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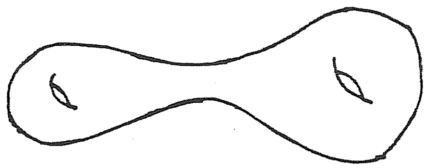
Two-Loop Computations in Superstring Theories—Playing Games with



Thesis Submitted for the Degree of
Magister Philosophiae

Candidate:
Zhu Chuan-Jie

Supervisor:
Roberto Iengo

Academic Year 1987/88

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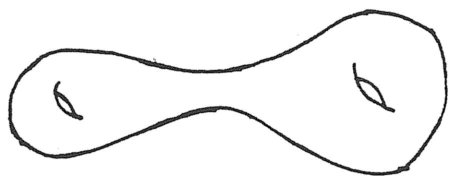

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Contents

Acknowledgemets

I would like to thank Prof. Roberto Iengo who guides me in the study of Two-Loop Computation in Superstring Theories. This thesis is based on works done together, and I also thank him for careful reading of the manuscript and suggesting many improvements.

Thanks also go to Drs. E. Gava, G. Sotkov and F. Ferrari for discussions. This thesis was written during my staying in Beijing where there was a Workshop on Recent Developments in Field Theory at this time, and I would like to thank the organizers: Profs. J. F. Tang, B. Y. Hou, Y. B. Ding and H. Y. Guo for their invitation to present part of these works on the Workshop.

Finally I would also like to thank Mrs. M. L. Zhang for her hospitality at home and help in preparing the manuscript.

1 Introduction

In the last couple of years, there has been much progress in superstring theory, in particular the study of multiloop amplitudes[1] in covariant formulation [2] a la Polyakov [3]. From the explicit computation of scattering amplitudes in superstring theory [4], one sees that closed superstring theories are free of any divergences, i.e. they are finite theories. Due to some heuristic arguments, one believes that this finiteness should be persist to all orders in string perturbation expansion. In particular, one believes that a nonrenormalization theorem [5] is true. In spite of the efforts of many authors, it is very difficult to verify this theorem explicitly. Even in the case of cosmological constant, i.e. vacuum amplitude or partition function, this problem has not been completely solved. It now appears that major steps have been taken by Atick, Moore and Sen in the case of heterotic string theories [6], but it seems to us that much work remains to be done, in particular for understanding modular invariance in the general genus case.

The recent progress in the computation of multiloop amplitudes in superstring theories stems from the work of Verlinde and Verlinde [7] (see also [8]). In particular, explicit calculation becomes feasible by using of hyperelliptic language [9,10,11,12] at two loops. Although the computation at arbitrary loops (or arbitrary genus in connection with Riemann surface) may be a formidable task, we will see in this thesis that this is not the case at two-loops. This is possible only when one understands how to implement modular invariance at two-loops. This was achieved in [13]. Following [13], we computed N-particle amplitudes in [14] up to $N=3$ and four-particle

[1] See ref. [1] for reviews and references therein.

amplitude in [15]. In this thesis, we will present more details about our computation and some new results. It is organized as follows:

In section 2, we review briefly the general strategy of multiloop calculation following [16,7]. In order to do explict calculation, it is convenient first to do integration over supermoduli—leaving two insertions of super current at two-loops—and then over moduli.

In section 3, we give some relevant mathematical background about hyperelliptic Riemann surface. Some useful relations among Θ-constants, period matrix and branch points (see section 3), i.e. Thomae formula and variational formula etc. are also given. We also give the proof of a formula (which will be used in section 5) given by V. G. Knizhnik in [10] and some new results.

In section 4, N-particle amplitudes up to $N = 3$ are calculated at two-loops and nonrenormalization theorem, i.e. the vanishing of N-particle ($N < 4$) amplitudes, is verified explicitly. This verification is based on a set of identities called Lianzi identities in [14,15]. We give in this section the proof of Lianzi identities and also the derivation of a summation formula used in [15].

In section 5, we present the full details about the computation of four-particle amplitude. The contribution from ghost and superghost is also calculated. As we will see in section 6, this contribution is necessary to ensure the right properties of the four-particle amplitude.

In section 6, we verify the main properties and in particular the finiteness of the four-particle amplitude. All our discussions go through both heterotic string (HST) and type II superstring (SST II) theories although sometimes we discuss heterotic string theories only.

In the last section (section 7), we discuss the implication of our calculation at two-loops. Some unsolved problems and in particular the factorization of the four-particle amplitude which may be interest for further investigation are also pointed out.

An apology: Because of the lack of space and time, we discuss mainly the original works about two-loop computation in this thesis and neglect most and almost all the works on string perturbation theory although all of these works are important for our understanding of string perturbation theory in the general case and at two-loops in particular. We apologize to all these authors for our ignorance and hope to review their works in a Doctor thesis.

2 String Perturbation Theory

In this section, we review briefly the general strategy of multiloop calculation following [16,7]. We will discuss first what Polyakov's approach to string theory means. We discuss how to fix the gauge and reduce the functional integration to the integration over moduli space. Then we extend all these discussions to superstring theories and derive the two-loop measure for superstring theories (for both HST and SST II).

In the Polyakov approach to string theory, quantization is performed by summing the functional integration over all geometry and string coordinates. The vacuum amplitude (partition function) is then

$$Z = \sum_{topologies} \int \frac{D(geometry)D(string\ coordinates)}{Vol.(symmetry\ group)} e^{-S} \tag{II.1}$$

Please note that we always assume Wick rotation both for two-dimensional world-sheet and target space-time where string moves. S is the action. The N-particle amplitude is computed by inserting vertices on Riemann surface in the partition function, i.e. we have

$$\begin{aligned} A^N(p_i, \epsilon_i) = \sum_{topologies} \int & \frac{D(geometry)D(string\ coordinates)}{Vol.(symmetry\ group)} \times \\ & \times \int_{\Sigma_g} \prod_{i=1}^{N} d^2 z_i V(p_i, \epsilon_i, z_i) e^{-S} \end{aligned} \tag{II.2}$$

where $V(p_i, \epsilon_i, z_i)$ is the vertex for the emission of i-th particle.

For closed bosonic string, we have

$$S = \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X \tag{II.3}$$

where X are string coordinates describing the embedding of string in space-time and $g_{\alpha\beta}(g^{\alpha\beta} = (g^{-1})_{\alpha\beta})$ is the world-sheet metric ($\sigma^\alpha(\alpha = 1, 2)$ are local coordinates on the world-sheet). It is not difficult to see that this action (II.3) is invariant under the following reparametrization of the local coordinates:

$$\begin{aligned} \sigma^\alpha &\longrightarrow f^\alpha(\sigma) \\ g_{\alpha\beta} &\longrightarrow \frac{\partial f^\gamma(\sigma)}{\partial\sigma^\alpha}\frac{\partial f^\delta(\sigma)}{\partial\sigma^\beta} g_{\gamma\delta}(f(\sigma)) \end{aligned} \tag{II.4}$$

and under the following rescaling of the metric:

$$g_{\alpha\beta} \longrightarrow e^{\varphi(\sigma)} g_{\alpha\beta} \tag{II.5}$$

To quantize this theory properly, one should factorize out the volume of this symmetry group and get the correct measure for the path integral. We will follow Faddeev-Popov procedure to factorize out this (infinite) volume.

To choose a gauge condition, we would like to choose the conformal gauge in which the metric takes the form

$$g_{\alpha\beta} = e^{\varphi}\delta_{\alpha\beta} \tag{II.6}$$

But this extremely convenient gauge has some topological limitations. Let us discuss now both the derivation of (II.6) and these limitations.

The first naive argument which shows that (II.6) is possible is the following. The possibility of the choice (II.6) means that any metric $g_{\alpha\beta}$ can be given in the form

$$g_{\alpha\beta} = (e^{\varphi(\sigma)}\delta_{\alpha\beta})^f = e^{\varphi(f(\sigma))}\frac{\partial f^\gamma}{\partial\sigma^\alpha}\frac{\partial f^\gamma}{\partial\sigma^\beta} \tag{II.7}$$

where $\{f^\gamma(\sigma)\}$ defines the necessary coordinate transformation. Hence, the RHS of (II.7) depends on three arbitrary functions $f^1(\sigma)$, $f^2(\sigma)$ and $\varphi(\sigma)$. But $g_{\alpha\beta}(\sigma)$ also has three independent components. Therefore, the number of independent functions matches. However, this is not enough. We must show that the transformation (II.7) is nonsingular, i.e. the jacobian for passing to the variables (φ, f^{α}) is non zero. To show this we shall consider a small variation of (II.7):

$$\delta g_{\alpha\beta} = \delta\varphi g_{\alpha\beta} + \nabla_{\alpha} V_{\beta} + \nabla_{\beta} V_{\alpha} \tag{II.8}$$

where $V^{\alpha} = \delta f^{\alpha}$. The nonsigular nature of the transformation (II.7) will be proved if for any $\delta g_{\alpha\beta}$ we can find $\delta\varphi$ and V such that (II.8) will hold. In other words, we must be able to solve the equation:

$$\delta\varphi\delta_{\alpha\beta} + \nabla_{\alpha} V_{\beta} + \nabla_{\beta} V_{\alpha} = \delta g_{\alpha\beta} \equiv \gamma_{\alpha\beta} \tag{II.9}$$

or

$$(PV)_{\alpha\beta} = \nabla_{\alpha} V_{\beta} + \nabla_{\beta} V_{\alpha} - g_{\alpha\beta} \nabla^{\delta} V_{\delta} = \gamma_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \gamma^{\delta}_{\delta} \tag{II.10}$$

which is obtained from (II.9) by substracting the trace. The question, whether the conformal gauge is always accessible, is reduced now to the possibility of solving (II.10) which we shall rewrite symbolically:

$$PV = \gamma \tag{II.11}$$

Here we have denoted by P the differential operator, defined by (II.10) which takes vector fields into traceless tensors (notice that the number of independent components is the same). There exist a conjugate operator which acts in the opposite direction—transforming tensors into vectors. It is easy to realize that the equation (II.11) will be solvable if and only if the conjugate operator P^{+} doesn't have zero modes. On the other hand, the solution of eq. (II.11) is not unique if P has zero modes.

So, our conclusion is that zero modes of the operator P^{+} mean that the conformal gauge is not accessible, and zero modes of P that it is not unique (and one should further fix the remaining gauge freedom).

The number of zero modes is regulated by index theorem. We will not go into the details of these mathematics and only recall the results. We have

$$N_0(P) - N_0(P^+) = 3\chi = -(6g-6) \qquad \text{(II.12)}$$

where N_0 denotes the number of zero modes, χ is the Euler character of the Riemann surface Σ_g and g is the genus. In particular, we have the following list:

$$\begin{aligned} N_0(P) = 6, \quad N_0(P^+) = 0, &\qquad for \quad g = 0 \quad (sphere) \\ N_0(P) = 2, \quad N_0(P^+) = 2, &\qquad for \quad g = 1 \quad (torus) \\ N_0(P) = 0, \quad N_0(P^+) = 6g-6, &\qquad for \quad g \geq 2 \end{aligned} \qquad \text{(II.13)}$$

So we found that on a sphere we can always introduce a conformal gauge, which is defined modulo $SL(2,C)$ transformations (with six $(= N_0(P))$ real parameters) which requires extra gauge fixing, e.g. the fixing of three out of four complex $z_i, i = 1,2,3,4$ (the locations of the inserted vertices) in the case of four-particle amplitude at tree level. In the case of Riemann surface with higher genus we have topological obstructions for the conformal gauge. The best thing which can be done is the following choice of gauge

$$g_{\alpha\beta}(\sigma) = e^{\varphi(\sigma)} g^{(0)}_{\alpha\beta}(\sigma; \tau_1, \tau_2, \cdots, \tau_{6g-6}) \qquad \text{(II.14)}$$

where $g^{(0)}_{\alpha\beta}$ is a metric which depends on $6g-6$ extra (real) parameters and, e.g. which can be chosen to have constant negative curvature. Integration over all metrics (i.e. geometry) must include not only functional integration over $\varphi(\sigma)$ but also $6g-6$ dimensional integration over $\{\tau_i, i = 1, 2, \cdots, 6g-$

6}—the moduli space. Let us now derive the explicit measure for such integration. Before doing that, let us mention an important mathematical result. Roughly speaking, this moduli space is a complex space. So we will use complex coordinates for this moduli space and also for the Riemann surface. In complex coordinates, the metric tensor on the Riemann surface are given by the components $g_{z\bar{z}}, g_{zz}$ and $g_{\bar{z}\bar{z}}$. Then eq.(II.8) can be written in the following form:

$$
\begin{aligned}
\delta g_{zz} &= \nabla_z V_z \\
\delta g_{\bar{z}\bar{z}} &= \nabla_{\bar{z}} V_{\bar{z}} \\
\delta g_{z\bar{z}} &= \delta\varphi g_{z\bar{z}} + g_{z\bar{z}}(\nabla^z V_z + \nabla_z V^z)
\end{aligned}
\tag{II.15}
$$

Here ∇_z and ∇^z are covariant derivatives:

$$
\begin{aligned}
\nabla^z_{(1)} V_z &= g^{z\bar{z}} \partial_{\bar{z}} V_z \\
\nabla^{(1)}_z V_z &= g_{z\bar{z}} \partial_z (g^{z\bar{z}} V_z) \qquad etc.
\end{aligned}
\tag{II.16}
$$

and

$$
(\nabla^{(1)}_z)^+ = -g^{z\bar{z}} \partial_{\bar{z}} \tag{II.17}
$$

where we used index (n) to distinguish the covariant derivatives acting on different tensor fields (see, e.g. [17] for more details).

