ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Pair formalism for the fractional quantum Hall effect and the theory of edge waves at filling one.

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Matteo Marsili

SUPERVISORS

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SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
STUDI AVANZATI
TRIESTE
Strada Costiera 11

SISSA ISAS

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1 Introduction

The behaviour of a 2D electron gas in extremely high magnetic fields has been the subject of extensive works in the past years. While the origin of the quantization for integer and simple odd fractional fillings is now well understood, a theory of the fractional quantum Hall effect (FQHE) for general filling fractions is still missing. Much of the present understanding of the FQHE is based indeed on first quantized many electrons wave functions which are available only at special filling fractions. The strategy based on trial *variational* wave functions has proved to be very fruitful in this field while traditional approaches like the Hartree Fock approximation and standard many body theory have not really produced satisfactory results so far.

The basic reason for this is that in two dimensions and in the quantum limit (all the electrons in the $n = 0$ Landau level), kinetic energy is totally suppressed, while lack of Hilbert space completeness (only recovered for $B \to \infty$) makes the many body correlation effects entirely non classical and novel. In these circumstances no simple description of the correlation between electrons is possible in terms of a single particle picture.

In this paper we propose a new picture for the problem of the many body physics of the lowest Landau level. This is based entirely on the second quantization formalism. The main idea is that a 2D many body system in high magnetic fields (in the symmetric gauge) may have its natural description in terms of pair operators. All the two body operators, like the interaction potential or the correlation function, are expressed in terms of the number operator of particle pairs. The quantum numbers of these pairs are the total and relative angular momentum.

The main difficulty within this representation comes from the complicated commutation relations among pair operators, which deviate from perfect boson character. Nonetheless, we show it to represent a useful tool in the case of e.g. edge waves. The first part of this paper presents the formal-

ism and shows that many of the known results are easily translated in terms of pairs. The picture presented has the advantage of being intuitive and simple. We also briefly report about some abortive approach to the general problem of the FQHE using a BCS type pairing theory. The failure of these approaches suggests that pairwise correlation are not enough to *diagonalize* the problem.

The second part deals with an application to a specific problem for which an interesting picture has been recently proposed on the basis of both analytical and numerical work [6]. This concerns edge waves or the low lying excitations near incompressible states as the $\nu = 1$ or $\nu = 1/3$ states. We deal with the $\nu = 1$ case, and using the formalism precedently introduced we test the proposed picture in the scheme of many body theory and second quantization, a description which has not been applied up to now to this problem. We show that the reason for edge waves excitations at $\nu = 1 - \epsilon$ to behave as free bosonic waves can be undersood, and corrections quantified, using the pair formalism. The predictions of our approach agree very well with numerical results. This example shows the simplicity of the pair formalism in dealing with operators whose ugly form, in the single particle picture, has up to now forced researchers towards straight numerical techniques.

The concluding section contains a general discussion and some perspective on the possible extensions.

2 Pair description of the FQHE in the disk geometry (symmetric gauge)

Let us consider a general two body operator represented, in first quantization, by the function $V(z_1, z_2)$. The second quantized form of this operator is:

$$\hat{V} = \frac{1}{2} \sum_{T=1}^{\infty} \sum_{u,s=0}^{T} \langle T-s, s|V(z_1, z_2)|T-u, u\rangle c^+_{T-u} c^+_u c_s c_{T-s} \qquad (1)$$

here c^+ (c) are single particle creation (destruction) operators and the conservation of total angular momentum has been made explicit.

The matrix element is given by:

$$\langle T-s,s|V(z_1,z_2)|T-u,u\rangle = \frac{I_V(T,s,u)}{4\pi^2 2^T\sqrt{(T-s)!s!(T-u)!u!}} \tag{2}$$

with

$$I_V(T,s,u) = \int d^2z_1 d^2z_2\ \bar{z_1}^{T-s}\bar{z_2}^{s}V(z_1,z_2)z_1^{T-u}z_2^{u}e^{-(|z_1|^2+|z_2|^2)/2} \tag{3}$$

We consider in what follows only operators $V(z_1,z_2)$ that are separable in the relative ($\xi = z_1 - z_2$) and center of mass ($Z = (z_1+z_2)/2$) coordinates and that depend only on the moduli of these coordinates:

$$V(Z+\xi/2, Z-\xi/2) = w(Z)\cdot v(\xi) \tag{4}$$

The usual case of a translationally invariant operator $V(z_1-z_2)$ corresponds simply to $w(Z)=1$.

In the more general case 4 the integral in the matrix element can be performed as follows:

$$\begin{aligned} I_V(T,s,u) \;=\; & \sum_{\alpha=0}^{T-s}\sum_{\beta=0}^{s}\sum_{\gamma=0}^{T-u}\sum_{\sigma=0}^{u}\binom{T-s}{\alpha}\binom{s}{\beta}\binom{T-u}{\gamma}\binom{u}{\sigma}(-1)^{\beta+\sigma}\cdot \\ & \cdot\int d^2Z\,\bar{Z}^{T-\alpha-\beta}w(Z)Z^{T-\gamma-\sigma}e^{-|Z|^2}\cdot \\ & \cdot\int d^2\xi\,(\bar{\xi}/2)^{\alpha+\beta}v(\xi)(\xi/2)^{\gamma+\sigma}e^{-|\xi|^2/4}. \end{aligned}$$

The integrals vanish whenever $\alpha+\beta\neq\gamma+\sigma$ and otherwise depend only on $q=\alpha+\beta=\gamma+\sigma$. Using the identies

$$\delta_{k,j} = \int_{-\pi}^{\pi}\frac{dx}{2\pi}e^{i(k-j)x} \quad\text{and}\quad \delta_{\alpha+\beta,\gamma+\sigma} = \sum_{q=0}^{T}\delta_{q,\alpha+\beta}\cdot\delta_{q,\gamma+\sigma} \tag{5}$$

the summations on α, β, γ and σ can be carried out and

$$\begin{aligned} I_V(T,s,u) &= \sum_{q=0}^{T} \int_{-\pi}^{\pi} \frac{dx}{2\pi} \left(1+e^{-ix}\right)^{T-u} \left(1-e^{-ix}\right)^{u} e^{iqx} \cdot \\ &\quad \cdot \int_{-\pi}^{\pi} \frac{dy}{2\pi} \left(1+e^{iy}\right)^{T-s} \left(1-e^{iy}\right)^{s} e^{-iqy} \cdot I_\xi(q) \cdot I_Z(T,q) \\ \text{with} \qquad & I_\xi(q) = 4^{-q} \int d^2\xi \, v(\xi) |\xi|^{2q} e^{-|\xi|^2/4} \\ \text{and} \qquad & I_Z(T,q) = \int d^2 Z \, w(Z) |Z|^{2(T-q)} e^{-|Z|^2} \end{aligned}$$

