous estimate for v_{cont}, which again holds in the same way for countably many intervals, with a new flux at the boundary $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{cont}}(A_{[a,b]}^{\tau})$ which is the difference $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}} - \Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}}$: notice indeed that one can take advantage of

$$\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}} - \Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}} \le [\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}} - \Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}}]^{+} \le [\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}]^{+} \le \mu_{\nu}^{IC}.$$

Notice that the negative terms in $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out},\mathrm{jump}}$ cancel positive ones in $\Phi_{\bar{\imath},[a,b]}^{\nu,\mathrm{out}}$, without adding a positive contribution in the difference. The last equation in the statement holds then by definition of $\mu_{\bar{\imath},\nu}^{ICJ} = \mu_{\nu}^{IC} + |\mu_{\bar{\imath},\mathrm{jump}}^{\nu}|$, which controls $\mu_{\bar{\imath}}^{\nu} - \mu_{\bar{\imath},\mathrm{jump}}^{\nu}$. $\qquad\square$

5.3. **The decay estimate.** We prove in this section an estimate analogous to the decay of positive waves (Pages 210-216 in [Bre]), but for the negative part of $(v_{\bar{\imath}})_{\mathrm{cont}}$, under the assumption that the $\bar{\imath}$-characteristic field is genuinely non-linear. We bound from below the continuous part of the waves of a semigroup solution u on a Borel set B at time t by the wave\jump wave balance measures $\mu_{\bar{\imath}} \backslash \mu_{\bar{\imath},\mathrm{jump}}$ on a strip a strip around t of arbitrary height $\tau < t$ and the Lebesgue measure of B divided by τ: together with the previous result due to Bressan we obtain

$$|(v_{\bar{\imath}})_{\mathrm{cont}}|(t)(B) \leq \mathcal{O}(1)\mathcal{L}^1(B)/\tau + \mathcal{O}(1)\big(\mu_{\bar{\imath}}^{ICJ}\big)\big([t-\tau, t+\tau] \times \mathbb{R}\big) \qquad \forall t > \tau > 0.$$

We remark that here there is no sharpness purpose. We first prove an estimate from below for the 'absolutely continuous' part of $v_{\bar{\imath}}^\nu$. As a consequence of Corollary 4.7, and Remark 4.8, when passing to the limit we get the claim (Lemma 5.9).

Lemma 5.7 (Approximate decay estimate on intervals)**.** *Assume the $\bar{\imath}$-th characteristic field is genuinely non-linear. Then for any disjoint closed intervals $\{I_h\}_{h \in \mathbb{N}}$ and $\tau, t_0 > 0$ one has the bound*

$$-v_{\bar{\imath},\mathrm{cont}}^\nu(t_0)\big(I_1 \cup \cdots \cup I_M\big) \leq \mathcal{O}(1)\Big\{\mathcal{L}^1(I_1 \cup \cdots \cup I_M)/\tau + \mu_{\bar{\imath},\nu}^{ICJ}(A_{I_1\cup\cdots\cup I_M}^{t_0,\tau}) + \varepsilon_\nu + \varepsilon_1\Big\},$$

where $A_{I_1\cup\cdots\cup I_M}^{t_0,\tau}$ is the region bounded by generalized $\bar{\imath}$-characteristics in (5.5).

Remark 5.8. Under the assumptions of the above lemma, an analogous statement holds for any Borel set B if one considers its evolution by minimal generalized $\bar{\imath}$-th characteristics (or a different selection with the semigroup property). Denoting the generalized $\bar{\imath}$-th characteristics $\{y^\nu(t; t_0, x)\}_{x \in B}$ and defining $A_B = \{y^\nu(t; t_0, x)\}_{t_0 < t \leq t_0+\tau}^{x \in B}$ one has

$$-v_{\bar{\imath},\mathrm{cont}}^\nu(t_0)\big(B\big) \leq \mathcal{O}(1)\Big\{\mathcal{L}^1(B)/\tau + \mu_{\bar{\imath},\nu}^{ICJ}(\overline{A_B}) + \varepsilon_\nu + \varepsilon_1\Big\}.$$

Indeed, this holds true for countably many intervals by Lemma 5.7 and the observation that

$$\mu_{\bar{\imath},\nu}^{ICJ}(A_{[a,b]}^{t_0,\tau}) = \mu_{\bar{\imath},\nu}^{ICJ}(A_{[a,b]}) = \mu_{\bar{\imath},\nu}^{ICJ}(\overline{A_{[a,b]}} \cap \{t > t_0\}),$$

for any choice of characteristics starting out from $[a, b]$ and having the semigroup property. Since on the r.h.s. we have a nonnegative Radon measure, by inner/outer regularity it can be extended to Borel sets.