From the previous discussions and (II.17), we see that an arbitrary variation of δg_{zz} can be written in the following form:

$$
\delta g_{zz} = \nabla_z V_z + \delta\tau_i \phi^i_{zz} \tag{II.18}
$$

where $\tau_i, i = 1, 2, \cdots, 3g-3$ are the complex coordinates for moduli space and $\{\phi^i_{zz}\}$ are a basis of the zero modes of $(\nabla^{(1)}_z)^+$—the holomorphic 2-differentials. Similarly, we have

$$
\delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} V_{\bar{z}} + \delta\bar{\tau}_i \bar{\phi}^i_{\bar{z}\bar{z}} \tag{II.19}
$$

In order to find the integration measure, one defines a metric in the space of all metrics:

$$\| \delta g_{z\bar{z}} \|^2 = \int d^2 z g_{z\bar{z}} \delta g_{z\bar{z}} \delta g^{z\bar{z}}$$

$$\| \delta g_{zz} \|^2 = \int d^2 z g_{z\bar{z}} \delta g_{zz} \delta g^{zz} \tag{II.20}$$

$$= \int d^2 z g_{z\bar{z}} \nabla_z^{(1)} V_z \nabla_{(-1)}^z V^z + \delta\tau_i \delta\bar{\tau}_j \langle \phi^j, \phi^i \rangle \quad etc.$$

where $\langle \phi^i, \phi^j \rangle = \int d^2 z (g_{z\bar{z}})^{-1} \bar{\phi}^i_{\bar{z}\bar{z}} \phi^j_{zz}$. Then we have

$$Dg = D[\varphi V_z V_{\bar{z}}] \prod_{i=1}^{3g-3} d^2\tau_i det'(\nabla_z \nabla_{(-1)}^z) det\langle \phi^i, \phi^j \rangle \tag{II.21}$$

Notice that the functional integration over conformal factor $e^{\varphi(\sigma)}$ can be trivially factorize out in the critical dimension $d = 26$ for closed bosonic string as shown by Polyakov in [3], the partition function can be written as

$$Z = \sum_g \int_{M_g} \prod_i d^2\tau_i det'(\nabla_z \nabla_{(-1)}^z) det\langle \phi^i, \phi^j \rangle \int DX e^{-S} \tag{II.22}$$

where M_g is the moduli space. In this expression we decomposed the variation of δg_{zz} as in (II.18). In other words, this is a choice of gauge slice. We can also choose other gauge slice, e.g.

$$\delta g_{zz} = \nabla_z V_z' + \delta y_i \mu^i_{zz}$$

$$\delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} V_{\bar{z}}' + \delta \bar{y}_i \bar{\mu}^i_{\bar{z}\bar{z}} \tag{II.23}$$

as shown schematically in Fig.1, and where $\bar{\mu}^i_{\bar{z}\bar{z}} = g_{z\bar{z}} \mu^{iz}_{\bar{z}}, \mu^{iz}_{\bar{z}}$ are called Beltrami differentials.

From (II.18) and (II.23), we have

$$\nabla_z V_z + \delta\tau_i \phi^i_{zz} = \nabla_z V_z' + \delta y_i \mu^i_{zz} \tag{II.24}$$

or

$$\delta\tau_i\langle\phi^j,\phi^i\rangle = \delta y_i\langle\phi^j,\mu^i\rangle \tag{II.25}$$

Doing wedge product over $j = 1, 2, ..., 3g-3$ with eq.(II.25), we have

$$\prod_{i=1}^{3g-3} dy_i \cdot det\langle\phi^i,\mu^j\rangle = \prod_{i=1}^{3g-3} d\tau_i \cdot det\langle\phi^i,\phi^j\rangle \tag{II.26}$$

Substituting this expression into (II.22), we get

$$Z = \sum_g \int_{M_g} \prod_i d^2y_i \frac{|\, det\langle\phi^i,\mu^j\rangle\,|^2}{det\langle\phi^i,\phi^j\rangle} det'(\bigtriangledown_z \bigtriangledown^z_{(-1)}) \int DXe^{-S} \tag{II.27}$$

Following the standard Faddeev-Popov procedure, the gauge parameter V^z for reparametrization invariance can be replaced by an anticommuting ghost field c^z. Introducing its conjugate antighost field b_{zz}, we have the following reparametrization invariant ghost action

$$S_{gh} = \int d^2z\sqrt{g}b_{zz} \bigtriangledown^z c^z + C.C. \tag{II.28}$$

Then we can represent $det'(\bigtriangledown_z \bigtriangledown^z_{(-1)})$ by a path integral over ghost fields. We have

$$\int D[bc\bar{b}\bar{c}] \prod_i b(z_i)\bar{b}(\bar{z}_i)e^{-S_{gh}} = det'(\bigtriangledown_z \bigtriangledown^z_{(-1)}) \frac{|\, det\phi^i(z_k)\,|^2}{det\langle\phi^i,\phi^j\rangle} \tag{II.29}$$

Substituting $det'(\bigtriangledown_z \bigtriangledown^z_{(-1)})/det\langle\phi^i,\phi^j\rangle$ by the above expression, we found that the partition function (II.27) can be expressed as

$$\begin{aligned} Z &= \sum_g \int_{M_g} \prod_i d^2y_i \,\left|\, \frac{det\langle\phi^i,\mu^j\rangle}{det\phi^i(z_k)} \,\right|^2 \int D[Xbc\bar{b}\bar{c}] \prod_i b(z_i)\bar{b}(\bar{z}_i)e^{-(S+S_{gh})} \\ &= \sum_g \int_{M_g} \prod_i d^2y_i \int D[Xbc\bar{b}\bar{c}] \prod_i |\,\langle\mu^i,b\rangle\,|^2 \, e^{-(S+S_{gh})} \end{aligned} \tag{II.30}$$

where $\langle \mu^i, b \rangle$ is the standard notation for the pairing between b field and the Beltrami differentials:

$$\langle \mu^i, b \rangle = \int d^2 z \mu^{iz}_{\bar{z}} b_{zz} \tag{II.31}$$

All the above discussinos can be extended to supersymmetric string theories. Here the complication comes mainly from the fermion fields on Riemann surface. First, we have supersymmetric (2-dimensional) partners for all the bosonic fields in closed bosonic string theory and have to integrate over all these fields. The functional integration can be carried out straight forwardly. In the end, because of topological obstruction one should also integrate over a $2g-2$ dimensional space $\{\xi_a, a = 1, 2, \cdots, 2g-2\}$—the supermoduli space, in addition to the $6g-6$ dimensional moduli space. However, the integration over supermoduli space is a Grassmannian integration and can be explicitly carried out and we have the following expression for the partition function derived in [7]:

$$\begin{aligned} Z = \sum_g \int_{M_g} \prod_i d^2 m_i \int D[X\psi bc\beta\gamma] e^{-(S+S_{gh}(b,c,\beta,\gamma))} \times \\ \times \prod_a \delta(\langle \chi_a, \beta \rangle)(\langle \chi_a, J \rangle + \frac{\partial}{\partial \xi_a}) \prod_i \langle \mu^i, b \rangle \times (Left\ sector) \end{aligned} \tag{II.32}$$

where J is the total super current (see eq.(IV.6)), β and γ are the bosonic ghost fields for the super reparametrization transformation. Here $\frac{\partial}{\partial \xi_a}$ acts on $\prod_i \langle \mu^i, b \rangle$ as follows

$$\frac{\partial}{\partial \xi_a} \mu^i = \frac{\partial}{\partial m_i} \chi_a \tag{II.33}$$

We will not go into the details of the derivation of the above expressions.

Second, because fermion fields are half integer differentials, they can change a sign when travel around a non-contractible cycle (path) on

Riemann surface. In order to define a fermion field, one should specify its properties when traveling around all the non-contractible cycles on Riemann surface. A specification of this properties is called a spin structures, and there can be 2^{2g} different spin structures on a genus g Riemann surface. Because large reparametrization (those which cann't be continuously deformed to identity and are related to modular transformation) mixes spin structures (changing one spin structures to another one), one has to discuss all the spin structures and do the approprate summation over spin structures in patition function in order to get a sensible (e.g. supersymmetric, tachyon free, etc.) theory. Up to now, the problem of summation over spin structures was not completely solved. However, at two-loops this was solved in [13] by using of modular invariance and the cosmological constant was shown to be zero. We will discuss this solution in section 4 and also the nonrenormalization theorem there. Now we recall some mathematics in order to do explicit computation at two-loops by using of hyperelliptic language.

3 Hyperelliptic Riemann Surface

It is a well-known fact that every genus two Riemann surface can be realized as a hyperelliptic surface in CP^2

$$y^2 = \prod_{i=1}^{6}(z - a_i) \tag{III.1}$$

where $a_i (i = 1, 2, \cdots, 6)$ are the six branch points. From (III.1) one readily solves y in term of z:

$$y = y(z) = \pm\sqrt{\prod_{i=1}^{6}(z - a_i)} \tag{III.2}$$

Then every genus two Riemann surface can be thought of as a double covering of S^2 (the Riemann sphere) with cutting and gluing appropraitly. We will see this in connection with canonical homology basis soon (Fig. 2).

There are two independent holomorphic abelian differentials on a genus two Riemann surface:

$$\Omega_1 = \frac{dz}{y(z)}, \qquad \Omega_2(z) = \frac{zdz}{y(z)} \tag{III.3}$$

To see that $\Omega_1(z)$ and $\Omega_2(z)$ are holomorphic differentials, one recalls that the uniformizer coordinate near branch point is u:

$$z - a_i = u^2 \tag{III.4}$$

and the coordinate near infinite point is v:

$$z = \frac{1}{v} \tag{III.5}$$

Set $\Omega(z) = \frac{z-x}{y(z)}dz$, one sees that $\Omega(z)$ has two zeros: one $z = x$ on the upper sheet of S^2, one $z = x$ on the lower sheet. We denote it simply as $z = x\pm$.

On hyperelliptic Riemann surface, spin structures are in one-to-one correspondence with the splitting of branch point $\{a_i\}$ into two non-intersecting sets $\{A_k\}$ and $\{B_l\}$. In particular, the ten even spin structure (at genus two) are corresponded with the case when either set $\{A_k\}$ and $\{B_l\}$ has exact three elements. If we use the cononical homology basis as shown in Fig.2, the ten even spin structures are calculated to be

$$s_1 \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \{A_k\} = \{a_1 a_2 a_3\}\ ,\ \{B_l\} = \{a_4 a_5 a_6\}$$

which is abbreviated as $(123 \mid 456)$

$$\begin{aligned} &s_2 \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \sim (124|356)\ , \quad s_3 \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim (125|346)\ , \quad s_4 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim (126|345), \\ &s_5 \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \sim (134|256)\ , \quad s_6 \sim \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \sim (135|246)\ , \quad s_7 \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim (136|245), \\ &s_8 \sim \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \sim (145|236)\ , \quad s_9 \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \sim (146|235)\ , \quad s_{10} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim (156|234) \end{aligned} \qquad \text{(III.6)}$$

where the symbol $\begin{Bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{Bmatrix}$ is the standard symbol to denote spin structure in connection with Θ-function with characteritics (see, for example [18, 19]). The ordering of the ten even spin structures is arranged following the convention: $A_1 = a_1, A_2 = a_i, A_3 = a_j$ with $i < j$ and $s_1 < s_2$ if $i_1 \leq i_2$ or $j_1 < j_2$, which has been used in [10,13,14,15].

