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SCATTERING AMPLITUDES FOR BOSONIC OPEN AND CLOSED STRINGS

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INTRODUCTION

The dual string models are today seriously regarded as realistic candidates for unified theory of all interactions. This is due to the problems encountered in constituing unified theories based on supergravity models and to the proof that the type I superstring theory is anomaly free if the gauge group is SO(32). The lenght scale of the theory is 10^{-33} cm and the energy scale is the Planck mass 10^{19} GeV. So there is a problem for the connection of the string theories with high energy physics experiments that are performed at energies 10^{2} GeV. The other main problem for this connection is that the superstring theories are valid in a ten dimensional world; the compactification of the six extra dimensions has been postulated but it can occur in many possible ways and therefore one looses the predictivity of the original theory. But, apart from these problems, there are very interesting aspects in the string theories. They provide, in fact, a quite unique and probably finite quantum theory unifying the gauge theories with general relativity. Furthermore, unlike gauge theories or general relativity, where the gauge bosons can be eliminated by switching off the gauge interaction, in the string theories one gets the gauge symmetries for free and in an absolutely natural way. This makes the string theories very interesting.

In this thesis we consider the theory of bosonic strings in 26 dimensions; the attention has been over all devoted to the methods of constructing the vertex operators and hence of computing scattering amplitudes for open and closed strings. This is performed in an operator formalism. The basic ingredient is the representation theory of Virasoro algebra, which is a manifestation of the conformal symmetry of the theory. The vertex operators, through which the scattering amplitude are constructed, are determined , in fact, by requiring definite conformal transformation properties on the interaction action operator.

The thesis is organized as follows:

- the first chapter deals with the free bosonic string theory and its quantization in the light-cone gauge;
- the subject of the second chapter is essentially about the general properties of a two dimensional conformal invariant theory, the machinary of which can be applied to the string theory. In this chapter, moreover, is also considered the covariant quantization procedure comparing it with the light-cone one;
- the general results of the second chapter are utilized in the third chapter, where the problem of the interacting string is faced. After having discussed the problem of determining the vertex operators for a bosonic string, it is considered the problem of constructing scattering amplitudes for open and closed strings;
- in the last chapter are reported some explicit examples of calculations, by using the results of the chapter III.

Chapter I

FREE BOSONIC STRING AND ITS QUANTIZATION IN THE LIGHT-CONE GAUGE

§ I.1 - FREE BOSONIC STRING

A bosonic string is described by the following action [1] :

$$S[g^{\alpha\beta}, x^\mu] = -\frac{T}{2}\int d\tau \int_0^\pi d\sigma \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \qquad (I.1.1)$$

$x^\mu(\tau, \sigma)$ describes the position in the space-time of the string; the variables σ and τ are arbitrary variables that parametrize the world-sheet of the string: the spatial coordinate σ labels points along the string and it is conveniently restricted to the interval $0 \leq \sigma \leq \pi$, while τ is a time-like evolution parameter; $T = \frac{1}{2\pi\alpha' \hbar c^3}$ (in the following we will use units where $\hbar = c = 1$) is the string tension with the parameter α' , identified as a Regge slope, having dimensions of lenght squared; $\eta_{\mu\nu}$ is a space-time metric that is conveniently chosen in the D-dimensional flat Minkowski space:

$$\eta_{\mu\nu} = (1, 1, \ldots, 1, -1) ; \qquad (I.1.2)$$

$g_{\alpha\beta}(\sigma, \tau)$ is a two-dimensional auxiliary metric defined on the world-sheet of the string and g and $g^{\alpha\beta}$ represent respectively the determinant and the inverse of $g_{\alpha\beta}$. The indices α and β take the values 0 and 1 referring to the τ and σ directions. Correspondingly, derivatives ∂_α stand for $\partial/\partial\tau$ and $\partial/\partial\sigma$.

The action (I.1.1) can be regarded as describing a two-dimensional field theory with a set of D massless fields interacting with an external gravitational field. In this case the D-dimensional Lorentz index plays the role of a flavour index.

The fact that $(\tau, \sigma) \equiv (\xi^0, \xi^1)$ are arbitrary variables characterizing the world-sheet of the string is reflected in the invariance of the action (I.1.1) under an arbitrary reparametrization of them, that is to say, under the following transformations:

$$\delta x^\mu(\xi) = \epsilon^\alpha(\xi)\, \partial_\alpha x^\mu(\xi) \tag{I.1.3a}$$

$$\delta g_{\alpha\beta}(\xi) = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma} \tag{I.1.3b}$$

where $\epsilon^\alpha(\xi)$ are arbitrary infinitesimal functions of ξ^α. Varying (I.1.1) with respect to $g^{\alpha\beta}$ gives its classical equation of motion:

$$\vartheta_{\alpha\beta} \equiv \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x\, \partial_\delta x = 0 \tag{I.1.4}$$

which exprimes, therefore, the vanishing of the two-dimensional energy-momentum tensor. This is a consequence of the fact that, because of the reparametrization invariance, there is no physical degree of freedom in the two-dimensional space of the world-sheet of the string.

Taking the square root of the determinant of each side of (I.1.4) shows that $g_{\alpha\beta}$ may be eliminated algebraically via its equation of motion leaving the Nambu- Goto action [1] :

$$S(x^\mu) = -T \int d\tau \int_0^\pi d\sigma \sqrt{-\det(\partial_\alpha x)(\partial_\beta x)} \tag{I.1.5}$$

So the two classical actions (I.1.1) and (I.1.5) are completely equivalent. The action (I.1.1) has the advantage to be quadratic in the "matter field" x^μ and therefore the functional integration over in the quantum theory can be easily performed.

The action (I.1.5) has the physical meaning of being proportional to the area spanned by the string: it is the most natural extension to the string of the action describing a spinless free point particle:

$$S(x^\mu) = -mc \int \sqrt{-\dot{x}^2}\, d\tau \tag{I.1.6}$$

that is proportional to the lenght of its world line. τ is an arbitrary parameter describing the motion of the particle; it does not have any physical meaning since (I.1.6) is invariant under an arbitrary reparametrization $\tau \rightarrow f(\tau)$ and the identification of τ with some physical parameter corresponds to a gauge choice for (I.1.6). For example, in this case, a possible gauge corresponds to taking τ proportional to time. Hence also in the case of the string one can introduce the notion of a gauge invariance, identifying the latter with as precisely reparametrization invariance of (I.1.1). Using this property one can choose for (I.1.1) the conformal gauge characterized by:

$$g_{\alpha\beta} = \eta_{\alpha\beta}\,\rho(\xi) \qquad \text{(I.1.7)}$$

with

$$\eta_{00} = -\eta_{11} = -1 \qquad \text{(I.1.8)}$$

The justification of this choice lies in the fact that, although the action (I.1.1) describes a string moving in a D-dimensional Minkowski space, it can also be regarded as a general invariant two-dimensional theory so one can apply all the machinery of the two-dimensional field theories to the string theories. In the conformal gauge the vanishing of the two-dimensional energy-momentum tensor implies the conditions [3] :

$$\partial_\sigma x \cdot \partial_\tau x = 0 \qquad \text{(I.1.9a)}$$

$$\partial_\sigma x\, \partial_\sigma x + \partial_\tau x\, \partial_\tau x = 0 \qquad \text{(1.1.9b)}$$

This is equivalent to choose an orthonormal system of coordinates on the world-sheet of the string and this is the reason why the conformal gauge is also called the orthonormal gauge. In this gauge the Lagrangian in (I.1.1) linearizes, defining a conformal invariant theory:

$$L = -\frac{T}{2}\,\partial_\alpha x^\mu \partial^\alpha x_\mu \qquad \text{(I.1.10)}$$

From this Lagrangian one can derive the equation of motion:

$$\left(\frac{\partial^2}{\partial\sigma^2}-\frac{\partial^2}{\partial\tau^2}\right)x^\mu=0 \tag{I.1.11}$$

with the following boundary conditions:

$$\frac{\partial}{\partial\sigma}x^\mu(\tau,\sigma)\Big|_{\sigma=0}=\frac{\partial}{\partial\sigma}x^\mu(\tau,\sigma)\Big|_{\sigma=\pi}=0 \tag{I.1.12}$$

for an open string and

$$x^\mu(\tau,0)=x^\mu(\tau,\pi) \tag{I.1.13}$$

for a closed string.

The boundary conditions (I.1.12) and (I.1.13) are necessary in order to drop surface terms in obtaining the equation of motion.

Let us introduce now the notation:

$$q^\mu(\tau)=\frac{1}{\pi}\int_0^\pi x^\mu(\sigma,\tau)\,d\sigma \tag{I.1.14}$$

for the "centre of mass" coordinates of the string and let p^μ be be the total D-momentum of the string. Then, the most general solution of the equation of motion with the previous boundary conditions can be written in terms of the string normal mode expansion:

$$x^\mu(\sigma,\tau)=q^\mu+2\alpha' p^\mu\tau-i(2\alpha')^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{\left[a_n^{*\mu}(0)\,e^{in\tau}-a^\mu_n(0)\,e^{-in\tau}\right]}{\sqrt{2}}\cos n\sigma \tag{I.1.15}$$

for an open string and

$$\begin{aligned}x^\mu(\sigma,\tau)=q^\mu+2\alpha' p^\mu\tau&-\frac{i}{2}(2\alpha')^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[a_n^{*\mu}(0)\,e^{2in(\tau+\sigma)}-a^\mu_n(0)\,e^{-2in(\tau+\sigma)}\right]+\\&-\frac{i}{2}(2\alpha')^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[\bar{a}_n^{*\mu}(0)\,e^{2in(\tau+\sigma)}-\bar{a}^\mu_n(0)\,e^{-2in(\tau+\sigma)}\right]\end{aligned} \tag{I.1.16}$$

for a closed string. It is possible to introduce some new variables which allow a certain simplification of the notation:

$$\alpha^\mu_n=\begin{cases}(2\alpha')^{\frac{1}{2}}\,n^{\frac{1}{2}}\,a^\mu_n & \text{if } n>0 \quad \text{(I.1.17a)}\\ 2\alpha'\,p_0^\mu & \text{if } n=0 \quad \text{(I.1.17b)}\\ (2\alpha')^{\frac{1}{2}}\,|n|^{\frac{1}{2}}\,a^{*\mu}_n & \text{if } n<0 \quad \text{(I.1.17c)}\end{cases}$$

In terms of these variables and putting $\alpha' = \frac{1}{2}$, one gets:

$$x^\mu(\sigma,\tau) = q^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in\tau}\cos n\sigma \tag{I.1.18}$$

for an open string and

$$x^\mu(\sigma,\tau) = q^\mu + p^\mu\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left[\alpha^\mu_n e^{-2in(\tau+\sigma)} + \tilde{\alpha}^\mu_n e^{-2in(\tau-\sigma)}\right] \tag{I.1.19}$$

for a closed string, where the quantities $\tilde{\alpha}^\mu_n$ are defined in analogous way to the α^μ_n ones:

$$\tilde{\alpha}^\mu_n = \begin{cases} (2\alpha')^{\frac{1}{2}}\, n^{\frac{1}{2}}\, a^\mu_n & \text{if } n > 0 \quad \text{(I.1.20a)} \\ \alpha' p^\mu_0 & \text{if } n = 0 \quad \text{(I.1.20b)} \\ (2\alpha')^{\frac{1}{2}}\, |n|^{\frac{1}{2}}\, a^{*\mu}_n & \text{if } n < 0 \quad \text{(I.1.20c)} \end{cases}$$

The choice of the conformal gauge does not fix uniquely the gauge: one can indeed still perform gauge transformations preserving the conformal gauge. These are the conformal transformations which have the same form of (I.1.3), but are characterized by a parameter $\epsilon^\alpha(\xi)$ satisfying the conditions:

$$\partial^\alpha\epsilon^\beta + \partial^\beta\epsilon^\alpha - \eta^{\alpha\beta}\partial^\gamma\epsilon_\gamma = 0 \tag{I.1.21}$$

By introducing light-cone coordinates:

$$\xi^\pm = \xi^0 \pm \xi^1 \tag{I.1.22a}$$

$$\epsilon^\pm = \epsilon^0 \pm \epsilon^1 \tag{I.1.22b}$$

$$\frac{\partial}{\partial\xi^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial\xi^0} \pm \frac{\partial}{\partial\xi^1}\right) \tag{I.1.22c}$$

the conditions (I.1.20) can be written in the following way:

$$\frac{\partial}{\partial\xi^-}\epsilon^+ = \frac{\partial}{\partial\xi^+}\epsilon^- = 0 \tag{I.1.23}$$

This shows clearly that the transformations that preserve the conformal gauge are characterized by two arbitrary functions $\epsilon^+(\xi^+)$ and $\epsilon^-(\xi^-)$, that transform the variables $\xi^\pm$ as follows:

$$\delta\xi^+ = \epsilon^+(\xi^+) \tag{I 1.24a}$$

$$\delta\xi^- = \epsilon^-(\xi^-) \tag{I.1.24b}$$

In the case of an open string further restrictions must be imposed on these functions. The end points of an open string are indeed parametrized by the values $\sigma = 0, \pi$ and therefore one requires that this parametrization is not changed by a reparametrization. Since:

$$\delta(\tau+\sigma) = \delta(\tau) + \delta(\sigma) = \epsilon^+(\xi^+) \tag{I.1.25a}$$

$$\delta(\tau-\sigma) = \delta(\tau) - \delta(\sigma) = \epsilon^-(\xi^-) \tag{I.1.25b}$$

it follows:

$$\delta\sigma = \frac{1}{2}\left[\epsilon^+(\tau+\sigma) - \epsilon^-(\tau-\sigma)\right] \tag{I.1.26}$$

Requiring the condition:

$$\delta\sigma \Big|_{\sigma=0,\pi} = 0 \tag{I.1.27}$$

implies that the two functions ϵ^+ and ϵ^- are restricted by:

$$\epsilon^+(\tau) = \epsilon^-(\tau) \equiv \epsilon(\tau) \tag{I.1.28}$$

and

$$\begin{aligned} &\epsilon^-(\tau-\pi) = \epsilon^+(\tau+\pi) \Rightarrow \\ \Rightarrow\ &\epsilon(\tau-\pi) = \epsilon(\tau+\pi) \end{aligned} \tag{I.1.29}$$

The generators of the conformal transformations that leave unchanged the parametrization of the end points of the string can be written in terms of the two independent components of the two-dimensional energy-momentum tensor $\vartheta_{\alpha\beta}$:

$$L_\epsilon = \frac{1}{4\pi}\int_0^\pi d\sigma \left\{ \left(\frac{\partial x}{\partial\tau} + \frac{\partial x}{\partial\sigma}\right)^2 (\tau+\sigma)\, \epsilon(\tau+\sigma) + \left(\frac{\partial x}{\partial\tau} - \frac{\partial x}{\partial\sigma}\right)^2 (\tau-\sigma)\, \epsilon(\tau-\sigma) \right\} \tag{I.1.30}$$

One must observe that $\left(\frac{\partial x}{\partial\tau} \pm \frac{\partial x}{\partial\sigma}\right)^2$ are only functions of $\tau \pm \sigma$ respectively: in fact the equation of motion (I.1.11) implies:

$$\left(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\right)\left(\frac{\partial x}{\partial\sigma} - \frac{\partial x}{\partial\tau}\right) = 0 \tag{I.1.31a}$$

$$\left(\frac{\partial}{\partial\sigma} - \frac{\partial}{\partial\tau}\right)\left(\frac{\partial x}{\partial\sigma} + \frac{\partial x}{\partial\tau}\right) = 0 \tag{I.1.31b}$$

Hence:

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\sigma}\right)\left(\frac{\partial x}{\partial\tau}-\frac{\partial x}{\partial\sigma}\right)^2=0 \tag{I.1.32a}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\sigma}\right)\left(\frac{\partial x}{\partial\tau}+\frac{\partial x}{\partial\sigma}\right)^2=0 \tag{I.1.32b}$$

These equations express the conservation of the two-dimensional energy-momentum tensor. They imply also that L_ϵ is indepent of τ.