Proof of Lemma 5.7. We adapt the argument in [Bre] (§10.2) about the decay of positive waves, whose main steps we briefly recall here. Since the generalization to countably many intervals is analogous to the one of Lemma 5.6, we consider the case of a single interval $I = [a, b]$.

The heuristic is the following. Suppose $v_{\bar{\imath},\mathrm{cont}}^\nu(t_0)(I) < 0$ and there are only $\bar{\imath}$-fronts in I, which is the main case we would like to manage. Then initially $\mathcal{L}^1(I(t))$ decreases at a rate at least $v_{\bar{\imath}}^\nu(t_0)(I)/4$; if $\mathcal{L}^1(I(t))$ keeps on decreasing at least of that rate between times t_0, $t_0 + \tau$, one can estimate the initial value $\mathcal{L}^1(I)$ from below as stated. Otherwise, interactions must take place in order to decrease the rate, with a special care for the ones at the boundary.

For simplicity of notation, we set $t_0 = 0$ and we consider the $\bar{\imath}$-th wave measures omitting the indices $\bar{\imath}$. Let the interval $[a, b]$ evolve by the minimal forward characteristics: its length at time t is

$$z(t) = b(t) - a(t).$$

This function is absolutely continuous and it satisfies $\dot{z}(t) = \bar{\lambda}(t, b(t)) - \bar{\lambda}(t, a(t))$ for a.e. t. At Page 213 of [Bre] it is explicitly defined function Φ which contains the variation of speed due to the waves of other characteristic families. Φ is piecewise Lipschitz continuous, non-decreasing, with discontinuities only at interaction times. More precisely one has the estimate

$$|\dot{z}(t) + \xi(t) - v^\nu(t)| \leq \mathcal{O}(1)(\varepsilon_\nu + \dot{\Phi}(t)z(t)) \tag{5.9}$$

where $v^\nu(t)$ counts the variation of speed due to the waves of the same family, and ξ corrects it taking into account waves at the boundary: we adopt the notation

$$v^\nu(t) := v^\nu(t)([a(t), b(t)]),$$

$$\xi(t) := (\bar{\lambda}(t, a(t)) - \bar{\lambda}(t, a(t)^-)) + (\bar{\lambda}(t, b(t)^+) - \bar{\lambda}(t, b(t))).$$

Case 1. If $\dot{z}(t) - \dot{\Phi}(t)z(t) < v^\nu(0)/4$ for all t, then

$$d/dt\big(e^{-\int_0^t \dot{\Phi}} z(t)\big) = e^{-\int_0^t \dot{\Phi}}\big\{\dot{z}(t) - \dot{\Phi}(t)z(t)\big\} < e^{-\int_0^t \dot{\Phi}} v^\nu(0)/4 \leq v^\nu(0)/4$$

and integrating the inequality between times 0 and τ, since $z(t) \geq 0$, one has

$$-z(0) \leq e^{-\int_0^\tau \dot{\Phi}} z(\tau) - z(0) \leq \tau v^\nu(0)/4 \leq \tau v^\nu_{\text{cont}}(0)/4.$$

Case 2. Suppose instead $\dot{z}(t) - \dot{\Phi}(t)z(t) \geq v^\nu(0)/4$ at some time t. We first apply the balances in the proof of Lemma 5.6, obtaining

$$\dot{z}(t) - \dot{\Phi}(t)z(t) \stackrel{(5.9)}{\leq} v^\nu(t) - \xi(t) + \mathcal{O}(1)\varepsilon_\nu$$
$$\stackrel{(5.6b)}{\leq} v^\nu(0) + \mu^\nu\big(A^\tau_{[a,b]}\big) + \Phi^{\nu,\text{out}}_{[a,b]}(A^\tau_{[a,b]}) - \xi(t) + \mathcal{O}(1)\varepsilon_\nu.$$