It is easy to see that the holomorphic abelian differentials $\Omega_i(z)$ (eq. (III.3)) are not normalized in the standard way:

$$\oint_{\alpha_i} \omega_j = \delta_{ij}, \qquad \oint_{\beta_i} \omega_j = \tau_{ij} = \tau_{ji} \qquad \text{(III.7)}$$

where τ is the 2×2 period matrix. In fact, these differentials are related as follows:

$$\Omega_i = \sum_j \omega_j \oint_{\alpha_j} \Omega_i \qquad \text{(III.8)}$$

Set $(K)_{ij} = \oint_{\alpha_i} \Omega_j$, we have

$$\Omega_i = \omega_j K_{ji} \qquad \text{(III.9)}$$

It is not difficult to solve ω_i in terms of Ω_i:

$$\begin{aligned} \omega_1 &= \frac{K_{22}\Omega_1 - K_{21}\Omega_2}{detK} \\ \omega_2 &= \frac{-K_{12}\Omega_1 + K_{11}\Omega_2}{detK} \end{aligned} \qquad \text{(III.10)}$$

In what follows, we give some useful formulae which will be used later. The first one is the Thomae formula:

$$\Theta_s^4(0) = \pm det^2 K \prod_{i<j}^{3} A_{ij} B_{ij} \qquad \text{(III.11)}$$

where $A_{ij} = A_i - A_j, B_{ij} = B_i - B_j$. Because of the sign ambiguity of the above expression, we will use another quantity Q_s intead of $\Theta_s^4(0)$ [13]:

$$Q_s = \prod_{i<j}^{3} A_{ij} B_{ij} \qquad \text{(III.12)}$$

The second formula is

$$Q(z,\bar{w}) = \omega(z) \cdot (Im\tau)^{-1} \cdot \bar{\omega}(\bar{w}) = \frac{2}{T} \frac{1}{y(z)\bar{y}(\bar{w})} \int \frac{(z-u)(\bar{w}-\bar{u})}{\mid y(u) \mid^2} d^2u \qquad \text{(III.13)}$$

where

$$T = \int \mid \frac{z_1 - z_2}{y(z_1)y(z_2)} \mid^2 d^2z_1 d^2z_2 = 2 \mid detK \mid^2 detIm\tau \qquad \text{(III.14)}$$

Finally we have a variational formula [9]:

$$\frac{\partial \tau_{ij}}{\partial a_n} = \frac{i\pi}{2} \hat{\Omega}_i(a_n) \hat{\Omega}_j(a_n) \qquad \text{(III.15)}$$

where $\hat{\Omega}(a_i)$ is defined as follows:

$$\Omega(z) = (\hat{\Omega}(z_0) + \hat{\Omega}'(z_0)(z - z_0) + \cdots)dz$$
$$= 2u\Omega(u^2 + a_i)du \qquad \text{(III.16)}$$
$$\hat{\Omega}(a_i) = \lim_{u\to 0} 2u\Omega(u^2 + a_i)$$

where one should use the uniformizer coordinate u instead of z near the branch point.

All these formulas can be proved quite easily by explicit computation. The only trick is using the standard formula:

$$\int \omega_i \wedge \bar{\omega}_j = \sum_k \left\{ \oint_{\alpha_k} \omega_i \oint_{\beta_k} \bar{\omega}_j - (\alpha \longleftrightarrow \beta) \right\} = -2i(Im\tau)_{ij} \qquad \text{(III.17)}$$

and the explicit formulas for τ in terms of K and G: $G = \oint_\beta \Omega$. For example, we have

$$\tau_{12} = \frac{-G_{11}K_{12} + G_{12}K_{11}}{detK} \qquad \text{(III.18)}$$

To conclude this section, we recall another formula which is given by V. G. Knizhnik in [10]:

$$\langle \partial X^\mu(x_1)\partial X^\nu(x_2)\rangle = -g^{\mu\nu}\left\{\frac{1}{4(x_1-x_2)^2} + \frac{1}{4T}\cdot\frac{\partial}{\partial x_2}\left\{\frac{y(x_2)}{y(x_1)}\cdot\frac{1}{x_1-x_2}\times\right.\right.$$
$$\left.\left.\times\frac{(x_1-z_1)(x_1-z_2)}{x_2-z_1)(x_2-z_2)}\cdot \mid \frac{z_1-z_2}{y(z_1)y(z_2)}\mid^2 d^2z_1d^2z_2\right\} + (x_1 \leftrightarrow x_2)\right\} \qquad \text{(III.19)}$$

We now give a derivation of this quite important formula (see section 5). Because $\langle \partial X^\mu(x_1)\partial X^\nu(x_2)\rangle$ has a double pole when $x_1 = x_2$, and no simple pole, one can postulate the general form of it as:

$$\langle \partial X^\mu(x_1)\partial X^\nu(x_2)\rangle = -g^{\mu\nu}\left\{\frac{\partial}{\partial x_2}[\frac{1}{2(x_1-x_2)}(1+\frac{y(x_2)}{y(x_1)})] + \omega_i(x_1)C_{ij}\omega_j(x_2)\right\}$$

$$= -g^{\mu\nu}\left\{\frac{\partial}{\partial x_2}[\frac{1}{2(x_1-x_2)}(1+\frac{y(x_2)}{y(x_1)})] + \Omega_i(x_1)C'_{ij}\Omega_j(x_2)\right\} \qquad 0)$$

Here we include the factor $(1+\frac{y(x_2)}{y(x_1)})$ (instead of the factor 2) to cancell the pole when x_1 and x_2 are on different sheet of the Riemann sphere.

Introducing the following anti-holomorphic $(0,1)$ differantials:

$$\bar{P}^1(\bar{x}) = \frac{1}{\bar{y}(\bar{x})}\int \mid \frac{z}{y(z)} \mid^2 d^2z - \frac{\bar{x}}{\bar{y}(\bar{x})}\int \frac{z}{\mid y(z) \mid^2} d^2z$$
$$\bar{P}^2(\bar{x}) = -\frac{1}{\bar{y}(\bar{x})}\int \frac{\bar{z}}{\mid y(z) \mid^2} d^2z + \frac{\bar{x}}{\bar{y}(\bar{x})}\int \frac{1}{\mid y(z) \mid^2} d^2z \qquad \text{(III.21)}$$

we have

$$\int \bar{P}^i(\bar{x})\Omega_j(x)d^2x = \frac{1}{2}T\delta^i_j \qquad \text{(III.22)}$$

Then it is easy to find $C'_{ij}\Omega_j(x_2)$ from the holomorphicity of $\langle\partial X(x_1)\partial X(x_2)\rangle$:

$$\int \bar{P}^i(\bar{x}_1)\langle\partial X(x_1)\partial X(x_2)\rangle d^2x_1 = 0 \qquad \text{(III.23)}$$

i.e. we have

$$\int \bar{P}^i(\bar{x}_1)\left\{\frac{\partial}{\partial x_2}[\frac{1}{2(x_1-x_2)}(1+\frac{y(x_2)}{y(x_1)})] + \Omega_i(x_1)C'_{ij}\Omega_j(x_2)\right\} d^2x_1 = 0 \quad \text{(III.24)}$$

From the above expression, we get:

$$C'_{ij}\Omega_j(x_2) = -\frac{1}{T}\int \bar{P}^i(\bar{x}_1)\frac{\partial}{\partial x_2}\left\{\frac{1}{x_1-x_2}\cdot\frac{y(x_2)}{y(x_1)}\right\} d^2x_1 \qquad \text{(III.25)}$$

Then

$$\langle\partial X^\mu(x_1)\partial X^\nu(x_2)\rangle = -g^{\mu\nu}\left\{\frac{1}{2(x_1-x_2)^2} + \frac{1}{2}\frac{\partial}{\partial x_2}[\frac{1}{x_1-x_2}\cdot\frac{y(x_2)}{y(x_1)}] - \right.$$
$$\left. -\frac{1}{T}\Omega_i(x_1)\int \bar{P}^i(\bar{z}_1)\frac{\partial}{\partial x_2}[\frac{1}{z_1-x_2}\cdot\frac{y(x_2)}{y(z_1)}]d^2z_1\right\}$$

$$= -g^{\mu\nu}\left\{\frac{1}{2(x_1-x_2)^2}+\frac{1}{2T}\frac{\partial}{\partial x_2}\int\frac{y(x_2)}{y(x_1)}\cdot\frac{1}{x_1-x_2}\times\right.$$
$$\left.\times\frac{(x_1-z_1)(x_1-z_2)}{(x_2-z_1)(x_2-z_2)}\cdot\mid\frac{z_1-z_2}{y(z_1)y(z_2)}\mid^2 d^2z_1d^2z_2\right\} \tag{III.26}$$

Symmetrizing in x_1 and x_2, we get (III.19) as given by V. G. Knizhnik in [10] with minor modification.

From (III.26), one can also write the expression for $\sum_i \alpha_i\langle\partial X(z)X(w_i)\rangle$ with $\sum_i \alpha_i = 0$. It is

$$\sum_i \alpha_i\langle\partial X^\mu(z)X^\nu(w_i)\rangle = -g^{\mu\nu}\sum_i \alpha_i\left\{\frac{1}{2(z-w_i)}+\frac{1}{2T}\int\frac{y(w_i)}{y(z)}\times\right.$$
$$\left.\times\frac{1}{z-w_i}\cdot\frac{(z-z_1)(z-z_2)}{(w_i-z_1)(w_i-z_2)}\mid\frac{z_1-z_2}{y(z_1)y(z_2)}\mid^2 d^2z_1d^2z_2\right\} \tag{III.27}$$

which satisfies

$$\int\bar{P}^i(\bar{z})\sum_i \alpha_i\langle\partial X^\mu(z)X^\nu(w_i)\rangle d^2z = 0 \tag{III.28}$$

By differentiating (III.27) with respect to $\bar{w}_i$, we get

$$\langle\partial X^\mu(z)\bar{\partial}X^\nu(w)\rangle = -\pi\delta^{(2)}(z-w)+\frac{\pi}{T}\cdot\frac{1}{y(z)\bar{y}(\bar{w})}\int\frac{(z-u)(\bar{w}-\bar{u})}{\mid y(u)\mid^2}d^2u$$
$$= -\pi\delta^2(z-w)+\frac{\pi}{2}\omega(z)\cdot(Im\tau)^{-1}\cdot\bar{\omega}(\bar{w}) \tag{III.29}$$

Finally, we would like to give the expression of Segö keneral—the propagator of 1/2-differential field ψ. It is [18]

$$\langle\psi(x)\psi(y)\rangle_s = \frac{1}{x-y}\cdot\frac{u(x)+u(y)}{2\sqrt{u(x)u(y)}} \tag{III.30}$$

where

$$u(x) = \prod_{i=1}^{3}\sqrt{\frac{x-A_i}{x-B_i}} \tag{III.31}$$

All these formulas will be used later in two-loop computation in superstring theories. That completes our review of the mathematics about genus 2 hyperelliptic Riemann surface. Now we start to do computation at two-loops.

4 Nonrenormalization Theorem at Two-Loops

Since odd spin structures give trivially no contributions to the N-particle amplitudes up to $N = 4$, we shall consider only even ones. From (II.2), (II.30) and (II.32), we know the expression for two-loop N-particle amplitude in HST for a given choice of the spin structures:

$$A^N_{ss'} = \int d\mu(m_i, \bar{m}_i) \prod_{i=1}^{N} d^2 z_i (det Im\tau)^{-5} \bar{L}_{s'} R_s \qquad \text{(IV.1)}$$

To get the right amplitude, we have to perform the sum over all spin structures:

$$A^N = \int d\mu(m_i, \bar{m}_i) \prod_{i=1}^{N} d^2 z_i (det Im\tau)^{-5} \sum_{ss'} \eta_s \varphi_{s'} \bar{L}_{s'} R_s$$

$$= \int d\mu(m_i, \bar{m}_i) \prod_{i=1}^{N} d^2 z_i (det Im\tau)^{-5} \sum_{s'} \varphi_{s'} \bar{L}_{s'} \sum_{s} \eta_s R_s \qquad \text{(IV.2)}$$

where η_s and φ_s are phases.