Since:

$$\frac{\partial x^\mu}{\partial\tau}=p^\mu+\sum_{n=1}^{\infty}\sqrt{n}\left\{a^\mu_n e^{-in\tau}+a^{*\mu}_n e^{in\tau}\right\}\cos n\sigma \tag{I.1.33a}$$

and

$$\frac{\partial x^\mu}{\partial\sigma}=-i\sum_{n=1}^{\infty}\sqrt{n}\left\{a^\mu_n e^{-in\tau}-a^{*\mu}_n e^{in\tau}\right\}\sin n\sigma \tag{I.1.33b}$$

one gets:

$$(\dot{x}+x')^2(\tau,\sigma)=(\dot{x}-x')^2(\tau,-\sigma) \tag{I.1.34}$$

where

$$\dot{x}\equiv\frac{\partial x^\mu}{\partial\tau}\qquad\qquad x'\equiv\frac{\partial x^\mu}{\partial\sigma} \tag{I.1.35}$$

The generators L_ϵ can therefore be written in the following way:

$$L_\epsilon=\frac{\lambda}{4\pi}\int_{-\pi}^{\pi}d\sigma\,(\dot{x}+x')^2(\tau+\sigma)\,\epsilon(\tau+\sigma) \tag{I.1.36}$$

Since there is complete symmetry between τ and σ, it is possible to integrate over τ instead of σ and put $\sigma=0$. Furthermore using the boundary conditions $\left.\frac{\partial x^\mu}{\partial\sigma}(\tau,\sigma)\right|_{\sigma=0,\pi}=0$, one has:

$$L_\epsilon=\frac{\lambda}{4\pi}\int_{-\pi}^{\pi}d\tau\,\dot{x}^2(\tau)\epsilon(\tau) \tag{I.1.37}$$

By using now the expression (I.1.33a) written for $\sigma=0$ and choosing $\epsilon(\tau)=e^{in\tau}$, it is possible to write:

$$L_n=\frac{\lambda}{2}\sum_{m=-\infty}^{+\infty}\alpha^\mu_{n-m}\cdot\alpha^\mu_m \tag{I.1.38}$$

Finally in terms of the variable $z=e^{i\tau}$ one has:

$$L_n=\frac{\lambda}{2\pi i}\oint dz\,z^{n+1}\left\{-\frac{1}{2}\frac{\partial x^\mu}{\partial z}\frac{\partial x_\mu}{\partial z}\right\} \tag{I.1.39}$$

The conformal invariance is a residual gauge invariance corresponding to the reparametrizations leaving in the conformal gauge. In other words, by fixing the conformal gauge, one still has a partial freedom in performing parametrizations of the world-sheet. These parametrizations correspond to conformal transformations. The generators of this residual invariance are identically vanishing, as it can be deduces from the condirions (I.1.9). Therefore one has:

$$L_n = 0 \tag{I.1.40}$$

for any integer n.

For the closed string it is possible to make analogous considerations. In this case the generators of the conformal transformations are characterized by two independent functions $\epsilon^+(\tau+\sigma) \equiv \epsilon(\tau+\sigma)$ and $\epsilon^-(\tau-\sigma) \equiv \bar{\epsilon}(\tau-\sigma)$ and they have the following expression:

$$L_\epsilon = \frac{1}{8\pi}\int_0^\pi d\sigma\,(\dot{x}+x')^2(\tau+\sigma)\,\epsilon(\tau+\sigma) \tag{I.1.41a}$$

$$L_{\bar{\epsilon}} = \frac{1}{8\pi}\int_0^\pi d\sigma\,(\dot{x}-x')^2(\tau-\sigma)\,\bar{\epsilon}(\tau-\sigma) \tag{I.1.41b}$$

The previous generators can be written, in terms of the variables $z = e^{2i\xi^+}$ and $\bar{z} = e^{2i\xi^-}$, in the following way:

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\left[-2\left(\frac{\partial x}{\partial z}\right)^2\right] \tag{I.1.42a}$$

$$\bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1}\left[-2\left(\frac{\partial x}{\partial \bar{z}}\right)^2\right] \tag{I.1.42b}$$

where the relations:

$$\left(\frac{\partial x}{\partial \sigma}+\frac{\partial x}{\partial \tau}\right)^2 = -16 z^2\left(\frac{\partial x}{\partial z}\right)^2 \tag{I.1.43a}$$

$$\left(\frac{\partial x}{\partial \sigma}-\frac{\partial x}{\partial \sigma}\right)^2 = -16 \bar{z}^2\left(\frac{\partial x}{\partial \bar{z}}\right)^2 \tag{I.1.43b}$$

have been used and it has been put $\epsilon(\tau+\sigma) = z^n$ and $\bar{\epsilon}(\tau-\sigma) = \bar{z}^n$

In terms of the harmonic oscillators introduced in the explicit solution of the equations of the motion for a closed string, one has:

$$L_n = \frac{1}{2}\sum_m \alpha_m \cdot \alpha_{n-m} \tag{I.1.44a}$$

$$\bar{L}_n = \frac{1}{2}\sum_m \bar{\alpha}_m \cdot \bar{\alpha}_{n-m} \tag{I.1.44b}$$

where the zero mode is given by $\alpha_0^\mu = \bar{\alpha}_0^\mu = p^\mu/2$.

Finally also for a closed string the conformal generators are vanishing quantities:

$$L_n = \bar{L}_n = 0 \tag{I.1.45}$$

for any integer n.

§ I.2 - CLASSICAL MOTIONS OF OPEN AND CLOSED STRINGS

An allowed motion for an open string is described in the following example. Let us consider a straight open string of lenght 2a rigidly rotating in the plane 12 around its centre. The coordinates of the string are given by:

$$\begin{aligned} x_1 &= a \cos\sigma \cos\tau & \qquad x_3 &= 0 \\ x_2 &= a \cos\sigma \sin\tau & \qquad x_0 &= ct = a\tau \end{aligned} \tag{I.2.1}$$

where $r = a \cos\sigma$ is the coordinate along the string that ranges from $r = -a$ to $r = a$ if $0 \leq \sigma \leq \pi$.

The end points of the string move at the speed of light since:

$$\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 = \frac{r^2 c^2}{a^2} = c^2 \qquad \text{if } r^2 = a^2 \tag{I.2.2}$$

The equations (I.2.1) satisfy the equations of motion (I.1.11) and the boundary conditions (I.1.12) together with the orthonormal gauge constraints (I.1.9). It follows that the motion described by (I.2.1) are allowed for a string.

The energy per unit lenght of the rotating string is given by:

$$\Pi^0 = \frac{cT}{\sqrt{1 - r^2/a^2}} \qquad -a \leq r \leq a \tag{I.2.3}$$

corresponding to a mass:

$$m = \frac{1}{c}\int_{-a}^{a} dr\, \Pi^0(r) = \pi a T \tag{I.2.4}$$

On the other hand the angular momrntum of a rigidly rotating string is given by:

$$J = \int_{-r}^{r} \frac{cT(r^2/a)\,dr}{\sqrt{1 - \frac{r^2}{a^2}}} = \frac{a^2}{2} T\pi c \tag{I.2.5}$$

By comparing (I.2.4) with (I.2.5) one gets the following relation between mass and angular momentum:

$$J = \alpha' \hbar c^4 m^2 \tag{I.2.6}$$

with

$$\alpha' = 1/2\pi T \hbar c^3 \tag{1.2.7}$$

These relations have a general validity implying that the states of a string lie on linearly rising Regge trajectories. (I.2.7) gives the relation between the Regge slope and the string tension. When one exprimes T versus α' from (I.2.7) and substitutes the obtained relations in the original action (I.1.1), then it is possible to note that the Planck's constant $\hbar$ already appears in the classical motion.

Another interesting feature of a string is that, if one puts a charge g at one end point and computes the gyromagnetic ratio G one finds the result that G = 2. The string has therefore no anomalous magnetic moment. This property can be checked for a rigidly rotating string, that generates a current given by:

$$j = g \frac{c}{2\pi a} \tag{I.2.8}$$

corresponding to a dipole magnetic moment:

$$\mu = \frac{j}{c} A \tag{I.2.9}$$

where $A = \pi a^2$ is the area spanned by the string.
Inserting (I.2.7) in (I.2.8) one obtains:

$$\mu \equiv \frac{g}{2mc} G J = \frac{ga}{2} \tag{I.2.10}$$

Substituting the prevoius formulas for m and J, (I.2.4) and (I.2.5), one gets the final result:

$$G = 2 \tag{I.2.11}$$

This result holds for an arbitrary motion of the string. Finally it is possible to show the following relation between the slopes of open and closed strings:

$$\alpha'_{closed} = \frac{1}{2}\alpha'_{open} \qquad (I.2.12)$$

By following the same argument used for an open string one can easily see that an allowed motion for a closed string is the one consisting of two straight open strings attached at the end points and rotating together around their common centre. Since the energy density for such a closed string is twice the one of an open string, its mass will be four times the mass of an open string:

$$m^2_{closed} = 4m^2_{open} \qquad (I.2.13)$$

On the other hand, its angular momentum is twice of that of an open string:

$$J_{closed} = 2\,J_{open} \qquad (I.2.14)$$

Combining these two relations gives:

$$\alpha'_{closed} = \tfrac{1}{2}\,\alpha'_{open} \qquad (I.2.15)$$

As for the open string, these motions correspond to states lying on linearly rising Regge trajectories.

§ I.3 - QUANTIZATION OF THE FREE BOSONIC STRING IN THE LIGHT-CONE GAUGE

It has been already shown that the choice of the conformal gauge does not fix uniquely the gauge. In fact it is still possible to perform conformal transformations remaining in the conformal gauge. A possibility of quantizing the string theory is to first fix completely the gauge in the classical theory: this procedure reduces the number of degrees of freedom and eliminates all redundant variables. Then one has only to assume canonical Poisson brackets for the independent variables. This leads, when quantized, to a positive metric space, but the procedure is not explicitly covariant . A convenient way of fixing completely the gauge is choosing the light-cone gauge characterized by the condition:

$$x^+ = 2\alpha' p^+ \tau \tag{I.3.1}$$

where

$$x^\pm = \frac{x^D \pm x^{D-1}}{2} \tag{I.3.2}$$

This is a possible gauge choice inside the conformal gauge both for the open and for the closed string.

Let us show this first for a closed string. For a closed string the most general solution of the equation of motion (I.1.11) with the boundary conditions (I.1.13) can be written as follows:

$$x^\mu(\tau,\sigma) = \phi^\mu(\tau+\sigma) + \bar{\phi}^\mu(\tau-\sigma) + \frac{1}{2}p^\mu(\tau+\sigma) + \frac{1}{2}p^\mu(\tau-\sigma) \tag{I.3.3}$$

where $\phi^\mu(\tau+\sigma)$ and $\bar{\phi}^\mu(\tau-\sigma)$ are periodic functions of with a period equal to 2π. Under a conformal transformation $x^\mu(\tau,\sigma)$ transforms as follows:

$$\delta x^\mu = \epsilon^+(\tau+\sigma)\frac{\partial}{\partial\xi^+}x^\mu + \epsilon^-(\tau-\sigma)\frac{\partial}{\partial\xi^-}x^\mu \tag{I.3.4}$$

where $\epsilon^\pm(\tau\pm\sigma)$ are also periodic functions with a period equal to 2π. Therefore by performing a suitable conformal transformation with periodic functions $\epsilon^\pm(\tau \pm \sigma)$, it is possible to put one component of $x^\mu(\tau,\sigma)$, for example x^+, in a form where $\phi^+ = \bar{\phi}^+ = 0$, obtaining so the form proposed in (I.3.1). In the case of an open string the two functions appearing in (I.3.3) and (I.3.4) are not independent: they are both periodic with a period equal to 2π and they must be identified so $\phi(\tau) = \bar{\phi}(\tau)$ and $\epsilon^+(\tau) = \epsilon^-(\tau)$. Therefore it is possible to choose the gauge (I.3.1.) also for an open string.

In the light-cone gaugz the only independent degrees of freedom are the transverse ones, which can be assumed therefore as the only dynamical variables. This is so because through the constraints (I.1.9) and the choice (I.3.1) it is possible to fix x^- as a function of the transverse components x^i ($i = 1, \dots, D-2$), that are ortho gonal to both $\pm$ directions, in the following way:

$$\dot{x}^- = \frac{1}{2p^+} (\dot{x}_i^2 + x_i'^2) \tag{I.3.5a}$$

$$x'^- = \frac{1}{p^+} \dot{x}_i \, x'^i \tag{I.3.5b}$$

Hence when the $x_i(\sigma, \tau)$ are known, also x^-'s are known up to an integration constant.

Since the Hamiltonian of a system is the conjugate variable to the evolution parameter, in the light-cone gauge, where the evolution parameter τ is proportional to x^+, the Hamiltonian will be proportional to p^-, being the latter the conjugate variable to x^+. From the Lagrangian (I.1.10) one gets:

$$p^- = \frac{\dot{x}^-}{\pi} \tag{I.3.6}$$

Hence, considering (I.3) gives:

$$p^+ p^- = H = \frac{1}{2\pi} \int_0^\pi d\sigma \left[\dot{x}_i^2 + x_i'^2 \right] \tag{I.3.7}$$

This Hamiltonian can be obtained from the action:

$$S = -\frac{T}{2} \int d\tau \int_0^\pi d\sigma \, \eta^{\alpha\beta} \partial_\alpha x^i \tag{I.3.8}$$

that is equal to the action in the conformal gauge written only for the transverse degrees of freedom. By using the gauge choice (I.3.1) in the general solutions (I.1.14) and (I.1.15) and remembering the definition of α^μ_n (I.1.16), one gets:

$$\alpha^+_n = 0 \quad n \neq 0 \tag{I.3.9}$$

for an open string and

$$\alpha^+_n = \tilde{\alpha}^+_n = 0 \quad n \neq 0 \tag{I.3.10}$$

for a closed string.

Furthermore the oscillators α^-_n can be written in terms of the transverse ones by (I.3.5) in the following way:

$$\alpha^-_n = \frac{1}{2p^+} \sum_{m=-\infty}^{+\infty} \alpha^i_{n-m} \alpha^i_m \; ; \; n \neq 0 \tag{I.3.11}$$

for an open string and

$$\alpha^-_n = \frac{1}{2p^+} \sum_{m=-\infty}^{+\infty} \alpha^i_{n-m} \alpha^i_m \qquad n \neq 0 \tag{II3.12a}$$

$$\tilde{\alpha}^-_n = \frac{1}{2p^+} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}^i_{n-m} \tilde{\alpha}^i_m \tag{I.3.12b}$$

for a close string.

Furthermore the equations (I.1.39) and (I.1.44) will determine p^- as a function of p^+ and of the transverse degrees of freedom.

All this shows that the only independent oscillators are the transverse ones and therefore, in the light-cone gauge, the only independent degrees of freedom are the transverse oscillators supplemented by the centre of mass variables q^μ and p^μ. At this point one can proceed to quantize the string theory keeping only the physical degrees of freedom α^i_n, q^μ and p^μ.