It is now possible to carry out independently the summations on u and s in equation 1, so that the second quantization form of the operator $\hat{V}$ becomes:

$$\hat{V} = \sum_{T=1}^{\infty} \sum_{q=0}^{T} V_T(q) f_T^+(q) f_T(q). \tag{6}$$

The sum on u (s), which involves the first (second) integral and the pair of operators $c_{T-u}^+ c_u^+$ ($c_{T-s} c_s$), defines the new pair operator $f_T^+(q)$ ($f_T(q)$) apart from a normalization constant

$$f_T^+(q) = \sum_{u=0}^{T} b_T(u,q) c_{T-u}^+ c_u^+ \tag{7}$$

The normalization of the envelope function $b_T(u,q)$ is determined requiring that the state $f_T^+(q)|0\rangle$ is normalized to 1. This means that $\sum_u b_T^2(u,q) = 1/2$. After some algebra, using the theorems of residues, we finally get

$$\begin{aligned} b_T(u,q) &= \sqrt{\frac{\binom{T}{u}}{\binom{T}{q} 2^{T+1}}} \int_{-\pi}^{\pi} \frac{dx}{2\pi} \left(1+e^{-ix}\right)^{T-u} \left(1-e^{-ix}\right)^{u} e^{iqx} \\ &= \sqrt{\frac{\binom{T}{u}}{\binom{T}{q} 2^{T+1}}} \left[\frac{\partial_z^q}{q!} (1+z)^{T-u} (1-z)^{u}\right]_{z=0} \end{aligned} \tag{8}$$

and

$$V_T(q) = \frac{I_Z(T,q) \cdot I_\xi(q)}{2\pi^2 (T-q)! q!} \tag{9}$$

The following properties hold for the coefficients $b_T(u,q)$

$$b_T(T-u,q) = (-1)^q b_T(u,q) \tag{10}$$

$$b_T(q,u) = b_T(u,q) \tag{11}$$

$$2\sum_{u=0}^{T} b_T(u,q)b_T(u,p) = \delta_{q,p} \tag{12}$$

see appendix A for the derivation of these relations.

Before moving on to consider more properties of the pair representation it is useful to pause for some comments:

i) Start with just two particles in a strong magnetic field interacting through the potential $V(z_1,z_2)$. This problem is easily solved in first quantization [1]. Diagonalization of $\hat{V}$ coincides with the lowest order degenerate perturbation theory if only the lowest Landau level is considered. The parameter of the expansion is the ratio between the energy scale of $\hat{V}$ (in the case of Coulomb energy e^2/ℓ_o) and the separation $\hbar\omega_c$ between Landau levels. This parameter can also be expressed by the ratio a_o/ℓ_o between the Bohr radius and the magnetic length. Higher Landau levels, whose effect will be neglected in the following, yields second order corrections.

ii) From the solution of the two particle problem, one obtains $b_T(u,q)$ as the *wave function* in the space of Landau orbitals. The quantum numbers q and T turn out to represent the relative and the total angular momentum respectively.

iii) In the case $w(Z)=1$, $I_Z(T,q)=\pi(T-q)!$ and $V_T(q)$ is independent of T:

$$V_T(q) = \epsilon_q = \frac{1}{2\pi q!} I_\xi(q) \tag{13}$$

iv) Considering the commutation properties of the operators c^+ and c, we get $f_T^+(q) = \sum_{u\le T/2}(b_T(u,q) \mp b_T(T-u,q))c_{T-u}^+ c_u^+$ where the upper

(lower) sign refer s to fermions (bosons). Then $f_T^+(q) \equiv 0$ for even values of q for fermions, while for bosons the same holds for odd q values. *As a consequence of the Pauli exclusion principle the relative angular momentum q can take only odd (even) values for fermions (bosons).*

In what follows we will specialize to the Fermi case. Consequently all the sums over the relative angular momentum will always be assumed to run on odd values.

The pair creation and destruction operators provide a very simple expression for most of the operators of physical interest. In particular the Coulomb interaction $V(z_1, z_2) = 1/|z_1 - z_2|$ is given by (in magnetic units)

$$\hat{H}_c = \sum_{T=1}^{\infty} \sum_q \epsilon_q f_T^+(q) f_T(q) \tag{14}$$

where

$$\epsilon_q = \frac{\sqrt{\pi}}{2} \frac{(2q)!}{(q!2^q)^2} \tag{15}$$

note that for $q \gg 1$ $\epsilon_q \propto 1/\sqrt{q}$ as in the classical case.

The pair correlation function can be expressed in the form 1 by replacing $V(z_1, z_2)$ with $\delta(r - z_1 + z_2)$, yielding

$$\hat{g}(r) = \sum_{T=1}^{\infty} \sum_q \frac{r^{2q}}{q!4^q} f_T^+(q) f_T(q) e^{-r^2/4} \tag{16}$$

and so on.

Once the number of pairs with a definite relative and total angular momentum $\hat{N}_T(q) = \langle f_T^+(q) f_T(q) \rangle$ is known on a given state for all T and q we are in a position to evaluate all correlation functions. This makes the pair representation quite appealing.

The main problem is however that $f_T^+(q)$ does not represent a true bosonic operator. In fact the commutation rules for $f_T^+(q)$ is not bosonic,

but has a residual term which contains a density excitation

$$\left[f_T(q), f_R^+(p)\right] = \delta_{T,R}\delta_{q,p} - 4\sum_u b_T(u,q)b_R(u,p)c^+_{R-u}c_{T-u} \qquad (17)$$

Loosely speaking, this reflects the fact that in the pair formalism not all degrees of freedom $f_T^+(q)$ are independent.

Suppose we want to build the ground state of an hamiltonian such as $\hat{H}_c$ for a system of $2n$ electrons. Intuitively one should occupy the lowest *pair orbitals* ϵ_q with n pairs consistently with the constraints on the total angular momentum. If boson-type commutation relations were satisfied $\hat{N}_T(q)$ would indeed count the pairs that have been used to build the state thus giving the lowest energy. However it is easy to check that

$$\sum_{T=1}^{\infty}\sum_q \hat{N}_T(q) = \sum_{T=1}^{\infty}\sum_{u=0}^{T} c^+_u c_u c^+_{T-u}c_{T-u} = n(2n-1) \qquad (18)$$

this is simply because the number of ways for making a pair out of $2n$ particles is $n(2n-1)$. So that while only n pairs have been used to build the state, this contains much more pairs than n (of order $2n^2$). Our trial ground state will almost certainly not be a good approximation. As an example, figure 1 shows the distribution of relative angular momentum $N(q) = \sum_T N_T(q)$ for $\nu = 1/3$ and 6 electrons. The physics this function describes is perhaps more evident if one considers its relation with the correlation function 16. Full dots refers to the true ground state while open squares to the ground state obtained in the zeroth order approximation in which the second term of the commutator is neglected. Even if this zeroth order ground state is made of three pair creation operators with $T = q = 15$ acting on the vacuum, it contains a lot of pairs with small relative angular momentum, including angular momentum one. The true ground state is instead characterized by a minimal number of pairs with the smallest value of q, since these give the largest contribution to the repulsive energy. In particular $N(1) = 0$ is fulfilled for $\nu = 1/3$ by the Laughlin state (open dots), as implied by $g(r) \propto r^6$