Then going on by the inequality $\dot{z}(t) - \dot{\Phi}(t)z(t) \geq v^\nu(0)/4$ one obtains

$$3v^\nu(0)/4 \geq -\mu^\nu\big(A^\tau_{[a,b]}\big) - \Phi^{\nu,\text{out}}_{[a,b]}(A^\tau_{[a,b]}) + \xi(t) - \mathcal{O}(1)\varepsilon_\nu.$$

Observing that $\bar{\lambda}(t, a(t))$ is a suitable mean of $\lambda(t, a(t)^\pm)$ and $v^\nu_{\text{jump}}((t, a(t))) = \lambda(t, a(t)^+) - \lambda(t, a(t)^-)$ by (2.4a), one derives the bound

$$\frac{4}{3}\xi(t) \;\geq\; v^\nu_{\text{jump}}((t, a(t))) + v^\nu_{\text{jump}}((t, b(t))) - 2\varepsilon_1$$

$$\geq\; v^\nu_{\text{jump}}(t) - 2\varepsilon_1$$

$$\stackrel{(5.8)}{\geq}\; v^\nu_{\text{jump}}(0) + \mu^\nu_{,\text{jump}}\big(A^\tau_{[a,b]}\big) + \Phi^{\nu,\text{out,jump}}_{[a,b]}(A^\tau_{[a,b]}) - 2\varepsilon_1,$$

this yields, subtracting $3v^\nu_{\text{jump}}(0)/4$ to $3v^\nu(0)/4$, that $3v^\nu_{\text{cont}}(0)/4$ can be estimated from below by

$$-\mu^\nu(A^\tau_{[a,b]}) + 3\mu^\nu_{\text{jump}}\big(A^\tau_{[a,b]}\big)/4 - \Phi^{\nu,\text{out}}_{[a,b]}(A^\tau_{[a,b]}) + 3\Phi^{\nu,\text{out,jump}}_{[a,b]}(A^\tau_{[a,b]})/4 - 3\varepsilon_1/2 - \mathcal{O}(1)\varepsilon_\nu$$
$$\geq -\mu^\nu(A^\tau_{[a,b]})/4 - 3\mu^\nu_{\text{cont}}(A^\tau_{[a,b]})/4 - \Phi^{\nu,\text{out}}_{[a,b]}(A^\tau_{[a,b]})/4 - 3\Phi^{\nu,\text{cont}}_{[a,b]}(A^\tau_{[a,b]})/4 - 3\varepsilon_1/2 - \mathcal{O}(1)\varepsilon_\nu$$
$$\geq -\mathcal{O}(1)\mu^{ICJ}_{\bar{\imath},\nu}(A^\tau_{[a,b]}) - 3\varepsilon_1/2 - \mathcal{O}(1)\varepsilon_\nu.$$

Conclusion. Collecting the two cases one finds the bound uniform in ν

$$v^\nu_{\text{cont}}(0) \geq -\mathcal{O}(1)\bigg\{ (b-a)/\tau + \mu^{ICJ}_\nu(A^\tau_{[a,b]}) + \varepsilon_1 + \varepsilon_\nu \bigg\}.$$

This shows the claim for a single closed interval. Similarly to what is done in Lemma 5.6, the argument extends by direct computation to the union of countably many closed intervals. $\square$

We finally arrive to the decay estimate for the negative part of $(v_{\bar{\imath}})_{\text{cont}}$. The characteristics of the semi-group solution u we get in the limit in general are no more minimal, but just generalized $\bar{\imath}$-characteristics: this is why in $(3.2 : \bar{\imath})$ we have chosen to state it enlarging the set A_B up to an horizontal strip.

Lemma 5.9 (Decay estimate on intervals)**.** *Assume the $\bar{\imath}$-th characteristic field is genuinely non-linear. Then there is a choice of generalized $\bar{\imath}$-th characteristics $\{y(t; s, x)\}^{s \geq 0}_{x \in \mathbb{R}}$ of u such that for any Borel set $B \subset \mathbb{R}$ and $t_0 > \tau > 0$ the following estimate of the $\bar{\imath}$-th wave holds*

$$-(v_{\bar{\imath}})_{\text{cont}}(t_0)\big(B\big) \leq \mathcal{O}(1)\bigg\{ \mathcal{L}^1(B)/\tau + \mu^{ICJ}_{\bar{\imath}}(\overline{A_B}) \bigg\} \qquad A_B = \{y(t; t_0, x)\}^{x \in B}_{t_0 \leq t \leq t_0+\tau}.$$

Remark 5.10. By the Jordan decomposition of $(v_{\bar{\imath}})_{\text{cont}}(t_0)$ and the arbitrariness of B, this is precisely a lower bound for the negative part of $(v_{\bar{\imath}})_{\text{cont}}(t_0)$.