The open string can be quantized by imposing the following commutation relations:

$$[\alpha^i_n, \alpha^j_m] = n\delta^{ij}\delta_{n+m;0} \tag{I.3.13a}$$

$$[q^\mu, p^\nu] = i g^{\mu\nu} \tag{I.3.13b}$$

In the case of the closed string the following commutation relations must be added for the oscillators $\tilde{\alpha}^i_n$:

$$[\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = n\delta^{ij}\delta_{n+m;0} \tag{I.3.14}$$

The spectrum of the open strin is obtained from the Hamiltonian (I.3.7), that in terms of the oscillators is given by :

$$p^+p^- = \frac{\lambda}{2}\left[p^ip^i + 2\sum_{n=1}^{\infty} n\, a^{+i}_n a^i_n + c\right] \tag{I.3.15}$$

or, equivalently, from the condition:

$$L_0 = \frac{p^2}{2} + \sum_1^{\infty} n\, a^{+i}_n a^i_n = -c \tag{I.3.16}$$

where an arbitrary constant c has been introduced in order to take care of the normal ordering of the harmonic oscillators and, hence, of the quantum definition of L_0. The value of the constant c is given, as the usual theory of the harmonic oscillator suggests, by the zero pint energy. For the string this value is given by :

$$c = \frac{D-2}{2} \sum_1^\infty n \tag{I.3.17}$$

and therefore it is formally infinite; it rises out the problem of regularizing it. A useful regularization scheme is the one that makes use of the ζ-function regularization. This amounts to replace (I.3.17) with

$$c = \frac{D-2}{2} \lim_{s \to D-1} \sum_1^\infty n^{-s} = $$

$$= \frac{D-2}{2} \lim_{s \to D-1} \zeta_R(s) \tag{I.3.18}$$

where ζ_R (s) is the Riemann ζ-function, that is an analytic function for s= -1. Its value is given by [6] :

$$\zeta_R(-1) = -1/12 \tag{I.3.19}$$

Inserting this value in (I.3.18) gives, for the zero point energy, the following value:

$$c = -(D-2)/24 \tag{I.3.20}$$

and therefore one can rewrite (I.3.15) in the following form:

$$\alpha(M^2) = \sum_{n=1}^\infty n\, a^{+i}_n a^i_n \tag{I.3.21}$$

where

$$\alpha(M^2) = \alpha_0 + \alpha' M^2 \tag{I.3.22}$$

with

$$\alpha_0 = \frac{D-2}{24} \qquad M^2 = -p^2 \tag{I.3.23}$$

In the light-cone gauge the theory is not anymore manifestly Lorentz covariant and therefore one must write the Lorentz generators in terms of the transverse oscillators and check that they satisfy the Lorentz algebra.

The quantum Lorentz operators are given by:

$$J^{ij} = l^{ij} - i \sum_1^\infty \frac{1}{n}\left(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n\right) \tag{I.3.24a}$$

$$J^{+-} = l^{+-} \tag{I.3.24b}$$

$$J^{i+} = l^{i+} \tag{I.3.24c}$$

$$J^{i-} = l^{i-} - i\sum_{1}^{\infty}\frac{1}{n}(\alpha^{i}_{-n}\alpha^{-}_{n} - \alpha^{-}_{n}\alpha^{i}_{n}) \tag{I.3.24d}$$

where

$$l^{\mu\nu} \equiv q^{\mu} p^{\nu} - q^{\nu} p^{\mu} \tag{I.3.25}$$

and α^{-}_{n} is given by (I.3.11). The algebra of these operators coincides with the Lorentz one olny if:

$$\alpha_{o} = 1 \tag{I.3.26}$$

and

$$D = 26 \tag{I.3.27}$$

These are therefore the only values of α_{o} and D that permit to preserve the Lorentz invariance in the quantum theory. Taking into account these values, the spectrum of the string states masses is given by:

$$\alpha' M^{2} = N - 1 \tag{I.3.28}$$

where

$$N \equiv \sum_{n=1}^{\infty} n\, a^{+i}_{n}\, a^{i}_{n} \tag{I.3.29}$$

The lowest state N = 0 is given by the vacuum $|0\rangle$ of the oscillators and corresponds to a tachyon with $M^{2} = -1/\alpha'$. The following level N = 1, corresponding to $M^{2} = 0$, is given by the state $a^{+}_{1;i}|0\rangle$; this describes the transverse components of a massless spin 1 particle ("photon").

At the level N = 2 there are two states:

$$a^{+}_{1i}\, a^{+}_{1j}|0\rangle \tag{I.3.30a}$$

and

$$a^{+}_{2i}\,|0\rangle \tag{I.3.30b}$$

that describe a massive spin 2 particle with $M^{2} = 1/\alpha'$.

Because of the disappearance of the time component the space of the vectors

$$\prod_{n=1}^{\infty} (a^{+}_{n\, i_n})^{\lambda_n}\, |0\rangle \tag{I.3.31}$$

has definite positive metric. The states of the string are hence described by purely transverse oscillators.

Since Lorentz invariance holds in the centre of mass frame, the states of the various levels must be classified according to the representations of SO(D-1). The number of states appearing at the level N = 2 is given by:

$$\frac{(D-2)(D-1)}{2} + D - 2 = \frac{(D-2)(D+1)}{2} \tag{I.3.32}$$

and this coincides with the number of components of a spin 2 in SO(D-1) given by $\frac{D(D-1)}{2} - 1$.

The degeneracy of the states at an arbitrary level can be obtained from the partition function:

$$f(x) = \frac{1}{x} \prod_{n=1}^{\infty} \left(\frac{1}{1-x^n} \right)^{D-2} \tag{I.3.33}$$

that is obtained from

$$f(x) = \mathrm{Tr} \left(x^{\sum_1^\infty n a^{+i}_n a^i_n - 1} \right) \tag{I.3.34}$$

where the term -1 is associated to the zero point energy. From (I.3.34) it follows that, if the level N is fixed, the degeneracy $T_d(N)$ of the states at that level is the coefficient of the power x^{N-1} in the expansion of (I.3.33) in power series around x=0:

$$f(x) = \frac{1}{x} \sum_{N=0}^{\infty} T_d(N)\, x^N \tag{I.3.35}$$

where d = D - 2. The function f(x) is the so called "partitio numerorum".

In the quantum theory of a closed string one can proceed analogously as in the case of an open string, getting the spectrum from the conditions (1.1.45) that imply:

$$L_0 - 1 = \bar{L}_0 - 1 = 0 \tag{I.3.36}$$

where the arbitrary constant c and the space-time dimension D have been chosen as for the open string in order to have a Lorentz invariant theory.

By summing and subtracting between themselves the relations (I.3.36) and taking into account that

$$\alpha'_{closed} = \frac{\alpha'}{2} = \frac{1}{4} \qquad (I.3.37)$$

one gets the following equations characterizing the spectrum of a closed string:

$$2 + \alpha'_c m^2 = 2N \qquad (I.3.38a)$$

$$N = \bar{N} \qquad (I.3.38b)$$

where

$$N = \sum_1^\infty n a^{+i}_n a^i_n \qquad (I.3.39a)$$

and

$$\bar{N} = \sum_1^\infty n \bar{a}^{+i}_n \bar{a}^i_n \qquad (I.3.39b)$$

The lowest state of the spectrum is a tachyon with mass $m^2 = -2/\alpha'_c$ described by the vacuum $|0\rangle$.

The first excited level containing massless states is described by the states:

$$a^+_{1i} \bar{a}^+_{1j} |0\rangle \qquad (I.3.40)$$

The symmetric and traceless state corresponds to the graviton, the state $\sum_{i=1}^{D-2} a^+_{1i} \bar{a}^+_{1i} |0\rangle$ corresponds to a dilaton and finally the antisymmetric state describes an antisymmetric tensor.

In this case the degeneracy of an arbitrary level can be obtained from the "partition function":

$$F(\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta \, \frac{1}{|z|^2} \prod_{n=1}^\infty \left[\frac{1}{|1-z^n|^2}\right]^{D-2} \qquad (I.3.41)$$

where $z = \rho e^{i\vartheta}$.

Chapter II

TWO-DIMENSIONAL CONFORMAL INVARIANT THEORIES

§ II.1 - CONFORMAL INVARIANT THEORIES

A D-dimensional conformal invariant quantum field theory is characterized by the existence of a conserved symmetric and traceless energy-momentum tensor [7] :

$$\vartheta^{\alpha\beta} = \vartheta^{\beta\alpha} \tag{II.1.1a}$$

$$\partial_\alpha \vartheta^{\alpha\beta} = \vartheta^{\alpha}{}_{\alpha} = 0 \tag{II.1.1b}$$

Denoting by ξ^a , a = 1, 2, ... , D, the coordinates, one can show that the theory, under the conditions (II.1.1), is invariant under the coordinate transformations

$$\xi^a \longrightarrow \eta^a(\xi) \tag{II.1.2}$$

having the property that the metric tensor transforms as:

$$g_{ab} \longrightarrow \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} g_{a'b'} = \rho(\xi) g_{ab} \tag{II.1.3}$$

where $\rho(\xi)$ is a certain function. Coordinate transformations of this type constitute the conformal group. For two-dimensional theories, the conformal group is infinite-dimensional and it consists of the conformal analytical transformations. To describe this group it is convenient to introduce the complex coordinates:

$$z = \xi^1 + i\xi^2 \tag{II.1.4a}$$

$$\bar{z} = \xi^1 - i\xi^2 \tag{II.1.4b}$$

the metric having the form:

$$ds^2 = dz\, d\bar{z} \tag{II.1.5}$$

The conformal group of the two-dimensional space consists of all substitutions of the form:

$$z \longrightarrow \xi(z) \tag{II.1.6a}$$

$$\bar{z} \to \bar{\zeta}(\bar{z}) \tag{II.1.6b}$$

where ζ and $\bar{\zeta}$ are arbitrary analitical functions. In the complex case the conformal group G is therefore a direct product:

$$G = \Gamma \otimes \bar{\Gamma} \tag{II.1.7}$$

where Γ ($\bar{\Gamma}$) is a group of the analytical substitutions of the variable $z(\bar{z})$. The properties holding for the group Γ also hold for $\bar{\Gamma}$, therefore one can consider only the properties of the group Γ. Infinitesimal transformations of the group Γ are:

$$z \to z + \epsilon(z) \tag{II.1.8}$$

where $\epsilon(z)$ is an infinitesimal analytic function. It can be represented as an infinite Lourant series:

$$\epsilon(z) = \sum_{n=-\infty}^{+\infty} \epsilon_n z^{n+1} \tag{II.1.9}$$

Therefore the Lie algebra of the group Γ coincides with the algebra of differential operators:

$$l_n = z^{n+1} \frac{d}{dz} \qquad n=0, \pm 1, \pm 2, \ldots \tag{II.1.10}$$

the commutation relations having the form

$$[l_n, l_m] = (n - m)\, l_{n+m} \tag{II.1.11}$$

The generators $\bar{l}_n$ of the group $\bar{\Gamma}$ satisfy the same commutation relations, the operators l_n and $\bar{l}_m$ being commutative. Let $\mathcal{L}_o$ denote the algebra (II.1.11). The generators l_{-1}, l_o, l_{+1} form the subalgebra $sl(2,C) \subset \mathcal{L}_o$. The corresponding subgroup $SL(2,C) \subset \Gamma$ consists of the projective transformations:

$$z \to \zeta = (az + b)/(cz + d) \tag{II.1.12}$$

$$ad - bc = 1$$

The projective transformations are uniquely invertible mappings of the whole z-plane on itself and these are the only conformal transformations with this property.

It is possible to show that the components of the energy-momentum tensor represent the generators of the conformal group G in the quantum field theory and the algebra of these generators is the central extension of the algebra $\mathcal{L}_0$ and coincides with the Virasoro algebra. For this aim, let us consider the energy-momentum tensor $\vartheta_{\alpha\beta}$. Since it is symmetric and traceless it has only two independent components. By introducing the light-cone coordinates

$$\xi^{\pm} = \xi^0 \pm \xi^1 \qquad \text{(II.1.13)}$$

the two independent components can be conveniently chosen to be:

$$\vartheta^{++} = \vartheta^{0+1,0+1} = 2(\vartheta^{01} + \vartheta^{00}) \qquad \text{(II.1.14a)}$$

$$\vartheta^{--} = \vartheta^{0-1,0-1} = 2(\vartheta^{00} - \vartheta^{01}) \qquad \text{(II.1.14b)}$$

while

$$\vartheta^{+-} = \vartheta^{-+} = 0 \qquad \text{(II.1.15)}$$

The conservation equation (II.1.1b) implies the two equations for ϑ^{++} and ϑ^{--}:

$$\frac{\partial}{\partial \xi^{+}} \vartheta^{++} = \frac{\partial}{\partial \xi^{-}} \vartheta^{--} = 0 \qquad \text{(II.1.16)}$$

Therefore $\vartheta^{++}(\xi^-)$ $[\vartheta^{--}(\xi^+)]$ is only a function of $\xi^-[\xi^+]$. The symmetry of $\vartheta^{\alpha\beta}$ and the equations (II.1.1) imply that

$$\frac{\partial}{\partial \xi^{\beta}} \left[\epsilon_\alpha \vartheta^{\alpha\beta}\right] = 0 \qquad \text{(II.1.17)}$$

if ϵ_α satisfies the condition characterizing a conformal transformation (I.1.21). Since (II.1.17) holds, one can construct the following constants of motion:

$$L_\epsilon = \int d\sigma \, \epsilon^\alpha \vartheta_{\alpha 0} \qquad \text{(II.1.18)}$$

depending on a function ϵ^α satisfying (I.1.21). In terms of the light-cone variables (II.1.18) becomes:

$$L_\epsilon = \frac{1}{4}\int d\sigma \left\{ \epsilon^+ \vartheta^{--} + \epsilon^- \vartheta^{++} \right\} \qquad (II.1.19)$$

where σ is chosen to vary in the interval $0 \leq \sigma \leq 2\pi$ and the functions appearing are periodic functions of period 2π.

In string theories it is convenient to introduce the new variables:

$$z = e^{i\xi^+} \qquad \bar{z} = e^{i\xi^-} \qquad (II.1.20)$$
$$\xi^\pm = \tau \pm \sigma$$

related to the original ones by a conformal transformation. In the euclidean space where $\tau \to i\tau$ $\bar{z}$ becomes the complex conjugate of z.

At this point we want to introduce the notion of a primary field. This is defined as a field that transforms in the following way under a finite conformal transformation:

$$\phi(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi(w,\bar{w}) \qquad (II.1.21)$$

where $w = w(z)$ and $\bar{w} = \bar{w}(\bar{z})$. Here Δ and $\bar{\Delta}$ are real non-negative parameters. in fact the combinations $d = \Delta + \bar{\Delta}$ and $s = \Delta - \bar{\Delta}$ are the anomalous scale dimension and the dimension of the field ϕ, respectively. The quantities Δ and $\bar{\Delta}$ are often called left and right conformal dimensions of the field. The simplest example of a primary field is the identity operator. The following infinitesimal transformations are implied by (II.1.21):

$$\delta\phi(z,\bar{z}) = \left[\epsilon(z)\frac{\partial}{\partial z} + \Delta\epsilon'(z)\right]\phi(z,\bar{z}) +$$
$$+ \left[\bar{\epsilon}(\bar{z})\frac{\partial}{\partial \bar{z}} + \bar{\Delta}\bar{\epsilon}'(\bar{z})\right]\phi(z,\bar{z}) \qquad (II.1.22)$$

where $w = z + \epsilon(z)$ and $\bar{w} = z + \bar{\epsilon}(z)$, with $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ infinitesimal quantities.

The transformation (II.1.22) can be written in terms of the operator product expansion of the energy-momentum tensor with $\phi(z,\bar{z})$. First of all one can rewrite the first term in (II.1.19) as a function of the variables z and $\bar{z}$:

$$L_n = \frac{1}{2\pi i} \oint z^{n+1}\, T(z)\, dz \tag{II.1.23}$$

where $\epsilon^+(\tau+\sigma) = z^n$ and $\vartheta^{--}(\tau+\sigma) = \frac{2}{\pi} T(z) z^2$

Let us introduce now the notion of radially ordered o.p.e. of two fields in the euclidean space:

$$R\,(\,A(z)\,B(\zeta)\,) = \begin{cases} = A(z)\,B(\zeta) & \text{if } |z|>|\zeta| \\ = \pm\, B(\zeta)\,A(z) & \text{if } |\zeta|>|z| \end{cases} \tag{II.1.24}$$

where the minus sign holds only if both fields are fermions. R can be omitted if one assumes, as we will do in the following, that $|z| > |\zeta|$.