To be specific, we consider the gauge boson vertex in HST. That is, we take the following form of $V(k, \epsilon, z)$:

$$V = V_R \cdot V_L$$

$$V_R = \{\partial(\epsilon \cdot X) + ik \cdot \psi \epsilon \cdot \psi\} e^{ik \cdot X} \qquad \text{(IV.3)}$$

$$V_L = \lambda^I \lambda^J$$

where λ^I are the left moving (i.e. antiholomorphic) two dimenional spinors. Then we have

$$R_s = \int D[X\psi bc\beta\gamma]e^{-(S+S_{gh}(b,c,\beta,\gamma))}\prod_{i=1}^{N} V_R(k_i,\epsilon_i,z_i)\times$$

$$\times\{\prod_{a=1}^{N}\delta(\langle\chi_a,\beta\rangle)\langle\chi_a,J\rangle\prod_{j=1}^{3}\langle\mu^i,b\rangle+\sum_{j=1}^{3}\prod_{i\neq j}\langle\mu^i,b\rangle\times$$

$$\times[\langle\frac{\partial\chi_1}{\partial m_j},\beta\rangle\delta(\langle\chi_1,\beta\rangle)\langle\chi_2,J\rangle\delta(\langle\chi_2,\beta\rangle)+(1\leftrightarrow 2)]\} \qquad \text{(IV.4)}$$

$$\bar{L}_{s'} = \int D[\lambda\bar{b}\bar{c}]e^{-(S+S_{gh}(\bar{b},\bar{c})}\prod_{j=1}^{3}\langle\bar{\mu}^j,\bar{b}\rangle\prod_{i=1}^{N} V_L$$

where s refer to spin structures of ψ, β and γ; s' refers to spin structures of λ^I; $\eta_i, i = 1,2,3$ and $\chi_a, a = 1,2$ are Beltrami and super-Beltrami differentials respectively. All the scalar products are defined as

$$\langle\chi_a,\beta\rangle = \int d^2z\chi_a\beta, \qquad etc. \qquad \text{(IV.5)}$$

and

$$J(z) = \psi\cdot\partial X + 2c\partial\beta - \gamma b + 3\partial c\beta \qquad \text{(IV.6)}$$

is the total super current. In eq. (IV.4) we have assumed that the metric is independent on supermoduli but allowed the super-Beltrami differentials χ_a to depend on moduli. Due to the local world sheet supersymmtry, there is a freedom in choosing χ_a and different choices are related by total derivative on moduli space . In the following we shall make the choice that χ_a are δ-functions located in moduli independent points $x_a (a = 1,2)$ on the Riemann surface and μ^i are also δ-functions located in b_i. In particular we make the convienient choice [13] of taking $x_{1,2}$ to be the zeros of a holomorphic abelian differential $\Omega(z) = \frac{z-x}{y(z)}dz$ to simplify computations. Then $x_{1,2} = x\pm$, i.e. the two corresponding points in the upper and lower Riemann sheet. Then eq.(IV.4) simplifies to the form [13]:

$$R_s = (det'\bar\partial_2)^{-5}(det'\bar\partial_2)(det\bar\partial_{1/2})^5(det'\bar\partial_{3/2})^{-1}\times$$
$$\times\frac{\langle J(x_1)J(x_2)\prod_{i=1}^N V_R(k_i,\epsilon_i,z_i)\rangle_s}{det\varphi^a(x_b)} \tag{IV.7}$$

where $\varphi^a(z)$ are the holomorphic 3/2-differentials and $\langle J(x_1)J(x_2)\cdots\rangle_s$ denotes the normalized correlator (the spin structure dependent part in R_s):

$$\Lambda_s^N \equiv \langle J(x_1)J(x_2)\prod_{i=1}^N V_R(k_i,\epsilon_i,z_i)\rangle_s Q_s$$
$$\equiv \frac{\ll \prod_{a=1}^2 J(x_a\delta(\beta(x_a))\prod_{i=1}^3 b(b_i)\prod_{i=1}^N V_R(k_i,\epsilon_i,z_i)\gg_s}{\ll \prod_{a=1}^2\delta(\beta(x_a))\prod_{i=1}^3 b(b_i)\gg_s}Q_s \tag{IV.8}$$

Here the double bracket $\ll\cdots\gg$ indicate the functional integration over all the right fields (including X also).

To begin with, let us first consider the case $N=0$, i.e. the vacuum amplitude. Using the explicit form of the supercurrent, we can represent Λ^0 as a sum of a matter part

$$\Lambda_m^0 = \langle\psi_\mu(x_1)\psi_\nu(x_2)\rangle_s\langle\partial X^\mu(x_1)\partial X^\nu(x_2)\rangle_s Q_s \tag{IV.9}$$

and a ghost part

$$\Lambda_{gh}^0 = \langle J_{gh}(x_1)J_{gh}(x_2)\rangle_s Q_s \tag{IV.10}$$

where $J_{gh} = J - \psi\cdot\partial X$. We will not do the calculations done in [13] and only recall the relevant results which will be used in present and next section. We have (following the notation of [13]):

$$\Lambda_{gh}^0 = \Big\{-2\partial_2 P(x_1x_2)R(x_1x_2) - (\partial_2 R(x_2x_1) + 2\Lambda(x_2)R(x_2x_1)\frac{\partial\varphi_2(x_1)}{\varphi_2(x_2)} -$$

$$-2P(x_2x_1)R(x_2x_1)\frac{\partial\varphi_1(x_1)}{\varphi_1(x_1)} - (1 \longleftrightarrow 2)\Bigg\} Q_s \qquad \text{(IV.11)}$$

In [13], it is proved that all the terms in Λ^0_{gh} can be reduced, up to spin structure independent factors, to $\langle\psi(x_1)\psi(x_2)\rangle_s Q_s$ and $\langle\partial\psi(x_1)\psi(x_2)\rangle_s Q_s$. Then, baseed on modular invariance, the following unique determination of phases

$$\eta_1 = -\eta_2 = \eta_3 = -\eta_4 = \eta_5 = -\eta_6 = \eta_7 = \eta_8 = -\eta_9 = \eta_{10} = 1 \qquad \text{(IV.12)}$$

was found and the cosmological constant was proven to be identically equal to zero.

Notice that a modular transformation in hyperelliptic language simply corresponds to a permutation of the six branch points $a_i, i = 1, 2, \cdots, 6$. So what modular invariance means is that $\sum_s \eta_s R_s$ is invariant under all the permutations of a_i's. From [13], we knew that

$$R^0_s = F_1 Q_s \sum_{i=1}^{3}(A_i - B_i) + F_2 Q_s \sum_{i=1}^{3}(A_i^2 - B_i^2) \qquad \text{(IV.13)}$$

where we have set $x = \infty$ and the coefficients $F_{1,2}$ are independent of s (to be precise, they are antisymmetric for every interchange $a_i \longleftrightarrow a_j, i \neq j$). Then we must have $\sum_s \eta_s Q_s \sum_{i=1}^{3}(A_i^n - B_i^n), n = 1, 2$ to be antisymmetric for every interchange. Because these expressions are homogeneous polynominal (of degree 6 and 7 for $n = 1$ and 2 respectively) in a_i, it should be proportional to $P(a) = \prod_{i<j}(a_i - a_j) \equiv \prod_{i<j} a_{ij}$ (a homogeneous polynominal of degree 15 in a_i). One sees immediately that the powers of a_i cann't be matched. We have then

$$\sum_s \eta_s Q_s \sum_{i=1}^{3}(A_i^n - B_i^n) = 0, \qquad n = 1, 2 \qquad \text{(IV.14)}$$

To prove nonrenormalization theorem at two-loops, we have to study the following quantities

$$\Lambda_R^N = \sum_s \eta_s \langle \prod_{i=1}^{N} V_R(k_i, \epsilon_i, z_i) J(x_1) J(x_2) \rangle_s Q_s \qquad \text{(IV.15)}$$

Substituting (IV.3) into (IV.15), we can calculate Λ_R^N by using of Wick theorem. We consider first the contractions of ψ which are relevant for the summation over spin structures. There are two types of contractions:

Type A: contractions $\langle J(x\pm)\psi(z_i)\rangle_s$ appear

Type B: only the contraction $\langle J(x+)J(x-)\rangle_s$ appears,

$$i.e.\ \langle \prod_{i=1}^{N} V_R(k_i, \epsilon_i, z_i)\rangle_s \langle J(x_1)J(x_2)\rangle_s$$

Let us compute Λ_s^3:

$$\Lambda_s^3 = Q_s \langle \prod_{i=1}^{3} \{\epsilon_i \cdot \partial X(z_i) + ik_i \cdot \psi(z_i)\epsilon_i \cdot \psi(z_i)\} e^{ik_i \cdot X(z_i)} \cdot J(x_1)J(x_2)\rangle_s \qquad \text{(IV.16)}$$

We have

$$\Lambda_s^3 = Q_s\{A_s + B_s + C_s + D_s\} \qquad \text{(IV.17)}$$

where

$$\begin{aligned}
A_s &= \langle \epsilon_1 \cdot \partial X(z_1)\, \epsilon_2 \cdot \partial X(z_2)\, \epsilon_3 \cdot \partial X(z_3) \prod_{i=1}^{3} e^{ik_i \cdot X(z_i)}\, J(x_1) J(x_2) \rangle_s \\
B_s &= -i \langle \{ \epsilon_1 \cdot \partial X(z_1)\, \epsilon_2 \cdot \partial X(z_2)\, \epsilon_3 \cdot \psi(z_3)\, k_3 \cdot \psi(z_3) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \} \prod_{i=1}^{3} e^{ik_i \cdot X(z_i)} J(x_1) J(x_2) \rangle_s \\
C_s &= -\langle \{ \epsilon_1 \cdot \partial X(z_1)\, \epsilon_3 \cdot \psi(z_3)\, k_3 \cdot \psi(z_3)\, \epsilon_2 \cdot \psi(z_2)\, k_2 \cdot \psi(z_2) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \} \prod_{i=1}^{3} e^{ik_i \cdot X(z_i)} J(x_1) J(x_2) \rangle_s \\
D_s &= -i \langle \epsilon_1 \cdot \psi(z_1)\, k_1 \cdot \psi(z_1)\, \epsilon_2 \cdot \psi(z_2)\, k_2 \cdot \psi(z_2)\, \epsilon_3 \cdot \psi(z_3)\, k_3 \cdot \psi(z_3) \prod_{i=1}^{3} e^{ik_i \cdot X(z_i)}\, J(x_1) J(x_2) \rangle_s
\end{aligned} \qquad \text{(IV.18)}$$

For $A_s Q_s$, one sees that it is similar to R_s up to spin structure independent factor because $X(z, \bar{z})$ is independent of spin structure s. Then

$$\sum_s \eta_s A_s Q_s = 0 \tag{IV.19}$$

by using of eq. (IV.14).

As to B_s, C_s and D_s, they lead to the following various spin structure dependent factors:

$$\begin{aligned}
E1_s &= \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(x_2)\rangle_s \\
E2_s &= \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(x_2)\rangle_s \\
E3_s &= \langle\psi(z_1)\psi(z_2)\rangle_s^2 \times \begin{cases}\langle\psi(x_1)\psi(x_2)\rangle_s \\ \langle\partial\psi(x_1)\psi(x_2)\rangle_s\end{cases} \\
E4_s &= \langle\psi(x_1)\psi(z_1)\rangle_s\langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(x_2)\rangle_s \\
E5_s &= \langle\psi(z_1)\psi(z_2)\rangle_s\langle\psi(z_2)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(z_1)\rangle_s \times \begin{cases}\langle\psi(x_1)\psi(x_2)\rangle_s \\ \langle\partial\psi(x_1)\psi(x_2)\rangle_s\end{cases} \\
E6_s &= \langle\psi(z_1)\psi(z_2)\rangle_s^2\langle\psi(x_1)\psi(z_3)\rangle_s\langle\psi(z_3)\psi(x_2)\rangle_s
\end{aligned} \tag{IV.20}$$

Setting $x_{1,2} = \infty\pm$ and using (III.31):

$$\begin{gathered}
\langle\psi(z_1)\psi(z_2)\rangle_s = \frac{1}{z_1 - z_2}\cdot\frac{u(z_1)+u(z_2)}{2\sqrt{u(z_1)u(z_2)}}, \quad \langle\psi(z_1)\psi(x_1)\rangle_s = \frac{u(z_1)-1}{2\sqrt{u(z_1)}} \\
\langle\psi(z_2)\psi(x_2)\rangle_s = \frac{u(z_2)+1}{2\sqrt{u(z_2)}}, \quad \langle\psi(x_1)\psi(x_2)\rangle_s = \frac{1}{4}\sum_{i=1}^{3}(A_i - B_i)
\end{gathered} \tag{IV.21}$$

and

$$\langle\partial\psi(x_1)\psi(x_2)\rangle_s = \frac{1}{8}\sum_{i=1}^{3}(A_i^2 - B_i^2) \tag{IV.22}$$

one sees that:

1). $\sum_s \eta_s E1_s Q_s = 0$ leads to the following identity

$$\sum_s \eta_s \left\{ u(z) - \frac{1}{u(z)} \right\} Q_s = 0 \qquad \text{(IV.23)}$$

2). $\sum_s \eta_s E2_s Q_s = 0$ leads to the identity

$$\sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)} \right\} Q_s = 0 \qquad \text{(IV.24)}$$

and if (IV.23) is true.

3). $\sum_s \eta_s E3_s Q_s = 0$ leads to the following identities

$$\sum_s \eta_s Q_s \left\{ \frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right\} \sum_{i=1}^{3} (A_i^n - B_i^n) = 0, \quad n = 1, 2 \qquad \text{(IV.25)}$$

where we have used (IV.14).

4). $\sum_s \eta_s E4_s Q_s = 0$ if (IV.23) is true.

5). $\sum_s \eta_s E5_s Q_s = 0$ if (IV.25) is true.

6). $\sum_s \eta_s E6_s Q_s = 0$ leads to

$$\sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)u(z_3)} - \frac{u(z_2)u(z_3)}{u(z_1)} \right\} Q_s = 0 \qquad \text{(IV.26)}$$

and if (IV.23) is true.

So if we want to show that $\sum_s \eta_s \Lambda_s^3 = 0$ is true, it is sufficient (but not necessary) to show that (IV.23) — (IV.26) are true. What we will show below is that this is really the case. All these identities (called Lianzi identities in [14,15]) are true. In fact, (IV.26) implies (IV.23) and (IV.24) as it can be easily seen by setting $z_1 = z_3$ and $z_3 = \infty$ respectively. Moreover, we have one more general identity:

$$\sum_s \eta_s \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} Q_s = 0 \qquad \text{(IV.27)}$$

From this identity, one can easily derive (IV.26) by settin $z_4 = \infty$.