Proof. As usual, simplify notations by setting $t_0 = 0$ and omitting $\bar{\imath}$.

When t_0 was not a time in Θ of Theorem 4.2, by Remark 4.8 the 'continuous part' of $v^\nu(0)$ w^*-converge to the continuous part of v. Nevertheless, we do not know whether the positive and negative part $[v^\nu(0)]^\pm$ of $v^\nu(0)$, and $[v^\nu_{\bar{\imath},\text{cont}}(0)]^\pm$, converge to the positive and negative part in the Jordan decomposition of $v(0)$, and $(v(0))_{\text{cont}}$. However, if one defines measures $[\bar{v}]^\pm$ by the relation

$$[v^\nu_{\text{cont}}(0)]^\pm \xrightarrow{w^*} [\bar{v}]^\pm,$$

where the convergence is obtained by compactness and up to a subsequence, then

$$[\bar{v}]^+ - [\bar{v}]^- = (v(0))_{\text{cont}} \qquad [\bar{v}]^+ \geq [(v)_{\text{cont}}(0)]^+ \qquad [\bar{v}]^- \geq [(v)_{\text{cont}}(0)]^-.$$

In general $[\bar{v}]^\pm$ are not orthogonal.

Since the atoms of $[\bar{v}]^\pm$ are at most countably many, one can consider first finitely many intervals $I_1, \ldots, I_M$ whose boundary has 0 $|\bar{v}|$-measure, where $|\bar{v}| = [\bar{v}]^+ + [\bar{v}]^-$. Then the measures of $J = I_1 \cup \cdots \cup I_M$ converge:

$$[v^\nu_{\mathrm{cont}}]^\pm(0)\big(J\big) \to [\bar{v}]^\pm\big(J\big), \qquad (v^\nu)_{\mathrm{cont}}(0)\big(J\big) \to v_{\mathrm{cont}}(0)\big(I\big).$$

By Lemma 5.7 and the following remark, the inequality

$$-v^\nu_{\mathrm{cont}}(0)\big(J\big) \le \mathcal{O}(1)\Big\{ \mathcal{L}^1(J)/\tau + \mu_\nu^{ICJ}(\overline{A^\nu_J}) + \varepsilon_\nu + \varepsilon_1 \Big\}.$$

holds with the minimal generalized $\bar{\imath}$-characteristics $\{y^\nu(t; t_0, x)\}_{x \in J}$ of u^ν starting from J.

In the ν-limit, by Ascoli-Arzelaà theorem the selected ν-characteristics $\{y^\nu(t; t_0, x)\}_{x \in J}$ converge to some generalized $\bar{\imath}$-characteristics $\{y(t; t_0, x)\}_{x \in J}$ of u. By the upper continuity of Borel probability measures on compact sets one can derive

$$-v_{\mathrm{cont}}(0)\big(J\big) = -\lim_\nu v^\nu_{\mathrm{cont}}(0)\big(J\big)$$
$$\le \liminf_\nu \mathcal{O}(1)\Big\{ \mathcal{L}^1(J)/\tau + \mu_\nu^{ICJ}(\overline{A^\nu_J}) + \varepsilon_\nu + \varepsilon_1 \Big\}$$
$$\le \mathcal{O}(1)\Big\{ \mathcal{L}^1(J)/\tau + \mu^{ICJ}(\overline{A_J}) \Big\}.$$

The same results hold for open sets by approximation from the interior. Then outer regularity yields the inequality for any Borel set.