The o.p.e. of a primary field and the energy-momentum tensor T(z) is given by:

$$T(z)\,\phi(\zeta) = \frac{\partial/\partial\zeta\, \phi(\zeta)}{z-\zeta} + \Delta\, \frac{\phi(\zeta)}{(z-\zeta)^2} + \text{reg. terms} \tag{II.1.25}$$

This implies the transformation (II.1.22). In fact, since:

$$\delta\phi \equiv [L_\epsilon, \phi(\zeta)] \tag{II.1.26}$$

L_ϵ can be rewritten as in (II.1.23) with $\epsilon(z) = z^{N+1}$, getting so:

$$\delta\phi = \frac{1}{2\pi i}\left\{\oint_{|z|>|\zeta|} dz\, \epsilon(z) T(z) \phi(\zeta) - \oint_{|z|<|\zeta|} dz\, \epsilon(z) \phi(\zeta) T(z)\right\} =$$

$$= \frac{1}{2\pi i} \oint_{\zeta} dz\, \epsilon(z)\, T(z)\, \phi(\zeta) \tag{II.1.27}$$

where the integral is performed in the complex plane z around the point ζ . Therefore only the singular terms in the o.p.e. contribute reproducing the first term in the transformations (II.1.22). Hence the singular terms in the o.p.e. of T(z) and a primary field ϕ are completely fixed by the conformal invariance of the theory. The energy-momentum tensor is also a primary field with conformal dimension $\Delta = 2$, implying the following o.p.e.:

$$T(z)T(\zeta) = \frac{\partial/\partial\zeta\, T(\zeta)}{z-\zeta} + 2\,\frac{T(\zeta)}{(z-\zeta)^2} + \frac{c/2}{(z-\zeta)^4} \tag{II.1.28}$$

where the last term containing an arbitrary parameter c is allowed for a primary field with conformal dimension $\Delta = 2$ being consistent with the closure of the conformal algebra. From (II.1.28) one gets:

$$\delta T(\zeta) \equiv [L_\epsilon, T(\zeta)] = \oint_\zeta dz\, \epsilon(z) \left\{ \frac{\partial/\partial\zeta\, T(\zeta)}{z-\zeta} + 2\frac{T(\zeta)}{(z-\zeta)^2} + \frac{c/2}{(z-\zeta)^3} \right\} \tag{II.1.29}$$

and performing the integral gives:

$$[L_\epsilon, T(\zeta)] = \left[\epsilon(z)\frac{\partial}{\partial\zeta} + 2\epsilon'(z)\right] T(\zeta) + \frac{c}{12}\, \epsilon'''(\zeta) \tag{II.1.30}$$

that implies the Virasoro algebra [8] :

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}\, n(n^2-1)\, \delta_{n+m;0} \tag{II.1.31}$$

The value of the central charge c is the parameter of the theory. For a free massless bosonic theory described by the action:

$$S = -\frac{1}{4\pi} \int d\tau \int_0^{2\pi} d\sigma\, \partial_\alpha \phi\, \partial^\alpha \phi \tag{II.1.32}$$

the two independent components of the energy-momentum tensor are:

$$\vartheta^{++} = \frac{1}{2\pi} : (\dot\phi - \phi')^2 : \qquad \vartheta^{--} = \frac{1}{2\pi} : (\dot\phi + \phi')^2 : \tag{II.1.33}$$

and

$$T(z) = - : \left(\frac{\partial\phi}{\partial z}\right)^2 : \tag{II.1.34}$$

§ II.2 - COVARIANT QUANTIZATION

The treatment of a conformal invariant theory given in the pre vious section is very useful for the covariant quantization of the bosonic string.

Differently from the light-cone quantization, in the covariant quantization one does not reduce the number of degrees of freedom but the constraints are disregarded, all commutators for the oscillators are considered and then the constraints on the dynamical system are imposed. This procedure is covariant, but leads to an indefinite metric space. In this case, therefore, one imposes the following covariant commutation relations for the oscillators:

$$[a_{n\mu}, a^{+}_{m\nu}] = g_{\mu\nu}\,\delta_{nm} \qquad \text{(II.2.1)}$$

and for the centre of mass variables

$$[q^{\mu}, p^{\nu}] = i g^{\mu\nu} \qquad \text{(II.2.2)}$$

These commutation relations follow from requiring the canonical commutation relations:

$$[x^{\mu}(\tau,\sigma), P^{\nu}(\tau,\sigma')] = i g^{\mu\nu}\,\delta(\sigma-\sigma') \qquad \text{(II.2.3)}$$

where the four momentum density can be obtained starting from the Lagrangian (I.1.10):

$$P_{\mu}(\tau,\sigma) = \frac{\partial L}{\partial \dot{x}^{\mu}} = \frac{1}{\pi}\dot{x}_{\mu} \qquad \text{(II.2.4)}$$

The commutatio between (II.2.1) and (II.2.3) is obtained by using the following definition of the δ - function:

$$\sum_{n=-\infty}^{+\infty} \cos n\sigma \cos n\sigma' = \pi\,\delta(\sigma-\sigma') \qquad \text{(II.2.5)}$$

valid for functions expandable in a Fourier series of $\cos(n\sigma)$. The generators L_n of the conformal transformations given in the classical theory by (I.1.38) can be also defined in the quantum theory by introducing the normal ordered expression:

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z) \qquad \text{(II.2.6)}$$

where

$$T(z) = -\frac{1}{2} : \left(\frac{\partial x}{\partial z}\right)^2 : \qquad \text{(II.2.7)}$$

The Virasoro generators satisfy the Virasoro algebra with c=D, where D is the dimension of the space-time. In terms of the harmonic oscillators they assume the following form:

$$L_n = \sqrt{n}\, p \cdot a_n + \sum_{m=1}^{\infty} \sqrt{m(n+m)}\, a_{n+m}\, a^+_m + \frac{1}{2} \sum_{m=1}^{n-1} a_m \cdot a_{n-m} \sqrt{m(n-m)} \qquad n>0 \qquad \text{(II.2.8a)}$$

$$L_{-n} = L^+_n \qquad \text{(II.2.8b)}$$

$$L_0 = p^2/2 + \sum_{n=1}^{\infty} n\, a^+_n a_n \qquad \text{(II.2.8c)}$$

Because of the Lorentz covariance of the commutation relations (II.2.1), as in the Gupta-Bleuler quantization of Q.E.D., the space spanned by the oscillators in (II.2.1) contains negative norm states. The space of the on-shell physical states is a subspace of the entire linear space defined by (II.2.1). It is characterized by the conditions [9] :

$$(L_0 - 1) |\text{Phys}\rangle = 0 \qquad \text{(II.2.9a)}$$

$$L_n |\text{Phys}\rangle = 0 \qquad n>0 \qquad \text{(II.2.9b)}$$

Like in the quantization in the light-cone gauge, an arbitrary constant can be added to L_0, that in the covariant gauge must be chosen to be equal to 1 if one requires the same spectrum as in the light-cone gauge. The value 1 can be obtained only if one introduces the coordinates associated to the ghosts as a consequence of having fixed the conformal gauge: therefore this would be the correct way to proceed. But since our interest is devoted to constructing tree diagrams for strings, we can neglect the ghost coordinates. Furthermore, zero norm states that satisfy the conditions (II.2.9) and that are decoupled from the physical spectrum must be eliminated by hand from the solutions of (II.2.9). The spectrum is determined by the mass-shell condition (II.2.9a) and it is given by:

$$\left(1+\alpha' M^2 - \sum_{n=1}^{\infty} n\, a^{+\mu}_n a^{\mu}_n\right)|Phys\rangle = 0 \qquad \text{(II.2.10)}$$

This time all Lorentz components of the oscillators are present and not only the transverse oscillators. In order to eliminate the negative norm states present among the solutions of (II.2.10) one needs to impose the additional constraints (II.2.9b). The state with lowest mass is the vacuum state $|0\rangle$ that satisfies (II.2.9b) for any positive n and (II.2.10) if $M^2 = -1/\alpha'$. Therefore $|0\rangle$ corresponds to a tachyon. The following level is spanned by the states $a^+_{1\mu}|0\rangle$ corresponding to a massless photon-like particle. the only non trivial condition that one gets from (II.2.9b) on a combination of photon states comes from L_1 and reduces to:

$$L_1\, \alpha^\mu a^+_{1\mu}|0\rangle = (p\cdot q_1)\, \alpha^\mu a^+_{1\mu}|0\rangle \qquad \text{(II.2.11)}$$

where α^μ are arbitrary coefficients and p^μ is the four-momentum of the photon. (II.2.11) is the Lorentz condition imposed on the physical states in the Gupta-Bleuler quantization of Q.E.D. It requires a restriction of the parameters α^μ:

$$p^\mu \alpha_\mu = 0 \qquad \text{(II.2.12)}$$

If one chooses a refernce frame where the momentum of the photon is given by $p^\mu = (0, 0, \dots, 0, p, p)$, then (II.2.12) implies that the only physical states are:

$$\alpha^i a^+_{1i}|0\rangle + \alpha(a^+_{1,D-1} - a^+_{1,D})|0\rangle \qquad \text{(II.2.13)}$$

where α^i and α are arbitrary parameters. This is the most general state of the level $N = 1$, satisfying the conditions (II.2.9). The first term contains states with positive norm, while the last term corresponds to a state with zero norm, that is orthogonal to all other physical states since it can be written as follows:

$$(a^+_{1,D-1} - a^+_{1,D})|0\rangle = L^+_1 |0\rangle \qquad \text{(II.2.14)}$$

in the reference frame where $p^\mu = (0, 0, \ldots, p, p)$. Because of the previous property it is decoupled from the physical states together with its conjugate:

$$(a^+_{1,D-1} - a^+_{1,D})|0\rangle \qquad \text{(II.2.15)}$$

Therefore only transverse physical states survive and these are the same found at the level N = 1 in the light-cone gauge.

For the level N = 2 the most general state with $M^2 = 1/\alpha'$ is given by:

$$|\psi\rangle = (\alpha^{\mu\nu} a^+_{1\mu} a^+_{1\nu} + \beta^\mu a^+_{2\mu})|0\rangle \qquad \text{(II.2.16)}$$

with $\alpha^{\mu\nu}$ and β^μ arbitrary parameters. In the centre of mass frame where $p^\mu = (\vec{0}, M)$, the most general physical state satisfying the conditions (II.2.9) is given by:

$$|Phys\rangle = \alpha^{ij}\left[a^+_{1i} a^+_{1j} - \frac{\delta_{ij}}{D-1}\sum_{n=1}^{D-1} a^+_{1n} a^+_{1n}\right]|0\rangle + \\ + \beta^i \left[a^+_{2i} + a^+_{1D} a^+_{1i}\right]|0\rangle + \\ + \left(\sum_i \alpha_{ii}\right)\left[\sum_{i=1}^{D-1} a^+_{1i} a^+_{1i} + \frac{D-1}{5}\left(a^{+2}_{1D} - 2 a^+_{2D}\right)\right]|0\rangle \qquad \text{(II.2.17)}$$

where the indices i,j run over the D - 1 space components. The first term corresponds to a spin 2 in D - 1 dimensional space and has a positive norm. The second term has zero norm and it is orthogonal to the other physical state since it can be written in the form $L^+_1 a^+_{1i} |0\rangle$. It must be therefore eliminated from the physical spectrum together with its conjugate. Finally the last state is spinless and has a norm given by:

$$2(D-1)(26-D) \qquad \text{(II.2.18)}$$

If D< 26 [10],[11] it corresponds to a physical spin zero particle with positive norm. If D > 26 it is a ghost. Finally if D = 26 [12] it has a zero norm and is also orthogonal to the other physical states since it can be written in the form:

$$(2L_2^+ + 3L_1^{+2})|0\rangle \tag{II.2.19}$$

It does not belong to the physical spectrum. In conclusion if D=26 one finds at the level N = 2 the same number of physical states as in the light-cone gauge. If instead D< 26 the transverse oscillators are not sufficient to reproduce the full degeneracy; one has also to add the so-called Brower's states.

By following a similar procedure, it is possible to quantize the closed string. One gets in this case two sets of harmonic oscillators and of conformal generators. The on-shell physical states are characterized by the following conditions:

$$L_n|\text{Phys}\rangle = \bar{L}_n|\text{Phys}\rangle = 0 \quad n>0 \tag{II.2.20a}$$

$$(L_0+\bar{L}_0-2)|\text{Phys}\rangle = (L_0-\bar{L}_0)|\text{Phys}\rangle = 0 \tag{II.2.20b}$$

and for D = 26 one gets the same number of physical states as in the light-cone gauge.

In conclusion, for D = 26 the covariant and the light-cone quantization give the same spectrum of states. If D< 26 it is not possible to quantize the string in the light-cone gauge keeping Lorentz invariance. On the other hand the covariant procedure developed in this section seems to work also for D< 26 and one needs also non transverse oscillators (Brower's state) in order to describe the full spectrum.

The apparent disagreement between the covariant and light-cone quantization is due to the fact that the procedure followed in this section is not quite correct because, fixing the conformal gauge, we have neglected the contribution of the ghosts, that is an important ingredient anytime one quantizes a gauge theory covariantly. One can show that the inclusion of the ghosts eliminates any contradiction between covariant and light-cone quantization.

Chapter III

INTERACTING BOSONIC STRING: TREE DIAGRAMS

§ III.1 - VERTEX OPERATORS FOR OPEN STRINGS

The interaction among strings can be constructed by adding to the free action, discussed in the chapter I, a term describing the interaction of a string with an external field [13]:

$$S_{INT} = \int d^D y \; \phi_L(y) J_L(y) \tag{III.1.1}$$

where $\phi_L(y)$ is the external field and $J_L(y)$ is the current generated by the string. The index L stands for possible Lorentz indices that are saturated in order to have a Lorentz invariant action.

S_{INT} will describe the interaction among strings because the only external fields that can consistently interact with a string are exactly those that correspond to the various states of the string, as it will be shown later. This follows from the fact that, for the sake of consistency, the following restrictions on S_{INT} must be required:

1) after quantizing the theory, it must be a well defined operator in the space spanned by the string oscillators;
2) it must preserve the invariances of the free string theory and, in particular, in the conformal gauge it must be conformal invariant;

3) in the case of an open string the interaction occurs at the end points of a string, say at $\sigma = 0$; this follows from the fact that two open strings interact attaching to each other at the end points.

Let us first consider the open string interaction.

The simplest scalar current generated by the motion of a string can be written as follows:

$$J(y) = \int d\tau \int d\sigma\, \delta(\sigma)\, \delta^{(D)}\left[y^\mu - x^\mu(\tau,\sigma)\right] \tag{III.1.2}$$

where $\delta(\sigma)$ has been introduced according the requirement 3). In (III.1.2) a coupling constant g has been omitted for the sake of simplicity. Let us suppose that the external field is a plane wa ve, $\phi(y) = e^{iky}$. Then inserting (III.1.2) in (III.1.1) gives:

$$S_{INT} = \int d\tau :e^{ik^\mu x_\mu(\tau,0)}: \tag{III.1.3}$$

where the normal ordering has been introduced in order to have a well defined operator according to 1). Another requirement is that S_{INT} must be conformal invariant, that is to say, S_{INT} must be invariant under a conformal transformation $\omega \to \tau = f(\omega)$ This implies the following relation is satisfied

$$S_{INT} = \int d\tau :e^{ik^\mu x_\mu(\tau,0)}: = \int d\omega \left(:e^{ik^\mu x_\mu(\omega,0)}:\right)^T \tag{III.1.4}$$

where $\left(:e^{ik^\mu x_\mu(\omega,0)}:\right)^T$ denotes the transformed of $:e^{ik^\mu x_\mu(\tau,0)}:$ under the previous transformation. By considering the jacobian $f'(\omega)$ of the latter, one can write (III.1.4) as follows:

$$\left(:e^{ik^\mu x_\mu(w,0)}:\right)^T = f'(w):e^{ik^\mu x_\mu(\tau,0)}: \tag{III.1.5}$$

Taking into account the transformation properties (II.1.21) under a conformal transformation, the requirement (III.1.5) implies that the vertex operator $:e^{ik^\mu x_\mu(\tau,0)}:$ must transform as a conformal field characterized by $\Delta = 1$.