All these identities (IV.23) — (IV.27) can be proved quite easily. Let us see, for example, (IV.24). Substituting $u(z_1)$ and $u(z_2)$ by (III.31) into (IV.24), we have:

$$LHS \;\; of \;\; (IV.24) = \sum_s \eta_s \left\{ \prod_{i=1}^{3} \sqrt{\frac{(z_1 - A_i)(z_2 - B_i)}{(z_2 - A_i)(z_1 - B_i)}} - (z_1 \leftrightarrow z_2) \right\} Q_s$$

$$= \sum_s \eta_s \frac{\prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (z_1 \leftrightarrow z_2)}{\sqrt{\prod_{i=1}^{6}(z_1 - a_i)(z_2 - a_i)}} Q_s \qquad \text{(IV.28)}$$

$$= \frac{1}{y(z_1)y(z_2)} \sum_s \eta_s \left\{ \prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (A \leftrightarrow B) \right\} Q_s$$

An important point is that this expression is modular invariant in the sense of that whenever we interchange a_i and a_j $(i \neq j)$ we get a minus sign for this expression. So this expression should be proportional to $P(a)$. By simple power counting, one sees that $\sum_s \eta_s(\prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (A \leftrightarrow B)) \cdot Q_s$ is a homogeneous polynominal of degree $6 + 6 = 12$ in a_i and z_j. But the degree of $P(a)$ is 15. So we must have

$$\sum_s \eta_s \left\{ \prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) - (A \leftrightarrow B) \right\} Q_s = 0 \qquad \text{(IV.29)}$$

That completes the proof of (IV.24) (and also of (IV.23) by setting $z_2 = \infty$). This same argument can also be used to prove (IV.25). We have

$$LHS \;\; of \;\; (IV.25) = \frac{1}{y(z_1)y(z_2)} \sum_s \eta_s Q_s \left\{ \prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) + \right.$$

$$+(A \leftrightarrow B)\}\sum_{i=1}^{3}(A_i^n - B_i^n) \qquad \text{(IV.30)}$$

One easily sees that this expression is also modular invariant and should be proportional to $P(a)$. But the degree of $\sum_s \eta_s Q_s\{\prod_{i=1}^3 (z_1 - A_i)(z_2 - B_i) + (A \leftrightarrow B)\}\sum_{i=1}^3 (A_i^n - B_i^n)$ is at most 14 ($n = 1, 2$). So we must have

$$\sum_s \eta_s Q_s\{\prod_{i=1}^{3}(z_1 - A_i)(z_2 - B_i) + (A \leftrightarrow B)\}\sum_{i=1}^{3}(A_i^n - B_i^n) = 0, \quad n = 1, 2 \qquad \text{(IV.31)}$$

To prove (IV.27), one follows the same strategy as above. We have

$$LHS \ of \ (IV.27) = \frac{1}{\prod_{i=1}^4 y(z_i)}\sum_s \eta_s \left\{\prod_{i=1}^{3}(z_1 - A_i)(z_2 - A_i)\times \right.$$
$$\left. \times (z_3 - B_i)(z_4 - B_i) - (A \leftrightarrow B)\right\} Q_s \qquad \text{(IV.32)}$$

Notice that when $z_1 = z_3$ or z_4, or $z_2 = z_3$ or z_4, (IV.27) is true because of (IV.24) (which has been proved). Then the last factor $\sum_s \eta_s(\cdots)Q_s$ in (IV.30) should be proportional to $P(a)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)$: a homogeneous polynominal of degree 19 in a_i and z_j. But the degree of $\sum_s \eta_s(\cdots)Q_s$ is $3 \times 4 + 6 = 18$. We have then

$$\sum_s \eta_s\{\prod_{i=1}^{3}(z_1 - A_i)(z_2 - A_i)(z_3 - B_i)(z_4 - B_i) - (A \leftrightarrow B)\}Q_s = 0 \qquad \text{(IV.33)}$$

In summary, we have proved the following Lianzi identities:

$$\sum_s \eta_s \left\{u(z) - \frac{1}{u(z)}\right\} Q_s = 0$$

$$\sum_s \eta_s \left\{\frac{u(z_1)}{u(z_2)} - \frac{u(z_2)}{u(z_1)}\right\} Q_s = 0$$

$$\sum_s \eta_s \left\{ \frac{u(z_1)}{u(z_2)u(z_3)} - \frac{u(z_2)u(z_3)}{u(z_1)} \right\} Q_s = 0$$

$$\sum_s \eta_s \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} - \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} Q_s = 0 \qquad \text{(IV.34)}$$

$$\sum_s \eta_s Q_s \left\{ \frac{u(z_1)}{u(z_2)} + \frac{u(z_2)}{u(z_1)} \right\} \sum_{i=1}^{3} (A_i^n - B_i^n) = 0, \quad n = 1, 2$$

Using this set of identities, one readily proves th nonrenormalization theorem, i.e. the vanishing of N-particle amplitude up to $N = 3$. We have

$$\sum_s \eta_s \Lambda_s^N \propto \sum_s \eta_s R_s^N = 0, \quad N = 1, 2, 3 \qquad \text{(IV.35)}$$

For example, for $N = 1$ we have

$$\sum_s \Lambda_s^2 = \sum_s \eta_s \langle J(x_1) J(x_2) V_R(k, \epsilon, z) \rangle_s Q_s$$

$$= F_1' \sum_s \eta_s \langle \psi(x_1)\psi(x_2) \rangle_s Q_s + F_s' \sum_s \eta_s \langle \partial\psi(x_1)\psi(x_2) \rangle_s Q_s +$$

$$+ F'' \sum_s \eta_s \langle \psi(x_1)\psi(z) \rangle_s \langle \psi(z)\psi(x_2) \rangle_s Q_s \qquad \text{(IV.36)}$$

$$= -\frac{F''}{4} \sum_s \eta_s \left\{ u(z) - \frac{1}{u(z)} \right\} Q_s = 0$$

To conclude this section, we would like to prove the following summation formula:

$$\sum_s \eta_s \left\{ \frac{u(z_1)u(z_2)}{u(z_3)u(z_4)} + \frac{u(z_3)u(z_4)}{u(z_1)u(z_2)} \right\} \sum_{i=1}^{3} (A_i^n - B_i^n) Q_s$$

$$= \frac{2P(a)}{\prod_{i=1}^4 y(z_i)} (z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4) \times \begin{cases} 1 & n = 1 \\ \sum_{i=1}^6 a_i - \sum_{k=1}^4 z_k & n = 2 \end{cases} \qquad \text{(IV.37)}$$

which has been used in [15].

For $n = 1$, we have

$$LHS\ of\ (IV.37) = \frac{1}{\prod_{i=1}^4 y(z_i)} \sum_s \eta_s \left\{ \prod_{i=1}^3 (z_1 - A_i)(z_2 - A_i)(z_3 - B_i)(z_4 - B_i) + \right.$$
$$\left. + (A \leftrightarrow B) \right\} \sum_{i=1}^3 (A_i - B_i) Q_s \tag{IV.38}$$

$$= \frac{1}{\prod_{i=1}^4 y(z_i)} \times \{a\ homogeneous\ polynominal\ of\ degree\ 19\ in\ a_i\ and\ z_j\}$$

From Lianzi identities (IV.25), this expression vanishes when $z_1 = z_3$ or z_4, or $z_2 = z_3$ or z_4. It should be proportional to $(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)$. It is also modular invariant and should be proportional to $P(a)$. Then we have

$$\sum_s \eta_s(\cdots) \sum_{i=1}^3 (A_i - B_i) Q_s = \frac{cP(a)}{\prod_{i=1}^4 y(z_i)} (z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4) \tag{IV.39}$$

where c is a constant and can be calculated to be: $c = 2$.

For $n = 2$, we should have

$$\sum_s \eta_s(\cdots) \sum_{i=1}^3 (A_i^2 - B_i^2) Q_s = \frac{cP(a)}{\prod_{i=1}^4 y(z_i)} (z_1 - z_3)(z_1 - z_4) \times$$
$$\times (z_2 - z_3)(z_2 - z_4) \left\{ \sum_{i=1}^6 a_i + F(z_i) \right\} \tag{IV.40}$$

from the above experience (and power counting). Here $F(z_i)$ is a linear function of z_i (without constant term). From the symmetry of the original expression, we have $F(z_i) = a \sum_i z_i$ and $a = -1$ (by explicit computation). That completes the proof of (IV.37). Let us now turn to the computation of four-particle amplitude which is presumably non-vanishing.

5 Four-Particle Amplitude at Two-Loops

In this section we are going to compute the following expression

$$\begin{aligned} A^4 &= \sum_s \eta_s \langle \prod_{i=1}^{4} V_R(k_i, \epsilon_i, z_i) \cdot J(x_1) J(x_2) \rangle_s Q_s \\ &= \sum_s \eta_s \langle \prod_{i=1}^{4} k_i \cdot \psi(z_i) \epsilon_i \cdot \psi(z_i) e^{ik \cdot X(z_i} \cdot J(x_1) J(x_2) \rangle_s Q_s \end{aligned} \tag{V.1}$$

because of nonrenormalization theorem which was proved in the last section.

First, we want to show that Type A contractions give zero contributions, i.e. we have

$$\sum_s \eta_s (\cdots) \langle \psi^\mu(x_1) \psi^\nu(z_i) \rangle_s \langle \psi^\rho(x_2) \psi^\sigma(z_j) \rangle_s Q_s = 0 \tag{V.2}$$

In fact, all the contractions are the following kinds:

$$\begin{aligned} A4_s &= \langle \psi(x_1) \psi(z_1) \rangle_s \langle \psi(z_1) \psi(z_2) \rangle_s \langle \psi(z_2) \psi(z_3) \rangle_s \langle \psi(z_3) \psi(z_4) \rangle_s \langle \psi(z_4) \psi(x_2) \rangle_s \\ B4_s &= \langle \psi(z_1) \psi(z_2) \rangle_s \langle \psi(z_2) \psi(z_3) \rangle_s \langle \psi(z_3) \psi(z_1) \rangle_s \langle \psi(z_4) \psi(x_1) \rangle_s \langle \psi(z_4) \psi(x_2) \rangle_s \\ C4_s &= \langle \psi(z_1) \psi(z_2) \rangle_s^2 \langle \psi(x_1) \psi(z_3) \rangle_s \langle \psi(z_3) \psi(z_4) \rangle_s \langle \psi(z_4) \psi(x_2) \rangle_s \end{aligned} \tag{V.3}$$

or sometimes the expressions permutated among z_1, z_2, z_3 and z_4. By using of the explicit formula of $\langle \psi(z_1) \psi(z_2) \rangle_s$ etc., one readily shows that

$$\sum_s \eta_s A4_s Q_s = \sum_s \eta_s B4_s Q_s = \sum_s \eta_s C4_s Q_s = 0 \tag{V.4}$$

by using of Lianzi identities (IV.32). Here one should use the more general identity (IV.27) which is not needed in the verification of nonrenormalization therom.