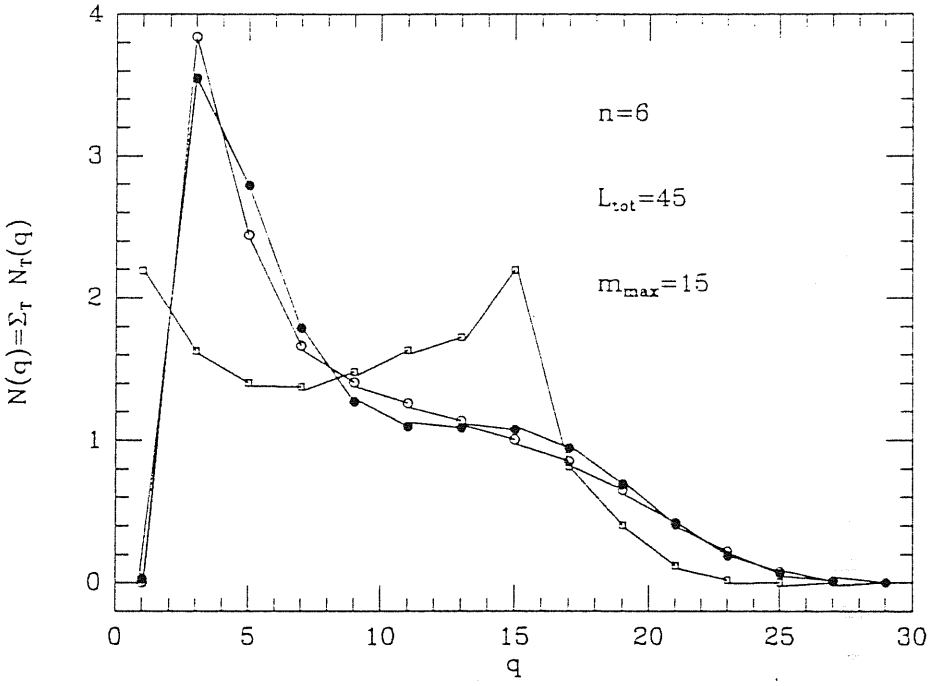

Figure 1: Plot of the distribution of relative angular momentum, $N(q) = \sum_T N_T(q)$, for a 6 fermion system at effective filling 1/3. L_{tot} is the total angular momentum, m_{max} is the highest Landau orbital considered. Lines are drawn only for guiding eyes. The full dots (•) refers to the exact ground state, open dots (○) to the $m = 3$ Laughlin state while the squares (□) refers to the state built with three pair creation operators of highest relative angular momentum.

and from equation 16. The approximate pair wave function does not satisfy $N(1) = 0$, so it has $g(r) \propto r^2$ which is energetically bad and fails to yield quantization at $\nu = 1/3$.

Equation 16 expresses a connection between the coefficients of different powers of r in $g(r)$ and the number of pairs with a definite relative angular momentum. These coefficients have been studied extensively by Yoshioka [3], in rectangular geometry (Landau gauge), who has found that the coefficient of r^2 and of r^4 (which is non zero in this gauge) decreases by decreasing ν and vanishes for $\nu \geq 1/3$. The same happens to the coefficient of r^6 and r^8 for $\nu \simeq 1/5$ and so on. In the pair picture this result suggests that the quantization in the FQHE is a consequence of the successive elimination of all the pairs with the smallest relative angular momentum. Loosely speaking approaching $\nu = 1/3$ from higher fillings, the degrees of freedom increase in the system and pairs with $q = 1$ are suddenly eliminated at the cost of introducing a large number of pairs with $q = 3$. For $\nu \leq 1/3$ there is no pair left in the $q = 1$ state. For lower fillings the same mechanism is applied for the pairs with $q = 3$ which are eliminated totally at $\nu = 1/5$. All of this is well known, and is just made more transparent by the pair language.

The hamiltonian itself can be approximated by the terms with $q = 1$ for $\nu \geq 1/3$. This approximation is the same as that of considering only the first component of Haldane [2] pseudopotentials. The interaction parameters V_m are exactly the correlation energies of pairs of particles with relative angular momentum m.

The $q = 1$ approximation coincides with replacing the Coulomb interaction with the hard-core potential $V(z_1, z_2) = 2\pi\nabla^2\delta(z_1 - z_2)$. Indeed performing the integral of equation 13 for this potential, we find $\epsilon_q = \delta_{q,1}$. We then recover the well known result [4] that Laughlin state is the exact ground state of this hamiltonian.

The simple form of two particles operators, such as the hamiltonian and the correlation function, suggests not only that pairs provide a relevant description in the FQHE, but that genuine pairing may take place in some filling regime. However it is obvious from our fresh example that simple pair condensation into a single T state (analogous to $K_{tot} = 0$ in the BCS case) will not work. Possible attempts, including extended Gutzwiller projectors, are now under study, aimed at eliminating the excessive near-neighbour repulsion implied by the unphysically large $N(q = 1)$ in the paired state.

3 Edge waves in the quantum Hall effect

We turn now to a specific problem where the pair formalism turns out to be helpful. We shall consider the spectrum of low lying excitations on the $\nu = 1$ quantum Hall state in disk geometry. These excitations have been called edge waves because they involve density fluctuations of the two dimensional electron gas at the boundary of the system. The characterization of these excitations is under current activities by many authors. For a review on the subject the interested reader is referred to a recent paper of X.G.Wen [5] where a general theory for edge excitations is discussed. The starting point of Wen's theory is a classical hydrodynamical approch where coordinates and canonical momenta yield, upon quantization, the creation operators of the edge modes. These turn out to behave as free bosons.

A different approach, based on first quantized operators, has been used by M.Stone. In a recent paper M.Stone et al. [6] have analyzed the energy spectrum by exact diagonalization for systems of up to 400 particles. If strictly only the lowest Landau level orbitals are considered, the ground state for $\nu = 1$ is that of the filled Landau level, whose first quantized form is (apart from the gaussian factors) just the Vandermonde determinant

$$\Psi_o(z_1, \ldots, z_n) = \prod_{i<j}(z_i - z_j) \tag{19}$$

whose total angular momentum is $L_o = \frac{1}{2}n(n-1)$.

Consider an excitation of angular momentum M. The Hilbert space of the system with total angular momentum $L = L_o + M$ is spanned, in first quantization, by the wave functions obtained by multiplying Ψ_o by symmetric polynomials of degree M. These in turn can be expressed in a unique way in terms of the *power sums*

$$\mathcal{S}_k(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^k \tag{20}$$

In their work M.Stone et al. [6] conjecture that these polynomials provide a bosonic description of the excitations of the quantum Hall system, in the sense that they correspond to the bosonic creation operators of edge modes. Moreover their numerical result strongly supports the expectation that edge excitations behave as free bosons, i.e. the energy spectrum for $L = L_o + M$ reduces, with excellent accuracy, to

$$E = E_o - \sum_k n_k \omega_k \tag{21}$$

where n_k are (integer) bosonic occupation numbers such that $\sum_k n_k k = M$ and $\omega_k \geq 0$ are *single particle* energies.

In the following we are going to reformulate the problem in the language of second quantization and to verify the validity of this picture. Our approach differs from the one of X.G.Wen [5]; our starting point are the electron creation operators. This method allows to recover qualitatively and quantitatively the results of M.Stone et al. [6].

First of all we notice that S_k correspond to single particle ladder operators

$$S_k^+ = \sum_{m=0}^{\infty} \sqrt{\frac{(m+k)!}{m!}} c_{m+k}^+ c_m \tag{22}$$

The conjugate operators S_k^- are easily defined. It is easy to check that in general this set of operators can not be interpreted as bosonic creation and

destruction operators, because of their commutation rules

$$[S_k^-, S_{k+j}^+] = \sum_{m=0}^{\infty} \left[\frac{(m+k+j)!}{(m+j)!} - \frac{m!}{(m-k)!} \right] \sqrt{\frac{(m+j)!}{m!}} c_{m+j}^+ c_m. \qquad (23)$$

So these operators do not create in general an orthogonal basis of states for a given M.

However we now show that, in the limit $n \to \infty$ and for $\nu = 1-\epsilon$, they do form a bosonic set of creation operators in the sense that the overlap between states with different occupation numbers (i.e. with different combinations of S_k^+) vanish as $n \to \infty$.