By introducing the variable $z = e^{i\tau}$ and considering the vertex operator as a function of it, (III.1.5) becomes:

$$:e^{ik^\mu x_\mu(\tau,0)}: = iz\left(:e^{ik^\mu x_\mu(z,0)}:\right)^T \tag{III.1.6}$$

In the following we will omit to write explicitely the dependence on σ, putting $x_\mu(\tau,0) \equiv x_\mu(\tau)$.

As already seen in the chapter III, the transformation properties of a conformal field can be deduced from its o.p.e. with the energy-momentum tensor characterizing the theory. Hence the transformation properties of $:e^{ik^\mu x_\mu(z)}:$ are determined by its o.p.e. with the energy-momentum tensor (II.2.7).

By using the propagator:

$$\langle x^\mu(z)\, x^\nu(\zeta)\rangle = -g^{\mu\nu}\log(z-\zeta) \tag{III.1.7}$$

and its derivatives, one has:

$$T(z):e^{ik\cdot x(\zeta)}: = \frac{\partial/\partial\zeta :e^{ik\cdot x(\zeta)}:}{z-\zeta} + \frac{k^2/2 :e^{ik\cdot x(\zeta)}:}{(z-\zeta)^2} + \text{reg. terms} \tag{III.1.8}$$

Inserting in (III.1.8) the definition of L_n in terms of $T(z)$

(II.2.6) gives:

$$\delta_n (:e^{ik\cdot x(\zeta)}:) \equiv [L_n, :e^{ik\cdot x(\zeta)}:] =$$

$$= \left[\zeta^{n+1}\frac{\partial}{\partial\zeta} + \frac{k^2}{2}(n+1)\zeta^n\right] :e^{ik\cdot x(\zeta)}: \quad \text{(III.1.9)}$$

from which one can deduce that $\Delta = k^2/2$.

Having previously shown that $:e^{ik\cdot x(z)}:$ is a conformal field with $\Delta = 1$, one can conclude that S_{INT} in (III.1.3) is conformal invariant only if the external field is on shell with $k^2 = 2$, corresponding to the tachyonic lowest state of the bosonic string.

The tachyonic state can be obtained from the vertex operator in the following way:

$$\lim_{z\to 0} :e^{ik\cdot x(z)}:|0\rangle = |0,k\rangle \quad \text{(III.1.10)}$$

and by using the following explicit formula in terms of the harmonic oscillators:

$$:e^{ik\cdot x(z)}: = e^{k\cdot\sum_1^\infty \frac{a_n^+}{\sqrt{n}} z^n} e^{ik\cdot q} e^{k\cdot p \log z} e^{-k\sum_1^\infty \frac{a_n}{\sqrt{n}} z^{-n}} \quad \text{(III.1.11)}$$

where $|0\rangle$ is the vacuum of the oscillators and of the zero mode p^μ, while $|0,k\rangle = e^{ik\cdot q}|0\rangle$. The latter represents a state with momentum k for the zero mode and the vacuum of modes n = 1, 2, ..., i.e. it satisfies:

$$p_\mu |0,k\rangle = k_\mu |0,k\rangle \quad \text{(III.1.12)}$$

$$a_{n\mu} |0,k\rangle = 0 \qquad n = 1,2\ldots \quad \text{(III.1.13)}$$

Furthermore the vertex operator (III.1.11) satisfies the follo wing hermicity property:

$$\left[: e^{-ik^\mu x_\mu(\frac{1}{z})} : \right]^+ = z^2 : e^{ik^\mu x_\mu(z)} : \tag{III.1.14}$$

It holds the following relation:

$$\lim_{z \to 0} \langle 0 | \left[: e^{ik\cdot x(z)} : \right]^+ = \langle 0, k | \tag{III.1.15}$$

As a function of τ , the vertex operator can be written in the following way:

$$: e^{ik\cdot x(\tau)} : \equiv : e^{ik\cdot Q(z)} : =$$
$$= e^{ik^\mu Q_\mu^{(+)}(z)} \, e^{ik^\mu Q_\mu^{(0)}(z)} \, e^{ik^\mu Q_\mu^{(-)}(z)} \tag{III.1.16}$$

In terms of the creation and annihilation operators $Q^+(z)$, $Q^-(z)$ and $Q^0(z)$ are given by:

$$Q^+(z) = \sum_{n=1}^{\infty} \frac{a_n^+}{\sqrt{n}} z^n \qquad Q^-(z) = \sum_{n=1}^{\infty} \frac{a_n^-}{\sqrt{n}} z^{-n} \tag{III.1.17}$$

$$Q^{(0)}(z) = q - i p \cdot \log z \tag{III.1.18}$$

and therefore they can be interpreted as the terms with positi ve, negative and zero frequencies of the fiel $Q_\mu(z)$ introduced by Fubini and Veneziano [17] :

$$Q_\mu(z) = q_\mu - i p_\mu \log z + i \sum_1^{\infty} \frac{1}{\sqrt{n}} (a_n z^{-n} - a_n^+ z^n) \tag{III.1.19}$$

Another simple current generated by the string is given by:

$$J_\mu(y) = \int d\tau \int d\sigma \, \delta(\sigma) \, \dot{x}_\mu(\tau,\sigma) \, \delta^{(D)}(y^\mu - x^\mu(\tau,\sigma)) \tag{III.1.20}$$

Inserting this in (III.1.1) gives [13]:

$$S_{INT} = \int d\tau \, \dot{x}_\mu(\tau,0)\, \epsilon^\mu e^{ik\cdot x(\tau,0)} \qquad \text{(III.1.21)}$$

if one chooses a plane wave for $\phi_\mu(y) = \epsilon_\mu e^{ik\cdot y}$. In order to check the conformal invariance of (III.1.21) the following o.p.e. must be computed:

$$T(z)V(\zeta;k) = \frac{\partial/\partial\zeta\, V(\zeta,k)}{z-\zeta} + \left(1+\frac{k^2}{2}\right)\frac{V(\zeta;k)}{(z-\zeta)^2} + \frac{(\epsilon\cdot k) :e^{ik\cdot x(\zeta)}:}{(z-\zeta)^3} + \text{reg. terms} \qquad \text{(III.1.22)}$$

where

$$V(\zeta,k) = \epsilon^\mu x'_\mu(\zeta)\, e^{ik\cdot x(\zeta)} \qquad \text{(III.1.23)}$$

and

$$x'_\mu(z) = \frac{dx_\mu}{dz} = -\frac{i}{z}\left[p_\mu + \sum_1^\infty \sqrt{n}\,(a_n z^{-n} + a_n^+ z^n)\right] \qquad \text{(III.1.24)}$$

The requirement 2) of conformal invariance implies:

$$k^2 = \epsilon\cdot k = 0 \qquad \text{(III.1.25)}$$

This shows that the external vector must be the massless photon state of the string. The photonic state can be obtained from the vertex operator in the following limits:

$$\lim_{z\to 0} \frac{dx^\mu}{dz}\epsilon_\mu e^{ik\cdot x(z)}|0\rangle = (-i)\,\epsilon\cdot a_1^+ |0,k\rangle \qquad \text{(III.1.26)}$$

$$\lim_{z\to 0} \langle 0| \left[\frac{dx^\mu}{dz}\epsilon_\mu e^{ik\cdot x(z)}\right]^+ = i\langle 0,k|\epsilon\cdot a_1 \qquad \text{(III.1.27)}$$

The photon vertex operator (III.1.23) satisfies the following hermicity property:

$$\left[V\left(\frac{1}{z}; -k\right) \right]^{+} = -z^{2} V(z; k) \qquad \text{(III.1.28)}$$

From these examples one can deduce that an open string can interact with an external field in a consistent way only if it corresponds to an on shell state of the string.

It is possible to write the most general current generated by the string as a combination of terms of the type [14], [15]:

$$J^{(1)}_{\mu_1 \cdots \mu_{n_1}}{}^{(1)}{}_{;\,\mu_1}{}^{(2)} \cdots {}_{\mu_{n_2}}{}^{(2)}{}_{;\,\cdots}(y) =$$

$$= \int d\tau\, \delta(y^{\mu} - x^{\mu}(\tau, 0)) \left[\frac{\partial x_{\mu_1^{(1)}}}{\partial z} \cdots \frac{\partial x_{\mu_{n_1}}}{\partial z} \right] \left[\frac{\partial^2 x_{\mu_1}}{\partial z^2} \cdots \frac{\partial^2 x_{\mu_2}}{\partial z^2} \right] \ldots \qquad \text{(III.1.29)}$$

The corresponding vertex operator is a combination of terms of the type:

$$V_{\alpha}(z; k) = :\left(\frac{\partial x^{\mu_1^{(1)}}}{\partial z} \cdots \frac{\partial x^{\mu_{n_1}^{(1)}}}{\partial z} \right) \left(\frac{\partial^2 x^{\mu_1^{(2)}}}{\partial z^2} \cdots \frac{\partial^2 x^{\mu_{n_2}^{(2)}}}{\partial z^2} \right) \cdots$$

$$\cdots \epsilon^{(1)}_{\mu_1 \cdots \mu_{n_1}}{}^{(1)}{}_{\mu_1}{}^{(2)} \cdots {}_{\mu_{n_2}}{}^{(2)} \cdots e^{ik\cdot x} : \qquad \text{(III.1.30)}$$

with the same amount of Lorentz indices and with the restriction $\sum_i n_i = N$ in order to describe states at the same level. In (III.1.30) the normal ordering has been inserted in order to have a well defined operator and the polarization tensor $\epsilon^{(1)}_{\mu_1 \cdots \mu_{n_1}}{}^{(1)}{}_{\mu_2}{}^{(2)} \cdots {}_{\mu_{n_2}}{}^{(2)}{}_{\mu_1}{}^{(3)} \cdots {}_{\mu_{n_3}}{}^{(3)} \cdots$ must be chosen to be orthogonal to k_{μ} for those indices that are not saturated.

The requirement of the conformal invariance implies the following o.p.e.:

$$T(z)\, V_{\alpha}(\zeta; k) = \frac{\partial/\partial\zeta\, V_{\alpha}(\zeta; k)}{z - \zeta} + \frac{V_{\alpha}(z; k)}{(z-\zeta)^2} + \text{reg. terms} \qquad \text{(III.1.31)}$$

In general for a term of the type (III.1.30) higher singularities will be present and they must be cancelled by taking suitable combinations of terms of the type (III.1.30). The coefficient of the term $(z-\zeta)^2$ is the same for each term and is given by:

$$\frac{k^2}{2} + \sum_i i n_i = \frac{k^2}{2} + N \qquad \text{(III.1.32)}$$

Conformal invariance then implies that:

$$\frac{k^2}{2} + N = 1 \qquad \text{(III.1.33)}$$

This is the mass-shell condition for an arbitrary state of the string. In addition (III.1.31) implies that:

$$[L_n, V_\alpha(z;k)] = \frac{d}{dz}\left[z^{n+1} V_\alpha(z;k)\right] \qquad \text{(III.1.34)}$$

and therefore the state:

$$|\alpha\rangle = \lim_{z\to 0} V_\alpha(z;k)|0\rangle \qquad \text{(III.1.35)}$$

satisfies the conditions (II.2.9) for an on-shell physical state of the string.

We can conclude by saying that the requirements 1), 2) and 3) imply that the external field must be one of the on-shell physical states of the string, the interaction of which with the string is described by a vertex operator $V_\alpha(z;k)$ satisfying the following conditions:

$$\lim_{z\to 0} V_\alpha(z;k)|0\rangle = |\alpha;k\rangle \qquad \text{(III.1.36a)}$$

$$\lim_{z\to 0} \langle 0|[V_\alpha(z;k)]^+ = \langle\alpha;k| \qquad \text{(III.1.36b)}$$

$$V_\alpha^+\left(\frac{1}{z};-k\right) = (-1)^N z^2 V_\alpha(z;k) \qquad \text{(III.1.36c)}$$

$$[L_n, V_\alpha(z;k)] = \frac{d}{dz}\left[z^{n+1} V_\alpha(z;k)\right] \qquad \text{(III.1.36d)}$$

In the previous chapters it has been shown that for D = 26 the physical states are given by an infinite set of transverse harmonic oscillators. It is possible to construct explicitly a complete and orthoneormal set of oscillators that generate all the physical transverse states at the critical dimension D = 26. At these dimensions, in fact, it is possible to write explicitly the vertex operator for an arbitrary physical state by using only the tachyon and the photon vertex operators. It is given by the following expression:

$$V_{\{N_j;i_j\}}(z,\pi) = z_{\{N_j,i_j\}} \prod_j \left[\frac{1}{2\pi i}\oint_z dz_j\, x'_\mu(z_j)\,\epsilon^\mu_{i_j}\, e^{-iN_j k\cdot x(z_j)}\right] \cdot V(z;p) \qquad \text{(III.1.37)}$$

where the integrals over the variables z_j are evaluated along a curve of the complex plane z_j containing the point z. In order to have only pole singularities this condition must be satisfied:

$$2\alpha' p\cdot k = 1 \qquad \text{(III.1.38)}$$

V(z;p) is the tachyon vertex and the momentum π of the operator (III.1.37) is:

$$\pi = p - \sum_j N_j k \qquad \text{(III.1.39)}$$

This satisfies the properties (III.1.36c) and (III.1.36d) and it reproduces the transverse states:

$$\lim_{z\to 0} V_{\{N_j;i_j\}}(z,\pi)|0\rangle = \prod_j A_{i_j,N_j}|0,p\rangle \qquad \text{(III.1.40)}$$

where

$$A_{i,N} = \frac{1}{2\pi i} \oint dz\, x'_\mu(z)\, \epsilon_i^\mu\, e^{-iNK\cdot x(z)} \tag{III.1.41}$$

The operators (III.1.41) satisfy the algebra of the non relativistic harmonic oscillators [15]:

$$[A_{n,i}\,;\, A_{m,j}] = n\,\delta_{ij}\,\delta_{n+m;0} \tag{III.1.42}$$

and they commute with the gauge operators L_n:

$$[L_m, A_{n,i}] = 0 \tag{III.1.43}$$

The transverse states (III.1.40) form a complete and orthogonal basis in the subspace of the physical states if D = 26. They are also orthonormal if one chooses the normalization in front of (III.1.37) as follows:

$$z_{\{N_j;i_j\}} = \prod_h \frac{1}{\sqrt{\lambda_h!}} \prod_j \frac{1}{\sqrt{N_j}} \tag{III.1.44}$$

where λ_h is the multiplicity of the operator A_h in the product in (III.1.40).

In conclusion it has been constructed a complete and orthonormal basis in the space of the physical states and the corresponding vertex operators [18],[19],[20] that can be used to compute scattering amplitudes for any physical state of the string.

§ III.2 - VERTEX OPERATORS FOR CLOSED STRINGS

For a closed string one can follow the same procedure as in the case of an open string with the only difference that the property 3) of the section I does not hold. The properties 1) and 2) keep on being valid.

As in the case of the open string, let us introduce the sim plest scalar current generated by the closed string:

$$J(y) = \int d\tau \int d\sigma \, \delta^{(D)}(y^\mu - x^\mu(\tau,\sigma)) \tag{III.2.1}$$

Inserting (III.2.1) in (III.1.1) gives the following vertex operator:

$$S_{INT} = \int d\tau \int d\sigma \, : e^{iK\cdot x(\tau,\sigma)} : \tag{III.2.2}$$

Also here it has been introduced the normal ordering prescrition in order to have a well defined operator. By considering (I.1.16) one can write $x_\mu(\tau, \sigma)$ as follows:

$$x_\mu(\tau,\sigma) = x_\mu(\xi^+) + \bar{x}_\mu(\xi^-) \tag{III.2.3}$$

where

$$x^\mu(\xi^+) = \frac{1}{2}\left[q^\mu + p^\mu \xi^+ + i \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-2in\xi^+}\right] \tag{III.2.4}$$

and

$$\bar{x}^\mu(\xi^-) = \frac{1}{2}\left[q^\mu + p^\mu \xi^- + i \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-2in\xi^-}\right] \tag{III.2.5}$$

with $\xi^\pm = \tau \pm \sigma$.