When t_0 is one of the countably many points of Θ in Theorem 4.2, one should take into account that $t \mapsto (v_{\bar{\imath}})_{\mathrm{jump}}(t)$ is continuous from the right, and thus $(v_{\bar{\imath}})_{\mathrm{cont}}$. The thesis can thus be obtained by a diagonal argument for a sequence $t_k \downarrow t_0$. $\qquad \square$

Corollary 5.11. *Consider a strictly hyperbolic system of conservation laws as* (1.1) *and let the $\bar{\imath}$-th characteristic field be genuinely non-linear,* $(2.1 : \bar{\imath}\mathrm{GN})$. *By Lemma 5.9 and the decay of positive waves in Theorem 10.3 of* [Bre], *one has that there exists a positive constant C such that for every $t > \tau > 0$, every Borel subset B of $\mathbb{R}$ and every solution u obtained as a limit of the front tracking approximation, the continuous part of the measures $v_{\bar{\imath}}$ satisfies*

$$|(v_{\bar{\imath}})_{\mathrm{cont}}|(t)(B) \le C\mathcal{L}^1(B)/\tau + C\big(\mu_{\bar{\imath}}^{ICJ}\big)\big([t-\tau, t+\tau] \times \mathbb{R}\big).$$

Appendix A. Table of notations

$\mathcal{L}^1, \mathcal{L}^2$	The Lebesgue measure of dimension respectively 1 and 2
$\mathcal{O}(1)$	A positive constant which can be uniformly bounded in the limiting index
$[\eta]^+, [\eta]^-, |\eta|$	Respectively positive part, negative part and variation of a measure η
$(\eta)_{\mathrm{jump}}, (\eta)_{\mathrm{cont}}$	The jump part and the continuous part of a measure η
Ω	Open subset of $\mathbb{R}^N$
f	Flux function for a strictly hyperbolic system of conservation laws
A	The Jacobian matrix $A = \mathrm{D}f$
$\lambda_i(u)$	The i-th eigenvalue of $A = \mathrm{D}f$, by strict hiperbolicity $\lambda_1 < \cdots < \lambda_N$
$\lambda_i^{(\nu)}(t, x), \bar{\lambda}_i(t, x)$	λ_i composed with $u^{(\nu)}$ and a suitable mean at jump points (Page 5 and (4.3))
$l, \tilde{l}^{(\nu)}, r, \tilde{r}^{(\nu)}$	Left and right eigenvectors of $A = \mathrm{D}f$, suitably normalized (Page 5)
$\sigma_1, \ldots, \sigma_N, \Lambda, \Psi_i$	Strengths of the waves and maps in the Riemann problem, see Page 4 and (2.4)
u	Mainly the variable in $\mathbb{R}^N$ or the semigroup solution to (1.1)
u^ν	The ν-front-tracking approximation, see recalls in Section 4
u_x, u_t	The (measure) derivatives of the $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ function u
$u(t)$	The restriction of the function u at time $t \ge 0$, see Page 4
$v_1^{(\nu)}, \ldots, v_N^{(\nu)}$	Measures of the wave decomposition (2.5) of $u_x^{(\nu)}$ along $\tilde{r}_1^{(\nu)}, \ldots, \tilde{r}_N^{(\nu)}$
$v_i^{(\nu)}(t)$	Conditional measures of $v_i^{(\nu)}$ in the disintegration w.r.t. time, see e.g. Page 7
$v_{i,\mathrm{jump}}^\nu, v_{i,\mathrm{cont}}^\nu$	Parts of v_i^ν converging to the jump\continuous part of v_i (Section 4, Page 16)
$\mathcal{J}_{(\varepsilon_0, \varepsilon_1)}^{\nu, i}, \mathcal{J}$	The 'jump sets' of u and u^ν of Section 4
μ_ν^I, μ_ν^{IC}	Interaction and interaction-cancellation measures, see (3.1)
μ^I, μ^{IC}	A w^*-limit of μ_ν^I, μ_ν^{IC}

$\mu_i^{(\nu)}, \mu_{i,\text{jump}}^{(\nu,(\varepsilon_0,\varepsilon_1))}$	Balance and wave balance measures defined in Section 5.1
$\mu_{(\bar{\imath},\nu)}^{ICJ}$	The interaction-cancellation-jump balance measures in (5.4), and a relative w^*-limit
$\Phi_{\bar{\imath},[a,b]}^{\nu,\text{out},(\text{jump}\backslash\text{cont})}$	The fluxes introduced in the proof of Lemma 5.6
$A_{[a,b]}^{t_0,\tau}$	The region bounded by selected (e.g. minimal) i-characteristics in (5.5)

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