Let us consider the state $|k\rangle$ defined as

$$|k\rangle = A_k(n) S_k^+ |0\rangle$$

$A_k(n)$ is a constant defined by the normalization condition

$$\begin{aligned} \langle k|k\rangle &= A_k^2(n) \langle 0| S_k^- S_k^+ |0\rangle = A_k^2(n) \sum_{m=n-k}^{n-1} \frac{(m+k)!}{m!} = 1 \\ \Rightarrow \quad A_k(n) &= \left[\sum_{m=n-k}^{n-1} \frac{(m+k)!}{m!} \right]^{-1/2} \end{aligned}$$

so that $A_k(n) \propto n^{-k/2}$ for $n \to \infty$. For the state $|k,j\rangle = A_{k,j}(n) S_k^+ S_j^+ |0\rangle$, we easily get that $A_{k,j}(n) \propto n^{-(k+j)/2}$. Now consider the overlap

$$\begin{aligned} \langle k,j|k+j\rangle &= A_{k,j}(n) A_{k+j}(n) \langle 0| S_j^- S_k^- S_{k+j}^+ |0\rangle = \\ &= A_{k,j}(n) A_{k+j}(n) \langle 0| \left[S_j^-, \left[S_k^-, S_{k+j}^+ \right] \right] |0\rangle = \ldots \\ &= A_{k,j}(n) A_{k+j}(n) \sum_{m=n-j}^{n-1} \left[\frac{(m+k+j)!}{m!} - \frac{(m+j)!}{(m-k)!} \right] \end{aligned}$$

the leading term in the sum is not m^{k+j}, but is m^{k+j-1}, so that for $n \gg k, j$ the sum is proportional to n^{k+j-1} and the matrix element $\langle j,k|k+j\rangle \propto$

$n^{-1} \to 0$. The same thermodynamic limit orthogonality between states with different combinations of S_k can be verified for generic states

$$|n_k\rangle = A_{\{n_k\}}(n) \prod_k \left(S_k^+\right)^{n_k} |0\rangle$$

The reason is that the norm of this state can be evaluated using a sort of Wick theorem, i.e. contracting all operators into commutators in all possile ways. The leading power of n in $A^{-2}_{\{n_k\}}(n)$ is obtained when all operators S_k^+ are coupled to their conjugates S_k^-. This term of $A^{-2}_{\{n_k\}}(n)$ is of order n^M with $M = \sum_k kn_k$, all other terms are of order at least n^{-1} with respect to this. On calculating the overlap between two different states this leading term cannot occur so that the overlap is at least of order n^{-1} smaller.

We conclude that in the limit $n \to \infty$ and for $M \ll n$ the operators S_k^+ and their conjugate can then be regarded as creation and destruction operators of *particles* obeying bose statistics.

Consider now the hamiltonian

$$\hat{H} = \sum_{T=1}^{\infty} \sum_q V_T(q) \hat{N}_T(q)$$

The operator $\hat{N}_T(q) = f_T^+(q) f_T(q)$ in the $\nu = 1$ state has the following value

$$N_T(q) = \langle 0|\hat{N}_T(q)|0\rangle = 2 \sum_u b_T^2(u, q) \tag{24}$$

where the sum runs from $\max(0, T-n+1)$ to $\min(T, n-1)$ so that $N_T(q) = 1$ for $T < n$ and $N_T(q) = 0$ for $T > 2n - 3$.

If $\hat{H}$ were the hamiltonian of a system of free bosons we would have

$$\left[H, b_k^+\right] = \omega_k b_k^+$$

where b_k^+ is the boson creation operator and ω_k its spectrum. We have then to evaluate the commutator of the hamiltonian with S_k^+ to test the free

boson picture proposed in ref. [6]:

$$\left[\hat{H}, S_k^+\right] = \sum_{T=1}^{\infty} \sum_q V_T(q) \left(f_T^+(q) \left[f_T(q), S_k^+ \right] + \left[f_T^+(q), S_k^+ \right] f_T(q) \right) =$$

$$= \sum_{T,q} \left(V_{T+k}(q) f_{T+k}^+(q) \left[f_{T+k}(q), S_k^+ \right] + V_T(q) \left[f_T^+(q), S_k^+ \right] f_T(q) \right) \quad (25)$$

Let us evaluate the first commutator:

$$\left[f_{T+k}(q), S_k^+ \right] = \sum_{u=0}^{T+k} \sum_{m=0}^{\infty} b_{T+k}(u,q) \sqrt{\frac{(m+k)!}{m!}} \left[c_u c_{T+k-u}, c_{m+k}^+ c_m \right] =$$

$$= \ldots = 2 \sum_{u=0}^{T} b_{T+k}(u+k,q) \sqrt{\frac{(u+k)!}{u!}} c_u c_{T-u} \quad (26)$$

We now define coefficients $G_{T,k}(p,q)$ such that

$$2 b_{T+k}(u+k,q) \sqrt{\frac{(u+k)!}{u!}} = \sum_p G_{T,k}(p,q) b_T(u,p) \quad (27)$$

so that by definition

$$\left[f_{T+k}(q), S_k^+ \right] = \sum_p G_{T,k}(p,q) f_T(p). \quad (28)$$

To find these coefficients we multiply equation 27 by $2b_T(u,r)$ and sum over u. Performing the summation on u in the second term, using the relation 12, we get a delta function $\delta_{p,r}$ so that

$$G_{T,k}(r,q) = 4 \sum_{u=0}^{T} b_T(u,r) \sqrt{\frac{(u+k)!}{u!}} b_{T+k}(u+k,q) \quad (29)$$

With a similar procedure we find

$$\left[f_T^+(q), S_k^+ \right] = -2 \sum_{u=0}^{T} b_T(u,q) \sqrt{\frac{(u+k)!}{u!}} c_{T-u}^+ c_{u+k}^+$$

this takes a form very similar to equation 28 using the coefficients $G_{T,k}(q,p)$. The only difference is now that the sum runs only from 0 to T, so in order

to reconstruct $f^+_{T+k}(p)$ we have to add and subtract the terms with $u = T+1, \ldots, T+k$. Carrying out these operations, after a little algebra we get:

$$\left[f^+_T(q), S^+_k\right] = -\sum_p G_{T,k}(q,p) f^+_{T+k}(p) + \sum_{u=0}^{k-1} b_T(u,q) \sqrt{\frac{(T+k-u)!}{(T-u)!}} c^+_{T+k-u} c^+_u$$

Note that the coefficients in the first term are the same of equation 28, but the indexes p and q are interchanged. We can now evaluate the commutator of equation 25 and after some simple algebra we get

$$\left[\hat{H}, S^+_k\right] = \sum_{T=1}^{\infty} \sum_{q,p} G_{T,k}(q,p) \left[V_{T+k}(p) - V_T(q)\right] f^+_{T+k}(p) f_T(q) + \hat{C}_2 \quad (30)$$

with

$$\hat{C}_2 = \sum_{T,q} V_T(q) \sum_{u=0}^{k-1} \sum_{s=0}^{T} b_T(u,q) b_T(s,q) \sqrt{\frac{(T+k-u)!}{(T-u)!}} c^+_{T+k-u} c^+_u c_s c_{T-s} \quad (31)$$

Here the sum on u is restricted only to $u = 0, \ldots, k-1 \ll n$, and these orbitals are always occupied so that $\hat{C}_2$ can be non zero only if $s = u$ or $T - s = u$. The second term of the commutator is then

$$\hat{C}_2 = 2 \sum_{T,q} \sum_{u=0}^{k-1} V_T(q) b^2_T(u,q) \sqrt{\frac{(T+k-u)!}{(T-u)!}} c^+_{T+k-u} c_{T-u}$$

when this is applied to a $\nu = 1 - \epsilon$ state, the relevant terms in the sum are those for $T \sim n$. In the case of the hard core potential only the $q = 1$ term occurs, and $\hat{C}_2$ can be expressed explicitely as

$$\hat{C}_2 = 2\pi \sum_{T=1}^{\infty} \sum_{u=T-k+1}^{T} \binom{T}{u} 2^{-T} \frac{(T-2u)^2}{T} \sqrt{\frac{(u+k)!}{u!}} c^+_{u+k} c_u$$

it is easy to check that the coefficient in front of $\sqrt{\frac{(u+k)!}{u!}} c^+_{u+k} c_u$ here is exponentially small in n, while we will see that the first part of the commutator gives a contribution of the same form whose coefficient vanish as a power of n.