The zero modes in (III.2.4) and (III.2.5) are the same, but the non zero modes are completely independent. The decomposition (III.2.3) implies that (III.2.2) can be written as follows:

$$S_{INT} = \int d\xi^+ : e^{ik^\mu x_\mu(\xi^+)} \int d\xi^- e^{ik^\mu \bar{x}_\mu(\xi^-)} : \tag{III.2.6}$$

Conformal invariance requires that the vertex operator transforms as a primary field with both right and left dimensions $\Delta = \bar{\Delta} = 1$

These conditions imply:

$$\frac{k^2}{8} = 1 \tag{III.2.7}$$

or, by introducing α' that has been considered equal to ½:

$$\alpha'_c k^2 = 2 \quad , \quad \alpha'_c = \frac{\alpha'}{2} = \frac{1}{4} \tag{III.2.8}$$

This shows that the external field must have the same mass as the tachyonic state of a closed string. Introducing the two variables $z = e^{2i\xi^+}$ and $\bar{z} = e^{2i\xi^-}$ the tachyon of a closed string is described by the following vertex operator:

$$: e^{ik\cdot x(z)} \, e^{ik\,\bar{x}(\bar{z})} : \tag{III.2.9}$$

where the normal ordering for the zero modes is defined with the q-operator on the left of the p-operator as in (III.1.11). Hence the vertex of a tachyon of the closed string is the product of two vertices of the tachyon of the open string, that are functions of the variables z and $\bar{z}$ respectively.

Let us consider now the possibility to have the following current generated by the string:

$$J_{\mu\nu}(y) = \int d\sigma \int d\tau \, \frac{\partial x_\mu}{\partial \xi^+} \frac{\partial x_\nu}{\partial \xi^-} \, \delta(y_\mu - x_\mu(\tau,\sigma)) \tag{III.2.10}$$

from which, by using a plane wave $\phi_{\mu\nu}(y) = \epsilon_{\mu\nu} e^{ik\cdot y}$ for the external field, one gets the following interaction action:

$$S_{INT} = \int d\xi^+ \int d\xi^- \, \epsilon^{\mu\nu} \frac{\partial x_\mu}{\partial \xi^+} \frac{\partial x_\nu}{\partial \xi^-} \, e^{ik\cdot x(\xi^+, \xi^-)} \tag{III.2.11}$$

where $\epsilon^{\mu\nu}$ is the polarization tensor of the external field. From the decomposition (III.2.3) one has:

$$S_{INT} = \epsilon^{\mu\nu} \left(\int d\xi^+ \frac{\partial x_\mu}{\partial \xi^+} e^{ik^\mu x_\mu(\xi^+)} \right) \int d\xi^- \frac{\partial x_\nu}{\partial \xi^-} e^{ik^\mu x_\mu(\xi^-)} \tag{III.2.12}$$

This is conformal invariant if $k^2 = 0$. Therefore the external field corresponds to a state of the massless level of a closed string. If $\epsilon^{\mu\nu}$ is symmetric, then one gets the interaction of a string with an external gravitational field, while if $\epsilon^{\mu\nu}$ is antisymmetric one gets the interaction with an external antisymmetric tensor field. The conformal invariance, however, implies $k^\mu \epsilon_{\mu\nu} = k^\nu \epsilon_{\mu\nu} = 0$. If, finally, $\epsilon^{\mu\nu} = \eta^{\mu\nu}$ one gets the interaction of a string with an external dilaton field. It is possible to express the vertex for a massless state of a closed string in terms of the variables z and $\bar{z}$:

$$: \frac{dx_\mu(z)}{dz} e^{ik\cdot x(z)} \frac{d\bar{x}_\nu(\bar{z})}{d\bar{z}} e^{ik\cdot x(\bar{z})} : \tag{III.2.13}$$

The examples just considered show that the most general vertex operator for a closed string is, in general, a product of two vertex operators of an open string:

$$V_{\alpha\beta}(z,\bar{z},k) = : V_\alpha(z;k) V_\beta(\bar{z};k) \tag{III.2.14}$$

Also in this case, the normal ordering is defined so that the q-operator always appears on the left of the p-operator.

Furthermore, the condition (I.3.38b) implies that the two open string states α and β must be chosen to belong to the same level.

§ III.3 - SCATTERING AMPLITUDES FOR OPEN STRINGS

Our interest is now devoted to compute the probability amplitude for the emission of N-2 external fields from a string. In the perturbation theory the S-matrix for the emission of the external field from the string is given by:

$$S = \lim_{\substack{\tau_i \to -\infty \\ \tau_f \to +\infty}} T\left[e^{iS_{INT}}\right] =$$

$$= \lim_{\substack{\tau_i \to -\infty \\ \tau_f \to +\infty}} T\left[e^{i\int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \mathcal{L}_{INT}}\right] \tag{III.3.1}$$

where

$$\int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \mathcal{L}_{INT} = \int_{\tau_i}^{\tau_f} d\tau\, V_\alpha(e^{i\tau}; k) \tag{III.3.2}$$

V_α is the vertex operator corresponding to a certain external field. The amplitude for the emission of N-2 external fields is given by a sum of (N-2)! terms corresponding to the different terms of the T-ordered product in (III.3.1). A single term is given by:

$$A(\alpha_1,k_1;\dots;\alpha_N,k_N) = \\ = \int_0^{\infty} \prod_{i=3}^{N-2} d\tau_i\, \vartheta(\tau_{i+1}-\tau_i) \langle\alpha_1,k_1| \prod_{i=2}^{N-1} V_{\alpha_i}(e^{i\tau_i};k_i)|\alpha_N,k_N\rangle \tag{III.3.3}$$

where the variable τ_2 has been put equal to zero because of the translational invariance of the matrix element in (III.3.3). The integral in $d\tau_i$ is performed along the positive real axis. But the vertex operator depends on $e^{i\tau_i}$ and the integral in (III.3.3) is not well defined. In order to make it convergent a Wick rotation $\tau \to i\tau$ must be performed. Introducing the Koba-Nielsen variables $z_i = e^{-\tau_i}$ (III.3.3) can be rewritten as

$$A(\alpha_1,k_1;\dots;\alpha_N,k_N) = \\ = \int_0^{1} \prod_{i=3}^{N-2} \left[dz_i\, \vartheta(z_i-z_{i+1})\right] \langle\alpha_1,k_1| \prod_{i=2}^{N-1} V_{\alpha_i}(z_i,k_i)|\alpha_N,k_N\rangle \tag{III.3.4}$$

where

$$V_\alpha(z;k) = iz V_\alpha(e^{i\tau_i};k) \tag{III.3.5}$$

Performing the Wick rotation makes the integral well defined.

The scattering amplitude (III.3.4) can be written in a more symmetric way by introducing the Koba-Nielsen variables z_1,

z_2 and z_N for the states α_1, α_2 and α_N [20] as:

$$A(\alpha_1, k_1; \ldots; \alpha_N, k_N) = \\ = \int_{-\infty}^{+\infty} \underbrace{\prod_{i=1}^{N} dz_i}_{dV_{abc}} \vartheta(z_1 - z_2) \ldots \vartheta(z_{N-1} - z_N) \langle 0 | \prod_{i=1}^{N} V_{\alpha_i}(z_i; k_i) | 0 \rangle \tag{III.3.6}$$

where

$$dV_{abc} = \frac{dz_a \, dz_b \, dz_c}{(z_a - z_b)(z_a - z_c)(z_b - z_c)} \tag{III.3.7}$$

The three variables z_a, z_b, z_c can be fixed arbitrarily because the expression under the integral in (III.3.6) is invariant under the projective transformations, already introduced in the section II.1:

$$z_i \to z'_i = \frac{Az_i + B}{Cz_i + D} \quad , \quad AD - BC = 1 \tag{III.3.8}$$

dV_{abc} is left invariant under (III.3.8); the vertex operator is a primary field with $\Delta = 1$ and therefore its transformation is:

$$V_{\alpha_i}\left(\frac{Az_i + B}{Cz_i + D}; k_i\right) = (Cz_i + D)^2 \, V_{\alpha_i}(z_i; k_i) \tag{III.3.9}$$

In addition one has also:

$$dz'_i = \frac{dz_i}{(Cz_i + D)^2} \tag{III.3.10}$$

Hence $V_\alpha(z;k)\,dz$ is projective invariant. It is possible to have again the expression (III.3.3) fixing $z_a = z_1 = +\infty$, $z_b = z_2 = 1$ and $z_c = z_N = 0$.

For identical particles the scattering amplitude (III.3.6) is invariant under a cyclic permutation of the external legs:

$$A(1, 2, \ldots, N) = A(N, 1, 2, \ldots, N-1) \tag{III.3.11}$$

and under an anticyclic permutation :

$$A(1,2,\ldots,N) = A(N,N-1,\ldots,1) \qquad \text{(III.3.12)}$$

Therefore in order to get a crossing symmetric amplitude one must sum only over the (N - 1)!/2 permutations that are not cyclic or anticyclic:

$$A = \sum_{\substack{\{i_1 \ldots i_N\} \\ \text{non cyclic or} \\ \text{anticyclic}}} A(\alpha_{i_1}, k_{i_1}; \ldots; \alpha_{i_N}, k_{i_N}) \qquad \text{(III.3.13)}$$

This sum restaures the symmetry between the first and the last particle and the other N - 2 particles, since they have been treated differently. In fact the first and the last particles in (III.3.3) have been treated as states of the string, while the others as external fields. (III.3.13) restaures so this symmetry.

Furthermore, until now, it has not been considered an internal symmetry. This can be done by associating a matrix $(\lambda_i)_{ab}$ with the i-th external string state and by defining a so-called Chan-Paton factor[21] multiplying the amplitude (III.3.6):

$$\mathrm{tr}\,(\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_N}) \qquad \text{(III.3.14)}$$

This factor provides an invariant coupling that shares the cyclic symmetry properties of the amplitudes that they multiply. After having introduced in the theory internal symmetries, the string theory reduces to a Yang-Mills theory in the low energy (or "zero slope") limit in D = 26 dimensions.

§III.4 - SCATTERING AMPLITUDES FOR CLOSED STRINGS

The amplitude for the emission of N-2 external fields from a closed string can be obtained by using the general formula (III.3.1), getting:

$$A(\alpha_1,\beta_1,k_1;\alpha_2,\beta_2,k_2;\dots;\alpha_N,\beta_N,k_N)=$$
$$=\int_0^\infty \prod_{i=3}^{N-1} d\tau_i \int_0^\pi \prod_{i=3}^{N-1} d\sigma_i \langle\alpha_1,\beta_1,k_1|T\left(\prod_{i=2}^{N-1} V_{\alpha_i,\beta_i}(e^{2i\xi_i^+},e^{2i\xi_i^-};k_i)\right)|\alpha_N,\beta_N,k_N\rangle \tag{III.4.1}$$

where the variables τ_2 and σ_2 have been taken equal to zero because of the translational invariance of the matrix element. As in the case of an open string, also here a Wick rotation $\tau \to i\tau$ must be performed. So doing, the two variables z and $\bar{z}$ become one the complex conjugate of the other:

$$z = e^{-2\tau}\, e^{2i\sigma} \tag{III.4.2a}$$
$$\bar{z} = e^{-2\tau}\, e^{-2i\sigma} \tag{III.4.2b}$$

In terms of the variables z and $\bar{z}$ (III.4.1) becomes:

$$A(\alpha_1,\beta_1,k_1;\alpha_2,\beta_2,k_2;\dots;\alpha_N,\beta_N,k_N)=$$
$$=\int \prod_{i=3}^{N-1} d^2 z_i \langle\alpha_1,\beta_1,k_1|R\left(\prod_{i=2}^{N-1} V_{\alpha_i\beta_i}(z_i,\bar{z}_i,k_i)\right)|\alpha_N,\beta_N,k_N\rangle \tag{III.4.3}$$

where the T-ordering becomes now an ordering on the modulus of z and the integrals are performed over the entire complex plane of variables z_i.

§III.5 - SCATTERING AMPLITUDES FOR OPEN AND CLOSED STRINGS

In this section our interest is devoted to processes involving the emission of a closed string from an open one [23],[24] .Ac cording to the approach until now followed in order to describe the interacting string, one can regard the closed string as an external field. This implies therefore that one must add to the free string action an interaction term similar to (III.3.2), in which the vertex operator is given by (III.2.14).

Our aim is now the write down the amplitude for a process in which an initial open string state $|\alpha_1, p_1\rangle$ emits at the time τ_i a state of open string α_i of momentum p_i and at the time $\tilde{\tau}_j$ a state of closed string of momentum k_j, and finally, after the emission of N-1 states of open string and M of closed string, jumps in the open string state $|\alpha_N p_N\rangle$. The total amplitude for this process is, unifying the results obtained separately for open and closed strings:

$$\langle\alpha_1,p_1|\prod_{i=2}^{N-2}\prod_{j=1}^{M}\int_0^1 dx_i\, d^2z_j\, \vartheta(x_{i+1}-x_i)\langle\alpha_1,p_1|\, T^*\big(V_{\alpha_i}(x_i\, p_i)\cdot$$
$$\cdot W_{\beta_j}(z_j,\bar{z}_j,k_j)\big).V_{\alpha_{N-1}}(x_{N-1},p_{N-1})|\alpha_N,p_N\rangle \tag{III.5.1}$$

where W_{β_j} denotes the vertex operator of closed string. Furthermore $x_i = e^{-\tau_i}$ and $z_j = e^{-2\tau_j}e^{2i\sigma_j}$ with $\bar{z}_j$ being its complex conjugate. The open string variables x_i are integrated over the real axis, while the closed string variables z_j are integrated over the upper half complex plane. The T* prescrition now refers to the ordering of the vertex operators of open string among themselves and with respect to the ones of closed string according to the moduli of their variables z and x. (III.5.1) can be written , as already done in the previous sections, in a more symmetric form introducing the Koba-Nielsen variables, obtaining:

$$\int\frac{dV}{dV_{abc}}\langle 0|\, T^*\Big(\prod_{i=1}^{N}\prod_{j=1}^{M}V_{\alpha_i}(x_i,p_i)W_{\beta_j}(z_j,\bar{z}_j,k_j)\Big)|0\rangle \tag{III.5.2}$$

where

$$dV=\prod_{i=1}^{N}\prod_{j=1}^{M}dx_i\, d^2z_j\,\tilde{\vartheta}(x_{i+1}-x_i)$$

The x_i are integrated over the real axis: the $\Pi\tilde{\vartheta}$ means that the x_i are ordered in the projective sense. The volume dV_{abc} depends on what variables are kept fixed.

Chapter IV

SCATTERING AMPLITUDES FOR OPEN AND CLOSED STRINGS: EXAMPLES

In this chapter we give some results coming from explicit calculations of scattering amplitudes for open and closed strings.