Using (V.2), we have

$$\Lambda^4 = \sum_s \eta_s \langle \prod_{i=1}^{4} k_i \cdot \psi(z_i) \epsilon_i \cdot \psi(z_i) \rangle_s \langle J(x_1) J(x_2) \prod_{i=1}^{4} e^{ik \cdot X(z_i)} \rangle_s Q_s \quad \text{(V.5)}$$

By using of the summation formula (IV.37) derived in the last section, we have

$$\begin{aligned} &\sum_s \eta_s \langle \psi(z_1)\psi(z_2)\rangle_s \langle \psi(z_2)\psi(z_3)\rangle_s \langle \psi(z_3)\psi(z_4)\rangle_s \langle \psi(z_4)\psi(z_1)\rangle_s \sum_{i=1}^{3} (A_i^n - B_i^n) Q_s \\ &= \sum_s \eta_s \langle \psi(z_1)\psi(z_2)\rangle_s^2 \langle \psi(z_3)\psi(z_4)\rangle_s^2 \sum_{i=1}^{3} (A_i^n - B_i^n) Q_s \qquad \text{(V.6)} \\ &= \frac{P(a)}{8\prod_{i=1}^{4} y(z_i)} \times \begin{cases} 1 & n=1 \\ \sum_{i=1}^{6} a_i - \sum_{k=1}^{4} z_k & n=2 \end{cases} \end{aligned}$$

Then one can do the summation over spin structures in (V.5). We have

$$\begin{aligned} &\sum_s \eta_s \langle \prod_{i=1}^{4} k_i \cdot \psi(z_i) \epsilon_i \cdot \psi(z_i) \rangle_s \prod_{i=1}^{3} (A_i^n - B_i^n) Q_s \\ &= \frac{P(a)}{4\prod_{i=1}^{4} y(z_i)} \cdot K(k,\epsilon) \times \begin{cases} 1 & n=1 \\ \sum_{i=1}^{6} a_i - \sum_{k=1}^{4} z_k & n=2 \end{cases} \qquad \text{(V.7)} \end{aligned}$$

where the kinematic factor $K(k,\epsilon)$ is computed to be:

$$\begin{aligned} K(k,\epsilon) = &-\frac{1}{4}(st\,\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4 + su\,\epsilon_2\cdot\epsilon_3\,\epsilon_1\cdot\epsilon_4 + tu\,\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4) \\ &+\frac{1}{2}s(\epsilon_1\cdot k_4\,\epsilon_3\cdot k_2\,\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_3\,\epsilon_4\cdot k_1\,\epsilon_1\cdot\epsilon_3 + \epsilon_1\cdot k_3\,\epsilon_4\cdot k_2\,\epsilon_2\cdot\epsilon_3 + \epsilon_2\cdot k_4\,\epsilon_3\cdot k_1\,\epsilon_1\cdot\epsilon_4) \\ &+\frac{1}{2}t(\epsilon_2\cdot k_1\,\epsilon_4\cdot k_3\,\epsilon_3\cdot\epsilon_1 + \epsilon_3\cdot k_4\,\epsilon_1\cdot k_2\,\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_4\,\epsilon_1\cdot k_3\,\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot k_1\,\epsilon_4\cdot k_2\,\epsilon_2\cdot\epsilon_1) \qquad \text{(V.8)} \\ &+\frac{1}{2}u(\epsilon_1\cdot k_2\,\epsilon_4\cdot k_3\,\epsilon_3\cdot\epsilon_2 + \epsilon_3\cdot k_4\,\epsilon_2\cdot k_1\,\epsilon_1\cdot\epsilon_4 + \epsilon_1\cdot k_4\,\epsilon_2\cdot k_3\,\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot k_2\,\epsilon_4\cdot k_1\,\epsilon_1\cdot\epsilon_2) \\ = &\; t^{ijklmnpq} k_1^i k_2^j k_1^k k_4^l \epsilon_1^m \epsilon_2^n \epsilon_3^p \epsilon_4^q \end{aligned}$$

which coincides with the standard kinematic factor at tree and one-loop level [4].

Recalling the relevant computations done in [13], we have

$$I(x) = \langle J(x+)J(x-)\prod e^{ik\cdot X}\rangle = -\frac{1}{4}g^{\mu\nu}\langle\partial X_\mu(x+)\partial X_\nu(x-)\times \times\prod e^{ik\cdot X}\rangle + \langle\prod e^{ik\cdot X}\rangle I_{gh}(x) \tag{V.9}$$

where $I_{gh}(x) = \langle J_{gh}(x+)J_{gh}(x-)\rangle$ is the contribution from the ghost part. See [13] and (IV.11) for explicit expression.

The various factors appearing in (V.9) can be calculated following [13] by Taylor expansion. The calculations are tedious and sometimes very complicated but straightforward. We only give the results of all the calculation for completeness.

1): x_1, x_2 are the zeros of $\Omega_1(z) = \frac{dz}{y(z)}$, $x_{1,2} = \infty\pm$.

2):
$$P(x,y) \equiv \frac{1}{\Omega_2(x)}\langle\psi(x)\psi(y)\rangle\Omega_2(y)$$

$$P(x_1x_2) = -\langle\psi(x_1)\psi(x_2)\rangle = -P(x_2x_1)$$

$$\partial_1 P(x_2x_1) = -\partial_2 P(x_1x_2) = \langle\partial\psi(x_1)\psi(x_2)\rangle - \Lambda(x_2)\langle\psi(x_1)\psi(x_2)\rangle$$

where $\Lambda(x)$ is the finite part of $P(x,y)$ when $y\to x$ and $\Lambda(x_1) = \Lambda(x_2) = -\frac{1}{2}\sum_{i=1}^{6} a_i$

3):
$$\varphi_i(x) = \pm\Omega_1(x)\langle\psi(x)\psi(x_i)\rangle, \quad \varphi_i(x_j) = \delta_{ij} \quad i = 1,2$$

$$\partial\varphi_2(x_1) = -\partial\varphi_1(x_2) = -\langle\psi(x_1)\psi(x_2)\rangle$$

$$\partial\varphi_1(x_1) = \partial\varphi_2(x_2) = \frac{1}{2}\sum_{i=1}^{6} a_i$$

4):
$$R(xy) = -\langle c(y)b(x)\textstyle\prod_{j=1}^{3} b(b_j)\rangle$$

$$R(x_2x_1) = -\frac{1}{4}\left(\sum_{i=1}^{6} a_i - 2\sum_{i=1}^{3} b_i\right) + \frac{1}{2}\Sigma(b)$$

$$\partial_2 R(x_2 x_1) = -\frac{1}{16}(5\sum_{i=1}^{6} a_i^2 + 6\sum_{i<j}^{6} a_i a_j) + \frac{1}{2}\sum_{i=1}^{6} a_i \sum_{i=1}^{3} b_i -$$

$$-\frac{1}{2}\sum_{i<j}^{3} b_i b_j + \frac{1}{4}\left\{\sum_{i=1}^{6} a_i \Sigma(b) - 2\Sigma'(b)\right\}$$

where

$$\Sigma(b) = \frac{y(b_1)}{(b_1 - b_2)(b_1 - b_3)} + (123 \to 231) + (123 \to 312)$$

$$\Sigma'(b) = \frac{(b_2 + b_3) y(b_1)}{(b_1 - b_2)(b_1 - b_3)} + (123 \to 231) + (123 \to 312)$$

5): $\langle \psi(x_1)\psi(x_2)\rangle = \frac{1}{4}\sum_{i=1}^{3}(A_i - B_i)$

$$\langle \partial\psi(x_1)\psi(x_2)\rangle = \frac{1}{8}\sum_{i=1}^{3}(A_i^2 - B_i^2)$$

where b_i are the locations of the Beltrami differentials. Putting the above results together and doing some computation, we get

$$I_{gh} = -\frac{1}{8}(\sum_{i=1}^{6} a_i - 2\sum_{i=1}^{3} b_i)\sum_{i=1}^{3}(A_i^2 - B_i^2) -$$
$$-\frac{1}{32}(\sum_{i=1}^{6} a_i^2 - 2\sum_{i<j}^{6} a_i a_j + 8\sum_{i<j}^{3} b_i b_j)\sum_{i=1}^{3}(A_i - B_i) \tag{V.10}$$

The factors $\sum_i (A_i^2 - B_i^2)$ and $\sum_i (A_i - B_i)$ appearing in (V.10) will have to be substituted by $\sum_i a_i - \sum_k z_k$ and 1 respectively in (V.9) due to the summation formula (IV.37).

By using of the formula (III.19), we have

$$\langle \partial X^\mu(x_1)\partial X^\nu(x_2)\rangle = \frac{g^{\mu\nu}}{2T}\int\left\{-\frac{1}{8}\sum_{i=1}^{6} a_i^2 + \frac{1}{4}\sum_{i<j}^{6} a_i a_j + u_1^2 + u_1 u_2 + u_2^2 -\right.$$

$$-\frac{1}{2}(u_1+u_2)\sum_{i=1}^{6}a_i\Big\} \mid \frac{u_1-u_2}{y(u_1)y(u_2)} \mid^2 d^2u_1d^2u_2$$

$$= \frac{g^{\mu\nu}}{2}\left\{-\frac{1}{8}\sum_{i=1}^{6}a_i^2+\frac{1}{4}\sum_{i<j}^{6}a_ia_j+\sum_{i=1}^{6}a_i^3\frac{\partial}{\partial a_i}ln(T\prod_{j=1}^{6}a_j)\right\} \tag{V.11}$$

Putting all these results together and doing some algebraic calculation, we have

$$I(x=\infty) = -\frac{1}{2}\sum_{i<j}^{6}a_ia_j - \frac{1}{4}\sum_{i<j}^{3}b_ib_j + \frac{1}{4}\sum_{i=1}^{6}a_i\sum_{i=1}^{3}b_i$$
$$+\frac{1}{8}(\sum_{i=1}^{6}a_ia_i - 2\sum_{i=1}^{3}b_i)\sum_{k=1}^{4}z_k - \frac{5}{4}\sum_{i=1}^{6}a_i^3\frac{\partial}{\partial a_i}ln(T\prod_{j=1}^{6}a_j) \tag{V.12}$$

where we write only the leading terms when $k\rightarrow 0$, i.e. we put $\prod e^{ik\cdot X}\rightarrow 1$ in (V.9). We see that, for generic b_i, $I(x)$ is symmetric for the permutations of a_i, i.e. it is modular invariant. In the following we will take $a_{1,2,3}$ to be the moduli, i.e. our integration variables over moduli, and therefore we fix $b_i = a_i$ for $i = 1, 2, 3$.

Notice that we have presented the expression for I(x) when $x_{1,2} = \infty\pm$, i.e. the zeros of Ω_1. To get the generic case $x_{1,2} = x\pm$, one simply perform a Möbius transformation: $z, a \longrightarrow -1/(z-x), -1/(a-x)$. Then

$$\prod_{i=1}^{4}dz_i\Omega(z_i)I(x) = \prod_{i=1}^{4}dz_k\frac{z_k-x}{y(z_k)}\cdot\left\{-\frac{1}{2}\sum_{i<j}^{6}\frac{1}{a_i-x}\frac{1}{a_j-x}-\right.$$
$$-\frac{1}{4}\sum_{i<j}^{3}\frac{1}{b_i-x}\frac{1}{b_j-x}+\frac{1}{4}\sum_{i=1}^{6}\frac{1}{a_i-x}\sum_{i=1}^{3}\frac{1}{b_i-x}+ \tag{V.13}$$
$$\left.+\frac{1}{8}(\sum_{i=1}^{6}\frac{1}{a_i-x}-2\sum_{i=1}^{3}\frac{1}{b_i-x})\sum_{i=1}^{4}\frac{1}{z_i-x}+\frac{5}{4}\sum_{i=1}^{6}\frac{1}{a_i-x}\frac{\partial}{\partial a_i}lnT\right\}$$

Putting everything together (see [13,15]), we get finally the following expression for the four-particle two-loop amplitude for HST (choosing $SO(32)$ and when $k \to 0$):

$$A(K) = cK(k,\epsilon) \int d^2a_1 d^2a_2 d^2a_3 \mid a_{45}a_{56}a_{64} \mid^2 \times$$
$$\times \frac{1}{T^5 \prod_{i<j}^6 a_{ij}\bar{a}_{ij}^3} \times \int \prod_{k=1}^4 d^2 z_k \Omega(z_k) I(x) \sum_s \langle \prod V_L \rangle \bar{Q}_s^4 \tag{V.14}$$

where the integration runs over the complex plane, c is an undetermined constant and s denotes the spin structures of the Left sector[2]. We will discuss the various properties of this amplitude (V.14) in the next section.

Similar calculation can be done for SST II. Here the relevant super current insertion is

$$\langle J(r+)\bar{J}(\bar{s}+)J(r-)\bar{J}(\bar{s}-)\rangle_s \tag{V.15}$$

where we have taken $x_{1,2} = r\pm$ for Right sector and $\bar{x}_{1,2} = \bar{s}\pm$ for the Left sector. We remark that it is necessary to take $r \neq s$ to get ride of some sigularities arising by simply taking $r = s$.

By using of eq.(III.29), it is an easy matter to arrive at the following expression for the four-particle[3] amplitude at two-loops ($k \to 0$):

$$AII(k) = c'\tilde{K} \int d^2a_1 d^2a_2 d^2a_3 \mid a_{45}a_{56}a_{64} \mid^2 \cdot \frac{1}{T^5 \prod_{i<j}^6 \mid a_{ij} \mid^2} \times$$
$$\times \int \prod_{i=1}^4 d^2 z_i \frac{(z_i - r)(\bar{z}_i - \bar{s})}{\mid y(z_i) \mid^2} \left\{ I(r)\bar{I}(\bar{s}) + \frac{5}{4}\left(\frac{\pi}{T}\frac{1}{y(r)\bar{y}(\bar{s})} \int \frac{d^2v(v-r)(\bar{v}-\bar{s})}{\mid y(v) \mid^2}\right)^2 \right\} \tag{V.16}$$

[2] Needless to say, $\langle \prod V_L \rangle$ can also be explicity expressed in terms of $\langle \lambda\lambda \rangle = \langle \bar{\psi}\psi \rangle$.
[3] The vertex function is $V(k,\epsilon,z) = (\partial X^\mu + ik\cdot\psi\psi^\mu)\epsilon_{\mu\nu}(\bar{\partial}X^\nu + ik\cdot\bar{\psi}\bar{\psi}^\nu)e^{ik\cdot X}$.

where $\tilde{K}$ is the same kinematic factor as in tree and one-loop level, see ref.[4].

In the next section we will discuss the various properties and in particular the finiteness of the four-particle amplitudes (V.14) and (V.16).