We conclude that for the hard core potential $\hat{C}_2$ is totally irrelevant. It can be shown that this is also the case for $q \ll n$. However if this is the case also for all q values is a more complex task. Motivated by the fact that the relevant part of the hamiltonian is the one with $q \ll n$ we will assume this to hold in general and we will neglect the second term of equation 31 in the following. In this approximation

$$\left[\hat{H}, S_k^+\right] = \sum_{T=1}^{\infty} \sum_{q,p} G_{T,k}(q,p) \left[V_{T+k}(p) - V_T(q)\right] f_{T+k}^+(p) f_T(q) \qquad (32)$$

Before continuing with the general case we spend some word on the particular role of the ladder operators $S_1^{\pm}$. Consider in fact the commutator of S_1^+ with the hamiltonian. We need to evaluate explicitly

$$\begin{aligned} G_{T,1}(q,p) &= 4 \sum_{u=0}^{T} b_T(u,q) \sqrt{u+1} b_{T+1}(u+1,p) = \\ &= \frac{\sqrt{2(T-q)!q!(T+1-p)!p!}}{T!2^T} \frac{\partial_z^p}{p!} \frac{\partial_w^q}{q!} (1-z) \cdot \\ &\quad \cdot \sum_{u=0}^{T} \binom{T}{u} (1+z+w+zw)^{T-u} (1-z-w+zw)^u \Big|_{z=w=0} = \\ &= \frac{\sqrt{2(T-q)!q!(T+1-p)!p!}}{T!} \frac{\partial_z^p}{p!} \frac{\partial_w^q}{q!} \cdot (1-z)(1+zw)^T \Big|_{z=w=0} \end{aligned}$$

carring out the derivatives we find that $G_{T,1}(q,p) \neq 0$ only if $p = q$ or $p = q+1$. The second possibility is ruled out by the fact that p and q must both be odd integers, so that we get

$$G_{T,1}(q,p) = \delta_{p,q} \sqrt{2(T-q+1)}$$

when this is inserted in equation 32 we find:

$$\left[\hat{H}, S_1^+\right] = \sum_{T=1}^{\infty} \sum_{q} \sqrt{2(T-q+1)} \left(V_{T+k}(q) - V_T(q)\right) f_{T+1}^+(q) f_T(q)$$

so that, provided $V_T(q) = \epsilon_q$ does not depend on T, or equivalently $V(z_1, z_2)$ depends only on $|z_1 - z_2|$, this expression vanishes. Only the $\hat{C}_2$ term contributes to the commutator, yielding an operator with exponentially small

matrix elements for $q \ll n$

$$\begin{aligned}\left[\hat{H}, S_1^+\right] &= 2\sum_{T=1}^{\infty}\sum_q \epsilon_q b_T^2(0,q)\sqrt{T+1}c_{T+1}^+ c_T \\ &= \sum_{T=1}^{\infty}\sum_q \epsilon_q \binom{T}{q} 2^{-T}\sqrt{T+1}c_{T+1}^+ c_T\end{aligned}$$

Then *in the approximation of equation 32 and if $V_T(q)$ does not depend on T, the operator S_1^+ commutes with the hamiltonian. This means that S_1^+ corresponds to a boson creation operator with $\omega_1 = 0$.* When this operator acts on an eigenstate of $\hat{H}$ for $L = L_o + M$, it produces an eigenstate for $M' = M + 1$ with the same eigenvalue. This result has been observed numerically [4, 6]. The excitation mode produced by S_1^+ concerns the motion of the center of mass ($\mathcal{S}_1(z_1, \ldots, z_n)$ is indeed the coordinate of the center of mass). Even if the interaction lifts the degeneracy of the lowest Landau levels for the electrons, the energy levels of the center of mass remain degenerate; $S_k^\pm$ are the ladder operators for these levels. S_1^- plays also another important role. *It can be shown (see appendix B) that S_1^- identifies homogeneous states.* For homogeneous state we mean those states which are described in first quantization by a wave function whose polynomial part has the property that

$$\Psi(z_1 + c, \ldots, z_n + c) = \Psi(z_1, \ldots, z_n).$$

This relation in second quantized form becomes

$$S_1^-|\Psi\rangle \equiv 0. \tag{33}$$

The operators $S_1^\pm$ characterize the translational properties of the system in the same way as the quantum number L [2] does in spherical geometry.

Finally note that the commutation rules of these operators with the others take a simple form

$$\left[S_1^-, S_k^+\right] = kS_{k-1}^+ \text{ for } k \geq 1. \tag{34}$$

Let us now return to equation 32. This does not have the form typical of free bosons because the right hand is not proportional to S_k^+. Note however that

$$\begin{aligned} f_{T+k}^+(p) f_T(q) &= f_T(q) f_{T+k}^+(p) - \left[f_T(q), f_{T+k}^+(p) \right] = \qquad (35) \\ &= f_T(q) f_{T+k}^+(p) + 4 \sum_{u=0}^{T} b_T(u,q) b_{T+k}(u+k,p) c_{u+k}^+ c_u \end{aligned}$$

The range of T which is relevant at $\nu = 1$ is from n to $2n-3$ because two electrons must be destroyed by $f_T(q)$ and two have to be created by $f_{T+k}^+(p)$. When the above operator is evaluated for $\nu \simeq 1$ the first term is very small because $f_{T+k}^+(p)$ has no room to create a pair of electrons. On the full Landau level this is possible only if $T > 2n-k$. The left hand side then can be approximated with the second term in the right hand side, neglecting the first which only introduces small corrections to the energy. When this approximation is used in equation 32 we find

$$\begin{aligned} \left[\hat{H}, S_k^+ \right] &= 4 \sum_{T=n}^{2n-3} \sum_{u=0}^{T} \left\{ \sum_p V_{T+k}(p) \left[\sum_q G_{T,k}(q,p) b_T(u,q) \right] b_{T+k}(u+k,p) + \right. \\ &- \left. \sum_q V_T(q) \left[\sum_p G_{T,k}(q,p) b_{T+k}(u+k,p) \right] b_T(u,q) \right\} c_{u+k}^+ c_u \end{aligned}$$

using equation 27 to simplify the expression in the first square brackets and a similar equation for the second term, we find:

$$\begin{aligned} \left[\hat{H}, S_k^+ \right] &= 8 \sum_u \left[\sum_{T,q} V_{T+k}(q) b_{T+k}^2(u+k,q) - \sum_{T,q} V_T(q) b_T^2(u,q) \right] \cdot \\ &\cdot \sqrt{\frac{(u+k)!}{u!}} c_{u+k}^+ c_u = \\ &= \sum_u (W_{n+k}(u+k) - W_n(u)) \cdot \sqrt{\frac{(u+k)!}{u!}} c_{u+k}^+ c_u \qquad (36) \\ &\text{where } W_n(u) = 8 \sum_{T=n}^{2n-3} \sum_q V_T(q) b_T^2(u,q) \qquad (37) \end{aligned}$$