§ IV.1 - SCATTERING AMPLITUDE FOR FOUR TACHYONS OF OPEN STRING

The scattering amplitude involving an arbitrary state of open string is given by the Koba-Nielsen formula (III.3.6), that in the case of four tachyons becomes:

$$A(k_1,k_2,k_3,k_4)=\int \frac{\prod_{i=1}^{4} dz_i}{dV_{abc}}\, \vartheta(z_1-z_2)\vartheta(z_2-z_3)\vartheta(z_3-z_4)\langle 0|\prod_{i=1}^{4} :e^{ik_i\cdot x(z_i)}:|0\rangle \qquad \text{(IV.1.1)}$$

The invariance of the theory under the three- parameter projective group transformations (III.3.8) allows one to fix any three variables. Let us perform the following choice:

$$z_1 = +\infty \qquad z_2 = 1 \qquad z_3 = z \qquad z_4 = 0 \qquad \text{(IV.1.2)}$$

Since:

$$\lim_{z_4\to 0} :e^{ik_4 x(z_4)}:|0\rangle = |\alpha_4,k_4\rangle \qquad \text{(IV.1.3a)}$$

$$\lim_{z_1\to +\infty} \langle 0|z_1^2 :e^{ik_1 x(z_1)} = \langle \alpha_1,-k_1| \qquad \text{(IV.1.3b)}$$

the amplitude (IV.1.1) becomes, after having put $\alpha_1=\alpha_4=0$:

$$A(k_1,k_2,k_3,k_4)=\int dz \langle 0,-k_1| :e^{ik_2 x(1)} : e^{ik_3 x(z_3)} :|0,k_4\rangle \qquad \text{(IV.1. 4)}$$

By using the explicit expression (III.1.11) of $e^{ik\cdot x(z)}$ in terms of the harmonic oscillators and the Baker-Hausdorf formula:

$$e^{A} e^{B} = e^{B} e^{A} e^{[A,B]}$$

one gets the following:

$$:e^{ik_2 x(z_2)}::e^{ik_3 x(z_3)}: = :e^{ik_2 x(z_2)} e^{ik_3 x(z_3)}: (z_2 - z_3)^{k_2 \cdot k_3} \tag{IV.1.5}$$

This allows to compute the expectation value in (IV.1.1), obtaining:

$$A(k_1, k_2, k_3, k_4) = \int_0^1 dz\, z^{k_3 \cdot k_4} (1-z)^{k_2 \cdot k_3} \tag{IV.1.6}$$

In terms of the Regge trajectories in the s- and t- channel:

$$\alpha_s = 1 - \alpha' (k_3 + k_4)^2 \qquad \alpha_t = 1 - \alpha' (k_2 + k_3)^2 \tag{IV.1.7}$$

one gets:

$$A(k_1, k_2, k_3, k_4) = \frac{\Gamma(-\alpha_s)\,\Gamma(-\alpha_t)}{\Gamma(-\alpha_s - \alpha_t)} \tag{IV.1.8}$$

By summing over the permutations that are not cyclic or anticyclic, one has:

$$A(k_1, k_2, k_3, k_4) = \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s - \alpha_t)} + \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_u)}{\Gamma(-\alpha_s - \alpha_u)} + \frac{\Gamma(-\alpha_t)\Gamma(-\alpha_u)}{\Gamma(-\alpha_t - \alpha_u)} \tag{IV.1.9}$$

This is just the famous Veneziano amplitude relative to the scattering of four scalar particles .

§ IV.2 - SCATTERING AMPLITUDE FOR FOUR TACHYONS OF CLOSED STRING

The scattering ampliutde for arbitrary closed string states is given by (III.4.3), that for four tachyions becomes:

$$A(k_1, k_2, k_3, k_4) = \int \frac{\prod_{i=1}^{4} d^2 z_i}{dV_{abc}} \langle 0 | R \prod_{i=1}^{4} V_{\alpha_i}(z_i, \bar{z}_i; k_i) | 0 \rangle \tag{IV.2.1}$$

where the variables z and $\bar{z}$ are given by $z = e^{2i(\tau+\sigma)}$ and $\bar{z} = e^{2i(\tau-\sigma)}$ respectively. After a Wick rotation $\tau \to i\tau$ $\bar{z}$ becomes the complex conjugate of z. In terms of the variables z and $\bar{z}$ the tachyon vertex is given by:

$$:e^{ik\cdot x(z)}\, e^{ik\cdot \bar{x}(\bar{z})}: \tag{IV.2.2}$$

where

$$x^\mu(z) = \frac{1}{2}\left[q^\mu - \frac{i}{2}p^\mu \log z + i\sum_{n\neq 0} \frac{\alpha_n^\mu}{n} z^{-n}\right] \tag{IV.2.3a}$$

$$\bar{x}^\mu(\bar{z}) = \frac{1}{2}\left[q^\mu - \frac{i}{2}p^\mu \log \bar{z} + i\sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} \bar{z}^{-n}\right] \tag{IV.2.3b}$$

By fixing

$$z_1 = +\infty \ , \quad z_2 = 1 \ , \quad z_3 = z \ , \quad z_4 = 0 \tag{IV.2.4}$$

and by using the properties (III.1.36) the amplitude (IV.2.1) can be written as follows:

$$\int d^2z \, \langle 0, -k_1 | V_{\alpha_2}(z_2,\bar{z}_2,k_2) V_{\alpha_3}(z_3,\bar{z}_3,k_3) | 0, k_4 \rangle \tag{IV.2.5}$$

where

$$V_{\alpha_2}(z_2,\bar{z}_2,k_2) V_{\alpha_3}(z_3,\bar{z}_3,k_3) =$$

$$= :V_{\alpha_2}(z_2,\bar{z}_2,k_2) V_{\alpha_3}(z_3,\bar{z}_3,k_3): (z_2-z_3)^{\frac{1}{4}k_2\cdot k_3} (\bar{z}_2-\bar{z}_3)^{\frac{1}{4}k_2\cdot k_3} \tag{IV.2.6}$$

The condition $z_2 = 1$ allows one to write:

$$A(k_1,k_2,k_3,k_4) = \int d^2z \, |1-z|^{\frac{1}{2}k_2\cdot k_3} z^{\frac{1}{2}k_3\cdot k_4} \tag{IV.2.7}$$

In terms of the Regge trajectories

$$\alpha_s = -2 - \frac{1}{2} k_1\cdot k_2 \tag{IV.2.8a}$$

$$\alpha_t = -2 - \frac{1}{2} k_2\cdot k_3 \tag{IV.2.8b}$$

$$\alpha_u = -2 - \frac{1}{2} k_1\cdot k_3 \tag{IV.2.8c}$$

and using the formula:

$$\int d^2z\ |z|^{2\alpha}\ |1-z|^{2\beta} = \pi\ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(\alpha+\beta+2)} \qquad (IV.2.9)$$

finally one gets:

$$A(k_1,k_2,k_3,k_4) = \pi\ \frac{\Gamma(-\frac{\alpha_s}{2})\Gamma(-\frac{\alpha_t}{2})\Gamma(-\frac{\alpha_u}{2})}{\Gamma(\frac{\alpha_s}{2}+1)\Gamma(\frac{\alpha_t}{2}+1)\Gamma(\frac{\alpha_u}{2}+1)} \qquad (IV.2.10)$$

that reproduces the scattering amplitude proposed by Shapiro and Virasoro [27],[28] .

§ IV.3 - SCATTERING AMPLITUDE FOR THREE TACHYONS OF OPEN STRING AND ONE TACHYONE OF CLOSED STRING

The scattering amplitude for mixed states is given by (III.5.2). For three tachyons of open string and one of coosed string one has:

$$A = \int \frac{dV}{dV_{abc}} \langle 0| T^* \left(\prod_{i=1}^{3} V_{\alpha_i}(x_i,p_i)\, W_{\beta_1}(z_1,\bar{z}_1,k_1)\right)|0\rangle \qquad (IV.3.1)$$

By fixing:

$$x_1 = +\infty\ ,\quad x_2 = 1\ ,\quad z_1 = z\ ,\quad x_3 = 0 \qquad (IV.3.2)$$

one can write (IV.3.1) in the following way:

$$\int d^2z\ \langle 0, -p_1|\, V_{\alpha_2}(1,p_2)\, W_{\beta_1}(z_1,\bar{z}_1,k_1)|0,p_3\rangle \qquad (IV.3.3)$$

where

$$V_{\alpha_2}(x_2,p_2)\, W_{\beta_1}(z_1,\bar{z}_1,k_1) = :V_{\alpha_2}(x_2,p_2)\, W_{\beta_1}(z_1,\bar{z}_1,k_1):\ (x_2-z_1)^{\frac{1}{2}p_2\cdot k_1}$$
$$\cdot\, |z_1-\bar{z}_1|^2 \cdot x_2^{\frac{1}{2}p_2\cdot k_1}\ x_2^{p_2\cdot p_3}\ z_1^{\frac{1}{4}k_1\cdot p_3}\ \bar{z}_1^{\frac{1}{4}k_1\cdot p_3} \qquad (IV.3.4)$$

Taking into account (IV.3.4), the amplitude (IV.3.3) becomes:

$$A = \int d^2 z \, |1-z|^{\frac{1}{2} p_2 \cdot k_1} \, |z|^{\frac{1}{2} k_1 \cdot p_3} \, |z-\bar{z}|^2 \tag{IV.3.5}$$

that can be written in the following way

$$A = 2\pi \frac{\Gamma(-\frac{1}{2}\alpha_s)\,\Gamma(-\frac{1}{2}\alpha_t)\,\Gamma(-\frac{1}{2}\alpha_u)}{\Gamma(-\frac{1}{2}\alpha_s-\frac{1}{2}\alpha_t)\,\Gamma(-\frac{1}{2}\alpha_t-\frac{1}{2}\alpha_u)\,\Gamma(-\frac{1}{2}\alpha_u-\frac{1}{2}\alpha_s)} \tag{IV.3.6}$$

in terms of the Regge trajectories:

$$\alpha_s = 1 - \frac{1}{2}(p_1+p_2)^2 \tag{IV.3.7a}$$

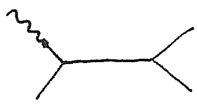
$$\alpha_t = 1 - \frac{1}{2}(p_2+p_3)^2 \tag{IV.3.7b}$$

$$\alpha_u = 1 - \frac{1}{2}(p_3+p_1)^2 \tag{IV.3.7c}$$

We want to observe that (IV.3.6) has been obtained considering the on-mass shell condition for the tachyon of closed string:

$$k^2 = 8 \tag{IV.3.8}$$

This leads to an absence of poles in k^2 in (IV.3.6), that is to say that under the condition (IV.3.8) one is considering the following picture:



If (IV.3.8) had not been fixed, then we would have found in (IV.3.6) a Γ function containing the poles in k^2, corresponding to the closed string tachyon - open string tachyon transition.

§ IV.4 - SCATTERING AMPLITUDE FOR THREE PHOTONS

The starting point for this computation is again (III.3.6) . For three photons, fixing:

$$z_1 = +\infty \ , \ z_2 = 1 \ , \ z_3 = 0 \qquad \text{(IV.4.1)}$$

the amplitude reduces simply to:

$$\lim_{\substack{z_1 \to +\infty \\ z_3 \to 0}} z_1^2 \langle 0 | \prod_{i=1}^{3} V_{\alpha_i}(z_i, k_i) | 0 \rangle \qquad \text{(IV.4.2)}$$

where

$$V_\alpha(z; k) = \epsilon_\nu \frac{P^\nu(z)}{z} e^{ik \cdot Q(z)} \qquad \text{(IV.4.3)}$$

with

$$Q_\mu = q_\mu - i p_\mu \log z + i \sum_1^\infty \frac{1}{\sqrt{n}} \left[a_{n\mu} z^{-n} - a^+_{n\mu} z^n \right] \qquad \text{(IV.4.4)}$$

$$P_\mu = -i \left[p_\mu + \sum_1^\infty \sqrt{n} \left(a_{n\mu} z^{-n} + a^+_{n\mu} z^n \right) \right] \qquad \text{(IV.4.5)}$$

Since:

$$\lim_{z_1 \to +\infty} z_1^2 \langle 0 | V_{\alpha_1}(z_1, k_1) = i \langle 0, -k_1 | \epsilon_1 \cdot a_1 \qquad \text{(IV.4.6)}$$

$$\lim_{z_3 \to 0} V_{\alpha_3}(z_3, k_3) | 0 \rangle = -i \left(\epsilon_3 \cdot a^+_1 \right) | 0, k_3 \rangle \qquad \text{(IV.4.7)}$$

one has for (IV.4.2) the following expression:

$$\epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \frac{i}{2} g_{\mu\nu} (k_{3\nu} - k_{1\nu}) \qquad \text{(IV.4.8)}$$

where the term $\frac{1}{2}(k_{3\nu} - k_{1\nu})$ follows from the hermicity of the operator q.

By adding to the symmetrized vertex just now computed the Chan-Paton factor:

$$\mathrm{Tr}\,(\lambda^a \lambda^b \lambda^c) \qquad \text{(IV.4.9)}$$

one gets:

$$\frac{i}{2}\,\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\{(k_{1\nu}-k_{2\nu})g_{\mu\rho}+(k_{3\mu}-k_{2\mu})\,g_{\nu\rho}+g_{\mu\nu}\cdot (k_{2\rho}-k_{1\rho})\}\cdot\delta(k_1+k_2+k_3) \qquad \text{(IV.4.10)}$$

that, apart from a constant factor, reproduces the three-gluon vertex in Q.C.D.

§ IV.5 - SCATTERING AMPLITUDE FOR FOUR PHOTONS

Starting from (III.3.6), one has for this case:

$$A(k_1,k_2,k_3,k_4)=\int_{-\infty}^{+\infty}\frac{\prod_{i=1}^{4}dz_i}{dV_{abc}}\,\vartheta(z_1-z_2)\,\vartheta(z_2-z_3)\,\vartheta(z_3-z_4)\cdot \langle 0|\,V_{\alpha_1}(z_1,k_1)V_{\alpha_2}(z_2,k_2)\cdot V_{\alpha_3}(z_3,k_3)\,V_{\alpha_4}(z_4,k_4)|0\rangle \qquad \text{(IV.5.1)}$$

Fixing, as usual, $z_a = z_1 = +\infty$

$z_b = z_2 = 1$

$z_c = z_N = 0$

and considering the relations (IV.4.6), one has:

$$\int_0^1 dz\,\langle 0,-k_1|\,V_{\alpha_2}(z_2;k_2)\,V_{\alpha_3}(z_3;k_3)\,|0,k_4\rangle \qquad \text{(IV.5.2)}$$

The photon vertex is defined by (IV.4.3) with

$$Q_\mu = q_\mu - i\,2\alpha' p_\mu \log z + i\sqrt{2\alpha'}\,\sum_1^\infty \frac{1}{\sqrt{n}}\left[a_{n\mu}z^{-n} - a^+_{n\mu}z^n\right] \qquad \text{(IV.5.3a)}$$

$$P_\mu = -i\left[p_\mu + \frac{1}{\sqrt{2\alpha'}} \sum_1^\infty \sqrt{n}\,(a_{n\mu} z^{-n} + a^+_{n\mu} z^n)\right] \tag{IV.5.3b}$$

Let us choose unit of mass such that $2\alpha' = 1$ and let us define:

$$Q^{(0)}_\mu(z) = q_\mu - i p_\mu \log z \tag{IV.5.4a}$$

$$Q^{(+)}_\mu(z) = -i \sum_1^\infty \frac{1}{\sqrt{n}} a^+_{n\mu} z^n \tag{IV.5.4b}$$

$$Q^{(-)}_\mu(z) = i \sum_1^\infty \frac{1}{\sqrt{n}} a_{n\mu} z^{-n} \tag{IV.5.4c}$$

and

$$P^{(-)}_\mu(z) = -i \sum_1^\infty \sqrt{n}\, a_{n\mu} z^{-n} \tag{IV.5.5a}$$

$$P^{(+)}_\mu(z) = -i \sum_1^\infty \sqrt{n}\, a^+_{n\mu} z^n \tag{IV.5.5b}$$

By using the following commutation relations:

$$\left[P^{(-)}_\mu(z),\, e^{ik_2\cdot Q^+(\zeta)}\right] = -\frac{\zeta}{z}\,\frac{ik_{2\mu}}{(z-\zeta)}\, e^{ik_2\cdot Q^+(\zeta)} \tag{IV.5.6a}$$

$$\left[e^{ik_1\cdot Q^-(z)},\, \frac{P^{(+)}_\nu(\zeta)}{\zeta}\right] = \frac{ik_{1\nu}}{z-\zeta}\, e^{ik_1\cdot Q^-(z)} \tag{IV.5.6b}$$

one has the following result about the product of two photon vertices:

$$\begin{aligned}
&\frac{\epsilon_1^\mu P_\mu(z)}{z}\, e^{ik_1\cdot Q(z)}\, \frac{\epsilon_2^\nu P_\nu(\zeta)}{\zeta}\, e^{ik_2\cdot Q(\zeta)} = \\
&= e^{i[k_1 Q^+(z) + k_2 Q^+(\zeta)]}\, e^{i(k_1+k_2)\cdot q}\, \epsilon_1^\mu \left[-\frac{ip_\mu}{z} + \frac{P^{(+)}_\mu(z)}{z} + \frac{P^{(-)}_\mu(z)}{z} + \right. \\
&\left. -\frac{ik_{2\mu}}{z-\zeta}\right] \cdot (z-\zeta)^{k_1\cdot k_2}\, z^{p\cdot k_1}\, \zeta^{p\cdot k_2}\, \epsilon_2^\nu \left[-\frac{ip_\nu}{\zeta} + \frac{P^{(+)}_\nu(\zeta)}{\zeta} + \frac{P^{(-)}_\nu(\zeta)}{\zeta} + \right. \\
&\left. + \frac{ik_{1\nu}}{z-\zeta}\right] \cdot e^{i[k_1 Q^-(z) + k_2 Q^-(\zeta)]}
\end{aligned} \tag{IV.5.7}$$