6 Finiteness of the Four-Particle Amplitude

The four-particle amplitudes calculated in the last section seem to depend on arbitary parameters, i.e. the choice for x and for the values of $a_{4,5,6}$. It is seen from the starting expression (it is actually simpler to do it before summing over spin structures) that $A(k)$ (eq.(V.14)) is invariant if we simultaneously make the same Möbius transformation for x and $a_{4,5,6}$. Since a generic Möbius transformation depends on three parameters, it follows that if we show that $A(k)$ is independent of x, then it will also be independent of $a_{4,5,6}$. The x independence is expected by a general argument [7] and indeed we can explicitly verify that it is true. In fact, the integrand on the RHS of (V.14) is a meromorphic function of x and the poles, i.e. $x \to a_i$, can be expressed as total derivatives in $a_{1,2,3}$ and $z_{1,2,3,4}$. The sigularities of the integrand for $x \to a_i$, $i = 1, 2, 3$, can be isolated in the form:

$$cK \sum_{i=1}^{3} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \cdot \left\{ \frac{1}{4} \frac{1}{(a_i - x)^2} - \frac{1}{4} \frac{1}{a_i - x} \sum_{j \neq i}^{6} \frac{1}{a_j - a_i} + \frac{1}{8} \frac{1}{a_i - x} \sum_{k=1}^{4} \frac{1}{z_k - a_i} + \right.$$

$$\left. + \frac{5}{4} \frac{2}{a_i - x} \frac{\partial}{\partial a_i} lnT \right\} \prod_{k<l}^{6} \frac{1}{a_{kl} \bar{a}_{kl}^3} \frac{1}{T^5} \sum_s \langle \prod V_L \rangle \bar{Q}_s^4 \qquad \text{(VI.1)}$$

$$= -\frac{1}{4} \sum_{i=1}^{3} \frac{\partial}{\partial a_i} \left\{ \frac{1}{a_i - x} \frac{1}{T^5 \prod_{k<l}^{6} a_{kl}} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \right\} \prod_{k<l}^{6} \bar{a}_{kl}^{-3} \sum_s \langle \prod V_L \rangle \bar{Q}_s^4$$

which are total derivatives in $a_{1,2,3}$. Similarly, the sigularities of the integrand of (V.14) for $x \to a_i$, $i = 4, 5, 6$, can be isolated in the form (choosing $x \to a_4$ for specifics):

$$cK \frac{1}{a_4 - x} \prod_{k=1}^{4} \frac{z_k - a_4}{y(z_k)} \cdot \left\{ \frac{1}{2} \cdot \frac{2a_4 - a_5 - a_6}{a_{45} a_{46}} - \frac{1}{4} \sum_{i=1}^{3} \frac{1}{a_i - a_4} + \right.$$

$$+\frac{1}{8}\sum_{k=1}^{4}\frac{1}{z_k-a_4}+\frac{5}{4}\frac{\partial}{\partial a_4}lnT\Bigg\}\prod_{k<l}^{6}a_{kl}^{-1}\bar{a}_{kl}^{-3}\cdot T^{-5}\sum_s\langle\prod V_L\rangle\bar{Q}_s^4 \quad \text{(VI.2)}$$

By making use of the formula

$$\frac{\partial}{\partial a_4}lnT=\frac{1}{a_{45}a_{46}}\left\{2(a_5+a_6)-\sum_{i=1}^{4}a_i-\sum_{i=1}^{4}a_{i5}a_{i6}\frac{\partial}{\partial a_i}lnT\right\} \quad \text{(VI.3)}$$

which can be derived from the projectve invariance of T, we can write (VI.2) as:

$$cK\left\{\frac{1}{4}\sum_{i=1}^{3}\frac{\partial}{\partial a_i}\left\{\frac{1}{a_4-x}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\prod_{i<j}^{6}a_{ij}^{-1}\cdot T^{-5}\prod_{k=1}^{4}\frac{z_k-a_4}{y(z_k)}\right\}+\right.$$

$$\left.+\frac{1}{2}\sum_{k=1}^{4}\frac{\partial}{\partial z_k}\left\{\frac{(z_k-a_5)(z_k-a_6)}{a_{45}a_{46}}\prod_{k=1}^{4}\frac{z_k-a_4}{y(z_k)}\right\}\prod_{i<j}^{6}a_{ij}^{-1}\cdot T^{-5}\right\}\times \quad \text{(VI.4)}$$

$$\times\prod_{i<j}^{6}\bar{a}_{ij}^{-3}\cdot\sum_s\langle\prod V_L\rangle\bar{Q}_s$$

which are also total derivatives in $a_{1,2,3}$ and $z_{1,2,3,4}$. When $x\to z_k$, one can easily see that there is no sigular terms because of the prefactor $\prod_k(z_k-x)$. There is also no sigular terms when $x\to\infty$. When we set $x=\infty$ in (V.13), we get precisely (V.12).

Actually, the above expressions for the sigularities in x can be generalized to include the $\prod e^{ik\cdot X}$ part of the vertices, by making use of the formula

$$\langle\partial X^\mu(x+)\partial X^\nu(x-)\prod e^{ik\cdot X}\rangle=\langle T(x+)\prod e^{ik\cdot X}\rangle+$$

$$+\langle T(x-)\prod e^{ik\cdot X}\rangle-\sum_{i<j}k_i\cdot k_j\frac{1}{x-z_i}\cdot\frac{1}{x-z_j}\langle\prod e^{ik\cdot X}\rangle \quad \text{(VI.5)}$$

where $T(x)$ is the (normal ordered) energy momentum tensor $T(x)=-\frac{1}{2}:\partial X(x)\cdot\partial X(x):$. This formula can be easily verified by calculating both side of (VI.5) explicitly by making use of the expressions of $\langle\partial X\partial X\rangle$ (eq.(III.19)) and $\langle X\partial X\rangle$ (eq.(III.27)). For $k\to 0$, we have

$$\lim_{x\to a_i}\langle\partial X(x+)\cdot\partial X(x-)\rangle = 5\left\{-\frac{1}{8}\frac{1}{(a_i-x)^2}+\right.$$
$$\left.+\frac{1}{4}\frac{1}{a_i-x}\sum_{j\neq i}^{6}\frac{1}{a_j-a_i}+\frac{1}{a_i-x}\frac{\partial}{\partial a_i}lnT\right\}+(regular\ terms) \tag{VI.6}$$

which has, implicitly, been used previously, see eq.(V.11) for $x=\infty$. By using of (VI.5), we have

$$\lim_{x\to a_i}\langle\partial X(x+)\cdot\partial X(x-)\prod e^{ik\cdot X}\rangle = 5\left\{-\frac{1}{(a_i-x)^2}\langle\prod e^{ik\cdot X}\rangle+\right.$$
$$\left.+\frac{1}{a_i-x}\frac{\partial}{\partial a_i}ln(T\prod_{k<l}^{6}\mid a_k-a_l\mid^{1/2}\langle\prod e^{ik\cdot X}\rangle)\cdot\langle\prod e^{ik\cdot X}\rangle\right\}+(regular\ terms) \tag{VI.7}$$

Then one easily sees that the above discussions go through by including also the factor $\prod e^{ik\cdot X}$. Therefore the present discussion holds for the general case.

The proof of independence of x is completed by checking that there are no boundary terms, i.e. that the total derivatives give vanishing contributions. We have checked that this is true, by studying in detail the potentially dangerous degenerate configurations in the moduli space. For instance, consider the integration of the expression (VI.1) or (VI.4) in the region $u\to 0$ where $u=a_2-a_1$. The boundary term will be proportional to

$$\lim_{n\to 0}\oint\frac{d\bar{u}}{\bar{u}^3}\cdot\mid u\mid^3\cdot F(u,\bar{u}) \tag{VI.8}$$

where we have taken into account that $T\to ln\mid u\mid$. The integration over d^2z_i is included in F: in the degeneration limit $dz/y(z)\sim dt/t$ in the uniformizer coordinate $t^2=z-a_1$. Since the left part is regular for $\bar{z}\to\bar{a}_1$, there is no sigularity coming from the integration over $d\bar{z}$. Therefore F is regular and the above expression (VI.8) vanishes.

Of course, when many z_k collide together, in particular in the point a_1, possible singularities have to be interpreted as physical singularities in the external momenta and one has to take into account the factor $\prod e^{ik\cdot X}$. As always in string theory, the integration by parts, like the ones we are discussing here, are meant to be done in the region of the momenta k where the integrand is regular, and then analytically continued everywhere [20].

As an extra exmple, in the "dividing" degeneration case $a_2 - a_1 = u$ $a_3 - a_1 = vu$, taking into account $T \to \frac{1}{|u|}$, we get the following boundary term for (VI.1) and (VI.4):

$$\lim_{n\to 0} \oint \frac{d\bar{u}}{\bar{u}^3} \cdot \mid u \mid^3 \cdot F(u, \bar{u}) \tag{VI.9}$$

In this degeneration limit $dz/y(z) \sim dt/t^2$ but the left part is regular (for all the spin structures but one, where however $\bar{Q}_s^4$ gives a further factor $(\bar{u})^8$) and F is regular so that (VI.9) vanishes. The conclusion is that $A(k)$ is independent of x, and therefore also of $a_{4,5,6}$.

The independence of the four-particle amplitude for SST II, eq.(V.16) on r (and $\bar{s}$) and $a_{4,5,6}$ can be discussed similarly. Let us compute the sigular terms when $r \to a_i$, $i = 1, 2, 3$ and $\bar{s}$ kept arbitary. They are

$$\begin{aligned} c\tilde{K} \prod_{k<l}^{6} \frac{1}{\bar{a}_{kl}} \cdot \prod_{k=1}^{4} \frac{\bar{z}_k - \bar{s}}{\bar{y}(\bar{z}_k)} \cdot \Bigg\{ & -\frac{1}{4} \frac{\partial}{\partial a_i} \Big(\frac{1}{a_i - r} \prod_{k<l}^{6} \frac{1}{a_{kl}} \frac{1}{T^5} \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \Big) \bar{I}(\bar{s}) + \\ & + \frac{5}{4} \frac{1}{a_i - r} \frac{1}{\prod_{j\neq i}^{6} (a_j - a_i) \prod_{k<l}^{6} a_{kl} \cdot T^5} \times \\ & \times \prod_{k=1}^{4} \frac{z_k - a_i}{y(z_k)} \Big(\frac{\pi}{T} \frac{1}{\bar{y}(\bar{s})} \int \frac{(v - a_i)(\bar{v} - \bar{s})}{\mid y(v) \mid^2} d^2 \Big)^2 \Bigg\} \end{aligned} \tag{VI.10}$$

$$\frac{1}{y(r)^2} \xrightarrow[r \to a_i]{} \frac{1}{a_i - r} \cdot \frac{1}{\prod_{j\neq i} (a_j - a_i)}$$

where we have used the results of the previous analysis. One notices that the only obstruction for the first term to be a total derivative in a_i, $i = 1, 2, 3$, is coming from $\bar{I}(\bar{s}) = \cdots + \frac{5}{4}\sum_j \frac{1}{\bar{a}_j - \bar{s}}\frac{\partial}{\partial \bar{a}_j} lnT$, the last term. What we will show below is that this gives a contribution which cancels the second term in (VI.10), i.e. we have

$$\frac{1}{4}\sum_{j=1}^{6} \frac{1}{\bar{a}_j - \bar{s}} \cdot \frac{\partial^2}{\partial \bar{a}_j \partial a_i} lnT + \frac{1}{\prod_{j \neq i}^{6}(a_j - a_i)} \left(\frac{\pi}{T}\frac{1}{\bar{y}(\bar{s})}\int \frac{(v - a_i)(\bar{v} - \bar{s})}{\mid y(v) \mid^2} d^2 v\right)^2 = 0 \quad \text{(VI.11)}$$

This identity can be proved as follows. From (III.15), we have

$$\frac{\partial \tau_{ij}}{\partial a_n} = \frac{i\pi}{2}\hat{\Omega}_i(a_n)\hat{\Omega}_j(a_n) \quad \text{(VI.12)}$$

Notice that $T = 2 \mid detK \mid^2 detIm\tau$, we have

$$\begin{aligned} \frac{\partial^2}{\partial a_i \partial \bar{a}_j} lnT &= \frac{\partial^2}{\partial a_i \partial \bar{a}_j} lndetIm\tau = Tr\left(\frac{\partial^2}{\partial a_i \partial \bar{a}_j} lnIm\tau\right) \\ &= -4\left(\frac{\pi}{T}\frac{1}{\hat{y}(a_i)\hat{\bar{y}}(\bar{a}_j)}\int \frac{(v - a_i)(\bar{v} - \bar{a}_j)}{\mid y(v) \mid^2}\right)^2 \end{aligned} \quad \text{(VI.13)}$$