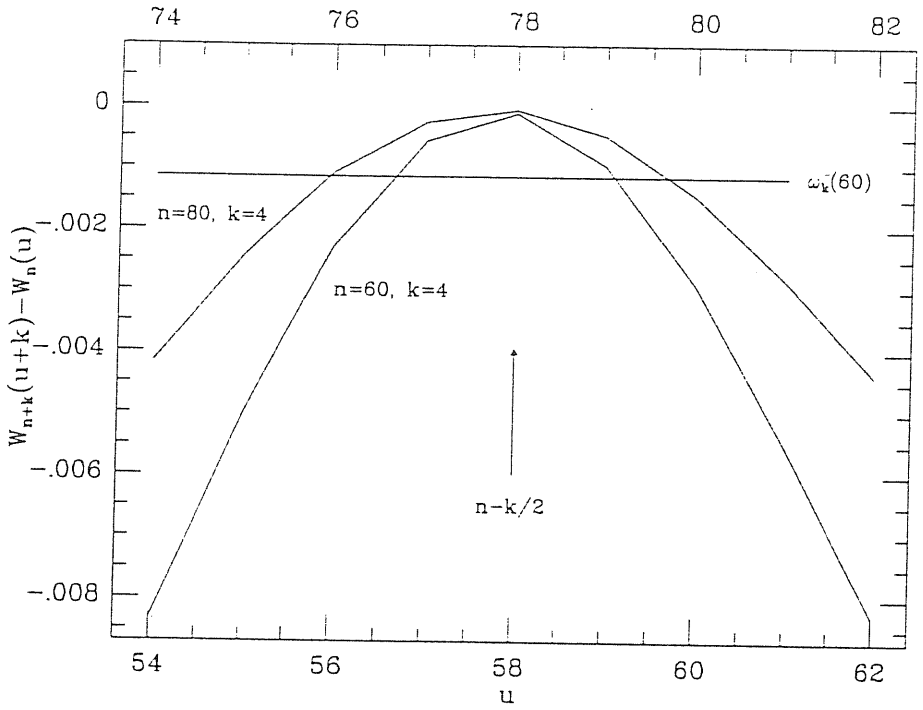

Figure 2: Plot of $W_{n+k}(u+k) - W_n(u)$ versus u for the hard core potential for $n = 60$ and $n = 80$. The horizontal line is the true frequency for $n = 60$. The data for $n = 80$ have been shifted in u to make the two maxima coincide. Note that this quantity is stationary at $u \sim n - k/2$, where it approximates well the true $\omega_k(n)$ for $n = 60$ and $k = 4$.

We have then arrived at something very similar to S_k^+. Provided that $W_{n+k}(u+k) - W_n(u)$ is almost constant in the range of interest in u (from $n-k$ to n), it can be moved out of the sum thus yielding

$$\left[\hat{H}, S_k^+\right] \simeq -\omega_k(n) S_k^+ \tag{38}$$

$$\text{with } \omega_k(n) = W_n(n) - W_{n+k}(n+k). \tag{39}$$

In figure 2 we show a plot of $W_{n+k}(u+k) - W_n(u)$ versus u for the hard core potential. We see that in the range of interest ($u \sim n - k/2$) this function

displays a maximum which becomes flatter and flatter as n increases. Moreover the numerical values around the maximum are close to the exact one for $n = 60$. We can therefore state that *in the limit* $M/n \ll 1$ *equation 38 is correct and the excitations of the electron gas can be described as a gas of free bosons whose creation operators are* S_k^+.

Figure 2 also shows that the definition of $\omega_k(n)$, equation 39, is not very precise. A better definition is given by

$$\begin{aligned} \omega_k^{th}(n) &= A_k^{-2}(n)\langle 0|S_k^- \left[\hat{H}, S_k^+\right]|n\rangle = \\ &= A_k^{-2}(n) \sum_{m=n-k}^{n-1} [W_n(u) - W_{n+k}(u+k)] \frac{(m+k)!}{m!} \end{aligned} \tag{40}$$

with $A_k(n)$ given by equation 24.

The frequencies ω_k are positive numbers and they are increasing function of k. This means that the ground state of the system with total angular momentum $L_o + M$ is given by $S_M^+|0\rangle$ as conjectured by Stone et al. [6]. Using equation 34 we can say also that this state will be homogeneous, in the sense of equation 33, for $n \to \infty$.

We can now compare the above result with exact numerical diagonalization results. We deal first with the hard core potential that allows to treat a relatively large number of particles. As shown in table 1 it is not necessary to go to very high number of particles to obtain a good test of the above picture. Already for 20 electrons the boson picture seems rather good and it becomes more and more precise for larger values of n. Clearly it get worse and worse increasing k as can be seen e.g. from the decreasing of the overlap.

The frequencies have been computed subracting the energy of the filled Landau level to the energies of the state $S_k^+|0\rangle$ and of the ground state with $L = L_o + k$. The last column contains the overlap between these states.

We will give later a justification for the deviation of theoretical values, ω_k^{th} in the table, obtained from equation 41, from the true values. Here we

k	$\omega_k(S_k^+\|0\rangle) \times 10^2$	$\omega_k(\|\text{true}\rangle) \times 10^2$	$\omega_k^{th} \times 10^2$	$\langle\text{true}\|S_k^+\|0\rangle$
$n = 20$				
1	0.0000	0.0000	-0.0003	1.0000
2	0.1832	0.1836	0.0567	0.9987
3	0.6854	0.6930	0.3789	0.9925
4	1.5533	1.5970	0.9966	0.9786
5	2.7246	2.8777	1.7698	0.9544
6	4.0445	4.4360	2.5447	0.9177
7	5.3212	6.1164	3.2983	0.8675
$n = 40$				
1	0.0000	0.0000	0.0000	1.0000
2	0.0328	0.0329	0.0110	0.9997
3	0.1274	0.1278	0.0749	0.9981
4	0.3042	0.3064	0.2181	0.9946
5	0.5724	0.5807	0.4450	0.9884
6	0.9278	0.9516	0.7393	0.9786
7	1.3533	1.4095	1.0735	0.9643
$n = 60$				
1	0.0000	0.0000	0.0000	1.0000
2	0.0120	0.0120	0.0040	0.9998
3	0.0470	0.0471	0.0276	0.9990
4	0.1140	0.1144	0.0823	0.9975
5	0.2192	0.2206	0.1740	0.9948
6	0.3651	0.3694	0.3014	0.9904
7	0.5504	0.5610	0.4580	0.9839

Table 1: Mode frequencies for $n = 20, 40, 60$ electrons interacting with an hard core potential. ω_k^{th} refers to equation 40. The last column contains the overlap between the ground state of the hard core potential $|\text{true}\rangle$ for $L = L_o + k$ and the state $|k\rangle = A_k(n)S_k^+|0\rangle$.

state	$(E_0 - E) \times 10^2$	overlap
$S_1^+ S_1^+ S_1^+ S_1^+ \|0\rangle$	0.0000	0.9129
$S_1^+ S_1^+ S_2^+ \|0\rangle$	0.0120	0.9992
$S_2^+ S_2^+ \|0\rangle$	0.0240	0.9997
$S_1^+ S_3^+ \|0\rangle$	0.0470	0.9983
$S_4^+ \|0\rangle$	0.1144	0.9975

Table 2: Eigenstates of the hard core hamiltonian classified according to the boson picture. E_0 is the full Landau level energy and E is the energy eigenvalue. The overlap is between the energy eigenstate and the state in the first column.

remark only the clear trend of convergence in n displayed by the data.