Using this result we get finally the following expression for the amplitude:

$$
\begin{aligned}
&\epsilon_1^{\mu}\,\epsilon_2^{\nu}\,\epsilon_3^{\rho}\,\epsilon_4^{\sigma}\Bigg\{\frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t)}K_{2\mu}K_{1\rho}K_{4\sigma}K_{3\nu}+\frac{\Gamma(-\alpha_s+1)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t+1)}\big(-g_{\mu\nu}K_{1\rho}K_{4\sigma}+\\
&+K_{2\mu}K_{1\rho}K_{4\sigma}K_{2\nu}+K_{2\mu}K_{1\rho}K_{3\nu}K_{1\sigma}+K_{2\mu}K_{4\rho}K_{2\sigma}K_{3\nu}+K_{3\mu}K_{1\rho}K_{4\sigma}K_{3\nu}-g_{\rho\sigma}K_{2\mu}K_{3\nu}\big)+\\
&+\frac{\Gamma(-\alpha_s+2)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t+2)}\big(K_{2\mu}K_{1\rho}K_{1\sigma}K_{2\nu}+K_{2\mu}K_{4\rho}K_{2\sigma}K_{2\nu}+K_{2\mu}K_{4\rho}K_{3\nu}K_{1\sigma}+\\
&+K_{3\mu}K_{1\rho}K_{4\sigma}K_{2\nu}+K_{3\mu}K_{3\nu}K_{1\rho}K_{1\sigma}+K_{3\mu}K_{4\rho}K_{4\sigma}K_{3\nu}-g_{\mu\nu}(K_{1\rho}K_{1\sigma}+K_{4\rho}K_{4\sigma})+\\
&-g_{\rho\sigma}(K_{2\mu}K_{2\nu}+K_{3\mu}K_{3\nu})+g_{\mu\nu}g_{\rho\sigma})\big)+\frac{\Gamma(-\alpha_s+3)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t+3)}\big(K_{2\mu}K_{4\rho}K_{1\sigma}K_{2\nu}+\\
&+K_{3\mu}K_{1\rho}K_{1\sigma}K_{2\nu}+K_{3\mu}K_{4\rho}K_{4\sigma}K_{2\nu}+K_{3\mu}K_{4\rho}K_{3\nu}K_{1\sigma}-g_{\mu\nu}K_{4\rho}K_{1\sigma}+\\
&-g_{\rho\sigma}K_{3\mu}K_{2\nu}\big)+\frac{\Gamma(-\alpha_s+4)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t+4)}K_{3\mu}K_{4\rho}K_{1\sigma}K_{2\nu}+\frac{\Gamma(-\alpha_s+1)\Gamma(-\alpha_t+1)}{\Gamma(-\alpha_s-\alpha_t+2)}\big(g_{\mu\sigma}K_{1\rho}K_{3\nu}+\\
&-g_{\mu\rho}(K_{1\sigma}K_{3\nu}+K_{4\sigma}K_{1\nu})+g_{\rho\nu}K_{2\mu}K_{4\sigma}-g_{\sigma\nu}(K_{2\mu}K_{4\rho}+K_{3\mu}K_{1\rho})\big)+\\
&+\frac{\Gamma(-\alpha_s+2)\Gamma(-\alpha_t+1)}{\Gamma(-\alpha_s-\alpha_t+3)}\big(g_{\mu\sigma}(K_{1\rho}K_{2\nu}+K_{4\rho}K_{3\nu})-g_{\mu\rho}K_{1\sigma}K_{2\nu}+g_{\rho\nu}(K_{2\mu}K_{1\sigma}+\\
&+K_{3\mu}K_{4\sigma})-g_{\sigma\nu}K_{3\mu}K_{4\rho}\big)+\frac{\Gamma(-\alpha_s+3)\Gamma(-\alpha_t+1)}{\Gamma(-\alpha_s-\alpha_t+4)}\big(g_{\mu\sigma}K_{4\rho}K_{2\nu}+g_{\rho\nu}K_{3\mu}K_{1\sigma}\big)+\\
&+\frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t+1)}{\Gamma(-\alpha_s-\alpha_t+1)}\big(-g_{\mu\rho}K_{4\sigma}K_{3\nu}-g_{\sigma\nu}K_{2\mu}K_{1\rho}\big)+\\
&+\frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t+2)}{\Gamma(-\alpha_s-\alpha_t+2)}g_{\mu\rho}g_{\sigma\nu}+\frac{\Gamma(-\alpha_s+2)\Gamma(-\alpha_t+2)}{\Gamma(-\alpha_s-\alpha_t+4)}g_{\mu\sigma}g_{\rho\nu}\Bigg\}\qquad\text{(IV.5.8)}
\end{aligned}
$$

For this amplitude we have performed a limit $\alpha' \to 0$; in this limit one would find the corresponding amplitude in Yang-Mills theories, after having introduced a Chan-Paton factor. In this limit the following terms survive:

$$
\begin{aligned}
& g_{\mu\nu} k_{1\rho} k_{4\sigma}\left(\frac{1}{t}+\frac{1}{s}\right) + g_{\rho\sigma}\left(\frac{1}{t}+\frac{1}{s}\right) k_{2\mu} k_{3\nu} - \left(1+\frac{s}{t}\right) g_{\mu\nu} g_{\rho\sigma} + \\
& - g_{\mu\nu} k_{4\rho} k_{1\sigma}\frac{1}{t} - g_{\rho\sigma} k_{3\mu} k_{2\nu}\frac{1}{t} - \left(\frac{1}{t}+\frac{1}{s}\right) g_{\mu\sigma} k_{1\rho} k_{3\nu} + \\
& + \left(\frac{1}{t}+\frac{1}{s}\right) g_{\mu\rho}\left(k_{1\sigma} k_{3\nu} + k_{4\sigma} k_{2\nu}\right) - \left(\frac{1}{t}+\frac{1}{s}\right) g_{\rho\nu} k_{2\mu} k_{4\sigma} + \\
& + \left(\frac{1}{t}+\frac{1}{s}\right) g_{\sigma\nu}\left(k_{2\mu} k_{4\rho} + k_{3\mu} k_{1\rho}\right) - g_{\mu\sigma}\left(k_{1\rho} k_{2\nu} + k_{4\rho} k_{3\nu}\right)\frac{1}{t} + \\
& + g_{\mu\rho} k_{1\sigma} k_{2\nu}\frac{1}{t} - g_{\rho\nu}\left(k_{2\mu} k_{1\sigma} + k_{3\mu} k_{4\sigma}\right)\frac{1}{t} + \\
& + g_{\sigma\nu} k_{3\mu} k_{4\rho}\frac{1}{t} - \frac{1}{t} g_{\mu\sigma} k_{4\rho} k_{2\nu} - \frac{1}{t} g_{\rho\nu} k_{3\mu} k_{1\sigma} + \\
& + \left(\frac{1}{t}+\frac{1}{s}\right) g_{\mu\rho} k_{4\sigma} k_{3\nu} + \left(\frac{1}{t}+\frac{1}{s}\right) g_{\sigma\nu} k_{2\mu} k_{1\rho} + \\
& - \left(1+\frac{t}{s}\right) g_{\mu\rho} g_{\sigma\nu} + g_{\mu\sigma} g_{\rho\nu}
\end{aligned}
\qquad \text{(IV.5.9)}
$$

Symmetrizing this expression leads to a complete vanishing of the amplitude, as it must be, since in the limit $\alpha' \to 0$ and without introducing some other internal symmetry, (IV.5.9) corresponds to the tree scattering amplitude of four photons. Let us introduce now a Chan-Paton factor:

$$\mathrm{Tr}\,(\lambda_a \lambda_b \lambda_c \lambda_d)$$

and let us concentrate, for the sake of simplicity, on the terms coinvolving two metric tensors. After having symmetrized also on theindices a, b, c and d one has:

$$
\begin{aligned}
& \left(g_{\mu\sigma} g_{\rho\nu} - \left(1+\frac{t}{s}\right) g_{\mu\rho} g_{\sigma\nu} - \left(1+\frac{s}{t}\right) g_{\mu\nu} g_{\rho\sigma}\right) \mathrm{Tr}\,(\lambda_a \lambda_b \lambda_c \lambda_d) + \\
& + \left(g_{\mu\rho} g_{\sigma\nu} - \left(1+\frac{t}{u}\right) g_{\mu\sigma} g_{\rho\nu} - \left(1+\frac{u}{t}\right) g_{\mu\nu} g_{\rho\sigma}\right) \mathrm{Tr}\,(\lambda_a \lambda_b \lambda_c \lambda_d) +
\end{aligned}
$$

$$+\left(g_{\mu\nu}g_{\rho\sigma}-\left(1+\frac{u}{s}\right)g_{\mu\rho}g_{\nu\sigma}-\left(1+\frac{s}{u}\right)g_{\mu\sigma}g_{\rho\nu}\right)\cdot \mathrm{Tr}(\lambda_a\lambda_b\lambda_c\lambda_d) \tag{IV.5.10}$$

For example, (IV.5.10) can be computed considering an internal symmetry SU(2). In this case one has:

$$\begin{aligned}&g^{\mu\rho}g^{\sigma\nu}\left[(-2\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{cb})+\frac{u-t}{s}(\delta_{ad}\delta_{bc}-\delta_{ac}\delta_{bd})\right]+\\&+g^{\mu\nu}g^{\rho\sigma}\left[(-2\delta_{ad}\delta_{bc}+\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd})+\frac{u-s}{t}(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd})\right]+\\&+g^{\mu\sigma}g^{\rho\nu}\left[(-2\delta_{ac}\delta_{bd}+\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc})+\frac{s-t}{u}(\delta_{ad}\delta_{bd}-\delta_{ab}\delta_{cd})\right]\end{aligned} \tag{IV.5.11}$$

On the other hand, in a Yang-Mills theory the invariant amplitude for four gluons is given by the sum of the following terms:

$$M_t=-g^2 f_{aed}f_{ebc}\,C^{\lambda\tau\sigma}(-q_1,q_1-q_4,q_4)\,\frac{g_{\tau\tau'}}{t}\cdot C^{\tau'\mu\nu}(q_2-q_3,-q_2,q_3) \tag{IV.5.12a}$$

$$M_u=-g^2 f_{aec}f_{ebd}\,C^{\lambda\tau\nu}(-q_1,q_1-q_3,q_3)\,\frac{g_{\tau\tau'}}{u}\cdot C^{\tau'\mu\sigma}(q_2-q_4,-q_2,q_4) \tag{IV.5.12b}$$

$$M_s=-g^2 f_{abe}f_{ecd}\,C^{\lambda\mu\tau}(-q_1,-q_2,q_1+q_2)\,\frac{g_{\tau\tau'}}{s}\cdot C^{\tau'\nu\sigma}(-q_3-q_4,q_3,q_4) \tag{IV.5.12c}$$

$$\begin{aligned}M_4=-g^2\,[&f_{abe}f_{cde}(g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\sigma}g^{\mu\nu})+\\&+f_{ace}f_{bde}(g^{\lambda\mu}g^{\nu\sigma}-g^{\lambda\sigma}g^{\mu\nu})+f_{ade}f_{cbe}(g^{\lambda\nu}g^{\mu\sigma}-g^{\lambda\mu}g^{\sigma\nu})]\end{aligned} \tag{IV.5.12d}$$

where

$$C^{\mu\lambda\nu}(q_1,q_2,q_3)=\left[(q_1-q_2)^\nu g^{\mu\lambda}+(q_2-q_3)^\mu g^{\lambda\nu}+(q_3-q_1)^\lambda g^{\mu\nu}\right] \tag{IV.5.13}$$

Taking, for the sake of simplicity, the expressions (IV.5.12) for SU(2) and isolating the terms coinvolving two metric tensors, it is possible to see that these terms give exactly (IV.5.11). Moreover it has been checked that also the complete amplitudes coincide.

The plain of our work foresees the extension of these computation techniques to other examples still involving bosonic strings, as, for example, the amplitude scattering between two photons and two gravitons, quite ultimated, but our final aim is essentially to apply this analysis to the spinning strings and superstrings. It will be also interesting to compare the amplitudes so obtained with the ones computed through effective action tecniques, following the works by E.S. Fradkin and A.A. Tseytlin .

REFERENCES

1. L. Brink, P. Di Vecchia and P. Howe, Phys. Lett. 65B (1976), 471.
2. T. Goto, Progr. Theor. Phys. 46 (1971), 1560.
3. P. Goddard, J. Goldstone, C. Rebbi and C . Thorn, Nucl. Phys. B56 (1973), 109.
4. J. Scherk, Rev. Mod. Phys. 47 (1975), 123.
5. L. Brink and H. B. Nielsen, Phys. Lett. 45B (1973), 332.
6. F. Gliozzi, unpublished.
   P. Di Vecchia in "Many Degrees of Freedom in Particle Physics", ed. by H. Satz (1978), 493.
7. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B224 (1984), 333.
8. M.A. Virasoro, Phys. Rev. D1 (1970), 2933.
9. E. Del Giudice and P. Di Vecchia, Nuovo Cimento 70A (1970), 579.
10. P. Goddard and C.B. Thorn, Phys. Lett. 40B (1972),235.
11. R.C. Brower, PHYS. Rev. D6 (1972), 1655.
12. E. Del Giudice, P. Di Vecchia and S. Fubini, Ann. Phys. 70 (1972), 378.
13. M. Ademollo, A. D'Adda, R. D'Auria, P. Di Vecchia, F. Gliozzi, R; Musto, E. Napolitano, F. Nicodemi and S. Sciuto, Nuovo Cimento 21A (1974), 77.
14. P. Campagna, S. Fubini, E. Napolitano and S. Sciuto, Nuovo Cimento 2A (1971), 911.
15. S. Weinberg, University of Texas, preprint UTTG-5-85 (1985).
16. R. Brower and P. Goddard, Nucl. Phys. B40 (1972), 437.
17. S. Fubini and G. Veneziano, Nuovo Cimento 67A, (1969), 29.
18. M. Ademollo, E. Del Giudice, P. Di Vecchia and S. Fubini, Nuovo Cimento 19A (1974), 181.

19. E. Cremmer and J.L. Gervais, Nucl. Phys. B76 (1974), 209.

20. S. Mendelstam, Nucl. Phys. B83 (1974), 413.

21. Z. Koba and H.B. Nielsen, Nucl. Phys. B10 (1969), 633.

22. J. Paton and Chan Hong-Mo, Nucl. Phys. B10 (1969), 519.

23. M. Ademollo, A. D'Adda, R. D'auria, E. Napolitano, P. Di Vecchia, F. Gliozzi and S. Sciuto, Nucl. Phys. B77 (1974), 189.

24. M. Ademollo, A. D'Adda, R. D'auria, F. Gliozzi, E. Napolitano, S. Sciuto and P. Di Vecchia, Nucl. Phys. B94 (1975) 221.

25. G. Veneziano, Nuovo Cimento 57A (1968), 190.

26. M.A. Virasoro, Phys. Rev. 177 (1969), 2309.

27. J. Shapiro, Phys. Lett. 33B (1970), 361.

28. S. Fubini and G. Veneziano, Nuovo Cimento 67A, 29.

29. E.S. Fradkin and A.A. Tseytlin, Phys. Lett. 158B (1985), 316.

30. E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B261 (1985), 27.

31. A.A. Tseytlin, Academy of Sciences of the USSR, preprint 95/86, (1986).