So we have

$$\begin{aligned} \frac{1}{4}\sum_{j=1}^{6} \frac{1}{\bar{a}_j - \bar{s}}\frac{\partial^2}{\partial a_i \partial \bar{a}_j} lnT &= -\sum_{j=1}^{6} \frac{1}{\bar{a}_j - \bar{s}}\left(\frac{\pi}{T}\frac{1}{\hat{y}(a_i)\hat{\bar{y}}(\bar{a}_j)}\int \frac{(v - a_i)(\bar{v} - \bar{a}_j)}{\mid y(v) \mid^2}\right)^2 \\ &= -\frac{1}{\hat{y}^2(a_i)}\left(\frac{\pi}{T}\frac{1}{\bar{y}(\bar{s})}\int \frac{(v - a_i)(\bar{v} - \bar{a}_j)}{\mid y(v) \mid^2}\right)^2 \end{aligned} \quad \text{(VI.14)}$$

This is precisely (VI.11). Then the sigular terms when $r \to a_i$, $i = 1, 2, 3$, are

$$-\frac{1}{4}c\tilde{K}\sum_{i=1}^{3} \frac{\partial}{\partial a_i}\left[\frac{1}{a_i - r}\frac{1}{T^5 \mid \prod_{i<j}^{6} a_{ij} \mid^2}\prod_{k=1}^{4} \frac{(z_k - a_i)(\bar{z}_k - \bar{s})}{\mid y(z_k) \mid^2}\bar{I}(\bar{s})\right] \quad \text{(VI.15)}$$

Similarly, the singular terms when $r \to a_4$ are

$$-\frac{1}{4}c\tilde{K}\sum_{i=1}^{3}\frac{\partial}{\partial a_i}\left[\frac{1}{a_4-r}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\frac{1}{T^5\mid\prod_{i<j}^{6}a_{ij}\mid^2}\prod_{k=1}^{4}\frac{(z_k-a_4)(\bar{z}_k-\bar{s})}{\mid y(z_k)\mid^2}\bar{I}(\bar{s})\right]$$
$$+(total\ \ derivatives\ \ in\ \ z_k) \tag{VI.16}$$

by making use of the formula

$$\sum_{i=1}^{3}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\frac{\partial^2}{\partial a_i\partial\bar{a}_j}lnT=-\frac{\partial^2}{\partial a_4\partial\bar{a}_j}lnT \tag{VI.17}$$

It is an easy matter to show that all these total derivatives eqs.(VI.15) and (VI.16) give zero contributions. That completes our verification of the independence of $AII(k)$ on r (and likewise on $\bar{s}$) and therefore also on $a_{4,5,6}$.

We metion that a form of the two-loop four-particle amplitude for SST II was conjectured in ref.[21]. While in some aspects it resembles our above formula eq.(V.16), in some others, in particular in the important ones related to the super current contribution, it does not seem to agree with the result of our explicit computation.

Next, we would like to discuss the finiteness of $A(k)$, considering for definiteness the HST case. In the "handle" case we consider the corner $u\to 0$ where $u=a_2-a_1$, and in the "dividing" case the corner $u\to 0$ and v keep fixed, where $u=a_2-a_1$, $vu=a_3-a_1$. We can read the corresponding expressions for HST from eq.(V.14). By taking the appropriate variable $y=u^2$ and doing some computations, we can put the "handle" degeneration expression in the canonical form [22,23]

$$\frac{d^2y}{\bar{y}^2y}\cdot(ln\mid y\mid)^{-5}. \tag{VI.18}$$

where one recognizes a tachyon $1/\bar{y}^2$ in the left sector and a massless state $1/y$ in the Right sector, as is expected in HST. For the "dividing" degeneration case the appropriate variable is $u = y^2$ and from (V.14) we get the canonical form

$$\frac{d^2y}{\bar{y}^2} \cdot y^2 \cdot F \tag{VI.19}$$

where we recognize again a tachyon in the Left sector and a level 3 massive excitation (y^2, compare with the zero level $1/y$) in the Right sector as is to be expected from the norenormalization theorems, implying that the one-loop tadpole vanishes if it is attached to a vertex $(\psi\psi)^n$ with $n < 4$. Of course, the integration over $\arg(y)$ will select the same contribution from the Left as it does from the Right, and therefore finally the degeneration expression will be

$$\frac{d^2y}{\mid y \mid^2}(ln \mid y \mid)^{-5} \cdot F' \qquad \textit{for the "handle" case} \tag{VI.20}$$

$$d^2y \mid y \mid^4 \cdot F' \qquad \textit{for the "dividing" case} \tag{VI.21}$$

The amplitude is thus finite, for generic values of the external momenta k_i, taking into account the part $\prod e^{ik\cdot X}$.

A more subtle question is whether the leading term which we have obtained for $A(k)$, i.e. the coefficient mutiplying $K(k,\epsilon)$ in eq.(V.14), where we dropped $\prod e^{ik\cdot X}$, is also finite. The question arises because in the "handle" case, taking $z - a_1 = t^2$ we get $y(z) \sim t \cdot (t^2 + u)^{1/2}$ and the integration over dz_i looks like $\prod_1^4 dt_i/(t_i^2 + u)^{1/2}$ which combined with an appropriate Left sector contribution could give $\sim (ln \mid y \mid)^4$, making the expression (VI.20) divergent [21] (notice that the divergence comes from the integration region $z_i \sim z_j$ and therefore disappears for generic k_i). We have not done the rather involved explicit computation for the Left sector, but we can nevertheless argue that even this divergence in $k_i \to 0$ is removed. In fact we can make use of the arbitrariness in x to choose[4] $x \to a_1$: we have then to take the finite part of (V.13) in this limit, and we can see that it contains a factor $(z_i - a_1) = t_i^2$ at least for two values of i. The resulting divergences from the integration over dz_i will then actually be

$$\prod_{i=1}^{2} \frac{dt_i}{\sqrt{t_i^2 + u}} \times (Left\ sector) \sim (ln \mid y \mid)^2 \tag{VI.22}$$

making the expression (VI.20) finite.

Similarly, in the "dividing" case: $z - a_1 = t^2$, $z - a_2 = t^2 + u$, $z - a_3 = t^2 = uv$, we put $x = a_1$ and we would get, from the corner $z \to a_1$, a divergence like

$$\prod_{i=1}^{2} \frac{dt_i}{t_i^2 + y^2} \times (Left\ sector) \sim \frac{1}{\mid y \mid^4} \tag{VI.23}$$

and the expression (VI.21) will remain finite.

In conclusion, we have obtained a finite integral representation of the two-loop amplitude for four massless particles in HST. The leading term for $k_i \to 0$ is written in eq.(V.14), where x and $a_{4,5,6}$ are arbitrary. In particular one can use the expression for $x \to \infty$:

$$\prod_{k=1}^{4} dz_k \Omega(z_k) \cdot I(x) \to \prod_{k=1}^{4} \frac{dz_k}{y(z_k)} \left\{ -\frac{1}{2} \sum_{i<j}^{6} a_i a_j - \frac{1}{4} \sum_{i<j}^{3} b_i b_j + \frac{1}{4} \sum_{i=1}^{6} a_i \sum_{i=1}^{3} b_i \right.$$
$$\left. + \frac{1}{8} (\sum_{i=1}^{6} a_i a_j - 2 \sum_{i=1}^{3} b_i) \sum_{k=1}^{4} z_k - \frac{5}{4} \sum_{i=1}^{6} a_i^3 \frac{\partial}{\partial a_i} ln(T \prod_{j=1}^{6} a_j) \right\} \tag{VI.24}$$

[4]This means choosing the super current inertion on a branch point, as conidered in ref.[10,24,25].

The terms in bracket on the RHS coincide with $I(\infty)$ defined in eq.(V.12). We have also obtained an expression for SST II and one can also take the limit $r = s = \infty$ in the SST II formula (V.16), since there is no sigularity in this limit, and we get

$$AII(k) = c'\tilde{K} \int d^2a_1 d^2a_2 d^2a_3 \mid a_{45}a_{56}a_{64} \mid^2 \cdot \frac{1}{T^5 \prod_{i<j}^{6} \mid a_{ij} \mid^2} \times \\ \times \int \prod_{i=1}^{4} d^2 z_i \frac{1}{\mid y(z_i) \mid^2} \left\{ \mid I(\infty) \mid^2 + \frac{5}{4} \left(\frac{\pi}{T} \frac{d^2 v}{\mid y(v) \mid^2} \right)^2 \right\} \tag{VI.25}$$

However, as we have seen it is more convenient to keep the arbitrariness of x, r, s in order to study the various properties of the amplitudes.

7 Conclusions

In this thesis, we have described the full details of the computation of the four-particle amplitudes for both HST and SST II at two-loops by using of the genus two hyperelliptic formalism. The obtained expressions, eqs.(V.14) and (V.16), are reminiscent to the Koba-Nielsen formula [26] for the tree amplitude. However, one should also integrate over moduli, which are described by $a_{1,2,3}$, in addition to the locations z_k, $k = 1, 2, 3, 4$, of the vertices. One virtue of the hyperelliptic formalism is the description of modular transformation. In this formalism, modular transformation is simply the permutation of the branch points, which form a finite group. So we needn't care about the fundamental region of the modular group and simply integrate $a_{1,2,3}$ over the where complex plane. The group factor $6! = 720$ can be reabsorbed in the overall factor c which should be determined from factorization or unitarity.

We have checked that the amplitude (V.14) and (V.16) have the correct properties. They are independent on the location of the supercurrent insertions. They are finite by itself. One sees that the contribution from the ghost part is very important to ensure the right properties of the amplitudes. Even though most of the string specialists believe that superstring theories are finite in perturbative expansion order by order, as shown by explicit one-loop calculation, this has not been checked explicitly beyond one-loop. The caculation we have done shows that this belief is really true at two-loops and one sees that superstring theories show miracles once again. The interplay between finiteness and the arbitrariness of the locations of supercurrents is very crucial to ensure the finiteness of the amplitudes. Probably this arbitrariness will also play a role to ensure the right factorization properties and unitarity.

The amplitudes we have obtained may have important physical consequences, for example, to study ultra high energy behaviour of scattering[27].

Another problem worth studying is to do similar calculation for four-dimensional superstring theories[28]. Here, all the relevant machinary are at hand, but we don't know whether the difficult is only complex.

As to higher loop computation, we have nothing to say. But we believe that our explicit computation may sheld some light on the general theory of high loop computation in superstring theories.

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Figure Captions

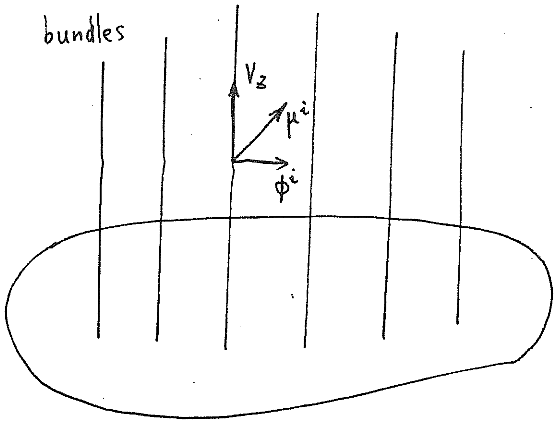

Fig. 1 Decomposition of δg_{zz} : $\delta g_{zz} = P_z V_z + \delta\tau_i \phi^i_{zz}$
$= P_z V'_z + \delta y_i \mu^i_{zz}$

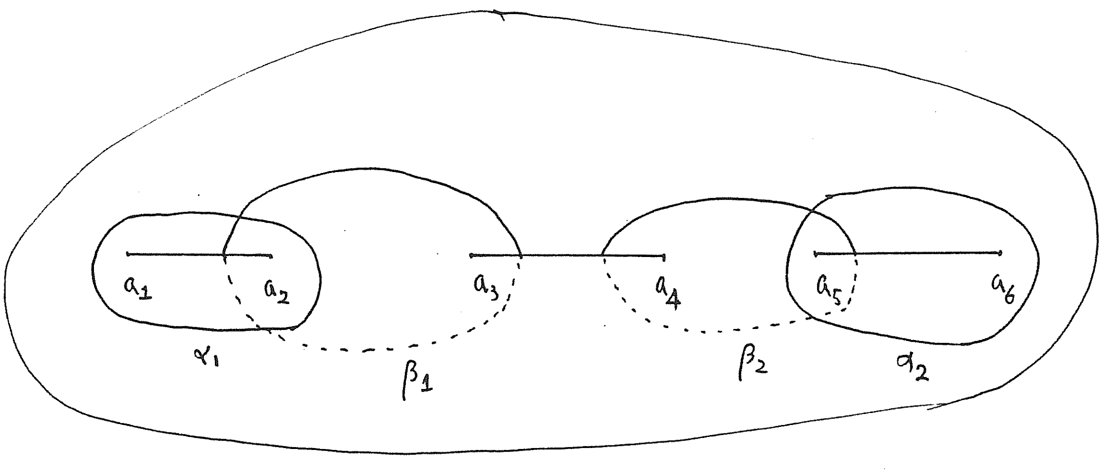

Fig. 2 Homology Basis on genus-two Riemann surface.
"- - - -" means that the path is on the lower sheet.