Table 1 refers only to the ground state. In table 2 we deal with all the eigenstate of the hamiltonian for $M = 4$. The overlaps of these states with those built with the operators S_k^+ are remarkably good. Also note that the energies correspond exactly with the sum of the frequencies of single bosons for composite states.

In the case of the hard core potential $\omega_k(n)$, using equation 39, is given by

$$\omega_k(n) = \frac{4}{\pi} \sum_{T=n}^{2n-1} \left[b_T^2(n,1) - b_{T+k}^2(n+k,1) \right] \tag{41}$$

We can extract the asymptotic behaviour of this frequency for a fixed k as $n \to \infty$. Note indeed that $2b_T^2(n,1) = \binom{T}{n}(T-2n)/T$ so that

$$2\left[b_T^2(n,1) - b_{T+k}^2(n+k,1) \right] = \binom{T}{n} 2^{-T} \left[\frac{(T-2n)^2}{T} - \frac{(T+k)!n!}{T!(n+k)!2^k} \frac{(T-2n-k)^2}{T+k} \right]$$

In the limit of large n we can make the following substitutions in the above formula:

$$\binom{T}{n} 2^{-T} \Rightarrow \sqrt{\frac{2}{\pi T}} \exp\left[-\frac{(n-T/2)^2}{T/2} \right] + O\left(\frac{k}{T}\right)$$

$$\begin{aligned} \frac{(T+k)!n!}{T!(n+k)!2^k} &\Rightarrow \left(\frac{T}{2n}\right)^k + O\left(\frac{k}{T}\right) \\ \frac{(T-2n-k)^2}{T+k} &\Rightarrow \frac{(T-2n)^2}{T} + O\left(\frac{k}{T}\right) \end{aligned} \tag{42}$$

with these relations, using the variable $x = T/2n$, we get

$$b^2_{2xn}(n,1) - b^2_{2xn+k}(n+k,1) \simeq \sqrt{\frac{n}{\pi}} \frac{(1-x^2)(1-x^k)}{x^{3/2}} e^{-2n\frac{(1-x)^2}{x}} \tag{43}$$

In order to obtain $\omega_k(n)$ we need to perform an integral on x from $1/2$ (i.e. $T = n$) to 1 (corrisponding to $T = 2n - 1$). The main contribution to this integral comes from the region $x \simeq 1$ because of the exponential function. In this interval $1 - x^k \simeq k(1-x) + O(1-x)^2$. The only dependence of $\omega_k(n)$ on k, at this stage of the calculation, is contained in this term. The predicted k dependence of $\omega_k(n)$ is then linear. A more complex dependence is expected using equation 41.

Changing the integration variable to $z = \sqrt{2n}(1-x)$ in equation 43 it is easy to see that

$$\omega_k(n) \propto n^{-3/2}.$$

If $\omega_k(n)$ is referred to single particle energies then $\omega_k(n) \propto n^{-5/2}$. We have then recovered analytically the result of ref. [6] for the asymptotic dependence of the frequencies on n. The main contribution to $\omega_k(n)$ comes from the region $T \sim 2n$. The approximation of neglecting the first term of the right hand side of equation 36 is relevant in this interval. So while the above approximation works very well for the total energy and for the eigenstate it may not work so well for $\omega_k(n)$ which is a difference of energies.

Let us now turn our attention to the Coulomb potential. The calculations are more difficult because $b_T(u,q)$ is a complex expression for a general $q > 1$. Moreover all q values are involved. The asymptotic behaviour of $\omega_k(n)$ with respect to n, obtained by Stone et al. [6], differs from the hard core case, having one extra power of n. This additional power may result from the

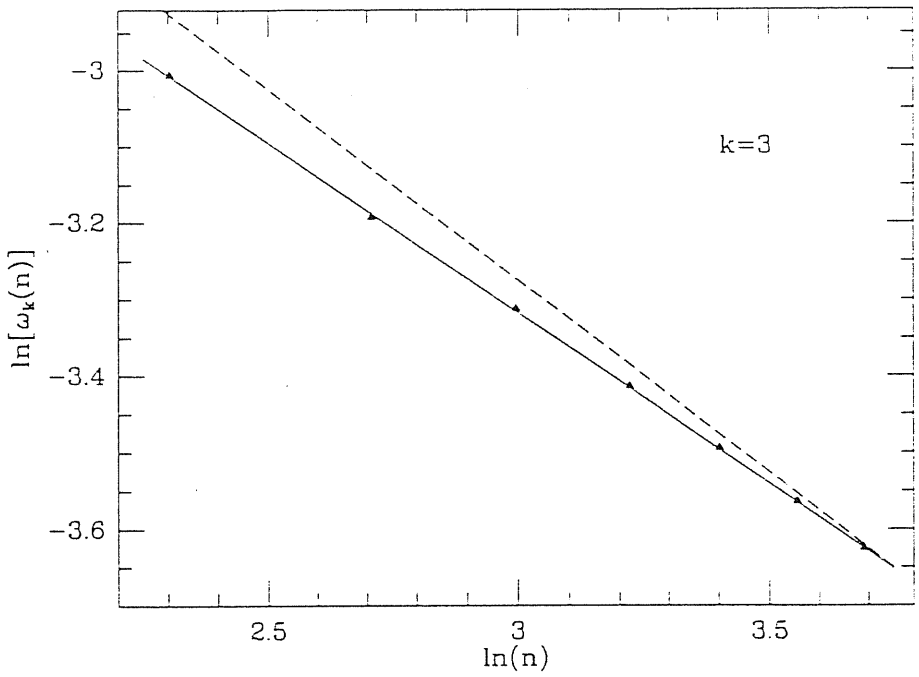

Figure 3: Log-log plot of $\omega_k(n)$ in the case of the Coulomb potential for $k = 3$. The range of n is from 10 to 40 the dashed line refers to $\omega_k(n) \sim sqrtn$.

extra summation over q for the Coulomb potential. We have computed $\omega_k(n)$ for the Coulomb interaction up to $n = 40$. For higher values of n round-off errors become significant because $b_T(u, q)$ is a sum of terms of different signs. In figure 3 we show a log-log plot of $\omega_k(n)$ calculated from equation 39 vs. n for $k = 3$. A linear fit of this plot yields

$$\omega_k(n) \sim n^{-0.4445}.$$

As before this refers to the total energy, so that the exponent $\gamma = 0.4445\ldots$ has to be increased by unity when referring to single particle energies. The power γ for different values of k is almost exactly the same. When this is

compared to the numerical result [6], $\gamma = 0.5$, we find a good agreement considering that the asymptotic value of γ is reached from below in ref. [6].

Note, in the end, that all the results of this section, apart from the propertier of $S_k^{\pm}$ and on numerical data, could have been obtained for a generic ladder operator $L_k^+ = \sum_m \ell_k(m) c_{m+k}^+ c_m$.

In particular the commutator of L_k^+ with the hamiltonian still yields, in the same approximations, an operator which is very similar to L_k^+ in the same sense of equation 37. The choice $\ell_k(m) = \sqrt{(m+k)!/m!}$ is however the only one related by second quantization to the first quantized one particle ladder operators $\mathcal{S}_k$. The properties of the envelope functions $b_T(u,q)$, more than those of $\ell_k(m)$, play instead a crucial role.

4 Conclusion

The pair formalism seems to be a very natural one for the description of the physics of interacting electrons in the lowest Landau level. The expressions of two body operators become very simple in this formalism. However the problem is still very difficult due to the ugly commutation rules between pair creation and destruction operators. This new perspective calls then for a different theoretical approach which has to deal with the correlations built in the commutator. These correlations contain more than the Pauli exclusion principle of single electrons (or the Bose statistic for bosons). It contains also the fact that $n(n-1)/2$ pairs are built using only n particles. The true independent degrees of freedom of the system pertain to real particles and this poses overwhelming restriction to the motion of pairs.

As we have shown in the previous section the pair formalism is nevertheless a very useful analytical tool. All the above complications are lumped up in the pair operators and in the coefficients $b_T(u,q)$. With the basic properties of these coefficients it is possible to handle expressions that would be very complicated otherwise. All this results in a very efficient tool for both

rigorous results and for approximation schemes. In particular the pair picture should eventually be relevant for the description of even denominator filling fractions and at $\nu = 1/2$. Also the extension of the above results on edge waves to other incompressible quantum states, as the $\nu = 1/3$ state, is under current investigation.

A The pair envelope function $b_T(u,q)$

In this appendix we analyze the properties of the coefficients $b_T(u,q)$. To enstablish the relation 10 between $b_T(u,q)$ and $b_T(T-u,q)$ it is sufficient to note that with the substitution $z \to z' = -z$ in equation 8 we obtain exactly $b_T(T-u,q)$ apart from a factor $(-1)^q$ which comes from the fact that $\partial_z^q = (-1)^q \partial_{z'}^q$.

To prove the second relation (11) we observe that

$$\frac{\partial_z^q}{q!}(1+z)^{T-u}(1-z)^u = \sum_{k=0}^{\min(u,q)} (-1)^k \binom{u}{k}\binom{T-u}{q-k}$$

which, once introduced in equation 8, yields

$$b_T(u,q) = \sum_{k=0}^{\min(u,q)} (-1)^k \frac{\sqrt{q!(T-q)!u!(T-u)!2^{-t-1}}}{k!(u-k)!(q-k)!(T-u-q+k)!}.$$

This is the same as $b_T(q,u)$ because the above expression is totally symmetric in u and q.

The last relation (12) follows from the equation

$$\begin{aligned} 2\sum_{u=0}^{T} b_T(u,q)b_T(u,p) &= \binom{T}{q}^{-1/2}\binom{T}{p}^{-1/2}\frac{\partial_z^q}{q!}\frac{\partial_w^p}{p!}2^{-T}\sum_{u=0}^{T}\binom{T}{u}\cdot \\ &\quad \cdot (1+z+w+zw)^u(1-z-w+zw)^{T-u}\Big|_{z=w=0} = \\ &= \binom{T}{q}^{-1/2}\binom{T}{p}^{-1/2}\frac{\partial_z^q}{q!}\frac{\partial_w^p}{p!}(1+zw)^T\Big|_{z=w=0} \end{aligned}$$

the p derivatives on w, evaluated at $w = 0$, leave a factor $\binom{T}{p} z^p$. This has to be derived q times with respect to z and then evaluated at $z = 0$. The result is clearly zero if $q \neq p$ and 1 if $q = p$.

B Homogeneous states

To prove equation 33 we first consider a Slater determinant of single particle noninteracting states. The polynomial part of such state is given by

$$\Psi_{\{\alpha_i\}}(z_1, \ldots, z_n) = \det \left\{ \frac{z_j^{\alpha_i}}{\sqrt{2\pi \alpha_i! 2^{\alpha_i}}} \right\}$$

the set of integers $\{\alpha_i, i = 1, \ldots, n\}$ identifies the state. Let us now analyze the state obtained substituting to each z_j in the above equation $z_j + x$:

$$\begin{aligned} \Psi_{\{\alpha_i\}}(z_1 + x, \ldots, z_n + x) &= \det \left\{ \frac{(z_j + x)^{\alpha_i}}{\sqrt{2\pi \alpha_i! 2^{\alpha_i}}} \right\} = \\ &= \det \left\{ \sum_{k_i=0}^{\alpha_i} \binom{\alpha_i}{k_i} \frac{x^{k_i} z_j^{\alpha_i - k_i}}{\sqrt{2\pi \alpha_i! 2^{\alpha_i}}} \right\} \end{aligned}$$

using the multilinear property of determinants [7], the sum on k_i can be moved out of the determinant thus yielding

$$\begin{aligned} \Psi_{\{\alpha_i\}}(z_1 + x, \ldots) &= \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_n=0}^{\alpha_n} \prod_{h=1}^{n} \frac{x^{k_h}}{k_h! 2^{k_h/2}} \sqrt{\frac{\alpha_h!}{(\alpha_h - k_h)!}} \cdot \\ &\quad \cdot \det \left\{ \frac{z_j^{\alpha_i - k_i}}{\sqrt{2\pi (\alpha_i - k_i)! 2^{\alpha_i - k_i}}} \right\} = \\ &= \sum_{Q=0}^{L} \frac{1}{Q!} \left(\frac{x}{\sqrt{2}} \right)^Q \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_n=0}^{\alpha_n} \frac{Q!}{k_1! \cdots k_n!} \prod_{h=1}^{n} \sqrt{\frac{\alpha_h!}{(\alpha_h - k_h)!}} \cdot \\ &\quad \cdot \det \left\{ \frac{z_j^{\alpha_i - k_i}}{\sqrt{2\pi (\alpha_i - k_i)! 2^{\alpha_i - k_i}}} \right\} \delta(Q - \sum_i k_i) \end{aligned}$$

Let us analyze the Q^{th} term of this sum. The determinant in this sum is, apart from the sign, a many body state $\Psi_{\{\alpha_i - k_i\}}(z_1, \ldots, z_n)$. This slater

determinant is obtained by the original one by lowering the orbitals α_i by k_i steps. This is obtained by applying a combination of lowering operator S_k^- to the original state. The coefficients $\sqrt{\alpha_i!/(\alpha_i - k_i)!}$ are indeed those of the ladder operators. The sum on all values of k_i consistent with the restriction $\sum_i k_i = Q$ indicate that this ladder operator acts in all possible ways on the single particle orbitals. The form of this combinations of ladder operators is given by the combinatorial coefficient $Q!/(k_1! \ldots k_n!)$. This counts the number of ways in which Q lowering operators can act k_1 times on α_1, k_2 times on α_2,... k_n times on α_n. This means that the above equation is the first quantization expression for the action of the operator $(S_1^-)^Q$ on the state $|\vec{\alpha}\rangle$ which corresponds to $\Psi_{\{\alpha_i\}}$. The above equation can be translated in the language of second quantization as follows:

$$\hat{T}_x|\Psi\rangle = \sum_{Q=0}^{L} \frac{1}{Q!}\left(\frac{xS_1^-}{\sqrt{2}}\right)^Q |\Psi\rangle = \exp\left[\frac{x}{\sqrt{2}}S_1^-\right]|\Psi\rangle$$

where $\hat{T}_x$ represent the *translation* operator. $|\Psi\rangle$ can be any many particle state in the lowest Landau level because $\hat{T}_x$ is a linear operator so that if the above equation holds for Slater determinants it will also hold for any linear combinations of them. From the above equation it is clear that a necessary and sufficient condition for $\Psi(z_1 + x, \ldots, z_n + x) = \Psi(z_1, \ldots, z_n)$ is that $S_1^-|\Psi\rangle = 0$.

We have to remember however that in all the above discussion the exponential part of the wavefunction has been neglected. The above results hold only for the polynomial part of many body wavefunctions in the lowest Landau level. In the plasma analogy the exponential part plays the role of the confining potential on the disk, while the polynomial part represents the interaction between particles. We may say that the operator $\hat{T}_x = \exp(xS_1^-/\sqrt{2})$ is a translation operator in the loose sense that it translate the electron gas by a constant x with respect to the ion background of the disk.

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